Linear Algebra Preliminary Exam 2012 - Retake

1. Find an orthonormal set $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ for which $\mathbf{q}_1$ and $\mathbf{q}_2$ span the column space of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

   Which fundamental subspace contains $\mathbf{q}_3$? What is the least squares solution of $\mathbf{Ax} = \mathbf{b}$ if $\mathbf{b} = [1,2,7]^T$?

2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   a) Find all eigenvalues and eigenvectors of $\mathbf{A}$.

   b) Write down a diagonalizing matrix $\mathbf{S}$ for $\mathbf{A}$, i.e. a matrix $\mathbf{S}$ for which $\mathbf{S}^{-1}\mathbf{AS}$ is diagonal.

   c) Write down a *different* diagonalizing matrix $\mathbf{S}'$ for $\mathbf{A}$, i.e. one whose columns are not all simply multiples of the columns of $\mathbf{S}$.

3. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

   be positive definite, i.e. $\mathbf{u}^T\mathbf{Au} > 0$, for any nonzero vector $(x,y)^T = \mathbf{u} \in \mathbb{R}^2$. Show that $\mathbf{A}$ is positive only if $a > 0, c > 0$ and $ac > b^2$. (Hint: Use appropriate choices of x and y).

4. Given the vectors:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -2 \\ -1 \\ a \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 4 \\ b \\ c \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

   Find the values of a, b, c such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ form an orthogonal set and $\mathbf{u}_3 \cdot \mathbf{v} > 0$.

5. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix and consider the function defined on $\mathbb{R}^n$

$$R(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{Ax}}{\mathbf{x}^T\mathbf{x}} \quad \mathbf{x} \neq 0$$

   Let A have eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots. \leq \lambda_n$. Show that for every $\mathbf{x} \in \mathbb{R}^n$, $\lambda_1 \leq R(\mathbf{x}) \leq \lambda_n$.

6. Consider a symmetric $n \times n$ matrix A for which $A_{ij} \in \{0,1\}$. Show that

$$0 \leq \sum_{i=1}^{n} \lambda_i^k \in \mathbb{N}$$

   for all $k \in \mathbb{N}$ where λ_i are the eigenvalues of A.