PRELIMINARY EXAM - LINEAR ALGEBRA 1/98

1. Let $A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$, $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solve the problem

$$Au = \lambda u$$

for the λ's and eigenvectors u.

2. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -3 \\ 1 & -1 & 3 \end{bmatrix}$$

(a) Find all α and β such that the system $Ax = b$ has a solution where $b = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$.

(b) Find a vector x which minimizes the expression $||Ax - c||_2$ where $c = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

(Note: $||x||_2 = \sqrt{x \cdot \bar{x}}$ is the 2-norm)

3. Compute

$$\lim_{n \to \infty} \frac{A^n x}{||A^n x||_2}$$

notice that [illegible]

where $A = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ and (a) $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and (b) $x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

[illegible] in terms of e-vectors

4. For each of the statements below, give a reason why true, or a counterexample if false.

   (a) If A is a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, then $||A||_2 = \max_i |\lambda_i|$.

   (b) The condition number of a matrix A, defined by $\text{cond}(A) = ||A||_2||A^{-1}||_2$, is always ≥ 1.

   as long as no pivots are zero [illegible]

   true

   (c) Any nonsingular matrix can be factored into $A = LU$ where L is lower triangular with ones on the diagonal and U is upper triangular.

   true (d) If A is Hermitian, then $A + iI$ is non-singular. $-i$ is not an evalue of A as all evs of A are real as A is hermitian

   (e) A matrix is invertible if and only if it is diagonalizable.

   False

   not all matrices have a basis of independent eigenvectors;

   [illegible] diagonalizable.

   [illegible]

1. $A = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix}$

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & -1/2 & -1/2 \\ 0 & 3/2-\lambda & 1/2 \\ 0 & 1/2 & 3/2-\lambda \end{bmatrix} = (1-\lambda)\left[(3/2-\lambda)(3/2-\lambda) - 1/4\right]$$

$$= (1-\lambda)\left[9/4 - 6/2\,\lambda + \lambda^2 - 1/4\right] = (1-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-1) = 0$$

$$\lambda = 1, 2$$

$\underline{\lambda = 2}$

$$\begin{bmatrix} -1 & -1/2 & -1/2 \\ 0 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$-x_1 - 1/2\,x_2 - 1/2\,x_3 = 0$

$x_2 = x_3$

$-x_1 = x_2$

$v_1 = \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}$

$\underline{\lambda = 1}$

$$\begin{bmatrix} 0 & -1/2 & -1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$x_2 = -x_3$

x_1 free

$v_2 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, $v_3 = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$

Eigenvalues: $\lambda = 2$ $\qquad\lambda = 1$

Eigenvectors: $\begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}$ $\qquad \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$

2. $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -3 \\ 1 & -1 & 3 \end{bmatrix}$ $b = \begin{Bmatrix} \alpha \\ \beta \\ 0 \end{Bmatrix}$

a. $Ax = b$

$$\begin{bmatrix} -1 & 1 & -3 \\ 2 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \beta \\ \alpha \\ 0 \end{Bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & -3 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \beta \\ \alpha + 2\beta \\ \beta \end{Bmatrix}$$

we have a solution provided that $\underline{\beta = 0}$

Then

$$3x_1 - 3x_2 + 9x_3 = 0$$
$$3x_2 - 6x_3 = \alpha$$
$$3x_1 + 3x_3 = \alpha$$

$\beta = 0$ and $\alpha = 3x_1 + 3x_3$

b. $c = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$ $\|x\|_2 = \sqrt{x \cdot \bar{x}}$

want $\min \|Ax - c\|_2$

can we get $\|Ax - c\|_2 = 0 \longrightarrow Ax = c$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ -1 & 1 & -3 & 0 \\ 1 & -1 & 3 & 2 \end{array}\right] \rightarrow \begin{bmatrix} -1 & 1 & -3 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & 3 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} -1 & 1 & -3 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

inconsistent! $0 \neq 2$!

c is not in the column space of A.

<u>Least Squares</u>

$A^TA\bar{x} = A^Tc$

$\bar{x} = (A^TA)^{-1}A^Tc$

$$A^TA = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -3 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 3 & -6 \\ 6 & -6 & 18 \end{bmatrix}$$

$\det A^TA = 6(3(18) - 36) + 6(-18)$

$= 6(2(18) -$

b. $Ax = c$ $\qquad c = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$

$A^T A x = A^T c$

$$A^T A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -3 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 3 & -6 \\ 6 & -6 & 18 \end{bmatrix}$$

$$A^T c = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -1 \\ 6 \end{Bmatrix}$$

$$[A^T A \mid A^T c] = \left[\begin{array}{ccc|c} 6 & 0 & 6 & 4 \\ 0 & 3 & -6 & -1 \\ 6 & -6 & 18 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & 0 & 6 & 4 \\ 0 & 3 & -6 & -1 \\ 0 & -6 & 12 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 6 & 0 & 6 & 4 \\ 0 & 3 & -6 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free

$3x_2 - 6x_3 = -1$

$3x_2 = -1 + 6x_3$

$6x_1 = 4 - 6x_3$

$$x = \begin{Bmatrix} 2/3 - x_3 \\ -1/3 + x_3 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2/3 \\ -1/3 \\ 0 \end{Bmatrix} + x_3 \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}$$

6. $A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda) + 1 = 0$

$15 - 8\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0 \qquad \lambda = 4$

$\underline{\lambda=4}$ $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ $\qquad v_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ only one linearly independent eigenvector

can't diagonalize A

Jordan form: $B = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad B^2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$

$B^3 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$

$B^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} = \begin{bmatrix} 4^n & n(4^{n-1}) \\ 0 & 4^n \end{bmatrix}$

To find A^n, we need a vector v_2 that satisfies $Av_2 = v_1$. Then, let $M = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$.

$\underline{A^n = M B^n M^{-1}}$