PRELIMINARY EXAM - LINEAR ALGEBRA January 2004

Please show all work. To get full credit for a problem you need to show your work and **CLEARLY** describe your calculations.

1. (40 points) The matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & -1 & 0 \\ 2 & -2 & k \end{bmatrix}$$

has the eigenvalues $0, 1, -1$.

(a) For what values of k is this true?

(b) Find the eigenvectors corresponding to each of the three eigenvalues.

(c) Is $\left(\sqrt{8}, \sqrt{2}, \sqrt{2}\right)^{\mathrm{T}}$ an eigenvector? Explain.

(d) Determine the eigenvalues of the matrix $\mathbf{A}^2$ without computing $\mathbf{A}^2$; explain and/or prove your arguments.

(e) Given the system of first order differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A} \cdot \mathbf{x}$$

where $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$. Determine the solution that satisfies the initial conditions $x_1(0) = 0,\ x_2(0) = 1,\ x_3(0) = 1/2$.

2. (20 points) Consider the matrix M

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ k & -1 & 1 \end{bmatrix},$$

where k is an arbitrary constant.

(a) For what values of k can you solve

$$\mathbf{M}\mathbf{x} = \mathbf{F}$$

for an arbitrary vector $\mathbf{F}$?

(b) For those values of k for which there is no solution to (a) when $\mathbf{F}$ is arbitrary, there may be a solution for a special class of $\mathbf{F}$. Identify this class of $\mathbf{F}$ and determine the general solution vector $\mathbf{x}$.

3. (40 points) Determine if the following are true or false. If true, **you must prove it**; if false, **give a counter-example**. Note: You may use the result of part (a) when proving (b)-(d), part (b) when proving (c)-(d), and part (c) when proving (d). Note: we write the determinant of the matrix $\mathbf{A}$ as $det\mathbf{A}$.

(a) If each element in one row (column) of a determinate is multiplied by a number c, the value of the determinant is multiplied by c.

(b) Suppose that v_1, v_2, ..., v_n, f_j are all $n \times 1$ column vectors. Consider the $n \times n$ square matrix $\mathbf{A}$ defined as

$$\mathbf{A} = [v_1 \cdots v_j + f_j \cdots v_n],$$

i.e., the columns of A are composed of the elements of the column vectors v_1, ... v_n, except for the jth column whose elements are the sum of v_j and f_j. Then

$$det\mathbf{A} = det[v_1 \cdots v_j \cdots v_n] + det[v_1 \cdots f_j \cdots v_n]$$

(c) If a matrix $\mathbf{B}$ is obtained from a square matrix $\mathbf{A}$ by interchanging two columns of $\mathbf{A}$, then $det\mathbf{A} = det\mathbf{B}$.

(d) If a matrix $\mathbf{B}$ is obtained from a square matrix $\mathbf{A}$ by adding to one row (column) vector of $\mathbf{A}$ a number c times a different row (column) vector, then $det\mathbf{A} = det\mathbf{B}$.