PRELIMINARY EXAM – LINEAR ALGEBRA 4/09

Note - all problems count equally

1. Let A and B be $n \times n$ matrices and let Δt be small compared to $\| A \|$ and $\| B \|$.

   (a) Show that
   $$\exp((A+B)\Delta t) = \exp(A\Delta t)\exp(B\Delta t) + O((\Delta t)^2).$$

   (b) Show that
   $$\exp((A+B)2\Delta t) = \exp(A\Delta t)\exp(B\Delta t)\exp(B\Delta t)\exp(A\Delta t) + O((\Delta t)^3).$$

   (c) Determine the most general condition on A and B so that
   $$\exp((A+B)\Delta t) = \exp(A\Delta t)\exp(B\Delta t).$$

2. Consider the matrix
   $$A = \begin{pmatrix} 3 & 3 \\ -\frac{5}{2} & -\frac{5}{2} \end{pmatrix}.$$
   Prove that $\lim_{n\to\infty} A^n = 0$.

3. Consider the matrix
   $$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix},$$
   where a, b, c, and d are arbitrary complex numbers. Determine the most general condition on a, b, c, d so that $\det(A) \neq 0$.

4. Consider the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

(a) Show that A is singular.

(b) Show that the eigenvalues of A are real and that the eigenvectors are orthogonal.

(c) Show further that all eigenvalues are nonpositive.

5. Consider the vectors

$$\begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.$$

Determine a basis for the space spanned by these vectors such that each vector in the basis consists only of the numbers $0, \pm 1, \pm 2$.

6. Consider the space of cubic polynomials $p(t) = a + bt + ct^2 + dt^3$. Determine a matrix representation for the operator

$$\frac{dtp}{dt}.$$

7. Consider the space spanned by the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

Determine an orthonormal basis for this set.