PRELIMINARY EXAM - ORDINARY DIFFERENTIAL EQUATIONS
WINTER, 2013

Note - all problems count equally

1. Solve the initial value problem

$$t\frac{dx}{dt} = x + t^3 \sin(\pi t^2), \quad x(1) = \frac{1}{\pi}.$$

2. Consider the differential equation

$$x'' + 4x = \sin(\omega t).$$

Determine the general solution for all $\omega > 0$.

3. Consider the equation

$$xy'' - (1+2x)y' + (1+x)y = 0, x > 0.$$

Determine the general solution. Hint: First try functions of the form $y = e^{ax}$.

4. Solve the initial value problem

$$x^2y'' - 2xy' + 2y = x^3, \quad 0 < x < \infty; \quad y(0) = 0, \; y'(0) = 2.$$

Is there anything unusual about this problem? Explain.

5. Solve the following eigenvalue problem,

$$y'' + \lambda y = 0, \quad 0 < x < 1; \quad y'(0) = 0, \; y'(1) - y(1) = 0.$$

Display the eigenvalues graphically if necessary.

6. Determine a constraint on the function $f(x)$ that allows the boundary value problem,

$$x^2y'' - 2y = f(x), \quad 0 < x < 1; \quad y(0) = 0, \; 2y(1) - y'(1) = 0,$$

to have a solution.

7. Consider the system

$$x' = xy^2 - x$$
$$y' = 2x - xy + 2 + y\,.$$

Find all critical points and determine their type and their stability. If any of the critical points turn out to be linear centers just state the possibilities for the nonlinear problem.

8. Consider the system

$$\begin{aligned} x' &= y^2, && (1)\\ y' &= x^2. && \end{aligned}$$

$$(2)$$

(a) Show that if $x(0) = y(0) = -1$, $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$.

(b) Show that if $x(0) = -1$ and $y(0) = -1.001$ both $x(t)$ and $y(t)$ are unbounded as t increases.