PRELIMINARY EXAM - DIFFERENTIAL EQUATIONS 1/97
20 points per problem

1. Consider the linear system of differential equations

$$\mathrm{x}' = A\mathrm{x}\,, \quad \mathrm{x} \in \mathbb{R}^2\,.$$

(A) Find the general solution, and (B) draw several trajectories in the $(x_1, x_2)$-phase plane for each of the following cases:

$$(i)\quad A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad (ii)\quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad (iii)\quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Your phase portraits should include arrows indicating the direction of increasing time along the trajectories.

2. Consider the vectors

$$\mathrm{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}\,; \quad \mathrm{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}\,.$$

(a) Compute the Wronskian of $\mathrm{x}^{(1)}$ and $\mathrm{x}^{(2)}$.

(b) In what intervals are $\mathrm{x}^{(1)}$ and $\mathrm{x}^{(2)}$ linearly independent?

(c) What conclusion can be drawn about the coefficients in the system of linear homogeneous differential equations satisfied by $\mathrm{x}^{(1)}$ and $\mathrm{x}^{(2)}$?

(d) Find this system of equations and verify the conclusions of part (c).

3. Consider the nonhomogeneous differential equation for $y = y(x)$:

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = g(x)$$

on $0 < x < 1$. Assume that $g(x)$ is an arbitrary continuous function.

(a) Given that one solution of the homogeneous problem is $y = e^x$, find the other.

(b) Find a particular solution of the nonhomogeneous problem.

(c) Find the solution of the initial val roblem when $y(0) = 0$ and $\frac{dy}{dx}(0) = 0$.

4. Consider the damped, forced spring system described by the equation

$$y'' + 2\epsilon y' + y = \cos(t)$$

where $y$ is the displacement of the block from equilibrium and $\epsilon > 0$ is constant. Find the general solution. Describe the limiting amplitude of the motion as time increases for fixed $\epsilon$. What happens to the solution $y(t)$ as $\epsilon \to 0$?

5. If a circular drum of radius $R$ is given initial conditions which depend only on distance from the center, then the motion can be written as

$$z_{tt} = z_{rr}, \; z_r(0,t) = 0, \; z(R,t) = 0$$

If the initial height of the drum is $z(r,0) = R-r$ and the initial velocity is $z_t(r,0) = 0$, compute the height of the drum for time $t > 0$.