Complex Variables Preliminary Examination
January 2011

1. (10 points) Suppose $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is an entire function.

   (a) Show that

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$r^2\frac{\partial^2 v}{\partial r^2} + r\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial \theta^2} = 0$$

   (b) Suppose $u(r,\theta) = r^3 \cos n\theta$. What values of n are permissable, and for each such n, find $v(r,\theta)$ such that $f(e^{i\pi/6}) = 0$.

2. (10 points) Show that if $|\beta| < 1$, then the linear fractional transformation

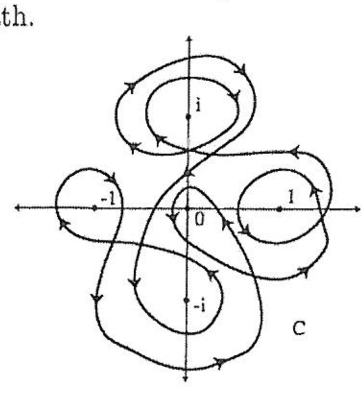
$$w(z) = \frac{z - \beta}{\bar{\beta}z - 1}$$

   maps the unit disk onto itself.

3. (10 points) Use contour integration to evaluate the integral

$$\int_0^\infty \frac{x^{-1/3}}{x^2 + 2x + 2}\,dx.$$

4. (10 points) Evaluate the integral $\int_C \frac{f(z)}{z^2+1}\,dz$ where $f(z)$ is analytic and C is the contour shown in the figure below. The arrows indicate the direction of the path.



5. (10 points) In the indicated region, determine a Laurent series (of the specified form) for

$$f(z) = \frac{z}{(z-3)(1-z)}$$

(a) $1 < |z| < 3$; Laurent series in powers of z.

(b) $|z-1| > 1$; Laurent series in powers of $(z-1)$.