Problem 1 (20 points) Solve the initial value problem

$$3y^2\frac{dy}{dx} = xy^3 + x + 3y^3 + 3, \qquad y(0) = 0.$$

Problem 2 (20 points) Solve the initial value problem

$$(x+3)\frac{dy}{dx} - (x+1)y = \frac{1}{x+3}, \qquad y(0) = 1.$$

Problem 3 (25 points) Solve the initial value problem

$$x'' + 2x' + x = 2\sin t, \quad x(0) = 0, \quad x'(0) = 1$$

Problem 4 (40 points) Using the method of undetermined coefficients write the form of a particular solution of the following equations. You **do not have** to compute the coefficients.

(a) $x'' + 5x' + 4x = e^t + t + \sin t$

(b) $x'' + 5x' + 4x = e^{-4t}$

(c) $x'' + 5x' + 4x = te^{-t}$

(d) $x'' + 4x = 2\sin 2t$

(e) $x'' + 4x = t\sin 2t$

Problem 5 (25 points) Solve the initial value problem

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Problem 6 (25 points) Solve the initial value problem

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Problem 7 (15 points) Find a particular solution of the system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

Problem 8 (30 points) Find all real critical points of the system of equations

$$\frac{dx}{dt} = (2+x)(y-x+2), \quad \frac{dy}{dt} = y+x.$$

Determine the type and stability of the critical points.

1. $3y^2 \dfrac{dy}{dx} = xy^3 + x + 3y^3 + 3$ $\qquad y(0)=0$

$3y^2 \dfrac{dy}{dx} = y^3(x+3) + (x+3)$

$3y^2 \dfrac{dy}{dx} = (y^3+1)(x+3)$

$\displaystyle\int \frac{3y^2}{(y^3+1)}\, dy = \int (x+3)\, dx$ $\qquad u = y^3+1$, $du = 3y^2\, dy$

$\displaystyle\int \frac{du}{u} = \int (x+3)\, dx$

$\ln(y^3+1) = \dfrac{x^2}{2} + 3x + C$

$y^3+1 = C\exp\left(\dfrac{x^2}{2} + 3x\right)$ $\qquad y(0)=0$

$0+1 = C\exp(0) \Rightarrow C=1$

$y = \left[e^{x^2/2 + 3x} - 1\right]^{1/3}$

2. $(x+3)\dfrac{dy}{dx} - (x+1)y = \dfrac{1}{x+3}$ $\qquad y(0)=1$

$u = x+3$
$du = dx$

$u\dfrac{dy}{du} + (2-u)y = \dfrac{1}{u}$

$\dfrac{dy}{du} + \left(\dfrac{2}{u} - 1\right)y = \dfrac{1}{u^2}$

$p(x) = \exp\left[\int\left(\dfrac{2}{u} - 1\right)du\right] = \exp\left[2\ln u - u\right] = e^{2\ln u - u}$

$= e^{-u}e^{2\ln u} = u^2e^{-u}$

$u^2e^{-u}\dfrac{dy}{du} + \left(2ue^{-u} - u^2e^{-u}\right)y = e^{-u}$

$\left(u^2e^{-u}y\right)' = e^{-u}$

$u^2e^{-u}y = -e^{-u} + C$

$(x+3)^2e^{-(x+3)}y = -e^{-(x+3)} + C$

$9e^{-3} = -e^{-3} + C \Rightarrow C = 10e^{-3}$

$(x+3)^2e^{-(x+3)}y = -e^{-(x+3)} + 10e^{-3}$

$(x+3)^2y = -1 + 10e^{x}$

$$\boxed{y = \dfrac{10e^{x} - 1}{(x+3)^2}}$$

$x'' + 2x' + x = 2\sin t \qquad x(0)=0 \qquad x'(0)=1$

$x_H'' + 2x_H' + x_H = 0$

$r^2 + 2r + 1 = 0$

$(r+1)(r+1) = 0$

$r = -1$

$x = C_1 e^{-t} + C_2 t e^{-t}$

$x_p = A\sin x + B\cos x$

$x_p' = A\cos x - B\sin x$

$x_p'' = -A\sin x - B\cos x$

$A\sin t - B\cos t + 2(A\cos t - B\sin t) + A\sin t + B\cos t = 2\sin t$

$\sin t(\cancel{-A} - 2B + \cancel{A}) + \cos t(\cancel{-B} + \cancel{2A} + \cancel{B}) = 2\sin t$

$-2B = 2 \qquad A = 0$

$B = -1$

$x_p = -\cos t$

$(t) = C_1 e^{-t} + C_2 t e^{-t} - \cos t$

$(0) = C_1 - 1 = 0 \Rightarrow C_1 = 1 \qquad x(t) = e^{-t} + C_2 t e^{-t} - \cos t$

$'(t) = -e^{-t} + C_2 e^{-t} - C_2 t e^{-t} + \sin t$

$'(0) = -1 + C_2 = 1 \Rightarrow C_2 = 2$

$\boxed{t) = e^{-t} + 2te^{-t} + \sin t}$

4. a. $x'' + 5x' + 4x = e^t + t + \sin t$

① $x'' + 5x' + 4x = e^t$
② $x'' + 5x' + 4x = t$
③ $x'' + 5x' + 4x = \sin t$

① $x_{p_1} = Ae^t$
② $x_{p_2} = B_0 + B_1 t$
③ $x_{p_3} = C\sin t + D\cos t$

$x_p = Ae^t + B_0 + B_1 t + C\sin t + D\cos t$

b. $x'' + 5x' + 4x = e^{-4t}$
$x_p = Ate^{-4t}$

see next page!

c. $x'' + 5x' + 4x = te^{-t}$
$x_p = (A_0 + A_1 t)e^{-t}$
$x_p' = -(A_0 + A_1 t)e^{-t} + A_1 e^{-t}$
$x_p'' = (A_0 + A_1 t)e^{-t} - 2A_1 e^{-t}$
$A_0 + \underline{A_1 t} - 2A_1 - 5A_0 - \underline{5A_1 t} + 5A_1 + 4A_0 + \underline{4A_1 t} = t$

$x_p = (A_0 t + A_1 t^2)e^{-t}$
$x_p' = (A_0 + 2A_1 t)e^{-t} - (A_0 t + A_1 t^2)e^{-t}$
$x_p'' = -2(A_0 + 2A_1 t)e^{-t} + 2A_1 e^{-t} + (A_0 t + A_1 t^2)e^{-t}$

d. $x'' + 4x = 2\sin 2t$
$x_p = A\sin 2t + B\cos 2t$

e. $x'' + 4x = t\sin 2t$
$x_p = (A_0 + A_1 t)\sin 2t + (B_0 + B_1 t)\cos 2t$

$x_p' = (A_0 + A_1 t) 2\cos(2t) + A_1 t \sin(2t)$
$\quad - (B_0 + B_1 t) 2\sin(2t) + B_1 t \cos(2t)$

$x_p'' = -4(A_0 + A_1 t)\sin(2t) + 2A_1 \cos(2t) + A_1 \sin(2t)$
$\quad + 2A_1 t \cos(2t)$
$\quad -4(B_0 + B_1 t)\cos(2t) - 2B_1 \sin(2t) + B_1 \cos(2t)$
$\quad - 2B_1 t \sin(2t)$

$x'' + 4x = -3(A_0 + A_1 t)\sin(2t) + 2A_1 \cos(2t) + A_1 \sin(2t$
$\quad + 2A_1 t \cos(2t) - 3(B_0 + B_1 t)\cos(2t) - 2B_1 \sin(2t)$
$\quad + B_1 \cos(2t) - 2B_1 t \sin(2t) \quad = t \sin 2t$

$-3A_1 - 2B_1 = 1$
$-3A_0 + A_1 - 2B_1 = 0$
$\vdots$

Check ©
$-2(A_0 + 2A_1 t)\cancel{e^{-t}} + 2A_1 \cancel{e^{-t}} + (A_0 t + A_1 t^2)\cancel{e^{-t}}$
$+ 5\left[(A_0 + 2A_1 t)\cancel{e^{-t}} - (A_0 t + A_1 t^2)\cancel{e^{-t}}\right] + 4(A_0 t + A_1 t^2)\cancel{e}$
$= t.$

$-2A_0 + 2A_1 + 5A_0 = 0 \Rightarrow 3A_0 + 2A_1 = 0$
$-4A_1 + A_0 + 10A_1 - 5A_0 + 4A_0 = 1 \Rightarrow -4A_1 + 10A_0 = 1$
$A_1 + 5A_1 + 4A_1 = 0 \Rightarrow A_1 = 0$ no!

$x_p = (A_0 t^2 + A_1 t^3)e^{-t}$

5. $\frac{d\vec{x}}{dt} = \underbrace{\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}}_{A} \vec{x}$ $\quad \bar{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 5-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (5-\lambda)(1-\lambda) + 4 = 0$

$5 - 6\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)(\lambda - 3) = 0$

$\lambda = 3$

$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$2\eta_1 + 2\eta_2 = 1 \quad v_2 = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{3t} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} e^{3t} \right]$

$\bar{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$c_1 = 0$

$-\cancel{c_1} + 1/2\, c_2 = -1 \Rightarrow c_2 = -2$

$\vec{x}(t) = -2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{3t} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} e^{3t} \right]$

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}}_{A}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$-\lambda I = \begin{pmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{pmatrix} \qquad \det(A-\lambda I) = 0 \Rightarrow (-1-\lambda)(-1-\lambda) + 9 = 0$$

$$\Rightarrow 1 + 2\lambda + \lambda^2 + 9 = 0 \Rightarrow \lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

$= -1 + 3i$

$$\begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -3i & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3ix_1 + 3x_2 = 0 \Rightarrow ix_1 = x_2 \qquad v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$-1-3i$

$$\begin{pmatrix} 3i & 3 \\ 3 & 3i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow ix_1 + x_2 = 0 \qquad v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$t) = c_1\begin{pmatrix} 1 \\ i \end{pmatrix}e^{(-1+3i)t} + c_2\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{(-1-3i)t}$$

$$0) = c_1\begin{pmatrix} 1 \\ i \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{B}{}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\cancel{\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ i & -i & 0 & 1 \end{array}\right)}$$

$$\rightarrow \cancel{\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2i & -i & 1 \end{array}\right)}$$

$$\rightarrow \cancel{\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2\, i \end{array}\right)}$$

$$\rightarrow \left(\begin{array}{cc|cc} \cancel{1} & \cancel{0} & \cancel{1/2} & \cancel{1/2\, i} \\ 0 & 1 & 1/2 & -1/2\, i \end{array}\right)$$

$$\cancel{B^{-1} = \frac{1}{2i}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}}$$

$$B_{11} = -i \qquad B_{12} = -1$$

$$B_{21} = -i \qquad B_{22} = 1$$

$$B^{-1} = \frac{1}{-2i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2\, i \\ 1/2 & -1/2\, i \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2\, i \\ 1/2 & -1/2\, i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2(1-i) \\ 1/2(1+i) \end{pmatrix}$$

$$t) = \frac{1}{2}(1-i)e^{(-1+3i)t} + \frac{1}{2}(1+i)e^{(-1-3i)t}$$

7. $$\frac{dx}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

$$x_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a_1 e^t \\ a_2 e^t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} 3a_1 - 2a_2 \\ 2a_1 - 2a_2 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

$$a_1 = 3a_1 - 2a_2$$

$$-2a_1 = -2a_2 \Rightarrow \underline{a_1 = a_2}$$

$$a_2 = 2a_1 - 2a_2 + 1$$

$$3a_2 = 2a_1 + 1$$

$$3a_2 - 2a_1 = 3a_2 - 2a_2 = a_2 = 1$$

$$x_p(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

Check: $x_p' = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cancel{e^t} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cancel{e^t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cancel{e^t}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\dot{x} = (2+x)(y-x+2)$

$\dot{y} = y+x$

$(2+x)(y-x+2) = 0 \rightarrow x = -2 \ , \ x-y = 2$

$y+x = 0 \rightarrow y = -x$

$x = -2 \Rightarrow y = 2 \qquad (-2, 2)$

$x-y = 2 \Rightarrow 2x = 2 \Rightarrow x = 1 \Rightarrow y = -1 \qquad (1, -1)$

$(-2, 2)$ and $(1, -1)$ are the critical points

$= (2+x)(y-x+2) = 2y - \cancel{2x} + 4 + xy - x^2 + \cancel{2x}$

$= 2y + xy - x^2 + 4$

$= y + x$

$$= \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} y-2x & 2+x \\ 1 & 1 \end{pmatrix}$$

$$(-2, 2) = \begin{pmatrix} -2-4 & -2+2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} -6-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (-6-\lambda)(1-\lambda) = 0$$

$\lambda = 1, -6$

$\Rightarrow (-2, 2)$ is an unstable saddle

$$(1, -1) = \begin{pmatrix} -1-2 & 2+1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} -3-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (-3-\lambda)(1-\lambda) - 3$$

$= -3 + 3\lambda - \lambda + \lambda^2 - 3 = 0$

$\lambda^2 + 2\lambda - 6 = 0$

$\lambda_1 > 0, \quad \lambda_2 < 0$

$\lambda = \dfrac{-2 \pm \sqrt{4+24}}{2} = \dfrac{-2 \pm 2\sqrt{7}}{2}$

$\Rightarrow (1, -1)$ is an unstable saddle

$= -1 \pm \sqrt{7}$