PRELIMINARY EXAM - LINEAR ALGEBRA 1/97
ALL PROBLEMS HAVE SAME WEIGHT

1. Consider the matrix A where

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \\ 4 & 8 & 2 \end{pmatrix}.$$

Find an LU factorization for A. Explicitly indicate all row operations and pivots. Using the LU factorization solve the problem $Ax = b$ where,

$$b = \begin{pmatrix} 6 \\ 10 \\ 2 \end{pmatrix}.$$

Explicitly indicate both the forward and backward steps in the solution process.

2. Consider a general $m \times n$ matrix A. Show that the number of linearly independent rows is the same as the number of linearly independent columns.

3. Find α so that

$$\exp(P) = I + \alpha P.$$

for any projection matrix P.

4. Let A be an $m \times n$ matrix. Assume that $m > n$ and that the rank of A is n.

   (a) Write down the general equation for the least squares solution to $Ax = b$.

   (b) Suppose that the columns of A are orthonormal. Write down an explicit formula for x in terms of the columns of A.

   (c) Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

   Determine all vectors which are in both the range of A and the left null space of A. Then find an orthonormal set of vectors from the columns of A.

5. (a) Find the determinant and all eigenvalues and eigenvectors associated with the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(b) Compute the determinant of the matrix

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & -3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

6. If the next number in a sequence is the average of the two previous numbers,

$$G_{k+2} = \frac{1}{2}(G_{k+1} + G_k),$$

and we define the vector

$$u_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix},$$

then u_k satisfies the difference equation

$$u_{k+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} u_k .$$

Using the general solution for u_k, find the general solution for G_k. Also, what is the long-term behavior in the specific case where $G_0 = 0$ and $G_1 = 1$?

PRELIMINARY EXAM: LINEAR ALGEBRA '9?

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \\ 4 & 8 & 2 \end{bmatrix}$

LU factorization:

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \\ 4 & 8 & 2 \end{bmatrix} \xrightarrow[-2r_1+r_3]{-2r_1+r_2} \begin{bmatrix} 2 & 3 & 4 \\ 0 & \overset{A_{22}'}{-1} & 2 \\ 0 & \underset{A_{32}'}{2} & -6 \end{bmatrix} \xrightarrow[2r_2+r_3]{} \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

$L_{11} = L_{22} = L_{33} = 1$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$

$L_{21} = \dfrac{A_{21}}{A_{11}}$, $L_{31} = \dfrac{A_{31}}{A_{11}}$

$L_{32} = \dfrac{A_{32}'}{A_{22}'}$

Check: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 10 \\ 4 & 8 & 2 \end{bmatrix}$

$Ax = b$ $\qquad b = \begin{Bmatrix} 6 \\ 10 \\ 2 \end{Bmatrix}$

$LUx = b$

$Lc = b$ $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 10 \\ 2 \end{Bmatrix}$

$c_1 = 6$

$2c_1 + c_2 = 10$

$2c_1 - 2c_2 + c_3 = 2$

$c = \begin{Bmatrix} 6 \\ -2 \\ -14 \end{Bmatrix}$

$c_2 = 10 - 2c_1 = -2$

$c_3 = 2 + 2c_2 - 2c_1$

$= 2 - 4 - 12 = -14$

$Ux = c$

$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -2 \\ -14 \end{Bmatrix}$

$2x_3 = -14$

$x_3 = -7$

$-x_2 + 2x_3 = -2$

$x_2 = 2x_3 + 2 = -14 + 2 = -12$

$2x_1 + 3x_2 + 4x_3 = 6$

$2x_1 - 36 - 28 = 6$

$x_1 = 35$

$x = \begin{Bmatrix} -35 \\ 16 \\ 7 \end{Bmatrix}$

2. Suppose A has r independent rows. Then,

$$A \sim \begin{bmatrix} a_1 & * & * & \cdots & * \\ & a_2 & & & \vdots \\ \text{zeros} & & & & a_r \end{bmatrix} = U$$

The first r rows are a basis for the row space A
now we show that the columns corresponding to the pivots are linearly independent, i.e, a basis for the column space of U.

$$c_1 \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} * \\ a_2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} * \\ * \\ a_3 \\ 0 \end{bmatrix} + \cdots + c_r \begin{bmatrix} * \\ \vdots \\ a_r \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

we want to show that $c_1 = c_2 = \cdots = c_r = 0$.
working backwards,

$$c_r a_r = 0 \implies c_r = 0 \quad (\text{since } a_r \neq 0).$$

$$c_{r-1} a_{r-1} + \cancel{c_r (*)} = 0 \implies c_{r-1} = 0 \quad (a_{r-1} \neq 0)$$

$$\vdots$$

$$c_1 a_1 + \cancel{c_2 (*)} + \cancel{c_3 (*)} + \cdots + \cancel{c_r (*)} = 0$$

$$c_1 a_1 = 0 \implies c_1 = 0 \quad (a_1 \neq 0)$$

$\therefore$ $c_1 = c_2 = \cdots = c_r = 0$ and the r columns are linearly independent.

3. $P^2 = P$
$P^T = P$

$$\exp(P) = I + P + \frac{P^2}{2!} + \frac{P^3}{3!} + \cdots + \frac{P^n}{n!}$$

$$= I + P\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)$$

$$= I + (e-1)P$$

$$\alpha = e - 1$$

4. $A = \underbrace{\begin{bmatrix} & & \end{bmatrix}}_{n} \}m$ $\quad m > n \quad \text{rank}(A) = n$

Then the system $AX=b$ is inconsistent

a. $A^TA\bar{X} = A^Tb$

$\bar{X} = (A^TA)^{-1}A^Tb$

b. $A^TA = I$

$\bar{X} = (A^TA)^{-1}A^Tb = I^{-1}A^Tb = A^Tb$

$\bar{X} = A^Tb$

c. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$R(A)$: multiples of $\begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$

$N(A^T)$: $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$ $\quad u=v=w=0$

$x = 1$

multiples of $\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$

$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$ is the only vector in both $R(A)$ and $N(A^T)$

$a = \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad b = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad c = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$

$q_2 = \dfrac{b'}{\|b'\|}$

$q_1 = \dfrac{a}{\|a\|} = \dfrac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$, $\quad b' = b - (q_1^Tb)q_1$

$c' = c - (q_1^Tc)q_1 - (q_2^Tc)q_2$

$q_3 = \dfrac{c'}{\|c'\|}$

5. a.

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\det A = 3(-1)^2 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 3(4) = 12$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ 4 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda)(2-\lambda)(2-\lambda) = 0$$

$\Rightarrow \lambda = 1, 2, 3$

$\quad\quad\quad\quad \uparrow$ twice

$\underline{\lambda = 1}$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} x_1 = 0 \\ x_3 = x_4 = 0 \\ x_2 = 1 \end{matrix} \quad V_1 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$\underline{\lambda = 3}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} x_4 = 0 \\ -x_3 + x_4 = 0 \Rightarrow x_3 = 0 \\ 4x_1 - 2x_2 = 0 \end{matrix} \quad V_2 = \begin{Bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{Bmatrix}$$

$\underline{\lambda = 2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ 4x_1 - x_2 = 0 \Rightarrow x_2 = 0 \\ x_4 = 0 \quad\quad x_3 = 1 \end{matrix} \quad V_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

only one independent ev associated with $\lambda = 2$!

$$\det \begin{bmatrix} 3 & 2 & 1 \\ 2 & -3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = 3[-9-4] - 2[6-2] + 1[4+3]$$

$$= 3(-13) - 2(4) + 7$$

$$= -39 - 8 + 7 = -40$$

6. $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$ $\quad u_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$

$$u_{k+1} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}}_{A} u_k$$

$$u_k = A^k u_0$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{bmatrix} = 0 \Rightarrow -\lambda(1/2 - \lambda) - 1/2 = 0$$

$$\lambda^2 - \lambda/2 - 1/2 = 0 \qquad \lambda = \frac{1/2 \pm \sqrt{1/4 + 4(1/2)}}{2}$$

$$= 1/4 \pm 3/4 = -1/2,\ 1$$

$\underline{\lambda = 1}$

$$\begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow v_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$\underline{\lambda = -1/2}$

$$\begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow v_2 = \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}$$

$$A = S\Lambda S^{-1} \Rightarrow A^k = S\Lambda^k S^{-1}$$

$$A^k = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}^k \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1/2)^k \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1/2)^k \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -(-1/2)^k \\ 1 & 2(-1/2)^k \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 + 1/3(-1/2)^k & 1/3 - 1/3(-1/2)^k \\ 2/3 - 2/3(-1/2)^k & 1/3 + 2/3(-1/2)^k \end{bmatrix}$$

$$u_k = \begin{bmatrix} 2/3 + 1/3(-1/2)^k & 1/3 - 1/3(-1/2)^k \\ 2/3 - 2/3(-1/2)^k & 1/3 + 2/3(-1/2)^k \end{bmatrix} u_0$$

$G_0 = 0$ $\quad\quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$G_1 = 1$

$$u_k = \begin{bmatrix} 2/3 + 1/3(-1/2)^k & 1/3 - 1/3(-1/2)^k \\ 2/3 - 2/3(-1/2)^k & 1/3 + 2/3(-1/2)^k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 + 1/3(-1/2)^k \\ 2/3 - 2/3(-1/2)^k \end{bmatrix}$$

$$\lim_{k \to \infty} \begin{bmatrix} 2/3 + 1/3(-1/2)^k \\ 2/3 - 2/3(-1/2)^k \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$