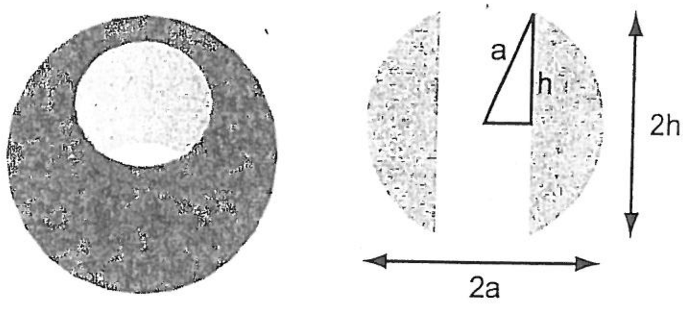
# Calculus Prelim, January 2009

1. A cylindrical hole of length $2h$ is drilled in a sphere of radius $a$. What is the volume of the remaining material?

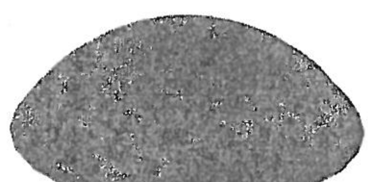



2. Suppose a gaseous spherical star of radius $a$ has a density $\rho = \rho_0 (1 - R^2/a^2)$, where $R$ is the distance from the center. Find its moment of inertia about a diameter. Does a star with the same mass, but constant density, have a larger or a smaller moment of inertia, and why?

3. Suppose NU proposes to cover its football stadium with a spherical shell of the form

$$(z + \frac{a}{2})^2 + x^2 + y^2 = a^2$$

with $z \geq 0$ (see below). Calculate the surface area of this shell.



4. A straight contour $C$ in the plane, connecting two points $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$, can be parametrized as $C = \{\mathbf{r}(t) = (1-t)\,\mathbf{r}_1 + t\,\mathbf{r}_2;\quad 0 \leq t \leq 1\}$. For a vector field $\mathbf{F}(\mathbf{r}) = -y\mathbf{i} + x\mathbf{j}$, show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = x_1 y_2 - x_2 y_1.$$

Now we define a new contour $C_p$ of a polygon $P$ consisting of straight lines that connect $N$ points $\mathbf{r}_1$,...,$\mathbf{r}_N$, i.e. the polygon has $N$ sides (Draw this and make sure you understand the geometry). Use Green's theorem to show that the area of $P$ for *any* polygon is given by

$$A = \frac{1}{2}\left[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + ... + (x_N y_1 - x_1 y_N)\right].$$

5. Suppose

$$\vec{F} = \frac{-y\hat{\imath} + x\hat{\jmath}}{x^2 + y^2}.$$

(a) Is $\vec{F}$ conservative? If so, find the function $f(x, y)$ such that

$$\vec{F} = \vec{\nabla} f\,?$$

(b) Suppose that $A(x_1, y_1) = (r_1, \theta_1)$ and $B(x_2, y_2) = (r_2, \theta_2)$ are two points in the right half-plane $x > 0$ and that $C$ is a smooth curve from $A$ to $B$. Explain why it follows that

$$\int_C \vec{F} \cdot d\vec{r} = \theta_2 - \theta_1\,.$$

(c) Suppose that $C_1$ is the upper half of the unit circle from $(1, 0)$ to $(-1, 0)$ and that $C_2$ is the lower half, oriented also from $(1, 0)$ to $(-1, 0)$. Show that

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \pi \quad \text{whereas} \quad \int_{C_2} \vec{F} \cdot d\vec{r} = -\pi\,.$$

Why does this not contradict results on path independence?

6. Suppose that the temperature $T(x, y, z, t)$ satisfies the diffusion equation

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

inside a volume $V$, where $D$ is a constant, and also that $T = 0$ on $S$, the boundary of $V$. Show that

$$\frac{d}{dt}\iiint_V T^2 dV = -\iiint_V 2D\left|\vec{\nabla}T\right|^2 dV\,.$$

[Hint: first use the diffusion equation and a vector identity to show that

$$\frac{\partial}{\partial t}\left(T^2\right) = 2DT\nabla^2 T = 2D\left\{\vec{\nabla}\cdot\left(T\vec{\nabla}T\right) - \left|\vec{\nabla}T\right|^2\right\}.]$$

7. Use Stokes' theorem to evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y^2\hat{\imath} + z^2\hat{\jmath} + x^2\hat{k}$ and where $C$ is the intersection of the plane $z = y$ and the cylinder $x^2 + y^2 = 2y$, oriented counterclockwise when viewed from above.