Linear Algebra Preliminary Exam 2007

1. Find all possible solutions to the system of equations $Ax = b$ where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 5 \\ 4 & -8 & 10 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

What are all possible solutions if A is unchanged but

$$b = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}?$$

In addition, find the rank of A, and the dimensions of and bases (if any) for each of the four fundamental subspaces associated with A (i.e., the column space, the row space, the nullspace and the left nullspace of A). For this A, what condition(s) on b must be true for $Ax = b$ to have solutions?

2. Consider the subspace $\mathbf{V}$ of $\mathbf{R}^3$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Are these vectors also a basis for $\mathbf{V}$? Why or why not? If they are not a basis, give one.

(b) What is a basis for the orthogonal complement of $\mathbf{V}$ (i.e., $\mathbf{V}^{\perp}$)?

3. Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p + q$, with p in the column space and q perpendicular to that space. Which of the four subspaces of A contains q?

4. Suppose $A = [\ a_1, \quad a_2, \quad a_3, \quad a_4\ ]$ is a (4x4) matrix with columns a_i, where det(A)=3. Find det(B) if

$$B = \begin{bmatrix} a_1 + 2a_2, & a_4, & -2a_3, & a_2 \end{bmatrix}.$$

5. Find α so that

$$\exp(P) = I + \alpha P,$$

for any projection matrix P.

6. Suppose there is an epidemic in which every month half of those who are well become sick, a quarter of those who are sick die, and a half of those who are sick survive and become immune. Find the general solution to the corresponding Markov process defined by

$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \\ i_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & 0 & 0 \\ 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \\ i_k \end{bmatrix}.$$

Here d_k, s_k, w_k and i_k are the fractions of people in month k who are dead, sick, well and immune. Does the solution eventually settle down to a steady-state? If so, and assuming initially everyone starts out well, what is the steady state? Does this make sense?