PRELIMINARY EXAM - LINEAR ALGEBRA 1/95

1. (15 pts.) Consider the matrix $A = ww^T$ where w is a column vector of length n. Note that w can also be considered to be a matrix of size $n \times 1$. Assume that $w \neq 0$.

   (i) Show that A is an n x n matrix and represents a multiple of a projection operator.

   (ii) Determine the range of ww^T and its rank.

   (iii) Determine all eigenvectors with nonzero eigenvalue.

   (iv) Write down a_{ij} in terms of the components of w.

2. (15 pts.) Let A be the matrix

$$A = \begin{pmatrix} \alpha & 1 & 0 \\ \beta & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Find all values of α and β so that (a) A is symmetric, (b) A is diagonally dominant and (c) A is non-singular.

3. (15 pts.) Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 2 & -1 \\ 1 & 2 & -3 \end{pmatrix}.$$

(i) State and then use the Fredholm alternative to determine for which vectors b, the equation Ax = b has a solution. (ii) If instead

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & a & -1 \\ 1 & 2 & -3 \end{pmatrix},$$

where $a \neq 2$, then Ax = b always has a solution. Justify this assertion.

4. (20 pts.) Let A be the matrix

$$A = \begin{pmatrix} -4 & -2 & -3 \\ 1 & 1 & 1 \\ 5 & 2 & 4 \end{pmatrix},$$

which has eigenvalues 1, 1, -1. (i) Find the eigenvectors/generalized eigenvectors of A. (ii) Find the matrix S such that $S^{-1}AS = J$, where

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is the Jordan form of the matrix A. (iii) Compute e^{Jt}.

5. (15pts.) Let A and B be symmetric matrices. Show that the largest eigenvalue of the matrix A − B is smaller than the largest eigenvalue of A if B is positive definite. (Hint: you can use Rayleigh's principle. Remember and A and B need not commute.)

6. (20 pts.) Consider the set $\mathcal{V}$ of polynomials $P(x)$ of the form

$$P(x) = \sum_{k=0}^{n} a_k x^{2k}.$$

i) Show that it forms a vector space.
ii) Consider the linear transformation T on that space given by

$$\mathrm{T}P(x) = \frac{d^2 P(x)}{dx^2}.$$

Give the matrix describing this transformation with respect to the basis 1, x^2, x^4, etc. What is its null-space and its column space (i.e., its range)? What do these spaces correspond to in terms of the polynomials?
iii) Consider the bilinear form

$$S(P_1, P_2) = \int_{-1}^{+1} x^\gamma P_1(x) P_2(x) dx.$$

For which integer values of γ does S define a scalar product on $\mathcal{V}$?
iv) For $n = 2$ and $\gamma = 0$ find an orthonormal basis of $\mathcal{V}$ using Gram-Schmidt.