Introduction to asymptotic expansions

First Year Foundations Workshop

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Motivation

- Many mathematical problems (e.g. differential equations) cannot be explicitly solved
- Physical problems often have very small or very large parameters, compared to other parameters in the problem
- Asymptotics is an analytic method of approximation
  - Try to have a problem where unperturbed problem is easily solved
  - Perturbation may create interesting results
  - Idea: Solving one problem at independent scales

Asymptotic expansions: definitions

- Definition: Consider a function $f(\epsilon)$ and a "gauge" function $\phi(\epsilon)$. The function $f(\epsilon)$ is said to be $o(\phi)$ as $\epsilon \to \epsilon_0^+$ if for every positive δ there exists an ϵ_2 (independent of ϵ) such that:

$$|f(\epsilon)| < \delta |\phi(\epsilon)|; \qquad \epsilon_0 < \epsilon < \epsilon_2$$

- A sequence of $\phi_j(\epsilon)$ (where $j \in \mathbb{N}$) is an *asymptotic sequence* as $\epsilon \to \epsilon_0^+$ iff $\phi_n(\epsilon) = o(\phi_m(\epsilon))$ as $\epsilon \to \epsilon_0^+$ for all m and n such that $m < n$.
- Suppose that $\phi_j(\epsilon)$ is an asymptotic sequence and $f(\mathbf{x};\epsilon)$ is a function. Then, $f(\mathbf{x};\epsilon)$ has an *asymptotic expansion to n terms* with respect to the sequence, iff

$$f(\mathbf{x};\epsilon) = \sum_{k=1}^{m} a_k(\mathbf{x}) \phi_k(\epsilon) + o(\phi_m); \qquad m = 1, \ldots, n; \epsilon \to \epsilon_0^+$$

More concrete, please!

- Suppose that $\epsilon_0 = 0$ in all cases here–this "should" always be the case in asymptotics class
- The little-o notation is better considered by a quotient:

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{\phi(\epsilon)} = L$$

  - If $L = 0$, then $f = o(\phi)$ as $\epsilon \to 0$. If L is bounded in ϵ, then $f = O(\phi)$.
    - Example: If $\phi(\epsilon) = \epsilon$, then $f(\epsilon) = 0.5\epsilon$ is $O(\phi)$ but NOT $o(\phi)$, but $f(\epsilon) = 10000\epsilon^{1+.0000000001}$ is $o(\phi)$.
- Asymptotic sequence often used: $\phi_j(\epsilon) = \epsilon^{\alpha_j}$ such that $\alpha_{j+1} > \alpha_j$
  - For example: $\phi_j(\epsilon) = (1, \epsilon, \epsilon^2, \epsilon^3, \ldots)$

Expansion of polynomial

- Consider expansion of a polynomial, where $\epsilon \ll 1$:

$$x^2 + 2\epsilon x - 1 = 0$$

- We can solve this exactly: $x = -\epsilon \pm \sqrt{\epsilon^2 + 1}$, but want to demonstrate asymptotic expansion
  - As aforementioned, expand x as an asymptotic series, for some large N, where we solve for x_j

$$x = \sum_{j=0}^{N} \epsilon^{j\alpha} x_j$$

Expansion of polynomial (cont.)

- Substitute ansatz into the original quadratic equation:

$$\left(x_0 + \epsilon^{\alpha} x_1 + \epsilon^{2\alpha} x_2 + ...\right)^2 + 2\epsilon\left(x_0 + \epsilon^{\alpha} x_1 + \epsilon^{2\alpha} x_2 + ...\right) - 1 = 0$$

- Simplify and combine powers of ϵ:

$$\begin{aligned} \left[x_0^2 - 1\right]\epsilon^0 + \left[2x_0\right]\epsilon + \left[2x_0 x_1\right]\epsilon^{\alpha} + \left[2x_1\right]\epsilon^{\alpha+1} & \\ + \left[x_1^2 + 2x_0 x_2\right]\epsilon^{2\alpha} + \left[2x_2\right]\epsilon^{2\alpha+1} & \\ + \left[2x_1 x_2\right]\epsilon^{3\alpha} + \left[x_2^2\right]\epsilon^{4\alpha} + ... & = 0 \end{aligned}$$

- Since $\epsilon \neq 0$, that implies that all like powers of ϵ must vanish
- Start with $O(1)$ problem:

$$x_0^2 - 1 = 0$$

  - Obviously, $x_0 = \pm 1$

Expansion of polynomial (cont.)

- Substitute x_0 into the equation and simplify, after dividing by ϵ:

$$\begin{aligned} \pm [2] \pm [2x_1]\,\epsilon^{\alpha-1} + [2x_1]\,\epsilon^{\alpha} + \left[x_1^2 \pm 2x_2\right]\epsilon^{2\alpha-1} & \\ + [2x_2]\,\epsilon^{2\alpha} + [2x_1 x_2]\,\epsilon^{3\alpha-1} + \left[x_2^2\right]\epsilon^{4\alpha-1} + \ldots &= 0 \end{aligned}$$

  - Leading order: "2" must balance with lowest remaining order of ϵ (else, $\pm 2 = 0$ \`)
  - Since $\alpha > 0$, $\epsilon^{\alpha-1}$ is lowest order here and forces $\alpha = 1$
  - Therefore, the next equation:

$$\begin{aligned} \pm 2 \pm 2x_1 &= 0 \\ \therefore x_1 &= -1 \end{aligned}$$

Expansion of polynomial (cont.)

- Substituting in $\alpha = 1$ and $x_1 = -1$ and again dividing by ϵ:

$$[-1 \pm 2x_2] + [2x_2 + 2x_1 x_2]\,\epsilon + \left[x_2^2\right]\epsilon^2 + \ldots \quad = \quad 0$$

- We find $x_2 = \pm\frac{1}{2}$. Can continue in this way, but let's review:

$$x = \pm 1 - \epsilon \pm \frac{1}{2}\epsilon^2 + \text{h.o.t.}$$

- Consider the exact solution, where the square root is Taylor-expanded about $\epsilon = 0$:

$$x = -\epsilon \pm \left\{1 + \frac{\epsilon^2}{2} + \mathrm{h.o.t.}\right\}$$

  - They match, to the order ϵ^2 (and if we continued in this way, we would get the same thing at higher orders)!

Boundary value problem

- This technique is also used for solving boundary value problems
- Consider the following:

$$\begin{aligned} y''(x) + \epsilon y'(x) - y(x) &= 1 \\ y(0) = y(1) &= 1 \end{aligned}$$

- Take the ansatz $y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots$ and substitute. Realizing that $1 = 1\epsilon^0 + 0\epsilon^1 + 0\epsilon^2 + \ldots$, it is easy to see that we get these problems:

$$\begin{aligned} y_0''(x) - y_0(x) &= 1; \\ y_0(0) = y_0(1) &= 0 \end{aligned}$$

$$\begin{aligned} y_j''(x) - y_j(x) &= -y_{j-1}(x); \quad & j \geq 1 \\ y_j(0) = y_j(1) &= 0; & j \geq 1 \end{aligned}$$

Boundary value problem (cont.)

- Solve the ODE $y_0''(x) - y_0(x) = 1$. The homogeneous solutions are $e^{\pm x}$, but write instead with cosh and sinh due to the presence of $x = 0$
  - Solve inhomogeneous problem with method of undetermined coefficients or variation of parameters
  - General solution:

$$y_0(x) = A\cosh(x) + B\sinh(x) - 1$$

  - Subbing into boundary conditions (i.e. $y(0) = y(1) = 1$), find A and B and now have, after simplifying:

$$y_0(x) = -1 + 2\cosh(x) - 2\frac{e-1}{e+1}\sinh(x)$$

Boundary value problem (cont.)

- The next order:

$$\begin{aligned} y_1''(x) - y_1'(x) &= 1 - 2\cosh(x) + 2\frac{e-1}{e+1}\sinh(x) \\ y_1(0) &= 0 \\ y_1(1) &= 0 \end{aligned}$$

- Find particular solution first. Of course, this is a case of resonance
  - Can use undet. coefs. or variation of parameters
- Whichever way you choose, the non-duplicating part of the particular solution is (constants written in terms of e):

$$\begin{aligned} y_{1p}(x) &= -x\sinh(x) + \frac{e-1}{e+1}\cosh(x) - 1 \\ \therefore y_1(x) &= A\cosh(x) + B\sinh(x) + y_{1p}(x) \end{aligned}$$

Boundary value problem (cont.)

- Use the homogeneous boundary conditions to find A and B: found to be $A = B = \frac{2}{e+1}$. So:

$$y_1(x) = \frac{2}{e+1}\cosh(x) + \frac{2}{e+1}\sinh(x) - x\sinh(x) + \frac{e-1}{e+1}\cosh(x) - 1$$

- Therefore, the full solution is:

$$\begin{aligned} y(x) &= -1 + 2\cosh(x) - 2\frac{e-1}{e+1}\sinh(x) \\ &\quad + \epsilon\left\{\frac{2}{e+1}\cosh(x) + \frac{2}{e+1}\sinh(x)\right. \\ &\quad \left. + -x\sinh(x) + \frac{e-1}{e+1}\cosh(x) - 1\right\} + O\left(\epsilon^2\right) \end{aligned}$$

- This can be carried out to further orders as well...

Early preview of class

- Regular perturbations, thus, although annoying, are pretty straightforward
- Singular perturbations will be seen at the start of the class
  - Ex: In the polynomial $\epsilon x^2 + 2x - 1$, notice that using a regular perturbation, the $O(1)$ problem has only one solution, not two
  - Ex: In the BVP $\epsilon y'' + y' + y$ with $y(0) = \alpha$ and $y(1) = \beta$, the $O(1)$ problem cannot simultaneously satisfy the BCs (overdetermined)

Summary

- Functions can be expanded in asymptotic series
- Asymptotics allow analytic approximations to a function for small parameters
- Regular perturbations: ansatz is often $f(\mathbf{x};\epsilon) = \sum_{j=0}^{N} f_j(\mathbf{x})\,\epsilon^j$
  - It's a domino effect on solving the problems
- Singular perturbations will be discussed more in class

Resources

- M.H. Holmes, "Introduction to Perturbation Methods"
- C. Bender & S.A. Orszag, "Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory (v. 1)"
- Richard Rand's notes: http://audiophile.tam.cornell.edu/randdocs/nlvibe52.pdf