Calculus Prelim, January 2009

1. Assume that $b > a > 0$ and $d > c > 0$. Consider the region R in the first quadrant that is bounded by the curves $y = ax^2$, $y = bx^2$, $x = cy^2$, $x = dy^2$. Make a change of coordinates $u = u(x, y)$, $v = v(x, y)$ that turns this region into a rectangular one in the (u, v)-plane. Use this coordinate transformation to determine the area of the region R.

2. A spherical pellet of radius a that is heated uniformly (per unit volume) by a chemical reaction has the equilibrium temperature distribution

$$T = \frac{1}{6}\frac{Q}{k}\left(a^2 - r^2\right),$$

where Q is the rate at which heat energy is added per unit volume, k is the coefficient of thermal conductivity and $r^2 = x^2 + y^2 + z^2$. The heat flux is given by $\vec{q} = -k\vec{\nabla}T$. Compute the total amount of heat flux out of the sphere using a surface integral and show that this is equal to the rate at which heat energy is being added to the entire sphere. What theorem corresponds to this situation, and why?

3. (a) For what value of α is

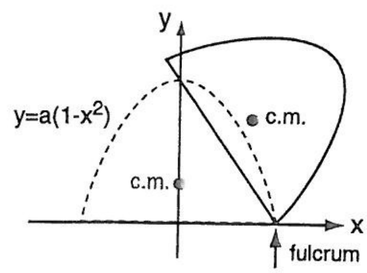
$$\vec{\mathbf{F}} = (x^2 + yz)\hat{\mathbf{i}} + \alpha(y + 2zx)\hat{\mathbf{j}} + (xy + z)\hat{\mathbf{k}}$$

a conservative vector field? For this value of α, find a potential Φ such that $\vec{\mathbf{F}} = \nabla\Phi$. (b) A particle is moved from the origin $(0, 0, 0)$ to the point $(1, 1, 1)$ along a straight-line path. Compute the work $\int \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$, for the vector field in part (a), assuming an arbitrary value of α. For which value of α is your answer independent of the path taken? Check your answer in this special case using a result from part (a).

4. Verify Stokes' theorem for the vector field $\vec{\mathbf{F}} = (x + y)\hat{\mathbf{i}} + 2(x + y)\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ and the surface S, where S is the portion of the sphere $x^2 + y^2 + z^2 = 25$ above the plane $z = 4$. (In order to do this you will need to evaluate both a surface integral and a line integral and show that they are equal.)

5. The variables x and y are functions of r and s determined by $x^2 - y^2 + 2r = 0$ and $xy - s = 0$. Find $\partial x/\partial r$ and $\partial x/\partial s$ in terms of x and y.

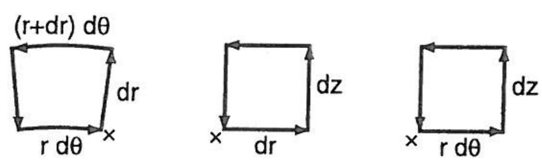
6. When an appliance is designed, one concern is how hard it will be to tip over. When tipped, it will right itself as long as its center of mass lies on the correct side of the *fulcrum*, the point about which the appliance is pivoting. Suppose the profile of an appliance with approximately constant density is parabolic (like an old-fashioned radio), so it fills the region $0 \le y \le a(1-x^2)$, $-1 \le x \le 1$, in the xy-plane (see the figure at right). What values of a will guarantee that the appliance has to be tipped more than 45° to fall over?



7. Use the definition for curl,

$$\hat{n} \cdot \vec{\nabla} \times \vec{F} = \lim_{S \to 0} \frac{1}{S} \oint \vec{F} \cdot d\vec{r}$$

to compute the formula for curl in cylindrical coordinates. [Hint: the idea is to write $\vec{F}$ in terms of coordinates appropriate for the coordinate system, e.g., $\vec{F} = F_1 \hat{r} + F_2 \hat{\theta} + F_3 \hat{k}$, where $\hat{r}$, $\hat{\theta}$ and $\hat{k}$ are unit vectors pointing in the r, θ and z directions, and then to compute the curl in terms of these components. Appropriate small surface elements are shown as below. Note that they have been oriented according to the right-hand rule, and that the starting point for the coordinates before being increased by the differentials has been indicated by a cross.] What is another way to derive this formula, and what is the advantage of using this method?



8. Extra credit: identify the following mathematicians: