PRELIMINARY EXAM - LINEAR ALGEBRA 4/99

1. Let $\mathbf{A} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solve the problem

$$\mathbf{Au} = \lambda \mathbf{Mu}$$

for the λ's and eigenvectors $\mathbf{u}$.

2. Let P be the space of all 2$^{\text{nd}}$ degree polynomials with basis $1 + x$, $x + x^2$, x.

   (a) Show these elements form a basis for P.

   (b) Let $\mathbf{D}$ be the differentiation operator on P. Write $\mathbf{D}$ as a matrix with respect to the above basis.

   (c) What is the dimension of the nullspace of $\mathbf{D}$?

3. Let $\mathbf{u}$, $\mathbf{v}$ be orthogonal unit vectors in $\mathbf{R}^3$. Define $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, $\mathbf{P}_1 = \mathbf{uu}^T$, $\mathbf{P}_2 = \mathbf{vv}^T$, and

$$\mathbf{R} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}^T$$

   (a) Compute $\mathbf{Ru}$, $\mathbf{Rv}$, and $\mathbf{Rw}$.

   (b) Show that for any vector $\mathbf{x}$,

$$\mathbf{P}_2\mathbf{R}\mathbf{P}_2^2\mathbf{R}\mathbf{P}_1^2\mathbf{x} = \mathbf{0}$$

4. Given vectors

$$\mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T, \qquad \mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}^T$$

let $\mathbf{P}_1 = \mathbf{uu}^T$ and $\mathbf{P}_2 = \mathbf{vv}^T$.

   (a) Compute $(\mathbf{P}_1\mathbf{P}_2)^n$ for any integer $n \geq 1$.

   (b) Show for any vector $\mathbf{x}$,

$$\lim_{n\to\infty} (\mathbf{P}_1\mathbf{P}_2)^n\mathbf{x} = \mathbf{0}$$

5. Let

$$\mathbf{A} = \begin{bmatrix} 6 & 3 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 9 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 & \alpha+1 & -1 \\ 0 & 1-\alpha & 1 \\ 0 & 0 & \alpha+4 \end{bmatrix}.$$

For which values of α can one find a basis in which both matrices are diagonal?

6. Find all solutions $\mathbf{x}$ of the system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -1 & 0 & 4 \\ 2 & -1 & -1 & 2 & 0 \\ 4 & -5 & -3 & 2 & 8 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$