Ordinary Differential Equations Preliminary Examination
January 2003

1. Experimental observations show that when a spherical drop in quiescent air evaporates, the rate at which the drop loses volume is proportional to its surface area. Suppose a drop with an initial radius $R = 1$ mm is is found after 1 minute to have a radius of $R = 0.9$ mm. Write down an equation for the dependence of the radius R on time, and determine the evaporation time; i.e., the time it takes for the droplet to totally evaporate.

2. Solve the problem
$$\frac{dy}{dx} + \tanh x\, y = \operatorname{sech} x\,, \quad y(0) = 0\,.$$

3. Determine the eigenvalues and eigenfunctions of the following boundary value problem:
$$\frac{d^2y}{dx^2} + \lambda y = 0\,, \quad \text{with} \quad \frac{dy}{dx}(0) = 0 \quad \text{and} \quad y(b) = 0\,.$$

4. Solve the following equation:
$$x\frac{dy}{dx} = -y + y^2\,.$$

5. Solve the following equation:
$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x} + \frac{1}{2}x\,.$$

6. Find the solution for the system of equations
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
with $x(0) = 3$ and $y(0) = 0$.

7. Find and classify all critical points of the system of equations
$$\begin{aligned}\frac{dx}{dt} &= 4x - 2x^2 + xy\,,\\ \frac{dy}{dt} &= 4y + xy - 2y^2\,.\end{aligned}$$