Complex Variables Exam

Monday, January 12, 2004 (12:00 – 2:00 PM)

Please show all work. To get full credit for a problem you need to **clearly** describe your calculations.

1. **(10 points)** Determine where $f'(z)$ exists and provide an expression for it:

$$f(z) = r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta), \quad z = re^{i\theta}.$$

2. **(20 points)** Use complex variables to evaluate the following integrals

$$\textbf{(a)} \quad \int_0^{2\pi} \frac{d\theta}{(5+3\cos\theta)^2}, \qquad \textbf{(b)} \quad \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}dx.$$

   Please give the answers in the real form (not as complex numbers).

3. **(10 points)** Evaluate the intergal

$$\oint_C \bar{z}^2(e^z - 1)\, dz,$$

   where $C: \; z = e^{i\theta}, \, 0 \le \theta < 2\pi.$

4. **(10 points)** How many zeros does the function

$$z^8 - 4z^5 + z^2 - 1$$

   have inside the unit circle $|z| = 1$?

5. **(10 points)** Verify that $v(x,y) = x + y - 3$ is harmonic everywhere in the plane and find all analytic functions $f(z)$, $z = x + iy$, such that $v = \text{Im}f(z)$.

6. **(10 points)** Find the image of the circle $x^2 + y^2 = 1$ under the transformation

$$w = \frac{z-a}{1-\bar{a}z} \quad (z = x + iy),$$

   where a is a complex number.