# PRELIMINARY EXAM – DIFFERENTIAL EQUATIONS 3/07

Note - all problems count equally

1. Consider the differential equation

$$(x^2-1)\frac{dy}{dx}+y^2=0. \tag{1}$$

   (a) Solve (1) for initial condition $y(0)=1$.

   (b) Indicate the range of $x$ values for which the solution in (a) is valid.

2. Find all values of $\lambda$ for which the boundary value problem

$$\frac{d^2u}{dt^2}+\lambda\frac{du}{dt}+2u=0, \quad u(0)=0, \quad u(1)=0,$$

   has a non-trivial solution, and write the corresponding solution.

3. Find the general solution of the equation

$$-\frac{1}{4}x^2\frac{d^2y}{dx^2}-\frac{1}{4}x\frac{dy}{dx}+y=0,$$

   given that $y=x^2$ is a solution.

4. Find the general solution of the following system of equations:

$$\frac{dx}{dt}=3x+4y+25t,$$
$$\frac{dy}{dt}=4x-3y-25t.$$

5. Consider the initial value problem

$$y'=-\alpha y+f_n(t) \quad y(0)=y_0, \tag{2}$$

   where $\alpha>0$ and

$$f_n(t)=\cos^n t.$$

   where $n$ is a positive integer.

(a) Show that $f_n(t)$ is periodic with period $T_n$ where $T_n = \pi$ if $n$ is even and $T_n = 2\pi$ if $n$ is odd.

(b) Determine the solution to (2) in terms of integrals of known functions. It is not necessary to actually perform the integration.

(c) Show that (2) can have at most one solution with period $T_n$.

(d) Determine the unique periodic solution to (2) in terms of integrals of known functions. It is not necessary to perform the integration.

(e) Show that when $\alpha = 0$ all solutions are periodic if $n$ is odd and no solutions are periodic if $n$ is even.

6. Consider the initial value problem

$$y' = y^n, \quad y(0) = -1,$$

where $n$ is a positive integer.

(a) Determine the solution for all $n > 0$.

(b) Show that $| \, y(t) \, |$ is bounded when $n$ is even and unbounded when $n$ is odd.

7. Find a particular solution to the equation

$$y'' + 4y = \cos^2 t.$$

Hint: Remember the trigonometric identity

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

for all $a$ and $b$.

8. Consider the system

$$\begin{aligned} x' &= (2 + x)(y - x), \\ y' &= (4 - x)(x + y). \end{aligned}$$

Determine the type of each critical point and its stability.