# Linear Algebra Preliminary Exam 2011

1. Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{pmatrix}.$$

2. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & k \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & k \end{pmatrix}$$

Find the values of $k$ such that

a) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

b) $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$

3. Consider the subspace $\mathbf{V}$ of $\mathbb{R}^4$ that is spanned by the vectors $\mathbf{v}_1 = (1,1,0,1)$ and $\mathbf{v}_2 = (0,0,1,0)$.

a) Find a basis for the orthogonal complement $\mathbf{V}^{\perp}$.

b) Find a projection matrix $\mathbf{P}$ onto $\mathbf{V}$.

c) Find the vector in $\mathbf{V}$ closest to the vector $\mathbf{b} = (0,1,0,-1)^T$ in $\mathbf{V}^{\perp}$.

d) Write the vector $\mathbf{c} = (1,-1,1,-1)$ in the form $\mathbf{c} = \mathbf{v}+\mathbf{w}$ where $\mathbf{v} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{V}^{\perp}$.

4. Suppose

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = (b_1, b_2, b_3)^T$$

a) What are the conditions on $\mathbf{b}$ for there to exist a solution to $\mathbf{Ax} = \mathbf{b}$?

b) Find a basis for the nullspace of $\mathbf{A}$.

c) Find a general solution to $\mathbf{Ax} = \mathbf{b}$, when a solution exists.

d) Find a basis for the column space of $\mathbf{A}$.

e) What is the rank of $\mathbf{A}^T$?

5. Let $\mathbf{A}$ be an $N \times N$ square matrix with elements

$$A_{nm} = a_n a_m$$

with positive real numbers $a_n > 0$, $n = 1, ..., N$. Show that $\mathbf{A}$ has two distinct non-negative eigenvalues $\lambda_1$ and $\lambda_2$ with corresponding eigenspace of dimension 1 and $N-1$, respectively.

6. Let $\mathbf{P}$ be a linear operator defined on a vector space $V$ with the property

$$\mathbf{P} = \mathbf{P}^2. \tag{0.1}$$

a) Show that $\mathbf{P}$ can only have eigenvalues 1 or 0.

b) Explain why an operator that fullfills property. (0.1) is called a projector.

c) Let $V = \mathbb{R}^n$ and $U$ be a subspace with orthonormal basis $\mathbf{u}_i$, $i = 1, ..., k$ and $k \leq n$. Show that $\mathbf{P}$ can be constructed explicitely as

$$\mathbf{P} = \sum_{i=1}^{k} \mathbf{P}_i \quad \text{with} \quad \mathbf{P}_i = \mathbf{u}_i \mathbf{u}_i^T, \tag{0.2}$$

i.e. show that $\mathbf{Pu} = \mathbf{u}$ and $\mathbf{Pw} = 0$ for $\mathbf{u} \in U$ and $\mathbf{w} \in U^{\perp}$ and $\mathbf{P}^2 = \mathbf{P}$. The notation $\mathbf{xy}^T$ is the outer product of two vectors, i.e. $\left(\mathbf{xy}^T\right)_{ij} = x_i y_j$ for $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$.

d) Show that all $\mathbf{P}_i$ are projectors as well and that $\mathbf{P}_i\mathbf{P}_j = 0$ for $i \neq j$.

e) Let $U$ be the plane in $\mathbb{R}^3$ orthogonal to the vector $\mathbf{v} = (a,b,c)^T$ in cartesian coordinates. Construct $\mathbf{P}$ explicitly that projects onto $U$.