Differential Equations
ESAM Preliminary Examination
January 8, 2009, 10-12

1. Solve the following differential equations for $y(x)$:

(a). $$(x + e^y)\frac{dy}{dx} = -y,$$

(b). $$xy'' - 2y' = \frac{1-2y}{x}.$$

2. **(a).** Consider the undamped forced linear oscillator describing the small vibrations of a mass m connected to a wall via a spring

$$m\ddot{x} + kx = F_0 \cos \omega t,$$

where $-kx$ is the linear restoring force of the spring (Hooke's law), and $F_0 \cos \omega t$ is the external force. Let the forcing frequency ω equal the natural frequency of vibration $\omega_0 = \sqrt{k/m}$. What is the form of the solution of the problem in this case, which is referred to as *resonance*.

(b). Suppose that a damping term $-c\dot{x}$, $c > 0$ due to friction is also present so that the differential equation becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t.$$

The solution now consists of two parts: the transient part which decays to zero as $t \to \infty$, and the steady state part which does not decay. Rather, it represents a particular solution of the nonhomogeneous problem. Find this steady state solution.

(c). If $\omega = \omega_0$ in the problem corresponding to (b), will the solution blow up as in the undamped case? Describe the behavior of the solution if $\omega = \omega_0$.

3. One solution of the equation

$$x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + 2y = 0$$

is $y_1(x) = e^{2x}$. Solve the initial value problem

$$x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + 2y = 1, \quad y(1) = \frac{1}{2}, \quad y'(1) = -4.$$

4. Solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

5. Consider the two systems of differential equations

$$\dot{x} = 14x - \frac{1}{2}x^2 - xy, \quad \dot{y} = 16y - \frac{1}{2}y^2 - xy,$$

and

$$\dot{x} = 14x - 2x^2 - xy, \quad \dot{y} = 16y - 2y^2 - xy$$

governing two species $x(t)$ and $y(t)$ which compete for the same resources. Only one of these systems allows for the two species to coexist, while in the other system one of the species must die out. Determine which system allows coexistence. Clearly explain your reasoning, supported by computations.

6. The boundary value problem

$$y'' + y = f(x), \quad 0 < x < \pi, \quad y(0) = 0, \ y(\pi) = 0$$

may or may not have a solution, depending on the function $f(x)$.
(a). Determine condition(s) on $f(x)$ which, if satisfied, guarantee the existence of a solution.
(b). If $f(x)$ satisfies the condition(s), then how many solutions exist?
(c). If $f(x) = \sin(x - \mu)$, where μ is a real parameter, solve, if possible, the problem. Comment on uniqueness and solvability of the problem depending on μ.
You may find the trig identities

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

useful.