Applied Mathematics Differential Equations Preliminary Exam

January 1[illegible], 201~~6~~
11

Name:____

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	70	

1. A water tank has a shape obtained by revolving the curve $y = x^{4/3}$ around the y axis. A plug in a hole at the bottom of the tank is removed at 12 noon, when the depth of the water is 2 meters. At 1:00 p.m., the depth of the water is 1 meter. When will the tank be empty? [Hint: assume that water flows out of the tank according to Torricelli's law, i.e., at a rate proportional to $h^{1/2}$.]

2. Consider the vectors

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} \frac{1}{3}t^3 \\ t^2 \end{pmatrix}.$$

(a) Compute the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

(b) In what intervals are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?

(c) What conclusion can be drawn about the coefficients in the system of linear homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?

(d) Find this system of equations and verify the conclusions of part (c).

3. Find the solution of the problem

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}, \quad y(0) = 1, \ \frac{dy}{dx}(0) = 1 .$$

4. Consider the nonhomogeneous differential equation for $y = y(x)$:

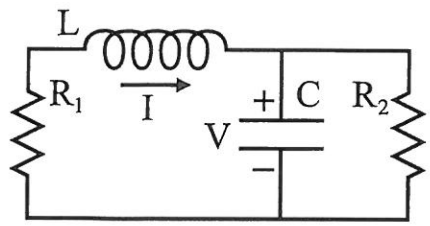
$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = g(x)$$

on $0 < x < 1$. Assume that $g(x)$ is an arbitrary continuous function.

(a) Given that one solution of the homogeneous problem is $y = e^x$, find the other.

(b) Find a particular solution of the nonhomogeneous problem.

(c) Find the solution of the initial value problem when $y(0) = 0$ and $dy/dx(0) = 0$.



5. For the above electrical circuit, show that the voltage across the capacitor and the current through the inductor satisfy the system of differential equations

$$\frac{d}{dt}\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_2 C} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R_1}{L} \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix}.$$

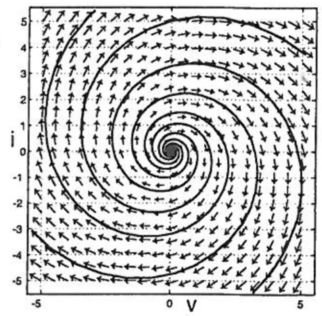
Assume that at $t = 0$, $V = 3$ volts and $I = 0$ amps.

(a) Use the eigenvalue method to find the solution for V and I if $R_1 = 3$ Ohms, $R_2 = 2$ Ohms, $L = 1$ Henry and $C = 1$ Farad (with the same initial conditions).

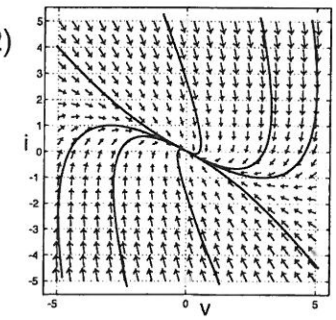
(b) Use the eigenvalue method to find the solution for V and I if $R_1 = R_2 = 1$ Ohm, $L = 1$ Henry and $C = 1$ Farad.

(c) Identify which of the following phase planes for V and I correspond to the solutions you found in parts (a) and (b) above. Make sure to explain how you arrive at your conclusion.

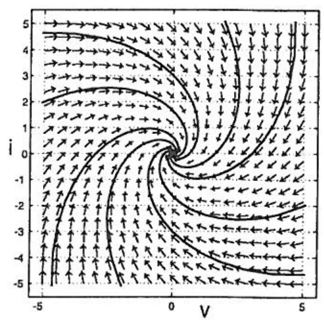
(1)



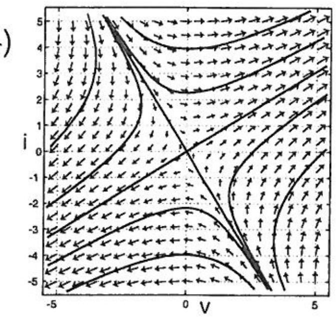
(2)



(3)



(4)



The phase plane that best corresponds to part (a) is: ____________.

The phase plane that best corresponds to part (b) is: ____________.

6. Find and classify all critical points of the system of equations

$$\begin{aligned}\frac{dx}{dt} &= x^2 - xy - x\\ \frac{dy}{dt} &= y^2 + xy - 2y\,.\end{aligned}$$

7. Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + \frac{\sqrt{2}}{2}\left(\frac{dy}{dt}\right)^2 + y = 0, \quad y(0) = 0, \ \frac{dy}{dt}(0) = 1 .$$