Elementary Differential Equation
ESAM Preliminary Examination
January 11, 2000, 3-5 p.m.

1. (20 points.) Solve the following differential equations:

$$x^3y'' - 2x^2y' + 2xy = 1,$$

$$\frac{d^8y}{dx^8} + 8\frac{d^4y}{dx^4} + 16y = 0.$$ #21 pg. 220

2. (15 points.) Verify that $y(x) = e^{-x^2}e^{2x}$ is a solution of

$$y'' + 4(x-1)y' + (4x^2 - 8x + 6)y = 0,$$

and solve the initial value problem

$$y'' + 4(x-1)y' + (4x^2 - 8x + 6)y = \frac{1}{x^2}e^{-(x-1)^2}, \quad x > 1, \quad y(1) = 0, \; y'(1) = 0.$$

3. (10 points.) Find the solution of the initial value problem

$$xyy' = x^2 + 3y^2, \quad x > 1, \quad y(1) = -1.$$

4. (15 points.) Solve the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

need more practice on these

5. (15 points.) Determine all critical points of the system

$$\frac{dx}{dt} = 1 - y, \quad \frac{dy}{dt} = x^2 - y^2$$

and discuss their type and stability.

6. (15 points.) Determine the eigenfunctions and eigenvalues (graphically if necessary) that satisfy

$$y'' + \lambda y = 0, \quad 0 < x < 1, \quad y(0) = 0, \; y(1) - y'(1) = 0.$$

7. (10 points.) Solve the boundary value problem for $u(x, y)$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \; 0 < y < \pi;$$

$$\frac{\partial u(0, y)}{\partial x} = \frac{\partial u(\pi, y)}{\partial x} = \frac{\partial u(x, 0)}{\partial y} = 0, \quad u(x, \pi) = 2 + \cos x.$$

PRELIM : DIFFERENTIAL EQUATIONS : 1/2000

$$x^3y'' - 2x^2y' + 2xy = 1$$

$$x^2y_H'' - 2xy_H' + 2y_H = 0$$

$$y_H = x^k \qquad y_H' = kx^{k-1} \qquad y_H'' = k(k-1)x^{k-2}$$

$$k(k-1)x^k - 2kx^k + 2x^k = 0$$

$$k(k-1) - 2k + 2 = 0$$

$$k^2 - k - 2k + 2 = 0$$

$$k^2 - 3k + 2 = 0$$

$$(k-1)(k-2) = 0$$

$$k = 1, 2 \qquad y_H = C_1x + C_2x^2$$

$$x^2y'' - 2xy' + 2y = \frac{1}{x} \qquad \frac{1}{x} = g(x)$$

$$y_1(x) = x \quad , y_2(x) = x^2$$

$$W(y_1(x), y_2(x)) = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2$$

$$= -y_1\int \frac{y_2(x)g(x)}{w(x)}\,dx + y_2\int \frac{y_1(x)g(x)}{w(x)}\,dx$$

$$= -x\int \frac{\cancel{x^2}(1/x)}{-\cancel{x^2}}\,dx + x^2\int \frac{\cancel{x}(1/\cancel{x})}{-x^2}\,dx$$

$$= x\int \frac{dx}{x} - x^2\int x^{-2}\,dx$$

$$= x\ln x - \frac{x^2x^{-1}}{-1} = x\ln x + x$$

$$= y_H + y_P$$

$$\boxed{= C_1x + C_2x^2 + x + \cancel{x\ln x}}$$

2. $y(x) = e^{-x^2} e^{2x}$

$y'(x) = -2xe^{-x^2}e^{2x} + 2e^{-x^2}e^{2x}$

$y''(x) = -2e^{-x^2}e^{2x} + 4x^2e^{-x^2}e^{2x} - 4xe^{-x^2}e^{2x} - 4xe^{-x^2}e^{2x} + 4e^{-x^2}e^{2x}$

$= 2e^{-x^2}e^{2x} - 8xe^{-x^2}e^{2x} + 4x^2e^{-x^2}e^{2x}$

$y'' + 4(x-1)y' + (4x^2 - 8x + 6)y = 0$

$(2e^{-x^2}e^{2x} - 8xe^{-x^2}e^{2x} + 4x^2e^{-x^2}e^{2x}) + 4x(2e^{-x^2}e^{2x} - 2xe^{-x^2}e^{2x})$

$-4(2e^{-x^2}e^{2x} - 2xe^{-x^2}e^{2x}) + 4x^2e^{-x^2}e^{2x} - 8xe^{-x^2}e^{2x}$

$+ 6e^{-x^2}e^{2x}$

$e^{-x^2}e^{2x} : 2 - 8 + 6 = 0$

$xe^{-x^2}e^{2x} : -8 + 8 + 8 - 8 = 0$

$x^2e^{-x^2}e^{2x} : 4 - 8 + 4 = 0$

$\Rightarrow y(x) = e^{-x^2}e^{2x}$ is a solution of the DE.

$y'' + 4(x-1)y' + (4x^2 - 8x + 6)y = \frac{1}{x^2} e^{-(x-1)^2}$ $\quad x > 1$

$y(1) = 0 \quad y'(1) = 0$

reduction of order

~~$y_2(x) = v(x)e^{-x^2}e^{2x}$~~

~~$y_2'(x) = v'(x)e^{-x^2}e^{2x} + v(x)(2e^{2x}e^{-x^2} - 2xe^{-x^2}e^{2x})$~~

~~$y_2''(x) = v''(x)e^{-x^2}e^{2x} + v'(x)(2e^{2x}e^{-x^2} - 2xe^{-x^2}e^{2x})$~~

~~$+ v'(x)(2e^{2x}e^{-x^2} - 2xe^{-x^2}e^{2x})$~~

~~$+ v(x)(4e^{2x}e^{-x^2} - 4xe^{2x}e^{-x^2} + 4x^2e^{-x^2}e^{2x} - 4xe^{-x^2}$~~

~~$= v''(x)e^{-x^2}e^{2x} + 2v'(x)(2e^{2x}e^{-x^2} - 2xe^{-x^2}e^{2x})$~~

~~$+ v(x)(4x^2e^{-x^2}e^{2x} - 8xe^{2x}e^{-x^2} + 4e^{2x}e^{-x^2})$~~

~~$y_H'' + 4(x-1)y_H' + (4x^2 - 8x + 6)y_H = 0$~~

~~$v''(x) + 2v'(x)(2 - 2x) + v(x)(4x^2 - 8x + 4) + 4(x-1)(v' + v(2-2x)$~~

~~$+ v(4x^2 - 8x + 6) = 0$~~

3. $xyy' = x^2 + 3y^2 \qquad x > 1 \qquad y(1) = -1$

$$y' = \frac{x}{y} + 3\frac{y}{x} \Rightarrow y' - \frac{3}{x}y = xy^{-1} \Rightarrow n=-1 \quad \text{Bernoulli Eqn}$$

$$v = y^2 \rightarrow \frac{dv}{dx} = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}\frac{dv}{dx}$$

$$\left(\frac{1}{2\sqrt{v}}\frac{dv}{dx} = \frac{x}{\sqrt{v}} + \frac{3\sqrt{v}}{x}\right) \Rightarrow \frac{dv}{dx} - \frac{6v}{x} = 2x$$

$$\mu(x) = \exp\left\{\int -\frac{6}{x}\,dx\right\} = \exp\{-6\ln x\} = x^{-6}$$

$$x^{-6}\frac{dv}{dx} - 6x^{-7}v = 2x^{-5}$$

$$(x^{-6}v)' = 2x^{-5} \Rightarrow x^{-6}v = \frac{2x^{-4}}{-4} = -\frac{1}{2}x^{-4} + C$$

$$v = -\frac{1}{2}x^2 + Cx^6$$

$$y^2 = -\frac{1}{2}x^2 + Cx^6$$

$$y = \left(-\frac{1}{2}x^2 + Cx^6\right)^{1/2} \qquad y(1) = -1$$

$$-1 = \left(-\frac{1}{2} + C\right)^{1/2} \Rightarrow 1 = -\frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$

$$y = \left(-\frac{1}{2}x^2 + \frac{3}{2}x^6\right)^{1/2}$$

<u>another way</u>

Homogeneous nonlinear Eqn: $\frac{dy}{dx} = F\left(\frac{x}{y}\right)$

$$y' = \frac{x}{y} + 3\frac{y}{x} \qquad y = vx, \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = \frac{1}{v} + 3v \quad \rightarrow \quad x\frac{dv}{dx} = \frac{1}{v} + 2v$$

$$u = 1+2v^2$$
$$du = 4v\,dv$$

$$x\,dv = \left(\frac{1+2v^2}{v}\right)dx \Rightarrow \int \frac{dx}{x} = \int \frac{v}{1+2v^2}\,dv$$
$$= \frac{1}{4}\int \frac{du}{u}$$

$$\ln x = \frac{1}{4}\ln(1+2v^2) + C$$

$$\ln x = \ln(1+2v^2)^{1/4} + C \Rightarrow Cx = (1+2v^2)^{1/4}$$

$$Cx = \left(1 + 2\frac{y^2}{x^2}\right)^{1/4} \qquad y(1) = -1$$

$$C = (1+2)^{1/4} = 3^{1/4}$$

$$3^{1/4}x = \left(1 + 2\frac{y^2}{x^2}\right)^{1/4} \Rightarrow 3x^4 = 1 + \frac{2y^2}{x^2} = \frac{x^2+2y^2}{x^2}$$

$$y = \left(-\frac{1}{2}x^2 + \frac{3}{2}x^6\right)^{1/2}$$ which is what we had before

4. $\dfrac{d\bar{x}}{dt} = \underbrace{\begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \bar{x}$

$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} -3-\lambda & 0 & -4 \\ -1 & -1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{bmatrix} = 0$

$\Rightarrow (-3-\lambda)(-1-\lambda)(1-\lambda) + 4(-1-\lambda) = 0$

$(-1-\lambda)\left[(-3-\lambda)(1-\lambda) + 4\right] = 0 \Rightarrow (-1-\lambda)\left[1 + 2\lambda + \lambda^2\right] = 0$

$(-1-\lambda)(1+\lambda)(1+\lambda) = 0 \Rightarrow \lambda = -1$ three times

$\begin{bmatrix} 1 & 0 & \overset{-4}{2} \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$a_1 + 2a_3 = 0 \qquad -a_1 - a_3 = 0$

$-a_1 = a_3 = 0$

$v_1 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$

$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$

$b_1 + 2b_3 = 0$

$-b_1 - b_3 = 1$

$b_3 = 1; \; b_1 = -2$

$v_2 = \begin{Bmatrix} -2 \\ 0 \\ 1 \end{Bmatrix}$

$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 0 \\ 1 \end{Bmatrix}$

$\cancel{2c_1 - 4c_3 = -2}$

$-c_1 - c_3 = 0 \Rightarrow c_3 = -c_1$

$c_1 + 2c_3 = 1$

$c_3 = 1$

$c_1 = -1$

$v_3 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$

$$\bar{x}(t) = C_1 \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} e^{-t} + C_2 \left[\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} t e^{-t} + \begin{Bmatrix} -2 \\ 0 \\ 1 \end{Bmatrix} e^{-t} \right]$$

$$+ C_3 \left[\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \frac{t^2}{2} e^{-t} + \begin{Bmatrix} -2 \\ 0 \\ 1 \end{Bmatrix} t e^{-t} + \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} e^{-t} \right]$$

2. $\dot{x} = 1 - y$
$\dot{y} = x^2 - y^2$

$\dot{x} = 0 \Rightarrow y = 1$

$\dot{y} = 0 \Rightarrow x^2 = y^2 \Rightarrow x = \pm 1$

critical points $(1, 1)$, $(-1, 1)$

$$A = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x & -2y \end{pmatrix}$$

$$A(1,1) = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix} \qquad |A(1,1) - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 2 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) + 2 = 0$$

$$2\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i = a \pm ib$$

$a < 0 \Rightarrow$ stable

$(1, 1)$ is an asymptotically stable spiral

$$A(-1,1) = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix} \qquad |A(-1,1) - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ -2 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 2\lambda - 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 + 8}}{2} = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

$\lambda_1 = -1 + \sqrt{3} > 0$

$\lambda_2 = -1 - \sqrt{3} < 0$

$\Rightarrow (-1, 1)$ is an unstable saddle

6. $y'' + \lambda y = 0 \qquad 0 < x < 1 \qquad y(0) = 0 \qquad y(1) - y'(1) = 0$

$\underline{\underline{\lambda > 0}}$

$y = \cancel{A\cos(\sqrt{\lambda}x)}^{\,0} + B\sin(\sqrt{\lambda}x)$

$y(0) = A = 0$

$y'(x) = B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$y'(1) = B\sqrt{\lambda}\cos\sqrt{\lambda}$

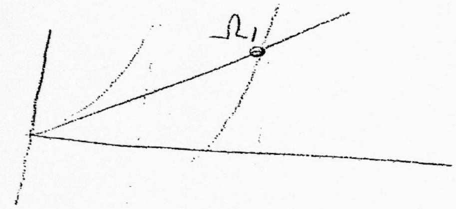
$y(1) = B\sin(\sqrt{\lambda})$

$B\sin\sqrt{\lambda} - B\sqrt{\lambda}\cos\sqrt{\lambda} = 0$ don't want $B = 0$

$\sin(\sqrt{\lambda}) = \sqrt{\lambda}\cos(\sqrt{\lambda})$

$\tan\sqrt{\lambda} = \sqrt{\lambda}$

$\Omega = \sqrt{\lambda} \Rightarrow \lambda = \Omega^2$



$\lambda_n = \Omega_n^2$

$y_n = \sin(\Omega_n x)$

$\underline{\underline{\lambda = 0}}$

$y'' = 0 \Rightarrow y = Ax + B$

$y(0) = 0 \Rightarrow B = 0$

$y = Ax$

$y' = A \qquad y(1) - y'(1) = A - A = 0 \qquad$ A arbitrary

$y_0 = x$ ← eigenfunction

$\lambda_0 = 0$

$\underline{\underline{\lambda < 0}}$

$y'' - |\lambda| y = 0$

$y = A\cosh(\sqrt{|\lambda|}x) + B\sinh(\sqrt{|\lambda|}x)$

$y(0) = A = 0$

$y'(x) = B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x)$

$y(1) - y'(1) = 0 \Rightarrow B\sinh(\sqrt{|\lambda|}) - B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}) = 0, \quad B \neq 0$

$\tanh(\sqrt{|\lambda|}) = \sqrt{|\lambda|}$

$\Omega = \sqrt{|\lambda|} \Rightarrow \Omega^2 = |\lambda|$

$= \sinh(\sqrt{|\lambda|}x)$