PRELIMINARY EXAM – ADVANCED CALCULUS JANUARY, 2011

Note - all problems count equally

1. P and Q & their first partials are continuous with $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ everywhere except at $(0,0)$, $(4,0)$ and $(-4,0)$. Given that

$$\oint_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y = 11 \qquad C_1:\ (x-2)^2 + y^2 = 9$$

$$\oint_{C_2} P\,\mathrm{d}x + Q\,\mathrm{d}y = 9 \qquad C_2:\ (x+2)^2 + y^2 = 9$$

$$\oint_{C_3} P\,\mathrm{d}x + Q\,\mathrm{d}y = 13 \qquad C_3:\ x^2 + y^2 = 25\ ,$$

find

$$\oint_{C_4} P\,\mathrm{d}x + Q\,\mathrm{d}y \qquad C_4:\ x^2 + y^2 = 1$$

2. Consider the partial differential equation

$$u_{rs} = 0\ .$$

Make the change of variables $r = t - x$ and $s = t + x$ and find the transformed equation satisfied by $u(x,t)$.

3. Consider a force field defined by

$$\vec{F} = 21x^2y\,\vec{\imath} + g(x,y)\,\vec{\jmath} + 0\,\vec{k}\ .$$

Determine $g(x,y)$ so that $\vec{F}$ is a conservative force and then find the associated potential function.

4. Find the critical points of $w = xyz$ subject to the condition $x^2 + y^2 + z^2 = 1$.

5. Evaluate

$$\iint_S \nabla \times \vec{F} \cdot \vec{n}\,\mathrm{d}S$$

both directly and by Stokes theorem, where $\vec{F} = y^2\vec{\imath} - xy\vec{\jmath} + z\vec{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

6. Let D be a simply connected domain in the x, y plane and let $\vec{w} = u\vec{\imath} - v\vec{\jmath}$ be the velocity vector of an irrotational, incompressible flow in D. Show that the following properties hold:

a) u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ in } D;$$

b) u and v are harmonic (i.e., satisfy Laplace's eq.) in D;

c) there is a vector $\vec{F} = \phi\vec{\imath} - \psi\vec{\jmath}$ in D such that

$$\frac{\partial \phi}{\partial x} = u = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -v = -\frac{\partial \psi}{\partial x},$$

where ϕ is the velocity potential and ψ is the stream function;

d) $\nabla \cdot \vec{F} = 0$ and $\nabla \times \vec{F} = 0$ in D;

e) ϕ and ψ are harmonic in D.