PRELIMINARY EXAM - LINEAR ALGEBRA - 1/03

Please show all work. To get full credit for a problem you need to show your work and clearly describe your calculations.

1. (20 points)

   (a) Given

   $$\mathrm{Ax} = \mathrm{b},\ \mathrm{x} \in \mathcal{R}^m,\ \mathrm{b} \in \mathcal{R}^n\ , n > m, \text{ where A is an } n \times m \text{ matrix,}$$

   show that the solution x_m with minimal error E, $E \equiv ||\mathrm{Ax} - \mathrm{b}||^2$ satisfies

   $$\mathrm{A}^T\mathrm{A}\mathrm{x}_m = \mathrm{A}^T\mathrm{b}\ .$$

   (b) In the above sense, determine the best solution for

   $$\mathrm{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\ .$$

2. (20 points) The solution of the boundary-value problem

$$\frac{d^2 f}{dx^2} = 1\ ,\ \ 0 < x < 1\ , \tag{1}$$

with

$$f(0) = f(1) = 0\ ,$$

is easily constructed. Now you will show that a finite-difference approximation using central differences always has a solution.

Let the step size be h, $h = \dfrac{1}{N+1}$ where $x_i = ih$, $i = 1, ..., N$, are the mesh points. A finite-difference approximation of (1) is

$$\mathrm{A}_N \mathrm{f}_N = \mathrm{b}_N, \tag{2}$$

where f_N, $\mathrm{b}_N \in \mathcal{R}^N$, $\mathrm{f}_N = [f(x_1), ..., f(x_N)]^T$, $\mathrm{b}_N = -[h^2, ..., h^2]^T$ and A_N is the tridiagonal $N \times N$ matrix,

$$\mathrm{A}_N = \begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & 0 \\ -1 & 2 & -1 & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ 0 & \cdot & \cdot & 0 & -1 & 2 \end{bmatrix}\ .$$

(a) Calculate the determinate of $\mathbf{A}_4$.

(b) Show and explain why for $N = 4$, the system (2) has a solution .

(c) Obtain a recursion formula for the determinant D_N of $\mathbf{A}_N$ in terms of D_{N-1} and D_{N-2}. By looking at D_1, D_2, D_3 etc. find D_N as a function of N.

(d) Explain why (2) has a solution for any value of N.

3. (60 points) Consider the matrix $\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & Q & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where Q is a constant.

(a) What is the trace of $\mathbf{M}$; what is the determinant of $\mathbf{M}$?

(b) Find the eigenvalues, eigenvectors and generalized eigenvectors (if appropriate) for all values of Q.

(c) State in words the possibilities that

$$\mathbf{M}\mathbf{x} = \mathbf{f}$$

has a solution for a given vector $\mathbf{f}$. Then calculate the general solution of

$$\mathbf{M}\mathbf{x} = \mathbf{f}$$

for all values of Q for which a solution exists.

(d) For the above $\mathbf{M}$, compute $\mathbf{S} = \sin \mathbf{M}$ for $Q = 0$.