

Complex Variables Preliminary Examination

1. Find the four roots of the quartic equation

$$z^4 + z^2 + 1 = 0.$$

2. Evaluate

$$\oint_C \frac{z^2 z}{z^2(z^2 + 4)} dz$$

where the closed contour C is given as:

$$(a) |z + 1| = 2 \quad (b) |z - 1| = 2 \quad (c) |z| = 3$$

3. Let $f(z)$ be analytic in the annular domain $1 \leq |z| \leq 3$, with $f(2i) = -3\pi i$. If it is known that

$$\oint_{|z|=1} \frac{f(z)}{z - 2i} dz = 2,$$

- (a) What can be said about the analyticity of $f(z)$ for $0 \leq |z| < 1$?
 (b) Determine the value of

$$\oint_{|z|=3} \frac{f(z)}{z - 2i} dz.$$

4. It is desired that $f(z) = U(r, \theta) + iV(r, \theta)$ be an entire function whose real part has the form

$$U(r, \theta) = r^k \sin 2\theta$$

- (a) Determine the admissible values of k .
 (b) For each admissible k , find $V(r, \theta)$ such that $f(3) = i$.
5. Find three separate Laurent expansions of the function

$$\frac{1}{(z + 1)(z - 2)}$$

about the point $z = 0$: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$. Make sure that you give a general formula for each term in the series (Hint: use partial fractions).

6. Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

(b) $\int_0^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx$

(c) $\int_0^{\infty} \frac{x^{1/2}}{x^2+x+1} dx$