A TUTORIAL ON OPTIMAL CONTROL THEORY

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ABSTRACT

Management science applications frequently involve problems of controlling continuous time dynamic systems, that is, systems which evolve over time. Optimal control theory, a relatively new branch of mathematics, determines the optimal way to control such a dynamic system. The purpose of this tutorial paper is to provide an elementary introduction to optimal control theory and to illustrate it by formulating a simple example. A reader who has covered this tutorial should be able to read most of this special issue of *INFOR* which contains articles applying optimal control theory to the solution of management science problems.

RÉSUMÉ

Les applications de la science de gestion comportent fréquemment des problèmes touchants au contrôle des systèmes dynamiques dans un temps continu, c'est à dire, dans des systèmes qui se déroulent avec le temps. La théorie de la commande optima, une nouvelle branche des mathématiques, détermine les moyens optima pour contrôler un tel système dynamique. Le but de cet article, un travail pratique, est à fournier au lecteur une introduction élémentaire à la théorie de la commande optimal et de l'illustrer en formulant un exemple simple. Le lecteur qui aura lu ce travail pratique devait pouvais lire dans cette parution de INFOR la plupart des articles qui appliquent la theorie de la command optima à la solution des problèmes de la science de gestion.

1 INTRODUCTION

Many management science applications involve the control of dynamic systems; that is, systems that evolve over time. They are called *continuous time systems* or *discrete time systems*, depending on whether the time variable t varies continuously or discretely. We shall deal only with continuous time systems in this paper.

Optimal control theory is a relatively new branch of mathematics developed to find the optimal way to control a dynamic system. The purpose of this paper is to give an elementary introduction to the mathematical theory, so that readers can absorb the papers in this issue which apply control theory to solve problems which arise in different fields of application. We have deliberately kept the level of mathematics

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as elementary as possible in order to make the paper accessible to a large audience. The only mathematical background needed to read this paper are calculus, including partial differentiation, some knowledge of vectors and matrices, and elementary differential equations.

The principal management science applications discussed by the authors of the papers in this special issue of *INFOR* are to the areas of finance, marketing and production. In [5] Bensoussan and Lesourne discuss the problem of optimal financing of a firm which is subject to the risk of bankruptcy. The correct balance of the two marketing variables, advertising level and product quality, are the subject of Tapiero's paper.[3]

The remaining papers are all in the area of production. Gruver and Narasimhan[20] study an optimal scheduling problem for multi-stage, continuous flow shops. The last two papers are formulated as stochastic optimal control problems. In [38] Vickson considers the problem of scheduling and controlling randomly drifting production sequences. Finally, Bernard, Haurie, and Missaoui[7] propose a model for managing efficiently a hydro-thermal power system.

There are numerous other applications of optimal control theory to these and other areas of management science and economics, see.[1,2,3,9,13,14,20,27,31,32,34,35]

2 What Is Optimal Control Theory?

We shall use the word *system* as a primitive term in this paper. The only property which we require of a system is that it is capable of existing in various *states*. Let the (real) variable $x(t)$ be the *state variable* of the system at time t. For example, $x(t)$ could measure the inventory level at time t, the amount of advertising goodwill at time t, the amount of unconsumed wealth or natural resources at time t, etc.

We assume that there is a way of controlling the state of the system. Let the (real) variable $u(t)$ be the *control variable* of the system at time t. For example, $u(t)$ could be the production rate at time t, the advertising rate at time t, the consumption rate at time t, etc.

When no confusion can arise, we shall usually suppress the time notation (t); thus, for example, $x(t)$ and $u(t)$ will be written as x and u. Furthermore, whether x denotes the state at time t or the entire state trajectory should be inferred from the context. A similar statement holds for u.

Given the values of the state variable $x(t)$ and the control variable $u(t)$, the *state equation*

$$\dot{x} = f(x, u, t), \; x(0) = x_0 \tag{1}$$

determines the instantaneous rate of change in the state variable, where f is a given function of x, u, and t, and x_0 is the initial value of x. If we know the initial value x_0 and the *control trajectory*, that is, the values of $u(t)$ over the whole time interval $0 \leqslant t \leqslant T$, then we can integrate (1) to get the *state trajectory*, that is, the values of $x(t)$ over the same time interval. We want to choose the control trajectory so that the state and control trajectories maximize the objective function

$$J = \int_0^T F(x, u, t)dt + S[x(T)]. \tag{2}$$

In (2), F is a given function of x, u, and t which could measure the negative of the cost of production, the benefits minus the costs of advertising, the utility of consumption, etc. Also in (2) the function S gives the "salvage value" of the ending state $x(T)$. The salvage value is needed so that the solution will make "good sense" at the end of the problem.

Usually the control variable $u(t)$ will be constrained. We indicate this as

$$u(t) \in \Omega(t), \tag{3}$$

where $\Omega(t)$ is the set of possible values of the control variable at time t.

Finally, we note that (1) and (3), when taken together limit the set of possible terminal values $x(T)$. We denote this by saying

$$x(T) \in X(T), \tag{4}$$

where $X(T)$ is called the *reachable set* of the state variable. Here $X(T)$ is the set of all possible terminal values which can be reached when $x(t)$ obeys (1) and the control variable obeys (3).

3 Formulation of an Advertising Control Model

We now formulate an advertising model as an example of an optimal control problem. The objective here is to identify and interpret for this model each of the variables and functions described above.

We consider a special case of the Nerlove-Arrow advertising model which is summarized in table 1. The problem is to determine the rate at which to advertise a product at each time t. Here the state variable is advertising "goodwill," $G(t)$, which measures how well the product is known. We assume there is a "forgetting" coefficient δ which measures the rate at which customers tend to forget the product. To counteract forgetting and to build new goodwill advertising is carried out at a rate measured by the control variable $u(t)$. Hence the state equation is $\dot{G} = u - \delta G$ with $G(0) = G_0$ giving the initial goodwill for the product.

TABLE 1

ADVERTISING EXAMPLE

State variable	$G(t)$ = Advertising goodwill
Control variable	$u(t)$ = Advertising rate
State equation	$\dot{G}(t) = u(t) - \delta G(t),\ G(0) = G_0$
Objective function	Maximize $\left\{ J = \int_0^\infty [\pi(G) - u]e^{-\rho t} dt \right\}$
Control constraint	$0 \leqslant u \leqslant Q$
Exogeneous function	$\pi(G)$ = Profit rate
Parameters	δ = Goodwill decay constant
	ρ = Discount rate
	Q = Upper bound on advertising rate
	G_0 = Initial goodwill level

The objective function J requires special discussion. Note that the integral defining J is from time $t = 0$ to $t = \infty$; we call a problem with upper time limit of ∞ an *infinite horizon problem*. Because of this upper limit the integrand of the objective function includes the discount factor $e^{-\rho t}$ where ρ is the (constant) discount rate. Without this discount factor the integral would (in most cases) diverge to infinity; hence, such discount factors are usually an essential part of infinite horizon models. The rest of the integrand in the objective function consists of the profit rate $\pi(G)$ which results from goodwill level G less (the cost of) the advertising level which is proportional to u (proportionality factor $= 1$); thus $\pi(G) - u$ is the net profit rate at time t. Also $[\pi(G) - u]e^{-\rho t}$ is the net discounted or present value profit rate. Hence J can be interpreted as the total value of discounted future profits, and is the quantity we are trying to maximize.

Note that there is no state constraint. (It can be seen from the state equation that G in fact never becomes negative.) However there is the control constraint $0 \leqslant u \leqslant Q$ where Q is the upper bound on the advertising level.

In the next section we describe the maximum principle which is required to solve this and other optimal control problems. The complete solution of the advertising problem appears in chapter 6 of.[34]

4 THE MAXIMUM PRINCIPLE

We state without proof the maximum principle, which gives the necessary conditions for optimality of the control problem stated in section 2. This is a problem without state constraints and is summarized in (5).

$$\underset{u(t)\in\Omega(t)}{\text{Maximize}}\left\{J = \int_0^T F(x, u, t)dt + S[x(T)]\right\},$$

subject to (5)

$$\dot{x} = f(x, u, t), x(0) = x_0.$$

To state the maximum principle, we introduce the *Hamiltonian* function.

$$H(x, u, \lambda, t) = F(x, u, t) + \lambda f(x, u, t), \tag{6}$$

where the *adjoint variable*, $\lambda(t)$, satisfies the adjoint differential equation

$$\dot{\lambda} = -H_x = -F_x - \lambda f_x \tag{7}$$

with the terminal or *transversality condition*

$$\lambda(T) = \left.\frac{\partial S(x)}{\partial x}\right|_{x=x(T)} = S_x[x(T)]. \tag{8}$$

Let u^* denote an *optimal control trajectory* and x^*, determined by the state equation, denote the corresponding *optimal state trajectory*. Then the necessary conditions for u^* to be an optimal control are given in (9).

$$\begin{gathered}\dot{x}^* = f(x^*, u^*, t), x^*(0) = x_0,\\ \dot{\lambda}^* = -H_x[x^*, u^*, \lambda, {}^*t], \lambda^*(T) = S_x[x^*(T)],\\ H[x^*(t), u^*(t), \lambda^*(t), t] \geqslant H[x^*(t), u(t), \lambda^*(t), t]\end{gathered} \tag{9}$$

for all admissible controls, i.e., piecewise-continuous controls such that

$$u(t) \in \Omega(t), t \in [0, T].$$

It should be emphasized that the state and the adjoint arguments of the Hamiltonian are $x^*(t)$ and $\lambda^*(t)$, respectively, on both sides of the Hamiltonian-maximizing condition in (9). Furthermore, $u^*(t)$ must provide a global maximum of the Hamiltonian $H[x^*(t), u(t), \lambda^*(t), t]$. For this reason the necessary conditions in (9) are called the *maximum principle*.

Note that to apply the maximum principle we must simultaneously solve two sets of differential equations with u obtained from the Hamiltonian maximizing condition in (9). With the control variable u so obtained the state equations for x depends on the initial value x_0, and the adjoint equations for λ depends on the terminal value $\lambda(T)$. Such a system of equations is called a *two-point boundary value problem*, see.[28]

Note also that if we can solve the Hamiltonian maximizing condition for an optimal control function in closed form as

$$u^*(t) = u[x^*(t), \lambda^*(t), t],$$

then we can substitute these into the state and adjoint equations to get a set of reduced differential equations which is the two-point boundary value problem.

The maximum principle stated in (9) is a statement of necessary optimality conditions. With an appropriate concavity condition to be stated later, these conditions are also sufficient for optimality. To state this we define a function H^0 called the *derived Hamiltonian* as follows:

$$H^0(x, \lambda, t) = \underset{u \in \Omega(t)}{\text{Maximum}}\, H(x, u, \lambda, t).$$

Then the sufficiency conditions are as follows.

Let $u^*(t)$ and the corresponding $x^*(t)$, $\lambda^*(t)$ satisfy the maximum principle necessary condition (14) for all $t \epsilon [0, T]$; then u^* is an optimal control if $H^0(x, \lambda^*, t)$ is concave in x for each t and $S(x)$ is concave in x. Since λ^* is not known a priori, it is usual to test H^0 for a stronger assumption, that is, check for the concavity of $H^0(x, \lambda, t)$ in x for each λ and each t. A good reference in this connection is.[30]

This sufficiency result is important from the management science viewpoint, since many applications in this area satisfy the required concavity hypothesis.

For rigorous statements and proofs of the above maximum principle, as well as more general versions, see.[7,8,10,15,21,26]

5 Economic Interpretations of the Maximum Principle

An important advantage of the maximum principle framework is the fact that the Hamiltonian and adjoint variables have useful economic and managerial interpretations. We discuss them next.

Recall from (5) that the objective function is

$$J = \int_0^T F(x, u, t)dt + S[x(T)],$$

where F will be considered to be the instantaneous profit rate measured in dollars per unit of time, and $S[x(T)]$ is the salvage value in dollars of the firm at time T when the terminal state is $x(T)$. For purposes of discussion it will be convenient to consider the state $x(t)$ as the stock of capital at time t.

Let $V(x, t)$ denote a function, called the *value function*, whose value is the maximum value of the objective function given that we start the system at time t and at state x. Then it can be shown [1] that

$$\lambda(t) = V_x(x, t), \tag{10}$$

so that we can interpret $\lambda(t)$ to be the per unit change in the value function $V(x, t)$ for small changes in the state variable x, which we term *capital stock* for convenience in interpretation. In other words, $\lambda(t)$ is the marginal value per unit capital at time t, and it is also referred to as the "price" or "shadow price" of a unit of capital. In particular, the value of $\lambda(0)$ is the marginal rate of change of the maximum value of J, (the objective function) with respect to the change in the initial state x_0.

The interpretation of the Hamiltonian function can now be derived. Multiplying (6) by dt gives

$$\begin{aligned} Hdt &= Fdt + \lambda f dt \\ &= Fdt + \lambda \dot{x} dt \\ &= Fdt + \lambda dx, \end{aligned}$$

where we made use of the state equation (1). The first term $F(x, u, t)dt$ represents the *direct contribution* to J in dollars from time t to $t + dt$ if we are in state x and we apply control u. The differential $dx = f(x, u, t)dt$ represents the change in capital stock from t to $t + dt$, when we are in state x and control u is applied. Therefore, the second term λdx represents the value in dollars of the incremental capital stock, dx, and hence can be considered as the *indirect contribution* to J in dollars. Thus Hdt can be interpreted as the *total contribution* to J from the interval t to $t + dt$ when $x(t) = x$ and $u(t) = u$.

With this interpretation of the Hamiltonian it is easy to see why the Hamiltonian must be maximized at each time t. If we were just to maximize F at each time t, we would not be maximizing J, because we would ignore the effect of control in changing the capital stock, which gives rise to indirect contributions to J. The maximum principle derives prices, the adjoint variables λ, in such a way that $\lambda(t)dx$ is the correct valuation for the indirect contribution from the interval t to $t + dt$. As a consequence the Hamiltonian maximizing problem can be treated as a static problem at each instant t. In other words, the maximum principle "decouples" the dynamic maximization problem (5) in the interval $[0, T]$ into a series of static maximization problems at each instant t in $[0, T]$. Therefore the Hamiltonian can be interpreted as a surrogate profit rate to be maximized at each time t.

The value of λ to be used in the maximum principle is given by (7) and (8), that is,

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x} - \lambda \frac{\partial f}{\partial x}\,;\ \lambda(T) = S_x[x(T)].$$

Rewriting the equation as

$$-d\lambda = H_x dt = (F_x dt) + \lambda(f_x dt),$$

we can observe that along the optimal path the decrease in the price of capital from t to $t + dt$, which can be considered as the *marginal cost of holding that capital*, equals the *marginal revenue of investing the capital* as is evident from the interpretation of the Hamiltonian given above. The marginal revenue, $H_x dt$, consists of the sum of direct marginal contribution, $F_x dt$, and the indirect marginal contribution, $\lambda f_x dt$. Thus the adjoint equation becomes the familiar economic equilibrium relation: *marginal cost equals marginal revenue.*

Further insight can be obtained by integrating the above adjoint equation from t to T as follows:

$$\begin{aligned}\lambda(t) &= \lambda(T) + \int_t^T H_x(x(\tau), u(\tau), \tau)d\tau \\ &= S_x[x(T)] + \int_t^T H_x d\tau.\end{aligned}$$

Note that the price $\lambda(T)$ of a unit of capital at time T is its marginal salvage value, $S_x[x(T)]$. The price $\lambda(t)$ of a unit of capital at time t is the sum of its terminal price, $\lambda(T)$, plus the integral of the marginal surrogate profit rate, H_x, from t to T.

The above interpretations show that the adjoint variables behave in much the same way as dual variables in linear (and non-linear) programming. Other interpretations are given in.[17,25] The differences are that here the adjoint variables are time dependent and satisfy derived differential equations.

6 The Maximum Principle with Inequality Constraints

Here we describe the Lagrangian form of the maximum principle which is needed when additional inequality constraints are imposed on the control problem in (5). The new problem is stated as in (11):

$$\text{Maximize}\left\{J = \int_0^T F(x, u, t)dt + S[x(T)]\right\} \tag{11}$$

subject to

$$\dot{x} = f(x, u, t),\ x(0) = x_0 \tag{11.1}$$

$$g(x, u, t) \geqslant 0 \tag{11.2}$$

$$x(T) \in Y \subseteq X, \tag{11.3}$$

where g is a differentiable function of x and u and must contain terms in u, and where X is the set of all feasible terminal states, also called the *reachable set*, and Y is a given convex subset of X. Note that the inequality constraints defined by g in (11) are assumed to include the control constraints (3).

To state the maximum principle we define the Lagrangian function as

$$L[x, y, \lambda, \mu, t] \triangleq H[x, u, \lambda, t] + \mu g(x, u, t), \tag{12}$$

where μ is called a *Lagrange multiplier*. The adjoint variable satisfies the differential equation

$$\dot{\lambda} = -L_x[x, u, \lambda, \mu, t] \tag{13}$$

with boundary conditions given in (14.1).

The maximum principle states that the necessary conditions for u^* to be an optimal control are that there exist λ^* and μ^* such that (14) holds.

$$\begin{aligned} \dot{x}^* &= f(x^*, u^*, t), x^*(0), x^*(T) \in Y \subset X \\ \dot{\lambda}^* &= -L_x[x^*, u^*, \lambda^*, \mu^*, t], \end{aligned} \tag{14}$$

with the terminal condition

$$[\lambda^*(T) - S_x]\,[y - x^*(T)] \geqq 0 \text{ for all } y \in Y, \tag{14.1}$$

the Hamiltonian maximizing condition

$$H[x^*(t), u^*(t), \lambda^*(t), t] \geqslant H[x^*(t), u(t), \lambda^*(t), t] \tag{14.2}$$

for all admissible controls, i.e., piecewise-continuous controls such that for $t \in [0, T]$,

$$g[x^*(t), u(t), t] \geqq 0, \tag{14.3}$$

and the Lagrange multipliers μ^* are such that

$$\left.\frac{\partial L}{\partial u}\right|_{u=u^*} = 0, \tag{14.4}$$

and the complementary slackness conditions

$$\mu^* \geqq 0 \text{ and } \mu^* g(x^*, u^*, t) = 0 \text{ hold.} \tag{14.5}$$

In connection with the Lagrangian form of the maximum principle we make three important remarks.

Remark 1 *Unspecified terminal time*

If the terminal time is unspecified, there is an additional necessary transversality condition for T^* to be optimal, namely,

$$H[x^*(T^*), u^*(T^*), \lambda^*(T^*), T^*] + S_T[x^*(T^*), T^*] = 0. \tag{15}$$

This condition is equivalent to taking the derivative of the optimal $J(T)$ for a given T, and setting the result to 0.

Remark 2 *Pure state variable inequality constraints*
In many management science and economics problems, it is common to have a non-negativity state space constraint of the form

$$x(t) \geqq 0 \text{ for } t \epsilon [0, T]. \tag{16}$$

Since the control variables do not enter directly into (16), these are called *pure state variable inequality constraints* [1]. To deal with such constraints, we proceed as follows.

At any point where $x(t) > 0$ the corresponding constraint (16) is not binding and can be ignored. In any interval where $x(t) = 0$ we must have $\dot{x}(t) \geqslant 0$ so that x does not become negative. Hence the control must be constrained to satisfy $x = f \geqslant 0$, making $f \geqslant 0$ as constraint of type (11.2) over the interval. We can add the constraint

$$f(x, u, t) \geqq 0, \text{ whenever } x(t) = 0, \tag{17}$$

to the original set (11.2). We associate multipliers η with (17) whenever (17) must be imposed, that is, whenever $x(t) = 0$. A convenient way to do this is to impose an "either/or" condition, $\eta x = 0$. This will make $\eta = 0$ whenever $x > 0$. We can now form the Lagrangian,

$$L = H + \mu g + \eta f \tag{18}$$

and apply the maximum principle in (14) with the additional necessary conditions:

$$\eta^* \geqq 0, \eta^* x^* = 0, \eta^* f(x^*, u^*, t) = 0. \tag{19}$$

Remark 3 *Important special terminal conditions*
Terminal conditions which frequently arise in practice are special cases of (11.3). For these the terminal conditions on the adjoint variable specified in (14) are specialized in the four cases shown in table 2.

7 Extensions and Final Remarks

The purpose of this paper was to provide a simple tutorial on optimal control theory. We have left out many important variations and extensions of the maximum principle. We conclude by briefly mentioning some of the important references to these other topics.

For infinite horizon problems ($T = \infty$) see.[1,34] This usually involves a discounted objective function and a current value version of the maximum principle.

TABLE 2

SPECIAL TERMINAL CONDITIONS

Constraint on $x(T)$		Description	$\lambda^*(T)$	$\lambda^*(T)$ when $S \equiv 0$
(1)	$x(T) \epsilon Y = X$	Free-end point	$\lambda^*(T) = S_x[x^*(T)]$	$\lambda^*(T) = 0$
(2)	$x(T) = b \epsilon X$, i.e., $Y = \{b\}$	Fixed-end point	–	$\lambda^*(T) =$ constant to be determined
(3)	$x(T) \epsilon X \cap [b, \infty)$ i.e., $Y = [b, \infty)$	One-sided constraint	$\lambda^*(T) \geqslant S_x[x^*(T)]$ and $(\lambda^*(T) - S_x[x^*(T)])\,[b - x^*(T)] = 0$	$\lambda^*(T) = 0$ and $\lambda^*(T)[b - x^*(T)] = 0$
(4)	$x(T) \epsilon Y \subset X$	General constraints	$\{\lambda^*(T) - S_x[x^*(T)]\}\,\{y - x^*(T)\} \geqslant 0$ $\forall\, y \epsilon Y$	$\lambda^*(T)[y - x^*(T)] \geqslant 0$ $\forall\, y \epsilon Y$

For discrete time optimal control problems see.[12]

For stochastic optimal control problems see.[4,18,22,23] These problems have been found especially important in finance applications.

For distributed parameter control problems, see.[6,11,16,29]

We are aware of more than 1000 papers already published which give applications of optimal control theory in management science, economics, and related areas, and we expect this list to grow rapidly in the future.

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