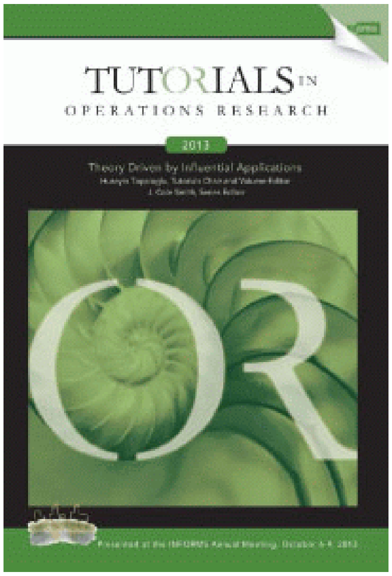

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Portfolio Optimisation: Models and Solution Approaches

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Abstract In this tutorial we review portfolio optimisation, with a focus on financial applications. Here, the problem is to decide the assets (a portfolio) to hold that have desired characteristics. Markowitz mean-variance portfolio optimisation is relatively well known but has been extended in recent years to encompass cardinality constraints. Less considered in the scientific literature are problems such as (1) index tracking, where the objective is to match the return achieved on a benchmark index such as the S&P 500; (2) enhanced indexation, where the objective is to exceed the return achieved on a benchmark index; here we may have a desired specified excess return, or we may simply wish to do better than the benchmark; and (3) absolute return/market neutral, where the objective is to achieve return irrespective of how the market, as represented by the benchmark index, performs. We will outline the mathematical optimisation models that can be adopted for portfolio problems such as these and the solution approaches that can be adopted.

Keywords enhanced indexation; index tracking; Markowitz mean-variance; portfolio optimisation

1. Key Concepts

The following summarise the key concepts/messages/take-home points from this tutorial:

- *Different types of portfolios require different mathematical models.* The purpose to which you intend to put your portfolio needs to be considered when you construct/use a mathematical model.
- *Even for portfolios intended for the same purpose, the model to use is not uniquely defined.* This point is not sufficiently well appreciated, either by academics or finance professionals. Even if we are considering a specific situation, there are different models, each with its own logic, that can be adopted. Which one might be best in practice is an open question.
- *Adopt an optimisation mind-set in building/choosing a model.* When you build/choose a model to construct a portfolio, you need to ask yourself, what is my objective? What am I trying to maximise/minimise?
- *Nonlinear models are more difficult to solve numerically than linear models.*

The first three of these points will hopefully become apparent by the end of this tutorial. The last point is not emphasised below, simply because it may be something you already explicitly know. Even if you do not already explicitly know that nonlinear models are more difficult to solve numerically than linear models, the point is simple to make; indeed, you (implicitly) already knew this back in your teenage years at school.

Remember all those years ago at school? Did your mathematics school teachers ever set you a problem of the form

$$2x + 3y = 7,$$
$$4x - 7y = 1,$$

i.e., simultaneous *linear* equations. I suspect they did, and if (like me) you were quantitatively skilled, you could easily solve these equations to get $x = 2$ and $y = 1$. However, suppose your mathematics school teacher set a problem of the form

$$2(x/y) + 3xy = 7,$$
$$4x - 7y = 1,$$

i.e., simultaneous *nonlinear* equations. Here, even when just one equation is nonlinear, the process of finding a solution is *much* more difficult. So hopefully the point is clear: *nonlinear models are more difficult to solve numerically than linear models.*

Obviously you may be wondering what such a simple back-to-school example is doing in a tutorial paper such as this. Well, the answer is that this tutorial will scale quite quickly from things that you (perhaps) already know to things that (I am fairly sure) you do not know.

2. Markowitz Models

2.1. Markowitz Mean-Variance Portfolio Optimisation (Purpose: Portfolios That Balance Risk and Return)

Ever since the pioneering work of Markowitz [23] in 1952, optimisation has been at the centre of work concerned with decisions relating to deciding the composition of financial portfolios. As such, both practitioners and academic researchers have been willing to trade off the disadvantages of optimisation (multiple optimal solutions, solution sensitivity) for its advantages (clear modelling framework, computational efficiency, algorithmic decision making).

To explain the basic Markowitz mean-variance portfolio optimisation model, we need some notation. Let

N be the number of assets (e.g., stocks) available,
μ_i be the expected (average, mean) return (per time period) of asset i,
ρ_{ij} be the correlation between the returns for assets i and j ($-1 \leq \rho_{ij} \leq +1$),
s_i be the standard deviation in return for asset i, and
R be the desired expected return from the portfolio chosen.

Then the decision variables are

w_i the proportion of the total investment associated with (invested in) asset i ($0 \leq w_i \leq 1$).

The proportion variables (w_i) are sometimes referred to as the weights.

Here we have imposed nonnegativity ($w_i \geq 0$). If we were to allow negative weights (so w_i can be positive or negative), then we would be allowing shorting. However, since portfolio problems often involve holding the decided portfolio for too long a time for shorting to be a practical possibility, we shall exclude shorting here.

Using the standard Markowitz mean-variance approach we have that the portfolio optimisation problem is

$$\text{minimise} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \rho_{ij} s_i s_j \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{N} w_i \mu_i = R, \tag{2}$$

$$\sum_{i=1}^{N} w_i = 1, \tag{3}$$

$$0 \leq w_i \leq 1, \quad i = 1, \ldots, N. \tag{4}$$

Here in Equation (1) we minimise the total variance (*risk*) associated with the portfolio. In the Markowitz framework, risk is equated to variance in portfolio return. Equation (2) ensures that the portfolio has an expected *return* of R. Equation (3) ensures that the proportions sum to one (so all of our money has be to invested in assets; we cannot keep any back and not invest it). This formulation is a simple nonlinear programming problem.

Often, instead of using standard deviations and correlation, we express the objective in terms of covariance. Let σ_{ij} be the covariance between the returns for assets i and j; then (since $\sigma_{ij} = \rho_{ij} s_i s_j$) we can express the above objective (Equation (1)) as

$$\text{minimise} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}. \tag{5}$$

Usually nonlinear problems are difficult to solve (as discussed above), but in this case, because the objective is quadratic and $[\sigma_{ij}]$ is positive semidefinite (a property of covariance matrices), computationally effective algorithms exist so that there is (in practice) little difficulty in calculating the optimal solution for any particular data set.

The point of the above optimisation problem is to construct an *efficient frontier* (*unconstrained efficient frontier* (*UEF*)), a smooth nondecreasing curve that gives the best possible trade-off of risk against return, i.e., the curve represents the set of *Pareto-optimal* (*nondominated*) portfolios.

One such efficient frontier is shown in Figure 1 for assets (stocks/shares/equities) drawn from the UK Financial Times Stock Exchange (FTSE) index of 100 top companies. Note how this nice smooth continuous curve runs from the minimum variance portfolio to the maximum return/maximum risk portfolio. Here we can choose to hold any of the portfolios on this efficient frontier. For this particular data set the minimum variance portfolio contained 30 out of the 100 assets.

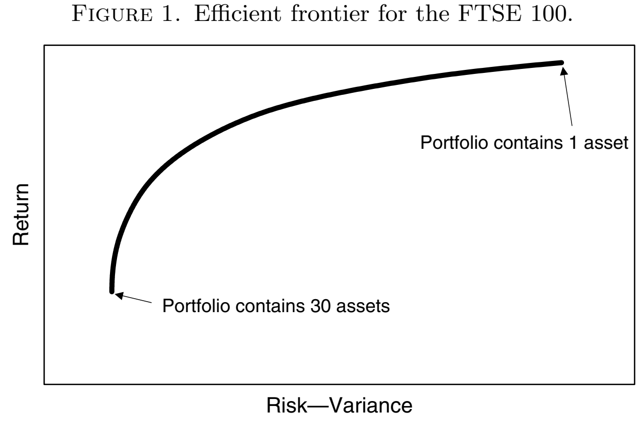
The approach followed in Markowitz mean-variance optimisation is as follows:

- look into the (immediate) past for relevant data (in-sample data);
- use that data to form a portfolio, as outlined in the model above, where the in-sample data is used to produce values for μ_i and σ_{ij} that are used in the optimisation model; and
- hold that portfolio into the (near) future (out-of-sample).

The underlying logic is that, since accurately forecasting future prices/returns for assets is extremely difficult, data from the immediate past are our best guide to construct a portfolio to hold into the immediate future.

One concept commonly associated with the Markowitz approach is that of *diversification*, "not putting all of your eggs into one basket," as we might say colloquially. Notice how, as the example in Figure 1 illustrates, this often arises naturally in Markowitz solutions,

FIGURE 1. Efficient frontier for the FTSE 100.



so here the minimum variance portfolio has spread our investment (in varying proportions) across 30 different assets.

2.2. Additional Constraints

Practical portfolio optimisation typically entails adding extra constraints (representing more realistic features) to the model. You should be clear here that, as the minimisation objective in the Markowitz model implies, adding any extra constraints can only increase (or leave unchanged) the in-sample risk as given by the Markowitz objective. The reason why we adopt such constraints is that we want to shape the optimised portfolio that we choose (and which we will hold out-of-sample), and we hope to reduce the risk (for a given return) we face out-of-sample as a result of adding such constraints.

Examples of extra constraints that might be added are considered below.

2.2.1. Restricting the Proportion Invested in Any Asset. Here, if δ_i is the maximum proportion that we are prepared to invest in asset i, then the constraint we add is

$$0 \leq w_i \leq \delta_i, \quad i = 1, \ldots, N, \tag{6}$$

where the only change from before is that whereas previously we had $0 \leq w_i \leq 1$, now we have $0 \leq w_i \leq \delta_i$. Note that we have only modelled a maximum proportion here. A minimum proportion has two meanings:

- always invest at least that proportion, so you must invest in the asset, or
- always invest at least that proportion *provided* you choose to invest in the asset (so you may choose not to invest in the asset at all).

This latter interpretation involves the addition of zero–one variables. This is perfectly possible, but since these will be introduced later below (when we come to consider cardinality constraints), we will not model a minimum proportion here.

2.2.2. Restricting the Proportion Invested in Sets of Assets (Class/Sector Constraints). Typically, here we assume that each asset can be classified as belonging to one (or more) sectors (e.g., energy, banking, telecommunications) and then constrain the total investment in any sector. So if we have a set S of assets in a particular sector, with $\delta(S)$ being the maximum proportion of the portfolio that we are prepared to hold in that sector, the constraint we add is

$$\sum_{i \in S} w_i \leq \delta(S). \tag{7}$$

Note here that for assets in the same sector we might expect to see positive correlation (i.e., they move together, up or down, with the sector). Hence we have, in an implicit way, already included sector constraints (do not make too much investment in the same sector) in our risk objective (which includes correlation, Equation (1)). But in practice we might do better to include such constraints explicitly, as we have done here in Equation (7).

2.2.3. Rebalancing an Existing Portfolio. So far we have talked purely in terms of the proportion invested in each asset. Indeed, you may have had the (correct) impression of a pile of cash waiting to be invested. However, a moment's thought will reveal that in virtually all practical situations we will have already invested in some existing portfolio of assets. We, when we come to portfolio optimisation, are seeking to change that existing portfolio—perhaps because we feel we can get better performance from a new portfolio. Changing an existing portfolio to a new portfolio is known as *rebalancing* the portfolio (or just rebalancing).

In practical situations we also need to consider

- investing new cash in our portfolio (e.g., if we are running an investment fund our investors might have made additional contributions that they expect us to invest), or

• taking cash out of the portfolio (e.g., to meet liabilities, such as investors in our investment fund cashing in their investment, necessitating selling some of the assets in our existing portfolio).

However, to ease the mathematical discussion here, we will neglect these issues.

To illustrate how to include rebalancing, in our model, suppose X_i is the number of units of asset i that we currently hold. As a result of portfolio rebalancing, we end up with x_i (≥ 0) units of asset i (where x_i is a variable that will be decided as a result of the optimisation). Suppose, for the sake of illustration, we wish to restrict the amount of trading in asset i that we do to T_i units—so we do not wish to trade more than T_i units of asset i away from our current holding (position) of X_i. Here T_i is our choice, not something the optimisation is deciding for us.

Assuming we have zero transaction cost, and the current price of asset i is P_i, then for trade constrained portfolio rebalancing the constraints we need to add are

$$X_i - T_i \leq x_i \leq X_i + T_i \quad i = 1, \dots, N, \tag{8}$$

$$w_i = (P_i x_i) \Big/ \sum_{j=1}^{N} (P_j X_j) \quad i = 1, \dots, N. \tag{9}$$

Equation (8) ensures that the number of units of asset i after rebalancing (x_i) lies within the desired amount T_i of the number of units X_i before rebalancing. Equation (9) defines the proportion (w_i) invested in each asset in terms of the number of units (x_i) of that asset held. Although strictly the variables $[x_i]$ should be integer since the sums of money involved are typically very large, we, without significant loss of generality, allow $[x_i]$ to take fractional values.

2.2.4. Transaction Cost. When rebalancing from an existing portfolio above we assumed zero transaction cost, so trading was free. Indeed, in the basic Markowitz model (Equations (1)–(4)) we also assumed trading was free. Clearly, in the real world, trading (buying/selling assets) may not be free, and so we need to account for transaction cost. This is easily done, as below in the context of rebalancing an existing portfolio. Let $C = \sum_{i=1}^{N} P_i X_i$ be the total value (worth) of our current portfolio, and let γ be the limit on the proportion of C that can be consumed by transaction cost (since we might well wish to limit the amount we spend in transaction cost when we trade from our current portfolio $[X_i]$ to our new portfolio $[x_i]$).

Then the constraints we add are

$$C_{\text{trans}} = \sum_{i=1}^{N} \text{ transaction cost}(X_i \to x_i \text{ given current price is } P_i), \tag{10}$$

$$C_{\text{trans}} \leq \gamma C, \tag{11}$$

$$\sum_{i=1}^{N} P_i x_i = C - C_{\text{trans}}. \tag{12}$$

Equation (10) defines the total transaction cost incurred. Depending on the nature of transaction cost, this may be a linear or nonlinear function. Equation (11) limits the total transaction cost incurred. Equation (12) is a balance constraint that says that the value of the portfolio after trading $(\sum_{i=1}^{N} P_i x_i)$ is equal to the value before trading (C), minus the transaction cost incurred. Equation (12), in its use of P_i on the left-hand side of the equation to value the portfolio after trading, assumes that the trading taking place, in moving from holding X_i to holding x_i, is not having an impact on the price P_i of asset i.

In the context of Markowitz mean-variance portfolio optimisation, the role of transaction cost is that it is the price we pay (now) to enable us to move from our existing portfolio $[X_i]$ to a new portfolio $[x_i]$ that will (on the basis of in-sample optimisation) have a better performance than our existing portfolio. As such, the role of the holding period, the number of time periods (H) for which we intend to hold the new portfolio, must be considered. In particular, a consequence of the Markowitz model in the presence of transaction cost is that trading

- reduces return, but also
- reduces risk.

For example, suppose I am holding 100% of my wealth in one asset, wake up one morning and sell all my holding, then a few minutes later buy it back (at the same price, but incurring transaction cost on the trade), so again having 100% of my wealth in the same asset. Then how has this affected the risk and return of my portfolio? Well, the return has decreased, but so too has the risk (variance in return) despite the fact that I am still holding 100% of my wealth in the same asset (for an explanation as to why this is true, if you need one, see Woodside-Oriakhi et al. [40]).

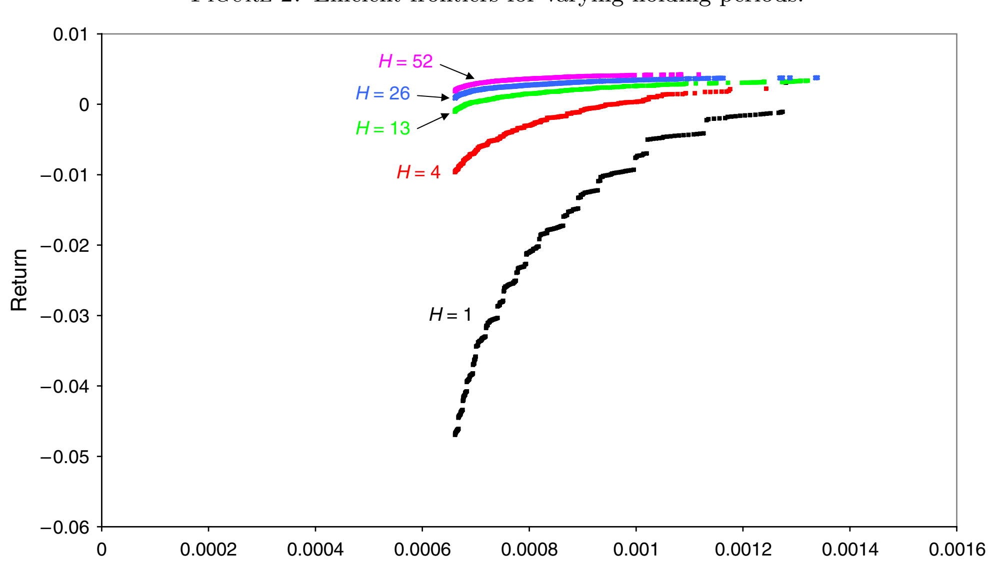
As an indication as to how the holding period effects the efficient frontier, Figure 2 shows efficient frontiers for a 31-asset example for a variety of holding periods (based on work reported in Woodside-Oriakhi et al. [40]). Here, the dotted nature of the efficient frontiers results from the fact that (for computational reasons) we have only examined a limited number of values of R (Equation (2)) in plotting the frontiers seen.

2.2.5. Statistical Considerations. Work relating to Markowitz mean-variance portfolio optimisation has also been presented in the literature from a statistical viewpoint. Here the idea is that the in-sample parameters that we are using in our optimisation model (namely, mean returns, μ_i, Equation (2); and covariances σ_{ij}, Equation (5)) can only be sample estimates of underlying population parameters. Therefore, since such estimates may be inaccurate, strategies such as

- ignore the estimated parameters, or
- use alternative estimators rather than the simple sample estimators, have been suggested.

With respect to ignoring the estimated parameters, the $1/N$ approach, where we invest an equal proportion in each of the N assets (i.e., $w_i = 1/N \ \ i = 1, \ldots, N$), has been used in the literature, e.g., DeMiguel et al. [10]. Also some authors have suggested focusing purely

FIGURE 2. Efficient frontiers for varying holding periods.



on the minimum variance portfolio (i.e., minimise Equation (5) subject to Equations (3) and (4)), so mean asset return μ_i plays no role.

Use of alternative estimators has focused on the covariance matrix. Here, there are a multiplicity of approaches that are possible (e.g., see Pantaleo et al. [29] for a study comparing nine different covariance estimators). Generic approaches used here include the following:

- Shrinkage, which produces a covariance matrix for use in Equation (5) by using a shrinkage constant to generate a linear combination of the sample covariance matrix with a target matrix. Ledoit and Wolf [16, 17], for example, use a target matrix composed of just the diagonal elements of the sample covariance matrix.
- Random matrix theory, which produces a correlation matrix for use in Equation (1) by adjusting the sample correlation matrix, based on examining and altering the eigenvalues associated with that matrix; see Gatheral [12].

2.2.6. Optimisation Objective. In the Markowitz framework, risk is equated to the in-sample variance in portfolio return, so that the portfolio is decided (optimised, via the solution to Equations (1)–(4)) so as to minimise risk. Clearly, risk can be defined in different ways. For example, a downside risk framework would equate risk with portfolio return falling below a predefined target. So here the objective in our optimisation problem would be changed. Depending on the precise mathematical nature of the risk objective adopted, the mathematical program that results might be easy/hard to solve.

Our optimisation model for portfolio construction contains two factors: risk and return. There are well-known functions defined in the academic literature, namely, the Sharpe and Sortino ratios (Sharpe [31], Sortino and Price [33]), that balance these two factors, and so we might optimise on one or other of these ratios subject to Equations (3) and (4). Here, the problem would be a continuous nonlinear program, but packages for such problems have been advancing and are now increasingly capable.

3. Markowitz Mean-Variance Portfolio Optimisation with Cardinality Constraints (Purpose: Portfolios That Balance Risk and Return and That Allow Us to Control the Number of Assets Held)

Imposing a cardinality constraint to restrict the number of assets (K) in which we can invest can only (for a given level of return) increase historic (in-sample) risk. This is because in the UEF as considered above, for a given level of return, the risk is at a minimum (as the mathematics for the Markowitz model, Equations (1)–(4), explicitly requires). The practical reason why we might impose a cardinality constraint is that we may find it more convenient to have a portfolio with just a few assets, or simply that we desire a degree of control to shape the optimised portfolio with respect to the number of assets that it contains.

Introducing zero–one decision variables,

$$\begin{aligned} z_i &= 1 \quad \text{if any of asset } i \text{ is held,} \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

the cardinality constrained portfolio optimisation problem is as follows:

minimise Equation (5), subject to Equations (2), (3), and

$$\sum_{i=1}^{N} z_i = K, \tag{13}$$

$$0 \le w_i \le z_i \quad i = 1, \ldots, N, \tag{14}$$

$$z_i \in [0, 1] \quad i = 1, \ldots, N. \tag{15}$$

Here, Equation (13) ensures that we choose precisely K assets. Equation (14) links the proportion variables (w_i) to the zero–one variables (z_i) and ensures that if we choose not to invest in an asset (so $z_i = 0$), then the associated proportion is also zero.

Algorithmically, the above mathematical program for the *cardinality constrained efficient frontier* (*CCEF*) is hard to solve (because it is a mixed-integer quadratic program). A mixed-integer program is one in which some variables take continuous (fractional) values, some take integer values.

Given that we have a hard problem to solve metaheuristics such as

- genetic algorithms (evolutionary algorithms, population-based approaches),
- simulated annealing,
- tabu search, and
- variable neighbourhood search

have been applied to the problem of finding the CCEF.

For those unfamiliar with the terms being used here,

- an *optimal algorithm* is one which (mathematically) guarantees to find the optimal solution (to the optimisation problem under consideration),
- a *heuristic algorithm* has no such guarantee, and
- a *metaheuristic* is a "framework" within which you design a heuristic for the problem you are considering.

3.1. Frontier Shape

For the portfolio optimisation problem as considered above (Equations (1)–(4)), the frontier was a nice, smooth, continuous curve, as in Figure 1.

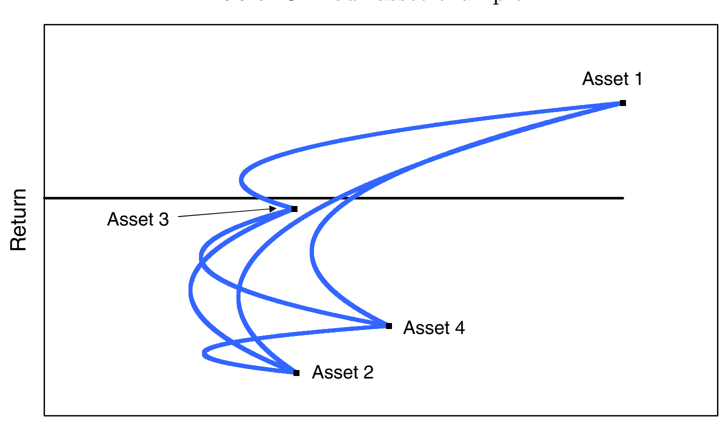
In the presence of cardinality constraints, the efficient frontier may become *discontinuous*, where the discontinuities imply that there are certain returns that no rational investor would consider (since there exist portfolios with less risk and higher return).

To illustrate this point, Figure 3 shows four assets (stocks) drawn from the FTSE 100 (data as in Chang et al. [7]). In that figure all possible portfolios involving *exactly* two assets ($K = 2$) are shown. For four assets there are $4\times3/2 = 6$ possible choices of pairs of assets, each of which leads to a different curve in Figure 3.

Suppose that we are interested in the return shown on the horizontal line in Figure 3. There are only three possible portfolios that achieve this return:

- one where the horizontal line intersects the curve between assets 3 and 1 (so a portfolio consisting of those two assets in some proportion),

Figure 3. Four-asset example.



- one where the horizontal line intersects the curve between assets 2 and 1 (so a portfolio consisting of those two assets in some proportion), and
- one where the horizontal line intersects the curve between assets 4 and 1 (so a portfolio consisting of those two assets in some proportion).

Are these three portfolios dominated or not?

Figure 4 shows the efficient frontier as derived from Figure 3. Note the discontinuities, where there are return values for which there is no portfolio having that return that is not dominated.

As an illustration of what can be achieved with cardinality constrained portfolio optimisation, Figure 5 shows some trade-off curves for assets chosen from the Deutscher Aktien Index (DAX).

Recent work, other than that discussed above, dealing with cardinality constrained portfolio optimisation can be found in Anagnostopoulos and Mamanis [1], Branke et al. [3], Chang et al. [6], Cura [9], Metaxiotis and Liagkouras [26], Pai and Michel [28], Soleimani et al. [32], and Woodside-Oriakhi et al. [39].

FIGURE 4. Cardinality constrained efficient frontier, four-asset example.

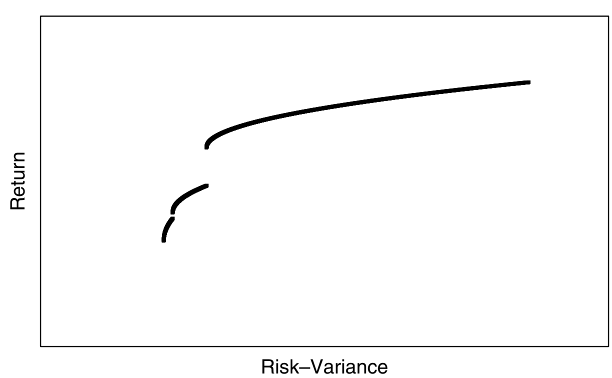
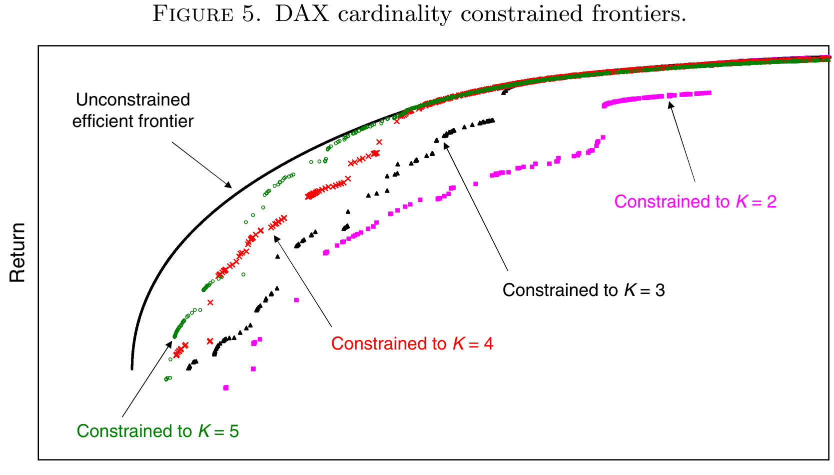


FIGURE 5. DAX cardinality constrained frontiers.



4. Consistent Portfolios (Purpose: Portfolios Which Perform Out-of-Sample in a Fashion Consistent with Their In-Sample Performance)

The logic behind Markowitz approaches is that we use known data (in-sample data) to construct frontiers of the types you have seen above and then choose a portfolio from the frontier to invest in. Our portfolio then varies in value as we hold it out-of-sample.

Criticisms of Markowitz approaches essentially often focus around the fact that, out-of-sample, portfolios chosen from the (in-sample-derived) frontier do not behave in a fashion that their position on the frontier would imply. In other words, based on in-sample data we choose a portfolio from the frontier with a particular mean and variance in return. However, out-of-sample we find that the mean and variance in return for the portfolio chosen are very different from the in-sample values.

If you decide to invest in a portfolio (e.g., one chosen from the Markowitz efficient frontier), then you are explicitly investing in a particular in-sample distribution of returns. A key question is,

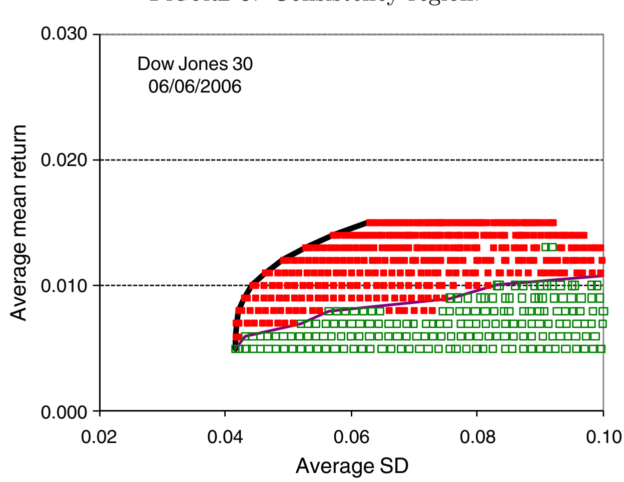
> If you consider the returns you get out-of-sample from your chosen portfolio, would you like these returns to be drawn from the same distribution as the in-sample returns or not?

To illustrate the point let us take a personal analogy. You are single and meet a promising life partner. You share the same interests, laugh together, all is great. So you decide to marry and on the day of the ceremony you stand, in front of your family and friends, ready to be married. Reflect, you are about to go out-of-sample. You have experienced life with your partner in-sample up to now. Do you hope that the distribution of experiences you have had in-sample will be repeated out-of-sample? Of course you do.

So returning to our portfolio context, we would like, in choosing a portfolio, to choose one that out-of-sample gives us the same distribution of returns that we had in-sample, since (obviously) that in-sample distribution played a strong role in our choice of a portfolio to hold out-of-sample.

Figure 6, based on work in Meade and Beasley [25], shows an approach that attempts to illuminate this. In the green *consistency region* (shown using hollow squares) we have portfolios where we will typically get returns out-of-sample consistent with in-sample behaviour. The red region, shown using solid squares, exhibits inconsistent behaviour. Note that for this particular example most of the portfolios on the efficient frontier exhibit inconsistent behaviour.

FIGURE 6. Consistency region.



In Figure 6 it is important to note that we are no longer just considering portfolios on the efficient frontier but are also considering portfolios that are not on the efficient frontier.

5. Index Tracking (Purpose: Portfolios That Give the Same Return as a Specified Benchmark Market Index)

Indices such as the S&P 500 in New York or the FTSE 100 in London are all stock market (equity) indices, which, in an easy-to-digest form, tell you how the stocks (companies) represented in those indices have changed in value over time.

If you want you can invest your money in index tracking funds ("trackers"), which aim to reproduce the performance of the index over time, perhaps by investing in all of the stocks that make up the index.

If a fund invests in all of the stocks in the index in such a way that its investment in each stock mirrors index composition (e.g., if a stock makes up 10% of the index, then it makes up 10% of the investment), then the fund is said to be following a *full/complete replication* strategy.

Full replication is possible—but as the number of stocks in the index grows, it can be an expensive strategy in terms of transaction cost. This is because

- stocks typically enter/leave the index at regular intervals, and so the entire fund must be rebalanced as this occurs to mirror the index as it changes, and
- any new money that is invested in (or money taken out of) the fund must be spread across all stocks to mirror the index.

For these reasons it is common, especially for indices where the number of stocks in the index is high, not to adopt full/complete replication when constructing an index tracking portfolio (TP).

Suppose therefore that we do not wish to adopt full replication. Then in essence we can view the index tracking problem as a *decision problem*, namely, to decide the *subset* of stocks to choose so as to (hopefully perfectly) mirror/reproduce the performance of the index over time. We call the subset of stocks we choose a *tracking portfolio*.

Suppose that we observe over time $0, 1, 2, \ldots, T$ the value of N stocks, as well as the value of the index we want to track. Further suppose that we are interested in deciding the best set of K stocks to hold (where $K < N$), as well as their appropriate quantities. In index tracking we want to answer the following question:

> What will be the best set of K stocks to hold, as well as their appropriate quantities, so as to best track the index in the future (from time T onward)?

Our basic approach in index tracking is a historic look-back approach: to ask the historic question,

> What would have been the best set of K stocks to have held, as well as their appropriate quantities, so as to have best tracked the index in the past (i.e., over the time period $[0, T]$)?

and then hold the stocks that answer this question into the future. Note how we adopt here the same approach as we actually adopted above in Markowitz mean-variance: use data from the immediate past to construct a portfolio, then hold that portfolio into the immediate future.

Although it is common to discuss index tracking in the context of stock (equity) indices, there are other financial indices (e.g., associated with bonds or commodities) that we might be interested in tracking, and so in the discussion below we use the word *asset* rather than *stock*.

Let

V_{it} be the value (price) of one unit of asset i at time t,
I_t be the value of the index at time t,

C be the amount we have to invest at time T,
ε_i be the minimum proportion that we must invest in asset i (if we hold any of the asset), and
δ_i be the maximum proportion that we can invest in asset i (if we hold any of the asset).

Then our decision variables are

$x_i =$ the number of units of asset i that we choose to hold in the TP, and
$z_i =$ 1 if any of asset i is held in the TP, $=0$ otherwise.

Without significant loss of generality, we allow $[x_i]$ to take fractional values.

Then the constraints associated with the index tracking problem are

$$\sum_{i=1}^{N} z_i = K, \tag{16}$$

$$w_i = V_{iT} x_i / C, \quad i = 1, \ldots, N, \tag{17}$$

$$\varepsilon_i z_i \le w_i \le \delta_i z_i, \quad i = 1, \ldots, N, \tag{18}$$

$$\sum_{i=1}^{N} V_{iT} x_i = C, \tag{19}$$

$$x_i \ge 0, \quad i = 1, \ldots, N, \tag{20}$$

$$z_i \in [0, 1], \quad i = 1, \ldots, N. \tag{21}$$

Equation (16) ensures that we choose exactly K assets, whereas Equation (17) defines the proportion variables in terms of the holding in each asset. Equation (18) links the proportion variables to the zero–one variables; in particular, it ensures that if we invest in an asset (so $z_i = 1$), the proportion held lies within the limits $[\varepsilon_i, \delta_i]$ defined. Equation (19) is a balance constraint that says that the value of the portfolio after trading is equal to the amount we have to invest (C). Here we have assumed zero transaction cost. Note here that Equations (17) and (19) together imply that $\sum_{i=1}^{N} w_i = 1$ (as in Equation (3)).

In time period t we get a return associated with the index, $R_t = \log_e(I_t / I_{t-1})$, where we define return using continuous time. If, in each and every time period, the return associated with the TP,

$$r_t = \log_e \left[\left[\sum_{i=1}^{N} V_{it} x_i \right] \Big/ \left[\sum_{i=1}^{N} V_{it-1} x_i \right] \right], \tag{22}$$

was *exactly* equal to R_t, then this might seem ideal. A possible objective in terms of index tracking is therefore

$$\text{minimise} \ \sum_{t=1}^{T} (r_t - R_t)^2 \Big/ T, \tag{23}$$

i.e., minimise the average squared difference between tracking portfolio return and index return.

The index tracking problem, as formulated above—minimise Equation (23) subject to Equations (16)–(22)—is a nonlinear mixed-integer program, and so it is hard to solve. Algorithmically, metaheuristics can be applied (e.g., see Beasley et al. [2]).

Note also here, echoing one of the key concepts introduced at the start of this tutorial, how we adopted an *optimisation mind-set* in constructing this index tracking model.

5.1. Alternatives

One consideration when working in the finance area as it relates to portfolio optimisation is that *problems are* (*mathematically*) *typically not uniquely defined*; that is, there are different mathematical models that take *different* (*but valid*) views of the problem. Taking Markowitz mean-variance (for example, where we are seeking portfolios that balance risk and return), could we not define risk using some measure other than variance? Indeed, we discussed that very issue above.

With regard to index tracking as considered here, an alterative view relates to regression. Suppose we perform a linear regression of the return from the tracking portfolio against the return from the index, i.e., the regression $r_t = \alpha + \beta R_t$. What intercept α and slope β would you expect to get if you perfectly track the index?

Clearly, we would get intercept $\alpha = 0$ and slope $\beta = 1$. This therefore gives us insight into a suitable optimisation model, produce a model that minimises $|\alpha - 0|$ and $|\beta - 1|$. This is easily done (albeit in a number of different ways since we have two factors of interest). As an illustration as to how this can be done, consider the approach below.

Let $\hat{\alpha}_i$ and $\hat{\beta}_i$ be the ordinary least-squares regression intercept and slope, respectively, when we regress the returns from asset i against the index returns R_t. Then when we regress the returns from the portfolio against index returns we will have intercept $\sum_{i=1}^{N} w_i \hat{\alpha}_i$ and slope $\sum_{i=1}^{N} w_i \hat{\beta}_i$. Hence we now wish to minimise $\left|\sum_{i=1}^{N} w_i \hat{\alpha}_i - 0\right| = \left|\sum_{i=1}^{N} w_i \hat{\alpha}_i\right|$ and $\left|\sum_{i=1}^{N} w_i \hat{\beta}_i - 1\right|$. Here we shall adopt a two-stage approach (as in Canakgoz and Beasley [5]) and first minimise $\left|\sum_{i=1}^{N} w_i \hat{\alpha}_i\right|$ and then minimise $\left|\sum_{i=1}^{N} w_i \hat{\beta}_i - 1\right|$. Although the modulus objectives here are nonlinear, they can be linearised in a standard way by introducing variables D and E, where

$$D \geq \sum_{i=1}^{N} w_i \hat{\alpha}_i, \tag{24}$$

$$D \geq -\sum_{i=1}^{N} w_i \hat{\alpha}_i, \tag{25}$$

$$E \geq \left[\sum_{i=1}^{N} w_i \hat{\beta}_i - 1,\right], \tag{26}$$

$$E \geq -\left[\sum_{i=1}^{N} w_i \hat{\beta}_i - 1\right]. \tag{27}$$

Then a mixed-integer linear programming formulation for the index tracking problem is in the first stage:

$$\begin{aligned} \text{minimise} \quad & D \\ \text{subject to} \quad & \text{Equations (16)–(21), (24)–(25).} \end{aligned}$$

Because this is a mixed-integer linear program we would expect that (computationally) finding an optimal solution might well be possible (e.g., using a package such as CPLEX).

If the optimal solution value is D*, then the second-stage program is

$$\begin{aligned} \text{minimise} \quad & E \\ \text{subject to} \quad & \text{Equations (16)–(21), (24)–(27) and } D = D^*, \end{aligned}$$

where the constraint $D = D^*$ ensures that at the second stage we retain the optimal value of D that we achieved at the first stage.

If you look back at what we did above, we have constructed *two different mathematical optimisation models for index tracking*:

- one a nonlinear model based on minimising the average squared difference between tracking portfolio return and index return, and

• one a linear model based on regression, seeking to achieve regression coefficients for the tracking portfolio, when its return is regressed against index return, as close to ideal values (intercept zero, slope one) as possible.

Each of these models had their own logic and each, by itself, seems perfectly reasonable. However, there is limited overlap between them. One uses regression coefficients, for example, and the other does not. This is an illustration of one of the key concepts we introduced at the start of this tutorial: *even for portfolios intended for the same purpose, the model to use is not uniquely defined.*

It is clear that (computationally) these two index tracking models might well produce different tracking portfolios when given the same in-sample data. The question therefore arises as to which one we might use, or more colloquially, which one might be better to use in practice. Here, the academic literature in finance is somewhat lacking, for two reasons:

• The finance field, unlike more traditional operations research, is one where different authors in the literature tend to use their own data. So in comparing two models (or papers), any scientific comparison is clouded by the fact that different in-sample/out-of-sample data have been used. I am old enough to remember when in operations research this was an issue, but the advent of data set repositories such as OR-Library (http://people.brunel.ac.uk/∼mastjjb/jeb/info.html) and others since the 1990s has eliminated this. So now in operations research, for many of the problems we consider, there are standard data sets (test problem instances) used by different authors in the literature, and this makes scientific comparison of different papers much easier. The finance field, for whatever reason, has not adopted the same approach.

• Even if we were take to take our two models, solve them both optimally using the same in-sample data, and evaluate them using the same out-of-sample data, what can we conclude? Here we face a methodological problem. Both models achieve optimal values in-sample, but on different objectives. Note how the portfolio constraints, Equations (16)–(21), are actually common to both models, so that is not a discriminator. With regard to out-of-sample performance, should we not evaluate each model with regard to its own objective? So, for example, if we take the first model, which minimises in-sample squared return differences, is not squared return differences the numeric factor that we should be examining out-of-sample? Is it valid to evaluate this first model (which does not include explicitly any regression information) with respect to its regression performance out-of-sample?

5.2. Alternatives, Again

Above we have given a model for index tracking based on ordinary least-squares regression, which was developed in the late 1800s, over 100 years ago (Stanton [34]). So is that the only game in town? Or is there a new kid on the block in terms of regression?

The answer is that there is a new kid on the block, *quantile regression.*

To digress slightly, most readers of this tutorial will probably be familiar with standard least-squares regression. In graphical form that involves plotting a dependent variable Y against an independent variable X and then fitting a straight line of the form $Y = \alpha + \beta X$ to the data. More technically, a regression of this type assumes a distribution of possible Y-values at each X-value (one realisation for a Y-value from the distribution that exists at each X-value being observed), and the regression relationship captures the linear relationship between the mean Y-values at each X-value.

In quantile regression, as first defined by Koenker and Bassett [14] in 1978, a linear equation relates how the quantiles of the dependent variable vary with the independent variable. Quantiles are values that divide the cumulative probability distribution. So, for example, the 50% quantile corresponds to the median of a distribution. The lower and upper quartiles of a distribution correspond to the 25% and 75% quantiles, respectively. Computationally, the coefficients in the linear equation relating how the quantiles of the dependent variable vary with the independent variable cannot be derived in a closed-form fashion (unlike standard least-squares regression), but instead are found as a result of solving a linear program.

So in our regression-based model for index tracking above, we have used regression coefficients based on the least-squares *mean* regression line. Could we not use coefficients based on *median* regression (so the 50% quantile)? Of course we could, and perhaps we should; after all, are not median values less susceptible to outliers than mean values? Work examining the application of quantile regression to index tracking can be found in Mezali and Beasley [27].

Recent work, other than that discussed above, dealing with index tracking can be found in Chen and Kwon [8], Garcia et al. [11], Guastaroba and Speranza [13], Krink et al. [15], Maringer [22], Ruiz-Torrubiano and Suarez [30], van Montfort et al. [37], and Wang et al. [38].

6. Enhanced Indexation

6.1. Enhanced Indexation—Specified Excess Return (Purpose: Portfolios That Achieve a Specified Excess Return with Respect to a Given Benchmark Market Index)

Enhanced indexation deals with the situation where we want to track the index (so getting the market return), as well as exceed it. Here we seek *relative return* (return relative to the index) rather than *absolute return* (of which more below). For example, we might want a stock (equity) portfolio that exceeds the return on a specified index by 1% per year. So if the index rises by 5% in a year, we want a portfolio that rises by 6% over the same time period. Alternatively, if the index falls by 9% in a year, we want a portfolio that only falls by 8%. Note here how our desired portfolio return is relative to the index return.

As we shall see below, finding an enhanced indexation portfolio can be accomplished using a mathematical model that is closely related to the index tracking model that we considered before. Here, for convenience, we use the term *tracking portfolio* to signify the enhanced indexation tracking portfolio.

How then can we construct TPs that "both track the index and exceed it"? One way is to

- take the return R_t given by the benchmark index,
- create an *artificial* (*enhanced return*) *index* whose return is $A_t = R_t + R^*$, where R^* is the desired excess return per time period (R^* being our choice, not something that the optimisation decides for us), and
- track this enhanced return index A_t.

So here, for example, we could take our first index tracking model above (minimise Equation (23) subject to Equations (16)–(22)) and apply it in an obvious fashion,

$$\text{minimise } \sum_{t=1}^{T} (r_t - A_t)^2 \Big/ T, \tag{28}$$

subject to Equations (16)–(22), to find an enhanced indexation portfolio.

Of course, there are alternative ways of constructing enhanced indexation portfolios:

1. use least-squares mean regression, as above for index tracking, but with respect to A_t;
2. use median quantile regression, as above for index tracking, but with respect to A_t;
3. use least-squares mean regression, as above for index tracking with respect to R_t, but seek a regression slope of one and a regression intercept $= R^*$; or
4. use median quantile regression, as above for index tracking with respect to R_t, but seek a regression slope of one and a regression intercept $= R^*$.

The first two of these approaches seek to track the artificial (enhanced return) index A_t.

Table 1. Out-of-sample excess returns, Russell 3000.

Desired level of excess return R^* (% per year)	Actual excess return out-of-sample (% per year)
3	36.9
4	34.3
5	34.9
10	35.0

The last two of these approaches use regression on index returns R_t, but with the regression intercept playing the role of excess return.

Table 1 (from the work reported in Canakgoz and Beasley [5]) shows for the Russell 3000 the out-of-sample excess return achieved (percentage in excess of the index per annum) when we find an enhanced indexation portfolio (with 70 stocks) using the third approach above.

Scientifically, the results in Table 1 are questionable—since out-of-sample excess return seems to have little relationship with desired excess return. Of course, if you were a practical investor you might not care, instead choosing to focus purely on the out-of-sample return achieved.

My experience has been that (subject to certain qualifications) it is not difficult to develop decision models based on optimisation that can find portfolios that (out-of-sample) outperform an index. Typically, what you cannot do (or at least I cannot do) is to have precise control over the level of outperformance.

As an aside here, if you consider Equation (28), can you see how to use this to decide the composition of a basket portfolio for an exchange traded fund that gives a multiplier λ of the return on a specified index?

The answer is obvious: set $A_t = \lambda R_t$. For example, we might consider values such as $\lambda = 2$, $\lambda = 1.5$, and $\lambda = -1$.

6.2. Enhanced Indexation—Outperform (Purpose: Portfolios That Do Better Than a Given Benchmark Market Index)

Sometimes we may not have a specified excess return (R^* above). Rather, we simply want to "do better than the index." Here, one objective that can be used is a modified Sortino ratio, namely,

$$\text{maximise} \quad \left(\sum_{t=1}^{T} r_t \Big/ T - R^{\text{mean}}\right) \Bigg/ \sqrt{\left[\sum_{t=1}^{T} (\min(0, r_t - R^{\text{mean}}))^2 \Big/ T\right]}, \tag{29}$$

$$\text{subject to} \quad (16)\text{–}(22),$$

where $R^{\text{mean}} = (\sum_{t=1}^{T} R_t/T)$ is the mean return on the index.

The Sortino ratio, as modified from its original definition (Sortino and Price [33]) here, balances return over and above a target return (here, the target being R^{mean}) with a downside risk term relating to falling below the target return.

Table 2 shows some results for this objective, solved using a metaheuristic algorithm (Meade and Beasley [24]). Here, we consider the S&P Global 1200 index where we are choosing a portfolio of 100 stocks and holding that portfolio for a specified period before rebalancing.

As an alternative here, one can construct a model for producing enhanced indexation portfolios using quantile regression. Details of such a model (a mixed-integer linear program) can be found in Mezali and Beasley [27], and Table 3 shows the annual (percentage) excess return for this model for the S&P 500 ($K = 40$) and the Russell 2000 ($K = 90$) for varying

Table 2. Out-of-sample excess returns, S&P Global 1200.

Holding period in weeks	Out-of-sample return (% per year) in excess of the index
4	10.5
12	12.3
24	11.2
36	12.0
48	9.5

Table 3. Out-of-sample excess returns, quantile regression.

Value of τ	0.45	0.40	0.35	0.30	0.25	0.20	0.15	0.10	0.05
S&P 500 excess return	4.0	−4.8	−0.4	9.5	6.8	9.2	7.2	5.8	5.5
Russell 2000 excess return	4.7	9.5	10.9	10.5	11.0	12.8	12.2	11.8	38.8

values of τ, the quantile of interest. So for the S&P 500 with $\tau = 0.30$ the annual percentage excess return (return over and above the index) is 9.5%.

Recent work, other than that discussed above, dealing with enhanced indexation can be found in Lejeune [18], Lejeune and Samatli-Pac [19], and Li et al. [21].

7. Absolute Return/Market Neutral Portfolios (Purpose: Portfolios That Do Well Irrespective of How a Benchmark Market Index Performs)

The idea here is to construct a portfolio of assets that gives a return independent of that of a benchmark market index, as the underlying index rises and falls (varies).

The terms *absolute return portfolio* and *market neutral portfolio* tend to be used fairly interchangeably in finance. Here we adopt an academic perspective and distinguish between absolute return/market neutral in the sense that

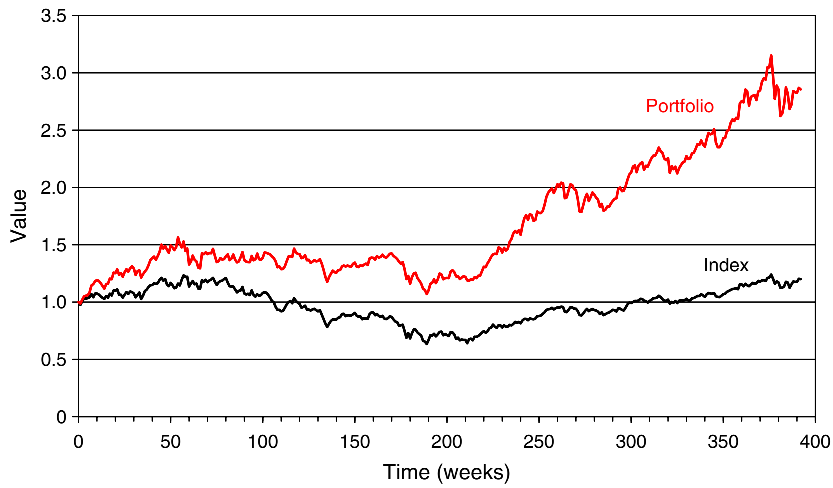
- we regard an absolute return portfolio as one that (ideally) gives a constant return per period, and
- we regard a market neutral portfolio as one that (ideally) has zero correlation between portfolio return and index return.

7.1. Absolute Return Portfolio

In terms of an absolute return portfolio, suppose that we perform a linear regression of the return from the portfolio against time, i.e., the regression $r_t = \alpha + \beta t$. What intercept α and slope β would you like if you are seeking an absolute return portfolio that gives a constant return per period? Clearly, we would like slope $\beta = 0$ and intercept α as large as possible. This therefore gives us insight into a suitable optimisation model, produce a model that minimises $|\beta - 1|$ and maximises α. We will not give details here, but we can proceed in the same manner as we did for the regression model associated with index tracking above to produce a mixed-integer linear program.

Figure 7 (based on work reported in Valle et al. [35]) shows some results for this approach for the S&P Global 1200 index. Here we are choosing a portfolio of 250 stocks and holding that portfolio for 26 weeks before rebalancing.

FIGURE 7. Absolute return portfolio.



7.2. Market Neutral Portfolio

In terms of a market neutral portfolio, suppose that we calculate the correlation coefficient Γ between portfolio return (r_t, Equation (22)) and index return R_t. What value for the correlation coefficient would you like if you are seeking a market neutral portfolio? Clearly, we would like a value for Γ of zero. This therefore gives us insight into a suitable optimisation model, produce a model that minimises $|\Gamma|$ so as to achieve a value for Γ as close to zero as possible. Therefore our model for a market neutral portfolio is minimise $|\Gamma|$ subject to Equations (16)–(22). Although the modulus in this objective is capable of linearisation (in the same manner as seen above when we had our regression model for index tracking), we do here have a mixed-integer nonlinear program (since correlation is itself a nonlinear function).

Mixed-integer nonlinear programs are by their nature algorithmically and computationally challenging, since they combine discrete and continuous variables and are nonlinear. However, packages for such programs have been improving (e.g., see Bussieck and Vigerske [4]), and certainly our computational experience with Minotaur (Leyffer et al. [20]) for producing a market neutral portfolio is that (with certain qualifications) it is capable of solving quite large problems.

As an indication as to what can be done, Table 4, based on work reported in Valle et al. [36], shows out-of-sample statistics for market neutral portfolios with respect to a number of benchmark indices. In that table we show results for long-only portfolios and for portfolios with a fixed mix of long/short positions (so for 130/30, we have 30% of the investment in short positions, 130% in long positions). The table shows the out-of-sample values (based on a holding period of 13 weeks) for correlation, portfolio, return, and portfolio excess return (return over and above the index) over 30 rebalances.

TABLE 4. Out-of-sample statistics, market neutral portfolios.

	Long only			130/30		
	Correlation	Return (% p.a.)	Excess return (% p.a.)	Correlation	Return (% p.a.)	Excess return (% p.a.)
S&P Europe 350	0.448	24.8	20.5	0.263	14.1	10.2
S&P US 500	0.494	11.4	11.5	0.319	16.9	17.1
S&P Global 1200	0.330	34.6	32.0	0.264	−7.8	−9.5

Note. p.a., per annum

8. Conclusions

8.1. Commonalities

Above we have considered a multitude of problems that fall within the context of financial portfolio optimisation. All seemed drawn from different contexts, aiming to derive optimised portfolios designed to achieve different purposes. But were there any commonalities? The answer is yes—the *constraints*.

Did you notice above how often the same set of constraints seemed to appear each time we did an optimisation? A moment's thought will indicate that since we are always dealing with financial portfolios, albeit designed for different purposes, the same underlying constraints must always apply: for instance,

- if we are restricting the number of assets, Equations (13)–(15);
- if we have constraints on the proportion held, Equations (6), (17), and (18);
- if we are rebalancing an existing portfolio, Equations (8) and (9); and
- if we have transaction costs, Equations (10)–(12).

8.2. Key Concepts

In this tutorial we have tried to illustrate the key concepts we set out at the start:

- *Different types of portfolios require different mathematical models.* The purpose to which you intend to put your portfolio needs to be considered when you construct/use a mathematical model. Compare, for example, the models for index tracking with the Markowitz mean-variance model to balance risk and return. Notice how different they are.
- *Even for portfolios intended for the same purpose, the model to use is not uniquely defined.* Even if we are considering a specific situation, there are different models, each with their own logic, that can be adopted. Compare, for example, the three models for index tracking (without regression but using squared return differences, with least-squares mean regression, and with median quantile regression) discussed above. Did not each of them, when considered individually, seem a perfectly reasonable model for index tracking?
- *Adopt an optimisation mind-set in building/choosing a model.* When you build/choose a model to construct a portfolio you need to ask yourself, what is my objective? Hopefully this became apparent as we outlined the models we presented above.

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