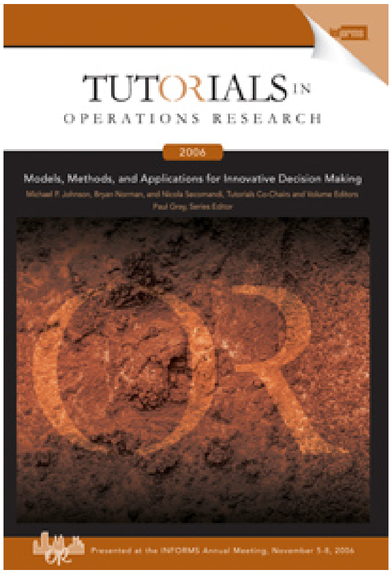

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Game Theory in Supply Chain Analysis*

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Abstract Game theory has become an essential tool in the analysis of supply chains with multiple agents, often with conflicting objectives. This chapter surveys the applications of game theory to supply chain analysis and outlines game-theoretic concepts that have potential for future application. We discuss both noncooperative and cooperative game theory in static and dynamic settings. Careful attention is given to techniques for demonstrating the existence and uniqueness of equilibrium in noncooperative games. A newsvendor game is employed throughout to demonstrate the application of various tools.

Keywords game theory; noncooperative; cooperative; equilibrium concepts

1. Introduction

Game theory (hereafter GT) is a powerful tool for analyzing situations in which the decisions of multiple agents affect each agent's payoff. As such, GT deals with interactive optimization problems. While many economists in the past few centuries have worked on what can be considered game-theoretic models, John von Neumann and Oskar Morgenstern are formally credited as the fathers of modern game theory. Their classic book "Theory of Games and Economic Behavior," (von Neumann and Morgenstern [102]), summarizes the basic concepts existing at that time. GT has since enjoyed an explosion of developments, including the concept of equilibrium by Nash [68], games with imperfect information by Kuhn [51], cooperative games by Aumann [3] and Shubik [86], and auctions by Vickrey [100] to name just a few. Citing Shubik [87], "In the '50s... game theory was looked upon as a curiosum not to be taken seriously by any behavioral scientist. By the late 1980s, game theory in the new industrial organization has taken over... game theory has proved its success in many disciplines."

This chapter has two goals. In our experience with GT problems, we have found that many of the useful theoretical tools are spread over dozens of papers and books, buried among other tools that are not as useful in supply chain management (hereafter SCM). Hence, our first goal is to construct a brief tutorial through which SCM researchers can quickly locate GT tools and apply GT concepts. Due to the need for short explanations, we omit all proofs, choosing to focus only on the intuition behind the results we discuss. Our second goal is to provide ample but by no means exhaustive references on the specific applications of various GT techniques. These references offer an in-depth understanding of an application where necessary. Finally, we intentionally do not explore the implications of GT analysis on supply chain management, but rather we emphasize the means of conducting the analysis to keep the exposition short.

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1.1. Scope and Relation to the Literature

There are many GT concepts, but this chapter focuses on concepts that are particularly relevant to SCM and, perhaps, have already found their applications in the literature. We dedicate a considerable amount of space to the discussion of static noncooperative, nonzero sum games, the type of game which has received the most attention in the recent SCM literature. We also discuss cooperative games, dynamic/differential games, and games with asymmetric/incomplete information. We omit discussion of important GT concepts covered in Simchi-Levi et al. [88]: auctions in Chapters 4 and 10, principal-agent models in Chapter 3, and bargaining in Chapter 11.

The material in this chapter was collected predominantly from Friedman [37], Fudenberg and Tirole [38], Moulin [62], Myerson [66], Topkis [96], and Vives [101]. Some previous surveys of GT models in management science include Lucas's [57] survey of mathematical theory of games, Feichtinger and Jorgensen's [35] survey of differential games, and Wang and Parlar's [105] survey of static models. A recent survey by Li and Whang [55] focuses on application of GT tools in five specific OR/MS models.

2. Noncooperative Static Games

In noncooperative static games, the players choose strategies simultaneously and are thereafter committed to their chosen strategies, i.e., these are simultaneous move, one-shot games. Noncooperative GT seeks a rational prediction of how the game will be played in practice.[1] The solution concept for these games was formally introduced by John Nash [68], although some instances of using similar concepts date back a couple of centuries.

2.1. Game Setup

To break the ground for the section, we introduce basic GT notation. A warning to the reader: to achieve brevity, we intentionally sacrifice some precision in our presentation. See the texts by Friedman [37] and Fudenberg and Tirole [38] if more precision is required.

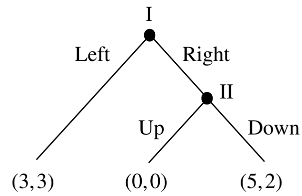
Throughout this chapter, we represent games in the normal form. A game in the normal form consists of (1) *players* indexed by $i = 1, \ldots, n$, (2) *strategies* or more generally a set of strategies denoted by x_i, $i = 1, \ldots, n$ available to each player, and (3) *payoffs* $\pi_i(x_1, x_2, \ldots, x_n)$, $i = 1, \ldots, n$ received by each player. Each strategy is defined on a set X_i, $x_i \in X_i$, so we call the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ the *strategy space*. Each player may have a unidimensional strategy or a multidimensional strategy. In most SCM applications, players have unidimensional strategies, so we shall either explicitly or implicitly assume unidimensional strategies throughout this chapter. Furthermore, with the exception of one example, we will work with continuous strategies, so the strategy space is R^n.

A player's strategy can be thought of as the complete instruction for which actions to take in a game. For example, a player can give his or her strategy to someone who has absolutely no knowledge of the player's payoff or preferences, and that person should be able to use the instructions contained in the strategy to choose the actions the player desires. As a result, each player's set of feasible strategies must be independent of the strategies chosen by the other players, i.e., the strategy choice by one player is not allowed to limit the feasible strategies of another player. (Otherwise, the game is ill defined and any analytical results obtained from the game are questionable.)

In the normal form, players choose strategies simultaneously. Actions are adopted after strategies are chosen and those actions correspond to the chosen strategies. As an alternative to the one-shot selection of strategies in the normal form, a game can also be designed in the extensive form. With the extensive form, actions are chosen only as needed, so sequential

[1]Some may argue that GT should be a tool for choosing how a manager should play a game, which may involve playing against rational or semirational players. In some sense there is no conflict between these descriptive and normative roles for GT, but this philosophical issue surely requires more in-depth treatment than can be afforded here.

Figure 1. Extensive vs. normal form game representation.



		I: Left	I: Right
II	Up	(3,3)	(0,0)
II	Down	(3,3)	(5,2)

choices are possible. As a result, players may learn information between the selection of actions, in particular, a player may learn which actions were previously chosen or what the outcome of a random event was. Figure 1 provides an example of a simple extensive form game and its equivalent normal form representation: There are two players: player I chooses from $\{\text{Left}, \text{Right}\}$ and player II chooses from $\{\text{Up}, \text{Down}\}$. In the extensive form, player I chooses first, then player II chooses after learning player I's choice. In the normal form, they choose simultaneously. The key distinction between normal and extensive form games is that in the normal form, a player is able to commit to all future decisions. We later show that this additional commitment power may influence the set of plausible equilibria.

A player can choose a particular strategy or a player can choose to randomly select from among a set of strategies. In the former case, the player is said to choose a *pure strategy*, whereas in the latter case, the player chooses a *mixed strategy*. There are situations in economics and marketing that have used mixed strategies: see Varian [99] for search models and Lal [52] for promotion models. However, mixed strategies have not been applied in SCM, in part because it is not clear how a manager would actually implement a mixed strategy. For example, it seems unreasonable to suggest that a manager should "flip a coin" among various capacity levels. Fortunately, mixed strategy equilibria do not exist in games with a unique pure strategy equilibrium. Hence, in those games, attention can be restricted to pure strategies without loss of generality. Therefore, in the remainder of this chapter, we consider only pure strategies.

In a *noncooperative* game, the players are unable to make binding commitments before choosing their strategies. In a *cooperative* game, players are able to make binding commitments. Hence, in a cooperative game, players can make side-payments and form coalitions. We begin our analysis with noncooperative static games. In all sections, except the last one, we work with the games of *complete information*, i.e., the players' strategies and payoffs are common knowledge to all players.

As a practical example throughout this chapter, we utilize the classic newsvendor problem transformed into a game. In the absence of competition, each newsvendor buys Q units of a single product at the beginning of a single selling season. Demand during the season is a random variable D with distribution function F_D and density function f_D. Each unit is purchased for c and sold on the market for $r > c$. The newsvendor solves the following optimization problem

$$\max_Q \pi = \max_Q E_D[r\min(D,Q) - cQ],$$

with the unique solution

$$Q^* = F_D^{-1}\left(\frac{r-c}{r}\right).$$

Goodwill penalty costs and salvage revenues can easily be incorporated into the analysis, but for our needs, we normalized them out.

Now consider the GT version of the newsvendor problem with two retailers competing on product availability. Parlar [75] was the first to analyze this problem, which is also one of the first articles modeling inventory management in a GT framework. It is useful to consider only the two-player version of this game because then graphic analysis and interpretations are feasible. Denote the two players by subscripts i and j, their strategies (in this case, stocking quantities) by Q_i, Q_j, and their payoffs by π_i, π_j.

We introduce interdependence of the players' payoffs by assuming the two newsvendors sell the same product. As a result, if retailer i is out of stock, all unsatisfied customers try to buy the product at retailer j instead. Hence, retailer i's total demand is $D_i + (D_j - Q_j)^+$: the sum of his own demand and the demand from customers not satisfied by retailer j. Payoffs to the two players are then

$$\pi_i(Q_i, Q_j) = E_D[r_i \min(D_i + (D_j - Q_j)^+, Q_i) - c_i Q_i], \quad i, j = 1, 2.$$

2.2. Best Response Functions and the Equilibrium of the Game

We are ready for the first important GT concept: *Best response functions.*

Definition 1. Given an n-player game, player i's best response (function) to the strategies x_{-i} of the other players is the strategy x_i^* that maximizes player $i's$ payoff $\pi_i(x_i, x_{-i})$:

$$x_i^*(x_{-i}) = \arg\max_{x_i} \pi_i(x_i, x_{-i}).$$

($x_i^*(x_{-i})$ is probably better described as a correspondence rather than a function, but we shall nevertheless call it a function with an understanding that we are interpreting the term "function" liberally.) If π_i is quasi-concave in x_i, the best response is uniquely defined by the first-order conditions of the payoff functions. In the context of our competing newsvendors example, the best response functions can be found by optimizing each player's payoff functions w.r.t. the player's own decision variable Q_i while taking the competitor's strategy Q_j as given. The resulting best response functions are

$$Q_i^*(Q_j) = F^{-1}_{D_i + (D_j - Q_j)^+}\left(\frac{r_i - c_i}{r_i}\right), \quad i, j = 1, 2.$$

Taken together, the two best response functions form a *best response mapping* $R^2 \to R^2$, or in the more general case, $R^n \to R^n$. Clearly, the best response is the best player i can hope for given the decisions of other players. Naturally, an outcome in which all players choose their best responses is a candidate for the noncooperative solution. Such an outcome is called a Nash equilibrium (hereafter NE) of the game.

Definition 2. An outcome $(x_1^*, x_2^*, \ldots, x_n^*)$ is a Nash equilibrium of the game if x_i^* is a best response to x_{-i}^* for all $i = 1, 2, \ldots, n$.

Going back to competing newsvendors, NE is characterized by solving *a system* of best responses that translates into the system of first-order conditions:

$$Q_1^*(Q_2^*) = F^{-1}_{D_1 + (D_2 - Q_2^*)^+}\left(\frac{r_1 - c_1}{r_1}\right),$$

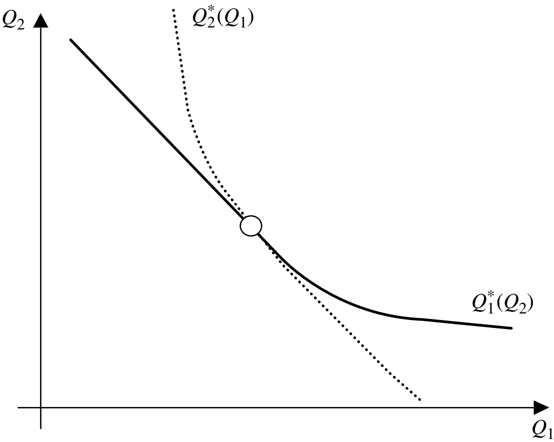
$$Q_2^*(Q_1^*) = F^{-1}_{D_2 + (D_1 - Q_1^*)^+}\left(\frac{r_2 - c_2}{r_2}\right).$$

When analyzing games with two players, it is often helpful to graph the best response functions to gain intuition. Best responses are typically defined implicitly through the first-order conditions, which makes analysis difficult. Nevertheless, we can gain intuition by finding out how each player reacts to an increase in the stocking quantity by the other player (i.e., $\partial Q_i^*(Q_j)/\partial Q_j$) through employing implicit differentiation as follows:

$$\frac{\partial Q_i^*(Q_j)}{\partial Q_j} = -\frac{\partial^2 \pi_i / \partial Q_i \partial Q_j}{\partial^2 \pi_i / \partial Q_i^2} = -\frac{r_i f_{D_i + (D_j - Q_j)^+ \mid D_j > Q_j}(Q_i) \Pr(D_j > Q_j)}{r_i f_{D_i + (D_j - Q_j)^+}(Q_i)} < 0. \tag{1}$$

The expression says that the *slopes* of the best response functions are negative, which implies an intuitive result that each player's best response is monotonically decreasing in the other

FIGURE 2. Best responses in the newsvendor game.



player's strategy. Figure 2 presents this result for the symmetric newsvendor game. The equilibrium is located on the intersection of the best responses, and we also see that the best responses are, indeed, decreasing.

One way to think about an NE is as a *fixed point* of the best response mapping $R^n \to R^n$. Indeed, according to the definition, NE must satisfy the system of equations $\partial\pi_i/\partial x_i = 0$, all i. Recall that a fixed point x of mapping $f(x)$, $R^n \to R^n$ is any x such that $f(x) = x$. Define $f_i(x_1, \ldots, x_n) = \partial\pi_i/\partial x_i + x_i$. By the definition of a fixed point,

$$f_i(x_1^*, \ldots, x_n^*) = x_i^* = \partial\pi_i(x_1^*, \ldots, x_n^*)/\partial x_i + x_i^* \to \partial\pi_i(x_1^*, \ldots, x_n^*)/\partial x_i = 0, \quad \forall\, i.$$

Hence, x^* solves the first-order conditions if and only if it is a fixed point of mapping $f(x)$ defined above.

The concept of NE is intuitively appealing. Indeed, it is a self-fulfilling prophecy. To explain, suppose a player were to guess the strategies of the other players. A guess would be consistent with payoff maximization and therefore would be reasonable only if it presumes that strategies are chosen to maximize every player's payoff given the chosen strategies. In other words, with any set of strategies that is not an NE there exists at least one player that is choosing a nonpayoff maximizing strategy. Moreover, the NE has a self-enforcing property: No player wants to unilaterally deviate from it because such behavior would lead to lower payoffs. Hence, NE seems to be the necessary condition for the prediction of any rational behavior by players.[2]

While attractive, numerous criticisms of the NE concept exist. Two particularly vexing problems are the nonexistence of equilibrium and the multiplicity of equilibria. Without the existence of an equilibrium, little can be said regarding the likely outcome of the game. If multiple equilibria exist, then it is not clear which one will be the outcome. Indeed, it is possible the outcome is not even an equilibrium because the players may choose strategies from different equilibria. For example, consider the normal form game in Figure 1. There are two Nash equilibria in that game $\{\text{Left}, \text{Up}\}$ and $\{\text{Right}, \text{Down}\}$: Each is a best response to the other player's strategy. However, because the players choose their strategies simultaneously, it is possible that player I chooses Right (the second equilibrium) while player II chooses Up (the first equilibrium), which results in $\{\text{Right}, \text{Up}\}$, the worst outcome for both players.

[2]However, an argument can also be made that to predict rational behavior by players it is sufficient that players not choose dominated strategies, where a dominated strategy is one that yields a lower payoff than some other strategy (or convex combination of other strategies) for all possible strategy choices by the other players.

In some situations, it is possible to rationalize away some equilibria via a refinement of the NE concept: e.g., trembling hand perfect equilibrium (Selten [83]), sequential equilibrium (Kreps and Wilson [50]), and proper equilibria (Myerson [66]). These refinements eliminate equilibria that are based on noncredible threats, i.e., threats of future actions that would not actually be adopted if the sequence of events in the game led to a point in the game in which those actions could be taken. The extensive form game in Figure 1 illustrates this point. $\{\text{Left}, \text{Up}\}$ is a Nash equilibrium (just as it is in the comparable normal form game) because each player is choosing a best response to the other player's strategy: Left is optimal for player I given player II plans to play Up and player II is indifferent between Up or Down given player I chooses Left. But if player I were to choose Right, then it is unreasonable to assume player II would actually follow through with UP: UP yields a payoff of 0 while Down yields a payoff of 2. Hence, the $\{\text{Left}, \text{Up}\}$ equilibrium is supported by a noncredible threat by player II to play Up. Although these refinements are viewed as extremely important in economics (Selten was awarded the Nobel Prize for his work), the need for these refinements has not yet materialized in the SCM literature. However, that may change as more work is done on sequential/dynamic games.

An interesting feature of the NE concept is that the system optimal solution (i.e., a solution that maximizes the sum of players' payoffs) need not be an NE. Hence, decentralized decision making generally introduces inefficiency in the supply chain. There are, however, some exceptions: see Mahajan and van Ryzin [59] and Netessine and Zhang [73] for situations in which competition may result in the system-optimal performance. In fact, an NE may not even be on the *Pareto frontier*: The set of strategies such that each player can be made better off only if some other player is made worse off. A set of strategies is *Pareto optimal* if they are on the Pareto frontier; otherwise, a set of strategies is *Pareto inferior*. Hence, an NE can be Pareto inferior. The prisoner's dilemma game (Fudenberg and Tirole [38]) is the classic example of this: Only one pair of strategies when both players "cooperate" is Pareto optimal, and the unique Nash equilibrium is when both players "defect" happens to be Pareto inferior. A large body of the SCM literature deals with ways to align the incentives of competitors to achieve optimality. See Cachon [17] for a comprehensive survey and taxonomy. See Cachon [18] for a supply chain analysis that makes extensive use of the Pareto optimal concept.

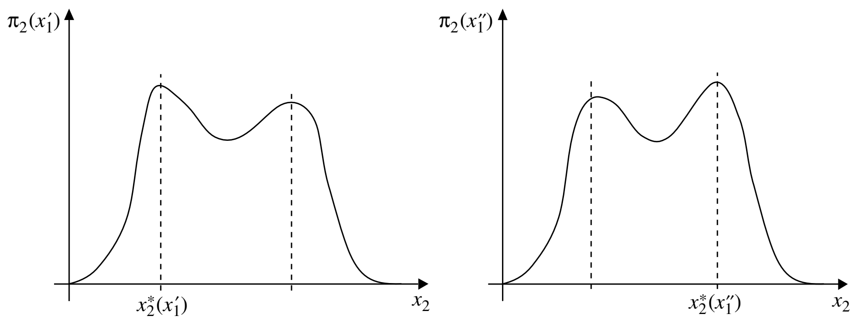
2.3. Existence of Equilibrium

An NE is a solution to a system of n first-order conditions; therefore, an equilibrium may not exist. Nonexistence of an equilibrium is potentially a conceptual problem because in this case the outcome of the game is unclear. However, in many games an NE does exist, and there are some reasonably simple ways to show that at least one NE exists. As already mentioned, an NE is a fixed point of the best response mapping. Hence, fixed-point theorems can be used to establish the existence of an equilibrium. There are three key fixed-point theorems, named after their creators: Brouwer, Kakutani, and Tarski, see Border [13] for details and references. However, direct application of fixed-point theorems is somewhat inconvenient, and hence generally not done. For exceptions, see Lederer and Li [54] and Majumder and Groenevelt [60] for existence proofs that are based on Brouwer's fixed point theorem. Alternative methods, derived from these fixed-point theorems, have been developed. The simplest and the most widely used technique for demonstrating the existence of NE is through verifying concavity of the players' payoffs.

Theorem 1 (Debreu [29]). *Suppose that for each player, the strategy space is compact*[3] *and convex and the payoff function is continuous and quasiconcave with respect to each player's own strategy. Then, there exists at least one pure strategy NE in the game.*

[3]Strategy space is compact if it is closed and bounded.

FIGURE 3. Example with a bimodal objective function.



If the game is symmetric in a sense that the players' strategies and payoffs are identical, one would imagine that a symmetric solution should exist. This is indeed the case, as the next Theorem ascertains.

Theorem 2. *Suppose that a game is symmetric, and for each player, the strategy space is compact and convex and the payoff function is continuous and quasiconcave with respect to each player's own strategy. Then, there exists at least one symmetric pure strategy NE in the game.*

To gain some intuition about why nonquasiconcave payoffs may lead to nonexistence of NE, suppose that in a two-player game, player 2 has a bimodal objective function with two local maxima. Furthermore, suppose that a small change in the strategy of player 1 leads to a shift of the global maximum for player 2 from one local maximum to another. To be more specific, let us say that at x_1', the global maximum $x_2^*(x_1')$ is on the left (Figure 3 left) and at x_1'', the global maximum $x_2^*(x_2'')$ is on the right (Figure 3 right). Hence, a small change in x_1 from x_1' to x_1'' induces a jump in the best response of player 2, x_2^*. The resulting best response mapping is presented in Figure 4, and there is no NE in pure strategies in this game. In other words, best response functions do not intersect anywhere. As a more specific example, see Netessine and Shumsky [72] for an extension of the newsvendor game to the situation in which product inventory is sold at two different prices; such a game may not have an NE because both players' objectives may be bimodal. Furthermore, Cachon and Harker [20] demonstrate that pure strategy NE may not exist in two other important settings: Two retailers competing with cost functions described by the economic order quantity (EOQ) model, or two service providers competing with service times described by the $M/M/1$ queuing model.

The assumption of a compact strategy space may seem restrictive. For example, in the newsvendor game, the strategy space R_+^2 is not bounded from above. However, we could

FIGURE 4. Nonexistence of NE.

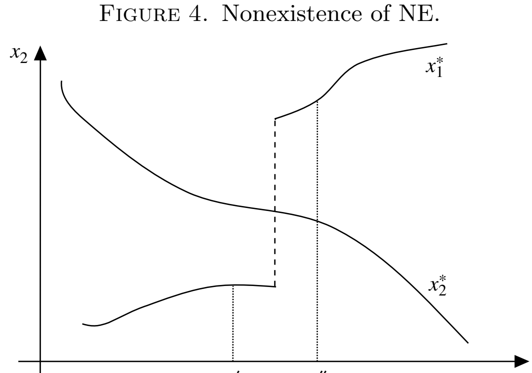
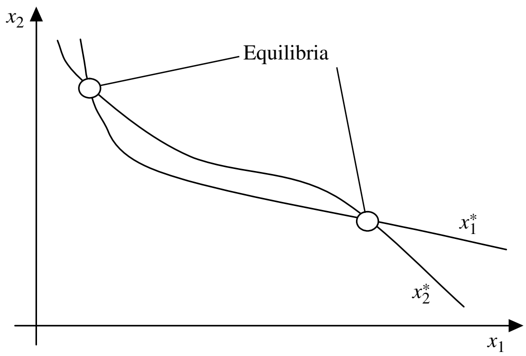


FIGURE 5. Nonuniqueness of the equilibrium.



easily bound it with some large enough finite number to represent the upper bound on the demand distribution. That bound would not impact any of the choices, and, therefore, the transformed game behaves just as the original game with an unbounded strategy space. (However, that bound cannot depend on any player's strategy choice.)

To continue with the newsvendor game analysis, it is easy to verify that the newsvendor's objective function is concave and, hence, quasiconcave w.r.t. the stocking quantity by taking the second derivative. Hence, the conditions of Theorem 1 are satisfied, and an NE exists. There are virtually dozens of papers employing Theorem 1. See, for example, Lippman and McCardle [56] for the proof involving quasiconcavity, and Mahajan and van Ryzin [58] and Netessine et al. [74] for the proofs involving concavity. Clearly, quasiconcavity of each player's objective function only implies uniqueness of the best response but does not imply a unique NE. One can easily envision a situation in which unique best response functions cross more than once so that there are multiple equilibria (see Figure 5).

If quasiconcavity of the players' payoffs cannot be verified, there is an alternative existence proof that relies on Tarski's [93] fixed-point theorem and involves the notion of supermodular games. The theory of supermodular games is a relatively recent development introduced and advanced by Topkis [96].

Definition 3. A twice continuously differentiable payoff function $\pi_i(x_1, \ldots, x_n)$ is supermodular (submodular) iff $\partial^2\pi_i/\partial x_i \partial x_j \geq 0$ (≤ 0) for all x and all $j \neq i$. The game is called supermodular if the players' payoffs are supermodular.

Supermodularity essentially means complementarity between any two strategies and is not linked directly to either convexity, concavity, or even continuity. (This is a significant advantage when forced to work with discrete strategies, e.g., Cachon [16].) However, similar to concavity/convexity, supermodularity/submodularity is preserved under maximization, limits, and addition and, hence, under expectation/integration signs, an important feature in stochastic SCM models. While in most situations the positive sign of the second derivative can be used to verify supermodularity (using Definition 3), sometimes it is necessary to utilize supermodularity-preserving transformations to show that payoffs are supermodular. Topkis [96] provides a variety of ways to verify that the function is supermodular, and some of these results are used in Cachon and Lariviere [22], Corbett [26], Netessine and Rudi [69, 71]. The following theorem follows directly from Tarski's fixed-point result and provides another tool to show existence of NE in noncooperative games:

Theorem 3. *In a supermodular game, there exists at least one NE.*

Coming back to the competitive newsvendors example, recall that the second-order cross-partial derivative was found to be

$$\frac{\partial^2\pi_i}{\partial Q_i \partial Q_j} = -r_i f_{D_i+(D_j-Q_j)^+|D_j>Q_j}(Q_i)\Pr(D_j > Q_j) < 0,$$

so that the newsvendor game is submodular, and, hence, existence of equilibrium cannot be assured. However, a standard trick is to redefine the ordering of the players' strategies. Let $y = -Q_j$ so that

$$\frac{\partial^2 \pi_i}{\partial Q_i \partial y} = r_i f_{D_i+(D_j+y)^+|D_j>Q_j}(Q_i) \Pr(D_j > -y) > 0,$$

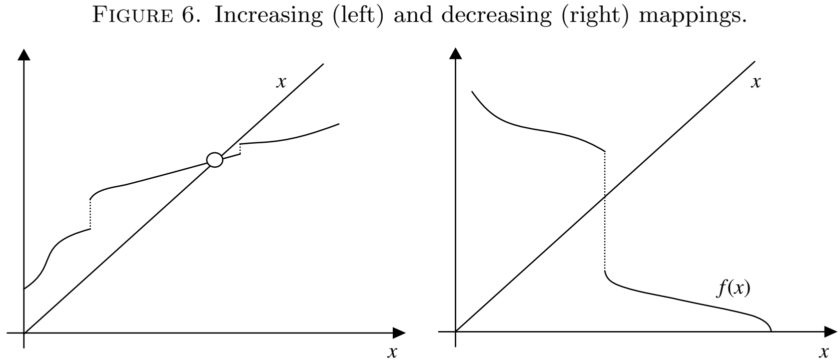
and the game becomes supermodular in (x_i, y), therefore, existence of NE is assured. Notice that we do not change either payoffs or the structure of the game, we only alter the ordering of one player's strategy space. Obviously, this trick only works in two-player games, see also Lippman and McCardle [56] for analysis of the more general version of the newsvendor game using a similar transformation. Hence, we can state that, in general, NE exists in games with decreasing best responses (submodular games) with two players. This argument can be generalized slightly in two ways that we mention briefly, see Vives [101] for details. One way is to consider an n-player game in which best responses are functions of aggregate actions of all other players, that is, $x_i^* = x_i^*(\sum_{j\neq i} x_j)$. If best responses in such a game are decreasing, then NE exists. Another generalization is to consider the same game with $x_i^* = x_i^*(\sum_{j\neq i} x_j)$ but require symmetry. In such a game, existence can be shown even with nonmonotone best responses, provided that there are only jumps up, but on intervals between jumps, best responses can be increasing or decreasing.

We now step back to discuss the intuition behind the supermodularity results. Roughly speaking, Tarski's fixed-point theorem only requires best response mappings to be nondecreasing for the existence of equilibrium and does not require quasiconcavity of the players' payoffs and allows for jumps in best responses. While it may be hard to believe that nondecreasing best responses is the only requirement for the existence of an NE, consider once again the simplest form of a single-dimensional equilibrium as a solution to the fixed-point mapping $x = f(x)$ on the compact set. It is easy to verify after a few attempts that if $f(x)$ is nondecreasing but possibly with jumps up, then it is not possible to derive a situation without an equilibrium. However, when $f(x)$ jumps down, nonexistence is possible (see Figure 6).

Hence, increasing best response functions is the only major requirement for an equilibrium to exist; players' objectives do not have to be quasiconcave or even continuous. However, to describe an existence theorem with noncontinuous payoffs requires the introduction of terms and definitions from lattice theory. As a result, we restricted ourselves to the assumption of continuous payoff functions, and in particular, to twice-differentiable payoff functions.

Although it is now clear why increasing best responses ensure existence of an equilibrium, it is not immediately obvious why Definition 3 provides a sufficient condition, given that it only concerns the sign of the second-order cross-partial derivative. To see this connection, consider separately the continuous and the discontinuous parts of the best response $x_i^*(x_j)$.

Figure 6. Increasing (left) and decreasing (right) mappings.



When the best response is continuous, we can apply the implicit function theorem to find its slope as follows

$$\frac{\partial x_i^*}{\partial x_j} = -\frac{\partial^2 \pi_i / \partial x_i \partial x_j}{\partial^2 \pi_i / \partial x_i^2}.$$

Clearly, if x_i^* is the best response, it must be the case that $\partial^2 \pi_i / \partial x_i^2 < 0$ or else it would not be the best response. Hence, for the slope to be positive, it is sufficient to have $\partial^2 \pi_i / \partial x_i \partial x_j > 0$, which is what Definition 3 provides. This reasoning does not, however, work at discontinuities in best responses because the implicit function theorem cannot be applied. To show that only jumps up are possible if $\partial^2 \pi_i / \partial x_i \partial x_j > 0$ holds, consider a situation in which there is a jump down in the best response. As one can recall, jumps in best responses happen when the objective function is bimodal (or more generally multimodal). For example, consider a specific point $x_j^\#$ and let $x_i^1(x_j^\#) < x_i^2(x_j^\#)$ be two distinct points at which first-order conditions hold (i.e., the objective function π_i is bimodal). Further, suppose $\pi_i(x_i^1(x_j^\#), x_j^\#) < \pi_i(x_i^2(x_j^\#), x_j^\#)$, but $\pi_i(x_i^1(x_j^\# + \varepsilon), x_j^\# + \varepsilon) > \pi_i(x_i^2(x_j^\# + \varepsilon), x_j^\# + \varepsilon)$. That is, initially, $x_i^2(x_j^\#)$ is a global maximum, but as we increase $x_j^\#$ infinitesimally, there is a *jump down*, and a smaller $x_i^1(x_j^\# + \varepsilon)$ becomes the global maximum. For this to be the case, it must be that

$$\frac{\partial \pi_i(x_i^1(x_j^\#), x_j^\#)}{\partial x_j} > \frac{\partial \pi_i(x_i^2(x_j^\#), x_j^\#)}{\partial x_j},$$

or, in words, the objective function rises faster at $(x_i^1(x_j^\#), x_j^\#)$ than at $(x_i^2(x_j^\#), x_j^\#)$. This, however, can only happen if $\partial^2 \pi_i / \partial x_i \partial x_j < 0$ at least somewhere on the interval $[x_i^1(x_j^\#), x_i^2(x_j^\#)]$, which is a contradiction. Hence, if $\partial^2 \pi_i / \partial x_i \partial x_j > 0$ holds, then only jumps up in the best response are possible.

2.4. Uniqueness of Equilibrium

From the perspective of generating qualitative insights, it is quite useful to have a game with a unique NE. If there is only one equilibrium, then one can characterize equilibrium actions without much ambiguity. Unfortunately, demonstrating uniqueness is generally much harder than demonstrating existence of equilibrium. This section provides several methods for proving uniqueness. No single method dominates; all may have to be tried to find the one that works. Furthermore, one should be careful to recognize that these methods assume existence, i.e., existence of NE must be shown separately. Finally, it is worth pointing out that uniqueness results are only available for games with continuous best response functions and, hence, there are no general methods to prove uniqueness of NE in supermodular games.

2.4.1. Method 1. Algebraic Argument. In some rather fortunate situations, one can ascertain that the solution is unique by simply looking at the optimality conditions. For example, in a two-player game, the optimality condition of one player may have a unique closed-form solution that does not depend on the other player's strategy, and, given the solution for one player, the optimality condition for the second player can be solved uniquely (Hall and Porteus [43], Netessine and Rudi [70]). In other cases, one can assure uniqueness by analyzing geometrical properties of the best response functions and arguing that they intersect only once. Of course, this is only feasible in two-player games. See Parlar [75] for a proof of uniqueness in the two-player newsvendor game and Majumder and Groenevelt [61] for a supply chain game with competition in reverse logistics. However, in most situations, these geometrical properties are also implied by the more formal arguments stated below. Finally, it may be possible to use a contradiction argument: Assume that there is more than one equilibrium and prove that such an assumption leads to a contradiction, as in Lederer and Li [54].

2.4.2. Method 2. Contraction Mapping Argument. Although the most restrictive among all methods, the contraction mapping argument is the most widely known and is the most frequently used in the literature because it is the easiest to verify. The argument is based on showing that the best response mapping is a contraction, which then implies the mapping has a unique fixed point. To illustrate the concept of a contraction mapping, suppose we would like to find a solution to the following fixed point equation:

$$x = f(x), \quad x \in R^1.$$

To do so, a sequence of values is generated by an iterative algorithm, $\{x^{(1)}, x^{(2)}, x^{(3)}, \ldots\}$ where $x^{(1)}$ is arbitrarily picked and $x^{(t)} = f(x^{(t-1)})$. The hope is that this sequence converges to a unique fixed point. It does so if, roughly speaking, each step in the sequence moves closer to the fixed point. One could verify that if $|f'(x)| < 1$ in some vicinity of x^*, then such an iterative algorithm converges to a unique $x^* = f(x^*)$. Otherwise, the algorithm diverges. Graphically, the equilibrium point is located on the intersection of two functions: x and $f(x)$. The iterative algorithm is presented in Figure 7. The iterative scheme in Figure 7 left is a contraction mapping: It approaches the equilibrium after every iteration.

Definition 4. Mapping $f(x)$, $R^n \to R^n$ is a contraction iff $\|f(x_1) - f(x_2)\| \le \alpha \|x_1 - x_2\|$, $\forall x_1, x_2, \alpha < 1$.

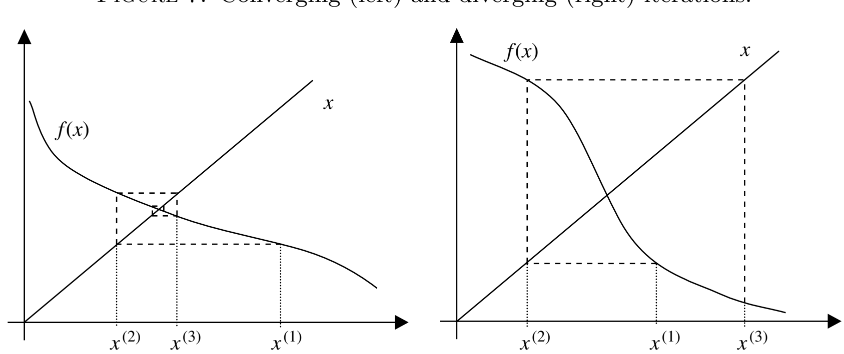
In words, the application of a contraction mapping to any two points strictly reduces (i.e., $\alpha = 1$ does not work) the distance between these points. The norm in the definition can be any norm, i.e., the mapping can be a contraction in one norm and not a contraction in another norm.

Theorem 4. *If the best response mapping is a contraction on the entire strategy space, there is a unique NE in the game.*

One can think of a contraction mapping in terms of iterative play: Player 1 selects some strategy, then player 2 selects a strategy based on the decision by player 1, etc. If the best response mapping is a contraction, the NE obtained as a result of such iterative play is *stable* but the opposite is not necessarily true; i.e., no matter where the game starts, the final outcome is the same. See also Moulin [62] for an extensive treatment of stable equilibria.

A major restriction in Theorem 4 is that the contraction mapping condition must be satisfied everywhere. This assumption is quite restrictive because the best response mapping may be a contraction locally, say, in some not necessarily small ε-neighborhood of the equilibrium, but not outside of it. Hence, if iterative play starts in this ε-neighborhood, then it converges to the equilibrium, but starting outside that neighborhood may not lead to the equilibrium (even if the equilibrium is unique). Even though one may wish to argue that it is reasonable for the players to start iterative play close to the equilibrium, formalization of such an argument is rather difficult. Hence, we must impose the condition that the entire

FIGURE 7. Converging (left) and diverging (right) iterations.



strategy space be considered. See Stidham [90] for an interesting discussion of stability issues in a queuing system.

While Theorem 4 is a starting point toward a method for demonstrating uniqueness, it does not actually explain how to validate that a best reply mapping is a contraction. Suppose we have a game with n players each endowed with the strategy x_i and we have obtained the best response functions for all players, $x_i = f_i(x_{-i})$. We can then define the following matrix of derivatives of the best response functions:

$$A = \begin{vmatrix} 0 & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & 0 & \cdots & \dfrac{\partial f_2}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & 0 \end{vmatrix}.$$

Further, denote by $\rho(A)$ the spectral radius of matrix A and recall that the spectral radius of a matrix is equal to the largest absolute eigenvalue $\rho(A) = \{\max |\lambda| : Ax = \lambda x, x \neq 0\}$ (Horn and Johnson [46]).

Theorem 5. *The mapping $f(x)$: $R^n \to R^n$ is a contraction if and only if $\rho(A) < 1$ everywhere.*

Theorem 5 is simply an extension of the iterative convergence argument we used above into multiple dimensions, and the spectral radius rule is an extension of the requirement $|f'(x)| < 1$. Still, Theorem 5 is not as useful as we would like it to be: Calculating eigenvalues of a matrix is not trivial. Instead, it is often helpful to use the fact that the largest eigenvalue and, hence, the spectral radius is bounded above by any of the matrix norms (Horn and Johnson [46]). So, instead of working with the spectral radius itself, it is sufficient to show $\|A\| < 1$ for any one matrix norm. The most convenient matrix norms are the maximum column-sum and the maximum row-sum norms (see Horn and Johnson [46] for other matrix norms). To use either of these norms to verify the contraction mapping, it is sufficient to verify that no column sum or no row sum of matrix A exceeds 1,

$$\sum_{i=1}^{n} \left| \frac{\partial f_k}{\partial x_i} \right| < 1 \quad \text{or} \quad \sum_{i=1}^{n} \left| \frac{\partial f_i}{\partial x_k} \right| < 1, \qquad \forall\, k.$$

Netessine and Rudi [69] used the contraction mapping argument in this most general form in the multiple-player variant of the newsvendor game described above.

A challenge associated with the contraction mapping argument is finding best response functions, because in most SC models, best responses cannot be found explicitly. Fortunately, Theorem 5 only requires the derivatives of the best response functions, which can be done using the implicit function theorem (from now on, IFT, see Bertsekas [12]). Using the IFT, Theorem 5 can be restated as

$$\sum_{i=1,\, i\neq k}^{n} \left| \frac{\partial^2 \pi_k}{\partial x_k \partial x_i} \right| < \left| \frac{\partial^2 \pi_k}{\partial x_k^2} \right|, \quad \forall\, k. \tag{2}$$

This condition is also known as "diagonal dominance" because the diagonal of the matrix of second derivatives, also called the Hessian, dominates the off-diagonal entries:

$$H = \begin{vmatrix} \dfrac{\partial^2 \pi_1}{\partial x_1^2} & \dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 \pi_2}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \pi_2}{\partial x_2^2} & \cdots & \dfrac{\partial^2 \pi_1}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 \pi_n}{\partial x_n \partial x_1} & \dfrac{\partial^2 \pi_n}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 \pi_n}{\partial x_n^2} \end{vmatrix}. \tag{3}$$

Contraction mapping conditions in the diagonal dominance form have been used extensively by Bernstein and Federgruen [7, 8, 9, 11]. As has been noted by Bernstein and Federgruen [10], many standard economic demand models satisfy this condition.

In games with only two players, the condition in Theorem 5 simplifies to

$$\left|\frac{\partial f_1}{\partial x_2}\right| < 1 \quad \text{and} \quad \left|\frac{\partial f_2}{\partial x_1}\right| < 1, \tag{4}$$

i.e., the slopes of the best response functions are less than one. This condition is especially intuitive if we use the graphic illustration (Figure 2). Given that the slope of each best response function is less than one everywhere, if they cross at one point then they cannot cross at an additional point. A contraction mapping argument in this form was used by Van Mieghem [97] and by Rudi et al. [81].

Returning to the newsvendor game example, we have found that the slopes of the best response functions are

$$\left|\frac{\partial Q_i^*(Q_j)}{\partial Q_j}\right| = \left|\frac{f_{D_i+(D_j-Q_j)^+|D_j>Q_j}(Q_i)\Pr(D_j>Q_j)}{f_{D_i+(D_j-Q_j)^+}(Q_i)}\right| < 1.$$

Hence, the best response mapping in the newsvendor game is a contraction, and the game has a unique and stable NE.

2.4.3. Method 3. Univalent Mapping Argument. Another method for demonstrating uniqueness of equilibrium is based on verifying that the best response mapping is one to one: That is, if $f(x)$ is a $R^n \to R^n$ mapping, then $y = f(x)$ implies that for all $x' \neq x$, $y \neq f(x')$. Clearly, if the best response mapping is one to one, then there can be at most one fixed point of such mapping. To make an analogy, recall that, if the equilibrium is interior,[4] the NE is a solution to the system of the first-order conditions: $\partial \pi_i / \partial x_i = 0$, $\forall i$, which defines the best response mapping. If this mapping is single-dimensional $R^1 \to R^1$, then it is quite clear that the condition sufficient for the mapping to be one to one is quasiconcavity of π_i. Similarly, for the $R^n \to R^n$ mapping to be one to one, we require quasiconcavity of the mapping, which translates into quasidefiniteness of the Hessian:

Theorem 6. *Suppose the strategy space of the game is convex and all equilibria are interior. Then, if the determinant $|H|$ is negative quasidefinite (i.e., if the matrix $H + H^T$ is negative definite) on the players' strategy set, there is a unique NE.*

[4]Interior equilibrium is the one in which first-order conditions hold for each player. The alternative is boundary equilibrium in which at least one of the players select the strategy on the boundary of his strategy space.

Proof of this result can be found in Gale and Nikaido [40] and some further developments that deal with boundary equilibria are found in Rosen [80]. Notice that the univalent mapping argument is somewhat weaker than the contraction mapping argument. Indeed, the restatement (2) of the contraction mapping theorem directly implies univalence because the dominant diagonal assures us that H is negative definite. Hence, it is negative quasidefinite. It immediately follows that the newsvendor game satisfies the univalence theorem. However, if some other matrix norm is used, the relationship between the two theorems is not that specific. In the case of just two players, the univalence theorem can be written as, according to Moulin [62],

$$\left|\frac{\partial^2 \pi_2}{\partial x_2 \partial x_1} + \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2}\right| \le 2\sqrt{\left|\frac{\partial^2 \pi_1}{\partial x_1^2} \cdot \frac{\partial^2 \pi_2}{\partial x_2^2}\right|}, \quad \forall x_1, x_2.$$

2.4.4. Method 4. Index Theory Approach. This method is based on the Poincare-Hopf index theorem found in differential topology (Guillemin and Pollak [42]). Similar to the univalence mapping approach, it requires a certain sign from the Hessian, but this requirement need hold only at the equilibrium point.

Theorem 7. *Suppose the strategy space of the game is convex and all payoff functions are quasiconcave. Then, if $(-1)^n|H|$ is positive whenever $\partial\pi_i/\partial x_i = 0$, all i, there is a unique NE.*

Observe that the condition $(-1)^n|H|$ is trivially satisfied if $|H|$ is negative definite, which is implied by the condition (2) of contraction mapping, i.e., this method is also somewhat weaker than the contraction mapping argument. Moreover, the index theory condition need only hold at the equilibrium. This makes it the most general, but also the hardest to apply. To gain some intuition about why the index theory method works, consider the two-player game. The condition of Theorem 7 simplifies to

$$\begin{vmatrix} \dfrac{\partial^2 \pi_1}{\partial x_1^2} & \dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 \pi_2}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \pi_2}{\partial x_2^2} \end{vmatrix} > 0 \quad \forall x_1, x_2 : \frac{\partial \pi_1}{\partial x_1} = 0, \frac{\partial \pi_2}{\partial x_2} = 0,$$

which can be interpreted as meaning the multiplication of the slopes of best response functions should not exceed one at the equilibrium:

$$\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} < 1 \quad \text{at } x_1^*, x_2^*. \tag{5}$$

As with the contraction mapping approach, with two players, the Theorem becomes easy to visualize. Suppose we have found best response functions $x_1^* = f_1(x_2)$ and $x_2^* = f_2(x_1)$ as in Figure 2. Find an inverse function $x_2 = f_1^{-1}(x_1)$ and construct an auxiliary function $g(x_1) = f_1^{-1}(x_1) - f_2(x_1)$ that measures the distance between two best responses. It remains to show that $g(x_1)$ crosses zero only once because this would directly imply a single crossing point of $f_1(x_1)$ and $f_2(x_2)$. Suppose we could show that every time $g(x_1)$ crosses zero, it does so *from below*. If that is the case, we are assured there is only a single crossing: It is impossible for a continuous function to cross zero more than once from below because it would also have to cross zero from above somewhere. It can be shown that the function $g(x_1)$ crosses zero only from below if the slope of $g(x_1)$ *at the crossing point* is positive as follows

$$\frac{\partial g(x_1)}{\partial x_1} = \frac{\partial f_1^{-1}(x_1)}{\partial x_1} - \frac{\partial f_2(x_1)}{\partial x_1} = \frac{1}{\partial f_2(x_2)/\partial x_2} - \frac{\partial f_2(x_1)}{\partial x_1} > 0,$$

which holds if (5) holds. Hence, in a two-player game condition, (5) is sufficient for the uniqueness of the NE. Note that condition (5) trivially holds in the newsvendor game because each slope is less than one, and, hence, the multiplication of slopes is less than one as well *everywhere*. Index theory has been used by Netessine and Rudi [71] to show uniqueness of the NE in a retailer-wholesaler game when both parties stock inventory and sell directly to consumers and by Cachon and Kok [21] and Cachon and Zipkin [24].

2.5. Multiple Equilibria

Many games are just not blessed with a unique equilibrium. The next best situation is to have a few equilibria. The worst situation is either to have an infinite number of equilibria or no equilibrium at all. The obvious problem with multiple equilibria is that the players may not know which equilibrium will prevail. Hence, it is entirely possible that a nonequilibrium outcome results because one player plays one equilibrium strategy while a second player chooses a strategy associated with another equilibrium. However, if a game is repeated, then it is possible that the players eventually find themselves in one particular equilibrium. Furthermore, that equilibrium may not be the most desirable one.

If one does not want to acknowledge the possibility of multiple outcomes due to multiple equilibria, one could argue that one equilibrium is more reasonable than the others. For example, there may exist only one symmetric equilibrium, and one may be willing to argue that a symmetric equilibrium is more focal than an asymmetric equilibrium. (See Mahajan and van Ryzin [58] for an example). In addition, it is generally not too difficult to demonstrate the uniqueness of a symmetric equilibrium. If the players have unidimensional strategies, then the system of n first-order conditions reduces to a single equation, and one need only show that there is a unique solution to that equation to prove the symmetric equilibrium is unique. If the players have m-dimensional strategies, $m > 1$, then finding a symmetric equilibrium reduces to determining whether a system of m equations has a unique solution (easier than the original system, but still challenging).

An alternative method to rule out some equilibria is to focus only on the Pareto optimal equilibrium, of which there may be only one. For example, in supermodular games, the equilibria are Pareto rankable under an additional condition that each player's objective function is increasing in other players' strategies, i.e., there is a most preferred equilibrium by every player and a least preferred equilibrium by every player. (See Wang and Gerchak [104] for an example.) However, experimental evidence exists that suggests players do not necessarily gravitate to the Pareto optimal equilibrium as is demonstrated by Cachon and Camerer [19]. Hence, caution is warranted with this argument.

2.6. Comparative Statics in Games

In GT models, just as in the noncompetitive SCM models, many of the managerial insights and results are obtained through comparative statics, such as monotonicity of the optimal decisions w.r.t. some parameter of the game.

2.6.1. The Implicit Functions Theorem Approach.

This approach works for both GT and single decision-maker applications, as will become evident from the statement of the next theorem.

Theorem 8. *Consider the system of equations*

$$\frac{\partial \pi_i(x_1, \ldots, x_n, a)}{\partial x_i} = 0, \quad i = 1, \ldots, n,$$

defining $x_1^*, \ldots, x_n^*$ as implicit functions of parameter a. If all derivatives are continuous functions and the Hessian (3) evaluated at $x_1^*, \ldots, x_n^*$ is nonzero, then the function $x^*(a)\colon R^1 \to R^n$ is continuous on a ball around x^* and its derivatives are found as follows:

$$\begin{vmatrix} \dfrac{\partial x_1^*}{\partial a} \\ \dfrac{\partial x_2^*}{\partial a} \\ \cdots \\ \dfrac{\partial x_n^*}{\partial a} \end{vmatrix} = -\begin{vmatrix} \dfrac{\partial^2 \pi_1}{\partial x_1^2} & \dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 \pi_2}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \pi_2}{\partial x_2^2} & \cdots & \dfrac{\partial^2 \pi_1}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 \pi_n}{\partial x_n \partial x_1} & \dfrac{\partial^2 \pi_n}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 \pi_n}{\partial x_n^2} \end{vmatrix}^{-1} \begin{vmatrix} \dfrac{\partial \pi_1}{\partial x_1 \partial a} \\ \dfrac{\partial \pi_1}{\partial x_2 \partial a} \\ \cdots \\ \dfrac{\partial \pi_1}{\partial x_n \partial a} \end{vmatrix}. \tag{6}$$

Because the IFT is covered in detail in many nonlinear programming books and its application to the GT problems is essentially the same, we do not delve further into this matter. In many practical problems, if $|H| \neq 0$, then it is instrumental to multiply both sides of the expression (6) by H^{-1}. That is justified because the Hessian is assumed to have a nonzero determinant to avoid the cumbersome task of inverting the matrix. The resulting expression is a system of n linear equations, which have a closed-form solution. See Netessine and Rudi [71] for such an application of the IFT in a two-player game and Bernstein and Federgruen [8] in n-player games.

The solution to (6) in the case of two players is

$$\frac{\partial x_1^*}{\partial a} = -\frac{\dfrac{\partial^2 \pi_1}{\partial x_1 \partial a}\dfrac{\partial^2 \pi_2}{\partial x_2^2} - \dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_2}\dfrac{\partial^2 \pi_2}{\partial x_2 \partial a}}{|H|}, \tag{7}$$

$$\frac{\partial x_2^*}{\partial a} = -\frac{\dfrac{\partial^2 \pi_1}{\partial x_1^2}\dfrac{\partial^2 \pi_2}{\partial x_2 \partial a} - \dfrac{\partial^2 \pi_1}{\partial x_1 \partial a}\dfrac{\partial^2 \pi_2}{\partial x_2 \partial x_1}}{|H|}. \tag{8}$$

Using our newsvendor game as an example, suppose we would like to analyze sensitivity of the equilibrium solution to changes in r_1 so let $a = r_1$. Notice that $\partial^2\pi_2/\partial Q_2 \partial r_1$ and also that the determinant of the Hessian is positive. Both expressions in the numerator of (7) are positive as well, so that $\partial Q_1^*/\partial r_1 > 0$. Further, the numerator of (8) is negative, so that $\partial Q_2^*/\partial r_1 < 0$. Both results are intuitive.

Solving a system of n equations analytically is generally cumbersome, and one may have to use Kramer's rule or analyze an inverse of H instead, see Bernstein and Federgruen [8] for an example. The only way to avoid this complication is to employ supermodular games as described below. However, the IFT method has an advantage that is not enjoyed by supermodular games: It can handle constraints of any form. That is, any constraint on the players' strategy spaces of the form $g_i(x_i) \le 0$ or $g_i(x_i) = 0$ can be added to the objective function by forming a Lagrangian:

$$L_i(x_1, \ldots, x_n, \lambda_i) = \pi_i(x_1, \ldots, x_n) - \lambda_i g_i(x_i).$$

All analysis can then be carried through the same way as before with the only addition being that the Lagrange multiplier λ_i becomes a decision variable. For example, let us assume in the newsvendor game that the two competing firms stock inventory at a warehouse. Further, the amount of space available to each company is a function of the total warehouse capacity C, e.g., $g_i(Q_i) \le C$. We can construct a new game in which each retailer solves the following problem:

$$\max_{Q_i \in \{g_i(Q_i) \le C\}} E_D[r_i \min(D_i + (D_j - Q_j)^+, Q_i) - c_i Q_i], \quad i = 1, 2.$$

Introduce two Lagrange multipliers, λ_i, $i=1,2$ and rewrite the objective functions as

$$\max_{Q_i,\lambda_i} L(Q_i,\lambda_i,Q_j) = E_D[r_i \min(D_i+(D_j-Q_j)^+,Q_i) - c_iQ_i - \lambda_i(g_i(Q_i)-C)].$$

The resulting four optimality conditions can be analyzed using the IFT the same way as has been demonstrated previously.

2.6.2. Supermodular Games Approach. In some situations, supermodular games provide a more convenient tool for comparative statics.

Theorem 9. *Consider a collection of supermodular games on R^n parameterized by a parameter a. Further, suppose $\partial^2\pi_i/\partial x_i\partial a \geq 0$ for all i. Then, the largest and the smallest equilibria are increasing in a.*

Roughly speaking, a sufficient condition for monotone comparative statics is supermodularity of players' payoffs in strategies *and* a parameter. Note that, if there are multiple equilibria, we cannot claim that every equilibrium is monotone in a; rather, a set of all equilibria is monotone in the sense of Theorem 9. A convenient way to think about the last Theorem is through the augmented Hessian:

$$\begin{vmatrix} \dfrac{\partial^2\pi_1}{\partial x_1^2} & \dfrac{\partial^2\pi_1}{\partial x_1\partial x_2} & \cdots & \dfrac{\partial^2\pi_1}{\partial x_1\partial x_n} & \dfrac{\partial^2\pi_1}{\partial x_1\partial a} \\ \dfrac{\partial^2\pi_2}{\partial x_2\partial x_1} & \dfrac{\partial^2\pi_2}{\partial x_2^2} & \cdots & \dfrac{\partial^2\pi_1}{\partial x_2\partial x_n} & \dfrac{\partial^2\pi_1}{\partial x_2\partial a} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2\pi_n}{\partial x_n\partial x_1} & \dfrac{\partial^2\pi_n}{\partial x_n\partial x_2} & \cdots & \dfrac{\partial^2\pi_n}{\partial x_n^2} & \dfrac{\partial^2\pi_n}{\partial x_n\partial a} \\ \dfrac{\partial^2\pi_1}{\partial x_1\partial a} & \dfrac{\partial^2\pi_1}{\partial x_2\partial a} & \cdots & \dfrac{\partial^2\pi_n}{\partial x_n\partial a} & \dfrac{\partial^2\pi_n}{\partial a^2} \end{vmatrix}.$$

Roughly, if all *off-diagonal* elements of this matrix are positive, then the monotonicity result holds (signs of diagonal elements do not matter and, hence, concavity is not required). To apply this result to competing newsvendors, we will analyze sensitivity of equilibrium inventories (Q_i^*,Q_j^*) to r_i. First, transform the game to strategies (Q_i,y) so that the game is supermodular and find cross-partial derivatives

$$\frac{\partial^2\pi_i}{\partial Q_i\partial r_i} = \Pr(D_i+(D_j-Q_j)^+ > Q_i) \geq 0,$$

$$\frac{\partial\pi_j}{\partial y\partial r_i} = 0 \geq 0,$$

so that (Q_i^*,y^*) are both increasing in r_i, or Q_i^* is increasing and Q_j^* is decreasing in r_i just as we have already established using the IFT.

The simplicity of the argument (once supermodular games are defined) as compared to the machinery required to derive the same result using the IFT is striking. Such simplicity has attracted much attention in SCM and has resulted in extensive applications of supermodular games. Examples include Cachon [16], Corbett and DeCroix [27], and Netessine and Rudi [71] to name just a few. There is, however, an important limitation to the use of Theorem 9: It cannot handle many constraints as IFT can. Namely, the decision space must be a lattice to apply supermodularity, i.e., it must include its coordinatewise maximum and minimum. Hence, a constraint of the form $x_i \leq b$ can be handled, but a constraint $x_i + x_j \leq b$ cannot because points $(x_i,x_j)=(b,0)$ and $(x_i,x_j)=(0,b)$ are within the constraint but the coordinatewise maximum of these two points (b,b) is not. Notice that to avoid dealing with this issue in detail, we stated in the theorems that the strategy space should all be R^n. Because many SCM applications have constraints on the players' strategies, supermodularity must be applied with care.

3. Dynamic Games

While many SCM models are static—including all newsvendor-based models—a significant portion of the SCM literature is devoted to dynamic models in which decisions are made over time. In most cases, the solution concept for these games is similar to the backward induction used when solving dynamic programming problems. There are, however, important differences, as will be clear from the discussion of repeated games. As with dynamic programming problems, we continue to focus on the games of complete information, i.e., at each move in the game all players know the full history of play.

3.1. Sequential Moves: Stackelberg Equilibrium Concept

The simplest possible dynamic game was introduced by von Stackelberg [103]. In a Stackelberg duopoly model, player 1—the Stackelberg leader—chooses a strategy first, and then player 2—the Stackelberg follower—observes this decision and makes his own strategy choice. Because in many SCM models the upstream firm—e.g., the wholesaler—possesses certain power over the typically smaller downstream firm—e.g., the retailer—the Stackelberg equilibrium concept has found many applications in SCM literature. We do not address the issues of who should be the leader and who should be the follower; see Chapter 11 in Simchi-Levi et al. [88].

To find an equilibrium of a Stackelberg game, which often is called the Stackelberg equilibrium, we need to solve a dynamic multiperiod problem via backward induction. We will focus on a two-period problem for analytical convenience. First, find the solution $x_2^*(x_1)$ for the second player as a response to any decision made by the first player:

$$x_2^*(x_1): \quad \frac{\partial \pi_2(x_2, x_1)}{\partial x_2} = 0.$$

Next, find the solution for the first player anticipating the response by the second player:

$$\frac{d\pi_1(x_1, x_2^*(x_1))}{dx_1} = \frac{\partial \pi_1(x_1, x_2^*)}{\partial x_1} + \frac{\partial \pi_1(x_1, x_2)}{\partial x_2} \frac{\partial x_2^*}{\partial x_1} = 0.$$

Intuitively, the first player chooses the best possible point on the second player's best response function. Clearly, the first player can choose an NE, so the leader is always at least as well off as he would be in NE. Hence, if a player were allowed to choose between making moves simultaneously or being a leader in a game with complete information, he would always prefer to be the leader. However, if new information is revealed after the leader makes a play, then it is not always advantageous to be the leader.

Whether the follower is better off in the Stackelberg or simultaneous move game depends on the specific problem setting. See Netessine and Rudi [70] for examples of both situations and comparative analysis of Stackelberg versus NE; see also Wang and Gerchak [104] for a comparison between the leader versus follower roles in a decentralized assembly model. For example, consider the newsvendor game with sequential moves. The best response function for the second player remains the same as in the simultaneous move game:

$$Q_2^*(Q_1) = F^{-1}_{D_2 + (D_1 - Q_1)^+}\left(\frac{r_2 - c_2}{r_2}\right).$$

For the leader, the optimality condition is

$$\begin{aligned}\frac{d\pi_1(Q_1, Q_2^*(Q_1))}{dQ_1} &= r_1 \Pr(D_1 + (D_2 - Q_2)^+ > Q_1) - c_1 \\ &\qquad - r_1 \Pr(D_1 + (D_2 - Q_2)^+ < Q_1, D_2 > Q_2) \frac{\partial Q_2^*}{\partial Q_1} \\ &= 0,\end{aligned}$$

where $\partial Q_2^*/\partial Q_1$ is the slope of the best response function found in (1). Existence of a Stackelberg equilibrium is easy to demonstrate given the continuous payoff functions. However, uniqueness may be considerably harder to demonstrate. A sufficient condition is quasiconcavity of the leader's profit function, $\pi_1(x_1, x_2^*(x_1))$. In the newsvendor game example, this implies the necessity of finding derivatives of the density function of the demand distribution, as is typical for many problems involving uncertainty. In stochastic models, this is feasible with certain restrictions on the demand distribution. See Lariviere and Porteus [53] for an example with a supplier that establishes the wholesale price and a newsvendor that then chooses an order quantity and Cachon [18] for the reverse scenario in which a retailer sets the wholesale price and buys from a newsvendor supplier. See Netessine and Rudi [70] for a Stackelberg game with a wholesaler choosing a stocking quantity and the retailer deciding on promotional effort. One can further extend the Stackelberg equilibrium concept into multiple periods; see Erhun et al. [34] and Anand et al. [1] for examples.

3.2. Simultaneous Moves: Repeated and Stochastic Games

A different type of dynamic game arises when both players take actions in multiple periods. Because inventory models used in SCM literature often involve inventory replenishment decisions that are made over and over again, multiperiod games should be a logical extension of these inventory models. Two major types of multiple-period games exist: without and with time dependence.

In the multiperiod game without time dependence, the exact same game is played over and over again, hence, the term *repeated* games. The strategy for each player is now a sequence of actions taken in all periods. Consider one repeated game version of the newsvendor game in which the newsvendor chooses a stocking quantity at the start of each period, demand is realized, and then leftover inventory is salvaged. In this case, there are no links between successive periods other than the players' memories about actions taken in all the previous periods. Although repeated games have been extensively analyzed in economics literature, it is awkward in an SCM setting to assume that nothing links successive games; typically, in SCM, there is some transfer of inventory and/or backorders between periods. As a result, repeated games thus far have not found many applications in the SCM literature. Exceptions are Debo [28], Ren et al. [79], and Taylor and Plambeck [94] in which reputational effects are explored as means of supply chain coordination in place of the formal contracts.

A fascinating feature of repeated games is that the set of equilibria is much larger than the set of equilibria in a static game, and may include equilibria that are not possible in the static game. At first, one may assume that the equilibrium of the repeated game would be to play the same static NE strategy in each period. This is, indeed, an equilibrium, but only one of many. Because in repeated games the players are able to condition their behavior on the observed actions in the previous periods, they may employ so-called *trigger strategies*: The player will choose one strategy until the opponent changes his play, at which point the first player will change the strategy. This *threat* of reverting to a different strategy may even induce players to achieve the best possible outcome, i.e., the centralized solution, which is called an *implicit collusion*. Many such threats are, however, noncredible in the sense that once a part of the game has been played, such a strategy is not an equilibrium anymore for the remainder of the game, as is the case in our example in Figure 1. To separate out credible threats from noncredible, Selten [82] introduced the notion of a *subgame-perfect equilibrium*. See Hall and Porteus [43] and Van Mieghem and Dada [98] for solutions involving subgame-perfect equilibria in dynamic games.

Subgame-perfect equilibria reduce the equilibrium set somewhat. However, infinitely repeated games are still particularly troublesome in terms of multiplicity of equilibria. The famous Folk theorem[5] proves that any convex combination of the feasible payoffs is attainable in the infinitely repeated game as an equilibrium, implying that "virtually anything" is an equilibrium outcome.[6] See Debo [28] for the analysis of a repeated game between the wholesaler setting the wholesale price and the newsvendor setting the stocking quantity.

In time-dependent multiperiod games, players' payoffs in each period depend on the actions in the previous as well as current periods. Typically, the payoff structure does not change from period to period (so called stationary payoffs). Clearly, such setup closely resembles multiperiod inventory models in which time periods are connected through the transfer of inventories and backlogs. Due to this similarity, time-dependent games have found applications in SCM literature. We will only discuss one type of time-dependent multiperiod games, *stochastic games* or *Markov games*, due to their wide applicability in SCM. See also Majumder and Groenevelt [61] for the analysis of deterministic time-dependent multiperiod games in reverse logistics supply chains. Stochastic games were developed by Shapley [84] and later by Heyman and Sobel [45], Kirman and Sobel [48], and Sobel [89]. The theory of stochastic games is also extensively covered in Filar and Vrieze [36].

The setup of the stochastic game is essentially a combination of a static game and a Markov decisions process: In addition to the set of players with strategies—which is now a vector of strategies, one for each period, and payoffs—we have a set of states and a transition mechanism $p(s'|s,x)$, probability that we transition from state s to state s' given action x. Transition probabilities are typically defined through random demand occurring in each period. The difficulties inherent in considering nonstationary inventory models are passed over to the game-theoretic extensions of these models, therefore, a standard simplifying assumption is that demands are independent and identical across periods. When only a single decision maker is involved, such an assumption leads to a unique stationary solution (e.g., stationary inventory policy of some form: order-up-to, S-s, etc.). In a GT setting, however, things get more complicated; just as in the repeated games described above, nonstationary equilibria, e.g., trigger strategies, are possible. A standard approach is to consider just one class of equilibria—e.g., stationary—because nonstationary policies are hard to implement in practice and they are not always intuitively appealing. Hence, with the assumption that the policy is stationary, the stochastic game reduces to an equivalent static game, and equilibrium is found as a sequence of NE in an appropriately modified single-period game. Another approach is to focus on "Markov" or "state-space" strategies in which the past influences the future through the state variables but not through the history of the play. A related equilibrium concept is that of *Markov perfect equilibrium* (MPE), which is simply a profile of Markov strategies that yields a Nash equilibrium in every subgame. The concept of MPE is discussed in Fudenberg and Tirole [38], Chapter 13. See also Tayur and Yang [95] for the application of this concept.

To illustrate, consider an infinite-horizon variant of the newsvendor game with lost sales in each period and inventory carry-over to the subsequent period; see Netessine et al. [74] for complete analysis. The solution to this problem in a noncompetitive setting is an order-up-to policy. In addition to unit-revenue r and unit-cost c, we introduce inventory holding cost h incurred by a unit carried over to the next period and a discount factor β. Also, denote by x_i^t the inventory position at the beginning of the period and by y_i^t the order-up-to quantity. Then, the infinite-horizon profit of each player is

$$\pi_i(x^1) = E\sum_{t=1}^{\infty}\beta_i^{t-1}[r_i\min(y_i^t, D_i^t + (D_j^t - y_j^t)^+) - h_i(y_i^t - D_i^t - (D_j^t - y_j^t)^+)^+ - c_iQ_i^t],$$

[5]The name is due to the fact that its source is unknown and dates back to 1960; Friedman [37] was one of the first to treat Folk theorem in detail.

[6]A condition needed to insure attainability of an equilibrium solution is that the discount factor is large enough. The discount factor also affects effectiveness of trigger and many other strategies.

with the inventory transition equation

$$x_i^{t+1} = (y_i^t - D_i^t - (D_j^t - y_j^t)^+)^+.$$

Using the standard manipulations from Heyman and Sobel [45], this objective function can be converted to

$$\pi_i(x^1) = c_i x_i^1 + \sum_{t=1}^{\infty} \beta_i^{t-1} G_i^t(y_i^t), \quad i = 1, 2,$$

where $G_i^t(y_i^t)$ is a single-period objective function

$$\begin{aligned} G_i^t(y_i^t) = {} & E[(r_i - c_i)(D_i^t + (D_j^t - y_j^t)^+) - (r_i - c_i)(D_i^t + (D_j^t - y_j^t)^+ - y_i^t)^+ \\ & - (h_i + c_i(1 - \beta_i))(y_i^t - D_i^t - (D_j^t - y_j^t)^+)^+], \quad i = 1, 2, \, t = 1, 2, \ldots \, . \end{aligned}$$

Assuming demand is stationary and independently distributed across periods $D_i = D_i^t$, we further obtain that $G_i^t(y_i^t) = G_i(y_i^t)$ because the single-period game is the same in each period. By restricting consideration to the stationary inventory policy $y_i = y_i^t$, $t = 1, 2, \ldots$, we can find the solution to the multiperiod game as a sequence of the solutions to a single-period game $G_i(y_i)$, which is

$$y_i^* = F^{-1}_{D_i + (D_j - y_j^*)^+}\left(\frac{r_i - c_i}{r_i + h_i - c_i \beta_i}\right), \quad i = 1, 2.$$

With the assumption that the equilibrium is stationary, one could argue that stochastic games are no different from static games; except for a small change in the right-hand side reflecting inventory carry-over and holding costs, the solution is essentially the same. However, more elaborate models capture some effects that are not present in static games but can be envisioned in stochastic games. For example, if we were to introduce backlogging in the above model, a couple of interesting situations would arise: A customer may backlog the product with either the first or with the second competitor he visits if both are out of stock. These options introduce the behavior that is observed in practice but cannot be modeled within the static game (see Netessine et al. [74] for detailed analysis) because firms' inventory decisions affect their demand in the future. Among other applications of stochastic games are papers by Cachon and Zipkin [24] analyzing a two-echelon game with the wholesaler and the retailer making stocking decisions, Bernstein and Federgruen [10] analyzing price and service competition, Netessine and Rudi [70] analyzing the game with the retailer exerting sales effort and the wholesaler stocking the inventory, and Van Mieghem and Dada [98] studying a two-period game with capacity choice in the first period and production decision under the capacity constraint in the second period.

3.3. Differential Games

So far, we have described dynamic games in discrete time, i.e., games involving a sequence of decisions separated in time. *Differential games* provide a natural extension for decisions that have to be made continuously. Because many SC models rely on continuous-time processes, it is natural to assume that differential games should find a variety of applications in SCM literature. However, most SCM models include stochasticity in one form or another. At the same time, due to the mathematical difficulties inherent in differential games, we are only aware of deterministic differential GT models in SCM. Although theory for stochastic differential games does exist, applications are quite limited (Basar and Olsder [6]). Marketing and economics have been far more successful in applying differential games because deterministic models are standard in these areas. Hence, we will only briefly outline some new concepts necessary to understand the theory of differential games.

The following is a simple example of a differential game taken from Kamien and Schwartz [47]. Suppose two players indexed by $i = 1, 2$ are engaged in production and sales of the same product. Firms choose production levels $u_i(t)$ at any moment of time and incur total cost $C_i(u_i) = cu_i + u_i^2/2$. The price in the market is determined as per Cournot competition. Typically, this would mean that $p(t) = a - u_1(t) - u_2(t)$. However, the twist in this problem is that if the production level is changed, price adjustments are not instantaneous. Namely, there is a parameter s, referred to as the speed of price adjustment, so that the price is adjusted according to the following differential equation:

$$p'(t) = s[a - u_1(t) - u_2(t) - p(t)], \quad p(0) = p_0.$$

Finally, each firm maximizes discounted total profit

$$\pi_i = \int_0^\infty e^{-rt}(p(t)u_i(t) - C_i(u_i(t)))\,dt, \quad i = 1, 2.$$

The standard tools needed to analyze differential games are the calculus of variations or optimal control theory (Kamien and Schwartz [47]). In a standard optimal control problem, a single decision maker sets the control variable that affects the state of the system. In contrast, in differential games, several players select control variables that may affect a common state variable and/or payoffs of all players. Hence, differential games can be looked at as a natural extension of the optimal control theory. In this section, we will consider two distinct types of player strategies: *open loop* and *closed loop*, which is also sometimes called *feedback*. In the open-loop strategy, the players select their decisions or control variables once at the beginning of the game and do not change them, so that the control variables are only functions of time and do not depend on the other players' strategies. Open-loop strategies are simpler in that they can be found through the straightforward application of optimal control that makes them quite popular. Unfortunately, an open-loop strategy may not be subgame perfect. On the contrary, in a closed-loop strategy, the player bases his strategy on current time and the states of both players' systems. Hence, feedback strategies are subgame perfect: If the game is stopped at any time, for the remainder of the game, the same feedback strategy will be optimal, which is consistent with the solution to the dynamic programming problems that we employed in the stochastic games section. The concept of a feedback strategy is more satisfying, but is also more difficult to analyze. In general, optimal open-loop and feedback strategies differ, but they may coincide in some games.

Because it is hard to apply differential game theory in stochastic problems, we cannot utilize the competitive newsvendor problem to illustrate the analysis. Moreover, the analysis of even the most trivial differential game is somewhat involved mathematically, so we will limit our survey to stating and contrasting optimality conditions in the cases of open-loop and closed-loop NE. Stackelberg equilibrium models do exist in differential games as well but are rarer (Basar and Olsder [6]). Due to mathematical complexity, games with more than two players are rarely analyzed. In a differential game with two players, each player is endowed with a control $u_i(t)$ that the player uses to maximize the objective function π_i

$$\max_{u_i(t)} \pi_i(u_i, u_j) = \max_{u_i(t)} \int_0^T f_i(t, x_i(t), x_j(t), u_i(t), u_j(t))\,dt,$$

where $x_i(t)$ is a state variable describing the state of the system. The state of the system evolves according to the differential equation

$$x_i'(t) = g_i(t, x_i(t), x_j(t), u_i(t), u_j(t)),$$

which is the analog of the inventory transition equation in the multiperiod newsvendor problem. Finally, there are initial conditions $x_i(0) = x_{i0}$.

The open-loop strategy implies that each player's control is only a function of time, $u_i = u_i(t)$. A feedback strategy implies that each players' control is also a function of state variables, $u_i = u_i(t, x_i(t), x_j(t))$. As in the static games, NE is obtained as a fixed point of the best response mapping by simultaneously solving a system of first-order optimality conditions for the players. Recall that to find the optimal control, we first need to form a Hamiltonian. If we were to solve two individual noncompetitive optimization problems, the Hamiltonians would be $H_i = f_i + \lambda_i g_i$, $i = 1, 2$, where $\lambda_i(t)$ is an adjoint multiplier. However, with two players, we also have to account for the state variable of the opponent so that the Hamiltonian becomes

$$H_i = f_i + \lambda_i^1 g_i + \lambda_i^2 g_j, \quad i, j = 1, 2.$$

To obtain the necessary conditions for the open-loop NE, we simply use the standard necessary conditions for any optimal control problem:

$$\frac{\partial H_1}{\partial u_1} = 0, \quad \frac{\partial H_2}{\partial u_2} = 0, \tag{9}$$

$$\frac{\partial \lambda_1^1}{\partial t} = -\frac{\partial H_1}{\partial x_1}, \quad \frac{\partial \lambda_1^2}{\partial t} = -\frac{\partial H_1}{\partial x_2}, \tag{10}$$

$$\frac{\partial \lambda_2^1}{\partial t} = -\frac{\partial H_2}{\partial x_2}, \quad \frac{\partial \lambda_2^2}{\partial t} = -\frac{\partial H_2}{\partial x_1}. \tag{11}$$

For the feedback equilibrium, the Hamiltonian is the same as for the open-loop strategy. However, the necessary conditions are somewhat different:

$$\frac{\partial H_1}{\partial u_1} = 0, \quad \frac{\partial H_2}{\partial u_2} = 0, \tag{12}$$

$$\frac{\partial \lambda_1^1}{\partial t} = -\frac{\partial H_1}{\partial x_1} - \frac{\partial H_1}{\partial u_2}\frac{\partial u_2^*}{\partial x_1}, \quad \frac{\partial \lambda_1^2}{\partial t} = -\frac{\partial H_1}{\partial x_2} - \frac{\partial H_1}{\partial u_2}\frac{\partial u_2^*}{\partial x_2}, \tag{13}$$

$$\frac{\partial \lambda_2^1}{\partial t} = -\frac{\partial H_2}{\partial x_2} - \frac{\partial H_2}{\partial u_1}\frac{\partial u_1^*}{\partial x_2}, \quad \frac{\partial \lambda_2^2}{\partial t} = -\frac{\partial H_2}{\partial x_1} - \frac{\partial H_2}{\partial u_1}\frac{\partial u_1^*}{\partial x_1}. \tag{14}$$

Notice that the difference is captured by an extra term on the right when we compare (10) and (13) or (11) and (14). The difference is because the optimal control of each player under the feedback strategy depends on $x_i(t)$, $i = 1, 2$. Hence, when differentiating the Hamiltonian to obtain Equations (13) and (14), we have to account for such dependence (note also that two terms disappear when we use (12) to simplify).

As we mentioned earlier, there are numerous applications of differential games in economics and marketing, especially in the area of dynamic pricing, see Eliashberg and Jeuland [32]. Desai [30, 31] and Eliashberg and Steinberg [33] use the open-loop Stackelberg equilibrium concept in a marketing-production game with the manufacturer and the distributor. Gaimon [39] uses both open and closed-loop NE concepts in a game with two competing firms choosing prices and production capacity when the new technology reduces firms' costs. Mukhopadhyay and Kouvelis [64] consider a duopoly with firms competing on prices and quality of design and derive open- and closed-loop NE.

4. Cooperative Games

The subject of cooperative games first appeared in the seminal work of von Neumann and Morgenstern [102]. However, for a long time, cooperative game theory did not enjoy as much attention in the economics literature as noncooperative GT. Papers employing cooperative GT to study SCM had been scarce, but are becoming more popular. This trend is probably due to the prevalence of bargaining and negotiations in SC relationships.

Cooperative GT involves a major shift in paradigms as compared to noncooperative GT: The former focuses on the outcome of the game in terms of the value created through cooperation of a subset of players but does not specify the actions that each player will take, while the latter is more concerned with the specific actions of the players. Hence, cooperative GT allows us to model outcomes of complex business processes that otherwise might be too difficult to describe, e.g., negotiations, and answers more general questions, e.g., how well is the firm positioned against competition (Brandenburger and Stuart [14]). However, there are also limitations to cooperative GT, as we will later discuss.

In what follows, we will cover transferable utility cooperative games (players can share utility via side payments) and two solution concepts: The core of the game and the Shapley value, and also biform games that have found several applications in SCM. Not covered are alternative concepts of value, e.g., nucleous and the σ-value, and games with nontransferable utility that have not yet found application in SCM. Material in this section is based mainly on Moulin [63] and Stuart [91]. Perhaps the first paper employing cooperative games in SCM is Wang and Parlar [106] who analyze the newsvendor game with three players, first in a noncooperative setting and then under cooperation with and without transferable utility. See Nagarajan and Sosic [67] for a more detailed review of cooperative games including analysis of the concepts of dynamic coalition formation and farsighted stability—issues that we do not address here.

4.1. Games in Characteristic Form and the Core of the Game

Recall that the noncooperative game consists of a set of players with their strategies and payoff functions. In contrast, the cooperative game (which is also called the game in characteristic form) consists of the set of players N with subsets or *coalitions* $S \subseteq N$ and a *characteristic function* $v(S)$ that specifies a (maximum) value (which we assume is a real number) created by any subset of players in N, i.e., the total pie that members of a coalition can create and divide. The specific actions that players have to take to create this value are not specified: The characteristic function only defines the total value that can be created by utilizing all players' resources. Hence, players are free to form any coalitions beneficial to them, and no player is endowed with power of any sort. Furthermore, the value a coalition creates is independent of the coalitions and actions taken by the noncoalition members. This decoupling of payoffs is natural in political settings (e.g., the majority gets to choose the legislation), but it is far more problematic in competitive markets. For example, in the context of cooperative game theory, the value HP and Compaq can generate by merging is independent of the actions taken by Dell, Gateway, IBM, Ingram Micro, etc.[7]

A frequently used solution concept in cooperative GT is the *core of the game*:

Definition 5. The utility vector $\pi_1, \ldots, \pi_N$ is in the core of the cooperative game if $\forall\, S \subset N, \sum_{i\in S} \pi_i \geq v(S)$ and $\sum_{i\in N} \pi_i \geq v(N)$.

A utility vector is in the core if the total utility of every possible coalition is at least as large as the coalition's value, i.e., there does not exist a coalition of players that could make all of its members at least as well off and one member strictly better off.

As is true for NE, the core of the game may not exist, i.e., it may be empty, and the core is often not unique. Existence of the core is an important issue because with an empty core, it is difficult to predict what coalitions would form and what value each player would receive. If the core exists, then the core typically specifies a range of utilities that a player can appropriate, i.e., competition alone does not fully determine the players' payoffs. What utility each player will actually receive is undetermined: It may depend on details of the residual bargaining process, a source of criticism of the core concept. (Biform games, described below, provide one possible resolution of this indeterminacy.)

[7] One interpretation of the value function is that it is the minimum value a coalition can guarantee for itself assuming the other players take actions that are most damaging to the coalition. However, that can be criticized as overly conservative.

In terms of specific applications to the SCM, Hartman et al. [44] considered the newsvendor centralization game, i.e., a game in which multiple retailers decide to centralize their inventory and split profits resulting from the benefits of risk pooling. Hartman et al. [44] further show that this game has a nonempty core under certain restrictions on the demand distribution. Muller et al. [65] relax these restrictions and show that the core is always nonempty. Further, Muller et al. [65] give a condition under which the core is a singleton.

4.2. Shapley Value

The concept of the core, though intuitively appealing, also possesses some unsatisfying properties. As we mentioned, the core might be empty or indeterministic.[8] For the same reason it is desirable to have a unique NE in noncooperative games, it is desirable to have a solution concept for cooperative games that results in a unique outcome. Shapley [85] offered an axiomatic approach to a solution concept that is based on three axioms. First, the value of a player should not change due to permutations of players, i.e., only the role of the player matters and not names or indices assigned to players. Second, if a player's added value to the coalition is zero then this player should not get any profit from the coalition, or, in other words, only players generating added value should share the benefits. (A player's added value is the difference between the coalition's value with that player and without that player.) Those axioms are intuitive, but the third is far less so. The third axiom requires additivity of payoffs: If v_1 and v_2 are characteristic functions in any two games, and if q_1 and q_2 are a player's Shapely value in these two games, then the player's Shapely value in the composite game, $v_1 + v_2$, must be $q_1 + q_2$. This is not intuitive because it is not clear what is meant by a composite game. Nevertheless, Shapley [85] demonstrates that there is a unique value for each player, called the Shapley value, that satisfies all three axioms.

Theorem 10. *The Shapley value, π_i, for player i in an N-person noncooperative game with transferable utility is*

$$\pi_i = \sum_{S \subseteq N \setminus i} \frac{|S|!\,(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)).$$

The Shapley value assigns to each player his marginal contribution $(v(S \cup \{i\} - v(S))$ when S is a random coalition of agents preceding i and the ordering is drawn at random. To explain further (Myerson [66]), suppose players are picked randomly to enter into a coalition. There are $|N|!$ different orderings for all players, and for any set S that does not contain player i there are $|S|!\,(|N| - |S| - 1)!$ ways to order players so that all players in S are picked ahead of player i. If the orderings are equally likely, there is a probability of $|S|!\,(|N| - |S| - 1)!/|N|!$ that when player i is picked, he will find S players in the coalition already. The marginal contribution of adding player i to coalition S is $(v(S \cup \{i\}) - v(S))$. Hence, the Shapley value is nothing more than a marginal expected contribution of adding player i to the coalition.

Because the Shapley value is unique, it has found numerous applications in economics and political sciences. So far, however, SCM applications are scarce: Except for discussion in Granot and Sosic [41] and analysis in Bartholdi and Kemahlioglu-Ziya [5], we are not aware of any other papers employing the concept of the Shapley value. Although uniqueness of the Shapely value is a convenient feature, caution should surely be taken with Shapley value: The Shapley value need not be in the core; hence, although the Shapely is appealing from the perspective of fairness, it may not be a reasonable prediction of the outcome of a game (i.e., because it is not in the core, there exists some subset of players that can deviate and improve their lots).

[8] Another potential problem is that the core might be very large. However, as Brandenburger and Stuart [15] point out, this may happen for a good reason: To interprete such situations, one can think of competition as not having much force in the game, hence the division of value will largely depend on the intangibles involved.

4.3. Biform Games

From the SCM point of view, cooperative games are somewhat unsatisfactory in that they do not explicitly describe the equilibrium actions taken by the players that is often the key in SC models. *Biform games*, developed by Brandenburger and Stuart [15], compensate to some extent for this shortcoming.

A biform game can be thought of as a noncooperative game with cooperative games as outcomes, and those cooperative games lead to specific payoffs. Similar to the noncooperative game, the biform game has a set of players N, a set of strategies for each player, and also a cost function associated with each strategy (cost function is optional—we include it because most SCM applications of biform games involve cost functions). The game begins by players making choices from among their strategies and incurring costs. After that, a cooperative game occurs in which the characteristic value function depends on the chosen actions. Hopefully, the core of each possible cooperative game is nonempty, but it is also unlikely to be unique. As a result, there is no specific outcome of the cooperative subgame, i.e., it is not immediately clear what value each player can expect. The proposed solution is that each player is assigned a confidence index, $\alpha_i \in [0,1]$, and the α_is are common knowledge. Each player then expects to earn in each possible cooperative game a weighted average of the minimum and maximum values in the core, with α_i being the weight. For example, if $\alpha_i = 0$, then the player earns the minimum value in the core, and if $\alpha_i = 1$, then the player earns the maximum value in the core. Once a specific value is assigned to each player for each cooperative subgame, the first stage noncooperative game can be analyzed just like any other noncooperative game.

Biform games have been successfully adopted in several SCM papers. Anupindi et al. [2] consider a game where multiple retailers stock at their own locations as well as at several centralized warehouses. In the first (noncooperative) stage, retailers make stocking decisions. In the second (cooperative) stage, retailers observe demand and decide how much inventory to transship among locations to better match supply and demand and how to appropriate the resulting additional profits. Anupindi et al. [2] conjecture that a characteristic form of this game has an empty core. However, the biform game has a nonempty core, and they find the allocation of rents based on dual prices that is in the core. Moreover, they find an allocation mechanism in the core that allows them to achieve coordination, i.e., the first-best solution. Granot and Sosic [41] analyze a similar problem but allow retailers to hold back the residual inventory. Their model actually has three stages: Inventory procurement, decision about how much inventory to share with others, and finally the transshipment stage. Plambeck and Taylor [76, 77] analyze two similar games between two firms that have an option of pooling their capacity and investments to maximize the total value. In the first stage, firms choose investment into effort that affects the market size. In the second stage, firms bargain over the division of the market and profits. Stuart [92] analyze biform newsvendor game with endogenous pricing.

5. Signaling, Screening, and Bayesian Games

So far, we have considered only games in which the players are on "equal footing" with respect to information, i.e., each player knows every other player's expected payoff with certainty for any set of chosen actions. However, such ubiquitous knowledge is rarely present in supply chains. One firm may have a better forecast of demand than another firm, or a firm may possess superior information regarding its own costs and operating procedures. Furthermore, a firm may know that another firm may have better information, and, therefore, choose actions that acknowledge this information shortcoming. Fortunately, game theory provides tools to study these rich issues, but, unfortunately, they do add another layer of analytical complexity. This section briefly describes three types of games in which the information structure has a strategic role: Signaling games, screening games, and Bayesian games. Detailed methods for the analysis of these games are not provided. Instead, a general description is provided along with specific references to supply chain management papers that study these games.

5.1. Signaling Game

In its simplest form, a signaling game has two players, one of which has better information than the other, and it is the player with the better information that makes the first move. For example, Cachon and Lariviere [23] consider a model with one supplier and one manufacturer. The supplier must build capacity for a key component to the manufacturer's product, but the manufacturer has a better demand forecast than the supplier. In an ideal world, the manufacturer would truthfully share her demand forecast with the supplier so that the supplier could build the appropriate amount of capacity. However, the manufacturer always benefits from a larger installed capacity in case demand turns out to be high, but it is the supplier that bears the cost of that capacity. Hence, the manufacturer has an incentive to inflate her forecast to the supplier. The manufacturer's hope is that the supplier actually believes the rosy forecast and builds additional capacity. Unfortunately, the supplier is aware of this incentive to distort the forecast, and, therefore, should view the manufacturer's forecast with skepticism. The key issue is whether there is something the manufacturer should do to make her forecast convincing, i.e., credible.

While the reader should refer to Cachon and Lariviere [23] for the details of the game, some definitions and concepts are needed to continue this discussion. The manufacturer's private information, or type, is her demand forecast. There is a set of possible types that the manufacturer could be, and this set is known to the supplier, i.e., the supplier is aware of the possible forecasts, but is not aware of the manufacturer's actual forecast. Furthermore, at the start of the game, the supplier and the manufacturer know the probability distribution over the set of types. We refer to this probability distribution as the supplier's belief regarding the types. The manufacturer chooses her action first, which, in this case, is a contract offer and a forecast, the supplier updates his belief regarding the manufacturer's type given the observed action, and then the supplier chooses his action, which, in this case, is the amount of capacity to build. If the supplier's belief regarding the manufacturer's type is resolved to a single type after observing the manufacturer's action (i.e., the supplier assigns a 100% probability that the manufacturer is that type and a zero probability that the manufacturer is any other type), then the manufacturer has signaled a type to the supplier. The trick is for the supplier to ensure that the manufacturer has signaled her actual type.

While we are mainly interested in the set of contracts that credibly signal the manufacturer's type, it is worth beginning with the possibility that the manufacturer does not signal her type. In other words, the manufacturer chooses an action such that the action does not provide the supplier with additional information regarding the manufacturer's type. That outcome is called a *pooling equilibrium*, because the different manufacturer types behave in the same way, i.e., the different types are pooled into the same set of actions. As a result, Bayes' rule does not allow the supplier to refine his beliefs regarding the manufacturer's type.

A pooling equilibrium is not desirable from the perspective of supply chain efficiency because the manufacturer's type is not communicated to the supplier. Hence, the supplier does not choose the correct capacity given the manufacturer's actual demand forecast. However, this does not mean that both firms are disappointed with a pooling equilibrium. If the manufacturer's demand forecast is less than average, then that manufacturer is quite happy with the pooling equilibrium because the supplier is likely to build more capacity than he would if he learned the manufacturer's true type. It is the manufacturer with a higher-than-average demand forecast that is disappointed with the pooling equilibrium because then the supplier is likely to underinvest in capacity.

A pooling equilibrium is often supported by the belief that every type will play the pooling equilibrium and any deviation from that play would only be done by a manufacturer with a low-demand forecast. This belief can prevent the high-demand manufacturer from deviating from the pooling equilibrium: A manufacturer with a high-demand forecast would rather be treated as an average demand manufacturer (the pooling equilibrium) than a low-demand manufacturer (if deviating from the pooling equilibrium). Hence, a pooling equilibrium can indeed be an NE in the sense that no player has a unilateral incentive to deviate given the strategies and beliefs chosen by the other players.

While a pooling equilibrium can meet the criteria of an NE, it nevertheless may not be satisfying. In particular, why should the supplier believe that the manufacturer is a low type if the manufacturer deviates from the pooling equilibrium? Suppose the supplier were to believe a deviating manufacturer has a high-demand forecast. If a high-type manufacturer is better off deviating but a low-type manufacturer is not better off, then only the high-type manufacturer would choose such a deviation. The key part in this condition is that the low type is not better off deviating. In that case, it is not reasonable for the supplier to believe the deviating manufacturer could only be a high type, therefore, the supplier should adjust his belief. Furthermore, the high-demand manufacturer should then deviate from the pooling equilibrium, i.e., this reasoning, which is called the intuitive criterion, breaks the pooling equilibrium; see Kreps [49].

The contrast to a pooling equilibrium is a *separating* equilibrium, also called a *signaling* equilibrium. With a separating equilibrium, the different manufacturer types choose different actions, so the supplier is able to perfectly refine his belief regarding the manufacturer's type given the observed action. The key condition for a separating equilibrium is that only one manufacturer type is willing to choose the action designated for that type. If there is a continuum of manufacturer types, then it is quite challenging to obtain a separating equilibrium: It is difficult to separate two manufacturers that have nearly identical types. However, separating equilibria are more likely to exist if there is a finite number of discrete types.

There are two main issues with respect to separating equilibria: What actions lead to separating equilibrium, and does the manufacturer incur a cost to signal, i.e., is the manufacturer's expected profit in the separating equilibrium lower than what it would be if the manufacturer's type were known to the supplier with certainty? In fact, these two issues are related: An ideal action for a high-demand manufacturer is one that costlessly signals her high-demand forecast. If a costless signal does not exist, then the goal is to seek the lowest-cost signal.

Cachon and Lariviere [23] demonstrate that whether a costless signal exists depends on what commitments the manufacturer can impose on the supplier. For example, suppose the manufacturer dictates to the supplier a particular capacity level in the manufacturer's contract offer. Furthermore, suppose the supplier accepts that contract, and by accepting the contract, the supplier has essentially no choice but to build that level of capacity because the penalty for noncompliance is too severe. They refer to this regime as forced compliance. In that case, there exist many costless signals for the manufacturer. However, if the manufacturer's contract is not iron-clad, so the supplier could potentially deviate—which is referred to as voluntary compliance—then the manufacturer's signaling task becomes more complex.

One solution for a high-demand manufacturer is to give a sufficiently large lump-sum payment to the supplier: The high-demand manufacturer's profit is higher than the low-demand manufacturer's profit, so only a high-demand manufacturer could offer that sum. This has been referred to as signaling by "burning money": Only a firm with a lot of money can afford to burn that much money.

While burning money can work, it is not a smart signal: Burning one unit of income hurts the high-demand manufacturer as much as it hurts the low-demand manufacturer. The signal works only because the high-demand manufacturer has more units to burn. A better signal is a contract offer that is costless to a high-demand manufacturer but expensive to a low-demand manufacturer. A good example of such a signal is a minimum commitment. A minimum commitment is costly only if realized demand is lower than the commitment, because then the manufacturer is forced to purchase more units than desired. That cost is less likely for a high-demand manufacturer, so, in expectation, a minimum commitment is costlier for a low-demand manufacturer. Interestingly, Cachon and Lariviere [23] show that a manufacturer would never offer a minimum commitment with perfect information, i.e., these contracts may be used in practice solely for the purpose of signaling information.

5.2. Screening

In a screening game, the player that lacks information is the first to move. For example, in the screening game version of the supplier-manufacturer game described by Cachon and Lariviere [23], the supplier makes the contract offer. In fact, the supplier offers a menu of contracts with the intention of getting the manufacturer to reveal her type via the contract selected in the menu. In the economics literature, this is also referred to as *mechanism design*, because the supplier is in charge of designing a mechanism to learn the manufacturer's information. See Porteus and Whang [78] for a screening game that closely resembles this one.

The space of potential contract menus is quite large, so large, that it is not immediately obvious how to begin to find the supplier's optimal menu. For example, how many contracts should be offered, and what form should they take? Furthermore, for any given menu, the supplier needs to infer for each manufacturer type which contract the type will choose. Fortunately, the *revelation principle* (Kreps [49]) provides some guidance.

The revelation principle begins with the presumption that a set of optimal mechanisms exists. Associated with each mechanism is an NE that specifies which contract each manufacturer type chooses and the supplier's action given the chosen contract. With some equilibria, it is possible that some manufacturer type chooses a contract, which is not designated for that type. For example, the supplier intends the low-demand manufacturer to choose one of the menu options, but instead, the high-demand manufacturer chooses that option. Even though this does not seem desirable, it is possible that this mechanism is still optimal in the sense that the supplier can do no better on average. The supplier ultimately cares only about expected profit, not the means by which that profit is achieved. Nevertheless, the revelation principle states that an optimal mechanism that involves deception (the wrong manufacturer chooses a contract) can be replaced by a mechanism that does not involve deception, i.e., there exists an equivalent mechanism that is truth telling. Hence, in the hunt for an optimal mechanism, it is sufficient to consider the set of revealing mechanisms: The menu of contracts is constructed such that each option is designated for a type and that type chooses that option.

Even though an optimal mechanism may exist for the supplier, this does not mean the supplier earns as much profit as he would if he knew the manufacturer's type. The gap between what a manufacturer earns with the menu of contracts and what the same manufacturer would earn if the supplier knew her type is called an information rent. A feature of these mechanisms is that separation of the manufacturer types goes hand in hand with a positive information rent, i.e., a manufacturer's private information allows the manufacturer to keep some rent that the manufacturer would not be able to keep if the supplier knew her type. Hence, even though there may be no cost to information revelation with a signaling game, the same is not true with a screening game.

There have been a number of applications of the revelation principle in the supply chain literature: e.g., Chen [25] studies auction design in the context of supplier procurement contracts; Corbett [26] studies inventory contract design; Baiman et al. [4] study procurement of quality in a supply chain.

5.3. Bayesian Games

With a signaling game or a screening game, actions occur sequentially so information can be revealed through the observation of actions. There also exist games with private information that do not involve signaling or screening. Consider the capacity allocation game studied by Cachon and Lariviere [22]. A single supplier has a finite amount of capacity. There are multiple retailers, and each knows his own demand but not the demand of the other retailers. The supplier announces an allocation rule, the retailers submit their orders, and then the supplier produces and allocates units. If the retailers' total order is less than capacity, then each retailer receives his entire order. If the retailers' total order exceeds capacity, the supplier's allocation rule is implemented to allocate the capacity. The issue is the extent to which the supplier's allocation rule influences the supplier's profit, retailer's profit, and supply chain's profit.

In this setting, the firms with the private information (the retailers) choose their actions simultaneously. Therefore, there is no information exchange among the firms. Even the supplier's capacity is fixed before the game starts, so the supplier is unable to use any information learned from the retailers' orders to choose a capacity. However, it is possible that correlation exists in the retailers' demand information, i.e., if a retailer observes his demand type to be high, then he might assess the other retailers' demand type to be high as well (if there is a positive correlation). Roughly speaking, in a Bayesian game, each player uses Bayes' rule to update his belief regarding the types of the other players. An equilibrium is then a set of strategies for each type that is optimal given the updated beliefs with that type and the actions of all other types. See Fudenberg and Tirole [38] for more information on Bayesian games.

6. Summary and Opportunities

As has been noted in other reviews, operations management has been slow to adopt GT. But because SCM is an ideal candidate for GT applications, we have recently witnessed an explosion of GT papers in SCM. As our survey indicates, most of these papers utilize only a few GT concepts, in particular, the concepts related to noncooperative static games. Some attention has been given to stochastic games, but several other important areas need additional work: Cooperative, repeated, differential, signaling, screening, and Bayesian games.

The relative lack of GT applications in SCM can be partially attributed to the absence of GT courses from the curriculum of most doctoral programs in operations research/management. One of our hopes with this survey is to spur some interest in GT tools by demonstrating that they are intuitive and easy to apply for a person with traditional operations research training.

With the invention of the Internet, certain GT tools have received significant attention: Web auctions gave a boost to auction theory, and numerous websites offer an opportunity to haggle, thus making bargaining theory fashionable. In addition, the advent of relatively cheap information technology has reduced transaction costs and enabled a level of disintermediation that could not be achieved before. Hence, it can only become more important to understand the interactions among independent agents within and across firms. While the application of game theory to supply chain management is still in its infancy, much more progress will soon come.

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