Linear Algebra Preliminary Exam January 9, 2008

1. By elimination (or any other method) find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

2. For

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -2 & -3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$$

find the least-squares solution of $Ax = b$. Give both the least-squares solution for x and also the projection of b onto the column space of A.

3. True or false? Give a reason or a counterexample:

   (a) If the vectors $x_1, \ldots, x_m$ span a subspace S, then the dimension of $S = m$.

   (b) The intersection of two subspaces of a vector space cannot be empty.

   (c) If $Ax = Ay \neq 0$, then $x = y$. Here A is a $n \times m$-matrix, x and y are m-dimensional vectors, and 0 is the n-dimensional zero vector.

   (d) The row space of A has a unique basis that can be computed by reducing A to echelon form.

   (e) If a square matrix A has independent columns, so does A^2.

4. Consider the system of linear equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ \alpha \end{pmatrix}.$$

Depending on the parameter α, determine all solutions (if any)of this system of equations.

5. For the matrix

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

compute the matrix e^{At}, where t is a real variable.
Comment on the side: the matrix e^{At} would be useful to solve the differential equation $dy/dt = Ay$.

6. Consider the vector space $\mathcal{V}$ spanned by the polynomials

$$P_n = x^n \qquad n = 0, 1, 2$$

defined on the interval $[0, 1]$ with scalar product

$$\langle P_m, P_n \rangle = \int_0^1 P_m(x) P_n(x)\, dx.$$

(a) Use Gram-Schmidt to determine an orthogonal (but not necessarily normalized) set $\mathcal{B}$ of vectors that spans $\mathcal{V}$.

(b) Consider the linear transformation T on $\mathcal{V}$ defined by

$$T\,P_n(x) = \frac{d}{dx}P_n(x).$$

Represent T in terms of a matrix with respect to the set $\mathcal{B}$ determined in part a). What is the rank of T , what is its range and its null space?

Linear Algebra Preliminary Exam 2007

1. Find all possible solutions to the system of equations $Ax = b$ where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 5 \\ 4 & -8 & 10 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

What are all possible solutions if A is unchanged but

$$b = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}?$$

In addition, find the rank of A, and the dimensions of and bases (if any) for each of the four fundamental subspaces associated with A (i.e., the column space, the row space, the nullspace and the left nullspace of A). For this A, what condition(s) on b must be true for $Ax = b$ to have solutions?

2. Consider the subspace $\mathbf{V}$ of $\mathbf{R}^3$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Are these vectors also a basis for $\mathbf{V}$? Why or why not? If they are not a basis, give one.

(b) What is a basis for the orthogonal complement of $\mathbf{V}$ (i.e., $\mathbf{V}^{\perp}$)?

3. Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p + q$, with p in the column space and q perpendicular to that space. Which of the four subspaces of A contains q?

4. Suppose $A = [\ a_1, \quad a_2, \quad a_3, \quad a_4\ ]$ is a (4x4) matrix with columns a_i, where det(A)=3. Find det(B) if

$$B = \left[\ a_1 + 2a_2, \quad a_4, \quad -2a_3, \quad a_2\ \right].$$

5. Find α so that

$$\exp(P) = I + \alpha P,$$

for any projection matrix P.

6. Suppose there is an epidemic in which every month half of those who are well become sick, a quarter of those who are sick die, and a half of those who are sick survive and become immune. Find the general solution to the corresponding Markov process defined by

$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \\ i_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & 0 & 0 \\ 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \\ i_k \end{bmatrix}.$$

Here d_k, s_k, w_k and i_k are the fractions of people in month k who are dead, sick, well and immune. Does the solution eventually settle down to a steady-state? If so, and assuming initially everyone starts out well, what is the steady state? Does this make sense?

Linear Algebra Exam

Friday, January 6, 2006 (9:00 – 11:00 AM)

Please show all work. To get full credit you must **clearly** describe your calculations.

1. **(20 points)** Let S

$$S = \{(1,1,1)^T,\ (-1,0,-1)^T,\ (-1,2,3)^T\}$$

be a basis for Euclidean space R^3 with the standard inner product

$$(a_1,a_2,a_3)^T \cdot (b_1,b_2,b_3)^T = a_1b_1 + a_2b_2 + a_3b_3.$$

Use the Gram-Schmidt procedure to transform S to an orthonormal basis.

2. **(20 points)** Find the eigenvalues (and their multiplicity) and corresponding standard eigenvectors (you need not find generalized eigenvectors) of

$$A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$$

3. **(20 points)** Consider the system of equations

$$\begin{aligned} ax + by + cz &= r_1 \\ a^2x + b^2y + c^2z &= r_2 \\ a^3x + b^3y + c^3z &= r_3 \end{aligned}$$

(a) Determine condition(s) on the constants a, b, c that guarantee that the system has a unique solution for any r_1, r_2, r_3.

(b) If one of these conditions is not satisfied, state a condition (in words only) involving r_1, r_2, r_3 that will allow the system to have solutions. In this case how many solutions will the system have (you need not find such solutions, assuming they exist)?

4. **(20 points)** Transform the quadratic form

$$3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$$

to a sum (or difference) of squares

$$\alpha y_1^2 + \beta y_2^2 + \gamma y_3^2,$$

and determine α, β and γ.

5. **(20 points)** Let Q_1 and Q_2 be square matrices. If Q_1 and Q_2 are orthogonal matrices, show that Q_1Q_2 is also an orthogonal matrix.

Linear Algebra Exam

Friday, January 7, 2005 (9:00 – 11:00 AM)

Please show all work. To get full credit you must **clearly** describe your calculations.

1. (**20 points**) For what values of α and β does the set S span R^3?

$$S = \{(1,0,0),\ (1,1,0),\ (\alpha,1,\beta)\}.$$

2. (**20 points**) Find the eigenvalues (and their multiplicity) and corresponding eigenvectors of

$$A = \begin{pmatrix} -4 & 5 & 5 \\ -5 & 6 & 5 \\ -5 & 5 & 6 \end{pmatrix}$$

3. (**20 points**) Consider the system of equations

$$\begin{aligned} x - 3y + 2z &= 4 \\ 2x + y - z &= 1 \\ 3x - 2y + z &= \alpha \end{aligned}$$

(a) Determine a condition on the constant α that guarantees that the system has at least one solution.

(b) For this value of α determine the number of solutions that the system has and find this (these) solutions.

4. (**20 points**) Find a transformation $\mathbf{x} = \mathrm{P}\mathbf{y}$, $\mathbf{x} = (x_1, x_2, x_3)^\top$, $\mathbf{y} = (y_1, y_2, y_3)^\top$ which transforms the quadratic form

$$2x_1^2 + 4x_2^2 - 4x_3^2 + 6x_2x_3$$

to a sum (or difference) of squares

$$\alpha y_1^2 + \beta y_2^2 + \gamma y_3^2,$$

and determine α, β and γ.

5. (**20 points**) Prove that all eigenvalues of a Hermitian matrix are real and the eigenvectors corresponding to different eigenvalues are orthogonal.

6. (**20 points**) Compute A^n and $\exp(A)$ where A is the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}.$$

PRELIMINARY EXAM - LINEAR ALGEBRA January 2004

Please show all work. To get full credit for a problem you need to show your work and **CLEARLY** describe your calculations.

1. (40 points) The matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & -1 & 0 \\ 2 & -2 & k \end{bmatrix}$$

has the eigenvalues $0, 1, -1$.

(a) For what values of k is this true?

(b) Find the eigenvectors corresponding to each of the three eigenvalues.

(c) Is $\left(\sqrt{8}, \sqrt{2}, \sqrt{2}\right)^{\mathrm{T}}$ an eigenvector? Explain.

(d) Determine the eigenvalues of the matrix $\mathbf{A}^2$ without computing $\mathbf{A}^2$; explain and/or prove your arguments.

(e) Given the system of first order differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A} \cdot \mathbf{x}$$

where $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$. Determine the solution that satisfies the initial conditions $x_1(0) = 0,\ x_2(0) = 1,\ x_3(0) = 1/2$.

2. (20 points) Consider the matrix M

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ k & -1 & 1 \end{bmatrix},$$

where k is an arbitrary constant.

(a) For what values of k can you solve

$$\mathbf{Mx} = \mathbf{F}$$

for an arbitrary vector $\mathbf{F}$?

(b) For those values of k for which there is no solution to (a) when $\mathbf{F}$ is arbitrary, there may be a solution for a special class of $\mathbf{F}$. Identify this class of $\mathbf{F}$ and determine the general solution vector $\mathbf{x}$.

3. (40 points) Determine if the following are true or false. If true, **you must prove it**; if false, **give a counter-example**. Note: You may use the result of part (a) when proving (b)-(d), part (b) when proving (c)-(d), and part (c) when proving (d). Note: we write the determinant of the matrix $\mathbf{A}$ as $det\mathbf{A}$.

(a) If each element in one row (column) of a determinate is multiplied by a number c, the value of the determinant is multiplied by c.

(b) Suppose that v_1, v_2, ..., v_n, f_j are all $n \times 1$ column vectors. Consider the $n \times n$ square matrix $\mathbf{A}$ defined as

$$\mathbf{A} = [v_1 \cdots v_j + f_j \cdots v_n],$$

i.e., the columns of A are composed of the elements of the column vectors v_1, ... v_n, except for the jth column whose elements are the sum of v_j and f_j. Then

$$det\mathbf{A} = det[v_1 \cdots v_j \cdots v_n] + det[v_1 \cdots f_j \cdots v_n]$$

(c) If a matrix $\mathbf{B}$ is obtained from a square matrix $\mathbf{A}$ by interchanging two columns of $\mathbf{A}$, then $det\mathbf{A} = det\mathbf{B}$.

(d) If a matrix $\mathbf{B}$ is obtained from a square matrix $\mathbf{A}$ by adding to one row (column) vector of $\mathbf{A}$ a number c times a different row (column) vector, then $det\mathbf{A} = det\mathbf{B}$.

PRELIMINARY EXAM – DIFFERENTIAL EQUATIONS 1/08

Note - all problems count equally

1. Consider the differential equation
$$x^2y' + y = 1.$$
Determine the solution satisfying the initial condition $y(1/2) = 0$.

2. Consider the equation
$$y'' + \lambda y = 0,$$
with the boundary conditions
$$y(0) + y'(0) = 0, \quad y(\frac{\pi}{2}) + y'(\frac{\pi}{2}) = 0.$$
Determine all solutions for every value of λ $(-\infty < \lambda < \infty)$.

3. Consider the equation
$$x'' + 2x' + x = \exp(\alpha t).$$
Determine the general solution for all values of α.

4. Find the general solution to the equation
$$y' = \frac{1}{x+y}.$$

5. Find the solution to the problem
$$x^2y'' + xy' + y = 0, \quad y(1) = 2, y'(1) = 3.$$

6. Consider the system of equations

$$\begin{aligned} x' &= y, \\ y' &= -x - \delta y. \end{aligned}$$

(a) Determine the general solution for $\delta = 0, 1, 2, 3$.

(b) For each case determine the type and stability of the critical point at the origin.

(c) For $\delta = 1$ determine the general solution to the forced system

$$\begin{aligned} x' &= y + t, \\ y' &= -x - \delta y + 1. \end{aligned}$$

7. Consider the system

$$\begin{aligned} x' &= x^2 - 2x - xy, \\ y' &= y^2 - 4y + xy. \end{aligned}$$

Determine all critical points, their type and stability.

PRELIMINARY EXAM – DIFFERENTIAL EQUATIONS 1/07

Note - all problems count equally

1. Consider the differential equation
$$x(2-x)\frac{dy}{dx}+y=0. \tag{1}$$
   (a) Solve (1) for initial condition $y(1)=1$.
   (b) Solve (1) for initial condition $y(3)=1$.
2. Find all values of λ for which the boundary value problem
$$\frac{d^2u}{dt^2}+2\frac{du}{dt}+\lambda u=0, \quad u(0)=0, \quad u(1)=0,$$
   has a non-trivial solution, and write the corresponding solution(s).
3. Find the general solution of the equation
$$\frac{1}{2}x^2(x-1)\frac{d^2y}{dx^2}-x(x-2)\frac{dy}{dx}+(x-3)y=0,$$
   given that $y=x^2$ is a solution.
4. Find the general solution of the following system of equations:
$$\begin{aligned}\frac{dx}{dt}&=-7x-10y+4e^t,\\ \frac{dy}{dt}&=4x+5y-2e^t.\end{aligned}$$

5. Consider the initial value problem

$$y' = -\alpha y + f(t) \quad y(t_0) = y_0, \tag{2}$$

where $\alpha > 0$ and

$$f(t) = \frac{\sin t}{t}, \quad t \neq 0, \quad f(0) = 1.$$

(a) Show that $f(t)$ is continuous at $t = 0$.

(b) Determine the solution to (2) in terms of integrals of known functions. It is not necessary to actually perform the integration.

(c) Show that $\mid f(t) \mid \leq 1$ for all t.

(d) Show that for $t \geq t_0$

$$\mid y(t) \mid \leq \mid y_0 \mid + \frac{1}{\alpha}.$$

(e) Show that for a fixed value of α, (2) can have at most one solution that is bounded for $-\infty < t < \infty$.

(f) For a given fixed value of α, determine the unique bounded solution to (2) in terms of integrals of known functions. It is not necessary to perform the integration.

6. Consider the initial value problem

$$y'' + y = \sin \omega t, \quad y(0) = y'(0) = 0. \tag{3}$$

(a) Determine the solution when $\omega = 2$.

(b) Determine the solution when $\omega = 1$.

7. Consider the system

$$\begin{aligned} x' &= y, \\ y' &= xy, \quad x(0) = x_0, \ y(0) = y_0. \end{aligned}$$

(a) Determine all critical points.

(b) Sketch the trajectories in the phase plane. Make certain you show the direction of motion on the trajectories (use arrows).

(c) Consider the solution with initial conditions $x(0) = 0, y(0) = -4$. Determine $\lim_{t \to \infty} x(t), y(t)$.

Differential Equations
ESAM Preliminary Examination
January 9, 2006, 9-11 a.m.

1. Solve the following differential equations for $y(x)$:

   (a) $x^3y'' - 2x^2y' + 2xy = 1,$

   (b) $\dfrac{d^4y}{dx^4} - y = e^x \sin x,$

   (c) $\dfrac{dy}{dx} = xy(y-2) + x,$

   (d) $2\dfrac{dy}{dx} = \dfrac{x+6y-2}{x-2y-2},$

   (e) $y\dfrac{dy}{dx} = x(e^{x^2} - y^2).$

2. The functions $y_1(x) = 2x+1$ and $y_2(x) = -1$ satisfy the equation

$$x\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + y = -1.$$

   Solve the initial value problem

$$x\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + y = -1, \quad y(0) = 0, \; y'(0) = 0.$$

3. Find a particular solution of the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

4. Consider the equation

$$y' = -y + \sin\left(\frac{1}{t}\right),$$

   with the initial condition $y(1) = 0$. Show that $|\, y(t) \,| \leq 1$ for $t \geq 1$. It is not necessary to find the solution explicitly.

Differential Equations
ESAM Preliminary Examination
January 7, 2005, 1-3 p.m.

1. Solve the following differential equations for $y(x)$:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = \sinh x,$$

$$\frac{dy}{dx} = xy(y-2),$$

$$\frac{dy}{dx} = \frac{x+3y-5}{x-y-1}.$$

2. Solve the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

3. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= y^2, \\ \frac{dy}{dt} &= x^2. \end{aligned}$$

   (a) Sketch the phase portrait of the system.

   (b) What is the qualitative difference in behavior of the solution with $x(0) = y(0) = -1000$ and the solution with $x(0) = -1000, y(0) = -1000.00001$?

4. Consider the system

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t),$$

where $\vec{x}$ and $\vec{f}$ are vectors of size n and A is a constant $n \times n$ matrix. Characterize **all** matrices A so that for all periodic functions $\vec{f}$ (irrespective of period) there will be at most one periodic solution.

Preliminary Exam: Elementary Differential Equations January 2004

1. Find the general solution of the following equation:

$$\frac{d^2y}{dx^2} + y = \cos(x)\,.$$

2. Find the general solution to the differential equation

$$xy'' - 2y' + \frac{2}{x}y = 1.$$

3. According to *Torricelli's law*, water in an open tank will flow out through a small hole in the bottom with the speed it would acquire in falling freely from the water's surface to the hole. A cylindrical tank of radius R and height H is initially full of water, and a small circular hole of radius r is punched in the bottom at time $t = 0$. How long will it take for the tank to empty itself?

4. Consider the system

$$\begin{aligned} \dot{x} &= x + 3y + e^{2t}, \\ \dot{y} &= x - y. \end{aligned}$$

   a) Find the general solution of the associated homogeneous problem.
   b) Find a particular solution of the inhomogeneous problem.

5. Find the solution for the system of equations

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

   with $x(0) = 3$ and $y(0) = 0$.

6. Find and classify the critical points of the system of equations by their linear stability properties

$$\begin{aligned} \frac{dx}{dt} &= a - bx + x^2y - y \\ \frac{dy}{dt} &= bx - x^2y\,. \end{aligned}$$

   for all positive values of a and b.

7. Solve the partial differential equation

$$u_t = u_{xx} \qquad 0 < x < \pi \qquad 0 < t$$

   with boundary conditions $u(t, 0) = 0 = u(t, \pi)$ and initial condition $u(0, x) = \sin x + \sin 2x$.

Complex Variables Preliminary Examination

January 11, 2008

1. Evaluate
$$\left(-1+i\sqrt{3}\right)^{3/2}.$$

2. Expand the function
$$\frac{1}{z^2(z-1)}$$
in a Laurent series valid for $0<|z|<1$.

3. Use contour integration to evaluate the following integral. Explain carefully each step of the method you use for evaluation.
$$\int_0^\infty \frac{\log x}{1+x^2}\,dx$$

4. Let $f(z)=u(r,\theta)+iv(r,\theta)$ be an entire function. If $u(r,\theta)=r\sin\theta$ and $f(1)=i$, find $f(z)$.

5. Use conformal mapping to solve the boundary value problem,
$$u_{xx}+u_{yy}=0$$
in the domain $x^2+y^2>1$, with the boundary conditions
$$u=1+x \text{ on } x^2+y^2=1\,, \qquad u\to 1 \text{ as } x^2+y^2\to\infty$$
Hint: The mapping $\zeta=1/z$ conformally maps the exterior of the unit disk into its interior.

Complex Variables Preliminary Examination

January 11, 2007

1. Find two separate Laurent expansions of the function
$$\frac{1}{(iz+1)(z-2)}$$
about the point $z=0$: one valid for $1<|z|<2$ and one valid for $|z|>2$.

2. Suppose
$$f(z)=\frac{H(z)}{(z^2-1)(z-2)^2},$$
where $H(z)$ is an entire function. Determine the value of
$$\oint_C f(z)dz,$$
where C is taken counterclockwise around the circle
   (a) $|z-5|=2$
   (b) $|z-5|=5$

3. Use contour integration to evaluate the following integral. Explain carefully each step of the method you use for evaluation.
$$\int_0^{2\pi}\frac{\cos\theta}{2+\cos\theta}\,d\theta \quad \text{(you need not simplify the answer)}$$

4. Use contour integration to evaluate the following integral. Explain carefully each step of the method you use for evaluation.
$$\int_0^{\infty}\frac{x^{1/2}}{1+x^2}\,dx$$

5. Find the harmonic conjugate of
$$u(x,y)=xe^{-y}\cos x-ye^{-y}\sin x\,.$$
Find all complex analytic functions $f(z)$ of which u is the real part.

6. What is the image of the negative real line $\{z=x+i0 : x<0\}$ under the map $f(z)=1/(z+i)$?

Complex Variables Preliminary Examination

January 12, 2006

1. Suppose the exponential function is defined for complex arguments z by assuming a) e^z is an entire function; b) when $z = x$ is real, all of the normal properties of real functions hold; and c) for complex arguments, the product of two exponentials is the exponential of the sum of the exponents, i.e.,

$$e^{z_1}e^{z_2} = e^{z_1+z_2}\,.$$

Then we can write

$$e^z = e^{x+iy} = e^x e^{iy} \equiv e^x\left[C(y) + iS(y)\right]$$

for some functions $C(y)$ and $S(y)$. Show that the requirement that e^z be analytic, and that $e^0 = 1$, means that $C(y)$ and $S(y)$ must be $\cos y$ and $\sin y$.

2. Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

3. Evaluate the following integrals, explaining carefully each step of the method used for the evaluation.

$$\int_0^\infty \frac{x^2\,dx}{1+x^6}$$

$$\int_{-\infty}^\infty \frac{\sin x}{x(x^2+a^2)}\,dx$$

$$\int_0^\infty \frac{x^{1/2}}{(1+x)^2}\,dx$$

$$\int_0^{2\pi} \frac{d\varphi}{1+2a\sin\varphi+a^2}\,, \qquad |a|<1.$$

4. Suppose that:

   (a) $F(z)$ is analytic in the cut plane $|\arg z| < \pi$

   (b) $F(z) \to 0$ as $z \to \infty$ uniformly in $|\arg z| < \pi$

   (c) the jump across the cut is known, i.e.,

$$F(re^{\pi i-}) - F(re^{-\pi i+}) = -2i\rho(r)\,.$$

Show that

$$F(z) = \frac{1}{\pi}\int_0^\infty \frac{\rho(r)\,dr}{r+z}\,.$$

Complex Variables Preliminary Examination

January 10, 2005

1. Find three separate Laurent expansions of the function

$$\frac{1}{(iz+1)(z-2)}$$

about the point $z = 0$: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$.

2. Suppose

$$f(z) = \frac{H(z)}{(z^2-1)(z-2)^2},$$

where $H(z)$ is an entire function. Determine the value of

$$\oint_C f(z)dz,$$

where C is taken counterclockwise around the circle

(a) $|z-5| = 2$
(b) $|z-5| = 5$

3. Use contour integration to evaluate the following integral. Please explain carefully each step of the method used for the evaluation.

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx \tag{1}$$

4. Evaluate the following integral, explaining carefully each step of the method used. You needn't simplify the answer.

$$\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\, d\theta. \tag{2}$$

5. Evaluate the following integral, explaining carefully each step of the method used.

$$\int_0^{\infty} \frac{x^{1/2}}{1+x^2}\, dx. \tag{3}$$

6. Classify all complex singularities of the following two functions

$$\text{(a)}\quad \frac{\log z}{(z+1)^2} \qquad \text{(b)}\quad \frac{e^{-\frac{1}{z}}}{\sin z} \tag{4}$$

(i.e., locate all singularities and determine whether each is a branch point, a pole (giving its order), or an essential singularity).

7. By any method, find the harmonic conjugate of

$$u(x,y) = xe^{-y}\cos x - ye^{-y}\sin x\,.$$

Of what complex analytic function $f(z)$ is this the real part?

8. (a) Find a conformal map, $w = f(z)$ ($z = x + iy$, $w = \xi + i\eta$), that transforms the first quadrant of the complex plane ($x, y \geq 0$) into the strip $-\infty < \xi < +\infty$, $0 \leq \eta \leq \frac{\pi}{2}$.

   (b) Use this conformal map to solve Laplace's equation $\nabla^2 T = 0$ with boundary conditions $T = 0$ for $y = 0$, $x > 0$, and $T = 1$ for $x = 0$, $y > 0$.

Complex Variables Exam

Monday, January 12, 2004 (12:00 – 2:00 PM)

Please show all work. To get full credit for a problem you need to **clearly** describe your calculations.

1. **(10 points)** Determine where $f'(z)$ exists and provide an expression for it:

$$f(z) = r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta), \quad z = re^{i\theta}.$$

2. **(20 points)** Use complex variables to evaluate the following integrals

$$\text{(a)} \quad \int_0^{2\pi} \frac{d\theta}{(5+3\cos\theta)^2}, \qquad \text{(b)} \quad \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}dx.$$

   Please give the answers in the real form (not as complex numbers).

3. **(10 points)** Evaluate the intergal

$$\oint_C \bar{z}^2(e^z - 1)\,dz,$$

   where $C: \ z = e^{i\theta}, 0 \le \theta < 2\pi$.

4. **(10 points)** How many zeros does the function

$$z^8 - 4z^5 + z^2 - 1$$

   have inside the unit circle $|z| = 1$?

5. **(10 points)** Verify that $v(x, y) = x + y - 3$ is harmonic everywhere in the plane and find all analytic functions $f(z)$, $z = x + iy$, such that $v = \mathrm{Im} f(z)$.

6. **(10 points)** Find the image of the circle $x^2 + y^2 = 1$ under the transformation

$$w = \frac{z-a}{1-\bar{a}z} \quad (z = x + iy),$$

   where a is a complex number.

Advanced Calculus Preliminary Exam

Tuesday, January 8, 2008 (9 – 11:00 am)

CHOOSE **5** PROBLEMS OUT OF **6**

Please show all work. To get full credit you must **clearly** describe the details of all your calculations.

Problem 1. Consider the vector field

$$\vec{F} = (y^3 + 3x^2 y)\mathbf{i} + 3x\mathbf{j} + z^2\mathbf{k}$$

and the curve, C, that is the intersection of the cylinder $x^2 + y^2 = 1$ with the surface $z = g(x, y)$. Compute the work, W, done along C by $\vec{F}$,

$$W = \oint_C \vec{F} \cdot d\vec{r}$$

(C is oriented clockwise when viewed from above). Does W depend on the choice of $g(x, y)$?

Problem 2. In cylindrical coordinates (r, θ, z), a torus has equation

$$(r - a)^2 + z^2 = b^2, \quad (a > b > 0).$$

Write and evaluate an integral for the volume of the torus in cylindrical coordinates.

Problem 3. Evaluate

$$I = \int\int_D (x + y)^2 \, dx \, dy$$

where D is the region

$$(x - y)^2 + \frac{(x + y)^2}{4} \leq 1 .$$

Problem 4. Consider the surface defined by

$$F(x, y, z) = \frac{1}{2}(x + y)^2 + (y + z)^2 + (x + z)^2 = 9.$$

(a) Evaluate ∇F.

(b) Find the highest and lowest points on the surface (i.e. the points where z attains a maximum or minimum).

(c) The surface is illuminated from far above by light rays that are directed parallel to the z-axis. Find the shape of its shadow in a plane below the surface and parallel to the (x, y) coordinate plane.

Problem 5. Compute directly the flux,

$$\int\int \vec{F} \cdot \hat{n} \, dS$$

out of the cone

$$x^2 + y^2 = z^2, \qquad 0 \leq z \leq 1$$

for the vector field

$$\vec{F} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}.$$

Verify your answer with the divergence theorem.

Problem 6. Find the area of the plane region bounded by the curve

$$(x^2 + y^2)^3 = x^4 + y^4.$$

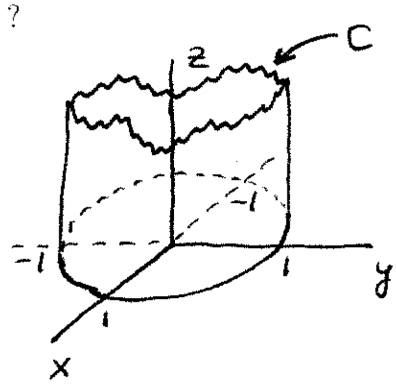
Advanced Calculus Exam

Friday, January 12, 2007 (12:00 – 2:00)

choose **5** *problems out of* **6**

Please show all work. To get full credit you must **clearly** describe your calculations.

1. **(20 points)** A broken wine bottle sits on the xy plane as shown. The bottle consists of a portion of a cylinder of radius 1 along the z axis, with the unit disc in the xy plane at the bottom (see the figure below). Let $\mathbf{C}$ be the path along the broken edge oriented counterclockwise when viewed from above. If $\mathbf{F}$ is the vector field $\mathbf{F} = \langle -y,\ 2x,\ 10z \rangle$, what is the value of $\oint \mathbf{F} \cdot dr$?



2. **(20 points)** Inconspicuously, James Bond put a drop of cyanide into a glass of whiskey that he saw the double agent X was drinking. The drop started to spread. Let $C(x, y, z, t)$ denote the concentration of cyanide in the whiskey. Fick's law states that the flux $\mathbf{q}$ of cyanide is negatively proportional to the gradient of C (the cyanide spreads from points of high concentration to points of low concentration), i.e. $\mathbf{q} = -k\nabla C$. Show that C satisfies the equation $\frac{\partial C}{\partial t} = k\nabla^2 C$.

3. **(20 points)**

   (a) For what values of α is the force field $\mathbf{F} = (e^x \sin y - y)\,\mathbf{i} + \alpha(e^x \cos y - x - 2)\,\mathbf{j}$ conservative? For each such value of α find a potential ϕ such that $\mathbf{F} = \nabla\phi$.

   (b) Consider a force field $\mathbf{F}$ derivable from a potential U, $\mathbf{F} = -\nabla U$. The work done by $\mathbf{F}$ on a particle moving from point A to point B is defined as $W = \int_A^B \mathbf{F} \cdot d\mathbf{r}$.

   (c) Show that $W = U(A) - U(B)$.

   (d) Let the particle move according to the Newton's Second law of motion, $\mathbf{F} = m\, d^2\mathbf{r}/dt^2$, where m is the mass of the particle and $\mathbf{r}$ is its position. Show that the sum of the potential and the kinetic energy, $U + m\,v^2/2$, where $v = |\mathbf{v}|$, $\mathbf{v} = d\mathbf{r}/dt$, is conserved (therefore such a force field is called conservative).

4. **(20 points)** Find the mass of the plane region R in the first quadrant of the (x, y)-plane bounded by the hyperbolas $xy = 1$, $xy = 2$, $x^2 - y^2 = 3$, $x^2 - y^2 = 5$. Assume the density at the point (x, y) is $\rho(x, y) = x^2 + y^2$.

5. **(20 points)** A function $y(x)$ is given in implicit form $F(x, y) = 0$. Find dy/dx and d^2y/dx^2 in terms of $\partial F/\partial x$ and $\partial F/\partial y$.

6. **(20 points)** Consider the force field $\mathbf{F} = v_0\mathbf{k}$, where $\mathbf{k}$ is the unit vector along the $+z$-direction.

   a) By direct computation find the flux of the vector field $\mathbf{F}$ across the hemispherical surface $|\mathbf{r}| = a$, $z > 0$ in the direction pointing away from the origin.

   b) Find the flux of the vector field $\mathbf{F}$ across the disk $\{x^2 + y^2 = a^2,\ z = 0\}$ in the $\mathbf{k}$ direction.

   c) Without appealing to Stokes theorem, present an argument showing why the fluxes in a) and b) must be equal.

PRELIMINARY EXAM - Calculus January 4, 2006

Please show *all* your work and **CLEARLY** describe your calculations.

1. Given the function

$$x\cos y + y\cos z + z\cos x = 1 \tag{1}$$

(a) Find the derivatives $\partial z/\partial x$ and $\partial z/\partial x$

(b) Write down an equation for the plane tangent to the surface $z = f(x, y)$, implicitly described by equation (1) at $x = y = 0$.

2. Find out whether or not the vector fields given below have a potential φ, and find φ if it exists.

(a) $\mathbf{V} = (5x^2y - 4xy)\mathbf{i} + (3x^2 - 2y)\mathbf{j}$

(b) $\mathbf{V} = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

3. Compute the maximum and minimum values of the function $f(x; y) = 6 - 4x - 3y$, on the circle of unit radius $x^2 + y^2 = 1$.

4. Calculate the flux of the vector field $\mathbf{F} = 1\mathbf{k}$ across the surface S in the direction away from the origin, where S is the upper unit hemisphere $x^2 + y^2 + z^2 = 1; z \geq 0$.

5. Show that if S is the surface bounding a volume V and f, g are scalar functions, then

$$\oiint_S (f(\nabla g \cdot \mathbf{n}) - g(\nabla f \cdot \mathbf{n}))\, dS = \iiint_V (f\nabla^2 g - g\nabla^2 f)\, dV.$$

6. Evaluate $\iint_S x^2y^2\, dS$ over the total surface (including the top and bottom) of the cylinder $x^2 + y^2 = a^2$, $z = 0$, $z = h$.

Advanced Calculus Preliminary Exam **January 5, 2005**

Please **choose five** of the following six problems to solve. Clearly **indicate on the cover of your exam, which problems you have selected.** Please show all work. To get full credit for a problem, you need to **CLEARLY** show the details of your calculations.

Problem 1 (20 pts) Use the change of variable $y + x = u$, $y - x = v$ to evaluate the following integral:

$$\iint_R e^{\frac{y-x}{y+x}}\, dx\, dy$$

where R is the triangular region in the x-y plane whose vertices are $(0,0)$, $(0,1)$, $(1,0)$. Sketch the region R in both the x-y and u-v planes.

Problem 2 (20 pts) Compute the maximum and minimum values of $f(x,y) = x^2y^2 - x^2 - 4y^2$ on the circle $x^2 + y^2 = 5$.

Problem 3 (20 pts) Consider a surface S in $\mathbb{R}^3$ defined by $g(\mathbf{x}) = 0$, and let $T(\mathbf{x})$ be a temperature field defined in $\mathbb{R}^3$. At a point $\mathbf{a} \in S$, compute (1) the direction of steepest ascent of the value of T on S, and (2) the rate of increase of T in that direction.

Problem 4 (20 pts) Evaluate the flux of the vector field $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ over the closed surface described by

$$z = 1 - (x^2 + y^2), \quad z \geq 0$$

where the flux is given by $\int_S \mathbf{v} \cdot \mathbf{n}\, dS$, and $\mathbf{n}$ is the unit outward normal. What happens if the unit normal $\mathbf{n}$ is chosen as the inward normal?

Problem 5 (20 pts) (a) Show that $\iint_S f\mathbf{n}\, dS = \iiint_V \nabla f\, dV$ for any piecewise smooth surface S enclosing the volume V and any differentiable scalar function $f(x,y,z)$. [Hint: consider $\iint_S \mathbf{a}f \cdot \mathbf{n}\, dS$ for fixed, but arbitrary vector $\mathbf{a}$.]

(b) Confirm your result for the function $f = x^2$ and the volume V corresponding to the box $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Problem 6 (20 pts) Let $\mathbf{F} = \dfrac{-y}{x^2+y^2}\hat{\mathbf{i}} + \dfrac{x}{x^2+y^2}\hat{\mathbf{j}}$.

(a) Compute directly (i.e. without using Green's theorem): $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle, $x^2 + y^2 = 1$, $z = 0$, taken in a counter-clockwise direction.

(b) Compute the integral in part (a) by applying the formula from Green's theorem.

(c) Compare your answers for parts (a), (b). Do they support or contradict Green's theorem? Explain.

Please **choose five** of the following six problems to solve. Clearly **indicate on the cover of your exam, which problems you have selected.** Please show all work. To get full credit for a problem, you need to **CLEARLY** show the details of your calculations.

Problem 1 (20 pts) Use Stokes' theorem to evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

where $\mathbf{F} = -yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$ and S is the area of the spherical cap $x^2 + y^2 + z^2 = 4$, $z \geq \sqrt{3}$. The vector $\mathbf{n}$ is the unit outward normal to the spherical cap.

Problem 2 (20 pts) For $\xi = x - y$, and $\eta = 2y$, convert the partial differential equation

$$x\frac{\partial^2 u}{\partial x^2} + 2(x+y)\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0$$

into one in terms of ξ and η.

Problem 3 (20 pts) Let $\mathbf{F}$ be a force field which is solenoidal ($\nabla \cdot \mathbf{F} = 0$) and consider a tube of force consisting of a surface with lateral side S and two end surfaces S_1 and S_2, with $\mathbf{F}$ directed along (parallel to) the lateral surface S. Show that the flux of $\mathbf{F}$ entering S_1 equals the flux of $\mathbf{F}$ leaving S_2.

Problem 4 (20 pts) Let P, Q be continuous functions with continuous partial derivatives and let $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ except at the points $(4,0)$, $(0,0)$, and $(-4,0)$. Let C_1 denote the circle $(x-2)^2+y^2 = 9$, let C_2 denote the circle $(x+2)^2+y^2 = 9$, let C_3 denote the circle $x^2+y^2 = 25$, and let C_4 denote the circle $x^2+y^2 = 1$. Given that $\oint_{C_1} P\,dx + Q\,dy = 11$, $\oint_{C_2} P\,dx + Q\,dy = 9$, and $\oint_{C_3} P\,dx + Q\,dy = 13$, find $\oint_{C_4} P\,dx + Q\,dy$.

Problem 5 (20 pts) Consider the curve $C = \{(x,y) : f(x,y) = 0\}$. Show that the sum of the squares of the directional derivatives of $f(x,y)$ at a point P on C along any two perpendicular directions, equals the square of the normal derivative of $f(x,y)$ at P.

Problem 6 (20 pts) Find the maximum and minimum values of the function

$$f(x,y) = 5x^2 + 6xy + 5y^2 - 6x - 6y$$

on the unit disk $\{(x,y) : x^2 + y^2 \leq 1\}$.