Numerical Analysis homework # 1

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I. Bisection Method Interval Width and Root Proximity

I-a

We can easily determine the initial interval width:

$$l_0 = 3.5 - 1.5 = 2 \tag{1}$$

Thus, let the interval width at the n-th step be l_n, it satisfies the following relation:

$$l_n = l_0 \times 2^{-n} = 2^{1-n} \tag{2}$$

I-b

We let the interval to be $[a_n, b_n]$, then the midpoint $c_n = \frac{a_n+b_n}{2}$. For the root $r \in [a_n, b_n]$, the distance between c_n and r satisfies:

$$|c_n - r| \leq \min\{r - a_n, b_n - r\} \leq \frac{b_n - a_n}{2} = \frac{l_n}{2} = 2^{-n} \tag{3}$$

Therefore the supermum distance between c_n and r is 2^{-n}.

II. Bisection Algorithm Accuracy Proof

Proof: the relative error at the n-th step is $\eta_n = \frac{|c_n - r|}{|r|}$. Using the similar method in , we have $|c_n - r| \leq (b_0 - a_0)2^{-n-1}$, thus we have:

$$\eta_n = \frac{|c_n - r|}{|r|} \leq \frac{(b_0 - a_0)2^{-n-1}}{r} \leq \frac{(b_0 - a_0)2^{-n-1}}{a_0} \tag{4}$$

In order to make the relative error less than ϵ, we get the inequality:

$$\frac{(b_0 - a_0)2^{-n-1}}{a_0} \leq \epsilon \tag{5}$$

By solving 5, we get the following inequality:

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1 \tag{6}$$

Thus we prove the inequality.

III. Newton's Method Iterations for Polynomial Equation

For the polynomial $p(x) = 4x^3 - 2x^2 + 3$, so $p' = 12x^2 - 4x$. Therefore, the iteration relation is:

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = x_n - \frac{4x_n^3 - 2x_n^2 + 3}{12x_n^2 - 4x_n} \tag{7}$$

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The iteration table is shown below:

n	x_n	$p(x_n)$
0	-1.000000000	-3.000000000
1	-0.812500000	-0.465820312
2	-0.770804196	-0.020137887
3	-0.768832384	-0.000043708
4	-0.768828086	-0.000000000
5	-0.768828086	0.000000000
6	-0.768828086	0.000000000

IV. Convergence of another Newton's Method

In this section, we will consider a variation of Newton's method and study its convergence properties, which is defined as follows:

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_0)} \tag{8}$$

We give some notations here: α is the root of the function $f(x)$, e_n is the error at the n-th step, $e_n = |x_n - \alpha|$. By Taylor expansion, we have:

$$f(x_n) = f(\alpha) + f'(\xi)(x_n - \alpha) = f'(\xi)(x_n - \alpha) \tag{9}$$

So we have:

$$e_{n+1} = |x_{n+1} - \alpha| = |1 - \frac{f'(\xi)}{f'(x_0)}|e_n, \quad \xi \in (x_n, \alpha) \ or \ \xi \in (\alpha, x_n) \tag{10}$$

Therefore $C = |1 - \frac{f'(\xi)}{f'(x_0)}|, \quad \xi \in (x_n, \alpha) \ or \ \xi \in (\alpha, x_n)$, $s = 1$

V. Convergence of A Certain Iterative Function

In this section, we will consider the convergence of the following iteration:

$$x_{n+1} = \tanh^{-1}(x_n), \quad x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \tag{11}$$

We need some properties of the function $f(x) = \tanh^{-1}(x)$. If $x > 0$, then $f(x) > 0$, if $x < 0$, then $f(x) < 0$. Moreover, when $x > 0$, $\tanh^{-1}(x) < x$, when $x < 0$, $\tanh^{-1}(x) > x$.

Therefore, if $x_0 > 0$, then $0 < x_1 < x_0$. By mathematical induction, we have $0 < x_n < x_{n-1} < \cdots < x_0$. So this is a monotonically decreasing and bounded sequence, thus it converges.

If $x_0 < 0$, then $0 > x_1 > x_0$. By mathematical induction, we have $0 > x_n > x_{n-1} > \cdots > x_0$. So this is a monotonically increasing and bounded sequence, thus it converges.

If $x_0 = 0$, then $x_1 = 0, \cdots, x_n = 0$, so it converges.

In conclusion, the sequence converges for all $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

VI. Value of A Continued Fraction

In this section, we will calculate the value of the continued fraction:

$$x = \frac{1}{p + \frac{1}{p + \frac{1}{p + \cdots}}}, p > 1 \tag{12}$$

To solve this, we define $x_1 = \frac{1}{p}, x_2 = \frac{1}{p + \frac{1}{p}}$ and so forth. Then we have a iteration relation:

$$x_{n+1} = \frac{1}{p + x_n}, \quad x_0 = \frac{1}{p} \tag{13}$$

So the value of the continued fraction is the limit of the sequence $\{x_n\}$. Next, we will prove the convergence of the sequence and calculate the limit of the sequence.

1. Convergence of the sequence

We have $x_1 > 0$, $x_{n+1} = \frac{1}{p+x_n}$, so by mathematical induction, we have $x_n > 0$. Consider the function $f(x) = \frac{1}{p+x}$, we know that $f' = -\frac{1}{(p+x)^2}$. We have $\forall x > 0, |f\prime(x)| = \frac{1}{(p+x)^2} < \frac{1}{p^2} < 1$.

By the convergence of contractions, we know that the sequence converges.

2. Calculate the limit of the sequence

Now we know that the sequence converges, let the limit be x, then we have:

$$\begin{aligned} x = f(x) &= \frac{1}{p+x} \\ \Rightarrow x &= \frac{\sqrt{p^2+4}-p}{2} \end{aligned} \tag{14}$$

In conclusion, the value of the continued fraction is $\frac{\sqrt{p^2+4}-p}{2}$.

VII. Bisection Method with Negative Initial Interval

Similar to , we can prove the relative error at the n-th step is $\eta_n \le \frac{|(b_0-a_0)2^{-n-1}|}{|r|}$.

So $\eta_n \le \epsilon \Rightarrow n \ge \frac{\log(b_0-a_0)-\log\epsilon-\log|r|}{\log 2} - 1$, r represents the root of the function.

There is no better way to imporve the inequality, because we don't know more information about the function and root.

Due to the root r may very close to 0 and the inequality is hard to compute, so the relative error is not an appropriate measure of the accuracy of the bisection method.

VII. Newton's Method with Multiple Zeros

VII-a

We first prove that Newton's method converges linearly when the root is multiple. We assume that the root is x^* and the multiplicity of the root is k. So we can write $f(x) = (x-x^*)^k g(x), g(x^*) \neq 0$. Let $\varphi(x) = x - \frac{f(x)}{f'(x)}$, then we have:

$$\varphi'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2} \tag{15}$$

$\varphi'(x^*) = 0$. Therefore, when x is close to x^*, $|\varphi'(x)| < 1$, by the convergence of contractions, we know that the Newton's method converges.

The convergence rate of the Newton's method is linear.

By Newton's method, we have:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n-x^*)^k g(x_n)}{k(x_n-x^*)^{k-1}g(x_n) + (x_n-x^*)^k g'(x_n)} \\ &= x_n - \frac{(x_n-x^*)g(x_n)}{kg(x_n)+(x_n-x^*)g'(x_n)} \\ x_{n+1} - x^* &= (x_n-x^*)(1 - \frac{g(x_n)}{kg(x_n)+(x_n-x^*)g'(x_n)}) \\ \lim_{n\to+\infty} \frac{x_{n+1}-x^*}{x_n-x^*} &= \frac{k-1}{k} \end{aligned} \tag{16}$$

By 16, we can easily know that $\lim_{n\to+\infty} |\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}| = \frac{k-1}{k}$

Therefore, the Newton's method converges linearly when the root is a multiple root.

So if we have the points $(x_n, f(x_n))$. We just calculate $\lim_{n\to+\infty} |\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}|$, if it converges to $\frac{k-1}{k}$, then it's a k-th root. If it converges to 0, then it's a single root.

VII-b

Similar to 16, we define $f(x) = (x-r)^k g(x), g(r) \neq 0$, so by 16, we have:

$$x_{n+1} - x^* = \frac{(x_n-x^*)^2 g'(x_n)}{kg(x_n)+(x_n-x^*)g'(x_n)} \tag{17}$$

Convergence of the new form of Newton's method
Define $\varphi(x) = x - k\frac{f(x)}{f'(x)}, f(x) = (x-r)^k g(x)$, then we have:

$$\varphi'(r) = 1 - k\frac{[f'(r)]^2 - f(r)f''(r)}{[f'(r)]^2} = 1 - k + k\frac{k(k-1)}{k^2} = 0 \tag{18}$$

So $|\varphi'(x)| < 1$ when x is close to r, by the convergence of contractions, the iteration converges.
The convergence is quadratic
By 17, we have:

$$\begin{aligned}
\frac{x_{n+1} - r}{(x_n - r)^2} &= \frac{g'(x_n)}{kg(x_n) + (x_n - r)g'(x_n)} \\
\Rightarrow \lim_{n\to+\infty} \frac{x_{n+1} - r}{(x_n - r)^2} &= \frac{g'(r)}{kg(r)}
\end{aligned} \tag{19}$$

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The section title is generated by **Kimi**, with a little revise.