Linear Algebra Preliminary Examination
January 2002

Each problem is worth the same number of points:

1. The matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

is associated with the quadratic form $3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$. Find the matrix S for the transformation $\mathrm{x} = \mathrm{Sy}$, $(\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)^T$ and $\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3)^T$) which will transform A to a diagonal matrix $\mathrm{D} = \mathrm{S}^{-1}\mathrm{AS}$, thus transforming the quadratic form to the sum of squares (sometimes referred to as a diagonal form) $\alpha y_1^2 + \beta y_2^2 + \gamma y_3^2$. Also, find D and determine the diagonal form (i.e., determine α, β, γ).

2. a) For the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

find the eigenvalues. For each eigenvalue determine the algebraic multiplicity (its multiplicity as a root of the characteristic equation), and determine the geometric multiplicity (the number of linearly independent eigenvectors corresponding to this eigenvalue). Is it possible to diagonalize the matrix A? Explain.

b) Repeat the question for the matrix

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

3. Consider the system of linear equations

$$\begin{aligned} x + y + z &= 4 \\ 2x + y + z &= 5 \\ 3x + 2y + 2z &= \alpha \end{aligned}$$

For each real α determine the number of solutions to the system and find all solutions for those α corresponding to which there are solutions.

4. a) Construct an orthonormal basis for $\mathbb{R}^3$ which includes a vector pointing in the same direction as $(1,1,1)^T$.

   b) Write the vector $e_1 = (1,0,0)^T$ in terms of the bases in part (a).

5. Consider the subspace V of $\mathbb{R}^4$ that is spanned by $(1,1,0,1)$ and $(0,0,1,0)$.

   a) Find a basis for the orthogonal complement $V^\perp$.

   b) Find a projection matrix P onto V.

   c) Find the vector in V closest to the vector $b = (0,1,0,-1)$ in $V^\perp$.

   d) Write the vector $c = (1,-1,1,-1)$ in the form $c = v + w$ where $v \in V$ and $w \in V^\perp$ (*i.e.*, find both v and w).

6. Consider the following two matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Compute (a) Q^{-1} and (b) $P^T(P^{-1}(QP^T)^{-1}Q)^{-1}P$.

(Hint: You should be able to do part b, even if you cannot do part a.)

$$P^T(P^{-1}(P^T)^{-1}\underbrace{Q^{-1}Q}_{I})^{-1}P$$

$$P^T(P^{-1}(P^T)^{-1})^{-1}P$$

$$= P^T(P^TP)P$$

1. $$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 = (3-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(2-\lambda)(3-\lambda) - 1\right] + (-1)(3-\lambda)$$

$$= (3-\lambda)\left[6 - 5\lambda + \lambda^2 - 1\right] + (-1)(3-\lambda)$$

$$= (3-\lambda)\left[\lambda^2 - 5\lambda + 4\right] = (3-\lambda)(\lambda-4)(\lambda-1) = 0$$

$$\Rightarrow \lambda = 1, 3, 4$$

$\underline{\underline{\lambda = 1}}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} -x_1 + x_2 - x_3 &= 0 \\ x_2 - 2x_3 &= 0 \\ x_2 &= 2x_3 \\ x_1 &= x_2 - x_3 \\ &= x_3 \end{aligned}$$

$$v_1 = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

$\underline{\underline{\lambda=3}}$ $\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow$ $-x_2 = 0$

$-x_1 - x_2 - x_3 = 0$

$x_1 + x_3 = 0$ $\quad x_1 = 1$

$x_3 = -1$

$V_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

$\underline{\underline{\lambda=4}}$ $\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ $\quad -x_1 = x_2$

$x_2 = -x_3$

$V_3 = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$

$S = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}}$

$\det S = 1 \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = -1 - (2+1) + (-2)$

$= -1 - 3 - 2 = -6$

$A_{11} = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1$ $\quad A_{12} = -3$ $\quad A_{13} = -2$

$A_{21} = -2$ $\quad A_{22} = 0$ $\quad A_{23} = 2$

$A_{31} = -1$ $\quad A_{32} = -(-1-2) = 3$ $\quad A_{33} = -2$

$S^{-1} = \frac{1}{\det S} A\omega\ell = -\frac{1}{6} \begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix}$

Check

$$S^{-1}S = -\frac{1}{6}\begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} = I$$

$D = S^{-1}AS$

$$= -\frac{1}{6}\begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

Check

$$-\frac{1}{6}\begin{bmatrix} -1 & -2 & -1 \\ -9 & 0 & 9 \\ -8 & 0 & -8 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -6 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -24 \end{bmatrix}$$ phew!

$\alpha = 1$, $\beta = 3$, $\gamma = 4$

2. a. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$$\begin{array}{l} 1-\lambda \\ 1-\lambda \\ \hline 1-2\lambda+\lambda^2 \\ 1-\lambda \\ \hline 1-2\lambda+\lambda^2 \\ -\lambda+2\lambda^2-\lambda^3 \end{array}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = 0 = ((1-\lambda)[(1-\lambda)^2 - 1] - (2(1-\lambda)) - 2$$

$$= (1-\lambda)^3 - (1-\lambda) - 2 + 2\lambda - 2$$

$$= (1-\lambda)^3 - (1-\lambda) + 2\lambda - 4$$

$$= \cancel{1} - 3\lambda + 3\lambda^2 - \lambda^3 \cancel{-1} + \lambda + 2\lambda - 4$$

$$= -\lambda^3 + 3\lambda^2 - 4$$

$$\begin{array}{r} -\lambda^2 + 4\lambda - 4 \\ \lambda + 1 \,\big)\, -\lambda^3 + 3\lambda^2 + 0\lambda - 4 \\ +\lambda^3 + \lambda^2 \\ \hline 4\lambda^2 + 0\lambda \\ -4\lambda^2 - 4\lambda \\ \hline -4\lambda - 4 \end{array}$$

$$= (\lambda+1)(-\lambda^2 + 4\lambda - 4)$$

$$= (\lambda+1)(-\lambda+2)(\lambda-2)$$

$\lambda = -1,\ 2$ (twice)

$\lambda = -1$ algebraic mult $= 1$

$\lambda = 2$ algebraic mult $= 2$

$\lambda = -1$: $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow x_2 = +2x_3$

$x_3 = 1$

$x_2 = 2$

$2x_1 + x_2 + x_3 = 0$

$2x_1 + 3 = 0 \quad x_1 = -3/2$

$v_1 = \begin{Bmatrix} -3/2 \\ 2 \\ 1 \end{Bmatrix}$ geometric mult $= 1$

$\lambda = 2$: $\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$-x_1 + x_2 + x_3 = 0$

$x_1 = 0$

$x_3 = 1$

$x_2 = -1$

$v_2 = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$ geometric mult $= 1$

There is not enough linearly independent eigenvectors to diagonalize A. The closest we can get to a diagonal matrix is the Jordan form

$$J = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

b. $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\qquad B - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$

$$\det(B - \lambda I) = -\lambda(\lambda^2 - 1) + (\lambda+1) + (\lambda+1)$$
$$= -\lambda(\lambda-1)(\lambda+1) + (\lambda+1) + (\lambda+1)$$
$$= (\lambda+1)[-\lambda(\lambda-1) + 2] = (\lambda+1)[-\lambda^2 + \lambda + 2]$$
$$= (\lambda+1)(-\lambda+2)(\lambda+1)$$

$\lambda = -1, \; 2$

alg mult = 2 (for -1), alg. mult = 1 (for 2)

$\underline{\lambda = -1}$ $\quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ $\qquad x_1 + x_2 + x_3 = 0$

x_2, x_3 free

$v_1 = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$ $\quad v_2 = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$ $\qquad$ geometric mult. = 2

$\underline{\lambda = 2}$ $\quad \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & -1 \\ 0 & 3 & -3 \end{bmatrix}$

$-2x_1 + x_2 + x_3 = 0$
$x_1 - 2x_2 + x_3 = 0$
$x_1 + x_2 - 2x_3 = 0$

$v_3 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ $\qquad$ geometric mult. = 1

Yes; B is diagonalizable ... since we have 3 linearly independent eigenvectors.

3. $x + y + z = 4$
$2x + y + z = 5$
$3x + 2y + 2z = \alpha$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 1 & 5 \\ 3 & 2 & 2 & \alpha \end{bmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & \alpha-12 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & \alpha-9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ \alpha-9 \end{Bmatrix}$$

There is a solution provided that $\alpha - 9 = 0 \Rightarrow \alpha = 9$

If $\alpha \neq 9$, there is no solution!

If $\alpha = 9$, we have

x_1, x_2 basic, x_3 free

$x_1 = 1$
$x_2 + x_3 = 3 \Rightarrow x_2 = 3 - x_3$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3-x_3 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 0 \end{Bmatrix} + x_3 \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$$

since x_3 is free, there are an infinite number of solutions

4. $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\boxed{q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

need two more vectors that are independent of a.

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

check for independence first

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_3 = 0,\ c_2 = 0,\ c_1 = 0$$

$$b' = b - (q_1^T b) q_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{3}}(1,1,1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \qquad \|b'\| = \sqrt{\frac{4}{3} + \frac{1}{3} + \frac{1}{3}} = \sqrt{2}$$

$$\boxed{q_2 = \frac{b'}{\|b'\|} = \frac{1}{3\sqrt{2}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}}$$

$$c' = c - (q_1^T c) q_1 - (q_2^T c) q_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left[\frac{1}{\sqrt{3}}(1,1,1)\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left[\frac{1}{3\sqrt{2}}(2,-1,-1)\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]\frac{1}{3\sqrt{2}}\begin{pmatrix} 2 \\ \cdot \\ \cdot \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{18}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/3 + 1/9 \\ -1/3 - 1/18 \\ -1/3 - 1/18 \end{pmatrix} = \begin{pmatrix} -2/9 \\ -7/9 \\ -7/9 \end{pmatrix}$$

$$= -\frac{1}{9}\begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}$$

$$c' = c - (q_1^T c)\, q_1 - (q_2^T c)\, q_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\left\{ \left\{ [1\ 1\ 1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} - \frac{1}{18}\left\{ [2\ -1\ -1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{18}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 - 1/3 + 1/9 \\ 0 - 1/3 - 1/18 \\ 1 - 1/3 - 1/18 \end{bmatrix} = \begin{bmatrix} -2/9 \\ -7/18 \\ 11/18 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -2 \\ -7/2 \\ 11/2 \end{bmatrix}$$

$$\|c'\| = \sqrt{\frac{1}{81}\left(4 + \frac{49}{4} + \frac{121}{4}\right)}$$

$$= \sqrt{\frac{174}{324}} = \sqrt{\frac{87}{162}}$$

$$q_3 = \sqrt{\frac{18}{29}} \begin{bmatrix} -2 \\ -7/2 \\ 11/2 \end{bmatrix}$$

Side work: $16 + 121 + 49 = 174$; $\frac{174}{324}$; $81 \times 4 = 324$; $174 \div 2 = 87$; $162 \div 3 = 54$; $54 \div 3 = 18$; $87 \div 3 = 29$

b. $\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

find c_1, c_2, c_3. Then

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{e}_1 = c_1 q_1 + c_2 q_2 + c_3 q_3$$

5. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$u + v + y = 0$ free: v, y

$x = 0$

a. basis for $V^\perp$: $v_1 = \begin{Bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$ $v_2 = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$

b. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} = B$

$$B^T B = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1 \\ 1/\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix} = P$$

c. $b = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}$

$$x = Pb = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

d. $c = \begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix}$ $c = v + w$

$v = Pc = \begin{Bmatrix} -1/3 \\ -1/3 \\ 1 \\ -1/3 \end{Bmatrix}$, $w = \begin{Bmatrix} 1 + 1/3 \\ -1 + 1/3 \\ 1 - 1 \\ -1 + 1/3 \end{Bmatrix} = \begin{Bmatrix} 4/3 \\ -2/3 \\ 0 \\ -2/3 \end{Bmatrix}$

$$c = \begin{Bmatrix} -1/3 \\ -1/3 \\ 1 \\ -1/3 \end{Bmatrix} + \begin{Bmatrix} 4/3 \\ -2/3 \\ 0 \\ -2/3 \end{Bmatrix}$$

6. $Q = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

a. Q^{-1}: $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\det Q = (1)(2)(1) = 2$

$A_{11} = (-1)^2(1) = 1$ $A_{12} = (-1)^3(-1) = 1$ $A_{13} = 0$

$A_{21} = (-1)^3(1) = -1$ $A_{22} = 1$ $A_{23} = 0$

$A_{31} = 0$ $A_{32} = 0$ $A_{33} = 2$

$$Q^{-1} = \frac{1}{\det Q} Q_{cof} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

b. $P^T(P^{-1}(QP^T)^{-1}Q)^{-1}P$

$= P^T(Q^{-1}(QP^T)^{-1}P)P$

$$QP^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\det QP^T = +2$

$A_{11} = 1$ $A_{12} = 0$ $A_{13} = 1$

$A_{21} = +1$ $A_{22} = 0$ $A_{23} = -1$

$A_{31} = 0$ $A_{32} = 2$ $A_{33} = 0$

$$(QP^T)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$Q^{-1}(QP^T)^{-1}P$

$$= \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$P^T\left(Q^{-1}(QP^T)^{-1}P\right)P$$

$$=\frac{1}{4}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}-2&1&1\\2&1&1\\0&2&-2\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}0&2&-2\\-2&1&1\\2&1&1\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-2&0&2\\1&-2&1\\1&2&1\end{bmatrix}$$

Better way

$$P^T\left(P^{-1}(QP^T)^{-1}Q\right)^{-1}P$$

$$=P^T\left(P^{-1}(P^T)^{-1}\underbrace{Q^{-1}Q}_{I}\right)^{-1}P$$

$$=P^T\left(P^{-1}(P^T)^{-1}\right)^{-1}P$$

$$=P^T\left(P^TP\right)P$$

$$=(P^T)^2P^2$$

$$(P^T)^2=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$P^2=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\boxed{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I}$$