

Complex Variables Preliminary Examination

January 10, 2005

1. Find three separate Laurent expansions of the function

$$\frac{1}{(iz+1)(z-2)}$$

about the point $z = 0$: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$.

2. Suppose

$$f(z) = \frac{H(z)}{(z^2 - 1)(z - 2)^2},$$

where $H(z)$ is an entire function. Determine the value of

$$\oint_C f(z) dz,$$

where C is taken counterclockwise around the circle

(a) $|z - 5| = 2$

(b) $|z - 5| = 5$

3. Use contour integration to evaluate the following integral. Please explain carefully each step of the method used for the evaluation.

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx \quad (1)$$

4. Evaluate the following integral, explaining carefully each step of the method used. You needn't simplify the answer.

$$\int_0^{2\pi} \frac{\cos \theta}{2 + \cos \theta} d\theta. \quad (2)$$

5. Evaluate the following integral, explaining carefully each step of the method used.

$$\int_0^{\infty} \frac{x^{1/2}}{1+x^2} dx. \quad (3)$$

6. Classify all complex singularities of the following two functions

$$(a) \frac{\log z}{(z+1)^2} \quad (b) \frac{e^{-\frac{1}{z}}}{\sin z} \quad (4)$$

(i.e., locate all singularities and determine whether each is a branch point, a pole (giving its order), or an essential singularity).

7. By any method, find the harmonic conjugate of

$$u(x, y) = xe^{-y} \cos x - ye^{-y} \sin x.$$

Of what complex analytic function $f(z)$ is this the real part?

8. (a) Find a conformal map, $w = f(z)$ ($z = x + iy$, $w = \xi + i\eta$), that transforms the first quadrant of the complex plane ($x, y \geq 0$) into the strip $-\infty < \xi < +\infty$, $0 \leq \eta \leq \frac{\pi}{2}$.
(b) Use this conformal map to solve Laplace's equation $\nabla^2 T = 0$ with boundary conditions $T = 0$ for $y = 0$, $x > 0$, and $T = 1$ for $x = 0$, $y > 0$.