DIFFERENTIAL EQUATIONS
ESAM Preliminary Examination
January 6, 2010, 10-12

1. Solve the following differential equations:

   (a) $xy''(x) + y'(x) = 1 \qquad (x > 0)$,

   (b) $y'(x) = \frac{x+3y-5}{x-y-1}$,

   (c) $(D^3 - 1)(D+1)^3(D^2+4)y(x) = 0$, where $D \equiv \frac{d}{dx}$.

2. One homogeneous solution of the equation

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = x^3e^x$$

   is $y = x$. Find the general solution.

3. Determine all the eigenfunctions and eigenvalues that satisfy

$$y'' + \lambda y = 0, \quad 0 < x < \pi, \quad y(0) = y'(\pi), \ y(\pi) = y'(0).$$

4. The population densities of two species, $u(t)$ and $v(t)$, which compete for the same food supply satisfy the system

$$\frac{du}{dt} = \frac{u}{2}(2 - 2u + v), \qquad \frac{dv}{dt} = \frac{v}{4}(10 - u - 6v).$$

   (a). Determine all possible equilibrium ($\frac{d}{dt} = 0$) solutions and their stability;

   (b). On a long time basis, will the two species coexist or will one population become extinct?

5. Find the solution of the initial value problem

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$