Introduction to Linear Programming

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Key Concepts:

- Linear Programs
- Compact Form
- Feasibility Region
- Graphical Intuition
- Duality and Shadow Price

1. Introduction

1.1. Mathematical Formulation

Linear programs are a class of optimization problem that can be formulated as (Bertsimas and Tsitsiklis, 1997)

General Optimization Problem	Linear Programming
Maximize $f(x, \theta)$	$Maximize\ \ c_1x_1 + \cdots + c_nx_n$
Subject to $g(x, \theta)$	
	$Subject\ to\ \ a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ for $i \in M_1$
with the notation	
	$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$ for $i \in M_2$
$f(.)$: Objective Function	
$g(.)$: Constraints	$a_{i1}x_1 + \cdots + a_{in}x_n = b_i$ for $i \in M_3$
x: Decision variables	
θ: Parameters	$x_j \geq 0$ for $j \in N_1$
	$x_j \leq 0$ for $j \in N_2$

where $x = \{x_1, \dots, x_n\}$ denotes the n decision variables and c, b and a capture the parameter of the decision problem (i.e., θ in the general formulation of an optimization problem). Specifically, both $f(.)$ and $g(.)$ are linear functions of the decision variables. Note that linear programs can be written more compactly as

Maximize	$c^T x$
Subject to	$Ax \leq b$

where c, b and x are vectors and A is a matrix.

1.2. Examples

For each of the example below, list the decision variables, constraints and objective functions and write the linear programs.

Example 1: Consider the following example (Bertsimas and Tsitsiklis, 1997, p. 7). A firm produces n different goods using m different raw materials. Let b_i, $i = 1, \dots, m$, be the available amount of the ith raw material. The jth good, $j = 1, \dots, n,$ requires a_{ij} units of the ith material and results in a revenue of $c_j$per unit produced. The firm faces the problem of deciding how much of each good to produce in order to maximize its total revenue.

Example 2: A bank wants to invest a budget C in two types of investments, e.g., 1 and 2. Let x_1 denote the part of budget invested in product 1 and x_2 be the same for investment in product 2. Product 1 yields 15% of returns, while product 2 yields 25%. The bank must invest at least a fourth of its budget to product 1. Furthermore, it cannot the amount invested in product 2 should be lower than double the investment in product 1. How should the bank invest?

Example 3: You are the manager of a Lego Furniture production facility. The current resources available in your factory are 6 large orange Lego blocks and 8 small green Lego blocks. You can make two kinds of furniture with your available Lego blocks. The first kind is a chair, which retails for $10 and takes 2 green blocks and 1 orange block to manufacture. The second one is a table, which retails for $16 and takes 2 green blocks and 2 orange blocks to manufacture. Both pieces are good sellers and you'll be able to sell anything you produce. What should you build in order to maximize revenues?

Example 4: A soft-drink producer must decide how to divide its spending between two forms of media: television advertising and magazine advertising. Each 30-second commercial on prime-time network television costs $120,000 and, by the company's estimate, will reach 10,000 viewers, 5,000 of whom are in the prime consumer age group, 15 to 25. A single-page ad in a leading human-interest weekly magazine costs $40,000 and reaches 5,000 individuals, 1,000 of whom are in the 15-to-25 age group. In addition, the company plans to hold a sweepstakes contest to promote its new soft drink. (A requirement for entry is to enclose the coded label from the new drink.) The company believes the print ad will be more effective in generating trial purchases and entries. Each magazine spot is expected to produce 500 entries and each television spot 250 entries. Finally, the company's goal in its promotion campaign is to reach at least 600,000 total viewers and 150,000 young (15 to 25) viewers and to generate 30,000 or more contest entrants. How many spots of each kind should it purchase to meet these three goals and do so at minimum cost?

2. Geometric Intuition of Linear Programming

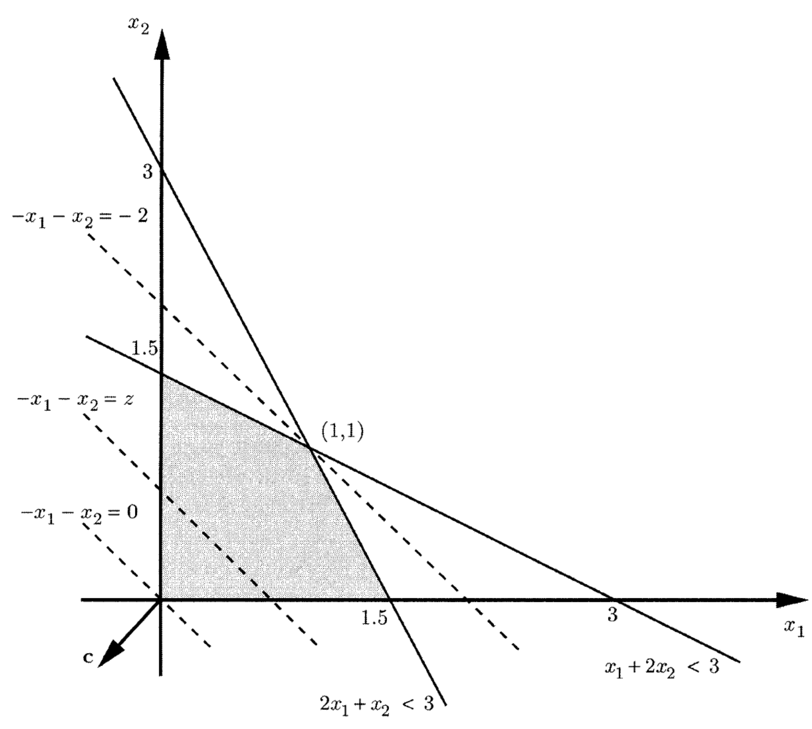
It is useful to provide geometric insights on the nature of linear programming problems. Consider the linear program (Bertsimas and Tsitsiklis, 1997)

$$
\begin{array}{ll}
\text{Minimize} & -x_1 - x_2 \\
\text{Subject to} & \begin{array}{rl} x_1 + 2x_2 & \leq 3 \\ 2x_1 + x_2 & \leq 3 \\ x_1, x_2 & \geq 0 \end{array}
\end{array}
$$

Exercise

1. Write this linear program in compact form.
2. What is the solution, i.e., $(x_1^*; x_2^*)$? Using the vector c (see Python code in the appendix) and $(x_1^*; x_2^*)$, provide the value of the objective function.

The constraints define a convex set (please see Chapter 2, pp. 19-69, in *Convex Optimization* by Boyd and Vandenberghe for the necessary background on convex sets). The geometric representation of the linear program is provided in the figure below, where the dashed lines represent the objective function $-x_1 - x_2$, while the bold lines represent the constraints. The shaded-area defines the **feasible** set, i.e., all the combinations (x_1, x_2) that satisfy the constraints.



Several remarks are important (which generalize to problems of higher dimensions, see Bertsimas and Tsitsiklis, 1997, Chapter 2)

(i) The shaded-area defines a *convex* set (see Chapter 2 in Boyd and Vandenberghe, 2009, for details on convex sets).

(ii) The convex set is characterized by corners or vertices that are referred to as the "extreme points".

(iii) These extreme points are important as the optimal solution to an LP problem can be found at an extreme point of the feasible region.

(iv) When looking for the optimal solution, you do not have to evaluate all feasible solution points. You have to consider only the extreme points of the feasible region.

(v) The constraints can be such that the feasible region is not convex.

(vi) The constraints can be such that a feasible region does not exist

One of the most efficient algorithms for finding the optimal solution of a linear program, the **Simplex Method**, exploits the fact that the optimal solution of the linear program is one of the vertices of the feasible region. The simplex method is an algorithm that determines how to jump from one extreme point to another such that to improve the objective function. This is one of the reasons why the convexity of the feasible set is important.

3. Duality[1]

The concept of duality is central to optimization. For business decisions, the concept of duality relates to the value of relaxing constraints faced by the decision maker, e.g., what is the price I would be willing to pay to relax my constraint a little bit. There are other advantages to duality (e.g., computational) but we will focus on the economic value of duality and most importantly of what is called "shadow prices".

To this end, we need to create the *dual* of the original optimization problem called the *primal*. Note that this terminology is important as it is standard for some optimization packages to provide the solution for both the dual and the primal, see for instance the code in the

[1] For a quick introduction: https://optimization.mccormick.northwestern.edu/index.php/Lagrangean_duality

appendix, the value of the objective function is obtained in CVXOPT by writing "print(sol['primal objective']).

3.1. Explanation (Bertsimas and Tsitsiklis, 1997, Section 4.1)

Mathematically, consider the optimization problem

Minimize	$c^T x$
Subject to	$Ax = b$
	$x \geq 0$

which we called the primal and let x^* be an optimal solution, assuming it exists. We introduced a relaxed problem in which the constraint $Ax = b$ is a replaced by a penalty $p^T(b - Ax)$, where p is a price *vector* of the same dimension as b. We are then faced with the problem

Minimize	$c^T x + p^T(b - Ax)$
Subject to	$x \geq 0$

Let $h(p)$ be the optimal cost for the relaxed problem, as a function of the price vector p. The relaxed problem allows for more options than those present in the primal problem and we expect $h(p)$ to be no larger than the optimal primal costs $c^T x^*$, i.e.,

$$h(p) = \min_{x>0}[c^T x + p^T(b - Ax)] \leq c^T x^* + p^T(b - Ax^* = c^T x^*).$$

Thus, each p leads to a lower bound $h(p)$ for the optimal cost $c^T x^*$. The problem

Maximize	$h(p)$
Subject to	No constraints

can be then interpreted as a search for the tightest possible lower bound of this type, and is known as the dual problem. The main result in duality theory asserts that the optimal cost in the dual problem is equal to the optimal costs $c^T x^*$ in the primal. In other words, when the prices are chosen according to an optimal solution for the dual, the option of violating the constraints $Ax = b$ is of no value.

Using the definition of $h(p)$, we have

$$h(p) = \min_{x>0}[c^T x + p^T(b - Ax)] = p^T b + \min_{x\geq 0}(c^T - p^T A)x.$$

As a result, the dual problem is the same as the linear program

Maximize	$p^T b$
Subject to	$p^T A \leq c^T$

3.2. The dual Problem

Primal Problem	**Dual Problem**
$Min\ \ c^T x$	$Max\ \ p^T b$
$Subject\ to\ \ a_i^T \geq b_i$ for $i \in M_1$	$Subject\ to\ \ p_i \geq 0$ for $i \in M_1$
$a_i^T \leq b_i$ for $i \in M_2$	$p_i \leq 0$ for $i \in M_2$
$a_i^T = b_i$ for $i \in M_3$	p_i free for $i \in M_3$
$x_j \geq 0$ for $j \in N_1$	$p^T A_j \leq c_j$ for $j \in N_1$
$x_j \leq 0$ for $j \in N_2$	$p^T A_j \geq c_j$ for $j \in N_2$
$x_j\ free$ for $j \in N_3$	$p^T A_j = c_j$ for $j \in N_3$

References

Bertsimas, Dimitris; Tsitsiklis, John N. *Introduction to Linear Optimization*. Athena Scientific.[2]

Boyd, Stephen; Vandenberghe, Lievem. *Convex Optimization*. 2009. Cambridge University Press

Appendix

```python
from cvxopt import matrix, solvers
import numpy
A = matrix([ [1.0, 2.0,-1.0,0.0], [2.0,1.0,0.0,-1.0] ])
b = matrix([ 3.0, 3.0,0.0,0.0 ])
c = matrix([ -1.0, -1.0 ])
sol=solvers.lp(c,A,b)
print(sol['x'])
print(sol['primal objective'])
```

Note that the code relies on the compact form of the linear program.

[2] If you google it, you might find it.