PRELIMINARY EXAM - LINEAR ALGEBRA 1/01

Please show all work. To get full credit for a problem you need to show your work and CLEARLY describe your calculations.

1. (20 points) Consider the three vectors

$$\mathrm{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathrm{c} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Use the Gram-Schmidt process to construct three orthonormal vectors $\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3$.

2. (20 points) Consider the matrix A

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

(a) Show that the polynominal equation

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$$

is the characteristic equation of the matrix A.

(b) If r is a characteristic root of the polynomial $p(\lambda)$, determine the corresponding characteristic vector (i.e., eigenvector) of the matrix A.

3. (40 points, note there are 4 parts to this problem) Consider the matrix M

$$\mathrm{M} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ Q & -1 & 1 \end{bmatrix}$$

where Q is an arbitrary constant.

(a) What is the trace of M? What is the determinant of M?

(b) For what values of Q can you solve

$$\mathrm{MX} = \mathrm{F}$$

for F arbitrary?

(c) For those values of Q for which there is no solution to (b), find the general solution for a class of F's.

(d) Is there a matrix N such that

$$\mathrm{MX} = \mathrm{NF}$$

always has a solution? If so, find one.

4. (20 points) A is a Hermitian matrix if it is equal to the transpose of its complex conjugate, i.e., $A = \bar{A}^T = A^H$.

(a) Show that if A is Hermitian, then for all complex vectors x, the number $x^H Ax$ is real.

(b) Show that every eigenvalue of a Hermitian matrix is real.

(c) Show that the eigenvectors of a Hermitian matrix , if they come from different eigenvalues, are orthogonal to one another, i.e., if x_i is an eigenvector of the eigenvalue λ_i, then $x_i^H x_j = 0$ for $\lambda_i \neq \lambda_j$.

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1. $a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Gram-Schmidt

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$b' = b - (q_1^T b) q_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \Big(\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}} \Big) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ -1 \end{bmatrix} \qquad \|b'\| = \sqrt{\frac{1}{4} + 1 + 1} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$q_2 = \frac{b'}{\|b'\|} = \frac{2}{3} \begin{bmatrix} -1/2 \\ -1 \\ -1 \end{bmatrix} = -\frac{2}{3} \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

$$b' = b - (q_1^T b) q_1$$

$$q_1^T b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} (3) = \frac{3}{\sqrt{2}}$$

$$(q_1^T b) q_1 = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$b' = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix} \qquad \|b'\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 4} = \sqrt{\frac{1}{2} + \frac{8}{2}} = \frac{3}{\sqrt{2}}$$

$$\boxed{q_2 = \frac{b'}{\|b'\|} = \frac{\sqrt{2}}{3} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}}$$

Check: $q_1^T \cdot q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \cdot \frac{\sqrt{2}}{3} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}$

$$= \frac{1}{3} \left[-\frac{1}{2} + \frac{1}{2} \right] = 0$$

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$$c' = c - (q_1^T c)\,q_1 - (q_2^T c)\,q_2$$

$$(q_1^T c)\,q_1 = \frac{1}{\sqrt{2}}(1,1,0)\cdot\begin{pmatrix}0\\1\\3\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$(q_2^T c)\,q_2 = \frac{\sqrt{2}}{3}\left(-\frac{1}{2},\frac{1}{2},2\right)\cdot\begin{pmatrix}0\\1\\3\end{pmatrix}\ \frac{\sqrt{2}}{3}\begin{pmatrix}-1/2\\1/2\\2\end{pmatrix}$$

$$= \frac{\cancel{2}}{9}\left(\frac{13}{\cancel{2}}\right)\begin{pmatrix}-1/2\\1/2\\2\end{pmatrix} = \frac{13}{9}\begin{pmatrix}-1/2\\1/2\\2\end{pmatrix}$$

$$c' = \begin{pmatrix}0\\1\\3\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\0\end{pmatrix} - \begin{pmatrix}-13/18\\13/18\\26/9\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\3\end{pmatrix} - \begin{pmatrix}-13/18\\13/18\\26/9\end{pmatrix}$$

$$= \begin{pmatrix}\frac{-9+13}{18}\\\frac{9-13}{18}\\\frac{27-26}{9}\end{pmatrix} = \frac{1}{9}\begin{pmatrix}2\\-2\\1\end{pmatrix} \qquad \|c'\| = \sqrt{\frac{4}{81}+\frac{4}{81}+\frac{1}{81}} = \sqrt{\frac{9}{81}} = \frac{3}{9} = \frac{1}{3}$$

$$q_3 = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

Check: $q_1^T\cdot q_3 = \frac{1}{\sqrt{2}}(1,1,0)\cdot\frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \frac{1}{3\sqrt{2}}(2-2) = 0$

2a. Induction

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n$$

n=2

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$$

$$p(\lambda) = \det\begin{bmatrix} -\lambda & 1 \\ -a_2 & -a_1-\lambda \end{bmatrix} = -\lambda(-a_1-\lambda) + a_2$$

$$= \lambda^2 + a_1\lambda + a_2$$

assume true for k, show true for k+1

$$A_{k+1} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & 0 & & \\ \vdots & \vdots & & 1 \\ -a_{k+1} & -a_k & \dots & -a_1 \end{bmatrix}$$

$$A_{k+1} - \lambda I = \begin{bmatrix} -\lambda & 1 & & \\ \vdots & -\lambda & & \\ \vdots & \vdots & & 1 \\ -a_{k+1} & -a_k & \dots & -a_1-\lambda \end{bmatrix}$$

$$\det(A_{k+1} - \lambda I) = p(\lambda) = (-\lambda)(-1)^2 \det\begin{bmatrix} -\lambda & 1 & \\ \vdots & & \\ -a_k & \dots & -a_1-\lambda \end{bmatrix}$$

$$+ (-a_{k+1})(-1)^{k+2} \det\begin{bmatrix} 1 & 0 & \\ -\lambda & 1 & \\ \vdots & \vdots & \ddots \; 1 \end{bmatrix}$$

$$= -\lambda \underbrace{\det\begin{bmatrix} -\lambda & 1 & \\ \vdots & & \\ -a_k & \dots & -a_1-\lambda \end{bmatrix}}_{\det(A_k - \lambda)} + (-1)^{k+2}(-a_{k+1})$$

$$= -\lambda(\lambda^k + a_1\lambda^{k-1} + \dots + a_k) + (-1)^{k+2}(-a_{k+1})$$

case 1: k odd

$$p(\lambda) = \lambda(\lambda^k + a_1\lambda^{k-1} + \dots + a_k) + a_{k+1}$$

$$= \lambda^{k+1} + a_1\lambda^k + \dots + a_{k+1} = 0$$

case 2: k odd

$$p(\lambda) = -\lambda^{k+1} - a_1\lambda^k - \dots - a_k - a_{k+1} = 0$$

$$\Rightarrow p(\lambda) = \lambda^{k+1} + a_1\lambda^k + \dots + a_{k+1} = 0$$

b. $\underline{k=2}$

$$\begin{bmatrix} -r & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \begin{bmatrix} -r & 1 \\ 0 & -a_1 - \frac{a_2}{r} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$= \frac{-a_1 r - a_2}{r} = 0 \Rightarrow a_1 r + a_2 = 0 \Rightarrow p(r) = 0$$

If we do induction, then we still get $p(r) = 0$ which is true since r is a root.

So the eigenvector is $v = \begin{Bmatrix} 1 \\ r \\ \vdots \\ r^{n-1} \end{Bmatrix}$

3. $m = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ Q & -1 & 1 \end{bmatrix}$

a. $tr(m) = 2 - 1 + 1 = 2$

$$\det m = 2\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 0 \\ Q & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ Q & -1 \end{vmatrix}$$

$$= 2(-1-0) - (-1) + (1+Q) = -2 + 1 + 1 + Q = Q$$

b. $mx = F$ can be solved for arbitrary F is $\det m \neq 0$.

$\Rightarrow Q \neq 0$

$Q = 0$

c. $$\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ Q & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 - 2F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \\ F_1 - 2F_2 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \\ F_1 - 2F_2 + 3F_3 \end{Bmatrix}$$

There is a solution provided that

$$4x_3 = F_1 - 2F_2 + 3F_3$$

$$-x_2 + x_3 = F_3$$

$$-x_1 - x_2 = F_2$$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_1 \\ F_3 \end{Bmatrix} \qquad 2r_1 + r_2$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ 2F_2 + F_1 \\ F_3 \end{Bmatrix} \qquad -r_2 + r_3$$

$$\begin{bmatrix} \underline{-1} & -1 & 0 \\ 0 & \underline{-1} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ 2F_2 + F_1 \\ F_3 - 2F_2 - F_1 \end{Bmatrix}$$

there is a solution provided that $F_3 - 2F_2 - F_1 = 0$

x_3 free, x_1 and x_2 basic

$-x_1 - x_2 = F_2$

$-x_2 + x_3 = 2F_2 + F_1 \implies x_2 = x_3 - 2F_2 - F_1$

$x_1 = -x_2 - F_2 = -x_3 + 2F_2 + F_1 - F_2$

$\qquad = -x_3 + F_1 + F_2$

$$x = \begin{Bmatrix} -x_3 + F_1 + F_2 \\ x_3 - 2F_2 - F_1 \\ x_3 \end{Bmatrix} = x_3 \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1 + F_2 \\ -2F_2 - F_1 \\ 0 \end{Bmatrix}$$

a. $MX = NF$ $\qquad$ N invertible

$N^{-1}MX = F$ $\qquad$ $N^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$

$$N^{-1}M = \begin{bmatrix} 2a-b+cQ & a-b-c & a+c \\ 2d-e+fQ & d-e-f & d+f \\ 2g-h+kQ & g-h-k & g-k \end{bmatrix}$$

choose $c = f = 0$

$g = 1,\ h = 2 \qquad k = 1/2$

$e = d = a = 1 \qquad b = 2$

$$N^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1/2 \end{bmatrix}$$

$1 - 2 - 1/2$
$= -3/2$

$\det N^{-1} = 1(1/2) - 2(1/2) = 1/2 - 1 = -1/2 \neq 0$

$$N^{-1}M = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 3/2 & 1/2 \end{bmatrix}$$

$-1 - 1/2$

$\det(N^{-1}M) = +1\left(\frac{1}{2} - 0\right) + 1\left(\frac{3}{2}\right) = \frac{1}{2} + \frac{3}{2} = 1 \neq 0$

$$N^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1/2 \end{bmatrix}$$

DISREGARD! WRONG!

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we need $\det(N) \neq 0$

and $\det(N^{-1}M) \neq 0$

But $\det(N^{-1}M) = (\det N^{-1})(\det M) = 0$

when $Q = 0$

$\Rightarrow$ There $\nexists$ no matrix N such that $MY = NF$ always has a solution

4. $A = A' = A^H$

a. Let $y = a + ib$

$y = x^H A x$

$y^H = (x^H A x)^H = x^H A^H x^{HH} = x^H A^H x = x^H A x$

$\Rightarrow y = y^H$

$a + ib = a - ib \quad \Rightarrow b = 0$

$\Rightarrow y = a \in \mathbb{R}$

$$\begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \end{bmatrix}$$

b. $Ax = \lambda x \qquad x \in \mathbb{C}, \quad A = A^H$

$x^H (Ax = \lambda x)$

$x^H A x = \lambda x^H x = \lambda \|x\|^2$

$x^H A x \in \mathbb{R}$ and $\|x\|^2 \in \mathbb{R} \quad \Rightarrow \lambda \in \mathbb{R}$

c. $Ax_1 = \lambda_1 x_1, \quad Ax_2 = \lambda_2 x_2 \qquad \lambda_1, \lambda_2 \in \mathbb{R} \quad A = A^H$

want $x_1^H x_2 = 0$

$(\lambda_1 x_1)^H = (Ax_1)^H$

$(\lambda_1 x_1)^H x_2 = (Ax_1)^H x_2$

$\lambda_1 x_1^H x_2 = x_1^H A^H x_2 = x_1^H A x_2 = x_1^H \lambda_2 x_2 = \lambda_2 x_1^H x_2$

$\lambda_1 x_1^H x_2 = \lambda_2 x_1^H x_2$

$\lambda_1 \neq \lambda_2 \quad \Rightarrow \underline{\underline{x_1^H x_2 = 0}}$