PRELIMINARY EXAM - LINEAR ALGEBRA 1/00

Please show all work. To get full credit for a problem you need to show your work and **CLEARLY** describe your calculations.

1. **(40 points)** Consider the matrix M

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & Q \end{bmatrix}$$

where Q is an arbitrary constant.

(a) What is the trace of $\mathbf{M}$? What is the determinant of $\mathbf{M}$?

(b) For what values of Q can you solve

$$\mathbf{MX} = \mathbf{F}$$

for $\mathbf{F}$ arbitrary?

(c) For those values of Q for which there is no solution to (b), find the general solution for a class of $\mathbf{F}$'s.

(d) Is there a matrix $\mathbf{N}$ such that

$$\mathbf{MX} = \mathbf{NF}$$

always has a solution? If so, find one.

2. **(20 points)** Project the vector $\mathbf{c} = [1, 2]$ onto two vectors that are not orthogonal, $\mathbf{a} = [1, 0]$ and $\mathbf{b} = [1, 1]$. Show that unlike the orthogonal case, the sum of the two one-dimensional projections does not equal $\mathbf{c}$.

3. **(20 points)** Suppose the given vectors are

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find the orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$.

4. **(20 points)** Compute $\mathbf{A}^{100}$ and $\exp(\mathbf{A})$ where

$$\mathbf{A} = \begin{bmatrix} 14 & 9 \\ -16 & -10 \end{bmatrix}.$$

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \quad \begin{matrix} 2y_1 = y_2 \\ y_1 = -y_3 \\ y_1 = y_2 + y_3 \end{matrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

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1. $m = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & Q \end{bmatrix}$ Q arbitrary constant

a. $tr(m) = 2 - 1 + Q = 1 + Q$

$$\det(m) = 2\begin{vmatrix} -1 & 0 \\ -1 & Q \end{vmatrix} - 1\begin{vmatrix} -1 & 0 \\ 0 & Q \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}$$
$$= 2(-Q) - (-Q) + (1)$$
$$= -2Q + Q + 1 = 1 - Q$$

another way to get det(m)

$$\det(m) = \det\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & Q \end{bmatrix} = -\det\begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & Q \end{bmatrix}$$

$$= -\det\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & Q \end{bmatrix} = -\det\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & Q-1 \end{bmatrix}$$

$$= (-1)(Q-1) = 1 - Q$$

b. $mx = F$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & Q \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$\det(m) = 0 \Rightarrow$ singular

singular when $Q = 1$, nonsingular otherwise

for arbitrary F, $mx = F$ is solveable if $Q \neq$

case #1: $Q \neq 1$

Then M is invertible and the $Mx = F$ can be solved for arbitrary F.

c. case #2: $Q = 1$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 - 2F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -F_2 \\ -F_1 + 2F_2 \\ -F_3 + F_1 - 2F_2 \end{Bmatrix}$$

There is a solution provided that $-F_3 + F_1 - 2F_2 =$

x_3 is free

$x_2 + x_3 = 2F_2 - F_1 \implies x_2 = 2F_2 - F_1 - x_3$

$x_1 + x_2 = -F_2 \implies x_1 = -F_2 - x_2$

$= -F_2 - 2F_2 + F_1 + x_3$

$= F_1 - 3F_2 + x_3$

$$x = \begin{Bmatrix} F_1 - 3F_2 + x_3 \\ 2F_2 - F_1 - x_3 \\ x_3 \end{Bmatrix} = x_3 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1 - 3F_2 \\ 2F_2 - F_1 \\ 0 \end{Bmatrix}$$

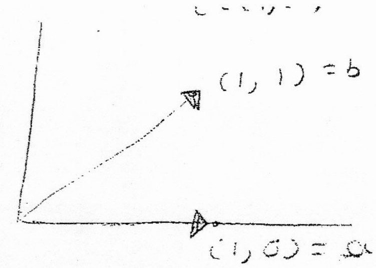
$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & Q \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [N] \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

no! If there is always a solution, then $N^{-1}m$ must always be invertible, Then $\det(N^{-1}m) \neq 0$

$$\det(N^{-1}m) = \det(N^{-1}) \underbrace{\det(m)}_{=0 \text{ for some } Q} \neq 0$$

$c = (1,2)$

$a = (1,0)\ ,\quad b = (1,1)$



ojection of c onto a:

$$P_{c/a} = \frac{(1,0)\cdot\binom{1}{2}}{(1,0)\cdot\binom{1}{0}}\ (1,0)$$

$$= \frac{1}{1}\ (1,0) = (1,0)$$

rojection of c onto b:

$$P_{c/b} = \frac{b^T c}{b^T b}\ b = \frac{\binom{1}{1}(1,2)}{\binom{1}{1}(1,1)}\ (1,1) = \frac{3}{2}\ (1,1)$$

$$= \left(\tfrac{3}{2}, \tfrac{3}{2}\right)$$

$$P_{c/a} + P_{c/b} = (1,0) + \left(\tfrac{3}{2}, \tfrac{3}{2}\right) = \left(\tfrac{5}{2}, \tfrac{3}{2}\right) \neq \underset{=c}{(1,2)}$$

$a \cdot b = 1 + 0 = 1 \neq 0$

Let $d = (0,1)$

$a \cdot d = 0$

$P_{c/a} = (1,0)$

$$P_{c/d} = \frac{\binom{0}{1}\cdot(1,2)}{\binom{0}{1}(0,1)}\ (0,1) = \frac{2}{1}\ (0,1) = (0,2)$$

$$P_{c/a} + P_{c/d} = (1,0) + (0,2) = \underline{(1,2)} = c$$

$a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$b' = b - (q_1^T b)\, q_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -1/2 \\ 3/2 \end{pmatrix}$$

$$q_2 = \frac{b'}{\|b'\|} = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{9}{4}}} \begin{pmatrix} 1/2 \\ -1/2 \\ 3/2 \end{pmatrix} = \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$c' = c - (q_1^T c)\, q_1 - (q_2^T c)\, q_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{11}} (2) \left(\frac{1}{\sqrt{11}}\right) \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \quad \frac{2}{11} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/11 \\ -2/11 \\ 6/11 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 - 2/11 \\ 1/2 + 2/11 \\ 1 - 6/11 \end{pmatrix} = \begin{pmatrix} -15/22 \\ 15/22 \\ 5/11 \end{pmatrix} = \frac{5}{11} \begin{pmatrix} -3/2 \\ 3/2 \\ 1 \end{pmatrix}$$

$$q_3 = \frac{c'}{\|c'\|} = \frac{1}{\|c'\|} \frac{5}{11} \begin{pmatrix} -3/2 \\ 3/2 \\ 1 \end{pmatrix}$$

4. $A = \begin{bmatrix} 14 & 7 \\ -16 & -10 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 14-\lambda & 9 \\ -16 & -10-\lambda \end{bmatrix}$ $\quad \det(A - \lambda I) = 0 \Rightarrow \lambda = 2$ (twice)

$$\begin{bmatrix} 12 & 9 \\ -16 & -12 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{aligned} 12x_1 + 9x_2 &= 0 \\ 4x_1 + 3x_2 &= 0 \end{aligned} \qquad v = \begin{Bmatrix} 3 \\ -4 \end{Bmatrix}$$

only one independent eigenvector $\rightarrow$ A not diagonalizable

$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ one eigenvector, one Jordan block

$A = mJm^{-1} \quad \Rightarrow \quad A^k = mJ^km^{-1}$

$$\begin{bmatrix} 12 & 9 \\ -16 & -12 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -4 \end{Bmatrix} \Rightarrow \begin{aligned} 12y_1 + 9y_2 &= 3 \\ 4y_1 + 3y_2 &= 1 \\ y_2 = 0, \ y_1 &= \tfrac{1}{4} \end{aligned}$$

$$m = \begin{bmatrix} 3 & 1/4 \\ -4 & 0 \end{bmatrix} \qquad m^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix}$$

check:

$$A = mJm^{-1} = \begin{bmatrix} 3 & 1/4 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 7/2 \\ -8 & -4 \end{bmatrix}\begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 9 \\ -16 & -10 \end{bmatrix}$$

$-\frac{6}{4} + \frac{42}{4} = \frac{36}{4}$

$$A^{100} = \begin{bmatrix} 3 & 1/4 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{100}\begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix}$$

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad J^2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

$$J^3 = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 0 & 8 \end{bmatrix}$$

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$$

$$J^{100} = \begin{bmatrix} 2^{100} & 100(2^{99}) \\ 0 & 2^{100} \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 3 & 1/4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 2^{100} & 100(2^{99}) \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(2^{100}) & 300(2^{99}) + \frac{1}{4}(2^{100}) \\ -4(2^{100}) & -400(2^{99}) \end{bmatrix} \begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1200(2^{99}) + 2^{100} & \cancel{-3/4(2^{100})} + 900(2^{99}) + \cancel{3/4(2^{100})} \\ -1600(2^{99}) & 2^{100} - 1200(2^{99}) \end{bmatrix}$$

$$= \begin{bmatrix} 2^{99}(1202) & 2^{99}(900) \\ 2^{99}(-1600) & 2^{99}(-1198) \end{bmatrix}$$

$$e^{A} = M e^{J} M^{-1} = \begin{bmatrix} 3 & 1/4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^2 & 13/4\, e^2 \\ -4e^2 & -4e^2 \end{bmatrix} \begin{bmatrix} 0 & -1/4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 13e^2 & 9e^2 \\ -16e^2 & -11e^2 \end{bmatrix}$$