Linear Algebra Preliminary Exam January 9, 2008

1. By elimination (or any other method) find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

2. For

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$$

find the least-squares solution of $Ax = b$. Give both the least-squares solution for x and also the projection of b onto the column space of A.

3. True or false? Give a reason or a counterexample:

   (a) If the vectors $x_1, \ldots, x_m$ span a subspace S, then the dimension of $S = m$.

   (b) The intersection of two subspaces of a vector space cannot be empty.

   (c) If $Ax = Ay \neq 0$, then $x = y$. Here A is a $n \times m$-matrix, x and y are m-dimensional vectors, and 0 is the n-dimensional zero vector.

   (d) The row space of A has a unique basis that can be computed by reducing A to echelon form.

   (e) If a square matrix A has independent columns, so does A^2.

4. Consider the system of linear equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ \alpha \end{pmatrix}.$$

Depending on the parameter α, determine all solutions (if any)of this system of equations.

5. For the matrix

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

compute the matrix e^{At}, where t is a real variable.
Comment on the side: the matrix e^{At} would be useful to solve the differential equation $dy/dt = Ay$.

6. Consider the vector space $\mathcal{V}$ spanned by the polynomials

$$P_n = x^n \qquad n = 0, 1, 2$$

defined on the interval $[0, 1]$ with scalar product

$$\langle P_m, P_n \rangle = \int_0^1 P_m(x) P_n(x)\, dx.$$

(a) Use Gram-Schmidt to determine an orthogonal (but not necessarily normalized) set $\mathcal{B}$ of vectors that spans $\mathcal{V}$.

(b) Consider the linear transformation T on $\mathcal{V}$ defined by

$$T\,P_n(x) = \frac{d}{dx}P_n(x).$$

Represent T in terms of a matrix with respect to the set $\mathcal{B}$ determined in part a). What is the rank of T , what is its range and its null space?