Basic Data Structures: Dynamic Arrays and Amortized Analysis

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Data Structures
Data Structures and Algorithms

Outline

Problem: static arrays are static!

```
int my_array[100];
```

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Semi-solution: dynamically-allocated arrays:

```
int *my_array = new int[size];
```

Problem: might not know max size when allocating an array

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Solution: *dynamic arrays* (also known as *resizable arrays*)
Idea: store a pointer to a dynamically allocated array, and replace it with a newly-allocated array as needed.

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

---

*must be constant time

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- `Get`(i): returns element at location i*
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- PushBack(val): Adds val to the end

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- `Get`(i): returns element at location i*
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- PushBack(val): Adds val to the end
- `Remove`(i): Removes element at location i

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- `Get(`i`)`: returns element at location i^*
- `Set(`i, val`)`: Sets element i to val^*
- PushBack(val): Adds val to the end
- `Remove(`i`)`: Removes element at location i
- `Size()`: the number of elements

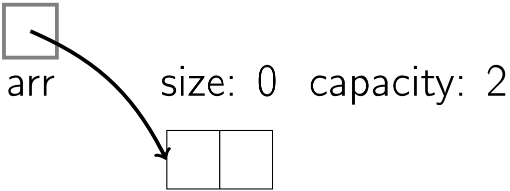
*must be constant time

Implementation

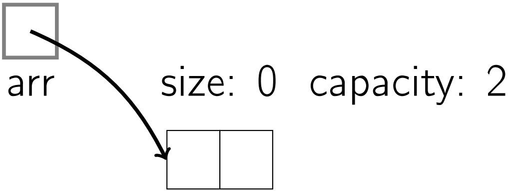
Store:

- `arr`: dynamically-allocated array
- `capacity`: size of the dynamically-allocated array
- `size`: number of elements currently in the array

Dynamic Array Resizing

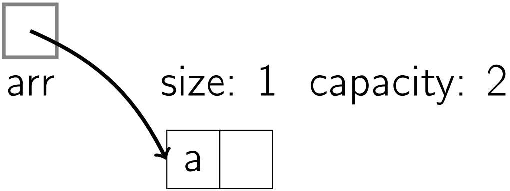


Dynamic Array Resizing



`PushBack(a)`

Dynamic Array Resizing



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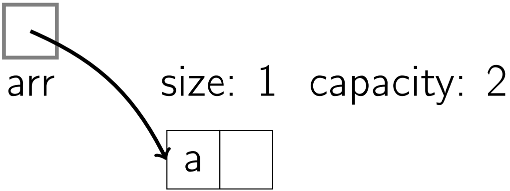

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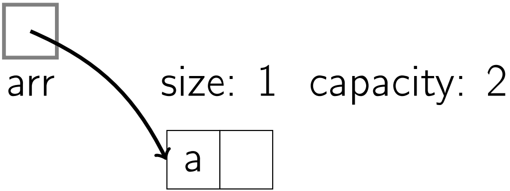
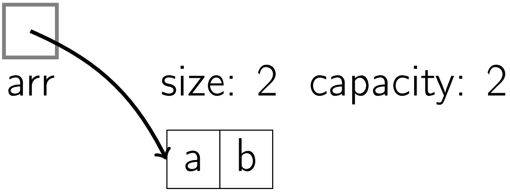


`PushBack(b)`

Dynamic Array Resizing



`PushBack(b)`

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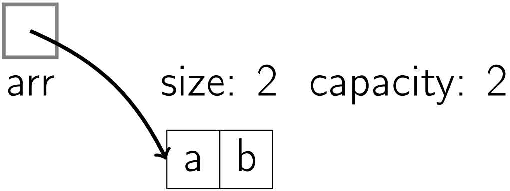

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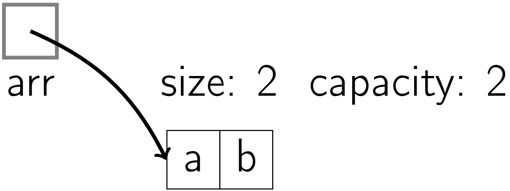
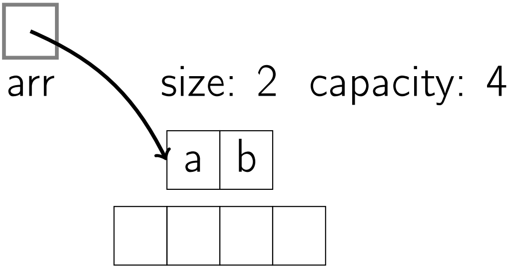


`PushBack(c)`

Dynamic Array Resizing



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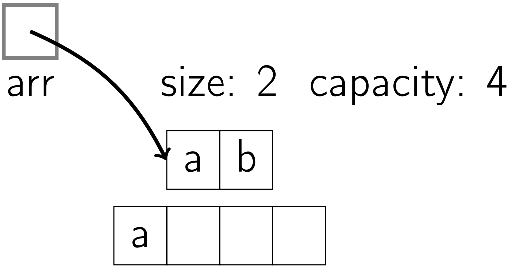

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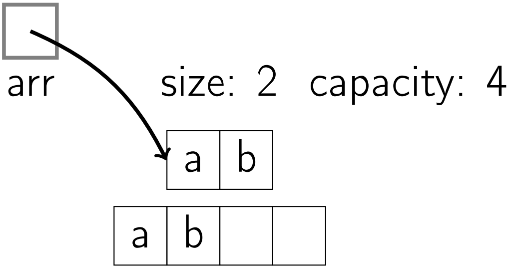

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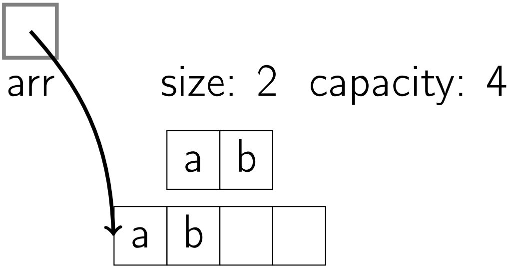

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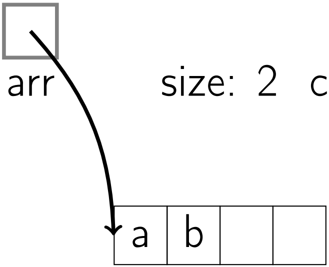
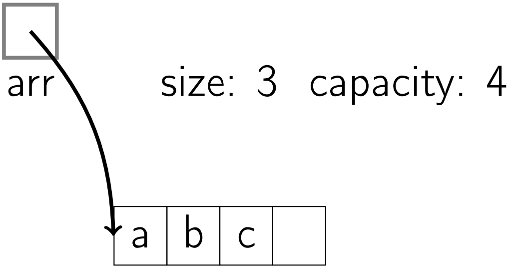


size: 2 capacity: 4

`PushBack(c)`

Dynamic Array Resizing



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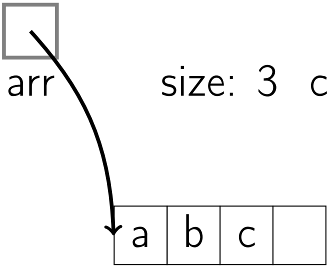
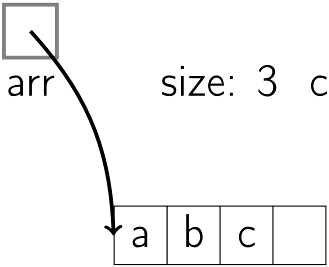


size: 3 capacity: 4

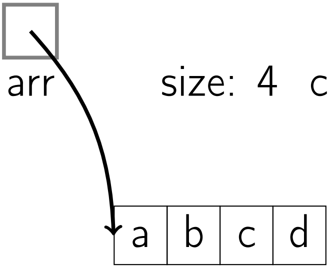
Dynamic Array Resizing



size: 3 capacity: 4

`PushBack(d)`

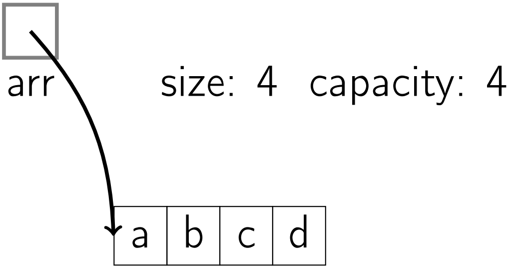
Dynamic Array Resizing



size: 4 capacity: 4

`PushBack(d)`

Dynamic Array Resizing



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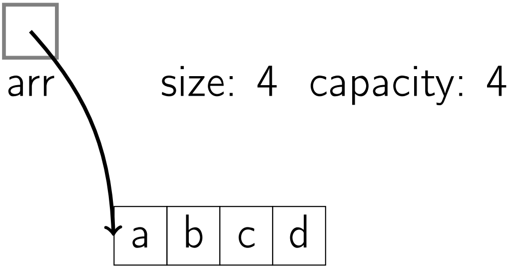
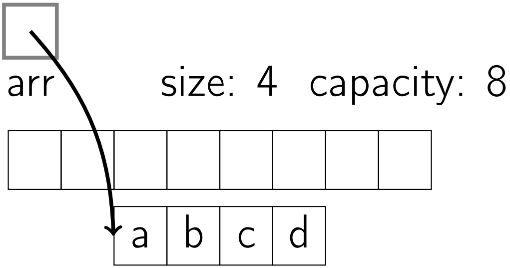


`PushBack(e)`

Dynamic Array Resizing



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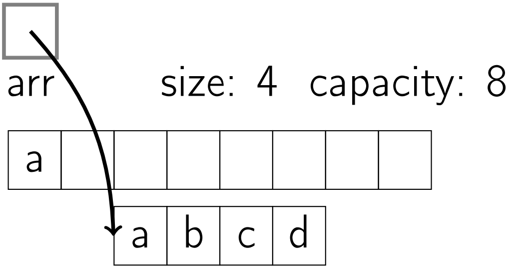
Dynamic Array Resizing



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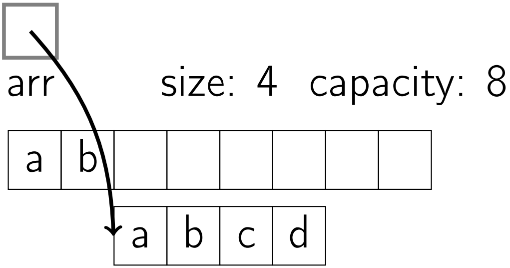

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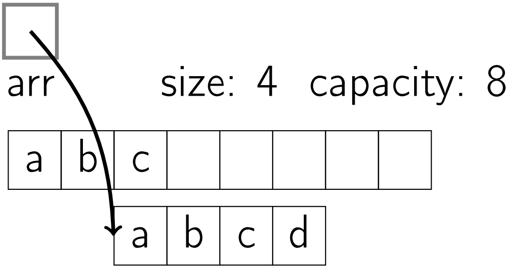

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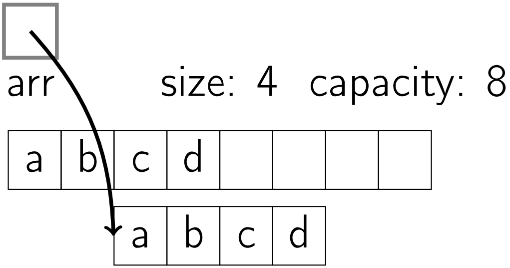

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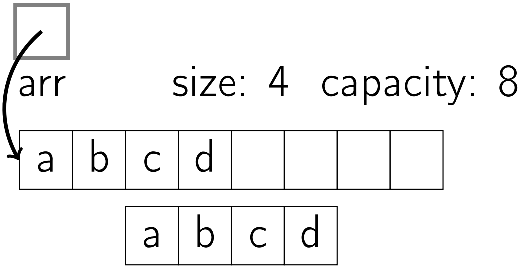

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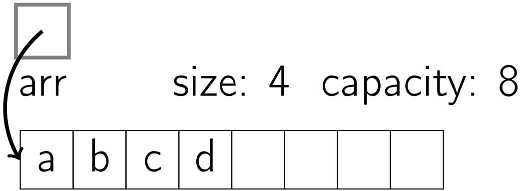

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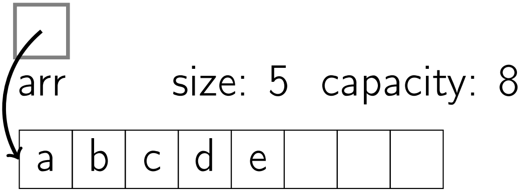

`PushBack(e)`

Get(i)

```
if i < 0 or i ≥ size:
   ERROR: index out of range
return arr[i]
```

$\texttt{Set}(i, val)$

```
if i < 0 or i ≥ size:
  ERROR: index out of range
arr[i] = val
```

PushBack(val)

```
if size = capacity:
    allocate new_arr[2 × capacity]
    for i from 0 to size − 1:
        new_arr[i] ← arr[i]
    free arr
    arr ← new_arr; capacity ← 2 × capacity
arr[size] ← val
size ← size + 1
```

Remove(i)

```
if i < 0 or i ≥ size:
  ERROR: index out of range
for j from i to size − 2:
  arr[j] ← arr[j + 1]
size ← size − 1
```

Size()

```
return size
```

Common Implementations

- **C++**: `vector`
- **Java**: `ArrayList`
- **Python**: `list` (the only kind of array)

Runtimes

$\texttt{Get}(i)$	$O(1)$

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Runtimes

$\texttt{Get}(i)$	$O(1)$
$\texttt{Set}(i, val)$	$O(1)$
$\texttt{PushBack}(val)$	$O(n)$
$\texttt{Remove}(i)$	$O(n)$
$\texttt{Size}()$	$O(1)$

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- Unlike static arrays, dynamic arrays can be resized.
- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.
- Some space is wasted—at most half.

Outline

1. Dynamic Arrays
2. Amortized Analysis—Aggregate Method
3. Amortized Analysis—Banker's Method
4. Amortized Analysis—Physicist's Method

Sometimes, looking at the individual worst-case may be too severe. We may want to know the total worst-case cost for a sequence of operations.

Dynamic Array

We only resize every so often.
Many $O(1)$ operations are followed by an $O(n)$ operations.
What is the total cost of inserting many elements?

Definition

Amortized cost: Given a sequence of n operations, the amortized cost is:

$$\frac{\text{Cost}(n \text{ operations})}{n}$$

Aggregate Method

Dynamic array: n calls to `PushBack`

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$$c_i = 1 + \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

Aggregate Method

Dynamic array: n calls to PushBack

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i-1 & \text{if } i-1 \text{ is a power of 2} \\ & \end{cases}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$$

Aggregate Method

Dynamic array: n calls to PushBack

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$$\frac{\sum_{i=1}^{n} c_i}{n}$$

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$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n}$$

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Banker's Method

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Like an amortizing loan.

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Charge 3 for each insertion: 1 token is the raw cost for insertion.

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- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.

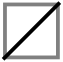
Banker's Method

Dynamic array: n calls to `PushBack`

Charge 3 for each insertion: 1 token is the raw cost for insertion.

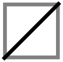
- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.
- Place one token on the newly-inserted element, and one token $\frac{capacity}{2}$ elements prior.

Dynamic Array Resizing



arr   size: 0   capacity: 0

Dynamic Array Resizing



arr size: 0 capacity: 0

PushBack(a)

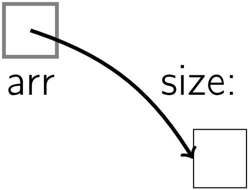
Dynamic Array Resizing

arr

size: 0 capacity: 1

PushBack(a)

Dynamic Array Resizing



size: 0 capacity: 1

PushBack(a)

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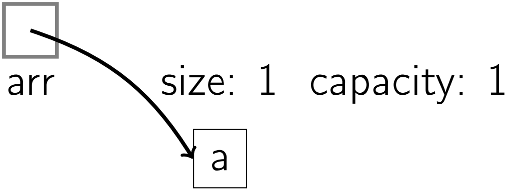

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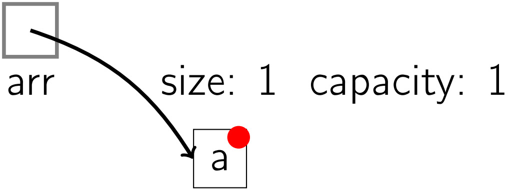

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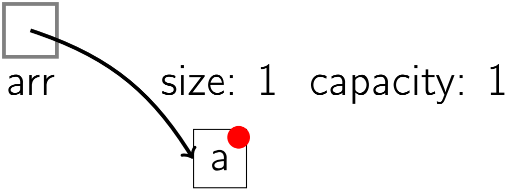

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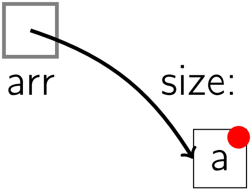
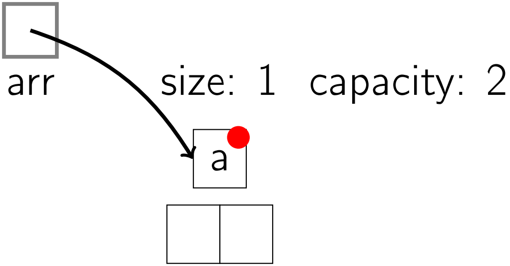


size: 1 capacity: 1

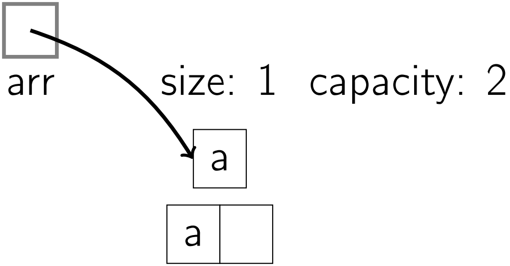
PushBack(b)

Dynamic Array Resizing



PushBack(b)

Dynamic Array Resizing



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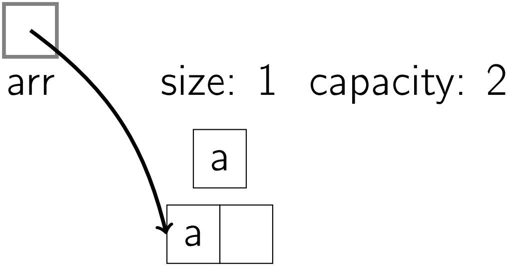

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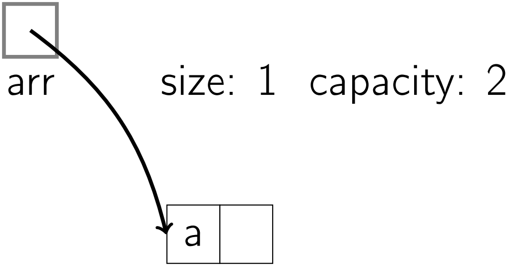

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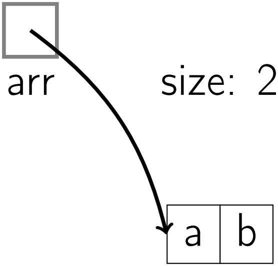
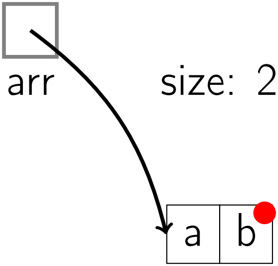


size: 2 capacity: 2

PushBack(b)

Dynamic Array Resizing



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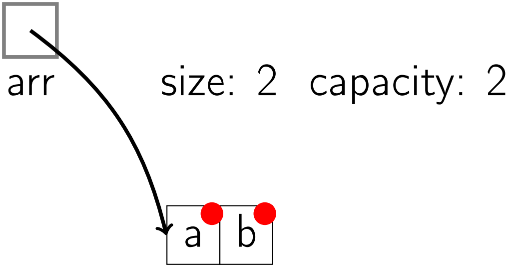

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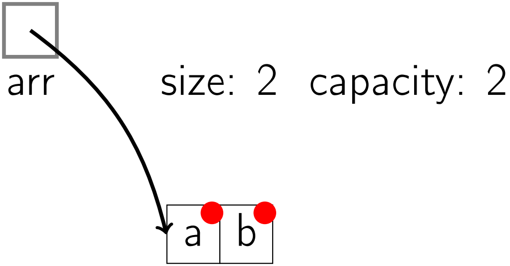

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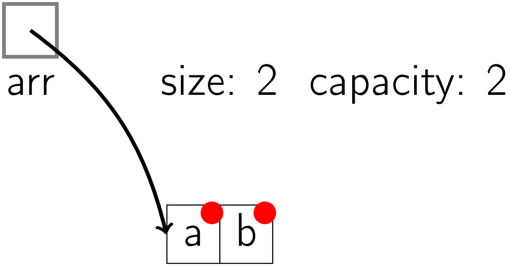

PushBack(c)

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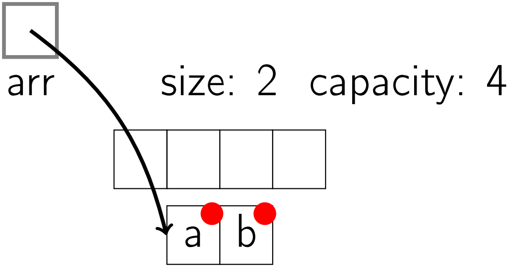

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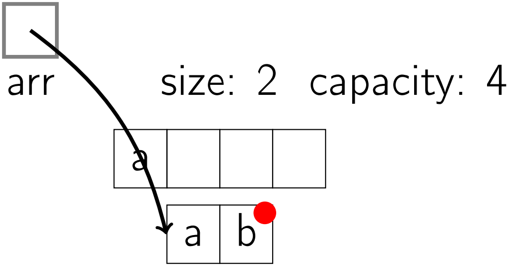
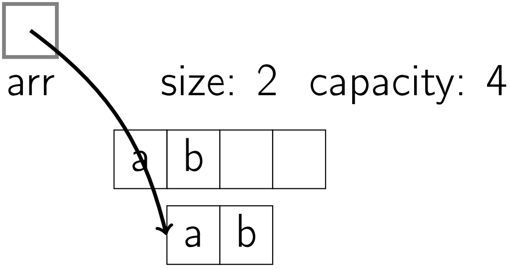


PushBack(c)

Dynamic Array Resizing



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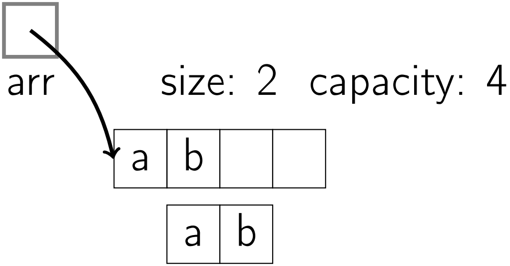

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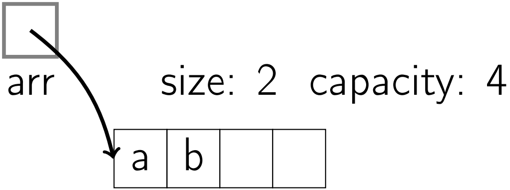

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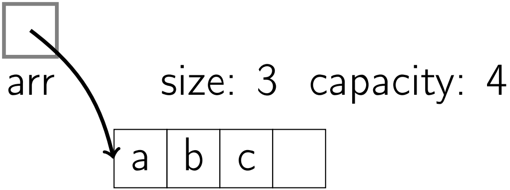

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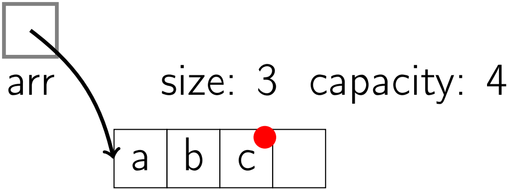

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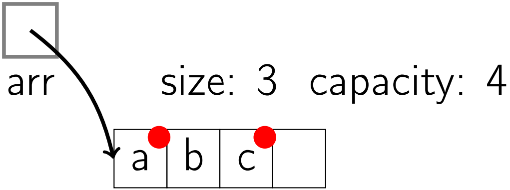

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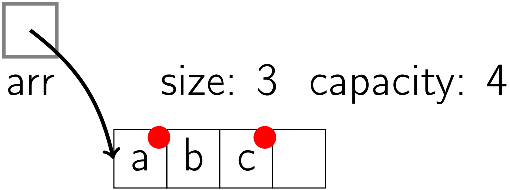

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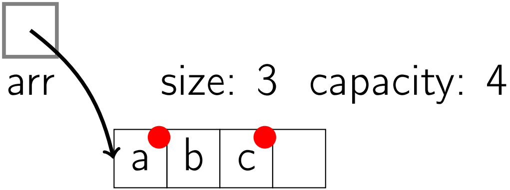
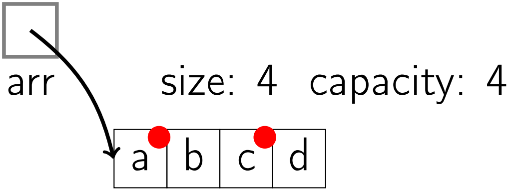


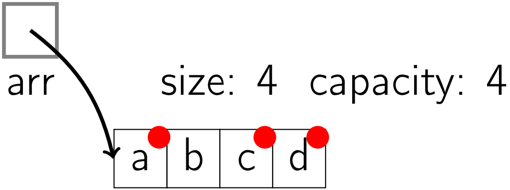
PushBack(d)

Dynamic Array Resizing



PushBack(d)

Dynamic Array Resizing



PushBack(d)

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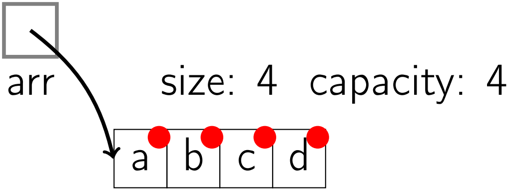

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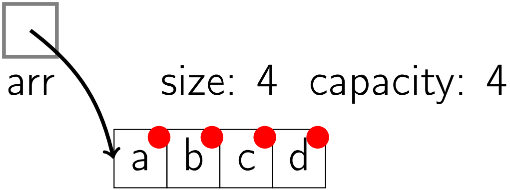

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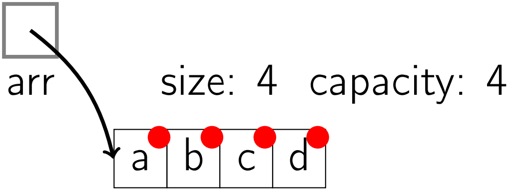

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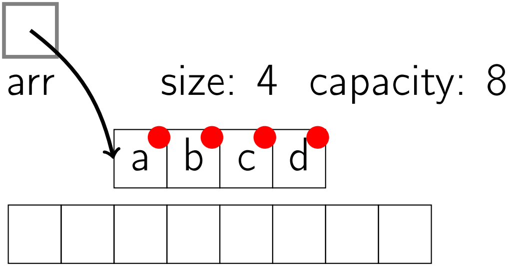

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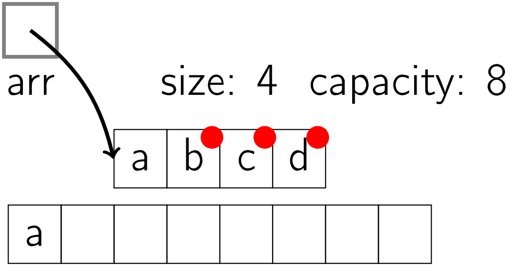

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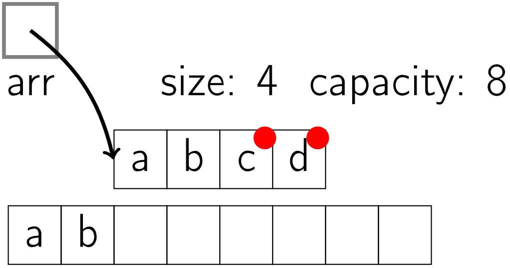

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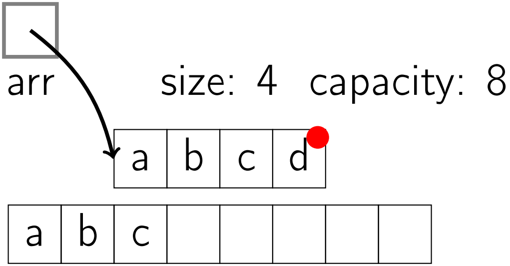

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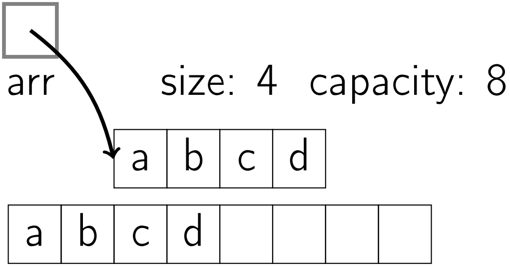

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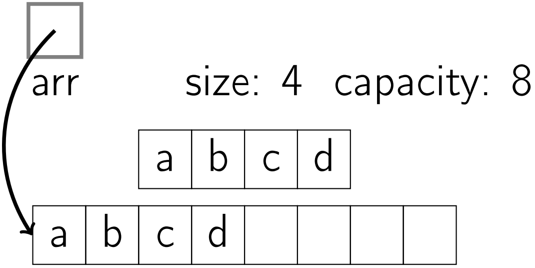

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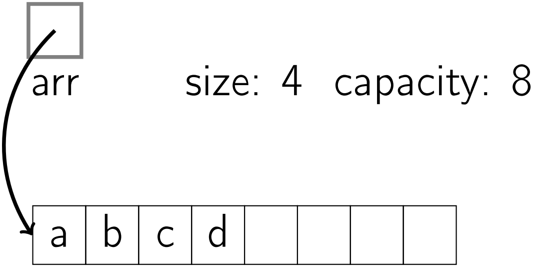

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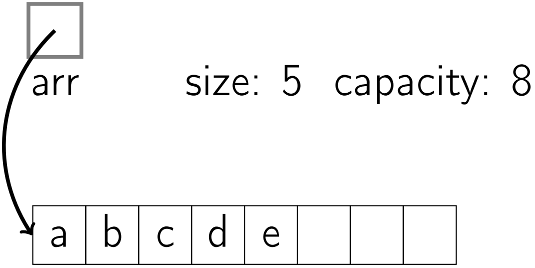

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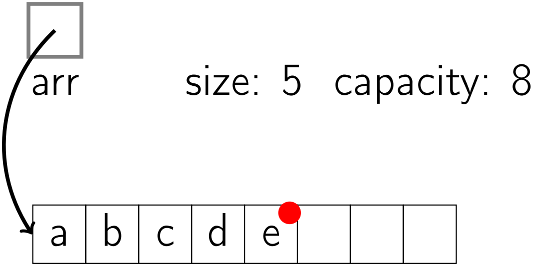

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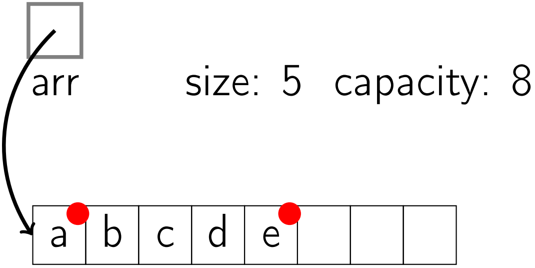
Dynamic Array Resizing



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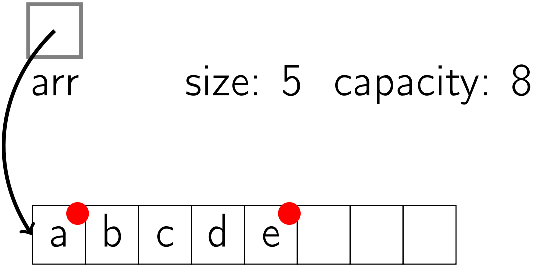

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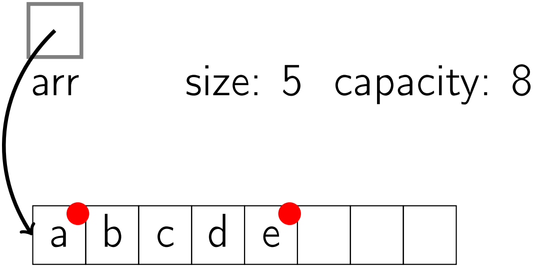

$O(1)$ amortized cost for each `PushBack`

Banker's Method

Dynamic array: n calls to `PushBack`

Charge 3 for each insertion. 1 coin is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the coin that's present on each element that needs to move.
- Place one coin on the newly-inserted element, and one coin $\frac{capacity}{2}$ elements prior.

Outline

1. Dynamic Arrays
2. Amortized Analysis—Aggregate Method
3. Amortized Analysis—Banker's Method
4. Amortized Analysis—Physicist's Method

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:

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Choose Φ so that:

- if c_t is small, the potential increases
- if c_t is large, the potential decreases by the same scale

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- The sum of the amortized costs is:

$$
\begin{aligned}
&\sum_{i=1}^{n}(c_i + \Phi(h_i) - \Phi(h_{i-1})) \\
=&c_1 + \Phi(h_1) - \Phi(h_0) + \\
&c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\
&c_n + \Phi(h_n) - \Phi(h_{n-1})
\end{aligned}
$$

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&c_n + \Phi(h_n) - \Phi(h_{n-1}) \\
=&\Phi(h_n) - \Phi(h_0) + \sum_{i=1}^{n} c_i
\end{aligned}$$

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- The sum of the amortized costs is:

$$\sum_{i=1}^{n} (c_i + \Phi(h_i) - \Phi(h_{i-1}))$$

$$= c_1 + \Phi(h_1) - \Phi(h_0) +$$

$$c_2 + \Phi(h_2) - \Phi(h_1) \cdots +$$

$$c_n + \Phi(h_n) - \Phi(h_{n-1})$$

$$= \Phi(h_n) - \Phi(h_0) + \sum_{i=1}^{n} c_i \geq \sum_{i=1}^{n} c_i$$

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Dynamic array: n calls to `PushBack`

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Dynamic array: n calls to PushBack

Let $\Phi(h) = 2 \times \textit{size} - \textit{capacity}$

- $\Phi(h_0) = 2 \times 0 - 0 = 0$
- $\Phi(h_i) = 2 \times \textit{size} - \textit{capacity} > 0$ (since $\textit{size} > \frac{\textit{capacity}}{2}$)

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

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Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$

$$= 1 + 2 \times size_i - cap_i - (2 \times size_{i-1} - cap_{i-1})$$

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

$$
\begin{aligned}
& c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
=& 1 + 2 \times \mathit{size}_i - \mathit{cap}_i - (2 \times \mathit{size}_{i-1} - \mathit{cap}_{i-1}) \\
=& 1 + 2 \times (\mathit{size}_i - \mathit{size}_{i-1})
\end{aligned}
$$

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

$$
\begin{aligned}
& c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
= & 1 + 2 \times size_i - cap_i - (2 \times size_{i-1} - cap_{i-1}) \\
= & 1 + 2 \times (size_i - size_{i-1}) \\
= & 3
\end{aligned}
$$

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Then:

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$\Phi(h_{i-1}) = 2size_{i-1} - cap_{i-1} = 2k - k = k$

$\Phi(h_i) = 2size_i - cap_i = 2(k+1) - 2k = 2$

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Amortized cost of adding element i:

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Amortized cost of adding element i:

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$
$$=(size_i) + 2 - k$$

Dynamic Array Resizing

With resize when adding element i

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\begin{aligned}
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Amortized cost of adding element i:

$$
\begin{aligned}
& c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
=& (size_i) + 2 - k \\
=& (k+1) + 2 - k \\
=& 3
\end{aligned}
$$

Alternatives to Doubling the Array Size

We could use some different growth factor (1.5, 2.5, etc.).
Could we use a constant amount?

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i'th insertion.

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If we expand by 10 each time, then:

Let $c_i =$ cost of i'th insertion.

$$c_i = 1 + \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let $c_i =$ cost of i'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ & \end{cases}$$

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If we expand by 10 each time, then:

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

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$$\frac{\sum_{i=1}^{n} c_i}{n}$$

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$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n}$$

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$$= \frac{n + 10O(n^2)}{n}$$

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$$= \frac{n + 10O(n^2)}{n} = \frac{O(n^2)}{n} = O(n)$$

Summary

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- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:
  - Aggregate method (brute-force sum)
  - Banker's method (tokens)
  - Physicist's method (potential function, Φ)
- Nothing changes in the code: runtime analysis only.