Advanced Calculus Preliminary Examination
January 2000

1. (a) Determine whether or not each of the following improper integrals converges. If it converges, evaluate the integral.

   (i) $I_1 = \int_0^\infty e^{-x}\, dx$

   (ii) $I_2 = \int_{-\infty}^0 \sin x\, dx$

   (iii) $I_3 = \int_{-\infty}^\infty \frac{dx}{1+x^2}$

   (b) Determine the values of α for which the following integrals converge:

   (i) $I_4 = \int_a^b \frac{dx}{(x-a)^\alpha}$

   (ii) $I_5 = \int_1^\infty \frac{dx}{x^\alpha}$

2. Consider a fluid dynamical field $(\rho, \vec{u})$ in a simply connected domain D. The law of conservation of mass implies the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$. Suppose that the fluid is incompressible so that the material derivative vanishes, i.e.,

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \vec{u} = 0.$$

   (i) Show that the fluid field is solenoidal, i.e. $\nabla \cdot \vec{u} = 0$.

   (ii) Suppose further that the circulation of the velocity field $\int_C \vec{u} \cdot d\vec{s}$ is independent of the path C. Show that the fluid field is irrotational, i.e., $\nabla \times \vec{u} = 0$.

   (iii) Show that if the field is irrotational, $\vec{u}$ is the gradient of a scalar velocity potential ϕ.

   (iv) Show that if the fluid field is both irrotational and incompressible, ϕ is a harmonic function, i.e., ϕ satisfies Laplace's equation $\nabla^2 \phi = 0$.

3. Let $F(x,y)$ and $G(x,y)$ be continuous and have continuous derivatives in a simply connected domain D. Let R be a region within D, whose boundary is a closed curve C. Let $\vec{n}$ be the unit outer normal of R and let $\frac{\partial F}{\partial n}$ and $\frac{\partial G}{\partial n}$ denote the normal derivatives of F and G, respectively, i.e., the directional derivatives in the direction of the normal $\vec{n}$. Show that:

   (i) $\int_C \frac{\partial F}{\partial n} ds = \int\int_R \nabla^2 F\, dx\, dy$

   (ii) $\int_C \nabla F \cdot d\vec{r} = 0$

   (iii) if F is harmonic, then $\int_C \frac{\partial F}{\partial n} ds = 0$

   (iv) $\int_C F \frac{\partial G}{\partial n} ds = \int\int_R F \nabla^2 G\, dx\, dy + \int\int_R (\nabla F \cdot \nabla G)\, dx\, dy$

   (v) $\int_C (F \frac{\partial G}{\partial n} - G \frac{\partial G}{\partial n})\, ds = \int\int_R (F \nabla^2 G - G \nabla^2 F)\, dx\, dy$

   Note that (iv) and (v) are referred to as Green's identities.

4. (a) Express the integral $\int\int_D f(x, y)\, dx\, dy$ with D a domain in in the (x, y)-plane in cylindrical coordinates.

(b) Calculate the integral

$$\int z^n \, dS,$$

over the hemisphere $z = \sqrt{1 - x^2 - y^2}$ for $n > -1$.

(c) Derive an expression for the Laplace operator $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ in cylindrical coordinates.

5. Calculate the rate of change of the function $f(x, y, z) = 3x^2 + y^2 + z^2$ at the point $(-2, 1, 3)$ along the line of intersection of the two surfaces given by

$$\frac{5}{4}x^2 + xz + y^2 = 0$$

and

$$x^2 y - 5z + 11 = 0.$$