Applied Mathematics Differential Equations Preliminary Exam

March 30, 2011

Name:____

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

1. Find the solution of the differential equation

$$(x^2+1)\frac{dy}{dx} + 3x^3 y = 6x\exp\left(-\tfrac{3}{2}x^2\right), \quad \text{with} \quad y(0) = 1\,.$$

2. A cake is removed from an oven at 400°F and left to cool at room temperature, which is 80°F. (This happens according to Newton's law of cooling, i.e., the rate of cooling is proportional to the temperature difference.) After 5 minutes the temperature of the cake is 240°F. When will it be 100°F?

3. Suppose that the population of stocked sport fish in a lake obeys the equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) - EP\,.$$

Here r is the linear growth rate for small populations, M is the carrying capacity, and the term EP represents the *yield* obtained for a given effort E of fishing. Assume $r > 0$, $M > 0$ and $E \geq 0$. (Note the yield is proportional to P, which means that the more fish there are, the easier it is to catch them — which isn't *too* unreasonable.)

(a) By finding the *stable* critical point, determine the sustainable harvested yield $Y = EP$ as a function of the effort E. What are the conditions on E to obtain a positive yield?

(b) Determine the value of the effort E that maximizes the yield Y.

(c) At maximum yield, by what fraction has the steady-state population been reduced from the un-harvested case?

4. Consider a mass-spring damping system,

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = 0\,,$$

with a 1 kg mass and spring constant of 25 N/m.

(a) Find the subsequent motion if the damping constant is 8 N/(m/sec) and the mass initially has zero displacement and velocity 1 m/sec.

(b) Repeat the calculation if the damping constant is increased to 10 N/(m/sec).

(c) Suppose an additional force of $40 \sin 5t$ Newtons is applied to the mass starting at $t = 0$ in case (a). What is the subsequent motion now?

5. Find the solution of the differential equation

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0, \quad y(0) = 2,\ y'(0) = 1\,.$$

6. Find a particular solution of the equation

$$y'' + y = \sec x\,.$$

7. Consider the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}\,.$$

Show that

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

are eigenvectors of the matrix **A**. What are the eigenvalues? Find the solution of the system of differential equations satisfying the initial condition $\mathbf{x}(0) = [\,1\;0\;0\,]^{\mathsf{T}}$.

8. For the system

$$\begin{aligned}\frac{dx}{dt} &= -x - 3y + 2x^2 + xy + y^2 \\ \frac{dy}{dt} &= x - y\end{aligned}$$

(a) find all critical points and determine their type and stability.

(b) See next page...

(b) Indicate by circling which of the following phase planes shown below best describes the phase plane of this system.