Complex Variables Preliminary Examination
April 4, 2011

1. (10 points) Determine where $f'(z)$ exists and provide an expression for it ($z = x + iy = re^{i\theta}$):
   (a) $f(z) = 2x(1-y) + i(x^2 - y^2 + 2y)$
   (b) $f(z) = \theta^2 - 2ir$

2. (10 points) Find a linear fractional transformation $w(z) = \dfrac{\alpha z - \sqrt{2}}{\gamma z + \delta}$ that maps the unit circle C onto the line L given by $z - i\bar{z} = 0$ with $w(e^{-i\pi/4}) = 0$ and such that w is the identity operator on the set $C \bigcap L$.

3. (10 points) For the function
$$f(z) = (z^2 + 1)^{-1/2},$$
   (a) Find the Taylor series expansion about $z = 0$, determine its radius of convergence, and explain why it has this value.
   (b) Fine the Laurent expansion of $f(z)$ about $z = 0$ valid for $|z| > 1$. Why is there no branch cut in this Laurent series?

4. (10 points) How many zeros does the function $z^8 - 4z^6 + z^2 - 1$ have inside the unit circle?

5. (20 points) Use contour integration to evaluate the following integrals. Please explain carefully each step of the method used for the evaluation.
   (a) $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}\,dx$
   (b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(1+\epsilon\cos\theta)^2}, \quad (0 < \epsilon < 1)$