Linear Algebra Exam   January 4, 1996

Please show all work. To get full credit for a problem you need to **CLEARLY** describe your calculations.

1. (15 pts.) Find the inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

2. (25 pts.) Answer the following questions *true* or *false* with a reason if true or a counterexample if false:

   (a) If the vectors $x_1, \ldots, x_m$ span a subspace S, then the dimension of $S = m$.

   (b) If $Ax = Ay$, then $x = y$.

   (c) If a square matrix A has independent columns, so does A^2.

   (d) The eigenvalues of a Hermitian matrix are all real.

   (e) The matrix exponential function satisfies the property

$$e^A e^B = e^{A+B}.$$

3. (20 pts.) Consider

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (a) Factor $A = QR$ where Q is orthogonal and R is upper triangular.

   (b) Use part (a) to find the solution x which minimizes $||Ax - b||^2$.

4. (20 pts.) The Ribonacci numbers satisfy the recursion relationship

$$R_{k+2} = R_{k+1} + 2R_k \qquad R_0 = 0,\ R_1 = 1\ .$$

(a) Write the recursion relationship in the form

$$U_{k+1} = AU_k$$

where

$$U_k = \begin{bmatrix} R_{k+1} \\ R_k \end{bmatrix}$$

and A is a matrix.

(b) Find a matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

(c) Compute a formula for R_k as the linear combination of two numbers raised to the kth power. Check your answer by computing R_5 using the recursion relationship and your formula.

5. (20 points) Consider a quadratic function

$$y = a_0 + a_1 x + a_2 x^2$$

(a) Find $y(x)$ that contains the points $(x, y) = (0, 4),\ (1, 1),\ (2, 0)$ by writing three linear equations for a_0, a_1, a_2 and solving.

(b) Find all quadratic functions that pass through the points $(1, 1), (2, 0)$ by writing two equations for a_0, a_1, a_2 and reducing them to upper echelon form.

(c) Find a condition on y_0, y_1, y_2, y_3 such that a quadratic can be found that passes through the points $(0, y_0), (1, y_1), (2. y_2), (3, y_3)$.

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1. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$A_{11} = (-1)^2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \qquad A_{12} = (-1) \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2 \qquad A_{13} = 1$

$A_{21} = (-1) \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2 \qquad A_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \qquad A_{23} = -1 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2$

$A_{31} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \qquad A_{32} = -1 \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = -2 \qquad A_{33} = 3$

$\det A = 2(3) - 2(1) = 6 - 2 = 4$

$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

2. a. $\dim(S) = m$ if $x_1, \ldots, x_m$ (which span) are also linearly independent.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ span $\mathbb{R}_2$ but $\dim(\mathbb{R}_2) = 2 \neq 3$

false

b. $Ax = Ay$

$x = A^{-1}Ay = Iy \Rightarrow x = y$ true

c.

d. true; $A = A^H$

$(Ax = \lambda x)x^H \qquad \underbrace{x^H A x}_{\in \mathbb{R}} = \lambda \underbrace{\|x\|^2}_{\in \mathbb{R}} \Rightarrow \lambda \in \mathbb{R}$

e. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad e^A e^B = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}$

$= I + A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad e^B = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$

e. $e^A e^B$

$$= \left(I + A + \frac{A^2}{2!} + \dots\right)\left(I + B + \frac{B^2}{2!} + \dots\right)$$

$$\approx (I + A)(I + B) = I^2 + IB + AI + AB$$

$$= I + A + B + AB$$

$$e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \dots$$

$$= I + A + B + \frac{A^2 + AB + BA + B^2}{2!} + \dots$$

$$\stackrel{?}{=} \left(I + A + \frac{A^2}{2!}\right)\left(I + B + \frac{B^2}{2!}\right)$$

$$= I + IB + \frac{IB^2}{2!} + IA + AB + \frac{AB^2}{2!} + \frac{A^2 I}{2!} + \frac{A^2 B}{2!} + \frac{A^2 B^2}{2!}$$

False!

3. $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$ $\quad a = \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}$ $\quad b = \begin{Bmatrix} 1 \\ 3 \\ 1 \end{Bmatrix}$

$$q_1 = \frac{1}{3}\begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} \qquad b' = b - (q_1^T b)\, q_1$$

$$= \begin{Bmatrix} 1 \\ 3 \\ 1 \end{Bmatrix} - \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$q_2 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \qquad = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$A = QR = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

Least Squares

$$Ax = b$$
$$A^T A x = A^T b$$
$$x = (A^T A)^{-1} A^T b$$
$$= \left[(QR)^T (QR) \right]^{-1} (QR)^T b$$
$$= \left[R^T Q^T Q R \right]^{-1} (R^T Q^T) b = R^{-1} \underbrace{Q^{-1} (Q^T)^{-1}}_{I} \underbrace{(R^T)^{-1} R^T}_{I} Q^T b$$
$$= R^{-1} Q^T b$$
$$= \frac{1}{3\sqrt{2}}\begin{bmatrix} \sqrt{2} & -3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}\begin{Bmatrix} 1 \\ 3 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 5/9 \end{Bmatrix}$$