/2002

Complex Variables Preliminary Examination

1. (a) Determine the value of

$$\oint_{|z|=1} \frac{(e^z)^z}{z-2}\, dz\,,$$

(b) and the value of

$$\oint_{|z|=3} \frac{(e^z)^z}{z-2}\, dz\,.$$

2. Evaluate the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\sqrt{x}dx}{1+x^2}$

(b) $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x^2+1}\, dx$

(c) $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$

3. Suppose $P_n(z)$ is an nth order polynomial ($n \geq 2$) with simple zeros z_k, $k = 1, \ldots, n$. Show that

$$\sum_{k=1}^{n} \frac{1}{P_n'(z_k)} = 0\,.$$

Hint: Evaluate $\int_C 1/P_n(z)\, dz$ on some contour C.

4. Find the Laurent series expansion of

$$f(z) = \log\left(\frac{1+z}{1-z}\right)$$

about $z = 0$ that is valid for large z. What is the radius of convergence of this series? Explain how the branch cuts of this function should be placed to be consistent with the Laurent expansion.

5. By using the conformal map $w = \ln z$ solve the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0, \qquad 1 < x^2 + y^2 < 4,\ y > 0\,, \\ \frac{\partial u}{\partial n} &= 0, \qquad x^2 + y^2 = 1,\ y > 0\,, \\ \frac{\partial u}{\partial n} &= 0, \qquad x^2 + y^2 = 4,\ y > 0\,, \\ u &= 0, \qquad 1 < x < 2,\ y = 0\,, \\ u &= 1, \qquad -2 < x < -1,\ y = 0\,. \end{aligned}$$

(Note: the notation $\partial u/\partial n$ means the directional derivative *normal* to the given curve.)

Complex Variables Preliminary Examination

January 2001

1. (10 points) Let $r_1, \ldots, r_n$ be distinct n^{th} roots of unity. Show

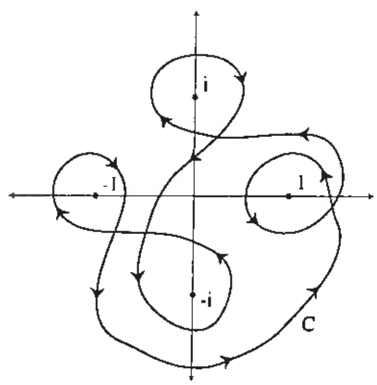
$$\prod_{k=1}^{n} r_k = (-1)^{n+1}.$$

2. (10 points) Let $f(t)$ be a continuous, complex valued function defined on the interval $[0,1]$, and define the function $g(z)$ by

$$g(z) = \int_0^1 f(t)e^{tz}\,dt, \qquad (z \in \mathbb{C})$$

Show that $g(z)$ is analytic in the entire complex plane.

3. (10 points) Evaluate the integral $\displaystyle\int_C \frac{f(z)}{z^4-1}\,dz$ where $f(z)$ is analytic and C is the contour shown in the figure below. The arrows indicate the direction of the path.



4. (10 points) Find a linear fractional transformation $w(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$ that maps the unit circle C onto the line L given by $z + i\bar{z} = 0$ with $w(e^{i\pi/4}) = 0$ and such that w is the identity operator on the set $C \bigcap L$.

Winding # ?

5. (10 points) Use the Argument Principle to determine how many roots $e^z + z = 0$ has in the strip $|Im(z)| \le \frac{\pi}{2}$.

Complex Variables Preliminary Examination
January 13, 2000

1. (10 points) Find the four roots of the quartic equation

$$z^4 + z^2 + 1 = 0\,.$$

Express each answer both in the form $re^{i\theta}$ and $a + ib$.

2. (15 points) Evaluate

$$\oint_C \frac{1}{z^2(z^2+4)}\,dz$$

where the closed contour C is given as:

(a) $|z+1| = 2$ (b) $|z-1| = 2$ (c) $|z| = 3$

3. (15 points) Let $f(z)$ be analytic in the annular domain $1 \leq |z| \leq 3$. with $f(2i) = -3\pi i$. If it is known that

$$\oint_{|z|=1} \frac{f(z)}{z-2i}\,dz = 2\,.$$

(a) What can be said about the analyticity of $f(z)$ for $0 \leq |z| < 1$?

(b) Determine the value of

$$\oint_{|z|=3} \frac{f(z)}{z-2i}\,dz\,.$$

$\boxed{4.}$ (20 points) Find three separate Laurent expansions of the function

$$\frac{1}{(z+1)(z-2)}$$

about the point $z = 0$: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$. Make sure that you give a general formula for each term in the series (Hint: use partial fractions).

5. (20 points) Suppose a complex function $f(z)$ is written in terms of polar coordinates using $z = re^{i\theta}$ in the form $f(z) = U(r,\theta) + iV(r,\theta)$. Suppose further that it is desired that this function be an entire function whose real part has the form

$$U(r,\theta) = r^k \sin 2\theta\,.$$

(a) Determine the admissible values of k.

(b) For each admissible k, find $V(r,\theta)$ such that $f(3) = i$.

Residues $\boxed{6.}$ (20 points) Use contour integration to evaluate the following integrals. Please explain carefully each step of the method used for the evaluation.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

✓(b) $\displaystyle\int_0^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx \quad (a > b > 0)$ pg. 214 # 1

(c) $\displaystyle\int_0^{\infty} \frac{x^{1/2}}{x^2+x+1}\,dx$

Complex Variables Exam Thursday, January 7, 1999 (10:00 – 12:00)

Please show all work. To get full credit for a problem you need to **CLEARLY** describe your calculations.

1. Suppose the functions

   (a) $3x^2y - y^3$,

   (b) $\cos(x)\cosh(y)$,

   of real variables are the real parts of analytic functions $f(z)$. Find the imaginary parts and the complex analytic function $f(z)$ itself.

2. Determine where $f'(z)$ exists and provide an expression for it.

   (a) $f(z) = \theta^2 - 2ir, \quad 0 \leq r < \infty, \quad 0 < \theta \leq 2\pi.$

   (b) $f(z) = e^{1/(z-1)}$

3. Let $f(z)$ be analytic. Use Cauchy's Theorem to show that

$$f(w) = \frac{1}{2\pi}\int_0^{2\pi} f(w + \pi e^{i\theta})d\theta \ .$$

4. Determine the value of

$$\oint_C f(z)dz \ ,$$

   if $f(z) = e^z g(z)/[(z-1)(z+3)(z-2)]$, where $g(z)$ is analytic for $|z| \leq 3$ and

   (a) C: $z = i + 2e^{i\theta}$, $0 \leq \theta \leq 2\pi$

   (b) C: $z = e^{i\pi/4} + e^{i\theta}$, $0 \leq \theta \leq 2\pi$

5. Use complex variables to evaluate

$$\text{(a)} \quad \int_0^\infty \frac{x \sin x}{1 + x^2}\, dx. \qquad \text{(b)} \quad \int_0^\infty \frac{x^{-1/10}}{1 + x}\, dx.$$

6. Find the Laurent expansion of the function

$$\cos \frac{z}{z+1}$$

   about the point $z = -1$.

Complex Variables Exam **Thursday, January 8, 1998 (12:00 – 2:00 p.m.)**

Please show all work. To get full credit for a problem you need to **CLEARLY** describe your calculations.

1. **(4×5=20 points)** Compute all possible values of the following

$$(-2+2i)^{1/4}, \quad 1^i, \quad \log(3), \quad \sin(\pi).$$

2. **(10+10=20 points)** Determine where $f'(z)$ exists and provide an expression for it.

   (a) $f(z) = (x^3 + 3xy^2 - 3x) + i(y^3 + 3x^2y - 3y)$,

   (b) $f(z) = e^{ix}e^{-|y|}$.

3. **(20+20=40 points)** Use complex variables to evaluate the following integrals

$$\text{(a)} \quad \int_0^\infty \frac{dx}{x^4 + 2x^2 + 1}, \qquad \text{(b)} \quad \int_0^\infty \frac{\ln x}{x^2+1}\, dx.$$

4. **(20 points)** Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D.
   Suppose that $u^2(x,y) + v^2(x,y) = 1$ for all $z = x + iy$ in D.
   Show that $f(z)$ is constant in D.
   (**Hint:** Show that $f'(z) = 0$ in D.)

5. **(10+10=20 points)** Suppose

$$f(z) = \frac{g(z)}{(z-2)(z-4)^2},$$

   where $g(z)$ is an entire function (i.e., analytic everywhere). Determine the value of

$$\oint_C f(z)dz,$$

   where (a) $C: \ z = 5 + 2e^{i\theta}$, $0 \le \theta \le 2\pi$; (b) $C: \ z = 5 + 4e^{i\theta}$, $0 \le \theta \le 2\pi$.

6. **(10+10+10=30 points)** Find three separate Laurent expansions of the function

$$\frac{1}{(iz+1)(z-2)}$$

   about the point $z = 0$: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$.

(total 150 points)

Preliminary Examination - Complex Variables
January 1997

1. Find all the values of $(-8)^{1/6}$, giving their real and imaginary parts, and plot them in the complex plane.

2. Evaluate each of the following integrals. Explicitly draw the contours employed in the integration and give all intermediate steps. Explain why a contribution vanishes if that should be the case.

a.

$$\int_C \frac{e^z}{z(z^2+9)}\,dz$$

with C denoting the path connecting the points (2,2), (-2,2), (-2,-2), and (2,-2) in counterclockwise direction.

b.

$$\int_C \frac{z^*}{z^2}\,dz$$

with C denoting a counterclockwise circular path around $z=0$ with radius 1. (z^* denotes the complex conjugate of z).

c.

$$\int_0^\infty \frac{x^{1/2}\log x}{(1+x)^2}\,dx$$

d.

$$\int_0^\infty \frac{x^2+1}{x^4+1}dx$$

4.

Calculate the inverse Fourier transform of

$$\hat{f}(\omega) = \frac{\omega}{(\omega+i\delta)^2}$$

i.e. calculate

$$f(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\hat{f}(\omega)\,d\omega.$$

Preliminary Examination - Complex Variables
March 1997

1. Verify that

$$u = \sin x \cosh y + 2 \cosh x \sinh y + x^2 - y^2 + 4xy$$

satisfies Laplace's equation and find $f(z)$, if $f(z)$ is a regular function whose real part is equal to u.

2. Show that
$$\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{a\pi}{(\sqrt{(a^2 - 1)})^3} \quad (a > 1)$$

3. Evaluate
$$\int_C \frac{dz}{(1 + z^4)}$$
where C is the ellipse $x^2 - xy + y^2 + x + y = 0$.

4. Calculate all values of $i^{(i)^{1/2}}$ and write them in the form $Re^{i\phi}$ with R and ϕ real.

5. Calculate the inverse Laplace transformation of $[1 + (z + 1)^2]^{-1}$; i.e. calculate the integral
$$\frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{zt} \frac{1}{1 + (z + 1)^2} dz$$
for arbitrary real values of t, where $\delta > 0$. Explain why should δ be chosen positive.

6. Find three separate Laurent expansions of the function
$$\frac{1}{(iz + 1)(z - 2)}$$
about the point $z = 0$: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$.
Hint: use partial fractions first and make use of the series
$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1.$$

COMPLEX VARIABLES PRELIMINARY EXAM - 1996

1. Calculate all values of

   (a) $\dfrac{(1+i)^5 - 1}{(1+i)^5 + 1}$ (b) $(-2+2i)^{1/3}$.

   (c) Is the function $f(z) = \dfrac{z-i}{1-i\bar{z}}$ analytic? Show why or why not. (Here $\bar{z}$ is the conjugate of z).

   (d) Consider the harmonic function $u(x,y) = e^x \cos y$. Find $v(x,y)$ so that $f = u + iv$ is an entire function of $z = x + iy$ and $f(0) = 1$.

2. Classify the singularities of each function:

   (a) $f(z) = e^{\frac{z}{z+1}}$ (b) $f(z) = \dfrac{1}{\sinh z}$

   (c) $f(z) = \dfrac{z}{z^3+1}$ (d) $f(z) = z^{1/2}e^z$.

3. Express $f(z) = \dfrac{5+z}{4z^3 - z^5}$ as two different infinite series in powers of z, one expanded about $z = 0$ and the other expanded about $z = \infty$. What is the radius of convergence of each?

4. Evaluate

   (a) $\displaystyle\int_0^\infty \frac{1}{x^4+1}dx$ (b) $\displaystyle\int_0^\infty \frac{dx}{x^{1/2}(1+x)}dx$ (c) $\displaystyle\int_0^\infty \frac{\cos kx}{x^2+1}dx$.

5. By using the conformal map $w = \ell n(z)$ solve the boundary value problem for $T = T(x,y)$,

$$\nabla^2 T = 0 \ , \qquad 1 < x^2 + y^2 < 4 \ , \qquad y > 0$$

$$T = 0 \ , \qquad x^2 + y^2 = 1 \ , \qquad y > 0$$

$$T = 1 \ , \qquad x^2 + y^2 = 4 \ , \qquad y > 0$$

$$\frac{\partial T}{\partial y} = 0 \ , \qquad 1 < x^2 < 4 \ , \qquad y = 0 \ .$$

1995

Complex Variables Preliminary Examination

1. Find the four roots of the quartic equation

$$z^4 + z^2 + 1 = 0\,.$$

2. Evaluate

$$\oint_C \frac{\sin z}{z^2(z^2+4)}\,dz$$

where the closed contour C is given as:

(a) $|z+1| = 2$ (b) $|z-1| = 2$ (c) $|z| = 3$

3. Let $f(z)$ be analytic in the annular domain $1 \leq |z| \leq 3$, with $f(2i) = -3\pi i$. If it is known that

$$\oint_{|z|=1} \frac{f(z)}{z-2i}\,dz = 2\,,$$

(a) What can be said about the analyticity of $f(z)$ for $0 \leq |z| < 1$?

(b) Determine the value of

$$\oint_{|z|=3} \frac{f(z)}{z-2i}\,dz\,.$$

4. It is desired that $f(z) = U(r,\theta) + iV(r,\theta)$ be an entire function whose real part has the form

$$U(r,\theta) = r^k \sin 2\theta$$

(a) Determine the admissible values of k.

(b) For each admissible k, find $V(r,\theta)$ such that $f(3) = i$.

5. Find three separate Laurent expansions of the function

$$\frac{1}{(z+1)(z-2)}$$

about the point $z = 0$: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$. Make sure that you give a general formula for each term in the series (Hint: use partial fractions).

6. Evaluate the following integrals:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

   (b) $\displaystyle\int_{0}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx$

   (c) $\displaystyle\int_{0}^{\infty} \frac{x^{1/2}}{x^2+x+1}\,dx$

1995

Complex Variables Preliminary Examination

1. Evaluate:

   (a) $(i)^{1/2}$ (b) $(-i)^{1/3}$

2. Determine where $f'(z)$ exists and provide an expression for it:

   (a) $f(z) = |z|^2$

   (b) $f(z) = (r^2 \cos 2\theta - 2r \sin\theta) + i(r^2 \sin 2\theta + 2r\cos\theta)$

3. For the function

$$f(z) = (z^2 + 1)^{-1/2},$$

   (a) Find the Taylor series expansion about $z = 0$, determine its radius of convergence, and explain why it has this value.

   (b) Find the Laurent expansion of $f(z)$ about $z = 0$ valid for $|z| > 1$. Why is there no branch cut in this Laurent series?

4. Evaluate the following integrals:

   (a) $$\int_0^\infty \frac{dx}{x^2 + 3x + 2}$$

   (Hint: introduce a factor $\log z$ into the numerator)

   (b) $$\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

   (Hint: convert it to an integral around the circle $|z| = 1$)

5. Evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{4x^2 - 1}\, dx$$

6. Use the conformal mapping $z = e^w$ to solve the boundary value problem for the harmonic function $u(x, y)$ which satisfies the boundary conditions

$$u = 0 \quad \text{on} \quad x^2 + y^2 = 1, \quad u = 1 \quad \text{on} \quad x^2 + y^2 = 4, \quad \text{both for } y > 0;$$

$$u_y = 0 \quad \text{on} \quad y = 0, \quad 1 \le |x| \le 2.$$

1994

Complex Variables Preliminary Examination

1. Determine where $f'(z)$ exists and provide an expression for it:

   (a) $f(z) = 2x(1-y) + i(x^2 - y^2 + 2y)$

   (b) $f(z) = \vartheta^2 - 2ir$

2. Evaluate the following contour integrals:

$$a) \oint_C \bar{z}\, dz \qquad C: x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$$

$$b) \oint_C \frac{dz}{z^4 + 2z^2 + 1} \qquad C: x = \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$$

3. Use the conformal mapping $z = e^w$ to help solve the boundary value problem

$$U_{xx} + U_{yy} = 0, \quad -\infty < x < \infty, \ \ y > 0;$$

$$U(x,0) = H(x), \quad -\infty < x < \infty$$

   Note: H(x) is the Heaviside function.

4. For the function $f(z) = (z^2 + 1)^{1/2}$:

   (a) What is the Taylor series expansion of $f(z)$ about $z = 0$? What is its radius of convergence?

   (b) What is the Laurent expansion of $f(z)$ which is valid for large values of $|z|$? What is its radius of convergence? Does this expansion have a branch cut? Why or why not?

5. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx$$

   Hint: $\cos x$ is the real part of e^{ix}.

6. Suppose

$$F(z) = \mathrm{Log}\left(\frac{z+1}{z}\right) = \mathrm{Log}(z+1) - \mathrm{Log}(z)\,,$$

   where $\mathrm{Log}(z)$ means the principal branch of the logarithm function.

   (a) Show that the branch cuts of $\mathrm{Log}(z+1)$ and $\mathrm{Log}(z)$ can be chosen so that F(z) only has a branch cut between $z = -1$ and $z = 0$ on the negative real z axis.

   (b) By closing the contour to the left, evaluate

$$f(t) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{zt}\, F(z)\, dz$$

   for $t > 0$.