Complex Variables Preliminary Examination

January 2012

1. (10 points) Use complex variable methods to solve the boundary-value problem
$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$
for $y > 0$, subject to the boundary condition
$$\lim_{y \to 0+} \Phi(x, y) = G(x) = \begin{cases} 0 & x < -1, \\ 1 & -1 < x < 1, \\ 2 & x > 1\,, \end{cases}$$
and Φ is bounded at infinity. Your answer should give Φ as an explicit function of x and y. Please clearly explain what you are doing.

2. (10 points) Compute $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx$, where $|a| < 1$.

3. (10 points)

   (a) Show that
   $$F(z) = \int_0^\infty (1+t)e^{-zt}\,dt$$
   converges only if $\mathrm{Re}(z) > 0$.

   (b) Find a function which is the analytic continuation of $F(z)$ into the left half plane. Please clearly explain why your answer is correct.

4. (10 points) Find a Laurent series expression for
$$\frac{z}{(z-3i)(z-4)}$$
for

   (a) $|z| < 3$

   (b) $3 < |z| < 4$

   (c) $|z| > 4$

5. (10 points) Suppose that both $f(z)$ and $\overline{f(z)}$ are analytic in a domain D. Show that $f(z)$ must be a constant in D.

6. (10 points) Construct a fractional linear (Möbius) transformation that takes the curves $|z| = 1$ and $|z - 1| = 5/2$ onto concentric circles centered at the origin.