Asymptotic + Perturbation Methods

420-1: Algebraic eqns 9-23-2008
Asymptotic series
Regular problems for ODES
Boundary Layers for ODES
Multiple Scales
420-2: Multiple Scales
Averaging
WKB method
Integrals
420-3: Asymptotic + Perturbation Methods
for PDEs
412: Nonlinear Analysis
← next year

Hwk: collected + graded
Midterm + final: take-home
No req'd book, but similar:
M.H. Holmes. Introduction to Perturbation Methods. Springer
Check Blackboard

Ex $x^2 - 2\varepsilon x - 1 = 0$. Solve for x.

ε is a small parameter, $0 < \varepsilon \ll 1$

Exact solution:

$$x_\pm = \varepsilon \pm \sqrt{1+\varepsilon^2}$$

Approximate:

$x_\pm \approx \pm 1$ (pretty rough)

Do better:

$$\sqrt{1+\varepsilon^2} = 1 + \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{8}\varepsilon^4 + \cdots$$

$\rightarrow$ Exact sol'n:

$$x_+ = 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{8}\varepsilon^4 + \cdots$$

$$x_- = -1 + \varepsilon - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{8}\varepsilon^4 + \cdots$$

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots$$

Plug into the eqn:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots)^2$$
$$-2\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) - 1 = 0$$

Collect like powers of ε:

$$x_0^2 - 1 + \varepsilon(2x_0 x_1 - 2x_0) + \varepsilon^2(x_1^2 + 2x_0 x_2 - 2x_1)$$
$$+ \varepsilon^3(2x_0 x_3 + 2x_1 x_2 - 2x_2) + \cdots = 0$$

ε^0: $x_0^2 - 1 = 0$

ε^1: $2x_0(x_1 - 1) = 0$

ε^2: $x_1^2 - 2x_1 + 2x_0 x_2 = 0$

ε^3: $2(x_0 x_3 + x_1 x_2 - x_2) = 0$

Use this to get x_0, then x_1, etc.

$x_0 = 1, -1$

$x_1 = 1, 1$

$x_2 = \tfrac{1}{2}, -\tfrac{1}{2}$

$x_3 = 0, 0$

$$\rightarrow x_+ = 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + 0\cdot\varepsilon^3 + \cdots$$
$$x_- = -1 + \varepsilon - \tfrac{1}{2}\varepsilon^2 + 0\cdot\varepsilon^3 + \cdots$$

Or $\begin{cases} x_+ = 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^4) \\ x_+ = 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + o(\varepsilon^3) \end{cases}$

$\rightarrow$ means the following terms $< c\varepsilon^4$ where c is a constant

means error is much smaller than ε^3

Def $f(\varepsilon) = O(\varphi(\varepsilon))$ if $|f(\varepsilon)| \le K|\varphi(\varepsilon)|$ for all $\varepsilon < \varepsilon_0$, K does not depend on ε.

Def $f(\varepsilon) = o(\varphi(\varepsilon))$ if

$$\lim_{\varepsilon \searrow 0} \frac{f(\varepsilon)}{\varphi(\varepsilon)} = 0$$

Note: $\lim_{\varepsilon \searrow 0}$ means $\varepsilon \to 0$ but $\varepsilon > 0$

Ex $\sin\varepsilon = O(\varepsilon)$

$\sin\varepsilon \le \varepsilon$ (for small ε)

so this works.

We could also say $\sin\varepsilon = O(1)$

What about $\sin\varepsilon - \varepsilon = o(\varepsilon^2)$?

this is also true:

$$\sin\varepsilon = \varepsilon - \frac{1}{3!}\varepsilon^3 + \frac{1}{5!}\varepsilon^5 - \cdots$$

$$\sin\varepsilon - \varepsilon = -\frac{1}{3!}\varepsilon^3 + \frac{1}{5!}\varepsilon^5 - \cdots$$

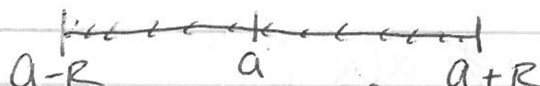
Then $$\lim_{\varepsilon \searrow 0} \frac{-\frac{1}{3!}\varepsilon^3 + \frac{1}{5!}\varepsilon^5 - \cdots}{\varepsilon^2} = 0$$

Taylor Series

$$f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a^2) + \frac{1}{3!}f'''(a)(x-a^3) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a^n)$$

works for $|x-a| < R$



converges for x in this interval.

Use most often for $a=0$:

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n$$

Ex: $e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \; R = \infty$

Radius of Convergence of $f(x) = \sum_{n=0}^{\infty} c_n x^n$

ratio test:

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

Ex $c_n = \frac{1}{n!}, \; c_{n+1} = \frac{1}{(n+1)!}$

$$\frac{c_n}{c_{n+1}} = \frac{(n+1)!}{n!} = n+1$$

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} n+1 = \infty$$

Ex $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$

Ex $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

Ex $(1+x)^\alpha = 1 + \frac{1}{1!}\alpha x + \frac{1}{2!}\alpha(\alpha-1)x^2 + \frac{1}{3!}\alpha(\alpha-1)(\alpha-2)x^3 + \cdots, \; R=1$

(for $\alpha \in \mathbb{N}$, it's a finite series)

$\alpha = \frac{1}{2}$: $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$

as we used before

$\alpha = -1: (1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3$
$= \sum_{n=0}^{\infty} (-1)^n x^n$

Ex: $\ln(1+x) = \int \frac{1}{1+x}$ integrate!
$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$
$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$

Ex: $\frac{1}{(1+x)^2} = -(-1 + 2x - 3x^2 + \dots)$ differentiate!
$= 1 - 2x + 3x^2 - \dots$

superposition:
Ex:
$e^{\sin x} = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \dots$
$= 1 + \sin x + \frac{1}{2}\sin^2 x + \frac{1}{6}\sin^3 x + \dots$ if you're disregarding x^4 etc
$= 1 + (x - \frac{1}{6}x^3) + \frac{1}{2}(x - \cancel{\frac{1}{6}x^3})^2 + \frac{1}{6}(x - \cancel{\frac{1}{6}x^3})^3 + \dots$
$= 1 + x + \frac{1}{2}x^2 + O(x^4)$
or $= 1 + x + \frac{1}{2}x^2 + O(x^3)$

Or:
$e^{\sin x} = e^{x - \frac{1}{6}x^3 + O(x^5)}$
$= 1 + (x - \frac{1}{6}x^3) + \frac{1}{2}(x)^2 + \frac{1}{6}(x^3) + \dots$
$= 1 + x + \frac{1}{2}x^2 + O(x^4)$

9-25 <u>Ex</u> $x^2 - 2\varepsilon x - 1 = 0$

$\rightarrow x_\pm = \varepsilon \pm \sqrt{1+\varepsilon^2}$

$x_+ = 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + O(\varepsilon^4)$

$x_- = -1 + \varepsilon - \frac{1}{2}\varepsilon^2 + O(\varepsilon^4)$

let $\varepsilon = 0.1$: exact sol'n: $0.1 + \sqrt{1.1} = 1.10499\ldots$

x_+ 3 term approx: $1 + 0.1 + \frac{1}{2}\cdot 0.1^2 = 1.105$

now $\varepsilon = 1$: exact: $1 + \sqrt{2} = 2.4142\ldots$

approx: $1 + 1 + \frac{1}{2}\cdot 1 = 2.5$

Ex: $x^3 - x + \varepsilon = 0$ Find 2-term expansion of x in powers of ε

Assume form of sol'n: $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$

(we expect $x_0^3 - x_0 = 0 \Rightarrow x = 0, +1, -1$)

$[x_0 + \varepsilon x_1 + O(\varepsilon^2)]^3 - [x_0 + \varepsilon x_1 + O(\varepsilon^2)] + \varepsilon = 0$

$x_0^3 + 3x_0^2 \varepsilon x_1 + O(\varepsilon^2) - x_0 - \varepsilon x_1 + \varepsilon = 0$

ε^0: $x_0^3 - x_0 = 0 \rightarrow x_0 = 0, 1, -1$

ε^1: $3x_0^2 x_1 - x_1 + 1 = (3x_0^2 - 1)x_1 + 1 = 0$

$\rightarrow x^{(1)} = 1 - \frac{1}{2}\varepsilon + O(\varepsilon^2)$

$x^{(2)} = -1 - \frac{1}{2}\varepsilon + O(\varepsilon^2)$

$x^{(3)} = \varepsilon + O(\varepsilon^2)$

$= \varepsilon + \varepsilon^2 x_2 + O(\varepsilon^3)$

$= \varepsilon + \varepsilon^3 + O(\varepsilon^4)$

how we got that last step:

$x = \varepsilon + x^3$

$\Rightarrow x = \varepsilon + \varepsilon^3 + o(\varepsilon^3)$

approx iteratively:

$x_{n+1} = \varepsilon + x_n^3, \; x_0 = 0, \; n = 0, 1, 2, 3, \ldots$

$x_1 = \varepsilon$

$x_2 = \varepsilon + \varepsilon^3$

$x_3 = \varepsilon + (\varepsilon + \varepsilon^3)^3$

$= \varepsilon + \varepsilon^3(1 + \varepsilon^2)^3 = \varepsilon + \varepsilon^3(1 + 3\varepsilon^2 + \cdots)$

$= \varepsilon + \varepsilon^3 + 3\varepsilon^5 + \cdots$

if this converges, we have the sol'n.

In general:

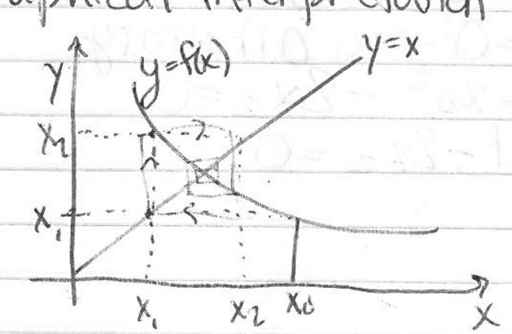
$x = f(x)$

then $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \cdots$ $x_0 = \square$

$\rightarrow x_0, x_1, x_2, x_3, \ldots \longrightarrow x_*$

$\rightarrow x_* = f(x_*)$

graphical interpretation:



converges if $|f'(x)| < 1$ for all x of interest

Ex: $(1-\varepsilon)x^2 - 2x + 1 = 0$ Find 3 term expansion

$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$ ignore ε^2 for now

$(1-\varepsilon)(x_0 + \varepsilon x_1 + O(\varepsilon^2))^2 - 2(x_0 + \varepsilon x_1 + O(\varepsilon^2)) + 1 = 0$

$(1-\varepsilon)(x_0^2 + 2\varepsilon x_0 x_1) - 2x_0 - 2\varepsilon x_1 + 1 + O(\varepsilon^2) = 0$

$\varepsilon^0: x_0^2 - 2x_0 + 1 = 0 \rightarrow (x_0 - 1)^2 = 0 \rightarrow x_0 = 1$ (double)

$\varepsilon^1: 2x_0 x_1 - x_0^2 - 2x_1 = 0$

$(x_0 - 1)2x_1 = x_0^2 \rightarrow 0 = 1 \Rightarrow\Leftarrow$

$\rightarrow$ form of sol'n is not correct

$$x_\pm = \frac{2 \pm \sqrt{4 - 4(1-\varepsilon)}}{2(1-\varepsilon)} = \frac{1 \pm \sqrt{\varepsilon}}{1-\varepsilon} = \frac{1 \pm \sqrt{\varepsilon}}{(1-\sqrt{\varepsilon})(1+\sqrt{\varepsilon})}$$

$$= \frac{1}{1 \mp \sqrt{\varepsilon}} = 1 \pm \sqrt{\varepsilon} + \varepsilon \pm \varepsilon^{3/2} + \cdots$$

so the powers aren't integers $\rightarrow$ that was the problem!

Now $x = x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + \varepsilon^{3/2} x_3 + \cdots$ will work.

$(1-\varepsilon)(x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + o(\varepsilon))^2$

$\quad - 2(x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + o(\varepsilon)) + 1 = 0$

$(1-\varepsilon)(x_0^2 + 2x_0 \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_1^2 + 2\varepsilon x_0 x_2)$

$\quad -2(x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + o(\varepsilon) + 1 = 0$

$x_0^2 + 2x_0 \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_1^2 + 2\varepsilon x_0 x_2$

$\quad - \varepsilon x_0^2 - 2x_0 - 2\varepsilon^{\frac{1}{2}} x_1 - 2\varepsilon x_2 + 1 + o(\varepsilon) = 0$

$\varepsilon^0: x_0^2 - 2x_0 + 1 = 0 \to x_0 = 1$

$\varepsilon^{1/2}: 2x_0 x_1 - 2x_1 = 0 \to x_1$ arbitrary

$\varepsilon^1: x_1^2 + 2x_0 x_2 - x_0^2 - 2x_2 = 0$

$\quad \to x_1^2 + 2x_2 - 1 - 2x_2 = 0$

$\quad \to x_1 = \pm 1$

(works so far.)

How would we know this is the correct form?

$x = 1 + \delta_1 + o(\delta_1)$

or $= 1 + \delta_1 + \delta_2 + \ldots$

where $\delta_1 = o(1)$ + $\delta_{i+1} = o(\delta_i)$

$\to (1-\varepsilon)(1 + \delta_1 + o(\delta_1))^2 - 2(1 + \delta_1 + o(\delta_1)) + 1 = 0$

$= (1-\varepsilon)(1 + 2\delta_1 + o(\delta_1) - 2 - 2\delta_1 + o(\delta_1) + 1 = 0$

$= 1 + 2\delta_1 - \varepsilon - \cancel{2\varepsilon\delta_1} + o(\delta_1) - 2 - 2\delta_1 = 0$

~~wait~~... this says $\varepsilon = o(\delta_1)$... not helpful.

$x = 1 + \delta_1 + \delta_2 + o(\delta_2)$

$(1-\varepsilon)(1 + 2\delta_1 + 2\delta_2 + \delta_1^2 + o(\delta_2))$

$\quad -2(1 + \delta_1 + \delta_2 + o(\delta_2)) + 1 = 0$

$\cancel{1} + \cancel{2\delta_1} + \cancel{2\delta_2} + \delta_1^2 + o(\delta_2)$

$- \varepsilon - 2\varepsilon\delta_1 - \varepsilon\delta_1^2 \cancel{-2} - \cancel{2\delta_1} - \cancel{2\delta_2} + \cancel{1} = 0$

$\underbrace{\delta_1^2 - \varepsilon}_{\text{largest terms}} - 2\varepsilon\delta_1 - \varepsilon\delta_1^2 + o(\delta_2) = 0$

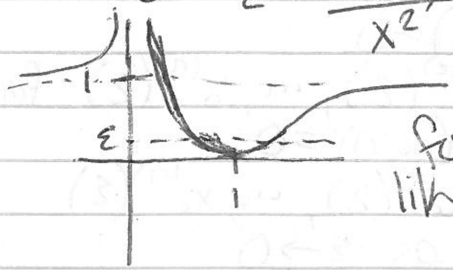
need $\delta_1 = \pm\sqrt{\varepsilon}$

rewrite: $x^2-2x+1-\varepsilon x^2=0$

$(x-1)^2=\varepsilon x^2$

Think graphically.

or $\varepsilon=\frac{(x-1)^2}{x^2}$



for ε small, behaves like a parabola (near $x=1$)

$(x-1)^2=\varepsilon$

$x-1=\pm\sqrt{\varepsilon}$

Ex: $x^4-4x^2+\varepsilon=0$ Find 2 term expansions

$x=x_0+\varepsilon x_1+O(\varepsilon^2)$ will this work?

find x_0: $x_0^4-4x_0^2=0 \rightarrow x_0^2(x_0^2-4)=0$

$\rightarrow x_0=\pm 2$, double 0.

So this will work for the simple roots, but not for the double roots.

What will it be like for the double roots?

$x^2=\frac{1}{4}\varepsilon+\frac{1}{4}x^4$ → this term is much smaller

$\rightarrow x=\pm\frac{1}{2}\sqrt{\varepsilon}$

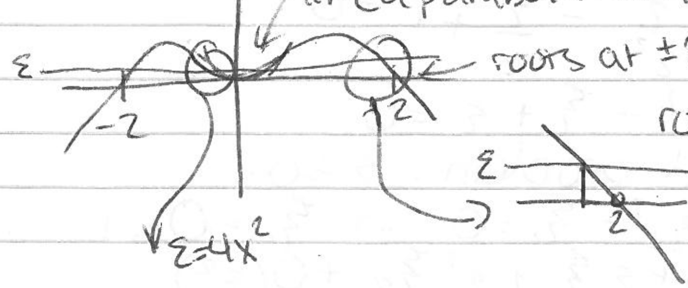
then you can use iteration because is a small addition

OR $\varepsilon=4x^2-x^4=x^2(4-x^2)$

$=x^2(2-x)(2+x)$

like a parabola near zero

roots at ± 2



$\varepsilon=4x^2$

root differs from 2 proportional to ε (linear)

Get local behavior by expanding in Taylor Series

9-30 Ex.: $\varepsilon x^2 - x + 1 = 0$

Reduced problem ($\varepsilon = 0$):

$$-x_0 + 1 = 0$$

Regular Problem: $P(x,\varepsilon) = 0$

solutions: $x^{(1)}(\varepsilon), x^{(2)}(\varepsilon), \dots, x^{(n)}(\varepsilon)$ for $\varepsilon > 0$

Reduced Problem: $P(x,0) = 0$

solutions: $x_0^{(1)}(\varepsilon), x_0^{(2)}(\varepsilon), \dots, x_0^{(n)}(\varepsilon)$

$$x^{(k)}(\varepsilon) \to x_0^{(k)} \text{ as } \varepsilon \to 0$$

(Reduced problem sol'ns are approximations of real solutions for small ε)

If not Regular, then Singular Problem

our example ↑ was singular.

$$\varepsilon x^2 - x + 1 = 0$$

$$x_\pm = \frac{1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon} = \frac{1 \pm (1 - 2\varepsilon - 2\varepsilon^2 + O(\varepsilon^3))}{2\varepsilon}$$

$$x_- = 1 + \varepsilon + O(\varepsilon^2)$$

$x_+ = \frac{1}{\varepsilon} - 1 + O(\varepsilon)$ → large as $\varepsilon \to 0$

convert into a regular problem:

let $x = \frac{\xi}{\varepsilon}$

$$\to \frac{\varepsilon \xi^2}{\varepsilon^2} - \frac{\xi}{\varepsilon} + 1 = 0$$

$$\to \xi^2 - \xi + \varepsilon = 0$$

Reduced problem: $\varepsilon = 0$

$$\xi_0^2 - \xi_0 = 0 \to \xi_0 = 0, 1$$

$$\xi = 0 + \varepsilon \xi_1 + \varepsilon^2 \xi_2 + O(\varepsilon^2)$$

Plug in: $\varepsilon^2 \xi_1^2 - \varepsilon \xi_1 - \varepsilon^2 \xi_2 + \varepsilon + O(\varepsilon^3) = 0$.

$$\xi_1 = 1 \qquad \xi_2 = 1$$

$$\xi = \varepsilon + \varepsilon^2 + O(\varepsilon^3)$$

$\rightarrow x_- = 1 + \varepsilon + O(\varepsilon^2)$

$\xi = 1 + \varepsilon\xi_1 + O(\varepsilon^2)$

Plug in: $1 + 2\varepsilon\xi_1 - 1 - \varepsilon\xi_1 + \varepsilon + O(\varepsilon^2$

$2\xi_1 - \xi_1 + 1 = 0 \rightarrow \xi_1 = -1$

$\xi = 1 - \varepsilon + O(\varepsilon^2)$

$\rightarrow x_+ = \frac{1}{\varepsilon} - 1 + O(\varepsilon)$

How to see the $x = \xi/\varepsilon$ substitution:

in $\varepsilon x^2 - x + 1 = 0$

find large terms: $-x$ is large

so εx^2 must balance it (1 is too small)

so set $\varepsilon x^2 = x$

$\rightarrow \varepsilon x = 1 \rightarrow x = \frac{1}{\varepsilon}$

Ex: $\varepsilon^2 x^3 - \varepsilon^3 x^2 + x - 1 = 0$

Reduced problem: $x_0 - 1 = 0$

$\rightarrow$ singular

large terms, x

either $\varepsilon^2 x^3$, $\varepsilon^3 x^2$, or both.

$\varepsilon^2 x^3$ is larger then $\varepsilon^3 x^2$ for x large, ε small.

So say $\varepsilon^2 x^3$ is of the order x

$\varepsilon^2 x^3 = x$

$\varepsilon^2 x^2 = 1$

$\varepsilon x = 1$

$x = \frac{1}{\varepsilon}$

$x = \xi/\varepsilon$

$\rightarrow \xi^3 - \varepsilon^2\xi^2 + \xi - \varepsilon = 0 \quad \rightarrow$ Regular

Reduced: $\xi_0^3 + \xi_0 = 0$

$\rightarrow \xi_0 = 0, \pm i$

$\xi = \xi_0 + \varepsilon \xi_1 + O(\varepsilon^2)$

$\rightarrow (\xi_0 + \varepsilon \xi_1)^3 + \xi_0 + \varepsilon \xi_1 - \varepsilon + O(\varepsilon^2) = 0$

$\xi_0^3 + 3\xi_0^2 \varepsilon \xi_1 + \xi_0 + \varepsilon \xi_1 - \varepsilon + O(\varepsilon^2) = 0$

$\varepsilon^0: \xi_0^3 + \xi_0 = 0 \rightarrow \xi = 0$ ✓

$\varepsilon^1 - 3\xi_0^2 \xi_1 + \xi_1 - 1 = 0 \rightarrow \xi_1 = \dfrac{1}{3\xi_0^2 + 1}$

$\xi_0 = i, \ \xi_1 = -\frac{1}{2}$

$\xi_0 = -i, \ \xi_1 = \frac{1}{2}$

$\xi_0 = 0, \ \xi_1 = 1$

$x^{(1)} = -i/\varepsilon - \frac{1}{2} + O(\varepsilon)$

$x^{(2)} = i/\varepsilon - \frac{1}{2} + O(\varepsilon)$

$x^3 = 1 + O(\varepsilon)$

could have used iterations instead:

$\xi^3 + \xi = \varepsilon + \varepsilon^2 \xi^2$

$\xi = \varepsilon + \varepsilon^2 \xi^2 - \xi^3 \equiv f(\xi)$

need $|f'(\xi)| < 1$

$|2\varepsilon^2 \xi - 3\xi^2| < 1$

or $\xi = \dfrac{\varepsilon + \varepsilon^2 \xi^2}{\xi^2 + 1} = \dfrac{\varepsilon(1 + \varepsilon \xi^2)}{\xi^2 + 1}$ ← might be better

back to before

$\xi_{n+1} = \varepsilon + \varepsilon^2 \xi_n^2 - \xi_n^3, \ \xi_0 = 0$

$\xi_1 = \varepsilon$

$\xi_2 = \varepsilon + \varepsilon^4 - \varepsilon^3$

$\xi_3 = \varepsilon + \varepsilon^2(\varepsilon + \varepsilon^4 - \varepsilon^3)^2 - (\varepsilon + \varepsilon^4 - \varepsilon^3)^3$

now for the $\xi_0 = i$ sol'n

can't use same iteration, since

$f'(\xi) = 2\varepsilon^2 \xi - 3\xi^2$ isn't small enough

$\xi(\xi + i)(\xi - i) = \varepsilon + \varepsilon^2 \xi^2$

$\xi = \cancel{i} = \dfrac{\varepsilon + \varepsilon^2 \xi^2}{\xi(\xi + i)} + i$ ← use this instead

Now $f'(\xi)$ is small.

Ex: $\varepsilon x^4+\varepsilon x^2+2\varepsilon^{\frac{1}{2}}x-1=0$

Singular problem

$x=\dfrac{\xi}{\delta}$ $\leftarrow$ δ is a function of ε, $\delta=o(1)$ $\quad \xi=O(1)$

Plug in:

$$\varepsilon\frac{\xi^4}{\delta^4}+\varepsilon\frac{\xi^2}{\delta^2}+2\varepsilon^{\frac{1}{2}}\frac{\xi}{\delta}-1=0$$

$$\rightarrow \xi^4+\delta^2\xi^2+2\delta^3\beta/\varepsilon^{\frac{1}{2}}-\delta^4/\varepsilon=0$$

$\xi^4 \gg \delta^2\xi^2$

So either 3rd or 4th term $=O(1)$

Say $\delta^3=\varepsilon^{\frac{1}{2}} \rightarrow \delta=\varepsilon^{\frac{1}{6}}$

$\rightarrow \delta^4/\varepsilon=\varepsilon^{-\frac{1}{3}} \rightarrow$ large

not balanced

So use 4th term: $\delta^4=\varepsilon \rightarrow \delta=\varepsilon^{\frac{1}{4}}$

$\rightarrow \delta^3/\varepsilon^{\frac{1}{2}}=\varepsilon^{\frac{1}{4}}$ fine.

$x=\xi/\varepsilon^{\frac{1}{4}}$

Plug in: $\xi^4+\varepsilon^{\frac{1}{2}}\xi^2+2\varepsilon^{\frac{1}{4}}\xi-1=0$

let $\varepsilon=0$

$\xi_0^4-1=0 \rightarrow \xi_0=\pm 1, \pm i$

let $\varepsilon_1=\varepsilon^{\frac{1}{4}}$

$\rightarrow \xi^4+\varepsilon_1^2\xi^2+2\varepsilon_1\xi-1=0$

So $\xi=\xi_0+\varepsilon^{\frac{1}{4}}\xi_1+O(\varepsilon^{\frac{1}{2}})$

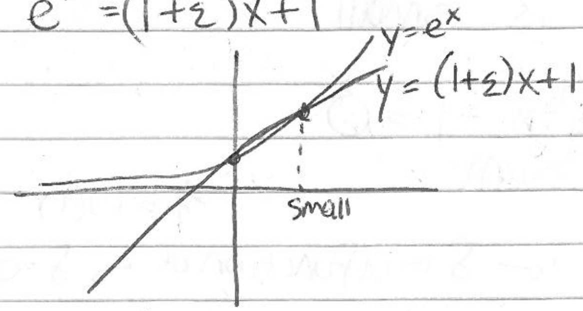
Plug in:

$$\xi_0^4+4\xi_0^3\varepsilon^{\frac{1}{2}}\xi_1+2\varepsilon^{\frac{1}{4}}\xi_0-1+O(\varepsilon^{\frac{1}{2}})=0$$

$$\rightarrow 4\xi_0^3\xi_1+2\xi_0=0$$

$$\xi_1=\frac{1}{2}\xi_0^2$$

Ex: $e^x = (1+\varepsilon)x + 1$



expand

$$1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + O(x^4) = (1+\varepsilon)x + 1$$

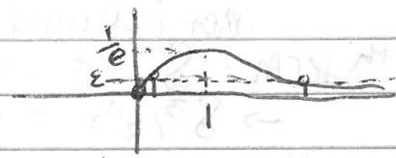
$$\tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + O(x^4) = \varepsilon x$$

$$\tfrac{1}{2}x + \tfrac{1}{6}x^2 + O(x^3) = \varepsilon$$

$$\to \quad x = 2\varepsilon - \tfrac{1}{3}x^2 + O(x^3)$$

$$\to x = 2\varepsilon - \tfrac{4}{3}\varepsilon^2 + O(\varepsilon^3)$$

Ex: $xe^{-x} = \varepsilon$



1st sol'n small, 2nd sol'n large

Reduced: $xe^{-x} = 0 \to x_0 = 0$

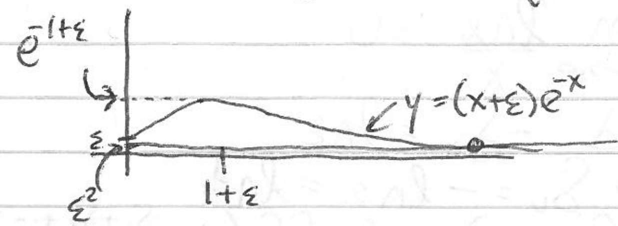
$\to$ singular

Ex: $(x+\varepsilon)e^{-x} = \varepsilon^2$

Reduced: $xe^{-x}=0$ only sol'n: $x=0$

graphically: $y=(x+\varepsilon)e^{-x}$

$\frac{dy}{dx} = e^{-x} - (x+\varepsilon)e^{-x} = 0$



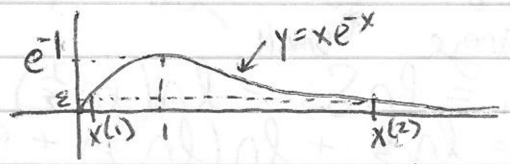
$x(\varepsilon) \to \infty$ as $\varepsilon \to 0$

but reduced problem had $x=0$

... okay maybe not quite right

if it went back up for $-x$, it would work

Ex: $xe^{-x}=\varepsilon$



two intersections: $x^{(1)}$ & $x^{(2)}$

$x^{(1)} = \varepsilon + \cdots$ "leading order term is ε

Rewrite: $x = \varepsilon e^{x}$

$x_{n+1} = \varepsilon e^{x_n}$, $x_0 = 0$

$x_1 = \varepsilon$

$x_2 = \varepsilon e^{\varepsilon} = \varepsilon(1+\varepsilon+\cdots)$

$= \varepsilon + \varepsilon^2 + \cdots$

could do $x = \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

better to plug into rewritten form

$x^{(2)} \to \infty$ as $\varepsilon \to 0$

can't use same kind of expansion

$x = \delta_0(\varepsilon) + \delta_1(\varepsilon) + \delta_2(\varepsilon) + \cdots$

all we know is $\delta_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$

and $\delta_1(\varepsilon) = o(\delta_0(\varepsilon))$

$\delta_2(\varepsilon) = o(\delta_1(\varepsilon)) \cdots$

Plug in: use $\ln x - x = \ln\varepsilon$

$$\underbrace{\ln(\delta_0+\delta_1+\delta_2+\cdots)}_{\text{also large, not as much}} \underbrace{-(\delta_0+\delta_1+\delta_2+\cdots)}_{\text{large negative}} = \ln\varepsilon$$

$\uparrow$ large negative

for $x \gg 1$, $\ln x \ll x$

i.e. $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

So we need $-\delta_0 = \ln\varepsilon$

$\rightarrow \delta_0 = -\ln\varepsilon = \ln\frac{1}{\varepsilon}$

$$\ln(\delta_0+\delta_1+\delta_2+\cdots) = \ln\left[\delta_0\left(1+\frac{\delta_1+\delta_2+\cdots}{\delta_0}\right)\right]$$

$$= \ln\delta_0 + \ln\left(1+\frac{\delta_1+\delta_2+\cdots}{\delta_0}\right)$$

note $\frac{\delta_1+\delta_2+\cdots}{\delta_0}$ is small

and $\ln(1+a) = a - \frac{1}{2}a^2 + \cdots$

So $= \ln\delta_0 + \frac{\delta_1+\delta_2+\cdots}{\delta_0} + \cdots$

$$\rightarrow \underbrace{\ln\delta_0}_{\text{large}} + \underbrace{\frac{\delta_1+\delta_2+\cdots}{\delta_0}}_{\text{small}} - \cancel{\delta_0} - \delta_1 - \delta_2 - \cdots = \cancel{\ln\varepsilon}$$

$\rightarrow \delta_1 = \ln\delta_0 = \ln\left(\ln\frac{1}{\varepsilon}\right)$

So $x^{(2)} = \ln\frac{1}{\varepsilon} + \ln\left(\ln\frac{1}{\varepsilon}\right) + \cdots$

Ex: $\tan\mu = \mu$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0<x<1,\ t>0,$$

IC: $u(x,0) = f(x)$

BC: $u(0,t) = 0$, $\frac{\partial u}{\partial x}(1,t) = u(1,t)$

Robin BC $\uparrow$

Separation of variables

$u(x,t) = X(x)\cdot T(t)$

Plug in:

$X(t)\,T'(t) = X''(x)\,T(t)$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$T' = -\lambda T \rightarrow T' + \lambda T = 0$

$X'' + \lambda X = 0$

Use BC's: $X(0)T(t)=0 \to X(0)=0$

similarly $X'(1)=X(1)$

$\to$ eigenvalue problem

$X(x)\equiv 0$ always works, but it's not what we want.

$\lambda>0$, so let $\lambda=\mu^2$, $\mu>0$

$X(x)=c_1\cos\mu x+c_2\sin\mu x$

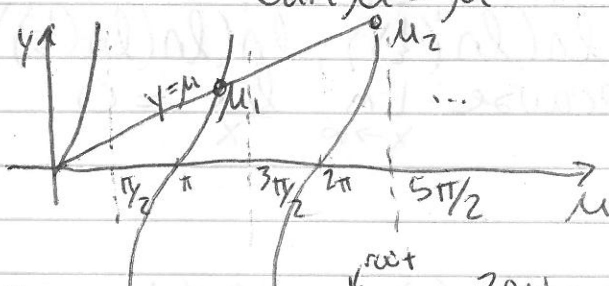
$X(0)=c_1=0$

$X'(x)=c_2\mu\cos\mu x$

$X'(1)=X(1) \to c_2\mu\cos\mu=c_2\sin\mu$

$\to \mu\cos\mu=\sin\mu$

$\to \tan\mu=\mu$



as $\mu\to\infty$ $\overset{\text{root}}{\mu_n}\to\frac{2n+1}{2}\pi$

$\mu_n=\underbrace{\frac{2n+1}{2}\pi}_{\text{large}}-\underbrace{\delta_n}_{\text{small}}$

$\tan\mu_n=\tan\left(\frac{2n+1}{2}\pi-\delta_n\right)$

$=\dfrac{\sin\left(\frac{2n+1}{2}\pi-\delta_n\right)}{\cos\left(\frac{2n+1}{2}\pi-\delta_n\right)}$

Recall $\sin(a+b)=\sin a\cos b+\cos a\sin b$

$\cos(a+b)=\cos a\cos b-\sin a\sin b$

$\to=\dfrac{\sin\left(\frac{2n+1}{2}\pi\right)\cos\delta_n}{\sin\left(\frac{2n+1}{2}\pi\right)\sin\delta_n}=\dfrac{\cos\delta_n}{\sin\delta_n}$

$=\dfrac{1-\frac{1}{2}\delta_n^2+\cdots}{\delta_n-\frac{1}{6}\delta_n^3+\cdots}=\dfrac{1}{\delta_n}\left(1+O(\delta_n^2)\right)$

$=RHS=\frac{2n+1}{2}\pi-\delta_n$

$\to \frac{1}{\delta_n}+O(\delta_n)=\frac{2n+1}{2}\pi-\delta_n$

$\delta_n=\dfrac{1}{\frac{2n+1}{2}\pi}+\cdots$

$$\mu_n = \frac{2n+1}{2}\pi - \frac{1}{\frac{2n+1}{2}\pi} + \cdots$$

Numerically: $u_1 = 4.4934 \cdots$

u_1 approx $= \frac{3}{2}\pi - \frac{1}{\frac{3}{2}\pi} = 4.5002$

Def: Functions $\varphi_1(\varepsilon), \varphi_2(\varepsilon), \cdots, \varphi_n(\varepsilon), \cdots$ form an asymptotic sequence as $\varepsilon \to 0$ iff $\varphi_{n+1}(\varepsilon) = o(\varphi_n(\varepsilon))$ for any n.

Ex's: ① $1, \varepsilon, \varepsilon^2, \varepsilon^3, \cdots$

② $\sin(\varepsilon), \sin^5(\varepsilon), \sin^5(\varepsilon), \cdots$

③ $\ln(\frac{1}{\varepsilon}), \ln(\ln(\frac{1}{\varepsilon})), \ln(\ln(\ln(\frac{1}{\varepsilon}))), \cdots$

because $\lim\limits_{x\to\infty} \frac{\ln x}{x} = 0$

Def: If $\varphi_1(\varepsilon), \varphi_2(\varepsilon), \cdots, \varphi_n(\varepsilon), \cdots$ is an asymptotic sequence as $\varepsilon \to 0$, then $f(\varepsilon)$ has an N-term asymptotic expansion with respect to $\{\varphi_n(\varepsilon)\}, n=1,2,\cdots$ if $f(\varepsilon) = \sum_{n=1}^{N} a_n \varphi_n(\varepsilon) + o(\varphi_N), \ \varepsilon \to 0$

Ex's: ① $\sin\varepsilon = \varepsilon - \frac{1}{6}\varepsilon^3 + o(\varepsilon^3), \ \varepsilon \to 0$

$\to \sin\varepsilon \sim \varepsilon - \frac{1}{6}\varepsilon^3$

② $e^{\varepsilon} \sim 1 + \varepsilon + \varepsilon^2 \leftarrow$ NOT CORRECT

since $e^{\varepsilon} - (1+\varepsilon+\varepsilon^2) = o(\varepsilon^2)$ is false

($e^{\varepsilon} \sim 1 + \varepsilon + \frac{1}{2}\varepsilon^2$ is correct)

③ $\sin(\sin\varepsilon) \sim \sin\varepsilon - \frac{1}{6}\sin^3\varepsilon, \ \varepsilon \to 0$

true since $\sin(\sin\varepsilon) - (\sin\varepsilon - \frac{1}{6}\sin^3\varepsilon)$

$= \frac{1}{5!}\sin^5\varepsilon = o(\sin^3\varepsilon)$

also $\sin(\sin\varepsilon) \sim \varepsilon - \frac{1}{3}\varepsilon^3$

Given $\varphi_1(\varepsilon), \varphi_2(\varepsilon), \cdots, \varphi_n(\varepsilon), \cdots, \quad \varepsilon \to 0$

Want $f(\varepsilon) \to a_1\varphi_1(\varepsilon) + a_2\varphi_2(\varepsilon) + \cdots + a_N\varphi_N(\varepsilon)$

$$\to \frac{f(\varepsilon)}{\varphi_1(\varepsilon)} \sim a_1 + a_2\frac{\varphi_2(\varepsilon)}{\varphi_1(\varepsilon)} + \cdots + a_N\frac{\varphi_n(\varepsilon)}{\varphi_1(\varepsilon)}$$

as $\varepsilon \to 0$, this term will $\to 0$ (as will the following terms)

So $a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varphi_1(\varepsilon)}$

$$a_2 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - a_1\varphi_1(\varepsilon)}{\varphi_2(\varepsilon)}$$

etc.

$\varphi_1(\varepsilon), \varphi_2(\varepsilon), \cdots, \varphi_n(\varepsilon), \cdots$ 10-7

is called an asymptotic sequence as $\varepsilon \to 0$

if $\varphi_{n+1}(\varepsilon) = o(\varphi_n(\varepsilon))$ as $\varepsilon \to 0$ for all $n = 1, 2, 3, \cdots$

Meaning $\lim_{\varepsilon \to 0} \frac{\varphi_{n+1}(\varepsilon)}{\varphi_n(\varepsilon)} = 0$

N-term asymptotic expansion of $f(\varepsilon)$ as $\varepsilon \to 0$

$f(\varepsilon) \sim a_1\varphi_1(\varepsilon) + \cdots + a_N\varphi_N(\varepsilon)$

Means $f(\varepsilon) = a_1\varphi_1(\varepsilon) + \cdots + a_N\varphi_N(\varepsilon) + o(\varphi_N(\varepsilon))$

Asymptotic series

$$f(\varepsilon) \sim \sum_{n=1}^{\infty} a_n\varphi_n(\varepsilon)$$

means each partial sum is an asymptotic expansion

Note that it's not $=$

$$f(\varepsilon) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\varepsilon^n$$

This is an asymptotic series, but it actually goes further - to equality

$$f(\varepsilon) \sim \sum_{n=1}^{N} a_n \varphi_n(\varepsilon)$$

$$cf(\varepsilon) \sim \sum_{n=1}^{N} ca_n \varphi_n(\varepsilon)$$

$$g(\varepsilon) \sim \sum_{n=1}^{N} b_n \varphi_n(\varepsilon)$$

$$f(\varepsilon) + g(\varepsilon) \sim \sum_{n=1}^{N} (a_n + b_n) \varphi_n(\varepsilon)$$

$$f(x,\varepsilon) \sim a_1(x)\varphi_1(\varepsilon) + \cdots + a_N(x)\varphi_N(\varepsilon)$$

$$\int_a^b f(x,\varepsilon)\,dx \sim \int_a^b a_1(x)\,dx\, \varphi_1(\varepsilon) + \cdots + \int_a^b a_N(x)\,dx\, \varphi_n(\varepsilon)$$

(differentiation may not work)

$$\frac{df(x,\varepsilon)}{dx} \sim b_1(x)\varphi_1(\varepsilon) + \cdots + b_N(x)\varphi_N(\varepsilon)$$

if this exists, then

$$b_n(x) = \frac{da_n}{dx}$$

Ex: $I(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1+\varepsilon t}\,dt \quad , \quad 0 \leq \varepsilon \ll 1$

$$I(0) = \int_0^\infty e^{-t}\,dt = 1$$

Integrate by parts: $u=$

$$I(\varepsilon) = -\int_0^\infty \frac{de^{-t}}{1+\varepsilon t} = \frac{-e^{-t}}{1+\varepsilon t}\Big|_{t=0}^{\infty} - \varepsilon \int_0^\infty \frac{e^{-t}\,dt}{(1+\varepsilon t)^2}$$

$$= 1 - \varepsilon \int_0^\infty \frac{e^{-t}\,dt}{(1+\varepsilon t)^2}$$

$$I_n(\varepsilon) = \int_0^\infty \frac{e^{-t}\,dt}{(1+\varepsilon t)^n} = -\int_0^\infty \frac{de^{-t}}{(1+\varepsilon t)^n} = 1 - n\varepsilon I_{n+1}(\varepsilon)$$

$$I(\varepsilon) = 1 - \varepsilon I_2(\varepsilon) = 1 - \varepsilon(1 - 2\varepsilon I_3(\varepsilon))$$

$$= 1 - \varepsilon + 2\varepsilon^2 I_3(\varepsilon)$$

$$= 1 - \varepsilon + 2\varepsilon^2(1 - 3\varepsilon I_4(\varepsilon))$$

$$= 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3 I_4(\varepsilon)$$

$= 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3(1 - 4\varepsilon I_5(\varepsilon))$

$= 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3 + 24\varepsilon^4 I_5(\varepsilon)$

$= 1 - \varepsilon + 2\varepsilon^2 - 3!\varepsilon^3 + 4!\varepsilon^4 I_5(\varepsilon)$

So $I(\varepsilon) = 1 - \varepsilon + 2\varepsilon^2 - 3!\varepsilon^3 + \cdots + (-1)^N N!\varepsilon^N + (-1)^{N+1}(N+1)!\varepsilon^{N+1} \cdot I_{N+2}(\varepsilon)$

This is an exact equation

$I(\varepsilon) \sim \sum_{n=0}^{N} (-1)^n n! \varepsilon^n$ ← this is an asymptotic expansion!

Claim: $(-1)^{N+1}(N+1)!\varepsilon^{N+1} I_{N+2}(\varepsilon) = o(\varepsilon^N)$

need $\lim_{\varepsilon \to 0} \left| \frac{(-1)^{N+1}(N+1)!\varepsilon^{N+1} I_{N+2}(\varepsilon)}{\varepsilon^N} \right|$

$= \lim_{\varepsilon \to 0} (N+1)!\varepsilon^1 I_{N+2}(\varepsilon)$

does this $\to 0$?

Need $I_{N+2}(\varepsilon)$ to have a finite limit

Look back at the integral,

$\lim_{\varepsilon \to 0} I_{N+2}(\varepsilon) = 1$

So the lim above $= 0$ ✓

Then $I_\varepsilon \sim \sum_{n=0}^{\infty} (-1)^n n! \varepsilon^n$ is an asymptotic series

R: radius of convergence

$0 < \varepsilon < R \to$ series converges

ratio test: $R = \lim_{n \to \infty} \left| \frac{(-1)^n n!}{(-1)^{n+1}(n+1)!} \right| = \lim_{n \to \infty} \left| \frac{1}{-1(n+1)} \right|$

$= \lim_{n \to \infty} \frac{1}{n+1} = 0$

So this series doesn't converge!

($n!$ increases too fast) (except at $\varepsilon = 0$)

let $S_N(\varepsilon) = \sum_{n=0}^{N} (-1)^n n! \varepsilon^n$

$I(0.1) = 0.9156\ldots$

$S_4(0.1) = 0.9164\ldots$

$S_{10}(0.1) = 0.9158\ldots$

why is it giving good approximations if it diverges?

$S_{20}(0.1) = 0.9319\ldots$

$S_{30}(0.1) = 200.27\ldots$

What's the difference between a convergent series + an asymptotic series?

$f(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$

converges for $0 \le \varepsilon < R$

the more terms you take, the more accurate the approximation will be

fix ε. To approx $f(\varepsilon)$, take $n \to \infty$

Asymptotic Series

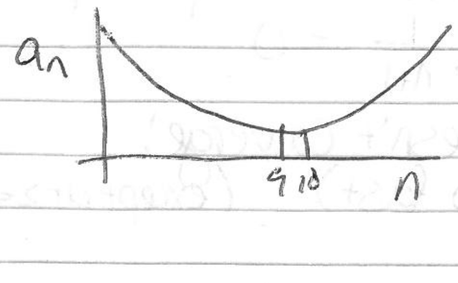
fix N and take $\varepsilon \to 0$ to approx $f(\varepsilon)$

Why did that blow up before?

look at $n!(0.1)^n$ next term is $0.1(n+1)$ times prev. term

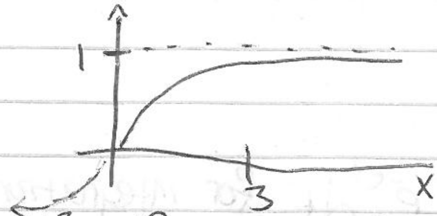
while $0.1(n+1) < 1$, it decreases

then when $0.1(n+1) > 1$, it blows up



take the # of terms that corresponds to the min here

Ex $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$



want to approx $\operatorname{erf}(3)$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (t^2)^n$$

$$\int_0^x t^{2n} dt = \left. \frac{t^{2n+1}}{2n+1} \right|_0^x$$

$$\int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

$$\to \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

Ratio test: converges for all $x > 0$

$$\operatorname{erf}(3) \approx \frac{2}{\sqrt{\pi}} \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{3^{2n+1}}{2n+1} \equiv S_N(3)$$

$\operatorname{erf}(3) = 0.9999779095\ldots$

$S_5(3) = -86.7144\ldots$

$S_{10}(3) = 68.586\ldots$

$S_{20}(3) = 1.12078\ldots$

$S_{100}(3) = 0.9999779097\ldots$

$S_{200}(3) = 0.9999779097\ldots$

another problem:

$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$

$x = 10$

the RHS will be large numbers, while the LHS, $\sin(10)$ we know is between -1 + 1

The computer has problems because we're interested in the #'s after the decimal, while it uses all its digit storage on the #'s before the decimal.

We can compute erf(x) with an Asymptotic expansion instead

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\int_0^x e^{-t^2} dt = \int_0^x \frac{2t}{2t} e^{-t^2} dt$$ for integration by parts

$$\text{erf}(x) = \frac{2}{\sqrt{x}} \int_0^x e^{-t^2} dt = \int_0^\infty e^{-t^2} dt - \int_x^\infty e^{-t^2} dt$$

$$= 1 - \text{erfc}(x)$$

integration by parts:

$$\int_x^\infty e^{-t^2} dt = \int_x^\infty \frac{2te^{-t^2}}{2t} dt = -\int_x^\infty \frac{de^{-t^2}}{2t}$$

$$= \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-t^2}}{2t^2} dt$$

$$I_n = \int_x^\infty \frac{e^{-t^2}}{t^n} dt = -\int_x^\infty \frac{de^{-t^2}}{2t^{n+1}} = \frac{e^{-x^2}}{2x^{n+1}} - \frac{n+1}{2} \int_x^\infty \frac{e^{-t^2}}{t^{n+2}} dt$$

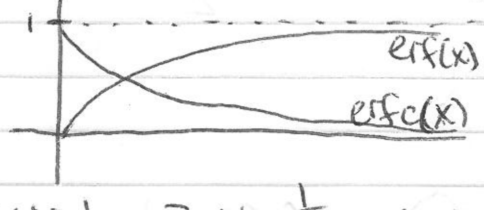
$$= \frac{e^{-x^2}}{2x^{n+1}} - \frac{1}{2}(n+1) I_{n+2}$$

(Continue from prev.)

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt = 1-\mathrm{erfc}(x)$$

$\mathrm{erf}(3) = ?$

expand about $x=\infty$ (asymptotic series)



$x \gg 1 \rightarrow x = \frac{1}{\varepsilon}$, ε a small parameter.

$$\int_x^\infty e^{-t^2}dt = -\int_x^\infty \frac{de^{-t^2}}{2t} = \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-t^2}}{2t^2}dt$$

$$I_n = \int_x^\infty \frac{e^{-t^2}}{t^n}dt = -\int_x^\infty \frac{de^{-t^2}}{2t^{n+1}} = \frac{e^{-x^2}}{2x^{n+1}} - \frac{1}{2}(n+1)\int_x^\infty \frac{e^{-t^2}}{t^{n+2}}dt$$

$$I_n = \frac{e^{-x^2}}{2x^{n+1}} - \frac{1}{2}(n+1)I_{n+2}$$

$$\int_x^\infty e^{-t^2}dt = \frac{e^{-x^2}}{2x} - \frac{1}{2}I_2 = \frac{e^{-x^2}}{2x} - \frac{1}{2}\left[\frac{e^{-x^2}}{2x^3} - \frac{1}{2}\cdot 3\cdot I_4\right]$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} + \frac{3}{2^2}\left[\frac{e^{-x^2}}{2x^5} - \frac{5}{2}I_6\right]$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} + \frac{3}{2^3}\frac{e^{-x^2}}{x^5} - \frac{3\cdot 5}{2^3}\left[\frac{e^{-x^2}}{2x^7} + \ldots\right.$$

$$= \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{3}{2^2x^4} - \frac{3\cdot 5}{2^3x^6} + \frac{3\cdot5\cdot7}{2^4x^8} - \ldots\right]$$

[Notation: $2\cdot4\cdot6\cdot8\cdot10 = 10!!$, $3\cdot5\cdot7\cdot9 = 9!!$]

$$= \frac{e^{-x^2}}{2x}\left[1 + \sum_{n=1}^\infty (-1)^n \frac{(2n-1)!!}{(2x^2)^n}\right]$$

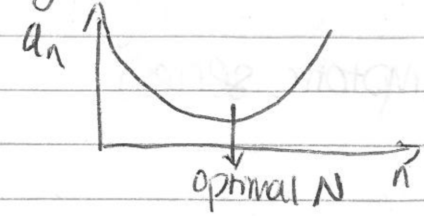
So, $\mathrm{erf}(x) \sim 1 - \dfrac{e^{-x^2}}{x\sqrt{\pi}}\left[1 + \sum_{n=1}^\infty (-1)^n \dfrac{(2n-1)!!}{(2x^2)^n}\right]$

(series diverges for all $x \rightarrow R=0$)

$\mathrm{erf}(3) = 0.9999779\ldots$

Approx with $N=3$: $0.999979\ldots$

larger values of x will give a better approx.



$$a_n = \frac{(-1)^n (2n-1)!!}{2^n} \quad \text{here}$$

Ex: $(x-1)(x-p) - \varepsilon x = 0$, $0 < \varepsilon \ll 1$

p: parameter, $0 < p < 1$

Reduced problem: $(x_0 - 1)(x_0 - p) = 0 \rightarrow x_0 = 1, p$

$x = 1 + \varepsilon x_1 + O(\varepsilon^2)$

Plug in: $\varepsilon x_1 (1-p) - \varepsilon + O(\varepsilon^2) = 0$

$x_1 = 1/1-p$

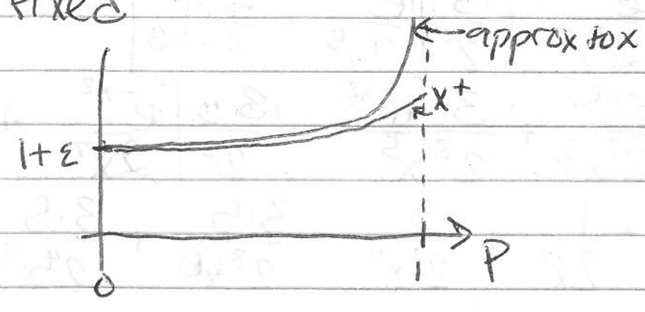
$x \sim 1 + \varepsilon/1-p$

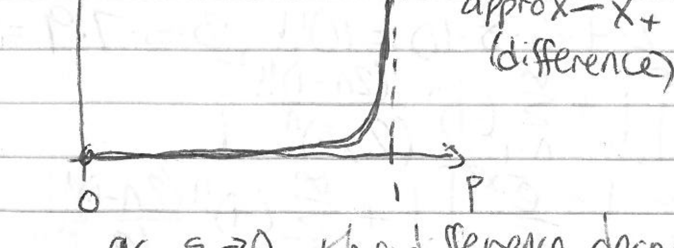
find exact sol'n:

$x^2 - (1+p+\varepsilon)x + p = 0$

$x_+ = \frac{1}{2}\left(1+p+\varepsilon + \sqrt{(1+p+\varepsilon)^2 - 4p}\right)$

ε fixed





as $\varepsilon \to 0$, the difference decreases, but is still ∞ at $p=0$

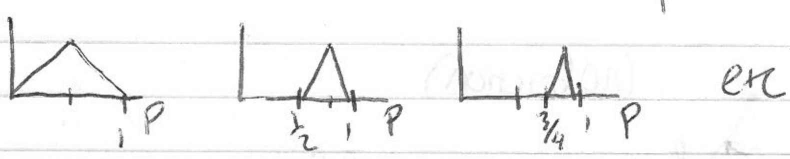
Called non-uniform convergence wrt parameter p
error goes to zero for any $p \in [0,1)$, but
it doesn't approach zero at the same rate.
But convergence is uniform for $p \in [0, 0.9]$

Def: $\varphi(p,\varepsilon)$ is called a uniform approximation for $f(p,\varepsilon)$ as $\varepsilon \to 0$ and $p \in I$ if

$$\lim_{\varepsilon\to 0} \underbrace{\left| \frac{f(p,\varepsilon) - \varphi(p,\varepsilon)}{\varphi(p,\varepsilon)} \right|}_{\text{call this } g(p,\varepsilon)} = 0 \quad \text{uniformly in } p \in I.$$

i.e. $\forall \delta > 0 \ \exists \varepsilon_* \text{ s.t. } \forall \varepsilon < \varepsilon_*, \ |g(p,\varepsilon)| < \delta$
where ε_* does not depend on p.

Ex: another non-uniform example



pointwise convergent everywhere, even bounded
but not uniformly convergent
difference btwn function shown above
and zero
The maximum of the function is always
one - that is the problem.

Ex: $\sin \varepsilon x$, $0 < x \ll 1$, $0 < x < 1$
is it true that $\sin \varepsilon x \sim \varepsilon x$ uniformly in x?

$$\lim_{\varepsilon\to 0} \left| \frac{\sin\varepsilon x - \varepsilon x}{\varepsilon x} \right| = 0 \quad ✓ \quad \text{but uniform in } x?$$

know $t - t^3/6 < \sin t < t$, $t > 0$ (graphically)

$\rightarrow \varepsilon x - (\varepsilon x)^3/6 < \sin \varepsilon x < \varepsilon x$

$\rightarrow -(\varepsilon x)^3/6 < \sin \varepsilon x - \varepsilon x < 0$

$\rightarrow -(\varepsilon x)^2/6 < \dfrac{\sin\varepsilon x - \varepsilon x}{\varepsilon x} < 0$

$$\rightarrow \left| \frac{\sin \varepsilon x - \varepsilon x}{\varepsilon x} \right| < \frac{(\varepsilon x)^2}{6} < \frac{\varepsilon^2}{6}$$ since $0<x<1$

Since this doesn't depend on x, and it $\to 0$ as $\varepsilon \to 0$, we're set!

<u>Ex</u>: $\sin \varepsilon/x \sim \varepsilon/x \quad \varepsilon \to 0, \; 0<x<1$

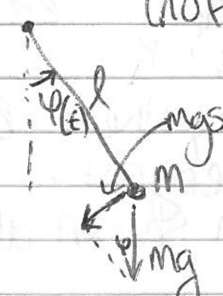
true uniformly?

$$\left| \frac{\sin \frac{\varepsilon}{x} - \frac{\varepsilon}{x}}{\frac{\varepsilon}{x}} \right| \quad \text{let } x = \varepsilon.$$

$$\rightarrow \left| \frac{\sin 1 - 1}{1} \right| = \text{const} \not\to 0$$

Not uniform.

<u>Ex</u>: (no friction)



$$m l \ddot{\varphi} = -mg \sin \varphi$$

$$\ddot{\varphi} + \omega^2 \sin \varphi = 0, \quad \omega^2 = g/l$$

$$\varphi(0) = \varphi_i, \; \dot{\varphi}(0) = 0$$

if φ_i is small, then $-\varphi_i \le \varphi(t) \le \varphi_i$

so $\varphi(t)$ is also small.

$\rightarrow \sin \varphi = \varphi + \dots$

$\rightarrow \ddot{\varphi} + \omega^2 \varphi = 0$

period $= 2\pi/\omega$

or we could say $\sin \varphi \sim \varphi - \varphi^3/6$

$\rightarrow \ddot{\varphi} + \omega^2 (\varphi - \varphi^3/6) = 0$

Let $u = \varphi/\varphi_i \rightarrow \varphi = \varphi_i u, \; \dot{\varphi} = \varphi_i \dot{u}, \; \ddot{\varphi} = \varphi_i \ddot{u}$

$\rightarrow \varphi_i \ddot{u} + \omega^2 (\varphi_i u - \frac{1}{6} \varphi_i^3 u^3) = 0$

$\varphi_i u(0) = \varphi_i, \; \varphi_i \dot{u}(0) = 0$

$\rightarrow \ddot{u} + \omega^2(u - \varepsilon u^3) = 0$ where $\varepsilon = \frac{1}{6}\varphi_i^2$

$u(0) = 1, \ \dot{u}(0) = 0$

$u(t) = u_0(t) + \varepsilon u_1(t) + O(\varepsilon^2)$

Plug in: $\ddot{u}_0 + \varepsilon \ddot{u}_1 + O(\varepsilon^2) + \omega^2[u_0 + \varepsilon u_1 - \varepsilon u_0^3]$

$u_0(0) + \varepsilon u_1(0) + O(\varepsilon^2) = 1$

$\dot{u}_0(0) + \varepsilon \dot{u}_1(0) + O(\varepsilon^2) = 0$

$\varepsilon^0: \ddot{u}_0 + \omega^2 u_0 = 0, \ u_0(0) = 1, \ \dot{u}_0(0) = 0$

$\rightarrow u_0(t) = \cos \omega t$

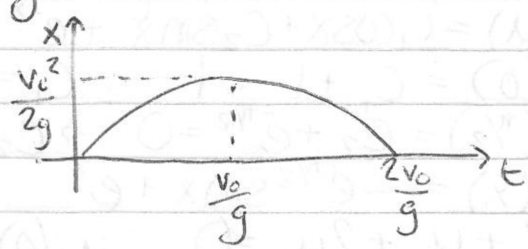
$\varepsilon^1: \ddot{u}_1 + \omega^2 u_1 = \omega^2 u_0^3, \ u_1(0) = 0, \ \dot{u}_1(0) = 0$

10-

$m\ddot{x} = -mg\dfrac{R^2}{(R+x)^2}$ $\quad x(0)=0,\ \dot{x}(0)=v_0$ 10-16

$m\ddot{x} = -mg$ $\quad x(0)=0,\ \dot{x}(0)=v_0$

$\rightarrow x(t) = -\frac{1}{2}gt^2 + v_0 t$



$x(t) = x_* y(\tau)$ $\quad \tau = t/t_*$

let $t_* = v_0/g$, $x_* = v_0^2/g$

$\rightarrow \ddot{y} = \dfrac{-1}{(1+\varepsilon y)^2}$, $y(0)=0,\ \dot{y}(0)=1$, $\varepsilon = x_*/R = v_0^2/Rg \ll 1$

$y(\tau) \sim y_0(\tau) + \varepsilon y_1(\tau)$

$\ddot{y}_0 + \varepsilon \ddot{y}_1 + O(\varepsilon^2) = \dfrac{-1}{(1+\varepsilon y_0)^2}$, $\quad y_0(0)+\varepsilon y_1(0) = 0$, $\dot{y}_0(0) + \varepsilon \dot{y}_1(0) = 1$

Taylor expand RHS: $-1 + 2\varepsilon y_0 + O(\varepsilon^2)$

ε^0: $\ddot{y}_0 = -1$, $y_0(0)=0$, $\dot{y}_0(0)=1$

$y_0(\tau) = -\frac{1}{2}\tau^2 + \tau$

corresponds to prev. sol'n

ε^1: $\ddot{y}_1 = 2y_0 = -\tau^2 + 2\tau$, $y_1(0)=0$, $\dot{y}_1(0)=0$

$y_1(\tau) = -\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3$

$\rightarrow y(\tau) = -\frac{1}{2}\tau^2 + \tau + \varepsilon\left(-\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3\right)$

Find T st $y(T)=0$ $\quad T \sim T_0 + \varepsilon T_1$

$y(T) = -\frac{1}{2}(T_0 + \varepsilon T_1)^2 + T_0 + \varepsilon T_1 + \varepsilon\left[-\frac{1}{12}T_0^4 + \frac{1}{3}T_0^3\right] = 0$

ε^0: $-\frac{1}{2}T_0^2 + T_0 = 0$ $\quad T_0 = 2, 0$ ✓

ε^1: $-T_0T_1 + T_1 - \frac{1}{12}T_0^4 + \frac{1}{3}T_0^3$

$T_1 = 8/3 - 16/12 = 4/3$

$T = 2 + 4/3\,\varepsilon$

Ex: BVP

$y'' + (1+2\varepsilon)y = 2e^{-x}, \quad 0<x<\tfrac{\pi}{2}, \; y(0)=1, \; y(\tfrac{\pi}{2})=0$

$y \sim y_0 + \varepsilon y_1$

$\varepsilon^0: \; y_0'' + y_0 = 2e^{-x}, \quad y_0(0)=1, \; y_0(\tfrac{\pi}{2})=0$

$y_0(x) = c_1\cos x + c_2 \sin x + e^{-x}$

$y_0(0) = c_1 + 1 = 1 \rightarrow c_1 = 0$

$y_0(\tfrac{\pi}{2}) = c_2 + e^{-\pi/2} = 0 \rightarrow c_2 = -e^{-\pi/2}$

$y_0(x) = -e^{-\pi/2}\sin x + e^{-x}$

$\varepsilon^1: \; y_1'' + y_1 + 2y_0 = 0, \quad y_1(0)=0, \; y_1(\tfrac{\pi}{2})=0$

$y_1'' + y_1 = 2e^{-\pi/2}\sin x - 2e^{-x}$

$y_1(x) = c_1\cos x + c_2\sin x + e^{-\pi/2}x\cos x - e^{-x}$

$y_1(0) = c_1 - 1 = 0 \rightarrow c_1 = 1$

$y_1(\tfrac{\pi}{2}) = c_2 - e^{-\pi/2} \rightarrow c_2 = e^{-\pi/2}$

$y_1(x) = \cos x - e^{-x} + e^{-\pi/2}(\sin x - x\cos x)$

$y \sim -e^{-\pi/2}\sin x + e^{-x} + \varepsilon\left(\cos x - e^{-x} + e^{-\pi/2}(\sin x - x\cos x)\right)$

Ex: Eigenvalue Problem

$y'' + (\lambda + \varepsilon q(x))y = 0, \quad a<x<b, \; y(a)=y(b)=0$

$y(x) = y_0(x) + \varepsilon y_1(x)$

$\lambda = \lambda_0 + \varepsilon\lambda_1$

$y_0'' + \varepsilon y_1'' + (\lambda_0 + \varepsilon\lambda_1 + \varepsilon q)(y_0 + \varepsilon y_1) + O(\varepsilon^2) = 0$

$\varepsilon^0: \; y_0'' + \lambda_0 y_0 = 0, \quad y_0(a) = y_0(b) = 0$

$\varepsilon^1: \; y_1'' + \lambda_0 y_1 = -(\lambda_1 + q)y_0 \quad y_1(a) = y_1(b) = 0$

$y_0(x) = c_1\cos\sqrt{\lambda_0}(x-a) + c_2\sin\sqrt{\lambda_0}(x-a)$

$y_0(a) = c_1 = 0$

$y_0(b) = c_2\sin\sqrt{\lambda_0}(b-a) = 0$

$\rightarrow \sqrt{\lambda_0}(b-a) = \pi n \quad n = 1, 2, \ldots$

$\lambda_0 = [\pi n/(b-a)]^2$

$y_0(x) = C\sin\left(\pi n \dfrac{x-a}{b-a}\right)$

Fredholm Alternative theorem

$y''+y=f(x)$, $0<x<\pi$, $y(0)=y(\pi)=0$

$\int_0^\pi (y''+y)\sin x\,dx=\int_0^\pi f(x)\sin x\,dx$

$\int_0^\pi y''\sin x\,dx = y'\sin x\Big|_{x=0}^{\pi} - \int_0^\pi y'\cos x\,dx$

$= -y\cos x\Big|_{x=0}^{\pi} - \int_0^\pi y\sin x\,dx$

So LHS above $=0$

$\rightarrow \int_0^\pi f(x)\sin x\,dx = 0$

Or in general

$\int_0^\pi (y''+y)v\,dx = \int_0^\pi f(x)v\,dx$

$\int_0^\pi (v''+v)dx = \int_0^\pi f(x)v\,dx$ if $v(0)=v(\pi)=0$

if $v''+v=0$ then $\int_0^\pi f(x)v\,dx=0$

$v''+v=0$, $v(0)=v(\pi)=0$ is called adjoined prob.

=

So, back to our example eigenvalue problem:

Need RHS $-(\lambda_1+q)y_0$ to be orthogonal to y_0.

$\int_a^b -(\lambda_1+q(x))y_0(x)\,y_0(x)\,dx=0$

$$\lambda_1 = -\frac{\int_a^b q(x)y_0^2(x)dx}{\int_a^b y_0^2(x)dx}$$

$$= -\frac{2}{b-a}\int_a^b q(x)\sin^2 \pi n\frac{x-a}{b-a}dx$$

So:

$$\lambda=\frac{\pi^2 n^2}{(b-a)^2} - \frac{2\varepsilon}{b-a}\int_a^b q(x)\sin^2\left(\pi n\frac{x-a}{b-a}\right)dx$$

Ex: $xu'' + u' + \lambda x u = 0$, $1<x<2$, $u(1)=u(2)=0$

$a_0(x)u'' + a_1(x)u' + a_2(x)u = 0$

let $u = y\, e^{\left(-\frac{1}{2}\int \frac{a_1(x)}{a_0(x)}dx\right)}$

So let $u = x^{-\frac{1}{2}} y$

$y'' + \left(\lambda + \frac{1}{4x^2}\right)y = 0 \qquad y(1) = y(2) = 0$

let $\frac{1}{4}\frac{1}{x^2} = \varepsilon q(x)$, $\varepsilon = 1/4$, $q(x) = 1/x^2$

Use answer from prev. example.

$\lambda \approx \pi^2 n^2 - 2 \cdot \frac{1}{4}\int_1^2 \frac{1}{x^2} \sin^2 \pi n(x-1)\, dx$

n	(exact) numerical	approximate
1	9.7537...	9.7534...
2	39.35599...	39.35599...

Ex: $y'' + \left(\lambda + \frac{10+11x^2}{1+x^2}\right)y = 0$, $y(0) = y(1) = 0$

don't do $\varepsilon = 1$

$\rightarrow y'' + \left(\lambda + 10 + \frac{x^2}{1+x^2}\right)y = 0$

let $\lambda_1 = \lambda + 10$ then you can let $\varepsilon = 1$

Ex: $y'' + (1+\varepsilon)y = 1, \quad y(0)=0, \; y(\pi)=0$

$y(x) \sim y_0(x) + \varepsilon y_1(x)$

$\varepsilon^0: \; y_0'' + y_0 = 1 \quad , \; y_0(0)=0, \; y_0(\pi)=0$

$y_0(x) = 1 + c_1 \cos x + c_2 \sin x$

$y_0(0) = 1 + c_1 = 0 \quad \to c_1 = -1$

$y_0(\pi) = 1 - c_1 = 0 \quad \to c_1 = 1$

$\to$ no solution

$y_H'' + y_H = 0 \quad , \; y_H(0)=0, \; y_H(\pi)=0$

$y_H(x) = c \sin x$

Then we'd need

$\int_0^\pi f(x) y_H(x) dx = 0$

$\int_0^\pi \sin x \, dx = -\cos x |_0^\pi = 1+1 = 2 \neq 0$

We can still find a sol'n of the original problem:

Exact solution

$y(x) = \frac{1}{1+\varepsilon} + c_1 \cos\sqrt{1+\varepsilon}\, x + c_2 \sin\sqrt{1+\varepsilon}\, x$

$y(0) = \frac{1}{1+\varepsilon} + c_1 = 0 \quad \to c_1 = \frac{-1}{1+\varepsilon}$

$y(\pi) = \frac{1}{1+\varepsilon} - \frac{1}{1+\varepsilon}\cos\sqrt{1+\varepsilon}\,\pi + c_2 \sin\sqrt{1+\varepsilon}\,\pi = 0$

$$c_2 = \frac{\frac{1}{1+\varepsilon}\left(\cos\sqrt{1+\varepsilon}\,\pi - 1\right)}{\sin\sqrt{1+\varepsilon}\,\pi}$$

$$y(x) = \frac{1}{1+\varepsilon}\left[1 - \cos\sqrt{1+\varepsilon}\,x + \frac{\cos\sqrt{1+\varepsilon}\,\pi - 1}{\sin\sqrt{1+\varepsilon}\,\pi}\sin\sqrt{1+\varepsilon}\,x\right]$$

$\varepsilon \to 0: \quad -2 / -\frac{\varepsilon\pi}{2} \sin x = \frac{4}{\varepsilon\pi}\sin x$

$\sin\sqrt{1+\varepsilon}\,\pi \sim \sin(1+\frac{\varepsilon}{2})\pi \sim -\frac{\varepsilon}{2}\pi$

$y_{exact}(x) \sim \frac{4}{\varepsilon\pi}\sin x$

So leading order term is $O(\frac{1}{\varepsilon})$, not $O(1)$

Rescale:

$y(x) = \frac{z(x)}{\delta} \quad , \; \delta = \delta(\varepsilon) = o(1)$

$\to z'' + (1+\varepsilon) z = \delta, \quad z(0) = z(\pi) = 0$

$z(x) = z_0(x) + v(x) \quad , \; v = o(z_0)$

$z_0'' + v'' + (1+\varepsilon)(z_0 + v) = \delta$

$z_0'' + z_0 = 0$

$z_0'(0) = z_0(\pi) = 0$

$z_0(x) = c\sin x$

$v'' + v = -\underset{\text{smaller than } \varepsilon z_0}{\varepsilon v} - \varepsilon z_0 + \delta$, $v(0) = v(\pi) = 0$

if $\delta >> \varepsilon$, no sol'n Try $\varepsilon >> \delta$

$v = \varepsilon v_2$

$v_2'' + v_2 = -z_0$, $v_2(0) = v_2(\pi) = 0$

doesn't work either

let $\delta = \varepsilon$ $(v(x) = z(x)/\varepsilon)$

So $v = \varepsilon z_1$

$z_1'' + z_1 = -z_0 + 1 = 1 - c\sin x$

$\int_0^\pi (1 - c\sin x)\, c\sin x\, dx$

$= \underbrace{\int_0^\pi \sin x\, dx}_{=2} - c\underbrace{\int \sin^2 x\, dx}_{=\pi/2} = 0$

$\rightarrow c = 4/\pi$

$\rightarrow y_0(x) = \frac{4}{\pi}\varepsilon \sin x$

Hwk * $z_1'' + z_1 = 1 - \frac{4}{\pi}\sin x$ $z_1(0) = z_1(\pi) = 0$

Find z_1

$z_H'' + z_H = 0$ $z_H(0) = z_H(\pi) = 0$

$z_H(x) = c\sin x$

(sol'n is not unique)

↑ find constant using Fredholm Alternative with z_2 problem

Ex: $y'' + y + \varepsilon y^3 = 1, \quad y(0) = y(\pi) = 0$

$y = z/\delta$

$z'' + z = -\frac{\varepsilon}{\delta^2} z^3 + \delta, \quad z(0) = z(\pi) = 0$

if ε/δ^2 small:

$z_0'' + z_0 = 0, \; z_0(0) + z_0(\pi) = 0 \rightarrow z_0(x) = c \sin x$

So $\varepsilon/\delta^2 = \delta \rightarrow \delta = \varepsilon^{1/3}$

$z(x) \sim z_0(x) + \varepsilon^{1/3} z_1$

$z_1'' + z_1 = -c^3 \sin^3 x + 1$

$\int_0^\pi (1 - c^3 \sin^3 x) \sin x \, dx = 0$

$\rightarrow$ get c

$c = \left(\frac{16}{3\pi}\right)^{1/3}$

Boundary Layers

Ex: $\varepsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \; y(1) = 1$

$y(x) = y_0(x) + \ldots$

$2y_0' + 2y_0 = 0, \quad y_0(0) = 0, \; y_0(1) = 1$

$y_0(x) = ce^{-x}$

$y_0(0) = 0 = c \qquad y_0(x) = 0$

$y_0(1) = ce^{-1} = 1 \rightarrow c = e \quad y_0(x) = e^{1-x}$ $\Big\} \Rightarrow \Leftarrow$

$\varepsilon m^2 + 2m + 2 = 0$

$\varepsilon = 0: \; 2m + 2 = 0$

$\rightarrow m_1 = \frac{1}{2\varepsilon}\left[-2 + \sqrt{4 - 8\varepsilon}\right] = \left(-1 + \sqrt{1 - 2\varepsilon}\right)/\varepsilon \sim -1$

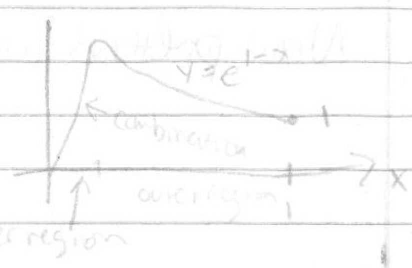
$m_2 = \frac{1}{2\varepsilon}\left[-2 - \sqrt{4 - 8\varepsilon}\right] \sim -2/\varepsilon$

$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y(x) = \dfrac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}}$ $\leftarrow$ satisfies bdry conditions

$\sim \dfrac{e^{-x} - e^{-2/\varepsilon x}}{e^{-1} - e^{-2/\varepsilon}}$ small

$= e^{1-x} - e^{1 - 2/\varepsilon x}$



inner region: boundary layer
Solve the problem seperately in
inner + outer regions
Then you need to match them

10-23 Ex: (from last time)

$\varepsilon y'' + 2y' + 2y = 0, \quad 0<x<1, \quad y(0)=0, \; y(1)=1$

Reduced problem: $2y_0' + 2y_0 = 0, \; y_0(0)=0, \; y_0(1)=1$

no sol'n

$\rightarrow$ original problem is singular

Exact solution:

$$\varepsilon m^2 + 2m + 2 = 0$$

$$m_1 = \frac{1}{\varepsilon}\left[-1-\sqrt{1-2\varepsilon}\right] \sim -1$$

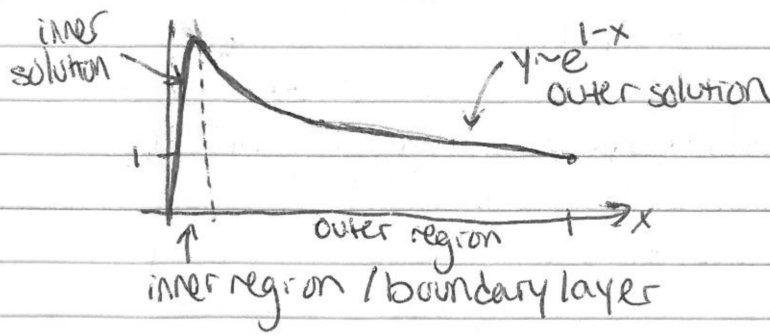
$$m_2 = \frac{1}{\varepsilon}\left[-1-\sqrt{1-2\varepsilon}\right] \sim -2/\varepsilon$$

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y(x) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}}$$

$$\sim \frac{e^{-x} - e^{-2/\varepsilon x}}{e^{-1} - e^{-2/\varepsilon}}$$

$$= e^{1-x} - e^{1-2/\varepsilon x}$$



Now pretend we couldn't find the exact sol'n.

Outer region:

$y(x) \sim y_0(x)$

$2y_0' + 2y_0 = 0, \quad y_0(1) = 1$

$y_0(x) = e^{1-x}$

Inner region:

$\xi = \frac{x}{\varepsilon}, \quad y(x) = Y(\xi) \quad \frac{dy}{dx} = \frac{dY}{d\xi}\frac{d\xi}{dx} = \frac{1}{\varepsilon}\frac{dY}{d\xi}$

$\frac{d^2y}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2Y}{d\xi^2}$

$\varepsilon \frac{1}{\varepsilon^2}\frac{d^2Y}{d\xi^2} + 2\frac{1}{\varepsilon}\frac{dY}{d\xi} + 2Y = 0$

$\frac{d^2Y}{d\xi^2} + 2\frac{dY}{d\xi} + 2\varepsilon Y = 0$

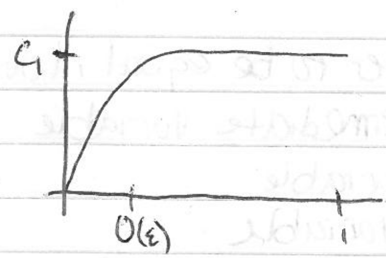
$Y(\xi) \sim Y_0(\xi)$

$Y_0'' + 2Y' = 0, \quad Y_0(0) = 0$

$Y_0(\xi) = C_1 + C_2 e^{1-2\xi}$

$Y_0(0) = C_1 + C_2 = 0 \rightarrow C_2 = -C_1$

$Y_0(\xi) = C_1(1 - e^{-2\xi})$



Need to match inner & outer

outer sol'n is basically constant in boundary layer

It has a value of $\approx e$

So we want the inner sol'n to saturate at e

Let outer sol'n at zero be the same as inner sol'n at saturation

$y_0|_{x=0} = Y_0|_{\xi=\infty}$

$Y_0|_{\xi=\infty} = C_1$

$y_0|_{x=0} = e$

$\rightarrow C_1 = e$

$Y_0(\xi) = e - e^{1-2\xi}$

common part

$y_{comp} = y_0(x) + Y_0(\xi) - e$

$= y_0(x) + Y_0(x/\varepsilon) - e$

y_{comp} = outer + inner − common

$$y_{comp} = e^{1-x} + e - e^{1-2\frac{x}{\varepsilon}} - e$$

$$= e^{1-x} - e^{1-2\frac{x}{\varepsilon}}$$

Scaling: we used $\xi = x/\varepsilon$

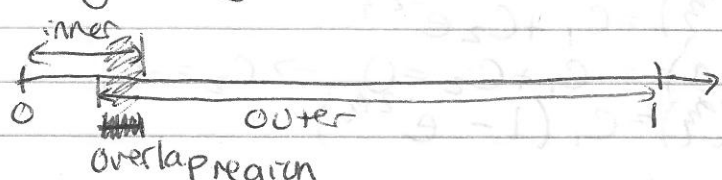
in general, use $\xi = x/\varepsilon^{\beta}$, $y(x) = Y(\xi)$, $\beta > 0$

$$\varepsilon^{1-2\beta} Y''(\xi) + 2\varepsilon^{-\beta} Y'(\xi) + 2Y = 0$$

$\rightarrow 1 - 2\beta = -\beta$ ← powers must be equal to balance large terms

$\rightarrow \beta = 1$

Matching using intermediate variables



want inner + outer to be equal in overlap region

$\eta = x/\varepsilon^{\alpha}$ ← intermediate variable, $0 < \alpha < 1$

$\alpha = 1$: inner variable

$\alpha = 0$: outer variable

Write outer solution in terms of η

$$y_0(x) = e^{1-x} = e^{1-\eta\varepsilon^{\alpha}} \sim e$$

$$Y_0(\xi) = C_1(1 - e^{-2\xi}) = C_1(1 - e^{-2\eta\varepsilon^{\alpha-1}}) \sim C_1$$

$\rightarrow C_1 = e$

Compare expansions: make leading orders equal

Solve for more terms:

Outer: $y_i(x) = y_0(x) + \varepsilon y_1(x)$

$$2y_0' + 2y_0 = 0 \quad y_0(1) = 1 \rightarrow y_0(x) = e^{1-x}$$

$$2y_1' + 2y_1 = -y_0'' \quad y_1(1) = 0$$

$$y_1(x) = \tfrac{1}{2}e^{1-x} - \tfrac{1}{2}xe^{1-x}$$

$$= \tfrac{1}{2}e^{1-x}(1-x)$$

$\rightarrow y(x) = e^{1-x} + \frac{1}{2}\varepsilon(1-x)e^{1-x}$

Inner Solution:

$y(x) = Y(\xi), \quad \xi = x/\varepsilon$

$Y'' + 2Y' + 2\varepsilon Y = 0 \qquad Y(0) = 0$

$Y(\xi) = Y_0(\xi) + \varepsilon Y_1(\xi)$

$Y_0'' + 2Y_0' = 0 \quad , \quad Y_0(0) = 0$

$Y_0(\xi) = c_1(1-e^{-2\xi})$

$Y_1'' + 2Y_1' = -2Y_0 \quad , \quad Y_1(0) = 0$

$Y_1'' + 2Y_1' = -2c_3(1-e^{-2\xi})$

$Y_1(\xi) = -c_3\xi(1+e^{-2\xi}) + c_4(1-e^{-2\xi})$

$Y(\xi) = c_3(1-e^{-2\xi}) + \varepsilon[c_4(1-e^{-2\xi}) - c_3\xi(1+e^{-2\xi})]$

Match:

$\eta = x/\varepsilon^\alpha \qquad (\frac{1}{2} < \alpha < 1)$

$x = \varepsilon^\alpha \eta$

outer:

$e^{1-\varepsilon^\alpha\eta} + \frac{1}{2}\varepsilon(1-\varepsilon^\alpha\eta)e^{1-\varepsilon^\alpha\eta}$

$= e(1-\varepsilon^\alpha\eta) + \frac{1}{2}\varepsilon e$ (disregard $\varepsilon^{2\alpha}, \varepsilon^{1+\alpha}$)

inner:

$\xi = x/\varepsilon = \varepsilon^{\alpha-1}\eta$ (large)

$c_3 + c_4\varepsilon - c_3\varepsilon^\alpha\eta$

$= c_3(1-\varepsilon^\alpha\eta) + c_4\varepsilon$

Compare outer + inner: $c_3 = e, \; c_4 = \frac{1}{2}e$

$y_c(x) = y_{outer}(x) + Y_{inner}(\frac{x}{\varepsilon}) -$ common part

$= e^{1-x} + \frac{1}{2}\varepsilon(1-x)e^{1-x} + (1-e^{-2\frac{x}{\varepsilon}})(e + \frac{1}{2}e\varepsilon)$

$\quad - e\,x/\varepsilon(1+e^{-2\frac{x}{\varepsilon}}) - e(1-x) - \frac{1}{2}\varepsilon e$

$y_c(x) = e^{1-x} + \frac{1}{2}\varepsilon(1-x)e^{1-x} - e^{-2x/\varepsilon}(e + \frac{1}{2}e\varepsilon)$

$\quad - e\,x\,e^{-2x/\varepsilon}$

$= e^{1-x}(1 + \frac{1}{2}\varepsilon(1-x)) - e^{1-2x/\varepsilon}[1 + \frac{1}{2}\varepsilon + x]$

10-28 Ex: $\varepsilon y'' - (x-1)y' - y = 0$, $y(0) = y(1) = 1$

Where is the boundary layer?

near zero:

outer: $y(x) \sim y_0(x)$

$$-(x-1)y_0' - y_0 = 0, \quad y_0(1) = 1$$

$$-[(x-1)y_0]' = 0$$

$$y_0(x) = \frac{c}{x+1}$$

$$y_0(1) = c/2 = 1 \to c = 2$$

$$y_0(x) = \frac{2}{x+1}$$

inner:

$\xi = \frac{x}{\varepsilon}$, $y(x) = Y(\xi)$

$$\to \frac{1}{\varepsilon}Y'' - (\varepsilon\xi + 1)\frac{1}{\varepsilon}Y' - Y = 0, \quad Y(0) = 1$$

$$Y'' - (\varepsilon\xi + 1)Y' - \varepsilon Y = 0, \quad Y(0) = 1$$

$$Y(\xi) \sim Y_0(\xi)$$

$$Y_0'' - Y_0' = 0, \quad Y_0(0) = 1$$

$$Y_0(\xi) = c_1 + c_2 e^{\xi}$$

$$Y_0(0) = c_1 + c_2 = 1$$

$$Y_0(\xi) = 1 - c_2 + c_2 e^{\xi}$$

match:

$\frac{2}{x+1} \to 2$ $\quad 1 - c_2 + c_2 e^{\xi} \to \infty$, so let $c_2 = 0$, $\to 1$

doesn't work.

(caused by sign on y')

boundary layer near center? at $x = a$, $0 < a < 1$ $\to$ internal layer

outer near 1: $y = \frac{2}{x+1}$

outer near 0: $y = \frac{1}{x+1}$

can we match these?

let $\xi = \frac{x-a}{\varepsilon}$, $y(x) = Y(\xi)$

$$\frac{1}{\varepsilon}Y'' - (a + \varepsilon\xi + 1)\frac{1}{\varepsilon}Y' - Y = 0 \qquad -\infty < \xi < \infty$$

$$Y'' - (a + 1 + \varepsilon\xi)Y' - \varepsilon Y = 0$$

need to match on L as $\xi \to -\infty$

on R as $\xi \to \infty$

$Y_0'' - (a+1)Y_0' = 0$

$Y_0 = c_1 + c_2 e^{(a+1)\xi}$

The exponential is growing.

we can't match it.

(for constant coeff, there won't be an inner layer)

near one:

Since the coeff of y' is negative, there is no BL near $x=0$. ($x=1$ may work)

outer: (near $x=0$)

$y(x) = y_0(x) + \varepsilon y_1(x)$

ε^0: $-(x+1)y_0' - y_0 = 0$, $y_0(0) = 1$

ε^1: $-(x+1)y_1' - y_1 = -y_0''$ $y_1(0) = 0$

$y_0(x) = \frac{1}{x+1}$

$((x+1)y_1)' = y_0''$

$(x+1)y_1 = y_0' + c = \frac{-1}{(x+1)^2} + c$

$y_1(x) = \frac{-1}{(x+1)^3} + \frac{c}{x+1}$ $c=1$

$y_1(x) = \frac{1}{x+1} - \frac{1}{(x+1)^3}$

$y(x) \sim \frac{1}{x+1} + \varepsilon\left[\frac{1}{x+1} - \frac{1}{(x+1)^3}\right]$

inner:

$\xi = \frac{x-1}{\varepsilon} < 0$, $y(x) = Y(\xi)$

$Y'' - (1 + \varepsilon\xi + 1)Y' - \varepsilon Y = 0$, $Y(0) = 1$, $\xi < 0$

$Y'' - (2 + \varepsilon\xi)Y' - \varepsilon Y = 0$, $\xi < 0$, $Y(0) = 1$

$Y(\xi) \sim Y_0 + \varepsilon Y_1$

ε^0: $Y_0'' - 2Y_0' = 0$ $Y_0(0) = 1$

$Y_0(\xi) = c_1 + c_2 e^{2\xi}$ $Y_0(0) = c_1 + c_2 = 1$

$= c_1 + (1-c_1)e^{2\xi}$

ε^1: $Y_1'' - 2Y_1' = \xi Y_0' + Y_0$, $Y_1(0) = 0$

$= (\xi Y_0)'$

$Y_1' - 2Y_1 = \xi Y_0 + c_3$

$$(Y_1 e^{-2\xi})' = \xi Y_0 e^{-2\xi} + c_3 e^{-2\xi} = c_1 \xi e^{-2\xi} + (1-c_1)\xi + c_3 e^{-2\xi}$$

$$Y_1 e^{-2\xi} = -\left(\tfrac{1}{2}\xi + \tfrac{1}{4}\right) e^{-2\xi} c_1 + \tfrac{1}{2}(1-c_1)\xi^2 - \tfrac{1}{2} c_3 e^{-2\xi} + c_4$$

$$Y_1 = \underbrace{-\tfrac{1}{2}\xi c_1 + \tfrac{1}{2}(1-c_1)\xi^2 e^{2\xi}}_{\text{particular sol'n}} + \underbrace{c_5 + c_4 e^{2\xi}}_{\text{homog. sol'n}}$$

$$Y_1(0) = c_5 + c_4 = 0 \rightarrow c_4 = -c_5$$

$$Y(\xi) \sim c_1 + (1-c_1)e^{2\xi} + \varepsilon\left[-\tfrac{1}{2}\xi c_1 + \tfrac{1}{2}(1-c_1)\xi^2 e^{2\xi} + c_5(1-e^{2\xi})\right]$$

Match:

<u>2-2 method</u>

$y_0(x) \rightarrow y_0(1+\varepsilon\xi) \rightarrow y_0(1)$

$Y_0(\xi) \rightarrow Y_0\left(\frac{x-1}{\varepsilon}\right) \rightarrow Y_0(-\infty)$ 1-1 matching as before

$$y(x) \sim \frac{1}{x+1} + \varepsilon\left[\frac{1}{x+1} - \frac{1}{(x+1)^3}\right]$$

$$= \frac{1}{2+\varepsilon\xi} + \varepsilon\left[\frac{1}{2+\varepsilon\xi} - \frac{1}{(2+\varepsilon\xi)^3}\right]$$

$$\sim \tfrac{1}{2}\left(1 - \tfrac{1}{2}\varepsilon\xi\right) + \tfrac{3}{8}\varepsilon + O(\varepsilon^2)$$

$Y(\xi)$: $e^{2\xi} \rightarrow e^{2\frac{x-1}{\varepsilon}}$ very small! $o(\varepsilon^2)$

$$Y(\xi) \sim c_1 + \varepsilon\left[-\tfrac{1}{2}\tfrac{x-1}{\varepsilon} c_1 + c_5\right] + O(\varepsilon^2)$$

$$= c_1 - \tfrac{1}{2}c_1(x-1) + c_5\varepsilon + O(\varepsilon^2)$$

$$y(x) \sim \tfrac{1}{2}\left(1 - \tfrac{1}{2}(x-1)\right) + \tfrac{3}{8}\varepsilon + O(\varepsilon^2)$$

$$\rightarrow c_1 = \tfrac{1}{2}, \quad c_5 = 3/8$$

$$y_c(x) = \frac{1}{x+1} + \varepsilon\left[\frac{1}{x+1} - \frac{1}{(x+1)^3}\right]$$
$$+ \cancel{\tfrac{1}{2}} + \tfrac{1}{2}e^{2\frac{x-1}{\varepsilon}} + \varepsilon\left[\cancel{-\tfrac{1}{4}\tfrac{x-1}{\varepsilon}} + \tfrac{1}{4}\left(\tfrac{x-1}{\varepsilon}\right)^2 e^{2\frac{x-1}{\varepsilon}} + \cancel{3/8} - \tfrac{3}{8}e^{2\frac{x-1}{\varepsilon}}\right]$$
$$\cancel{-\tfrac{1}{2}} \cancel{+\tfrac{1}{4}(x-1)} \cancel{-3/8\,\varepsilon}$$

$$y_c(x) = \frac{1}{x+1} + \varepsilon\left[\frac{1}{x+1} - \frac{1}{(x+1)^3}\right]$$
$$+ e^{2\frac{x-1}{\varepsilon}}\left[\tfrac{1}{2} + \tfrac{1}{4}\frac{(x-1)^2}{\varepsilon} - 3/8\,\varepsilon\right]$$

Ex: $\varepsilon y'' + a(x)y' + b(x)y = 0$, $\alpha < x < \beta$, $y(\alpha) = A$, $y(\beta) = B$

Suppose $a(x) > 0$, $\alpha \le x \le \beta$
$\to$ boundary layer near $x = \alpha$
outer sol'n satisfies $y(\beta) = B$ ①

Suppose $a(x) < 0$, $\alpha \le x \le \beta$
$\to$ boundary layer near $x = \beta$
outer sol'n satisfies $y(\alpha) = A$ ②

Suppose $a(x)$ changes sign
$\to$ more complicated

① $a(x) > 0 \to$ BL near $x = \alpha$

Outer:

$$a(x)y_0' + b(x)y_0 = 0, \quad y_0(\beta) = B$$

$$y_0(x) = ce^{-\int \frac{b(x)}{a(x)}dx}$$

$$y_0(x) = y_0(\alpha)\exp\left(-\int_\alpha^x \frac{b(s)}{a(s)}ds\right)$$

$$y_0(x) = c\exp\left(\int_x^\beta \frac{b(s)}{a(s)}ds\right)$$

$$\to c = B$$

$$y_0(x) = B\exp\left(\int_x^\beta \frac{b(s)}{a(s)}ds\right)$$

Inner:

$\xi = \frac{x-\alpha}{\varepsilon}$ $y(x) = Y(\xi)$

$$Y_0'' + a(\alpha)Y_0' = 0$$

$$Y_0(\xi) = c_1 + c_2 e^{-a(\alpha)\xi}$$

$$Y_0(0) = A = c_1 + c_2 \to c_2 = A - c_1$$

$$Y_0(\xi) = c_1 + (A - c_1)e^{-a(\alpha)\xi}$$

Match:

$$Y_0(\infty) = c_1$$

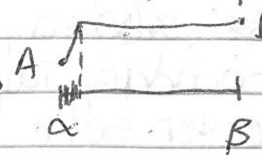
$$y_0(\alpha) = B\exp\left(\int_\alpha^\beta \frac{b(s)}{a(s)}ds\right) = c_1$$

$$y_c(x) \sim B\exp\left(\int_x^\beta \frac{b(s)}{a(s)}ds\right) + \left(A - B\exp\left(\int_\alpha^\beta \frac{b(s)}{a(s)}ds\right)\right)e^{-a(\alpha)\frac{x-\alpha}{\varepsilon}}$$

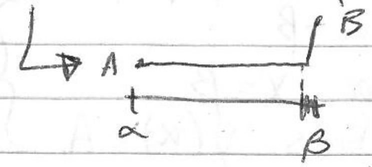
10-30 Ex: $\varepsilon y'' + a(x)y' + b(x)y = 0$, $\alpha < x < \beta$, $y(\alpha) = A$, $y(\beta) = B$

(from last time)

$a(x) > 0 \quad \alpha \le x \le \beta \longrightarrow$



$a(x) < 0 \quad \alpha \le x \le \beta$



$a(x) = 0$ at an end point

Ex: $\varepsilon y'' + x^\alpha y' + x^\beta y = 0$, $0 < x < 1$ $(\alpha > 0, \beta \ge 0)$

$y(0) = A$, $y(1) = B$

BL near $x = 0$

$y(x) = Y(\xi)$, $\xi = \frac{x}{\delta}$, $\delta = o(1)$

need to check scaling b/c x^α gives a different power of ε

before it was x/ε b/c we matched $y'' + y'$ terms

$\frac{\varepsilon}{\delta^2} Y'' + \xi^\alpha \delta^\alpha \frac{1}{\delta} Y' + \xi^\beta \delta^\beta Y = 0$

compare powers of δ

balance $Y'' + Y'$ when $Y' \gg Y \rightarrow \beta > \alpha - 1$

$\varepsilon / \delta^2 = \delta^{\alpha - 1} \rightarrow \varepsilon = \delta^{\alpha+1} \rightarrow \delta = \varepsilon^{\frac{1}{\alpha+1}}$

$\rightarrow \xi = x / \varepsilon^{\frac{1}{1+\alpha}}$

balance $Y'' + Y$, if $Y \gg Y' \rightarrow \beta < \alpha - 1$

$\varepsilon/\delta^2 = \delta^\beta \rightarrow \varepsilon = \delta^{\beta+2}$ $\quad \delta = \varepsilon^{\frac{1}{2+\beta}}$

$\xi = x / \varepsilon^{\frac{1}{\beta+2}}$

(if $\beta = \alpha - 1$ all three terms balance)

Ex: $\varepsilon y'' + x^{\frac12} y' - y = 0$, $y(0)=0$, $y(1)=e^2$

boundary layer near left endpt

Outer solution:

$$x^{\frac12} y_0' - y_0 = 0, \quad y_0(1) = e^2$$

$$y_0(x) = c\exp\left(\int x^{-\frac12}dx\right)$$

$$= c\exp(2x^{\frac12})$$

$$y_0(1) = ce^2 = e^2 \to c = 1$$

$$y_0(x) = e^{2\sqrt{x}}$$

BL near $x=0$:

$$\xi = \frac{x}{\delta}, \quad \delta = o(1), \quad y(x) = Y(\xi)$$

$$\underbrace{\varepsilon/\delta^2\, Y''}_{\text{must be large}} + \underbrace{\delta^{\frac12}\xi^{\frac12}\frac{1}{\delta}Y'}_{\text{large}} - Y = 0$$

$$\frac{1}{\delta^{\frac12}} = \frac{\varepsilon}{\delta^2} \to \delta^{3/2} = \varepsilon \to \delta = \varepsilon^{2/3}$$

So $\xi = \frac{x}{\varepsilon^{2/3}}$

$$\to \varepsilon^{-\frac13}Y'' + \varepsilon^{-\frac13}\xi^{\frac12}Y' - Y = 0$$

$$Y'' + \xi^{\frac12}Y' - \varepsilon^{\frac13}Y = 0$$

$$Y_0'' + \xi^{\frac12}Y_0' = 0, \quad Y_0(0) = 0$$

if we wanted two terms, $Y(\xi) \sim Y_0(\xi) + \varepsilon^{\frac13}Y_1(\xi)$ would probably work

$$Y_0'\exp\left(\tfrac23\xi^{3/2}\right) = c_1$$

$$Y_0' = c_1\exp\left(-\tfrac23\xi^{3/2}\right)$$

$$Y_0 = c_1\int_0^{\xi}\exp\left(-\tfrac23 s^{3/2}\right)ds$$

← lower limit 0 since $Y_0(0)=0$

$$y_0(0) = Y_0(\infty)$$

$$1 = c_1\int_0^{\infty}\exp\left(-\tfrac23 s^{3/2}\right)ds$$

$$y_c(x) \sim e^{2\sqrt{x}} + \frac{\int_0^{\xi}\exp\left(-\tfrac23 s^{3/2}\right)ds}{\int_0^{\infty}\exp\left(-\tfrac23 s^{3/2}\right)ds} - 1$$

Ex: $\varepsilon y'' - x^2 y' - y = 0, \quad y(0)=1, \quad y(1)=1$

minus sign $\to$ possible to have BL near $x=1$

can still have BL near $x=0$ because of x^2

Outer:

$x^2 y_0' + y_0 = 0$

$y_0' + x^{-2} y_0 = 0$

$y_0(x) = c\exp(-\int x^{-2}dx)$

$= c e^{1/x}$

So we need a BL near zero, since this can't satisfy $y(0)=1$

BL near $x=0$

$y(x) = Y(\xi), \quad \xi = x/\delta, \quad \delta = o(1)$

$\to \varepsilon/\delta^2 Y'' - \underbrace{\xi^2 \delta^2 \tfrac{1}{\delta} Y'}_{\text{small}} - \underbrace{Y}_{\text{larger}} = 0$

match $Y'' + Y \to \varepsilon/\delta^2 = 1 \to \delta = \sqrt{\varepsilon}$

$Y'' - \xi^2\sqrt{\varepsilon} - Y = 0$

$Y_0'' - Y_0 = 0, \quad Y_0(0) = 1$

$Y_0(\xi) = c_1 e^{\xi} + c_2 e^{-\xi}$

$Y_0(0) = c_1 + c_2 = 1$

need $c_1 = 0$ since exp blows up

$\to c_2 = 1$

$Y_0(\xi) = e^{-\xi}$

$Y_0(\infty) = 0$

won't match unless

$y_0(x) = ce^{1/x}$ must take $c=0$, so that $y_0(x) \equiv 0$

but then this doesn't satisfy $y(1)=1$

BL near $x=1$

$y(x) = Z(\eta), \quad \eta = \frac{x-1}{\varepsilon}$ ← b/c nothing weird going on near $x=1$

$x = 1 + \varepsilon\eta$

$\varepsilon \frac{1}{\varepsilon^2} Z'' - (1+\varepsilon\eta)^2 \frac{1}{\varepsilon} Z' - Z = 0, \quad \eta < 0, \quad Z(0)=1$

$Z_0'' - Z_0' = 0, \quad \eta < 0, \quad Z_0(0) = 1$

$Z_0(\eta) = c_1 + c_2 e^{\eta}$

$Z_0(0) = c_1 + c_2 = 1$

matching: $Z_0(-\infty) = y_{outer}(1) = 0$

$\rightarrow c_1 = 0 \rightarrow c_2 = 1$

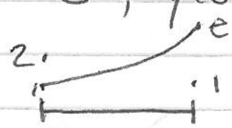
$Z_0(\eta) = e^{\eta}$

$y_c(x) = 0 + e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{\frac{x-1}{\varepsilon}}$

$y_c(x) = e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{\frac{x-1}{\varepsilon}}$

Ex: $\varepsilon^2 y'' + \varepsilon x y' - y = -e^x$, $y(0)=2$, $y(1)=1$

Outer: $y_o(x) = e^x$ $\rightarrow$ 2 BL?



BL near $x=0$:

$y(x) = Y(\xi)$, $\xi = \frac{x}{\delta}$, $\delta = o(1)$

$$\underbrace{\frac{\varepsilon^2}{\delta^2} Y''}_{} + \underbrace{\varepsilon \xi \delta \frac{1}{\delta} Y'}_{\text{small}} - \underbrace{Y}_{O(1)} = \underbrace{-e^{\xi\delta}}_{O(1)}$$

$\hookrightarrow$ large: nothing to balance

small: get outer sol'n again

so must be $O(1)$ $\rightarrow \delta = \varepsilon$

$Y'' + \varepsilon \xi Y' - Y = \underbrace{-e^{-\xi\varepsilon}}$ $\rightarrow$ expand

$Y_0'' - Y_0 = -1$ $Y_0(0) = 2$

$Y_0(\xi) = c_1 e^{\xi} + c_2 e^{-\xi} + 1$

$Y_0(0) = c_1 + c_2 + 1 = 2$

$Y_0(\infty) = y_o(0) = 1$ (common part $=1$)

$\rightarrow c_1 = 0 \rightarrow c_2 = 1$

$Y_0(\xi) = e^{-\xi} + 1$

BL near $x=1$:

$y(x) = Z(\eta)$, $\eta = \frac{x-1}{\delta}$, $\delta = o(1)$ $x = 1 + \delta\eta$

$$\underbrace{\frac{\varepsilon^2}{\delta^2} Z''}_{(\varepsilon/\delta)^2} + \underbrace{\varepsilon(1+\delta\eta)\frac{1}{\delta} Z'}_{(\varepsilon/\delta)} - \underbrace{Z}_{O(1)} = \underbrace{e^{1+\delta\eta}}_{O(1)}$$

need $(\varepsilon/\delta) = O(1)$ $\rightarrow \delta = \varepsilon$

$Z'' + (1+\varepsilon\eta) Z' - Z = e^{1+\varepsilon\eta}$, $Z(0) = 1$, $\eta < 0$

$$Z_0'' + Z_0' - Z_0 = -e \quad , \quad Z_0(0) = 1$$

matching: $Z_0(-\infty) = e$

$$Z_0(\eta) = e + C_1 e^{(\sqrt{5}-1)/2\,\eta} + C_2 e^{(-\sqrt{5}-1)/2\,\eta}$$

$C_2 = 0$

$$Z_0(0) = e + C_1 = 1 \rightarrow C_1 = 1 - e$$

$$Z_0(\eta) = e + (1-e)\exp\left(\tfrac{\sqrt{5}-1}{2}\eta\right)$$

common part $= e$

$$y_c(x) \sim e^x + (1 + e^{-\frac{x}{\varepsilon}}) - 1 + \left(e + (1-e)\exp\left(\tfrac{\sqrt{5}-1}{2} \cdot \tfrac{x-1}{\varepsilon}\right)\right) - e$$

$$= e^x + e^{-\frac{x}{\varepsilon}} + (1-e)\exp\left(\tfrac{\sqrt{5}-1}{2} \cdot \tfrac{x-1}{\varepsilon}\right)$$

11-4 <u>Ex</u>: $\varepsilon y'' + y(y' + 3) = 0, \quad y(0) = y(1) = 1$

nonlinear!

Outer: $y_0(y_0' + 3) = 0$

Either $y_0(x) \equiv 0$

or $y_0' + 3 = 0 \rightarrow y_0(x) = -3x + C$

Assume there is one BL, at $x = 0$

$\rightarrow$ not $y_0(x) \equiv 0$

$\rightarrow y_0(x) = 4 - 3x \qquad (y_0(1) = 1)$

BL near $x = 0$:

$$y(x) = Y(\xi), \quad \xi = \frac{x}{\delta}, \quad \delta = o(1)$$

$$\frac{\varepsilon}{\delta^2} Y'' + Y\left(\frac{1}{\delta} Y' + 3\right) = 0$$

$$\rightarrow \varepsilon/\delta^2 = \frac{1}{\delta} \rightarrow \delta = \varepsilon, \quad \xi = \frac{x}{\varepsilon}$$

$$\frac{1}{\varepsilon} Y'' + \frac{1}{\varepsilon} YY' + 3Y = 0$$

$$Y'' + YY' + 3\varepsilon Y = 0$$

$Y(\xi) \sim Y_0(\xi)$:

$$Y_0'' + Y_0 Y_0' = 0 \quad , \quad Y_0(0) = 1, \quad Y_0(+\infty) = 4$$

$$Y_0' + \frac{1}{2} Y_0^2 = C_1$$

$$\frac{1}{2}(4)^2 = 8 = C_1$$

$$\frac{dY_0}{16 - Y_0^2} = \frac{1}{2} d\xi$$

$$\frac{1}{16-Y_0^2} = \frac{1}{(4-Y_0)(4+Y_0)} = \frac{1/8}{4-Y_0} + \frac{1/8}{4+Y_0}$$

$$\frac{1}{8}\ln\left(\frac{4+Y_0}{4-Y_0}\right) = \frac{1}{2}\xi + c_1$$

$$Y_0(0) = 1 \rightarrow \frac{1}{8}\ln 5/3 = c_1$$

$$\ln\left(\frac{4+Y_0}{4-Y_0}\right) - \ln\frac{5}{3} = 4\xi$$

$$\frac{3}{5}\,\frac{4+Y_0}{4-Y_0} = e^{4\xi}$$

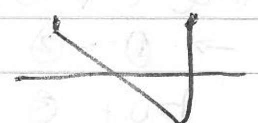
$$3/5\,(4+Y_0) = e^{4\xi}(4-Y_0)$$

$$Y_0(\xi) = \frac{4(5/3\,e^{4\xi} - 1)}{1 + 5/3\,e^{4\xi}}$$

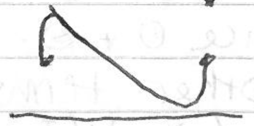
$$y_c(x) = \cancel{4} - 3x - \cancel{4} + 4 \cdot \frac{5 - 3e^{-4\xi}}{5 + 3e^{-4\xi}}$$

$$= -3x + 4\left(\frac{5 - 3e^{-4\frac{x}{\varepsilon}}}{5 + 3e^{-4\frac{x}{\varepsilon}}}\right)$$

if we had assumed a BL at $x=1$
→ sol'n in the form:



or another alternate:



Ex: $\varepsilon^3 y'' + x^3 y' - \varepsilon y = x^3$, $y(0)=1$, $y(1)=3$

Outer:

$y_0' = 1 \rightarrow y_0(x) = x + c_1$

Can BL exist near $x=1$?

$y(x) = Y(\xi)$, $\xi = \frac{x-1}{\delta}$ $\xi < 0$

$\frac{\varepsilon^3}{\delta^2} Y'' + (1+\xi\delta)^3 \frac{1}{\delta} Y' - \varepsilon Y = (1+\xi\delta)^3$

$Y_0'' + Y_0' = 0$

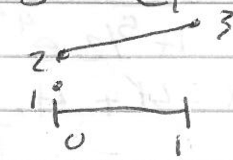
$Y_0(\xi) = c_1 + c_2 e^{-\xi}$

growing exponential $\rightarrow$ No!

So use $y(1)=3$ for outer sol'n

$y_0(1) = 1 + c_1 = 3 \rightarrow c_1 = 2$

$y_0 = x + 2$



$\rightarrow$ BL near $x=0$

BL near $x=0$:

$y(x) = Y(\xi)$, $\xi = \frac{x}{\delta}$, $\delta = o(1)$

$$\underset{①}{\frac{\varepsilon^3}{\delta^2} Y''} + \underset{②}{\xi^3\delta^3 \frac{1}{\delta} Y'} - \underset{③}{\varepsilon Y} = \underset{④}{\xi^3 \delta^3}$$

Balance ① + ②: $\frac{\varepsilon^3}{\delta^2} = \delta^2 \rightarrow \delta^4 = \varepsilon^3 \rightarrow \delta = \varepsilon^{3/4}$

$\rightarrow$ ① $\sim$ ② $\sim \varepsilon^{3/2}$

but ③ is larger $\rightarrow$ doesn't work

Balance ① + ③: $\frac{\varepsilon^3}{\delta^2} = \varepsilon \rightarrow \varepsilon^2 = \delta^2 \rightarrow \delta = \varepsilon$

other terms are smaller $\rightarrow$ good!

So $y(x) = Y(\xi)$, $\xi = \frac{x}{\varepsilon}$

$Y_0'' - Y_0 = 0$, $Y_0(0) = 1$

$Y_0(\xi) = c_1 e^{\xi} + c_2 e^{-\xi}$

$c_1 = 0$ for matching

$c_2 = 1$ from BC

$Y_0(\xi) = e^{-\xi}$

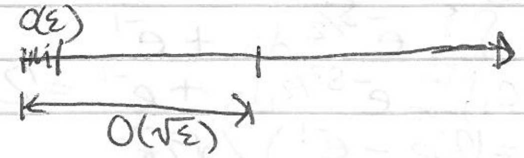
But matching doesn't work!

Maybe there's another way to balance

Balance ② + ③:

$$\delta^2 = \varepsilon \to \delta = \sqrt{\varepsilon}$$

BL within a BL

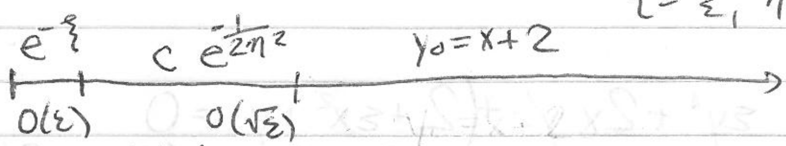


$$y(x) = Z(\eta), \quad \eta = \frac{x}{\sqrt{\varepsilon}}$$

$$\eta^3 Z_0' - Z_0 = 0$$

$$Z_0(\eta) = Ce^{-\frac{1}{2}\eta^{-2}}$$

$\xi = \frac{x}{\varepsilon}, \; \eta = \frac{x}{\sqrt{\varepsilon}}$



Matching: $Y_0(\infty) = Z_0(0)$

$$0 = 0 \quad ✓$$

$$Z_0(\infty) = y_0(0)$$

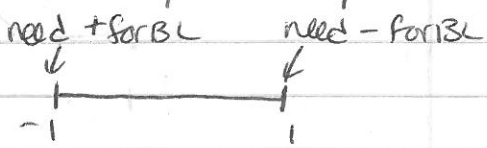
$$C = 2 \quad \leftarrow \text{common part}$$

$$y_c \sim x + 2e^{-\frac{\varepsilon}{2x^2}} + e^{-x/\varepsilon}$$

Ex: $\varepsilon y'' + xy' + xy = 0, \quad -1 < x < 1, \quad y(-1) = 1, \; y(1) = 2$

Outer: $y_0' + y_0 = 0$

$$y_0(x) = Ce^{-x}$$



Might have an internal layer:
→ two outer sol'ns

$$y_0(x) = \begin{cases} e^{-1-x} & -1 < x < 0 \\ 2e^{1-x} & 0 < x < 1 \end{cases}$$

Internal layer near $x = 0$:

$$y(x) = Y(\xi), \quad \xi = \frac{x}{\delta}, \quad \delta = o(1)$$

$$\frac{\varepsilon}{\delta^2} Y'' + \xi\delta \frac{1}{\delta} Y' + \xi\delta Y = 0$$

$$\to \frac{\varepsilon}{\delta^2} = 1 \quad \delta = \sqrt{\varepsilon}$$

$Y_0'' + \xi Y_0' = 0 \quad , \quad -\infty < \xi < \infty$

$Y_0(-\infty) = y_{0\,left}(0) = e^{-1}$

$Y_0(\infty) = y_{0\,right}(0) = 2e$

$Y_0'(\xi) = c_1 e^{-\xi^2/2}$

$Y_0(\xi) = c_1 \int_{-\infty}^{\xi} e^{-s^2/2} ds + e^{-1}$

$Y_0(+\infty) = c_1 \int_{-\infty}^{\infty} e^{-s^2/2} ds + e^{-1} = 2e$

$c_1 = (2e - e^{-1}) / \sqrt{2\pi}$

$Y_0(\xi) = (2e - e^{-1}) / \sqrt{2\pi} \int_{-\infty}^{\xi} e^{-s^2/2} ds + e^{-1}$

$$y_c \sim \begin{cases} e^{-1-x} + (2e - e^{-1})/\sqrt{2\pi} \int_{-\infty}^{x/\sqrt{\varepsilon}} e^{-s^2/2} ds & -1 < x < 0 \\ 2e^{1-x} - 2e + (2e - e^{-1})/\sqrt{2\pi} \int_{-\infty}^{x/\sqrt{\varepsilon}} e^{-s^2/2} ds + e^{-1} & 0 < x < 1 \end{cases}$$

Ex: $\varepsilon y'' + 2x y' + (2 + \varepsilon x^2) y = 0$

$-1 < x < 1 \quad , \quad y(-1) = 2 \, , \, y(1) = -2$

Similar: two outer + inner BL

Outer: $x y_0' + y_0 = 0$

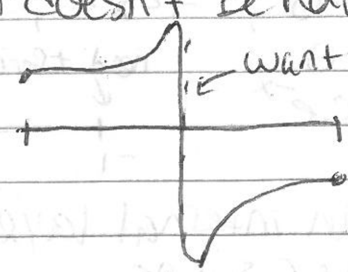
$(x y_0)' = 0$

$y_0(x) = c/x$

$y_0(x) = -2/x$

But this satisfies both BC!

(it doesn't behave well at $x=0$ though



Internal layer:

$y(x) = \varepsilon^{-\gamma} Y(\xi), \; \xi = x/\delta, \; \delta = o(1)$

$\gamma > 0$ to be determined

$\frac{\varepsilon}{\delta^2} Y'' + 2 \xi \delta \frac{1}{\delta} Y' + (2 + \varepsilon \xi^2 \delta^2) Y = 0$

$\delta = \sqrt{\varepsilon} \rightarrow Y_0'' + 2 \xi Y_0' + 2 Y_0 = 0$

Ex: continued from prev.

$\varepsilon y'' + 2xy' + (2 + \varepsilon x^2)y = 0$

Outer: $y_0(x) = -2/x$

Internal layer:

$y(x) = \varepsilon^{-\gamma} Y(\xi), \quad \xi = \frac{x}{\delta}$

$\uparrow$ we want $Y(\xi) = O(1)$, but it may need to be asymptotically large to match w/ outer

$\varepsilon/\delta^2 \, Y'' + 2\xi\delta \frac{1}{\delta} Y' + (2 + \varepsilon \xi^2 \delta^2) Y = 0$

$\varepsilon/\delta^2 = 1 \rightarrow \delta = \sqrt{\varepsilon}$

$Y'' + 2\xi Y' + (2 + \varepsilon^2 \xi^2) Y = 0$

$Y(\xi) \sim Y_0(\xi)$

$Y_0'' + 2\xi Y_0' + 2Y_0 = 0, \quad -\infty < \xi < \infty$

$Y_0'' + 2(\xi Y_0)' = 0$

$Y_0' + 2\xi Y_0 = c_1$

$(Y_0 e^{\xi^2})' = c_1 e^{\xi^2}$

$Y_0 e^{\xi^2} = c_1 \int_0^{\xi} e^{s^2} ds + c_2$

$Y_0(\xi) = c_1 e^{-\xi^2} \int_0^{\xi} e^{s^2} ds + c_2 e^{-\xi^2}$

Asymptotics of $Y_0(\xi)$ as $\xi \to \pm\infty$

$\int_0^{\xi} e^{s^2} ds \sim \int_1^{\xi} e^{s^2} ds = \int_1^{\xi} \frac{1}{2s} de^{s^2}$

$= e^{s^2}/2s \Big|_{s=1}^{\xi} + \underbrace{\int_1^{\xi} \frac{1}{2s^2} e^{s^2} ds}_{\text{much smaller than}}$

$\sim e^{\xi^2}/2\xi$

So $Y_0(\xi) \sim c_1/2\xi$ for $|\xi| \to \infty$

Van Dyke's 1-1 method

Outer: $y_0(x) = -2/x = -2/\xi\sqrt{\varepsilon}$

Inner: $\varepsilon^{-\gamma} Y_0(\xi) = \varepsilon^{-\gamma} c_1 e^{-\frac{x^2}{\varepsilon}} \int_0^{x/\sqrt{\varepsilon}} e^{s^2} ds + \varepsilon^{-\gamma} c_2 e^{-\frac{x^2}{\varepsilon}}$

$\sim \varepsilon^{-\gamma} c_1 / 2x/\sqrt{\varepsilon} = \varepsilon^{-\gamma + \frac{1}{2}} / 2x$

$\gamma = \frac{1}{2}, \quad c_1/2 = -2 \rightarrow c_1 = -4$

Can't determine c_2

natural to expect $c_2 = 0$

$y_c \sim -2/x + \varepsilon^{-\frac{1}{2}} \left[-4 e^{-\frac{x^2}{\varepsilon}} \int_0^{x/\sqrt{\varepsilon}} e^{s^2} ds + c_2 e^{-x^2/\varepsilon} \right. + 2/x$

expect sol'n to be odd

$\rightarrow c_2$ term must be zero

Example wouldn't have worked if, say,

$y(-1)=1, y_2(1)=-2$

Ex: $\varepsilon y'' + (x-\tfrac{1}{2})y' - y = 0$, $y(0)=2$, $y(1)=3$

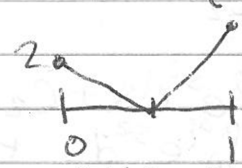
because of coeff of y', can't have BL at edges.

Outer solution:

$$(x-\tfrac{1}{2})y_0' - y_0 = 0$$

$$y_0(x) = c(x-\tfrac{1}{2})$$

$$y_0(x) = \begin{cases} -4(x-\tfrac{1}{2}) & 0 \le x < \tfrac{1}{2} \\ 6(x-\tfrac{1}{2}) & \tfrac{1}{2} < x \le 1 \end{cases}$$



$\rightarrow$ "Corner Layer"

want to smooth out the corner

Corner: $Y(\xi)$, δ $\quad \xi = \frac{x-1/2}{\delta}$

$$\varepsilon \frac{1}{\delta^2} Y'' + \delta \xi \frac{1}{\delta} Y' - Y = 0$$

$$\rightarrow \delta = \sqrt{\varepsilon}$$

$$Y_0'' + \xi Y_0' - Y_0 = 0$$

$Y_0(\xi) = \xi$ is a solution

Use reduction of order to determine the general solution

$$Y_0(\xi) = \xi u(\xi)$$

$$\xi u'' + [2+\xi^2]u' = 0$$

$$u'' + (2/\xi + \xi)u' = 0$$

$$u' = c_1 \exp(-\textstyle\int 2/\xi + \xi \, d\xi)$$

$$= c_1 \frac{1}{\xi^2} e^{-\frac{\xi^2}{2}}$$

$$u(\xi) = c_1 \int \frac{1}{\xi^2} e^{-\xi^2/2} d\xi + c_2$$

$$\int \frac{1}{\xi^2} e^{-\xi^2/2}\, d\xi = -\int e^{-\xi^2/2}\, d\frac{1}{\xi}$$
$$= -\frac{1}{\xi} e^{-\xi^2/2} - \int \frac{1}{\xi} e^{-\xi^2/2}\cdot \xi \, d\xi$$
$$= -\frac{1}{\xi} e^{-\xi^2/2} - \int e^{-\xi^2/2}\, d\xi$$
$$Y_0(\xi) = c_1\left[-e^{-\xi^2/2} - \xi\int^{\xi} e^{-\xi^2/2}\, d\xi\right] + c_2\xi$$
$$= c_3\left[e^{-\xi^2/2} + \xi\int_0^{\xi} e^{-\xi^2/2}\, d\xi\right] + c_2\xi$$
$\int_0^\infty e^{-\xi^2/2}\, d\xi$ let $t = \xi/\sqrt{2}$
$$= \sqrt{2}\int_0^\infty e^{-t^2}\, dt = \sqrt{\pi/2}$$
$Y_0(\xi)$, $\xi \to \infty$
$$\sim c_3\, \xi\sqrt{\pi/2} + c_2\xi, \quad \xi \to \infty$$
$$Y_0(\xi) \sim -c_3\, \xi\sqrt{\pi/2} + c_2\xi \quad \xi \to -\infty$$

Matching:

$$y_0(x) = \begin{cases} -4\xi\sqrt{\varepsilon} & 0 \le x \le \frac12 \to \xi < 0 \\ 6\xi\sqrt{\varepsilon} & \frac12 \le x \le 1 \to \xi > 0 \end{cases}$$
$$\varepsilon^\gamma Y_0(\xi) \sim \varepsilon^\gamma \xi\left(c_3\sqrt{\pi/2} + c_2\right)$$
$$= \varepsilon^\gamma \frac{x-\frac12}{\sqrt{\varepsilon}}\left(c_3\sqrt{\pi/2} + c_2\right) \qquad \tfrac12 < x \le 1$$
$$\varepsilon^\gamma Y_0(\xi) \sim \varepsilon^\gamma \xi\left(-c_3\sqrt{\pi/2} + c_2\right)$$
$$= \varepsilon^\gamma \frac{x-\frac12}{\sqrt{\varepsilon}}\left(-c_3\sqrt{\pi/2} + c_2\right) \qquad 0 \le x \le \tfrac12$$
$$y_0(x) = \begin{cases} -4(x-\frac12) & 0 \le x < \frac12 \\ 6(x-\frac12) & \frac12 < x \le 1 \end{cases}$$
Compare: $\gamma = \frac12$, $-4 = -c_3\sqrt{\pi/2} + c_2$
$$6 = c_3\sqrt{\pi/2} + c_2$$
$$2c_2 = 2 \to c_2 = 1$$
$$10 = 2\sqrt{\pi/2}\, c_3 \to c_3 = 5\sqrt{2/\pi}$$
$$Y_0(\xi) = 5\sqrt{2/\pi}\left[e^{-\xi^2/2} + \xi\int_0^{\xi} e^{-s^2/2}\, ds\right] + \xi$$

$$y_c(x) \sim \cancel{-4(x-\tfrac12)} + \sqrt{\varepsilon}\left[5\sqrt{\tfrac{2}{\pi}}\left(e^{-\frac{(x-\frac12)^2}{2\varepsilon}} + \frac{x-\frac12}{\sqrt{\varepsilon}}\int_0^{\frac{x-\frac12}{\sqrt{\varepsilon}}} e^{-s^2/2}\, ds\right) + \frac{x-\frac12}{\sqrt{\varepsilon}}\right] \cancel{+4(x-\tfrac12)}, \quad 0 \le x \le \tfrac12$$
$$= 5\sqrt{\frac{2\varepsilon}{\pi}}\left[e^{-\frac{(x-\frac12)^2}{2\varepsilon}} + \frac{x-\frac12}{\sqrt{\varepsilon}}\int_0^{\frac{x-\frac12}{\sqrt{\varepsilon}}} e^{-s^2/2}\, ds\right] + x - \frac12, \quad 0 \le x \le \tfrac12$$
Same for $\frac12 < x \le 1$

Ex: $\varepsilon y'' - xy' = 0$, $-a < x < b$, $a > 0, b > 0$

$y(-a) = \alpha$, $y(b) = \beta$

allowed to have BL at endpts

Outer: $-xy_0' = 0$

$y_0(x) = \text{const}$

But an internal layer won't work:

$y(x) = Y(\xi)$, $\xi = x/\delta$

$\varepsilon/\delta^2 Y'' - \xi\delta/\delta\, Y' = 0$

$\rightarrow \delta = \sqrt{\varepsilon}$

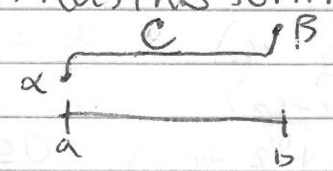
$Y_{i0}'' - \xi Y_{i0}' = 0$

$Y_{i0}' = c_1 e^{\xi^2/2}$

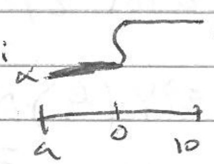
$Y_0(\xi) = c_1 \int_0^{\xi} e^{s^2/2} ds$

won't match

So it has this form:



not this:



BL near $x = -a$: $y(x) = Y(\xi)$, $\xi = \frac{x+a}{\delta}$

$\varepsilon/\delta^2 Y'' - (-a + \xi\delta)\frac{1}{\delta} Y' = 0$

$\delta = \varepsilon$

$Y_0'' + aY_0' = 0$, $Y_0(0) = \alpha$, $Y_0(+\infty) = C$

$Y_0(\xi) = c_1 + c_2 e^{-a\xi}$

$\rightarrow c_1 = C$

$Y_0(0) = c_1 + c_2 = \alpha$

$Y_0(\xi) = C + (\alpha - C) e^{-a\xi}$

BL near $x = b$:

$y(x) = Z(\eta)$, $\eta = \frac{x-b}{\varepsilon}$, $Z(0) = \beta$, $Z(-\infty) = C$

$Z_0(\eta) = C + (\beta - C)e^{b\eta}$

$y_c(x) \sim C + (\alpha - C)e^{-a\frac{x+a}{\varepsilon}} + (\beta - C)e^{b\frac{x-b}{\varepsilon}}$

C not determined

Ex: $\varepsilon y'' - xy' = 0$ (from last time) 11-11

$-a < x < b$, $a, b > 0$, $y(-a) = \alpha$, $y(b) = \beta$

$$y(x) \sim C + (\alpha - C)e^{-a\frac{x+a}{\varepsilon}} + (\beta - C)e^{b\frac{x-b}{\varepsilon}}$$

C arbitrary

Exact solution:

$$y'(x) = c_1 e^{\int \frac{x}{\varepsilon} dx} = c_1 e^{x^2/2\varepsilon}$$

$$y(x) - y(a) = c_1 \int_{-a}^{x} e^{s^2/2\varepsilon} ds$$

$$y(b) = \beta - \alpha = c_1 \int_{-a}^{b} e^{s^2/2\varepsilon} ds$$

$$c_1 = \frac{\beta - \alpha}{\int_{a}^{b} e^{s^2/2\varepsilon} ds}$$

$$y(x) = \alpha + (\beta - \alpha)\frac{\int_{-a}^{x} e^{s^2/2\varepsilon} ds}{\int_{-a}^{b} e^{s^2/2\varepsilon} ds}$$

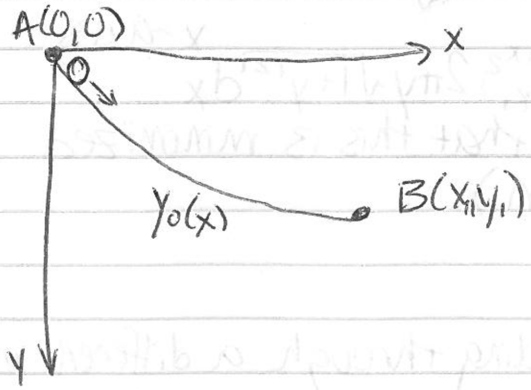
So, what is C?

Calculus of Variations

Brachistochrone Problem

Brachistos = shortest

Chronos = time



particle reaches B at shortest time, falling under gravity + no friction

Galileo (1638): arc of a circle

Johann Bernoulli (1696):

Jacob Bernoulli → developed more complicated version

Newton — solved problem — Euler (1760)

Leibnitz → calculus of variations

L'Hôpital — Lagrange

$\frac{1}{2}mv^2 + mg(-y) = \text{const} = 0 \leftarrow$ due to IC

$v = \sqrt{2gy}$

$s(t)$: distance traveled along curve

$v = ds/dt$

$ds/dt = \sqrt{2gy}$

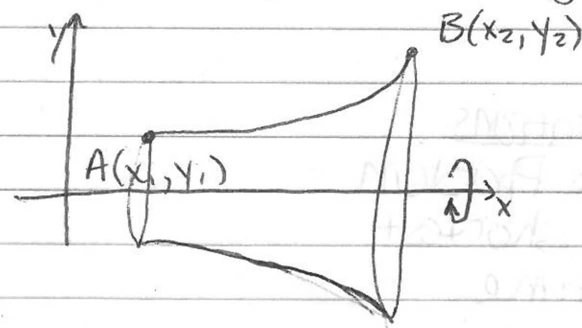
$$dt = \frac{ds}{\sqrt{2gy}}$$

$$dt = \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}} \leftarrow \text{arclength formula}$$

$$T = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx \leftarrow \text{want to minimize}$$

$y(x) = ?$

functional: input function, get number



want to min. area of the surface of revolution about x-axis

$$\int_{x_1}^{x_2} 2\pi y\,ds = \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}\,dx$$

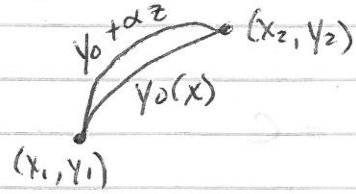
want $y(x)$ so that this is minimized (soap film)

more examples:

- light traveling through a different medium (air into water
- mechanics

$I(y)=\int_{x_1}^{x_2} L(x,y,y')\,dx$

Determine $y_0(x)$ such that $y_0(x_1)=y_1$
$y_0(x_2)=y_2$, $I(y_0)<I(y)$ for any
$y(x)$ such that $y(x_1)=y_1$, $y(x_2)=y_2$



$z(x):\ z(x_1)=0,\ z(x_2)=0$

$y_0(x)+\alpha z(x)$

$\alpha\in\mathbb{R}$

$I(y_0+\alpha z)=\varphi(\alpha)$

$\varphi(0)=\min$

$d/d\alpha\ \varphi(\alpha)\big|_{\alpha=0}=0$

$d/d\alpha \int_{x_1}^{x_2} L(x, y_0+\alpha z, y_0'+\alpha z')\,dx$

$=\int_{x_1}^{x_2} (\partial L/\partial y)\, z+(\partial L/\partial y')\, z'\,dx$

$=\int_{x_1}^{x_2} (\partial L/\partial y)\, z - d/dx\, \frac{\partial L}{\partial y'}\, z\ dx + \cancelto{0}{\frac{\partial L}{\partial y'}\, z\Big|_{x=x_1}^{x_2}}$

$=\int_{x_1}^{x_2} z\left[\partial L/\partial y - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]dx$

$=0$

$\rightarrow \frac{\partial L}{\partial y}-\frac{d}{dx}\frac{\partial L}{\partial y'}=0 \qquad (L=L(x,y_0,y_0'))$

$\partial L/\partial y - \partial^2 L/\partial x\partial y' - \partial^2 L/\partial y\partial y'\ y' - \partial^2 L/\partial y'^2\ y''=0$

If $L=L(y,y')$, i.e. doesn't depend on x
then Euler-Lagrange eqn simplifies:

$L-y'\,\partial L/\partial y'=\text{const}$

How:

$y'\,\partial L/\partial y - y'^2\,\partial^2 L/\partial y\partial y' - y'y''\,\partial^2 L/\partial y'^2=0$

$\frac{d}{dx}\left(L-y'\,\frac{\partial L}{\partial y'}\right)=0$

$\rightarrow L-y'\,\partial L/\partial y'=\text{const}$

Brachistochrone:

$$T = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

$$L = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

Use $L - y'\frac{\partial L}{\partial y'} = \text{const} = C$

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\frac{1}{\sqrt{y}}\frac{2y'}{2\sqrt{1+y'^2}} = C$$

$$\rightarrow \frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} = C$$

$$\rightarrow \frac{1+y'^2 - y'^2}{\sqrt{y}\sqrt{1+y'^2}} = C$$

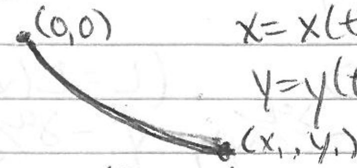
$$\rightarrow y(1+y'^2) = C_1$$

$$y' = \pm\sqrt{C/y - 1}$$

$$-\frac{dy}{\sqrt{C/y - 1}} = dx$$

... can get $x = f(y)$, but it's ugly

alternate method:



$x = x(t)$

$y = y(t)$

$dy/dx = \cot(t)$ (to get unique paramaterization)

$$y(1+y'^2) = C_1$$

$$y(1+\cot^2 t) = C_1$$

$$y\,\frac{\sin^2 t + \cos^2 t}{\sin^2 t} = C_1$$

$$y(t) = C_1 \sin^2 t$$

$$dy/dx = (dy/dt)/(dx/dt)$$

$$= \frac{C_1\, 2\sin t\cos t}{dx/dt} = \cot t = \frac{\cos t}{\sin t}$$

$$dx/dt = 2C_1\sin^2 t = C_1(1-\cos 2t)$$

$$x(t) = C_1(t - \tfrac{1}{2}\sin 2t) + \cancel{C_2}^{\,0}$$

$x(t) = c_1(t - \tfrac{1}{2}\sin 2t)$

$y(t) = c_1 \sin^2 t$

$x = \tfrac{1}{2}c_1(2t - \sin 2t)$

$y = \tfrac{1}{2}c_1(1 - \cos 2t)$

Let $\tau = 2t$, $c_2 = \tfrac{1}{2}c_1$

$x = c_2(\tau - \sin\tau)$

$y = c_2(1 - \cos\tau)$

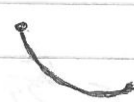
$\rightarrow$ cycloid

follow a pt. on a rolling wheel
but flip that over x-axis

$\left(\frac{x}{c_2} - \tau\right)^2 + \left(\frac{y}{c_2} - 1\right)^2 = 1$

a circle if τ was constant

mimize surface area problem:
like the shape of a heavy chain
hanging between two points



$\int_{-a}^{b} L\,dx$, $L = \tfrac{1}{2}\varepsilon y'^2 e^{-x^2/2\varepsilon}$

does depend on x

$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$ ← Euler-Lagrange

$-\tfrac{d}{dx}(\varepsilon y' e^{-x^2/2\varepsilon})$

$= -\varepsilon\left[y'' e^{-x^2/2\varepsilon} - y' e^{-x^2/2\varepsilon}\frac{2x}{2\varepsilon}\right]$

$= -e^{-x^2/2\varepsilon}\left[\varepsilon y'' - xy'\right] = 0$

$\rightarrow \varepsilon y'' - xy' = 0$

$y(x) = C + (\alpha - C)e^{-a\frac{x+a}{\varepsilon}} + (\beta - C)e^{b\frac{x-b}{\varepsilon}}$

plug into integral to find C

11-13 $I = \int_{-a}^{b} L(x, y(x), y'(x))\,dx \to \min$

$y(-a) = \alpha, \quad y(b) = \beta$

$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$ necessary, not sufficient

$L(x, y, y') = \varepsilon/2\, y'^2 e^{-x^2/2\varepsilon}$

$\to \varepsilon y'' - x y' = 0 \quad y(-a) = \alpha, \; y(b) = \beta$

$y(x) = C + (\alpha - C) e^{-a\frac{x+a}{\varepsilon}} + (\beta - C) e^{b\frac{x-b}{\varepsilon}}$

Determine C.

Plug back into integral.

$$I = \int_{-a}^{b} \frac{\varepsilon}{2} e^{-\frac{x^2}{2\varepsilon}} \left[(\alpha - C)\left(-\frac{a}{\varepsilon}\right) e^{-a\frac{x+a}{\varepsilon}} + (\beta - C)\left(\frac{b}{\varepsilon}\right) e^{b\frac{x-b}{\varepsilon}}\right]^2 dx$$

$$= \frac{\varepsilon}{2}(\alpha - C)^2 \frac{a^2}{\varepsilon^2} \int_{-a}^{b} e^{-\frac{x^2}{2\varepsilon} - 2a\frac{x+a}{\varepsilon}} dx + \frac{\varepsilon}{2}(\beta - C)^2 \frac{b^2}{\varepsilon^2} \int_{-a}^{b} e^{-\frac{x^2}{2\varepsilon} + 2b\frac{x-b}{\varepsilon}} dx - \frac{\varepsilon}{2} \cdot 2(\alpha - C)(\beta - C)\frac{a}{\varepsilon}\frac{b}{\varepsilon} \int_{-a}^{b} e^{-\frac{x^2}{2\varepsilon} - a\frac{x+a}{\varepsilon} + b\frac{x-b}{\varepsilon}} dx$$

$$= \frac{a^2}{2\varepsilon}(\alpha - C)^2 I_1 + \frac{b^2}{2\varepsilon}(\beta - C)^2 I_2 - \frac{ab}{\varepsilon}(\alpha - C)(\beta - C) I_3$$

$$I_1 = \int_{-a}^{b} e^{\frac{-x^2}{2\varepsilon} - 2a\frac{x+a}{\varepsilon}} dx$$

$$I_2 = \int_{-a}^{b} e^{\frac{-x^2}{2\varepsilon} + 2b\frac{x-b}{\varepsilon}} dx$$

$$I_3 = \int_{-a}^{b} e^{\frac{-x^2}{2\varepsilon} - a\frac{x+a}{\varepsilon} + b\frac{x-b}{\varepsilon}} dx$$

$$I_1 = \int_{-a}^{b} e^{-\frac{1}{2\varepsilon}(x^2 + 4ax + 4a^2)} = \int_{-a}^{b} e^{-\frac{1}{2\varepsilon}(x+2a)^2}$$

$[s = x+2a/\sqrt{2\varepsilon} \to ds = dx/\sqrt{2\varepsilon}]$

$$= \sqrt{2\varepsilon} \int_{a/\sqrt{2\varepsilon}}^{\frac{b+2a}{\sqrt{2\varepsilon}}} e^{-s^2} ds = \sqrt{2\varepsilon}\int_{a/\sqrt{2\varepsilon}}^{\infty} e^{-s^2} ds - \int_{\frac{b+2a}{\sqrt{2\varepsilon}}}^{\infty} e^{-s^2} ds$$

Know $\int_A^\infty e^{-s^2} ds \sim e^{-A^2}/2A, \; A \to \infty$

$$\sim \sqrt{2\varepsilon}\left[e^{-\frac{a^2}{2\varepsilon}} \frac{\sqrt{2\varepsilon}}{2a} - e^{-\frac{(b+2a)^2}{2\varepsilon}} \frac{\sqrt{2\varepsilon}}{2(b+2a)}\right]$$

$\sim (\varepsilon/a) e^{-a^2/2\varepsilon}$

Back to orig. eqn for I:

$a^2/2\varepsilon (\alpha - c)^2 \, \varepsilon/a \, e^{-a^2/2\varepsilon} + b^2/2\varepsilon (\beta - c)^2 \, \varepsilon/b \, e^{-b^2/2\varepsilon}$

$+ \ldots I_3$

$$I_3 = \int_{-a}^{b} e^{-\frac{1}{2\varepsilon}[x^2 + 2ax + 2a^2 - 2bx + 2b^2]} dx$$

$x^2 + 2x(a-b) + 2a^2 + 2b^2$

$x^2 + 2x(a-b) + (a-b)^2 - (a-b)^2 + 2a^2 + 2b^2$

$(x + (a-b))^2 + a^2 + b^2 + 2ab$

$(x + a - b)^2 + (a+b)^2$

$$I_3 = \int_{-a}^{b} e^{-\frac{1}{2\varepsilon}[(x+a-b)^2 + (a+b)^2]} dx$$

$$= e^{-\frac{(a+b)^2}{2\varepsilon}} \int_{-a}^{b} e^{-\frac{1}{2\varepsilon}(x+a-b)^2} dx$$

$[s = (x+a-b)/\sqrt{2\varepsilon} \quad ds = dx/\sqrt{2\varepsilon}]$

$$= e^{-\frac{(a+b)^2}{2\varepsilon}} \sqrt{2\varepsilon} \int_{-b/\sqrt{2\varepsilon}}^{a/\sqrt{2\varepsilon}} e^{-s^2} ds$$

$$\sim \sqrt{2\varepsilon\pi}\, e^{-\frac{(a+b)^2}{2\varepsilon}}$$

Back to orig. eqn.

$(\ldots) + (\ldots) \quad -(ab/\sqrt{\varepsilon}) \sqrt{2\pi}\, e^{-\frac{(a+b)^2}{2\varepsilon}} (\alpha - c)(\beta - c)$

magnitude of terms depends on exponentials

Suppose: $a > b$:

largest term has $(\beta - c)^2$

so to minimize, let $c = \beta$

$a < b$: $c = \alpha$

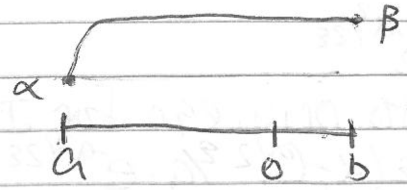
$a = b$: $a/2 (\alpha - c)^2 e^{-a^2/2\varepsilon} + a/2 (\beta - c)^2 e^{-a^2/2\varepsilon}$

$= (a/2) e^{-a^2/2\varepsilon} [(\alpha - c)^2 + (\beta - c)^2]$

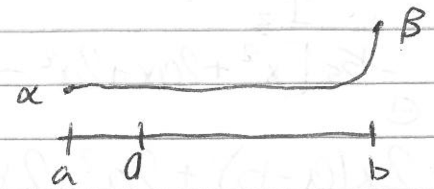
$[2c^2 - 2c(\alpha + \beta) + \alpha^2 + \beta^2]$

$c = \frac{\alpha + \beta}{2}$

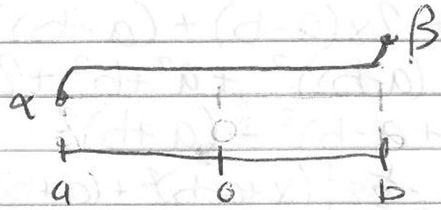
$a > b : C = \beta$



$a < b : C = \alpha$



$a = b : C = \frac{\alpha+\beta}{2}$



WKB Method (Wentzel Kramers Brillouin)

$\varepsilon u'' + p(x) u' + q(x) u = 0$

$u(x) = y(x) e^{-\frac{1}{2}\int \frac{p(x)}{\varepsilon} dx}$ ← eliminates 1st deriv.

$u' = y' e + y e (-\frac{1}{2} p/\varepsilon)$

$u'' = y'' e + 2y' e(-\frac{1}{2} \frac{p}{\varepsilon}) + y e (-\frac{1}{2} \frac{p}{\varepsilon})^2 + y e (-\frac{1}{2} \frac{p'}{\varepsilon})$

$\varepsilon y'' - \cancel{y' p} + y p^2/4\varepsilon - \frac{1}{2} y p' + \cancel{y' p} - \frac{1}{2} y p^2/\varepsilon + q y = 0$

$\varepsilon y'' - y[p^2/4\varepsilon + \frac{1}{2} p' - q] = 0$

$\varepsilon^2 y'' - y[p^2/4 + \frac{1}{2}\varepsilon p' - q\varepsilon] = 0$

$\varepsilon^2 y'' - q(x,\varepsilon) y = 0 \quad , \quad q(x,\varepsilon) = \frac{p^2}{4} + \frac{1}{2}\varepsilon p' - q\varepsilon$

$q(x,\varepsilon) \equiv q_0$ const

$y'' - \frac{1}{\varepsilon^2} q_0 y = 0$

$y(x) = a_0 e^{\frac{1}{\varepsilon}\sqrt{q_0}\, x} + b_0 e^{-\frac{1}{\varepsilon}\sqrt{q_0}\, x}$

$y'' - \frac{1}{\varepsilon^2} q(x) y = 0$ ($q(x)$ doesn't depend on ε)

Sol'n has the form:

$e^{\frac{1}{\varepsilon}\theta(x) + \theta_0(x) + \varepsilon\theta_1(x) + \cdots}$

$= e^{\frac{1}{\varepsilon}\theta(x)} e^{\theta_0(x)} e^{\varepsilon\theta_1(x) + \cdots}$

$y = e^{\frac{1}{\varepsilon}\theta(x)} [y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots]$

$$y' = \tfrac{1}{\varepsilon}\theta' e^{\theta/\varepsilon}\left[y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right]$$
$$\quad + e^{\theta/\varepsilon}\left[y_0' + \varepsilon y_1' + \cdots\right]$$
$$y'' = \tfrac{1}{\varepsilon}\theta'' e^{\theta/\varepsilon}\left[y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right]$$
$$\quad + \left(\tfrac{1}{\varepsilon}\theta'\right)^2 e^{\theta/\varepsilon}\left[y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right]$$
$$\quad + \tfrac{1}{\varepsilon}\theta' e^{\theta/\varepsilon}\left[y_0' + \varepsilon y_1' + \cdots\right]$$
$$\quad + \theta'/\varepsilon\, e^{\theta/\varepsilon}\left[y_0' + \varepsilon y_1' + \cdots\right] + e^{\theta/\varepsilon}\left[y_0'' + \varepsilon y_1'' + \cdots\right]$$

$$\rightarrow \tfrac{1}{\varepsilon}\theta''(y_0 + \varepsilon y_1) + \tfrac{1}{\varepsilon^2}\theta'^2(y_0 + \varepsilon y_1 + \varepsilon^2 y_2) + \tfrac{2}{\varepsilon}\theta'(y_0' + \varepsilon y_1')$$
$$+ y_0'' - \tfrac{1}{\varepsilon^2} q(y_0 + \varepsilon y_1 + \varepsilon^2 y_2) + O(\varepsilon) = 0$$

$$O(\tfrac{1}{\varepsilon^2}):\ \theta'^2 y_0 - q y_0 = 0$$
$$O(\tfrac{1}{\varepsilon}):\ \theta'' y_0 + \theta'^2 y_1 + 2\theta' y_0' - q y_1 = 0$$
$$O(1):\ \theta'' y_1 + \theta'^2 y_2 + 2\theta' y_1' + y_0'' - q y_2 = 0$$

$$\begin{cases} (\theta'^2 - q)\, y_0 = 0 \\ (\theta'^2 - q)\, y_1 + \theta'' y_0 + 2\theta' y_0' = 0 \\ (\theta'^2 - q)\, y_2 + \theta'' y_1 + 2\theta' y_1' + y_0'' = 0 \end{cases}$$

$1+\frac{1}{\varepsilon}\theta+\frac{1}{2}\frac{1}{\varepsilon^2}\theta^2$

11-18 $y''-\frac{1}{\varepsilon^2}q(x)y=0$

$y(x)=e^{\frac{1}{\varepsilon}\theta(x)}\left[y_0(x)+\varepsilon y_1(x)+\varepsilon^2 y_2(x)+\cdots\right]$

$O(\frac{1}{\varepsilon^2})$: $[\theta'^2(x)-q(x)]\,y_0(x)=0$

$O(\frac{1}{\varepsilon})$: $[\theta'^2(x)-q(x)]\,y_1+2\theta' y_0'+\theta'' y_0=0$

$O(1)$: $[\theta'^2-q]\,y_2+2\theta' y_1'+\theta'' y_1+y_0''=0$

Since $y_0(x)\neq 0$,

$\theta'^2(x)-q(x)=0$ (from $O(\frac{1}{\varepsilon^2})$)

$\theta'(x)=\pm\sqrt{q(x)}$

$\theta(x)=\pm\int^x\sqrt{q(s)}\,ds$

$O(\frac{1}{\varepsilon})$: $2\theta' y_0'+\theta'' y_0=0$

$y_0'+(\theta''/2\theta')\,y_0=0$

integrating factor $=\exp\int\frac{\theta''}{2\theta'}dx=\exp(\frac{1}{2}\ln|\theta'|)=\sqrt{|\theta'|}$

$\left[y_0\sqrt{|\theta'|}\right]'=0$

$y_0=\frac{c}{\sqrt{|\theta'|}}=\frac{c}{q^{1/4}(x)}$

$y(x)\sim\frac{1}{q^{1/4}(x)}\left[a_0 e^{-\frac{1}{\varepsilon}\int^x\sqrt{q(s)}ds}+b_0 e^{\frac{1}{\varepsilon}\int^x\sqrt{q(s)}ds}\right]$

$q(x_0)=0\ \rightarrow\ x_0$ is a turning point

Ex: $y''+\frac{1}{\varepsilon^2}e^{2x}y=0$

$\rightarrow q(x)=-e^{2x}$

$\theta(x)=\pm\int^x\sqrt{q(s)}\,ds=\pm\int^x i e^s ds=\pm i e^x$

$y(x)\sim e^{-\frac{x}{2}}\left[a_0 e^{-\frac{1}{\varepsilon}ie^x}+b_0 e^{\frac{1}{\varepsilon}ie^x}\right]$

$=e^{-\frac{x}{2}}\left[a_0\cos\frac{e^x}{\varepsilon}+b_1\sin\frac{e^x}{\varepsilon}\right.$

Ex: $y''+\lambda e^{2x}y=0$; $0<x<1$, $y(0)=y(1)=0$, $\lambda\gg1$

$\rightarrow \lambda=1/\varepsilon^2$

$y(x)=e^{-x/2}\left[a_1\cos\frac{e^x}{\varepsilon}+b_1\sin e^x/\varepsilon\right]$

$y(0)=a_1\cos\frac{1}{\varepsilon}+b_1\sin\frac{1}{\varepsilon}=0$

$y(1)=e^{-1/2}(a_1\cos\frac{e}{\varepsilon}+b_1\sin\frac{e}{\varepsilon})=0$

non-trivial sol'n $\rightarrow \begin{vmatrix} \cos\frac{1}{\varepsilon} & \sin\frac{1}{\varepsilon} \\ \cos e/\varepsilon & \sin e/\varepsilon \end{vmatrix} = 0$

$\rightarrow \sin e/\varepsilon \cos\frac{1}{\varepsilon} - \cos(e/\varepsilon)\sin 1/\varepsilon = 0$

$\rightarrow \sin\frac{e-1}{\varepsilon} = 0$

$\rightarrow \frac{e-1}{\varepsilon} = \pi n \quad , n \gg 1$

$\frac{1}{\varepsilon} = \pi n/(e-1)$

$\lambda_n = \dfrac{\pi^2 n^2}{(e-1)^2}$

Ex: $y'' + \lambda x^4 y = 0 \quad , \quad y(1) = y(2) = 0$

$\lambda = 1/\varepsilon^2 \quad , \quad q(x) = -x^4$

$y(x) = \frac{1}{x}\left[a_0 e^{-\frac{1}{\varepsilon}\frac{1}{3}ix^3} + b_0 e^{\frac{1}{\varepsilon}\frac{1}{3}ix^3}\right]$

$= \frac{1}{x}\left[a_1 \cos\frac{x^3}{3\varepsilon} + b_1 \sin\frac{x^3}{3\varepsilon}\right]$

$y(1) = a_1 \cos\frac{1}{3\varepsilon} + b_1 \sin\frac{1}{3\varepsilon} = 0$

$y(2) = \frac{1}{2}\left(a_1 \cos\frac{8}{3\varepsilon} + b_1 \sin\frac{8}{3\varepsilon}\right) = 0$

$\begin{vmatrix} \cos\frac{1}{3\varepsilon} & \sin\frac{1}{3\varepsilon} \\ \cos 8/3\varepsilon & \sin 8/3\varepsilon \end{vmatrix} = 0$

$\sin\left(8/3\varepsilon - \frac{1}{3\varepsilon}\right) = 0 \rightarrow \sin 7/3\varepsilon = 0$

$7/3\varepsilon = \pi n$

$\frac{1}{\varepsilon} = \frac{3}{7}\pi n$

$\lambda_n = (9/49)\pi^2 n^2 \quad , n \gg 1$

How far from roots of q, do we need to be?
↳ turning points

Look at y_1 correction term - it must be small.

$O(1): \; y_1' + \frac{\theta''}{2\theta'} y_1 = -y_0''/2\theta'$

$\left[y_1\sqrt{|\theta'|}\right]' = (-y_0''/2\theta')\sqrt{|\theta'|}$

recall $\sqrt{|\theta'|} = c/y_0$

$\left[y_1 \, c/y_0\right]' = -y_0''/(2c^2/y_0^2)\,\frac{c}{y_0}$

$\left[y_1/y_0\right]' = -y_0'' y_0/2c^2$

$y_1/y_0 = \int^x -\frac{1}{2c^2}\, y_0'' y_0 \, ds + c_1$

$$y_1(x) = -\tfrac{1}{2c^2}\, y_0 \int^x y_0'' y_0 \, ds + c_1 y_0$$
$$= -\tfrac{1}{2c^2}\, y_0 \left[y_0' y_0 - \int^x y_0'^2 \, ds \right] + c_1 y_0$$
$$= y_0(x) \Big[-\tfrac{1}{2c^2} \left(c/q^{1/4} \right) \left(-\tfrac{1}{4} c q^{-5/4} \right) q' + \tfrac{1}{2c^2} \int^x \tfrac{1}{16} c^2 q^{-5/2} q'^2 \, ds + c_1 \Big]$$
$$= y_0(x)\, w(x)$$
$$w(x) = \tfrac{1}{8}\, q'/q^{3/2} + \tfrac{1}{32} \int^x q'^2/q^{5/2} \, ds + c_1$$

want $|y_0(x)| \gg |\varepsilon y_1(x)| = |\varepsilon y_0(x) w(x)|$

$\to \quad |\varepsilon w(x)| \ll 1$

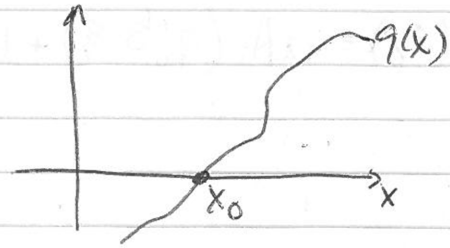
if we are close to turning pt x_0,

$$q(x) \sim q'(x_0)(x - x_0) \quad \text{as } x \to x_0$$
$$\to w(x) \sim \frac{1}{8} \frac{q'(x_0)}{q'(x_0)^{3/2} (x-x_0)^{3/2}} + \text{const} \cdot (x-x_0)^{-3/2} + c_1$$
$$w(x) \sim \frac{\text{const}}{(x-x_0)^{3/2}}$$
$$\varepsilon \cdot \frac{\text{const}}{(x-x_0)^{3/2}} \ll 1$$
$$\varepsilon \cdot \text{const} \ll (x-x_0)^{3/2}$$
$$|x - x_0| \gg \text{const} \cdot \varepsilon^{2/3}$$

So what do we do when there's a turning point in the interval of interest?

$\to$ internal layer!

Turning Points



$q(x) \sim q'(x_0)(x-x_0)$

as $x \to x_0$

$q'(x_0) = q_0' > 0$

$y'' - \frac{1}{\varepsilon^2} q(x) y = 0$

$$y(x) = \begin{cases} y_R(x) & x > x_0 \\ y_L(x) & x < x_0 \end{cases}$$

$$y_R(x) = \frac{1}{q^{1/4}}\left[a_R e^{-\frac{1}{\varepsilon}\int_{x_0}^{x}\sqrt{q(s)}\,ds} + b_R e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\sqrt{q(s)}\,ds}\right]$$

$$y_L(x) = \frac{1}{q^{1/4}}\left[a_L e^{+\frac{1}{\varepsilon}\int_{x}^{x_0}\sqrt{q(s)}\,ds} + b_L e^{-\frac{1}{\varepsilon}\int_{x}^{x_0}\sqrt{q(s)}\,ds}\right]$$

$y(x) = \varepsilon^{-\delta} Y(\xi), \quad \xi = \frac{x-x_0}{\varepsilon^\beta}, \quad x = \xi\varepsilon^\beta + x_0$

$\frac{1}{\varepsilon^{2\beta}} Y'' - \frac{1}{\varepsilon^2} q(x_0 + \varepsilon^\beta \xi) Y = 0$

$\frac{1}{\varepsilon^{2\beta}} Y'' - \frac{1}{\varepsilon^2}\left[q_0' \varepsilon^\beta \xi\right] Y = 0$

$\frac{1}{\varepsilon^{2\beta}} = \frac{\varepsilon^\beta}{\varepsilon^2} \qquad \varepsilon^{3\beta} = \varepsilon^2 \to \beta = 2/3$

$Y(\xi) \sim Y_0(\xi)$

$Y_0'' - q_0' \xi Y_0 = 0$

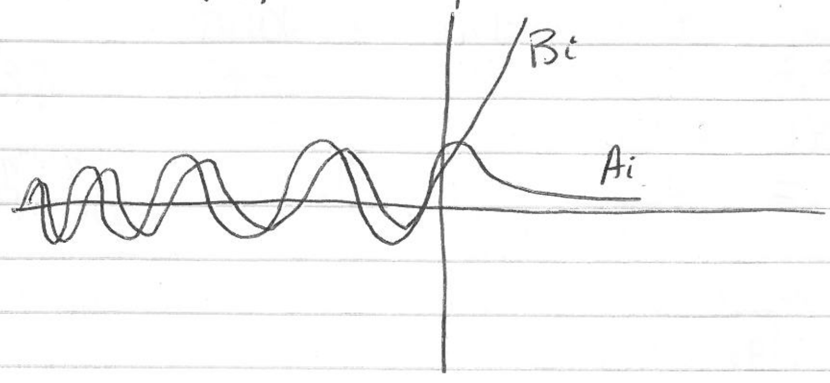
$Y_0(\xi) = Z(s), \quad s = q_0'^{\frac{1}{3}} \xi$

$q_0'^{2/3} Z'' - q_0'(s/q_0'^{\frac{1}{3}}) Z = 0$

$Z'' - sZ = 0$ ← Airy Equation

$Z(s) = a\,Ai(s) + b\,Bi(s)$

Ai, Bi : Airy functions of the 1^{st} + 2^{nd} kind



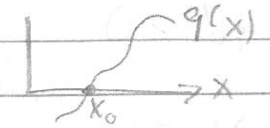
$$Y_0(\xi) = a\,Ai(q_0'^{1/3}\xi) + b\,Bi(q_0'^{1/3}\xi)$$

11-20 Turning Points:

$$y'' - \frac{1}{\varepsilon^2} q(x)\, y = 0$$

$$q(x) \sim q_0'(x - x_0), \; x \to x_0$$

$$q_0' > 0$$



$$y(x) = \begin{cases} y_R(x) & x > x_0 \\ y_L(x) & x < x_0 \end{cases}$$

$$y_R(x) = q(x)^{-1/4}\left\{ a_R e^{-\frac{1}{\varepsilon}\int_{x_0}^{x}\sqrt{q(s)}\,ds} + b_R e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\sqrt{q(s)}\,ds} \right\}$$

$$y_L(x) = q(x)^{-1/4}\left\{ a_L e^{\frac{1}{\varepsilon}\int_{x}^{x_0}\sqrt{q(s)}\,ds} + b_L e^{-\frac{1}{\varepsilon}\int_{x}^{x_0}\sqrt{q(s)}\,ds} \right\}$$

Internal layer:

$$y(x) = \varepsilon^{-\gamma} Y(\xi), \quad \xi = \frac{x - x_0}{\varepsilon^{2/3}}$$

$$\rightarrow Y_0'' - q_0'\xi Y_0 = 0$$

$$Y_0(\xi) = a\,A_i(q_0'^{1/3}\xi) + b\,B_i(q_0'^{1/3}\xi)$$

Asymptotics of Airy functions:

As $s \to -\infty$:

$$\left.\begin{matrix} Ai(s) \\ Bi(s) \end{matrix}\right\} \sim \frac{1}{\pi |s|^{1/4}} \cos\left(\frac{2}{3}|s|^{3/2} \mp \frac{\pi}{4}\right)$$

As $s \to +\infty$:

$$A_i(s) \sim \frac{1}{2\sqrt{\pi}\, s^{1/4}} e^{-2/3\, s^{3/2}}$$

$$Bi(s) \sim \frac{1}{\sqrt{\pi}\, s^{1/4}} e^{2/3\, s^{3/2}}$$

Matching using intermediate variable

$$\eta = \frac{x - x_0}{\varepsilon^\alpha}, \quad 0 < \alpha < 2/3, \quad x = x_0 + \varepsilon^\alpha \eta, \quad \xi = \eta\, \varepsilon^{\alpha - 2/3}$$

Match $Y(\xi)$ & $y_R(x)$:

$$y_R(x) = \frac{1}{q(x_0 + \varepsilon^\alpha \eta)^{1/4}}\left[a_R e^{-\frac{1}{\varepsilon}\int_{x_0}^{x_0+\varepsilon^\alpha\eta}\sqrt{q(s)}\,ds} + b_R e^{\frac{1}{\varepsilon}\int_{x_0}^{x_0+\varepsilon^\alpha\eta}\sqrt{q(s)}\,ds} \right]$$

$$\sim \frac{1}{[q_0'(\varepsilon^\alpha \eta)]^{1/4}}\left[\cdots \right.$$

$$\int_{x_0}^{x_0+\varepsilon^\alpha\eta}\sqrt{q(s)}\,ds \sim \int_{x_0}^{x_0+\varepsilon^\alpha\eta}\sqrt{q_0'}\sqrt{s-x_0}\,ds \qquad t=s-x_0$$

$$=\int_0^{\varepsilon^\alpha\eta}\sqrt{q_0'}\sqrt{t}\,dt = \sqrt{q_0'}\cdot\frac{2}{3}(\varepsilon^\alpha\eta)^{3/2}$$

$$y_R(x)\sim\frac{1}{(q_0'\varepsilon^\alpha\eta)^{1/4}}\left[a_R e^{-\frac{2}{3}\sqrt{q_0'}\,\varepsilon^{3/2\alpha-1}\eta^{3/2}}+b_R e^{\frac{2}{3}\sqrt{q_0'}\,\varepsilon^{3/2\alpha-1}\eta^{3/2}}\right]$$

$$\varepsilon^{-\gamma}Y_0\left(\tfrac{x}{\varepsilon}\right)=\varepsilon^{-\gamma}a\,Ai(q_0'^{1/3}\varepsilon^{\alpha-2/3}\eta)+\varepsilon^{-\gamma}b\,Bi(q_0'^{1/3}\varepsilon^{\alpha-2/3}\eta)$$

$$\sim\frac{1\cdot\varepsilon^{-\gamma}}{2\sqrt{\pi}(q_0'^{1/3}\varepsilon^{\alpha-2/3}\eta)^{1/4}}\left[a e^{-2/3\, q_0'^{1/2}\varepsilon^{3/2\alpha-1}\eta^{3/2}}+2b e^{2/3\, q_0'^{1/2}\varepsilon^{3/2\alpha-1}\eta^{3/2}}\right]$$

Compare: exponentials are the same

$\gamma=\frac{1}{6}$

$$\frac{a_R}{q_0'^{1/4}}=\frac{a}{2\sqrt{\pi}\,q_0'^{1/12}}\rightarrow a_R=\frac{q_0'^{1/6}}{2\sqrt{\pi}}a$$

$$b_R=\frac{q_0'^{1/6}}{\sqrt{\pi}}b$$

Match $Y(\frac{x}{\varepsilon})$ + $y_L(x)$:

$$y_L(x)=\frac{1}{[q(x_0+\varepsilon^\alpha\eta)]^{1/4}}\left[a_L e^{\frac{1}{\varepsilon}\int_{x_0+\varepsilon^\alpha\eta}^{x_0}\sqrt{q(s)}ds}+b_L e^{-\frac{1}{\varepsilon}\int_{x_0+\varepsilon^\alpha\eta}^{x_0}\sqrt{q(s)}ds}\right]$$

$$[q(x_0+\varepsilon^\alpha\eta)]^{1/4}\sim q_0'^{1/4}\varepsilon^{\alpha/4}|\eta|^{1/4}e^{\pi/4 i}$$

$$\int_{x_0+\varepsilon^\alpha\eta}^{x_0}\sqrt{q(s)}\,ds\sim\int_{x_0+\varepsilon^\alpha\eta}^{x_0}\sqrt{q_0'}\sqrt{s-x_0}\,ds \qquad t=x_0-s$$

$$=\int_{-\varepsilon^\alpha\eta}^{0}\sqrt{q_0'}\,i\sqrt{t}\,(-dt)=i\sqrt{q_0'}\int_0^{-\varepsilon^\alpha\eta}\sqrt{t}\,dt$$

$$=i\sqrt{q_0'}\,\tfrac{2}{3}t^{3/2}\Big|^{t=\varepsilon^\alpha|\eta|}=\tfrac{2}{3}i\sqrt{q_0'}(\varepsilon^\alpha|\eta|)^{3/2}$$

$$y_L(x)=\frac{e^{-\pi/4 i}}{(q_0'\varepsilon^\alpha|\eta|)^{1/4}}\left[a_L e^{2/3 i\sqrt{q_0'}\varepsilon^{3/2\alpha-1}|\eta|^{3/2}}+b_L e^{-2/3 i\sqrt{q_0'}\varepsilon^{3/2\alpha-1}|\eta|^{3/2}}\right]$$

$$\varepsilon^{-1/6}Y_0\left(\tfrac{x}{\varepsilon}\right)=\varepsilon^{-1/6}\left[a\,Ai(q_0'^{1/3}\varepsilon^{\alpha-2/3}\eta)+b\,Bi(q_0'^{1/3}\varepsilon^{\alpha-2/3}\eta)\right]$$

$$\sim\frac{\varepsilon^{-1/6}}{\sqrt{\pi}\,q_0'^{1/12}\varepsilon^{\alpha/4-1/6}|\eta|^{1/4}}\left[a\cos\left(\tfrac{2}{3}q_0'^{1/2}\varepsilon^{\frac{3}{2}\alpha-1}|\eta|^{3/2}-\pi/4\right)+b\cos\left(\tfrac{2}{3}q_0'^{1/2}\varepsilon^{3/2\alpha-1}|\eta|^{3/2}+\pi/4\right)\right]$$

$$=\frac{1}{\sqrt{\pi}\,q_0'^{1/12}\varepsilon^{\alpha/4}|\eta|^{1/4}}\left[\frac{a}{2}e^{i(\frac{2}{3}q_0'^{1/2}\varepsilon^{3/2\alpha-1}|\eta|^{3/2}-\pi/4)}+c.c.\right.$$

$\leftarrow$ cplx conj

$$\left.+\frac{b}{2}e^{i(\frac{2}{3}q_0'^{1/2}\varepsilon^{3/2\alpha-1}|\eta|^{3/2}+\pi/4)}+c.c.\right]$$

$$= \frac{1}{\sqrt{\pi}\, q_0'^{1/12}\, \varepsilon^{\alpha/4} |\eta|^{1/4}} \left[\frac{1}{2} e^{i\frac{2}{3} q_0'^{1/2} \varepsilon^{3/2\alpha - 1} |\eta|^{3/2}} \left(a e^{-i\pi/4} + b e^{i\pi/4} \right) + c.c. \right]$$

Compare:

$$\begin{cases} \dfrac{a e^{-i\pi/4} + b e^{i\pi/4}}{2\sqrt{\pi}\, q_0'^{1/12}} = \dfrac{a_L e^{-\pi/4 i}}{q_0'^{1/4}} \\[2ex] \dfrac{a e^{i\pi/4} + b e^{-i\pi/4}}{2\sqrt{\pi}\, q_0'^{1/12}} = \dfrac{b_L e^{-\pi/4 i}}{q_0'^{1/4}} \end{cases}$$

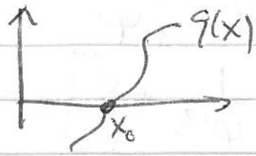
$$\rightarrow \begin{cases} a_L = \dfrac{(a + bi)\, q_0'^{1/6}}{2\sqrt{\pi}} \\[2ex] b_L = \dfrac{(ai + b)\, q_0'^{1/6}}{2\sqrt{\pi}} \end{cases}$$

$$\left.\begin{aligned} a_L &= a_R + \tfrac{1}{2} i b_R \\ b_L &= i a_R + \tfrac{1}{2} b_R \end{aligned}\right\} \text{connection formulas}$$

$$y(x) \sim \begin{cases} \dfrac{1}{q(x)^{1/4}} \left[a_R e^{-\frac{1}{\varepsilon} K(x)} + b_R e^{\frac{1}{\varepsilon} K(x)} \right] & x > x_0 \\[2ex] \dfrac{1}{|q(x)|^{1/4}} \left[2 a_R \cos\left(\dfrac{\theta(x)}{\varepsilon} - \dfrac{\pi}{4} \right) + b_R \cos\left(\dfrac{\theta(x)}{\varepsilon} + \dfrac{\pi}{4} \right) \right] & x < x_0 \end{cases}$$

$$K(x) = \int_{x_0}^{x} \sqrt{q(s)}\, ds, \quad \theta(x) = \int_{x}^{x_0} \sqrt{|q(s)|}\, ds$$

$y'' - \frac{1}{\varepsilon^2} q(x) y = 0$ 11-25



$q(x) \sim q_0'(x - x_0)$

$$y(x) \sim \begin{cases} \frac{1}{|q|^{1/4}}\left[2a_R \cos\left(\frac{\theta(x)}{\varepsilon} - \frac{\pi}{4}\right) + b_R \cos\left(\frac{\theta(x)}{\varepsilon} + \frac{\pi}{4}\right)\right] & x < x_0 \\ \frac{1}{q^{1/4}}\left[a_R e^{-\frac{1}{\varepsilon}K(x)} + b_R e^{\frac{1}{\varepsilon}K(x)}\right] & x > x_0 \end{cases}$$

$$\theta(x) = \int_x^{x_0} \sqrt{|q(s)|}\, ds \ , \quad K(x) = \int_{x_0}^{x} \sqrt{q(s)}\, ds$$

↑ Doesn't include inner layer (not uniformly valid)

$$y(x) \sim \frac{[K(x)]^{1/6}}{[q(x)]^{1/4}} \left\{ a_0 Ai\left(\left(\frac{3K}{2\varepsilon}\right)^{2/3}\right) + b_0 Bi\left(\left(\frac{3K}{2\varepsilon}\right)^{2/3}\right) \right\}$$

↑ Lager's Solution (1931)

<u>Lager's Transformation</u>

$y'' - \frac{1}{\varepsilon^2} q(x) y = 0$

$z = \varphi(x) \ , \quad v(z) = \psi(x) y(x)$

$y(x) = v(z)/\psi(x)$

$$\frac{dy}{dx} = \frac{dv}{dz}\frac{\varphi'}{\psi} - \frac{\psi'}{\psi^2} v$$

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dz^2}\frac{\varphi'^2}{\psi} + \left[\frac{\varphi''}{\psi} - 2\frac{\varphi'\psi'}{\psi^2}\right]\frac{dv}{dz} - \frac{\psi''}{\psi^2} v + 2\frac{\psi'^2}{\psi^3} v$$

$$\rightarrow \frac{\varphi'^2}{\psi}\frac{d^2v}{dz^2} + \left[\frac{\varphi''}{\psi} - 2\frac{\varphi'\psi'}{\psi^2}\right]\frac{dv}{dz} - \left[\frac{\psi''}{\psi^2} - 2\frac{\psi'^2}{\psi^3} + \frac{q}{\varepsilon^2}\frac{1}{\psi}\right] v = 0$$

$$\frac{d^2v}{dz^2} + \left[\frac{\varphi''}{\varphi'^2} - 2\frac{\psi'}{\psi\varphi'}\right]\frac{dv}{dz} - \left[\frac{\psi''}{\psi\varphi'^2} - 2\frac{\psi'^2}{\psi^2\varphi'^2} + \frac{q}{\varepsilon^2}\frac{1}{\varphi'^2}\right] v = 0$$

Need to choose φ + ψ to make eqn simpler.

→ get rid of first deriv.

$$\frac{\varphi''}{\varphi'^2} = 2\frac{\psi'}{\psi\varphi'} \rightarrow \frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi} \rightarrow (\ln|\varphi'|)' = (\ln \psi^2)'$$

$$\rightarrow \ln|\varphi'| = \ln \psi^2 \rightarrow \varphi' = \psi^2 \rightarrow \psi = \sqrt{\varphi'}$$

So let $\psi = \sqrt{\varphi'}$.

$$\frac{d^2v}{dz^2} - \left[\frac{q}{\varepsilon^2}\frac{1}{\varphi'^2} + \delta\right]v = 0$$

(*) $$\delta = \frac{\psi''}{\psi\varphi'^2} - 2\frac{\psi'^2}{\psi^2\varphi'^2}$$

choose φ:

let $\frac{q}{\varepsilon^2\varphi'^2} = 1 \to \varphi' = \frac{\sqrt{q}}{\varepsilon} \to \varphi(x) = \frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,ds$

$$\to \frac{d^2v}{dz^2} - \left[1 + \delta\right]v = 0, \quad \delta = \varepsilon^2 \frac{q''q - \frac{5}{4}q'^2}{q^3}$$

At the leading order:

$$\frac{d^2v}{dz^2} - v = 0 \to v(z) = a_1 e^{-z} + b_1 e^{z}$$

$$y(x) = \frac{v(\varphi(x))}{\psi(x)} = \frac{\varepsilon^{1/2}}{q^{1/4}}\left[a_1 e^{-\frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,ds} + b_1 e^{\frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,ds}\right]$$

matches prev. soln w/o turning pt

Now include a turning pt.

Restart at (*), choose φ diferently

$$\frac{q}{\varepsilon^2\varphi'^2} = z = \varphi(x) \to \varphi\varphi'^2 = \frac{q}{\varepsilon^2} \to \sqrt{\varphi}\,\varphi' = \frac{\sqrt{q}}{\varepsilon}$$

$$\tfrac{2}{3}\varphi^{3/2}(x) = \frac{1}{\varepsilon}\int_{x_0}^x \sqrt{q(s)}\,ds$$

$$\varphi(x) = \left[\frac{3}{2\varepsilon}\int_{x_0}^x \sqrt{q(s)}\,ds\right]^{2/3}$$

$$\psi = \varphi'^{\frac{1}{2}} = \left(\frac{\sqrt{q}}{\varepsilon\sqrt{\varphi}}\right)^{\frac{1}{2}} = \frac{q^{1/4}}{\varepsilon^{1/2}}\left[\frac{3}{2\varepsilon}\int_{x_0}^x \sqrt{q(s)}\,ds\right]^{-\frac{1}{6}}$$

$$= \frac{q^{1/4}}{\varepsilon^{1/3}\left(\frac{3}{2}\int_{x_0}^x \sqrt{q(s)}\,ds\right)^{1/6}}$$

$$\frac{d^2v}{dz^2} - (z+\delta)v = 0, \quad \delta = O(\varepsilon^{4/3})$$

$\frac{d^2v}{dz^2} - zv = 0$ at leading order

$v = a_1 Ai(z) + b_1 Bi(z)$

$$y = \frac{v(\varphi(x))}{\psi(x)} = \frac{[K(x)]^{1/6}}{q^{1/4}}\left[a_0 Ai\left(\left(\frac{3K(x)}{2\varepsilon}\right)^{2/3}\right) + b_0 Bi\left(\left(\frac{3K(x)}{2\varepsilon}\right)^{2/3}\right)\right]$$

$K = \int_{x_0}^{x} \sqrt{q(s)}\,ds$

check δ: (don't want denom $\to 0$)

What if we had tried z^2 instead of z?

$$\frac{q}{\varepsilon^2 \varphi'^2} = z^2 = \varphi^2(x) \rightarrow \varphi^2 \varphi'^2 = \frac{q}{\varepsilon^2}$$

$$\varphi\varphi' = \sqrt{q}/\varepsilon, \quad \tfrac{1}{2}(\varphi^2)' = \sqrt{q}/\varepsilon$$

$$\varphi^2 = 2\int_{x_0}^{x} \sqrt{q(s)}/\varepsilon\, ds$$

$$\varphi(x) = \sqrt{2/\varepsilon}\left[\int_{x_0}^{x} \sqrt{q(s)}ds\right]^{1/2}$$

$\rightarrow \delta = O(\varepsilon)$

but is it uniformly $O(\varepsilon)$?

$q(x) \sim q_0'(x-x_0)$

$$\rightarrow \varphi(x) \sim c_1 (x-x_0)^{3/4}$$

$$\varphi'(x) \sim c_2 (x-x_0)^{-1/4}$$

$$\psi(x) \sim c_3 (x-x_0)^{-1/8}$$

$$\psi'(x) \sim c_4 (x-x_0)^{-9/8}$$

$$\psi''(x) \sim c_5 (x-x_0)^{-17/8}$$

$$\frac{\psi''}{\psi\varphi'^2} \sim c\frac{(x-x_0)^{-17/8}}{(x-x_0)^{-1/8}(x-x_0)^{-1/2}} = c\,(x-x_0)^{-3/2}$$

Not uniformly small!

With prev. assumption (z not z^2)

it <u>is</u> uniformly small

12-2 Multiple Scales

Duffing Equation:

$\ddot{x} + \delta\dot{x} + x + \varepsilon x^3 = f\cos\omega t$

$x = x(t)$, $\delta \geq 0$, ε, f, ω const

like a pendulum.

$\delta = f = 0$:

$\ddot{x} + x + \varepsilon x^3 = 0$, $x(0) = A$, $\dot{x}(0) = 0$

$x(t) \sim x_0(t) + \varepsilon x_1(t)$

$O(1)$: $\ddot{x}_0 + x_0 = 0$, $x_0(0) = A$, $\dot{x}_0(0) = 0$

$x_0(t) = A\cos t$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -x_0^3$, $x_1(0) = 0$, $\dot{x}_1(0) = 0$

$\left[\cos^3 t = \left(\frac{e^{it} + e^{-it}}{2}\right)^3 = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right]$

$\ddot{x}_1 + x_1 = -A^3\left(\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right)$

$x_1(t) = c_1\cos t + c_2\sin t + A^3\frac{1}{32}\cos 3t$
$\quad - \frac{3}{8}A^3 t\sin t$

$x_1(0) = c_1 + A^3/32 = 0 \rightarrow c_1 = -A^3/32$

$\dot{x}_1(0) = c_2 = 0$

$x_1(t) = A^3/32(\cos 3t - \cos t) - \frac{3}{8}A^3 t\sin t$

$x(t) = A\cos t + (A^3/32)\varepsilon(\cos 3t - \cos t - 12t\sin t)$

Prove that $|x(t)| \leq A$:

$\dot{x}\ddot{x} + \dot{x}x + \varepsilon\dot{x}x^3 = 0$

$\frac{1}{2}\frac{d}{dt}(\dot{x}^2) + \frac{1}{2}\frac{d}{dt}(x^2) + \frac{\varepsilon}{4}\frac{d}{dt}(x^4) = 0$

$\rightarrow \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}\varepsilon x^4 = \text{const} = \frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4$

$\dot{x}^2 = A^2 - x^2 + \frac{1}{2}\varepsilon A^4 - \frac{1}{2}\varepsilon x^4$

$A^2 - x^2 + \frac{1}{2}\varepsilon A^4 - \frac{1}{2}\varepsilon x^4 \geq 0$

$(A^2 - x^2)(1 + \frac{1}{2}\varepsilon(A^2 + x^2)) \geq 0$

$A^2 - x^2 \geq 0$

$A^2 \geq x^2$

$|x(t)| \leq A$ ✓

But our approximation is <u>not</u> bounded!

Ex: $\ddot{x} + x + \varepsilon x = 0$, $x(0) = A$, $\dot{x}(0) = 0$

Approx. sol'n: $x(t) \sim x_0(t) + \varepsilon x_1(t)$

$O(1)$: $\ddot{x}_0 + x_0 = 0$, $x_0(0) = A$, $\dot{x}_0(0) = 0$

$x_0(t) = A\cos t$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -x_0$, $x_1(0) = 0$, $\dot{x}_1(0) = 0$

$x_1(t) = c_1\cos t + c_2 \sin t - \frac{1}{2}A t\sin t$

$x_1(0) = c_1 = 0$

$\dot{x}_1(0) = c_2 = 0$

$x_1(t) = -\frac{1}{2}A t\sin t$

$x(t) = A\cos t - \frac{1}{2}A\varepsilon t \sin t$

Exact sol'n:

$x(t) = A\cos\sqrt{1+\varepsilon}\,t$

$\sim A\cos t - \frac{1}{2}A\varepsilon t\sin t$ (As found)

Using this expansion, there is no way to avoid the factor of t in the sol'n.

Want a 2π-periodic sol'n.

$\ddot{x} + x + \varepsilon x^3 = 0$, $x(0) = A$, $\dot{x}(0) = 0$

$x = x(t)$, frequency $\omega(\varepsilon)$, period $\frac{2\pi}{\omega(\varepsilon)}$

let $\tau = \omega(\varepsilon)t$, $x(t) = y(\tau)$

$0 < t < 2\pi/\omega \rightarrow 0 < \tau < 2\pi$

$\frac{dx}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = \omega\frac{dy}{d\tau}$

$\frac{d^2x}{dt^2} = \omega^2 \frac{d^2y}{dt^2}$

$\omega^2\frac{d^2y}{d\tau^2} + y + \varepsilon y^3 = 0$, $y(0) = A$, $\omega\frac{dy}{d\tau}(0) = 0$

$y(\tau) \sim y_0(\tau) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)$

$\omega \sim \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2$

$\omega^2 \sim \omega_0^2 + 2\varepsilon\omega_0\omega_1 + 2\varepsilon^2(\omega_1^2 + 2\omega_1\omega_2)$

$[\omega_0^2 + 2\varepsilon\omega_0\omega_1 + 2\varepsilon^2(\omega_1^2 + 2\omega_1\omega_2)][\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2]$

$+ y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon(y_0^3 + 3y_0^2\varepsilon y_1) + O(\varepsilon^3) = 0$

$O(1)$: $\omega_0^2\ddot{y}_0 + y_0 = 0$, $y_0(0) = A$, $\dot{y}_0(0) = 0$

$y_0(t) = c_1\cos\tau/\omega_0 + c_2\sin\tau/\omega_0$

2π-periodic if $\omega_0 = 1$

$y_0(\tau) = A\cos\tau$

$O(\varepsilon)$: $\ddot{y}_1 + y_1 = f_1$, $y_1(0)=0$, $\dot{y}_1(0)=0$

$f_1 = -2\omega_1 \ddot{y}_0 - y_0^3$

$O(\varepsilon^2)$: $\ddot{y}_2 + y_2 = f_2$ $y_2(0)=0$, $\dot{y}_2(0)=0$

$f_2 = -2\omega_1\ddot{y}_1 - (\omega_1^2 + 2\omega_2)\ddot{y}_0 - 3y_0^2 y_1$

Solve $O(\varepsilon)$:

$f_1 = 2\omega_1 A\cos\tau - A^3\cos^3\tau$

$= 2\omega_1 A\cos\tau - \frac{3}{4}A^3\cos\tau - \frac{1}{4}A^3\cos 3\tau$

$= (2\omega_1 A - \frac{3}{4}A^3)\cos\tau - \frac{1}{4}A^3\cos 3\tau$

let $2\omega_1 A - \frac{3}{4}A^3 = 0$

$\frac{3}{4}A^2 = 2\omega_1$

$\frac{3}{8}A^2 = \omega_1$

$\ddot{y}_1 + y_1 = -\frac{1}{4}A^3\cos 3\tau$

$y_1(\tau) = c_1\cos t + c_2\sin\tau + A^3/32\ \cos 3\tau$

$y_1(0) = c_1 + A^3/32 = 0$

$\dot{y}_1(0) = c_2 = 0$

$y_1(\tau) = A^3/32\,(\cos 3t - \cos t)$

Solve $O(\varepsilon^2)$: (just want ω_2)

$f_2 = +2\omega_1 A^3/32\,(9\cos 3\tau - \cos\tau)$

$+ (\omega_1^2 + 2\omega_2)A\cos\tau$

$- 3A^2\cos^2\tau\, A^3/32\,(\cos 3\tau - \cos\tau)$

$$\left\{\begin{aligned} &\cos^2\tau(\cos 3\tau - \cos\tau) \\ &= \left(\frac{e^{i\tau}+e^{-i\tau}}{2}\right)^2\left[\frac{e^{3i\tau}+e^{-3i\tau}}{2} - \frac{e^{i\tau}+e^{-i\tau}}{2}\right] \\ &= \frac{1}{8}\left[(e^{2i\tau}+e^{-2i\tau}+2)(e^{3i\tau}+e^{-3i\tau}-e^{i\tau}-e^{-i\tau})\right] \\ &= \frac{1}{8}\big[\underbrace{-e^{i\tau}+e^{i\tau}-2e^{i\tau}}_{\downarrow} + c.c. + \underbrace{NST}_{\text{non-secular terms}}\big] \end{aligned}\right.$$

$-\frac{3}{32}A^5\cos\tau\left(-\frac{1}{2}\right) - 2\omega_1 A^3/32\cos\tau$

$+ A(\omega_1^2 + 2\omega_2)\cos t \quad + NST$

$\cos\tau\left[\frac{3}{64}A^5 - \frac{3}{128}A^5 + \frac{9}{64}A^5 + 2A\omega_2\right]$

$\to$ let $\omega_2 = -\frac{21}{256}A^4$

$\omega \sim 1 + \frac{3}{8}A^2\varepsilon - \frac{21}{256}A^4\varepsilon^2$

$y(\tau) \sim A\cos\tau + (A^3/32)\varepsilon(\cos 3\tau - \cos\tau)$

$x(t) = y(\omega t) =$

$= A\cos\left[\left(1+\frac{3}{8}A^2\varepsilon - \frac{21}{256}A^4\varepsilon^2\right)t\right]$

$+ \frac{A^3}{32}\varepsilon\left(\cos\left[\left(1+\frac{3}{8}A^2\varepsilon-\frac{21}{256}A^4\varepsilon^2\right)3t\right]\right.$

$\left. - \cos\left[\left(1+\frac{3}{8}A^2\varepsilon - \frac{21}{256}A^4\varepsilon^2\right)t\right]\right)$

$\frac{d^2x}{dt^2} + x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \frac{dx}{dt}\Big|_{t=0} = 0$ 12-4

$x(t) \sim A\cos(\omega t) + \frac{1}{32}A^3\varepsilon[\cos 3\omega t - \cos\omega t]$

$\omega \sim 1 + \frac{3}{8}A^2\varepsilon - \frac{21}{256}A^4\varepsilon^2$

used $\tau = \omega t$ (strained variable)
to avoid terms like $\varepsilon t \sin t$ (secular)

Accuracy?
we went to $O(\varepsilon^2)$ in ω, but only $O(\varepsilon)$ in solution.
which is better?

① $x(t) \sim A\cos t$

② $x(t) \sim A\cos t + \frac{1}{32}A^3\varepsilon[\cos 3t - \cos t]$

③ $x(t) \sim A\cos(1 + \frac{3}{8}A^2\varepsilon)t$

if we use ① for $t = O(\frac{1}{\varepsilon})$, the argument neglects an $O(1)$ quantity → no good

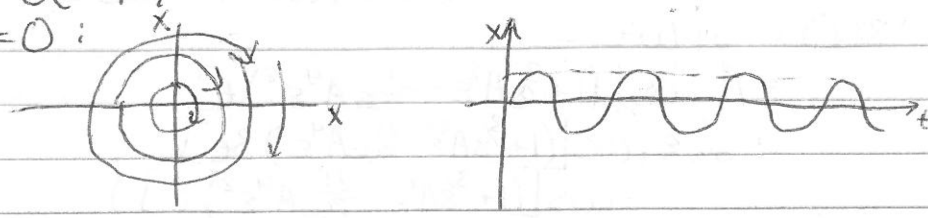
for ②, when $t = O(1)$ it's not better than ① b/c 2nd term $= O(\varepsilon)$ & we neglect $O(\varepsilon)$ from argument of cosine

for ③ if $t = O(1)$ we're fine. if $t = O(\frac{1}{\varepsilon})$, still good. Problem comes when $t = O(\frac{1}{\varepsilon^2})$.

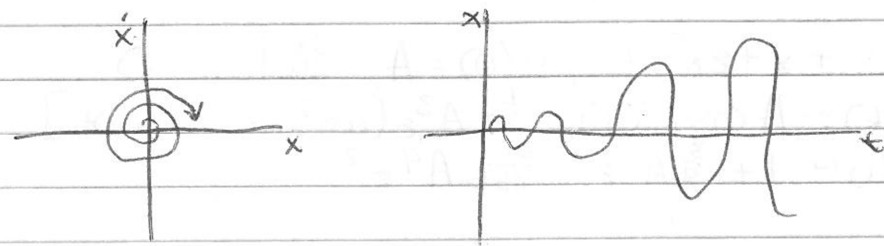
Rayleigh Oscillator

$\ddot{x} + \varepsilon(-1 + \frac{1}{3}\dot{x}^2)\dot{x} + x = 0$

$\varepsilon = 0$:



$\dot{x}$ small, $\varepsilon \neq 0$:

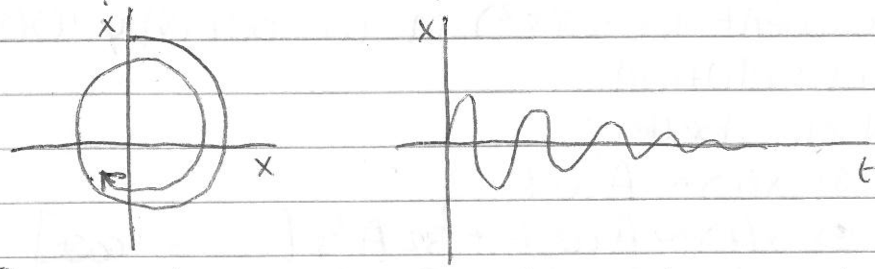


Think of it like a pendulum:

$\ddot{x} + \delta\dot{x} + x = 0$, $\delta = \varepsilon(-1 + \frac{1}{3}\dot{x}^2)$

if $\delta < 0$, get prev. case

if $\delta > 0 \rightarrow$ damped



So maybe it approaches a limit cycle

Poincare-Bendixon Theory:

$\dot{x} = f(x,y)$

$\dot{y} = g(x,y)$

If there's a closed region where the vector field points inward along the boundary and there are no stable critical pts inside the region, then there must be a limit cycle inside the region

Use this to find a solution.

$x(0)=A(\varepsilon), \dot{x}(0)=0$

let $\tau=\omega t$, $x(t)=y(\tau)$

$y(\tau)$ should be 2π-periodic

$\omega^2\ddot{y} - \varepsilon\omega\dot{y} + \frac{1}{3}\varepsilon\omega^3\dot{y}^3 + y = 0$

$y(0)=A$, $\omega\dot{y}(0)=0 \rightarrow \dot{y}(0)=0$

$y(\tau) \sim y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau)$

$\omega(\varepsilon) \sim 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2$

$A(\varepsilon) \sim A_0 + \varepsilon A_1 + \varepsilon^2 A_2$

$\rightarrow [1+2\varepsilon\omega_1 + \varepsilon^2(\omega_1^2+2\omega_2)][\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2]$

$- \varepsilon(1+\varepsilon\omega_1)(\dot{y}_0+\varepsilon\dot{y}_1) + \frac{1}{3}\varepsilon(1+3\varepsilon\omega_1)(\dot{y}_0^3+3\dot{y}_0^2\varepsilon\dot{y}_1)$

$+ y_0 + \varepsilon y_1 + \varepsilon_2 y_2 + O(\varepsilon^3) = 0$

& $y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2$

& $\dot{y}_0(0) + \varepsilon\dot{y}_1(0) + \varepsilon^2\dot{y}_2(0) = 0$

$O(1)$: $\ddot{y}_0 + y_0 = 0$, $y_0(0)=A_0$, $\dot{y}_0(0)=0$

$y_0(\tau) = A_0\cos\tau$

$O(\varepsilon)$: $\ddot{y}_1 + y_1 = f_1$, $y_1(0)=A_1$, $\dot{y}_1(0)=0$

$f_1 = -2\omega_1\ddot{y}_0 + \dot{y}_0 - \frac{1}{3}\dot{y}_0^3$

$= 2\omega_1 A_0\cos\tau - A_0\sin\tau + \frac{1}{3}A_0^3\sin^3\tau$

$[\sin^3 t = \frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau]$

$= 2\omega_1 A_0\cos\tau + \sin\tau(-A_0 + \frac{1}{4}A_0^3) - \frac{1}{12}A_0^3\sin 3\tau$

$\rightarrow -1 + \frac{1}{4}A_0^2 = 0 \rightarrow A_0^2 = 4 \rightarrow A_0 = \pm 2$

$\rightarrow 2\omega_1 A_0 = 0 \rightarrow \omega_1 = 0$ $\quad$ ← choose +

$f_1 = -\frac{2}{3}\sin 3\tau$

$y_1(\tau) = c_1\cos\tau + c_2\sin\tau + \frac{1}{12}\sin 3\tau$

$y_1(0) = c_1 = A_1$

$\dot{y}_1(0) = c_2 + \frac{1}{4} = 0 \rightarrow c_2 = -\frac{1}{4}$

$y_1(\tau) = A_1\cos\tau - \frac{1}{4}\sin\tau + \frac{1}{12}\sin 3t$

$O(\varepsilon^2)$: $\ddot{y}_2 + y_2 = f_2$; $y_2(0) = A_2$, $\dot{y}_2(0) = 0$

$$f_2 = -2\omega_2 \ddot{y}_0 + \dot{y}_1 - y_0^2 \dot{y}_1$$

$$= (4\omega_2 + \tfrac{1}{4})\cos\tau + 2A_1 \sin\tau - \tfrac{1}{2}\cos 3\tau$$

$$- A_1 \sin 3\tau + \tfrac{1}{4}\cos 5\tau$$

$$\rightarrow A_1 = 0, \quad \omega_2 = -\tfrac{1}{16}$$

420-2

The Mathieu Equation

$\ddot{u} + (\delta + \varepsilon \cos 2t)\, u = 0$, $u = u(t)$, $|\varepsilon| \ll 1$

like a pendulum whose length oscillates

parametric resonance

has unbounded or bounded solutions depending on the values of ε & δ.

The boundary in the ε-δ plane between bdd + unbdd are lines where the solution has period 2π.

looking for curves $\delta = \delta(\varepsilon)$ along which there are periodic sol'ns.

$u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2$

$\delta \sim \delta_0 + \varepsilon \delta_1 + \varepsilon^2 \delta_2$

$\rightarrow \ddot{u}_0 + \varepsilon \ddot{u}_1 + \varepsilon^2 \ddot{u}_2 + [\delta_0 + \varepsilon \delta_1 + \varepsilon^2 \delta^2 + \varepsilon \cos 2t] \cdot$
$\cdot (u_0 + \varepsilon u_1 + \varepsilon^2 u_2) + O(\varepsilon^3) = 0$

$O(1)$: $\ddot{u}_0 + \delta_0 u_0 = 0$

$O(\varepsilon)$: $\ddot{u}_1 + \delta_0 u_1 = -(\delta_1 + \cos 2t) u_0$

$O(\varepsilon^2)$: $\ddot{u}_2 + \delta_0 u_2 = -(\delta_1 + \cos 2t) u_1 - \delta_2 u_0$

need $u_1(t), u_2(t), \cdots$ to have 2π as a period

$O(1)$: period of u_0: $2\pi/\sqrt{\delta_0}$ (need $\delta_0 > 0$)

want $(2\pi/\sqrt{\delta_0})\, n = 2\pi \rightarrow \sqrt{\delta_0} = n \rightarrow \delta_0 = n^2$, $n = 0, 1, 2, \cdots$

case $n=0$:

$O(1)$: $\ddot{u}_0 = 0 \rightarrow u_0(t) = \text{const} = 1$

$O(\varepsilon)$: $\ddot{u}_1 = -(\delta_1 + \cos 2t)$

$u_1(t) = \frac{1}{4} \cos 2t + c_1$

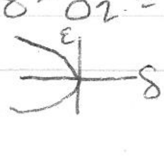
need $\delta_1 = 0$ to get periodic

$O(\varepsilon^2)$: $\ddot{u}_2 = -\frac{1}{4} \cos^2 2t - c_1 \cos 2t - \delta_2$

$= -\frac{1}{8} - \frac{1}{8} \cos 4t - c_1 \cos 2t - \delta_2$

need $-\frac{1}{8} - \delta_2 = 0 \rightarrow \delta_2 = -\frac{1}{8}$

$\rightarrow \delta \sim -\varepsilon^2/8$



case $n=1 \to \delta_0 = 1$

$O(1)$: $\ddot{u}_0 + u_0 = 0 \to u_0(t) = c_1 \cos t + c_2 \sin t$

$O(\varepsilon)$: $\ddot{u}_1 + u_1 = -(\delta_1 + \cos 2t)(c_1 \cos t + c_2 \sin t)$

$= -\delta_1 c_1 \cos t - \delta_1 c_2 \sin t$
$-(c_1/2)\cos 3t - (c_1/2)\cos t$
$-(c_2/2)\sin 3t + (c_2/2)\sin t$

$= -(\delta_1 + \tfrac{1}{2}) c_1 \cos t - (\delta_1 - \tfrac{1}{2}) c_2 \sin t - {}^{c_1}/_2 (\cos 3t + \sin 3t)$

case A:

$\delta_1 = \tfrac{1}{2},\ c_1 = 0$

case B:

$\delta_1 = -\tfrac{1}{2},\ c_2 = 0$

case A:

$u_0(t) = c_2 \sin t$

$\ddot{u}_1 + u_1 = -{}^{c_2}/_2 \sin 3t$

$u_1(t) = {}^{c_2}/_{16} \sin 3t + c_3 \cos t + c_4 \sin t$

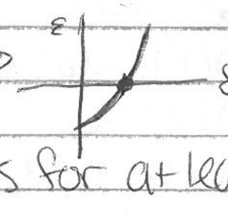
$O(\varepsilon^2)$: $\ddot{u}_2 + u_2 = -(\tfrac{1}{2} + \cos 2t)(\tfrac{c_2}{16} \sin 3t + c_3 \cos t + c_4 \sin t)$
$- \delta_2 c_2 \sin t$

need to expand products & get coefficients of $\cos t$ & $\sin t$ to be zero

$= -\tfrac{1}{2} c_3 \cos t - \cancel{\tfrac{1}{2} c_4 \sin t} - \delta_2 c_2 \sin t$
$- {}^{c_2}/_{32} \sin t - {}^{c_3}/_2 \cos t + \cancel{{}^{c_4}/_2 \sin t}$
$+$ terms with other arguments

$= -c_3 \cos t - c_2 \sin t (\delta_2 + \tfrac{1}{32}) + \cdots$

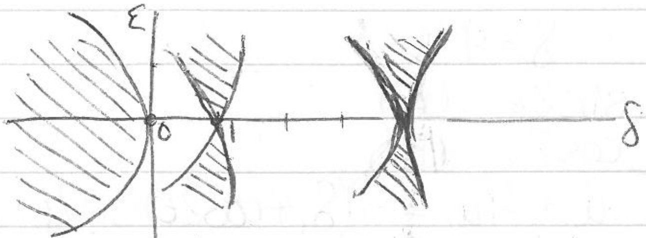
need $\delta_2 = -\tfrac{1}{32},\ c_3 = 0$

$\delta = 1 + \tfrac{1}{2}\varepsilon - \tfrac{1}{32}\varepsilon^2 \to$ 

u will have only sin terms for at least $O(1), O(\varepsilon), O(\varepsilon^2)$.

case B: (very similar)

$\delta \sim 1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{32}\varepsilon^2 \longrightarrow$ 



shaded regions have unbounded sol'ns.

Are there any functions that will work (2π-periodic) but are neither odd nor even?

Let $u(t)$ be a 2π-periodic function

so $u(-t)$ is also

so $u(t) + u(-t)$ is also

but $u(t)+u(-t)$ is even

and $u(t) - u(-t)$ is also; it is odd

So it is sufficient to look for only even and odd solutions

Simplify: want to normalize so we don't have to carry around arbitrary constants.

Suppose we impose the following condition for $n=1$, case A:

$$\int_0^{2\pi} u(t)\sin t\, dt = \pi$$

$$u_{1n} = c_4 \sin t$$

$$\int_0^{2\pi} (c_2 \sin t + \varepsilon c_4 \sin t + \varepsilon^2 c_6 \sin t + \cdots)\sin t\, dt = \pi$$

$$c_2\pi + \varepsilon c_4 \pi + \varepsilon^2 c_6 \pi + \cdots = \pi$$

$$\rightarrow c_2 = 1,\ c_4 = c_6 = \cdots = 0$$

case B: $\int_0^{2\pi} u(t)\cos t\, dt = \pi$

$n=2$: $\int_0^{2\pi} u(t)\sin 2t\, dt = \pi$, $\cos 2t$.

etc.

Case $n=2$: $\delta_0 = 4$

$u_0(t) = \begin{cases} \sin 2t & (A) \\ \cos 2t & (B) \end{cases}$

A: $O(\varepsilon)$: $\ddot{u}_1 + 4u_1 = -(\delta_1 + \cos 2t)\sin 2t$

$= -\delta_1 \sin 2t - \frac{1}{2}\sin 4t$

$u_1 = \frac{1}{24}\sin 4t$

need $\delta_1 = 0$

$u_h = 0$ due to looking only at odd functions and orthogonality condition.

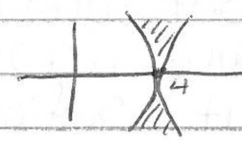
$O(\varepsilon^2)$: $\ddot{u}_2 + 4u_2 = -\frac{1}{24}\cos 2t \sin 4t - \delta_2 \sin 2t$

$= -\frac{1}{48}\sin 6t - \frac{1}{48}\sin 2t - \delta_2 \sin 2t$

need $\delta_2 = -\frac{1}{48}$

$\rightarrow \delta \sim 4 - \frac{1}{48}\varepsilon^2$

B: $\delta \sim 4 + \frac{5}{48}\varepsilon^2$



1-15 Floquet Theorey

$\frac{d^2u}{dt^2} + Q(t)u = 0$ Where $Q(t+T) = Q(t)$ for any t

Constant coeff: $u'' + u = 0$

$u_1(t) = \cos t$, $u_2(t) = \sin t$

$u_1(t+2\pi) = u_1(t)$, $u_2(t+2\pi) = u_2(t)$

But: $u'' - 2u' + 2u = 0$

$u_1 = e^t \cos t$, $u_2 = e^t \sin t$

$\rightarrow$ not periodic

$u_1(t+2\pi) = e^{t+2\pi}\cos^{t+2\pi} = e^{2\pi} e^t \cos t = e^{2\pi} u_1(t)$

$u(t+2\pi) = e^{2\pi} u(t) = \rho u(t)$

if $\rho = 1 \rightarrow$ periodic

if $\rho = -1 \rightarrow$ periodic w/ period $2T$

if $|\rho| < 1 \rightarrow$ not periodic, but decays to zero

if $|\rho| > 1 \rightarrow$ solution $\rightarrow \infty$

① $u(t) = c_1 u_1(t) + c_2 u_2(t)$ assume c_1 & c_2 not both $=0$

$$\frac{d^2 u_j}{dt^2} + Q(t)u_j = 0 \qquad j=1,2$$

$u_1(0)=1,\ u_1'(0)=0$

$u_2(0)=0,\ u_2'(0)=1$

$$W(u_1, u_2) = \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix} \quad \left(\text{doesn't depend on } t\right)$$

$$= \begin{vmatrix} u_1(0) & u_2(0) \\ u_1'(0) & u_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\rightarrow$ linearly independent.

② $u(t+T) = c_1 u_1(t+T) + c_2 u_2(t+T)$

note if $u(t)$ is a sol'n, then so is $u(t+T)$ since $Q(t+T) = Q(t)$.

$\rightarrow u_1(t+T)$ & $u_2(t+T)$ are also sol'ns.

But all sol'ns are linear combinations of $u_1(t)$ & $u_2(t)$.

$u_1(t+T) = \alpha u_1(t) + \beta u_2(t)$

$u_1(T) = \alpha u_1(0) + \beta u_2(0)$

$\rightarrow u_1(T) = \alpha$

$u_1'(T) = \alpha u_1'(0) + \beta u_2'(0)$

$\rightarrow u_1'(T) = \beta$

So $u_1(t+T) = u_1(T)u_1(t) + u_1'(T)u_2(t)$

Similarly, $u_2(t+T) = u_2(T)u_1(t) + u_2'(T)u_2(t)$

$$u(t+T) = c_1\left[u_1(T)u_1(t) + u_1'(T)u_2(t)\right] + c_2\left[u_2(T)u_1(t) + u_2'(T)u_2(t)\right]$$

$$= \left[c_1 u_1(T) + c_2 u_2(T)\right]u_1(t) + \left[c_1 u_1'(T) + c_2 u_2'(T)\right]u_2(t)$$

③ Plug into $u(t+T) = \rho u(t)$

$$\left[c_1 u_1(T) + c_2 u_2(T)\right]u_1(t) + \left[c_1 u_1'(T) + c_2 u_2'(T)\right]u_2(t) = \rho c_1 u_1(t) + \rho c_2 u_2(t)$$

$$\begin{cases} c_1 u_1(T) + c_2 u_2(T) - \rho c_1 = 0 \\ c_1 u_1'(T) + c_2 u_2'(T) - \rho c_2 = 0 \end{cases}$$

need det = 0

$$\begin{vmatrix} u_1(T)-\rho & u_2(T) \\ u_1'(T) & u_2'(T)-\rho \end{vmatrix}$$

$$\rho^2-\rho(u_1(T)+u_2'(T))+u_1(T)u_2'(T)$$
$$-u_2(T)u_1'(T)=0$$

Since $W=1$

$$\rho^2-\rho(u_1(T)+u_2'(T))+1=0$$
$$\rho^2-2k\rho+1=0,\quad 2k=u_1(T)+u_2'(T)$$
$$\rho=k\pm\sqrt{k^2-1}$$

Case A: $|k|<1$

$$\rightarrow \rho_{1,2}=k\pm i\sqrt{1-k^2}$$
$$|\rho_{1,2}|=\sqrt{k^2+1-k^2}=1$$
$$\rho_1=e^{i\alpha T},\quad \rho_2=e^{-i\alpha T}$$
where $\alpha\in\mathbb{R}$

Case B: $|k|>1$

$$\rightarrow \rho_{1,2}\in\mathbb{R}$$
$$\rho_1=e^{i\alpha T},\quad \rho_2=e^{-i\alpha T}$$
where α is imaginary

Case C: $|k|=1$

- $k=1$: $\rho_{1,2}=1$ $(\alpha=0)$
- $k=-1$: $\rho_{1,2}=-1$ $(\alpha T=\pi\rightarrow\alpha=\frac{\pi}{T})$
- $\rightarrow k=1$: one solution that is T-periodic
- $\rightarrow k=-1$: one solution that is $2T$-periodic

④ $|k|\neq 1$:

$$\begin{cases} u^{(1)}(t+T)=e^{i\alpha T}u^{(1)}(t) \\ u^{(2)}(t+T)=e^{-i\alpha T}u^{(2)}(t) \end{cases}$$

multiply by $e^{-i\alpha(t+T)}$ & $e^{i\alpha(t+T)}$, respectively:

$$\begin{cases} e^{-i\alpha(t+T)}u^{(1)}(t+T)=e^{-i\alpha t}u^{(1)}(t) \\ e^{i\alpha(t+T)}u^{(1)}(t+T)=e^{i\alpha t}u^{(2)}(t) \end{cases}$$

let $\begin{cases} U^{(1)}(t)=e^{-i\alpha t}u^{(1)}(t) \\ U^{(2)}(t)=e^{i\alpha t}u^{(2)}(t) \end{cases}$

$\rightarrow U^{(1)}(t)$ is T-periodic

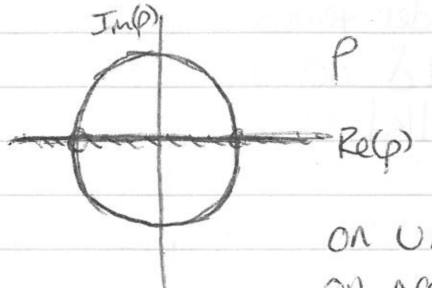
$\rightarrow \begin{cases} u^{(1)}(t) = e^{i\alpha t} U^{(1)}(t) \\ u^{(2)}(t) = e^{-i\alpha t} U^{(2)}(t) \end{cases}$

$u(t) = a_1 u^{(1)}(t) + a_2 u^{(2)}(t)$ ← general sol'n

So for Case A: exp. is periodic

$\rightarrow$ sol'n is bounded (quasi-periodic)

Case B: one sol'n decays, the other blows up.



ρ either on unit circle or real axis

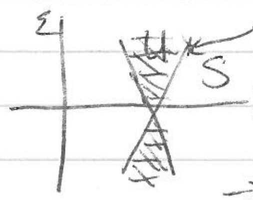
on unit circle: bdd sol'ns

on real axis, $\neq \pm 1$: unbdd

Mathieu eqn: $\ddot{u} + (\delta + \varepsilon \cos 2t) u = 0$

$\rho = \pm 1$ is boundary between bdd + unbdd solutions

$\rightarrow$ where there is a T or 2T-periodic sol'n



← boundaries are where period $= \pi$ or 2π

$\rightarrow$ agrees w/ prev. result.

finding k:

$k = \frac{1}{2}(u_1(T) + u_2'(T))$

$T = \pi$ for this eqn

$u_1(0) = 1$, $\dot{u}_1(0) = 0$

solve for $\delta = 1$, $\varepsilon > 0$

$u \sim u_0 + \varepsilon u_1 + \cdots$

$\ddot{u}_0 + u_0 = 0$, $u_0(0) = 1$, $\dot{u}_0(0) = 0$

$u_0 = \cos t$, $u_0(\pi) = -1$

+ for 2^{nd} sol'n $\rightarrow u_0'(\pi) = -1 \rightarrow k = -1$

need higher orders

$\ddot{u}_1 + u_1 = -\cos 2t \cos t \quad , \ u_1(0) = 0, \ \dot{u}_1(0) = 0$

$= -\frac{1}{2}\cos 3t - \frac{1}{2}\cos t$

$u_1(t) = c_1 \cos t + c_2 \sin t + \frac{1}{16}\cos 3t - \frac{1}{4} t \sin t$

$u_1(0) = c_1 + \frac{1}{16} = 0 \ \rightarrow c_1 = -\frac{1}{16}$

$\dot{u}_1(0) = c_2 = 0$

$u_1(t) = -\frac{1}{16}\cos t + \frac{1}{16}\cos 3t - \frac{1}{4} t \sin t$

$u \sim \cos t + \varepsilon\left(\frac{1}{16}(\cos 3t - \cos t) - \frac{1}{4} t \sin t\right)$

need higher order terms...

$O(\varepsilon^2)$ should do it.

should get $|k| > 1$

1-20 $\ddot{u} + (\delta + \varepsilon \cos 2t) u = 0$

look at solutions along $\delta = 1 + \delta_1 \varepsilon$, where $|\delta_1| < \frac{1}{2}$

$\rightarrow$ inside unstable region

$|\delta_1| > \frac{1}{2} \rightarrow$ outside unstable region

$u(t) = e^{i\alpha t} U(t)$

U is π-periodic

$\alpha = 1 + \beta, \quad |\beta| \ll 1$

$u(t) = e^{i\beta t}\Big(\underbrace{\underbrace{e^{it}}_{2\pi}\ \underbrace{U(t)}_{\pi}}_{2\pi\text{-periodic}}\Big)$

$= e^{i\beta t}\varphi(t)$, where β is small & φ is 2π-periodic

$\ddot{u} = [-\beta^2 \varphi + 2i\beta\dot{\varphi} + \ddot{\varphi}]e^{i\beta t}$

$\ddot{\varphi} + 2i\beta\dot{\varphi} - \beta^2\varphi + (\delta + \varepsilon\cos 2t)\varphi = 0$

$\delta = 1 + \varepsilon\delta_1$

$\beta \sim \varepsilon\beta_1$

$\varphi \sim \varphi_0 + \varepsilon\varphi_1 \quad , \ \varphi_0 \ \& \ \varphi_1 \ 2\pi\text{-periodic}$

$O(1): \ \ddot{\varphi}_0 + \varphi_0 = 0 \quad , \ \varphi_0(t) = c_1 \cos t + c_2 \sin t$

$O(\varepsilon): \ \ddot{\varphi}_1 + \varphi_1 = -2i\beta_1\dot{\varphi}_0 - (\delta_1 + \cos 2t)\varphi_0$

$= -2i\beta_1(-c_1 \sin t + c_2 \cos t) - \delta_1(c_1 \cos t + c_2 \sin t)$

$- c_1/2 \cos t + c_2/2 \sin t +$ NST

$$= \cos t\left[-2i\beta_1 c_2 - \delta_1 c_1 - \frac{c_1}{2}\right]$$
$$+ \sin t\left[2i\beta_1 c_1 - \delta_1 c_2 + c_2/2\right] + \text{NST}$$

$\rightarrow (\delta_1 + \frac{1}{2})c_1 + 2i\beta_1 c_2 = 0$

$2i\beta_1 c_1 + (\frac{1}{2} - \delta_1)c_2 = 0$

don't want $c_1 = c_2 = 0$

det $= 0 \rightarrow \frac{1}{4} - \delta_1^2 + 4\beta_1^2 = 0$

$\rightarrow \beta_1 = \pm \frac{1}{2}\sqrt{\delta_1^2 - \frac{1}{4}}$

confirms prev. result:

$|\delta_1| < \frac{1}{2} \rightarrow$ unstable

$|\delta_1| > \frac{1}{2} \rightarrow$ stable

$\delta_1^2 < \frac{1}{4}$: $\beta_1 = \pm\frac{1}{2} i\sqrt{\frac{1}{4} - \delta_1^2}$

$\beta_1 = +\frac{1}{2} i\sqrt{\frac{1}{4} - \delta_1^2}$

$$c_1 = \frac{-2i\beta_1}{\delta + \frac{1}{2}} c_2 = \frac{\sqrt{\frac{1}{4} - \delta_1^2}}{\delta_1 + \frac{1}{2}} c_2$$

$$= \frac{\sqrt{(\frac{1}{2} - \delta_1)(\frac{1}{2} + \delta_1)}}{(\delta_1 + \frac{1}{2})} c_2 = \frac{\sqrt{\frac{1}{2} - \delta_1}}{\sqrt{\frac{1}{2} + \delta_1}} c_2$$

$$u(t) \sim e^{i\beta t}\varphi_0(t) = e^{-\varepsilon\frac{1}{2}\sqrt{\frac{1}{4} - \delta_1^2}\, t}\left[\frac{\sqrt{\frac{1}{2} - \delta_1}}{\sqrt{\frac{1}{2} + \delta_1}} c_2 \cos t + c_2 \sin t\right]$$

$$= a e^{-\frac{1}{2}\varepsilon t\sqrt{\frac{1}{4} - \delta_1^2}}\left(\sqrt{\tfrac{1}{2} - \delta_1}\cos t + \sqrt{\tfrac{1}{2} + \delta_1}\sin t\right)$$

$u(t) \sim b e^{\frac{1}{2}\varepsilon\sqrt{\frac{1}{4} - \delta_1^2}}\left(\sqrt{\tfrac{1}{2} - \delta_1}\cos t - \sqrt{\tfrac{1}{2} + \delta_1}\sin t\right) \leftarrow$ from $-\beta$

Multiple Scales

$\ddot{x}+2\varepsilon\dot{x}+x=0$, $x(0)=0$, $\dot{x}(0)=1$

Regular perturbations:

$x(t)\sim x_0(t)+\varepsilon x_1(t)$

$O(1)$: $\ddot{x}_0+x_0=0$, $x_0(0)=0$, $\dot{x}_0(0)=1$

$\rightarrow x_0(t)=\sin t$

$O(\varepsilon)$: $\ddot{x}_1+x_1=-2\dot{x}_0=-2\cos t$

$x_1(0)=0$, $\dot{x}_1(0)=0$

$x_1(t)=c_1\cos t+c_2\sin t-t\sin t$

$x_1(0)=c_1=0$

$\dot{x}_1(0)=c_2=0$

$x_1(t)=-t\sin t$

$x(t)\sim\sin t-\varepsilon t\sin t$

not good (expected damping, ε-term should be small for all time)

Exact Solution:

$m^2+2\varepsilon m+1=0 \rightarrow m=-\varepsilon\pm i\sqrt{1-\varepsilon^2}$

$x(t)=e^{-\varepsilon t}(c_1\cos t\sqrt{1-\varepsilon^2}+c_2\sin t\sqrt{1-\varepsilon^2})$

$x(0)=0\rightarrow x(t)=c_1e^{-\varepsilon t}\sin(t\sqrt{1-\varepsilon^2})$

$\dot{x}(0)=1\rightarrow c_1=1/\sqrt{1-\varepsilon^2}$

$x(t)=(1-\varepsilon^2)^{-\frac{1}{2}}e^{-\varepsilon t}\sin(t\sqrt{1-\varepsilon^2})$

So the expansion was correct, but not what we wanted (can't expand exponential for times $O(\frac{1}{\varepsilon})$).

Use two timescales:

$\tau=\omega t$, ω is unknown function of ε → takes care of oscillations

$t_1=\varepsilon t$ → takes care of exponent

$x(t)=y(\tau,t_1)$

Treat the times τ & t_1 as if they were completely independent

$$\frac{dx}{dt} = \frac{\partial y}{\partial \tau}\cdot\omega + \frac{\partial y}{\partial t_1}\cdot\varepsilon$$

$$\frac{d^2x}{dt^2} = \omega^2\frac{\partial^2 y}{\partial\tau^2} + 2\omega\varepsilon\frac{\partial^2 y}{\partial\tau\partial t_1} + \varepsilon^2\frac{\partial^2 y}{\partial t_1^2}$$

Plug in:

$$\omega^2\frac{\partial^2 y}{\partial\tau^2} + 2\varepsilon\omega\frac{\partial^2 y}{\partial\tau\partial t_1} + \varepsilon^2\frac{\partial^2 y}{\partial t_1^2} + 2\varepsilon\omega\frac{\partial y}{\partial\tau} + 2\varepsilon^2\frac{\partial y}{\partial t_1} + y = 0$$

$$y(0)=0,\quad \omega\frac{\partial y}{\partial\tau}(0,0) + \varepsilon\frac{\partial y}{\partial t_1}(0,0) = 1$$

$$y \sim y_0(\tau,t_1) + \varepsilon y_1(\tau,t_1) + \varepsilon^2 y_2(\tau,t_1)$$

$$\omega \sim 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2$$

$O(1)$: $\omega^2 \sim 1 + 2\varepsilon\omega_1 + \varepsilon^2(\omega_1^2 + 2\omega_2)$

$$\rightarrow \frac{\partial^2 y_0}{\partial\tau^2} + y_0 = 0,\quad y_0(0,0)=0,\quad \frac{\partial y}{\partial\tau}(0,0)=1$$

$$y_0(\tau,t_1) = A_0(t_1)\cos\tau + B_0(t_1)\sin\tau$$

$$y_0(0,0) = A_0(0) = 0$$

$$y_{0\tau}(0,0) = B_0(0) = 1$$

$O(\varepsilon)$: $$\frac{\partial^2 y_1}{\partial\tau^2} + y_1 = -2\omega_1\frac{\partial^2 y_0}{\partial\tau^2} - 2\frac{\partial^2 y_0}{\partial\tau\partial t_1} - 2\frac{\partial y_0}{\partial\tau}$$

$$= +2\omega_1(A_0\cos\tau + B_0\sin\tau)$$
$$-2(-A_0'\sin\tau + B_0'\cos\tau)$$
$$-2(-A_0\sin\tau + B_0\cos\tau)$$
$$= 2\cos\tau[\omega_1 A_0 - B_0' - B_0]$$
$$+2\sin\tau[\omega_1 B_0 + A_0' + A_0]$$

$$\rightarrow \begin{cases} A_0' = -A_0 - \omega_1 B_0 \\ B_0' = \omega_1 A_0 - B_0 \end{cases}$$

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}' = \begin{pmatrix} -1 & -\omega_1 \\ \omega_1 & -1 \end{pmatrix}\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & -\omega_1 \\ \omega_1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 2\lambda + 1 + \omega_1^2 = 0 \qquad \lambda = \left[-2 \pm \sqrt{4 - 4(1+\omega_1^2)}\right]\tfrac{1}{2}$$

$$\lambda = -1 \pm i\omega_1$$

$\lambda = -1 - i\omega_1$:

$$\begin{pmatrix} i\omega_1 & -\omega_1 \\ \omega_1 & i\omega_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$i\omega_1 a - \omega_1 b = 0$$

$$\rightarrow a = 1,\ b = i \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rightarrow e^{\lambda t_1}\begin{pmatrix} a \\ b \end{pmatrix} = e^{(-1-i\omega_1)t_1}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= e^{-t_1}(\cos\omega_1 t_1 - i\sin\omega_1 t_1)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

Real part:

$$e^{-t_1}\begin{pmatrix} \cos\omega_1 t_1 \\ \sin\omega_1 t_1 \end{pmatrix}$$

Imag. part:

$$e^{-t_1}\begin{pmatrix} -\sin\omega_1 t_1 \\ \cos\omega_1 t_1 \end{pmatrix}$$

$$\rightarrow A_0 = e^{-t_1}(c_1\cos\omega_1 t_1 - c_2\sin\omega_1 t_1)$$

$$B_0 = e^{-t_1}(c_1\sin\omega_1 t_1 + c_2\cos\omega_1 t_1)$$

From before: $A_0(0) = 0$, $B_0(0) = 1$

$$A_0(0) = c_1 = 0$$

$$A_0(0) = c_2 = 1$$

$$\rightarrow A_0 = -e^{-t_1}\sin\omega_1 t_1$$

$$B_0 = e^{-t_1}\cos\omega_1 t_1$$

$$\rightarrow y_0(\tau, t_1) = e^{-t_1}\sin(\tau - \omega_1 t_1)$$

from trig identity

$$\rightarrow e^{-\varepsilon t}\sin(t + \varepsilon\omega_1 t - \omega_1 \varepsilon t)$$

$$= e^{-\varepsilon t}\sin(t + O(\varepsilon^2))$$

which agrees with prev. result

$\ddot{x}+2\varepsilon\dot{x}+x=0, \quad x(0)=0, \; \dot{x}(0)=1$ 1-22

$x(t)=y(\tau,t_1)$ where $\tau=\omega t, \; t_1=\varepsilon t$

$$\omega^2\frac{\partial^2 y}{\partial\tau^2}+2\omega\varepsilon\frac{\partial^2 y}{\partial\tau\partial t_1}+\varepsilon^2\frac{\partial^2 y}{\partial t_1^2}+2\varepsilon\left[\omega\frac{\partial y}{\partial\tau}+\frac{\varepsilon\partial y}{\partial t_1}\right]+y=0$$

$$y(0,0)=0, \quad \left[\omega\frac{\partial y}{\partial\tau}+\varepsilon\frac{\partial y}{\partial t_1}\right]_{(0,0)}=1$$

$y\sim y_0+\varepsilon y_1+\varepsilon^2 y_2, \quad \omega\sim 1+\varepsilon\omega_1+\varepsilon^2\omega_2$

$O(1)$: $\frac{\partial^2 y_0}{\partial\tau^2}+y_0=0, \quad y_0(0,0)=0, \; \frac{\partial y_0}{\partial\tau}(0,0)=1$

$\rightarrow y_0(\tau,t_1)=A_0(t_1)\cos\tau+B_0(t_1)\sin\tau$

IC's $\rightarrow A_0(0)=0, \; B_0(0)=1$

$O(\varepsilon)$: $\frac{\partial^2 y_1}{\partial\tau^2}+y_1=-2\omega_1\frac{\partial^2 y_0}{\partial\tau^2}-2\frac{\partial^2 y_0}{\partial\tau\partial t_1}-2\frac{\partial y_0}{\partial\tau}$

$$y_1(0,0)=0, \quad \left.\frac{\partial y_1}{\partial\tau}+\omega_1\frac{\partial y_0}{\partial\tau}+\frac{\partial y_0}{\partial t_1}\right|_{(0,0)}=0$$

$=2\omega_1(A_0\cos\tau+B_0\sin\tau)+2(A_0'\sin\tau-B_0'\cos\tau)$
$+2(A_0\sin\tau-B_0\cos\tau)$

$\rightarrow A_0'=-A_0-\omega_1 B_0$
$B_0'=2\omega_1 A_0-B_0, \quad A_0(0)=0, \; B_0(0)=1$

$\rightarrow A_0(t_1)=-e^{-t_1}\sin\omega_1 t_1$
$B_0(t_1)=e^{-t_1}\cos\omega_1 t_1$

$\rightarrow y_0(\tau,t_1)=e^{-t_1}\sin(\tau-\omega_1 t_1)$

$\rightarrow x(t)=y(\omega t,\varepsilon t)\sim e^{-\varepsilon t}\sin(\omega t-\omega_1\varepsilon t)$
$=e^{-\varepsilon t}\sin[(1+O(\varepsilon^2))t]$

Aside: $\cos(x+y)e^{x-y}$

$\hookrightarrow z=x-y \rightarrow x=z+y$
$\hookrightarrow \cos(z+2y)e^z$

can do this for any function

So instead of τ & t_1, we can use
$\tau - \omega_1 t_1$ & t_1
Or another way to think about it:
we alread have εt, don't need $\omega_1(\varepsilon t)$.

Solve $O(\varepsilon)$:

$$y_1(\tau, t_1) = A_1(t_1)\cos(\tau - \omega_1 t_1) + B_1(t_1)\sin(\tau - \omega_1 t_1)$$

$$y_1(0,0) = A_1(0) = 0$$

$$B_1(0) + \omega_1 - \omega_1 = B_1(0) = 0$$

$O(\varepsilon^2)$:

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = -2\omega_1 \frac{\partial^2 y_1}{\partial \tau^2} - (\omega_1^2 + 2\omega_2)\frac{\partial^2 y_0}{\partial \tau^2}$$

$$-2\omega_1 \frac{\partial_2 y_0}{\partial \tau \partial t_1} - 2\frac{\partial^2 y_1}{\partial \tau \partial t_1} - \frac{\partial^2 y_0}{\partial t_1^2}$$

$$-2\frac{\partial y_1}{\partial \tau} - 2\omega_1 \frac{\partial y_0}{\partial \tau} - 2\frac{\partial y_0}{\partial t_1}$$

$$= -\left[2\left(\frac{\partial}{\partial t_1} + \omega_1 \frac{\partial}{\partial \tau}\right) y_0 + \left(\frac{\partial}{\partial t_1} + \omega_1 \frac{\partial}{\partial \tau}\right)^2 y_0 \right.$$

$$\left. + 2\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial t_1} + \omega_1 \frac{\partial}{\partial \tau}\right) y_1 + 2\omega_2 \frac{\partial^2 y_0}{\partial \tau} + 2\frac{\partial y_1}{\partial \tau}\right]$$

And $\underbrace{\left(\frac{\partial}{\partial t_1} + \omega_1 \frac{\partial}{\partial \tau}\right)}_{} y_0 = \left(\frac{\partial}{\partial t_1} + \omega_1 \frac{\partial}{\partial \tau}\right) e^{-t_1}\sin(\tau - \omega_1 t_1)$

Notice $L \sin(\tau - \omega_1 t_1) = 0$
So you get $-y_0$
Then $L^2 y_0 = y_0$

$$L y_1 = A_1' \cos(\tau - \omega_1 t_1) + B_1' \sin(\tau - \omega_1 t_1)$$

$$\partial/\partial\tau (L y_1) = -A_1' \sin(\tau - \omega_1 t_1) + B_1' \cos(\tau - \omega_1 t_1)$$

$$\partial^2 y_0 / \partial \tau^2 = -y_0$$

$$\partial y_1 / \partial \tau = -A_1 \sin(\tau - \omega_1 t_1) + B_1 \cos(\tau - \omega_1 t_1)$$

$$\rightarrow \frac{\partial^2 y_2}{\partial \tau^2} + y_2 = -\left[\begin{array}{l} -2y_0 + y_0 - 2A_1'\sin(\tau - \omega_1 t_1) + 2B_1'\cos(\tau - \omega_1 t_1) \\ -2\omega_2 y_0 - 2A_1 \sin(\tau - \omega_1 t_1) + 2B_1 \cos(\tau - \omega_1 t_1) \end{array}\right.$$

$$= (-(1 + 2\omega_2)e^{-t_1} - 2A_1' - 2A_1)\sin(\tau - \omega_1 t_1)$$

$$(2B_1' + 2B_1)\cos(\tau - \omega_1 t_1)$$

$$\rightarrow \begin{cases} A_1' + A_1 + (\tfrac{1}{2} + \omega_2) e^{-t_1} = 0 \\ B_1' + B_1 = 0 \end{cases}$$

And $A_1(0) = B_1(0) = 0$

$\rightarrow B_1(t_1) = 0$

A_1: need to avoid secular terms

$\rightarrow \tfrac{1}{2} + \omega_2 = 0 \rightarrow \omega_2 = -\tfrac{1}{2}$

$\rightarrow A_1(t_1) = 0$

So $x(t) = y(\tau, t_1) = y(\omega t, \varepsilon t)$

$$\sim e^{-\varepsilon t} \sin(1 - \tfrac{1}{2}\varepsilon^2)t + O(\varepsilon^2)$$

Recall exact sol'n: $\dfrac{1}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon t} \sin\sqrt{1-\varepsilon^2}\, t$

Agrees.

Alternate method:

$X(t) = y(t_0, t_1, t_2, \ldots, t_N)$

where $t_0 = t,\ t_1 = \varepsilon t,\ t_2 = \varepsilon^2 t, \ldots, t_N = \varepsilon^N t$

Then $\dot{x} = \left(\dfrac{\partial}{\partial t_0} + \varepsilon \dfrac{\partial}{\partial t_1} + \varepsilon^2 \dfrac{\partial}{\partial t_2} + \cdots + \varepsilon^N \dfrac{\partial}{\partial t_N}\right) y$

$$\ddot{x} = \left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \cdots + \varepsilon^N \frac{\partial}{\partial t_N}\right)^2 y$$

Plug in: $\dfrac{\partial^2 y}{\partial t_0^2} + 2\varepsilon \dfrac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2\left(\dfrac{\partial^2 y}{\partial t_1^2} + 2\dfrac{\partial^2 y}{\partial t_0 \partial t_2}\right) + \cdots$
$+ 2\varepsilon\left(\frac{\partial y}{\partial t_1} + \varepsilon \frac{\partial y}{\partial t_1}\right) + y = 0$

$y(0, 0, \ldots, 0) = 0$

$$\left(\frac{\partial y}{\partial t_0} + \varepsilon \frac{\partial y}{\partial t_1} + \varepsilon^2 \frac{\partial y}{\partial t_2} + \cdots\right)\Bigg|_{(0,0,\ldots)} = 1$$

Consider $N=2$

$y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0$, $y_0(0,0,0) = 0$ $\left.\dfrac{\partial y_0}{\partial t_0}\right|_{(0,0,0)} = 1$

$y_0 = A(t_1, t_2)\cos t_0 + B(t_1, t_2)\sin t_0$

$\rightarrow A_0(0,0) = 0$, $B_0(0,0) = 1$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + y_1 = f_1$, $y_1(0,0,0)=0$

$$\left(\frac{\partial y_1}{\partial t_0} + \frac{\partial y_0}{\partial t_1}\right)\Bigg|_{(0,0,0)} = 0$$

$$-f_1 = 2\frac{\partial^2 y_0}{\partial t_0 \partial t_1} + 2\frac{\partial y_0}{\partial t_0}$$

$$= -2\frac{\partial A_0}{\partial t_1}\sin t_0 + 2\frac{\partial B_0}{\partial t_1}\cos t_0 - 2A_0 \sin t_0 + 2B_0 \cos t_0$$

$$\rightarrow \frac{\partial B_0}{\partial t_1} + B_0 = 0, \quad \frac{\partial A_0}{\partial t_1} + A_0 = 0$$

$$\rightarrow B_0(t_1, t_2) = b_0(t_2)e^{-t_1}$$

$$A_0(t_1, t_2) = a_0(t_2)e^{-t_1}$$

IC's: $b_0(0) = 1$, $a_0(0) = 0$

1-27 $\ddot{x} + 2\varepsilon\dot{x} + x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$

$t_0 = t$, $t_1 = \varepsilon t$, $t_2 = \varepsilon^2 t$

$x(t) = y(t_0, t_1, t_2)$

$$\dot{x} = \left[\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}\right]y$$

$$\ddot{x} = \left[\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}\right]^2 y$$

$$= \frac{\partial^2 y}{\partial t_0^2} + 2\varepsilon\frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2\left(\frac{\partial^2 y}{\partial t_1^2} + 2\frac{\partial^2 y}{\partial t_0 \partial t_2}\right) + \cdots$$

$y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0$, $y_0(0,0,0) = 0$, $\dfrac{\partial y_0}{\partial t_0}(0,0,0) = 1$

$$y_0(t_0, t_1, t_2) = A_0(t_1, t_2)\cos t_0 + B_0(t_1, t_2)\sin t_0$$

$A_0(0,0) = 0$

$B_0(0,0) = 1$

$O(\varepsilon^n)$: $\dfrac{\partial^2 y_n}{\partial t_0^2} + y_n = f_n$, $y_n(0,0,0) = 0$

$$\frac{\partial y_n}{\partial t_0} + \frac{\partial y_{n-1}}{\partial t_1} + \cdots + \frac{\partial y_0}{\partial t_n} = 0$$

$n=1$: $f_1 = -2\dfrac{\partial^2 y_0}{\partial t_0 \partial t_1} - 2\dfrac{\partial y_0}{\partial t_0}$

$$= -2\left(-\frac{\partial A_0}{\partial t_1}\sin t_0 + \frac{\partial B_0}{\partial t_1}\cos t_0\right)$$
$$-2(-A_0 \sin t_0 + B_0 \cos t_0$$

$\rightarrow \dfrac{\partial A_0}{\partial t_1} + A_0 = 0$, $\dfrac{\partial B_0}{\partial t_1} + B_0 = 0$

$A_0 = e^{-t_1} a_0(t_2)$, $B_0 = b_0(t_2) e^{-t_1}$

$a_0(0) = 0$ $\quad$ $b_0(0) = 1$

$\rightarrow y_0(t_0, t_1, t_2) = a_0(t_2) e^{-t_1} \cos t_0 + b_0(t_2) e^{-t_1} \sin t_0$

$y_1(t_0, t_1, t_2) = A_1(t_1, t_2)\cos t_0 + B_1(t_1, t_2)\sin t_0$

$n=2$: $f_2 = -2\dfrac{\partial^2 y_1}{\partial t_0 \partial t_1} - \dfrac{\partial^2 y_0}{\partial t_1^2} - 2\dfrac{\partial^2 y_0}{\partial t_0 \partial t_2}$

$$-2\frac{\partial y_1}{\partial t_0} - 2\frac{\partial y_0}{\partial t_1}$$

$$= -2\left[\frac{\partial A_1}{\partial t_1}(-\sin t_0) + \frac{\partial B_1}{\partial t_1}(\cos t_0)\right]$$
$$+ (a_0 e^{-t_1}\cos t_0 + b_0 e^{-t_1}\sin t_0)$$
$$-2(a_0' e^{-t_1}(-\sin t_0) + b_0' e^{-t_1}\cos t_0)$$
$$-2(-A_1 \sin t_0 + B_1 \cos t_0)$$

$\rightarrow (\partial A_1/\partial t_1) + A_1 + \frac{1}{2} b_0 e^{-t_1} + a_0' e^{-t_1} = 0$

$-\partial B_1/\partial t_1 - B_1 + \frac{1}{2} a_0 e^{-t_1} - b_0' e^{-t_1} = 0$

$\partial A_1/\partial t_1 + A = -(a_0' - \frac{1}{2} b_0) e^{-t_1}$

$\partial B_1/\partial t_1 + B_1 = (b_0' - \frac{1}{2} a_0) e^{-t_1}$

set secular-producing terms $= 0$

$$a_0' + \tfrac{1}{2} b_0 = 0 \;,\; b_0' - \tfrac{1}{2} a_0 = 0$$

$$a_0' = -\tfrac{1}{2} b_0$$

$$a_0'' = -\tfrac{1}{2} b_0' = -\tfrac{1}{4} a_0$$

$$a_0(t_2) = c_1 \cos \tfrac{1}{2} t_2 + c_2 \sin \tfrac{1}{2} t_2$$

$$b_0 = -2 a_0'$$

$$= c_1 \sin \tfrac{1}{2} t_2 - c_2 \cos \tfrac{1}{2} t_2$$

$$a_0(0) = c_1 = 0$$

$$b_0(0) = -c_2 = 1$$

$$a_0(t_2) = -\sin t_2/2$$

$$b_0(t_2) = \cos t_2/2$$

$$\rightarrow y_0(t_0, t_1, t_2) = -\sin \tfrac{1}{2} t_2 \, e^{-t_1} \cos t_0 + \cos \tfrac{1}{2} t_2 \, e^{-t_1} \sin t_0$$

$$= e^{-t_1} \sin (t_0 - \tfrac{1}{2} t_2)$$

$$x(t) = y(t, \varepsilon t, \varepsilon^2 t)$$

$$\sim y_0(t, \varepsilon t, \varepsilon^2 t)$$

$$= e^{-\varepsilon t} \sin (1 - \tfrac{1}{2} \varepsilon^2) t$$

Doesn't make sense to use more time terms than y terms (they will be disregarded)

It makes more sense to use extra y terms

Back to exact sol'n:

$$\frac{1}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon t} \sin \sqrt{1-\varepsilon^2}\, t$$

$$\sim e^{-\varepsilon t} \sin (t - \tfrac{1}{2} \varepsilon^2 t - \tfrac{1}{8} \varepsilon^4 t - \dots)$$

$$\cdot (1 + \tfrac{1}{2} \varepsilon^2 + \_\ \varepsilon^4)$$

if you only take 3 time terms, you'll never get but expanding y can get)

But you still shouldn't include it in sol'n because you don't have correction inside sin term

Doesn't always work nicely:
Equations with slowly varying coefficients

$\ddot{x}(t) + \omega^2(\varepsilon t)x = 0, \quad x(0) = a, \quad \dot{x}(0) = b$

Ex: $\omega(\varepsilon t) = 2 + \cos(\varepsilon t)$

Try multiple scales: $t_0 = t,\ t_1 = \varepsilon t$

$x(t) = y(t_0, t_1)$

$$\left(\frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1}\right)^2 y + \omega^2(t_1) y = 0$$

$$y(0,0) = a, \quad \frac{\partial y}{\partial t_0}(0,0) + \varepsilon \frac{\partial y}{\partial t_1}(0,0) = b$$

$y \sim y_0 + \varepsilon y_1$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + \omega^2(t_1) y_0 = 0, \quad y_0(0,0) = a, \quad \dfrac{\partial y}{\partial t_0}(0,0) = b$

$y_0(t_0, t_1) = A(t_1)\cos \omega(t_1)t_0 + B(t_1)\sin \omega(t_1) t_0$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + \omega^2(t_1) y_1 = f_1$

$$f_1 = -2\frac{\partial^2 y_0}{\partial t_0 \partial t_1}$$

$$= -2\,\partial/\partial t_1\left[-A\omega\sin(\omega t_0) + B\omega\cos(\omega t_0)\right]$$

$$= -2\left[-(A\omega)'\sin(\omega t_0) - A\omega t_0 \omega' \cos(\omega t_0) + (B\omega)'\cos(\omega t_0) - B\omega t_0 \omega' \sin(\omega t_0)\right]$$

$-(A\omega)' - B\omega t_0 \omega' = 0$

But A, B, ω functions only of t_1

you can't have t_0 in the eqn

Instead: $(A\omega)' = 0, \quad B\omega\omega' = 0$

$\Rightarrow A \equiv 0, \ B \equiv 0$

This method won't work

Period should depend on fast time

$\ddot{x} + x + \varepsilon f(x, \dot{x}, \dots) = 0$

This would work

need change of variables

$t_0 = f(t, \varepsilon)$

$t_1 = \varepsilon t$

$\ddot{x} + \omega^2(\varepsilon t)\, x = 0$

$x(t) = y(t_0, t_1)$, $t_0 = f(t, \varepsilon)$, $t_1 = \varepsilon t$

$$\dot{x} = \frac{\partial y}{\partial t_0}\frac{\partial t_0}{\partial t} + \frac{\partial y}{\partial t_1}\frac{\partial t_1}{\partial t}$$

$$= f_t \frac{\partial y}{\partial t_0} + \varepsilon \frac{\partial y}{\partial t_1}$$

$$\ddot{x} = f_{tt}\frac{\partial y}{\partial t_0} + f_t^2 \frac{\partial^2 y}{\partial t_0^2} + 2\varepsilon f_t \frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 y}{\partial t_1^2}$$

$$\rightarrow f_t^2 \frac{\partial^2 y}{\partial t_0^2} + f_{tt}\frac{\partial y}{\partial t_0} + 2\varepsilon f_t \frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 y}{\partial t_1^2}$$

$$+ \omega^2(t_1)\, y = 0$$

$\rightarrow$ need $f_t = \omega(t_1) = \omega(\varepsilon t)$

$\rightarrow f = \int_0^t \omega(\varepsilon s)\,ds = t_0$

$f_{tt} = \varepsilon \omega'(\varepsilon t)$

$$\frac{\partial^2 y}{\partial t_0^2} + y + \frac{\varepsilon \omega'(t_1)}{\omega^2(t_1)}\frac{\partial y}{\partial t_0} + \frac{2\varepsilon}{\omega(t_1)}\frac{\partial^2 y}{\partial t_0 \partial t_1} + \frac{\varepsilon^2}{\omega^2(t_1)}\frac{\partial^2 y}{\partial t_1^2} = 0$$

$y \sim y_0(t_0, t_1) + \varepsilon y_1(t_0, t_1)$

$x(0) = a \rightarrow y(0,0) = a$

$\dot{x}(0) = b \rightarrow \omega(0)\frac{\partial y}{\partial t_0}(0,0) + \varepsilon \frac{\partial y}{\partial t_1}(0,0) = b$

$$\frac{d^2x}{dt^2} + \omega^2(\varepsilon t)x = 0 \ , \ x(0)=a, \ \frac{dx}{dt}(0) = b$$ 1-29

$t_1 = \varepsilon t, \quad t_0 = f(t,\varepsilon)$ (Assume $\omega > 0$)

$f_t = \omega(\varepsilon t)$

$t_0 = f(t,\varepsilon) = \int_0^t \omega(\varepsilon s)ds$

$x(t) = y(t_0, t_1)$

$$\begin{cases} \dfrac{\partial^2 y}{\partial t_0^2} + y + \varepsilon \dfrac{\omega'}{\omega}\dfrac{\partial y}{\partial t_0} + 2\varepsilon \dfrac{\omega}{\omega^2}\dfrac{\partial^2 y}{\partial t_0 \partial t_1} + \dfrac{\varepsilon^2}{\omega^2}\dfrac{\partial^2 y}{\partial t_1^2} = 0 \\ y(0,0) = a, \ \left(\omega(0)\dfrac{\partial y}{\partial t_0} + \varepsilon\dfrac{\partial y}{\partial t_1}\right)\Big|_{(0,0)} = b \end{cases}$$

$y \sim y_0 + \varepsilon y_1$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0$, $y_0(0,0) = a$, $\omega(0)\dfrac{\partial y_0}{\partial t_0}(0,0) = b$

$y_0 = A(t_1)\cos t_0 + B(t_1)\sin t_0$

$A(0) = a$

$B(0) = b/\omega(0)$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + y_1 = f_1$

$$f_1 = -\frac{\omega'}{\omega^2}\frac{\partial y_0}{\partial t_0} - \frac{2\omega}{\omega^2}\frac{\partial^2 y_0}{\partial t_0 \partial t_1}$$

$$= -\frac{\omega'}{\omega^2}\left[-A\sin t_0 + B\cos t_0\right] - \frac{2\omega}{\omega^2}\left[-A'\sin t_0 + B'\cos t_0\right]$$

$$\rightarrow \begin{cases} 2\omega A' + \omega' A = 0 \\ 2\omega B' + \omega' B = 0 \end{cases}$$

$A' + \omega'/2\omega \, A = 0$

$A(t_1) = c_1 \exp(-\int \omega'/2\omega \, dt_1$

$A(t_1) = c_1 / \sqrt{\omega(t_1)}$

$B(t_1) = c_2 / \sqrt{\omega(t_1)}$

$A(0) = c_1/\sqrt{\omega(0)} = a \rightarrow c_1 = a\sqrt{\omega(0)}$

$B(0) = c_2/\sqrt{\omega(0)} = b/\omega(0) \rightarrow c_2 = b/\sqrt{\omega(0)}$

$$y_0(t_0,t_1) = \frac{a\sqrt{\omega(0)}}{\sqrt{\omega(t_1)}}\cos t_0 + \frac{b}{\sqrt{\omega(0)\omega(t_1)}}\sin t_0$$

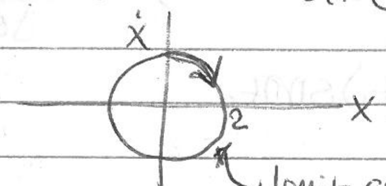
$$x(t) = y(t_0, t_1) = $$
$$\sim \frac{a\sqrt{\omega(0)}}{\sqrt{\omega(\varepsilon t}} \cos\int_0^t \omega(\varepsilon s)\,ds$$
$$+ \frac{b}{\sqrt{\omega(0)\omega(\varepsilon t)}} \sin\int_0^t \omega(\varepsilon s)\,ds$$

Rayleigh Equation

$$\frac{d^2x}{dt^2} + \varepsilon\left[-\frac{dx}{dt} + \frac{1}{3}\left(\frac{dx}{dt}\right)^3\right] + x = 0$$

Recall: solutions are

$$x(t) \sim 2\cos\left(1 - \frac{\varepsilon^3}{16}\right)t$$



limit cycle: circle of radius 2

let $x(0) = a$, $\dot{x}(0) = 0$

$\tau = \omega t$, $t_1 = \varepsilon t$

$y(\tau, t_1) = x(t)$

$$\begin{cases} \omega^2 \frac{\partial^2 y}{\partial \tau^2} + 2\varepsilon\omega \frac{\partial^2 y}{\partial \tau \partial t_1} + \varepsilon^2 \frac{\partial^2 y}{\partial t_1^2} \\ + \varepsilon\left[-\left(\omega\frac{\partial y}{\partial \tau} + \varepsilon\frac{\partial y}{\partial t_1}\right) + \frac{1}{3}\left(\omega\frac{\partial y}{\partial \tau} + \varepsilon\frac{\partial y}{\partial t_1}\right)^3\right] + y = 0 \\ y(0,0) = a, \quad \omega\frac{\partial y}{\partial \tau}(0,0) + \varepsilon\frac{\partial y}{\partial t_1}(0,0) = 0 \end{cases}$$

$\omega \sim 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2$, $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2$

because at leading order, sol'n is 2π-periodic

$\omega_1 = 0$ because t_1 encompasses slow time.

let's just go to $O(\varepsilon)$

$\rightarrow$ $\omega \sim 1$, $y \sim y_0 + \varepsilon y_1$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial \tau^2} + y_0 = 0$, $y_0(0,0) = a$, $\dfrac{\partial y_0}{\partial \tau}(0,0) = 0$

$y_0(\tau, t_1) = A(t_1)\cos\tau + B(t_1)\sin\tau$

$A(0) = a$, $B(0) = 0$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial \tau^2} + y_1 = f$

$$f = -2\frac{\partial^2 y_0}{\partial \tau \partial t_1} + \frac{\partial y_0}{\partial \tau} - \frac{1}{3}\left(\frac{\partial y_0}{\partial \tau}\right)^3$$

$$= -2(-A'\sin\tau + B'\cos\tau) + (-A\sin\tau + B\cos\tau) + \tfrac{1}{3}(A\sin\tau - B\cos\tau)^3$$

$\sin^3\tau \to \tfrac{3}{4}\sin\tau$

$\cos^3\tau \to \tfrac{3}{4}\cos\tau$

$(A\sin\tau - B\cos\tau)^3$

$$A^3\sin^3\tau - 3A^2B\underbrace{\sin^2\tau\cos\tau}_{\to \cos\tau - \cos^3\tau} + 3AB^2\overbrace{\sin\tau\cos^2\tau}^{\sin\tau - \sin^3\tau} - B^3\cos^3\tau$$

secular-producing terms:

$$\underset{\sin}{\tfrac{3}{4}A^3} - \underset{\cos}{\tfrac{3}{4}B^3} - \underset{\cos}{3A^2B(1-\tfrac{3}{4})} + \underset{\sin}{3AB^2(1-\tfrac{3}{4})}$$

$\sin\tau$: $2A' = A - \tfrac{1}{4}A^3 - \tfrac{1}{4}AB^2$

$\cos\tau$: $2B' = B - \tfrac{1}{4}B^3 - \tfrac{1}{4}A^2B$

$$\to \begin{cases} (A^2)' = A^2 - \tfrac{1}{4}A^2(A^2+B^2) \\ (B^2)' = B^2 - \tfrac{1}{4}B^2(A^2+B^2) \end{cases}$$

let $z(t_1) = A^2 + B^2$ + add the two eqns

$z' = z - \tfrac{1}{4}z^2$, $z(0) = a^2$

$$\frac{dz}{z - \tfrac{1}{4}z^2} = dt_1$$

$$\frac{dz}{z(1-\tfrac{1}{4}z)} = \frac{dz}{z} + \frac{\tfrac{1}{4}dz}{1-\tfrac{1}{4}z} = dt_1$$

$\ln z - \ln|1 - \tfrac{1}{4}z| = t_1 + c_1$

$$\ln\frac{z}{1-\tfrac{1}{4}z} = t_1 + c_1$$

$$\frac{z}{1-\frac{1}{4}z} = c_2 e^{t_1}$$

$$\frac{z}{1-\frac{1}{4}z} = \frac{a^2}{1-\frac{1}{4}a^2} e^{t_1}$$

$$z = \frac{\frac{a^2}{1-\frac{1}{4}a^2} e^{t_1}}{1+\frac{1}{4} a^2/(1-\frac{1}{4}a^2) e^{t_1}}$$

$$= \frac{4a^2 e^{t_1}}{4(1-\frac{1}{4}a^2) + a^2 e^{t_1}}$$

$$= \frac{4a^2}{a^2+(4-a^2)e^{-t_1}}$$

Then:

$$\begin{cases} 2A' = A - \frac{1}{4}Az \\ 2B' = B - \frac{1}{4}Bz \end{cases}$$

$$2A' = A \frac{(4-a^2)e^{-t_1}}{a^2+(4-a^2)e^{-t_1}}$$

$$A(t_1) = c_3 \exp\left(\frac{1}{2}\int \frac{(4-a^2)e^{-t_1}}{a^2+(4-a^2)e^{-t_1}} dt\right.$$

$$= c_3 \exp\left(-\frac{1}{2} \ln|a^2+(4-a^2)e^{-t_1}|\right.$$

$$= \frac{c_3}{\sqrt{a^2+(4-a^2)e^{-t_1}}}$$

$$A(0) = a \rightarrow \frac{c_3}{2} = a \rightarrow c_3 = 2a$$

$$A(t_1) = \frac{2a}{\sqrt{a^2+(4-a^2)e^{-t_1}}}$$

$$B(0) = 0 \rightarrow B(t_1) \equiv 0$$

$$x(t) = y(t, \varepsilon t) \sim y_0(t, \varepsilon t)$$

$$= \frac{2a \cos t}{\sqrt{a^2+(4-a^2)e^{-\varepsilon t}}}$$

so if we start at $a=4 \rightarrow 2\cos t$

$\rightarrow$ limit cycle

if $a^2 < 4$: coeff of exp is >0
the amount you add to a^2 decr. w/time
denom decr $\rightarrow$ amplitude of cos incr to 2
if $a^2 > 4$: denom increases $\rightarrow$ amplitude decr

$$y_0(\tau, t_1) = C(t_1)\cos(\tau - \varphi(t_1))$$
$$y_0(0,0) = C(0)\cos(\varphi(0)) = a$$
$$\frac{\partial y_0(0,0)}{\partial t} = -C(0)\sin(-\varphi(0)) = 0$$
$$C(0)\sin(\varphi(0)) = 0$$
want $C(0) \neq 0 \rightarrow \sin(\varphi(0)) = 0$
$\rightarrow \varphi(0) = 0$ or π
let $\varphi(0) = 0. \rightarrow C(0) = a$
(if $\varphi(0) = \pi \rightarrow C(0) = -a$)
$$f = +2\,\partial/\partial t_1 [C(t_1)\sin(\tau - \varphi(t_1))] - C(t_1)\sin(\tau - \varphi(t_1)) + \tfrac{1}{3}[C(t_1)\sin(\tau - \varphi(t_1))]^3$$
$$= 2C'\sin(\tau-\varphi) - 2C\varphi'\cos(\tau-\varphi) - C\sin(\tau-\varphi) + \tfrac{1}{3}C^3\tfrac{3}{4}\sin(\tau-\varphi) + NST$$
$\rightarrow 2C\varphi' = 0 \rightarrow \varphi' \equiv 0 \rightarrow \varphi = \text{const}$
but $\varphi(0) = 0 \rightarrow \varphi \equiv 0$
$$2C' - C + \tfrac{1}{4}C^3 = 0$$
$$2C' = C - \tfrac{1}{4}C^3$$
$$(C^2)' = C^2 - \tfrac{1}{4}C^4$$
$\rightarrow C^2 = z$
$$C = \sqrt{z} = \frac{2a}{\sqrt{a^2 + (4-a^2)e^{-t_1}}}$$ (same as before)
$$y_0(\tau, t_1) = \frac{2a\cos t}{\sqrt{a^2 + (4-a^2)e^{-t_1}}}$$

2-3 Continue with Rayleigh Eqn

$$\frac{d^2x}{dt^2} + \varepsilon\left[\frac{-dx}{dt} + \frac{1}{3}\left(\frac{dx}{dt}\right)^3\right] + x = 0, \quad x(0) = a, \quad \frac{dx(0)}{dt} = 0$$

$\tau = \omega t, \; t_1 = \varepsilon t$

$x(t) = y(\tau, t_1)$

$y \sim y_0 + \varepsilon y_1, \quad \omega \sim 1 + \varepsilon \cancel{\omega_1}^{\,0}$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial \tau^2} + y_0 = 0, \quad y_0(0,0) = a, \quad \dfrac{\partial y_0}{\partial \tau}(0,0) = 0$

$y_0(\tau, t_1) = A(t_1)e^{i\tau} + \bar{A}(t_1)e^{-i\tau}$

$y_0(0,0) = a = A(0) + \bar{A}(0)$

$\partial y_0/\partial\tau(0,0) = iA(0) - i\bar{A}(0) = 0$

$\rightarrow A(0) = \bar{A}(0) \rightarrow A$ is real

$\rightarrow A(0) = \frac{1}{2}a$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial\tau^2} + y_1 = f$

$$f = -2\frac{\partial^2 y_0}{\partial\tau\partial t_1} + \frac{\partial y_0}{\partial\tau} - \frac{1}{3}\left(\frac{\partial y_0}{\partial\tau}\right)^3$$

$$= -2(iA'e^{i\tau} - iA'e^{-i\tau}) + iAe^{i\tau} - iAe^{-i\tau} + \tfrac{1}{3}i(Ae^{i\tau} - Ae^{-i\tau})^3$$

$$= e^{i\tau}(-2iA' + iA - iA^3) + e^{-i\tau}(2iA' - iA + iA^3) + NST$$

should have used $A + \bar{A}$

$\rightarrow 2A' = A - A^2\bar{A}$

$A = \frac{1}{2}R(t_1)e^{i\theta(t_1)}$

$\rightarrow R'e^{i\theta} + Ri\theta'e^{i\theta} = \frac{1}{2}Re^{i\theta} - \frac{1}{8}R^3e^{i\theta}$

$\rightarrow R' + Ri\theta' = \frac{1}{2}R - \frac{1}{8}R^3$

$$\rightarrow \begin{cases} R\theta' = 0 & \text{Im part} \\ 2R' = R - \frac{1}{4}R^3 \end{cases}$$

let $z = R^2$

$\rightarrow z' = z - \frac{1}{4}z^2$

$A(0) = \frac{1}{2}R(0)e^{i\theta(0)} = \frac{1}{2}a$

let $\theta(0) = 0 \rightarrow R(0) = a \rightarrow z(0) = a^2$

$$z(t_1) = \frac{4a^2}{a^2 + (4-a^2)e^{-t_1}}$$

$$R(t_1) = \frac{2a}{\sqrt{a^2+(4-a^2)e^{-t_1}}}$$

Then since $R \neq 0$, $\theta'(0) = 0$

$\rightarrow \theta = \text{const}$, but $\theta(0) = 0 \rightarrow \theta(t_1) \equiv 0$

$$\rightarrow A = \tfrac{1}{2}R(t_1)$$

$$= \frac{a}{\sqrt{a^2+(4-a^2)e^{-t_1}}}$$

$$y_0 = \frac{2a\cos\tau}{\sqrt{a^2+(4-a^2)e^{-t_1}}}$$

same as before!

New Example: (more general)

$$\ddot{x} + x + \varepsilon f(x, \dot{x}) = 0, \quad x(0) = a, \ \dot{x}(0) = b$$

$t_0 = t, \ t_1 = \varepsilon t$

$x(t) = y(t_0, t_1)$

$$\frac{\partial^2 y}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 y}{\partial t_1^2} + y + \varepsilon f\left(y, \frac{\partial y}{\partial t_0} + \varepsilon \frac{\partial y}{\partial t_1}\right) = 0$$

$$y(0,0) = a, \quad \frac{\partial y}{\partial t_0}(0,0) + \varepsilon \frac{\partial y}{\partial t_1}(0,0) = b$$

$y \sim y_0 + \varepsilon y_1$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0$, $y_0(0,0) = a$, $\dfrac{\partial y_0}{\partial t_0} = b$

$y_0(t_0, t_1) = R(t_1)\cos(t_0 + \varphi(t_1))$

$y_0(0,0) = R(0)\cos(\varphi(0)) = a$

$\partial y_0/\partial t_0 (0,0) = -R(0)\sin(\varphi(0)) = b$

$$\rightarrow \begin{cases} R(0) = \sqrt{a^2+b^2} \\ \varphi(0) = -\tan^{-1}\frac{b}{a} \end{cases}$$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + y_1 = f_1$

$$f_1 = -2\frac{\partial^2 y_0}{\partial t_0 \partial t_1} - f\left(y_0, \frac{\partial y_0}{\partial t_0}\right)$$

$$= 2\,\partial/\partial t_1\left(R(t_1)\sin(t_0+\varphi(t_1))\right) - f\left(R\cos(t_0+\varphi), -R\sin(t_0+\varphi)\right)$$

let $\psi = t_0 + \varphi(t_1)$

$$= 2R'\sin\psi + 2R\varphi'\cos\psi - \underbrace{f(R\cos\psi, -R\sin\psi)}_{\text{expand in fourier series}}$$

$$f(R\cos\psi, -R\sin\psi) = A_0 + \sum_{n=1}^{\infty}(A_n\cos n\psi + B_n\sin n\psi)$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(R\cos\psi, -R\sin\psi)\,d\psi$$

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = \left\{\frac{1}{\pi}\int_0^{2\pi} F(R\cos\psi, -R\sin\psi)\begin{Bmatrix}\cos n\psi \\ \sin n\psi\end{Bmatrix} d\psi\right.$$

secular-producing terms are $n=1$

So we have

$$2R'\sin\psi + 2R\varphi'\cos\psi - A_1\cos\psi - B_1\sin\psi + NST$$

$$\rightarrow \begin{cases} 2R' = B_1(R) \\ 2R\varphi' = A_1(R) \end{cases}$$

$$\int \frac{2\,dR}{B_1(R)} = \int_0 dt_1$$

$$t_1 = 2\int_{R(0)}^{R(t_1)} \frac{ds}{B_1(s)} = 2\int_{\sqrt{a^2+b^2}}^{R(t_1)} \frac{ds}{B_1(s)}$$

$$R\frac{d\varphi}{dR} = \frac{A_1(R)}{B_1(R)}$$

$$\rightarrow \int d\varphi = \int \frac{A_1(R)}{R\,B_1(R)}\,dR$$

$$\varphi(t_1) - \varphi(0) = \int_{R(0)}^{R(t_1)} \frac{A_1(s)}{R\,B_1(s)}\,ds$$

$$\varphi(t_1) = \int_{\sqrt{a^2+b^2}}^{R(t_1)} \frac{A_1(s)}{R\,B_1(s)}\,ds - \arctan\left(\frac{b}{a}\right)$$

$$\frac{d^2x}{dt^2} + x + \varepsilon x^2 = 0, \quad x(0) = a, \quad \dot{x}(0) = 0$$

Try $t_0 = t$, $t_1 = \varepsilon t$

$x(t) = y(t_0, t_1)$, $y \sim y_0 + \varepsilon y_1$

$O(1)$: $\frac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0$, $y_0(0,0) = a$, $\frac{\partial y_0}{\partial t_0}(0,0) = 0$

$y_0(t_0, t_1) = R(t_1)\cos(t_0 + \varphi(t_1))$

$y_0(0,0) = R(0)\cos\varphi(0) = a$

$\partial y_0/\partial t_0(0,0) = -R(0)\sin\varphi(0) = 0$

$\rightarrow \varphi(0) = 0, \; R(0) = a$

$O(\varepsilon)$: $\frac{\partial^2 y_1}{\partial t_0^2} + y_1 = f$

$f = -2\frac{\partial^2 y_0}{\partial t_0 \partial t_1} - y_0^2$

$= 2R'\sin(t_0 + \varphi(t_1)) + 2R\varphi'\cos(t_0 + \varphi(t_1))$

$- \frac{1}{2}R^2 - \frac{1}{2}R^2\cos 2(t_0 + \varphi)$

$$\rightarrow \begin{cases} 2R' = 0 \rightarrow R = \text{const} \rightarrow R(t_1) \equiv a \\ R\varphi' = 0 \rightarrow \varphi = \text{const} \rightarrow \varphi(t_1) \equiv 0 \end{cases}$$

$y_0(t_0, t_1) = a\cos t_0$

$y_1(t_0, t_1) = -\frac{1}{2}a^2 + \frac{1}{6}a^2\cos 2t_0$

$+ \frac{1}{3}a^2\cos t_0 + \underline{\frac{5}{12}a^3 t_1 \sin t_0}$ not good

(after computing $O(\varepsilon^2)$, etc.)

So sol'n doesn't depend on t_1

$\rightarrow$ need t_2

But how do we know t_2 will work?

Need to look back at regular expansions

$\ddot{x} + x + \varepsilon x^2 = 0, \quad x(0) = a, \quad \dot{x}(0) = 0$

$x(t) \sim x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)$

$O(1)$: $\ddot{x}_0 + x_0 = 0$

$x_0(0) = a, \quad \dot{x}_0(0) = 0$

$x_0(t) = a\cos t$

$O(\varepsilon)$ $\ddot{x}_1 + x_1 = -x_0^2, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0$

$= -a^2\cos^2 t$

$= -\frac{1}{2}a^2 - \frac{1}{2}a^2\cos 2t$

$x_1 = c_1\cos t + c_2\sin t - \frac{1}{2}a^2 + \frac{1}{6}a^2\cos 2t$

$x_1(0) = c_1 - \frac{1}{2}a^2 + \frac{1}{6}a^2 = 0$

$\rightarrow c_1 = \frac{1}{3}a^2$

$\dot{x}_1(0) = c_2 = 0$

$\rightarrow x_1(t) = (a^2/6)(2\cos t - 3 + \cos 2t)$

$\rightarrow$ no secular terms! (so far)

$x(t) \sim a\cos t + \frac{1}{6}\varepsilon a^2(2\cos t + \cos 2t - 3)$

$O(\varepsilon^2)$ $\ddot{x}_2 + x_2 = -2x_0x_1$

$= -2a\cos t \, a^2/6(2\cos t - 3 + \cos 2t)$

$= -\frac{1}{3}a^3(1 + \cos 2t - 3\cos t + \frac{1}{2}\cos t + \frac{1}{2}\cos 3t)$

$= -\frac{1}{3}a^3(1 + \cos 2t + \frac{1}{2}\cos 3t - \underline{\frac{5}{2}\cos t})$

will get secular terms

$x_2(t) = \frac{5}{12}a^3 t\sin t + \cdots$

$\rightarrow x(t) \sim a\cos t + \frac{1}{6}\varepsilon a^2(2\cos t + \cos 2t - 3)$

$+ \varepsilon^2(\frac{5}{12}a^3 t\sin t + \cdots)$

So there is probably a term in the exact sol'n that is like $a\cos(1 - \frac{5}{12}a^2\varepsilon^2)t$

$\rightarrow$ need to expand time to $\varepsilon^2 t$ (because secular term appeared at $O(\varepsilon^2)$)

$\ddot{x} + x + \varepsilon x^2 = 0$, $x(0) = a$, $\dot{x}(0) = 0$ 2-5

$t_0 = t$, $t_2 = \varepsilon^2 t$, $x(t) = y(t_0, t_2)$

$$\frac{\partial^2 y}{\partial t_0^2} + 2\varepsilon^2 \frac{\partial^2 y}{\partial t_0 \partial t_2} + \varepsilon^4 \frac{\partial^2 y}{\partial t_2^2} + y + \varepsilon y^2 = 0$$

$$y(0,0) = a\ ,\quad \frac{\partial y}{\partial t_0}(0,0) + \varepsilon^2 \frac{\partial y}{\partial t_2}(0,0) = 0$$

$y \sim y_0 + y_1 + \varepsilon^2 y_2$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0$, $y_0(0,0) = a$, $\dfrac{\partial y_0}{\partial t_0}(0,0) = 0$

$y_0(t_0, t_2) = R(t_2)\cos(t_0 + \varphi(t_2))$

$\varphi(0) = 0$, $R(0) = a$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + y_1 = f_1$ $y_1(0,0) = 0$, $\dfrac{\partial y_1}{\partial t_0}(0,0) = 0$

$f_1 = -y_0^2 = -\frac{1}{2}R^2 - \frac{1}{2}R^2\cos 2(t_0 + \varphi)$

(no secular terms)

$y_1(t_0, t_1) = A(t_2)\cos(t_0 + \varphi(t_2)) + B(t_2)\sin(t_0 + \varphi(t_2))$
$\quad - \frac{1}{2}R^2 + \frac{1}{6}R^2\cos 2(t_0 + \varphi)$

$y_1(0,0) = A(0) - \frac{1}{2}a^2 + \frac{1}{6}a^2 = 0 \rightarrow A(0) = \frac{1}{3}a^2$

$\partial y_1/\partial t_0 (0,0) = B(0) = 0$

$O(\varepsilon^2)$: $\dfrac{\partial^2 y_2}{\partial t_0^2} + y_2 = f_2$

$$f_2 = -2\frac{\partial^2 y_0}{\partial t_0 \partial t_2} - 2y_0 y_1 = 2\frac{\partial}{\partial t_2}\left[R\sin(t_0+\varphi)\right] - 2R\cos(t_0+\varphi)\cdot$$
$$\cdot\left[A\cos(t_0+\varphi) + B\sin(t_0+\varphi) - \tfrac{1}{2}R^2 + \tfrac{1}{6}R^2\cos 2(t_0+\varphi)\right]$$
$$= 2R'\sin(t_0+\varphi) + 2R\varphi'\cos(t_0+\varphi)$$
$$+ R^3\cos(t_0+\varphi) - \tfrac{1}{6}R^3\cos(t_0+\varphi) + NST$$

$$\rightarrow \begin{cases} R' = 0 \text{ and } R(0) = a \rightarrow R(t_2) \equiv a \\ 2R\varphi' = -5/6\, R^3 \text{ and } \varphi(0) = 0 \end{cases}$$

$\varphi' = -\frac{5}{12}R^2 = -5/12\, a^2$

$\varphi = -\frac{5}{12}a^2 t_2 + \cancel{c}^{\,0}$

$y_0(t_0, t_2) = a\cos(t_0 - 5/12\, a^2 t_2)$

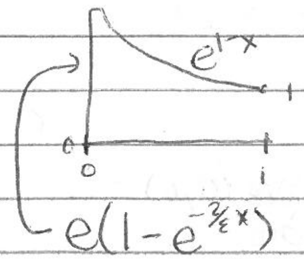
$x(t) = y(t, \varepsilon^2 t) \sim a\cos(t - 5/12\, a^2 \varepsilon^2 t_2)$

$= a\cos t(1 - 5/12\, a^2\varepsilon^2)$

$$\varepsilon \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

$0<x<1, \quad y(0)=0, \quad y(1)=1$

BL at left edge



$y_c(x) \sim e^{1-x} - e^{1-2/\varepsilon x}$

So we'd introduce:

$t_1 = x/\varepsilon, \quad t_2 = x$

$y(x) = z(t_1, t_2)$

$0<t_1<\infty, \quad 0<t_2<1$

$$\to \varepsilon\left[\frac{1}{\varepsilon^2}\frac{\partial^2 z}{\partial t_1^2} + \frac{2}{\varepsilon}\frac{\partial^2 z}{\partial t_1 \partial t_2} + \frac{\partial^2 z}{\partial t_2^2}\right] + 2\left[\frac{1}{\varepsilon}\frac{\partial z}{\partial t_1} + \frac{\partial z}{\partial t_2}\right] + 2z = 0$$

$$\frac{\partial^2 z}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 z}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2 z}{\partial t_2^2} + 2\frac{\partial z}{\partial t_1} + 2\varepsilon\frac{\partial z}{\partial t_2} + 2\varepsilon z = 0$$

$z(0,0)=0, \quad z(\infty,1)=1$

$z \sim z_0 + \varepsilon z_1$

$O(1)$: $\dfrac{\partial^2 z_0}{\partial t_1^2} + 2\dfrac{\partial z_0}{\partial t_1} = 0, \quad z_0(0,0)=0, \quad z_0(\infty,1)=1$

$z_0(t_1, t_2) = A(t_2) + B(t_2)e^{-2t_1}$

$A(0)+B(0)=0$

$A(1)=1$

$O(\varepsilon)$: $\dfrac{\partial^2 z_1}{\partial t_1^2} + 2\dfrac{\partial z_1}{\partial t_1} = f_1 = -2\dfrac{\partial^2 z_0}{\partial t_1 \partial t_2} - 2\dfrac{\partial z_0}{\partial t_2} - 2z_0$

$= 4B'e^{-2t_1} - 2A' - 2B'e^{-2t_1} - 2A - 2Be^{-2t_1}$

$= -2(A'+A) + 2e^{-2t_1}(B'-B)$

$$\to \begin{cases} A' + A = 0 & \to A(t_2) = c_1 e^{-t_2} \\ B' - B = 0 & \quad B(t_2) = c_2 e^{t_2} \end{cases}$$

$A(0) + B(0) = c_1 + c_2 = 0$

$A(1) = c_1 e^{-1} = 1 \quad \to c_1 = e \to c_2 = -e$

$z_0(t_1, t_2) = e^{1-t_2} - e^{1+t_2} e^{-2t_1}$

$= e^{1-t_2} - e^{1+t_2-2t_1}$

$y(x) = z(\frac{x}{\varepsilon}, x) \sim e^{1-x} - e^{1+x-2\frac{x}{\varepsilon}}$

slightly different than BL approach but they are both valid.

difference between the two:

$d = e^{1+x-2x/\varepsilon} - e^{1-2x/\varepsilon}$

take deriv $= 0$ & find max

(should be $O(\varepsilon)$)

$$\varepsilon \frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0$$

$t_1 = \frac{x}{\varepsilon}, \quad t_2 = x \quad \to \quad y(x) = z(t_1, t_2)$

$$\varepsilon\left[\frac{1}{\varepsilon^2}\frac{\partial^2 z}{\partial t_1^2} + \frac{2}{\varepsilon}\frac{\partial^2 z}{\partial t_1 \partial t_2} + \frac{\partial^2 z}{\partial t_2^2}\right] + a(t_2)\left[\frac{1}{\varepsilon}\frac{\partial z}{\partial t_1} + \frac{\partial z}{\partial t_2}\right] + b(t_2) z = 0$$

$$\frac{\partial^2 z}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 z}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2 z}{\partial t_2^2} + a(t_2)\frac{\partial z}{\partial t_1} + \varepsilon a(t_2)\frac{\partial z}{\partial t_2} + \varepsilon b(t_2) z = 0$$

$\to$ slowly-varying coefficients

should have chosen $t_1 = f(x, \varepsilon)$ and choose f so that leading order terms no longer have slowly-varying coefficients.

Forced Nonlinear Oscillators

$$\ddot{x} + \underbrace{\hat{\beta}\dot{x}}_{\text{damping}} + \Omega_0^2 x + \underbrace{\hat{\alpha} x^3}_{\text{non linearity}} = \underbrace{\hat{f}\cos(\Omega t)}_{\text{forcing}}$$

$x(0) = \delta_1, \quad \dot{x}(0) = 0$

$$\hat{\alpha} = 0: \quad x(t) = x_h(t) + \frac{(\Omega_0^2 - \Omega^2)\cos\Omega t + \hat{\beta}\Omega \sin\Omega t}{(\Omega_0^2 - \Omega^2)^2 + \hat{\beta}^2\Omega^2}\hat{f}$$

As $t \to \infty$, $x_h \to 0$ due to damping.

Recall $A\cos\Omega t + B\sin\Omega t$

$$= \sqrt{A^2 + B^2}\cos(\Omega t - \varphi)$$

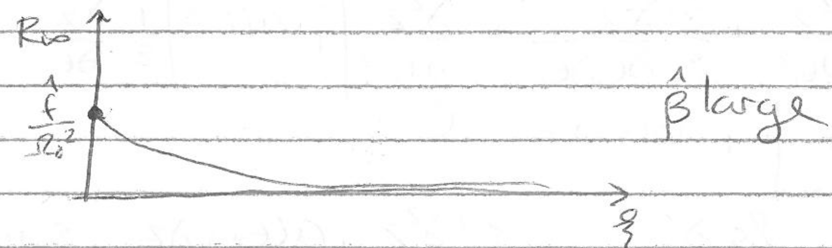
So amplitude of SS ($t \to \infty$

$$R_\infty = \hat{f}\frac{1}{\sqrt{(\Omega_0^2 - \Omega^2)^2 + \hat{\beta}^2\Omega^2}}$$

$$= \frac{\hat{f}}{\Omega_0^2}\frac{1}{\sqrt{(1 - \Omega^2/\Omega_0^2)^2 + (\hat{\beta}^2/\Omega_0^2)(\Omega^2/\Omega_0^2)}}$$

$$= \frac{\hat{f}}{\Omega_0^2}\frac{1}{\sqrt{(1-\xi)^2 + \tilde{\beta}^2\xi}}$$

where $\underbrace{\xi = \Omega^2/\Omega_0^2}_{\text{nondimensional frequency}}$, $\tilde{\beta} = \hat{\beta}/\Omega_0$



$(1-\xi)^2 + \tilde{\beta}^2\xi$

$\xi = 1 - \frac{1}{2}\tilde{\beta}^2$ is location of min. of denom.

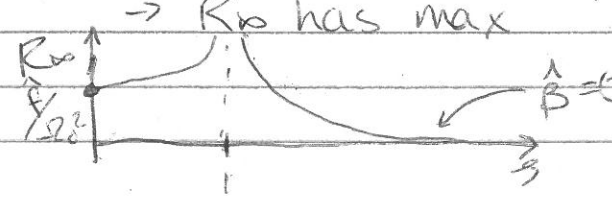
for large damping, min will occur for $\xi < 0$

but we're only interested in $\xi > 0$

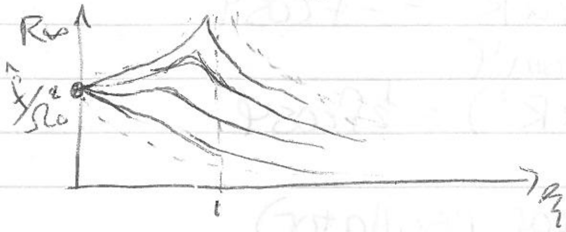
$\to$ denom. monotone incr $\to$ R_∞ decr.

Otherwise, denom has min for $\xi > 0$

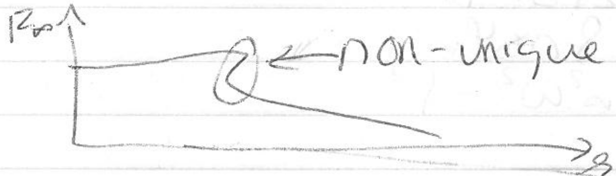
$\to$ R_∞ has max



$0 < \tilde{\beta} <$ large:



non linear: get shapes like



$$\ddot{x} + \hat{\beta}\dot{x} + \Omega_0^2 x + \hat{\alpha}x^3 = \hat{f}\cos\Omega t, \quad x(0)=\delta, \dot{x}(0)=0 \quad \text{2-10}$$

$$\hat{\beta} = \varepsilon\beta, \ \hat{\alpha} = \varepsilon\alpha, \ \hat{f} = \varepsilon f; \ \Omega_0 = 1; \ \Omega = \Omega_0 + \varepsilon\omega$$
$$= 1 + \varepsilon\omega$$

$$t_0 = t, \ t_1 = \varepsilon t, \ x(t) = y(t_0, t_1)$$

$$\begin{cases} \dfrac{\partial^2 y}{\partial t_0^2} + 2\varepsilon\dfrac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2\dfrac{\partial^2 y}{\partial t_1^2} + \varepsilon\beta\left(\dfrac{\partial y}{\partial t_0} + \varepsilon\dfrac{\partial y}{\partial t_1}\right) + y \\ \quad + \varepsilon\alpha y^3 = \varepsilon f\cos(t_0 + \omega t_1) \\ y(0,0) = \delta, \ \dfrac{\partial y}{\partial t_0}(0,0) + \varepsilon\dfrac{\partial y}{\partial t_1}(0,0) = 0 \end{cases}$$

$$y \sim y_0 + \varepsilon y_1$$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0, \quad y_0(0,0) = \delta, \ \dfrac{\partial y_0}{\partial t_0}(0,0) = 0$

$$y_0(t_0, t_1) = R(t_1)\cos(\underbrace{t_0 + \omega t_1 - \varphi(t_1)}_{=\psi})$$
$$R(0) = \delta, \quad \varphi(0) = 0$$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0} + y_1 = -2\dfrac{\partial^2 y_0}{\partial t_0 \partial t_1} - \beta\dfrac{\partial y_0}{\partial t_0} - \alpha y_0^3 + f\cos(t_0 + \omega t_1)$

$$= +2\,\partial/\partial t_1(R\sin\psi) + \beta R\sin\psi - \alpha R^3\cos^3\psi + f\cos(\psi + \varphi)$$

$$= 2R'\sin\psi + 2R(\omega - \varphi')\cos\psi + \beta R\sin\psi$$
$$-\alpha R^3 \tfrac{3}{4}\sin\psi + f\cos\psi\cos\varphi - f\sin\psi\sin\varphi + NST$$

$$\begin{cases} 2R' + \beta R = f\sin\varphi \\ 2R(\omega - \varphi') - \tfrac{3}{4}\alpha R^3 = -f\cos\varphi \end{cases}$$

$$\begin{cases} R' + \tfrac{1}{2}\beta R = \tfrac{1}{2}f\sin\varphi \\ R\varphi' - R(\omega - \tfrac{3}{8}\alpha R^2) = \tfrac{1}{2}f\cos\varphi \end{cases}$$

when $\alpha = 0$, (linear oscillator)

steady state:

$$\begin{cases} \tfrac{1}{2}\beta R_\infty = \tfrac{1}{2}f\sin\varphi_\infty \\ -R_\infty\omega = \tfrac{1}{2}f\cos\varphi_\infty \end{cases}$$

$$\beta^2 R_\infty^2 + 4R_\infty^2\omega^2 = f^2$$

$$R_\infty = \frac{f}{\sqrt{4\omega^2+\beta^2}}$$

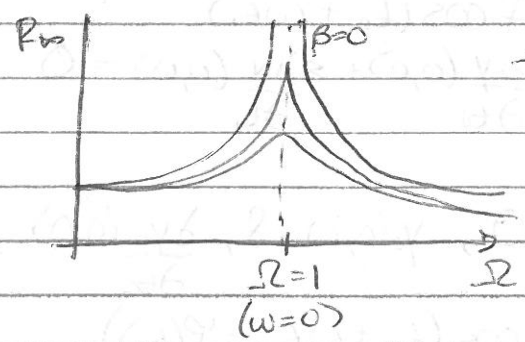
previously, we had

$$R_\infty = \frac{\hat{f}}{\Omega_0^2}\frac{1}{\sqrt{(\xi-1)^2+\tilde{\beta}^2\xi}}, \quad \xi = \frac{\Omega^2}{\Omega_0^2}, \quad \tilde{\beta} = \frac{\hat{\beta}}{\Omega_0^2}$$

$\rightarrow \xi = \Omega^2 = (1+\varepsilon\omega)^2 \sim 1 + 2\varepsilon\omega$

$\rightarrow \tilde{\beta}^2 = \hat{\beta}^2 = (\varepsilon\beta)^2 = \varepsilon^2\beta^2$

$$\rightarrow R_\infty \sim \frac{\varepsilon f}{\sqrt{4\varepsilon^2\omega^2 + \varepsilon^2\beta^2}} = \frac{f}{\sqrt{4\omega^2+\beta^2}}$$



$\rightarrow$ maxima are at $\Omega = 1$ for this approximation

want to examine stability

$$\begin{cases} R(t_1) = R_\infty + \tilde{R}(t_1) \\ \varphi(t_1) = \varphi_\infty + \tilde{\varphi}(t_1) \end{cases}$$ where $\tilde{R}$ & $\tilde{\varphi}$ are small perturbations

$$\rightarrow \begin{cases} \tilde{R}' + \tfrac{1}{2}\beta\tilde{R} = \tfrac{1}{2}f(\cos\varphi_\infty)\tilde{\varphi} \\ R_\infty\tilde{\varphi}' - \omega\tilde{R} = -\tfrac{1}{2}f(\sin\varphi_\infty)\tilde{\varphi} \end{cases}$$

$$\begin{cases} \tilde{R}' + \frac{1}{2}\beta\tilde{R} = -\omega R_\infty \tilde{\varphi} \\ R_\infty \tilde{\varphi}' - \omega\tilde{R} = -\frac{1}{2}\beta R_\infty \tilde{\varphi} \end{cases}$$

let $u = \tilde{R}/R_\infty$

$$\rightarrow \begin{cases} u' + \frac{1}{2}\beta u = -\omega\tilde{\varphi} \\ \omega\tilde{\varphi}' - \omega^2 u = -\frac{1}{2}\beta\tilde{\varphi}\omega \end{cases}$$

$$\rightarrow -u'' - \tfrac{1}{2}\beta u' - \omega^2 u = +\tfrac{1}{2}\beta(u' + \tfrac{1}{2}\beta u)$$

$$u'' + \beta u' + (\omega^2 + \tfrac{1}{4}\beta^2)u = 0$$

$$r^2 + \beta r + \omega^2 + \tfrac{1}{4}\beta^2$$

$$r = (-\beta \pm \sqrt{\beta^2 - 4(\omega^2 + \tfrac{1}{4}\beta^2)})\tfrac{1}{2}$$

$$= -\tfrac{1}{2}(\beta \pm \sqrt{-4\omega^2})$$

$$= -\tfrac{1}{2}\beta \pm i\omega$$

$$u = e^{-\frac{1}{2}\beta t_1}(\cos\omega t_1 + \sin\omega t_1)$$

So R approaches limiting value with oscillations

now look at non-linear: $\alpha > 0$

let $R = R_1/\sqrt{\alpha}$, $f = f_1/\sqrt{\alpha}$

$\rightarrow$ rescale to make $\alpha = 1$

$$\begin{cases} \frac{1}{2}\beta R_\infty = \frac{1}{2} f \sin\varphi_\infty \\ -R_\infty(\omega - \frac{3}{8}R_\infty^2) = \frac{1}{2} f\cos\varphi \end{cases}$$

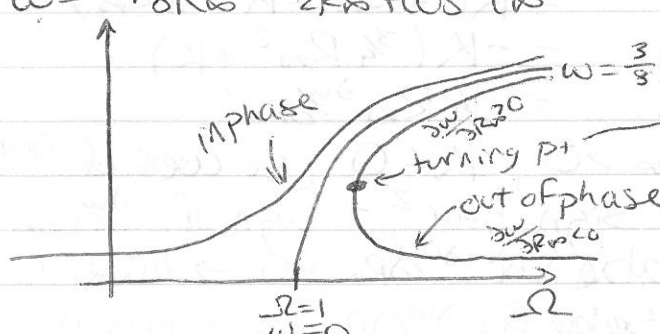
if $\beta = 0 \rightarrow$ no damping

$\rightarrow \sin\varphi_\infty = 0$

$\rightarrow \varphi_\infty = 0$ or π

$$\begin{cases} \varphi_\infty = 0: \text{in phase oscillations} \\ \varphi_\infty = \pi: \text{out of phase} \end{cases}$$

$$\omega = \tfrac{3}{8}R_\infty^2 - \tfrac{1}{2R_\infty} f\cos\varphi_\infty$$



$\frac{d\omega}{dR_\infty} = \frac{3}{4}R_\infty - \frac{f}{2R_\infty^2} = 0$

$R_\infty^* = \left(\frac{2}{3}f\right)^{1/3}$

$\omega^* = \ldots$

inphase: $\omega = \frac{3}{8}R_\infty^2 - \frac{f}{2R_\infty}$, out of phase: $\omega = \frac{3}{8}R_\infty^2 + \frac{f}{2R_\infty}$

2-12 $\ddot{x}+\varepsilon\beta\dot{x}+x+\varepsilon x^3=\varepsilon f\cos(1+\varepsilon\omega)t$

$x(t)=y(t_0,t_1)\quad,\quad t_0=t,\ t_1=\varepsilon t$

$y_0(t_0,t_1)=R(t_1)\cos(t_0+\omega t_1-\varphi(t_1))$

$R'+\frac{1}{2}\beta R=\frac{1}{2}f\sin\varphi$

$R\varphi'-(\omega-\frac{3}{8}R^2)R=\frac{1}{2}f\cos\varphi$

$\beta=0$: undamped oscillator

critical pts: $\begin{cases}0=\frac{1}{2}f\sin\varphi_\infty\\ -(\omega-\frac{3}{8}R_\infty^2)R_\infty=\frac{1}{2}f\cos\varphi_\infty\end{cases}$

$\omega=\frac{3}{8}R_\infty^2-\underbrace{\frac{f}{2R_\infty}\cos\varphi_\infty}_{=K}$

$\frac{\partial\omega}{\partial R_\infty}=\frac{3}{4}R_\infty+\frac{f}{2R_\infty^2}\cos\varphi_\infty$

$R=R_\infty+\tilde{R}\quad\varphi=\varphi_\infty+\tilde{\varphi}$

$\begin{cases}\tilde{R}'=\frac{1}{2}f\cos\varphi_\infty\tilde{\varphi}\\ R_\infty\tilde{\varphi}'-\omega\tilde{R}+\frac{9}{8}R_\infty^2\tilde{R}=-\frac{1}{2}\cancel{f\sin\varphi_\infty}^{\,0}\tilde{\varphi}\end{cases}$

$u=\tilde{R}/R_\infty$

$\rightarrow\begin{cases}u'=K\tilde{\varphi}\\ \tilde{\varphi}'-\omega u+\frac{9}{8}R_\infty^2u=0\end{cases}$

$K\tilde{\varphi}'-K\omega u+\frac{9}{8}KR_\infty^2u$

$\rightarrow u''-K\omega u+\frac{9}{8}KR_\infty^2u=0$

$u''-Ku(\omega-\frac{9}{8})R_\infty^2=0$

$r^2=K(\omega-\frac{9}{8}R_\infty^2$

if this $>0\rightarrow$ unstable

if $<0\rightarrow$ oscillatory $\rightarrow$ stable

$=K(\frac{3}{8}R_\infty^2-K-\frac{9}{8}R_\infty^2)$

$=-K(\frac{3}{4}R_\infty^2+K)$

$=-KR_\infty\,\partial\omega/\partial R_\infty$

$R_\infty>0,\ K<0$, so coef of $\partial\omega/\partial R_\infty>0$

sign of r^2 = sign of $\partial\omega/\partial R_\infty$

$\Rightarrow$ stable for $\partial\omega/\partial R_\infty<0\rightarrow$ lower branch

unstable for $\partial\omega/\partial R_\infty>0\rightarrow$ intermediate branch

Damped oscillator: $\beta > 0$

critical pts:

$$\tfrac{1}{4}\beta^2 R_\infty^2 + (\omega - \tfrac{3}{8}R_\infty^2)^2 R_\infty^2 = \tfrac{1}{4}f^2$$

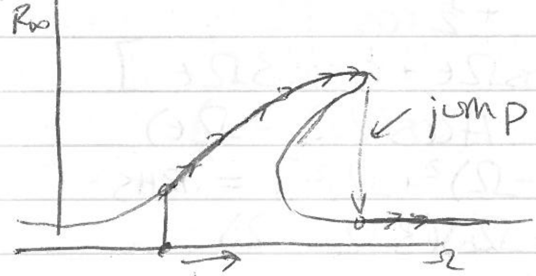
$$(\omega - \tfrac{3}{8}R_\infty^2)^2 = \frac{f^2}{4R_\infty^2} - \frac{\beta^2}{4}$$

$$\omega = \frac{3}{8}R_\infty^2 \pm \frac{1}{2}\sqrt{\frac{f^2}{R_\infty^2} - \beta^2}$$

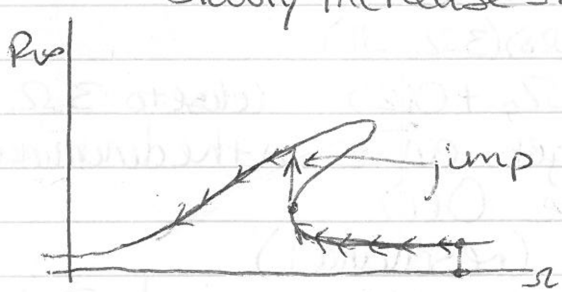
curve must be inside $\beta = 0$ case

<u>Hysterisis</u>

system "remembers" previous states



slowly increase Ω



slowly decrease Ω

$$\ddot{x} + \varepsilon\beta\dot{x} + \Omega_0^2 x + \varepsilon x^3 = f\cos\Omega t$$

$$x(t) \sim x_0(t) + \varepsilon x_1(t) \qquad \frac{1}{O(\varepsilon)}$$

$$x(0) = 1, \quad \dot{x}(0) = 0$$

$O(1)$: $\ddot{x}_0 + \Omega_0^2 x_0 = f\cos\Omega t$, $x_0(0) = 1$, $\dot{x}_0(0) = 0$

$$x_0(t) = \cos\Omega_0 t + \boxed{\frac{f}{\Omega_0^2 - \Omega^2}}(\cos\Omega t - \cos\Omega_0 t) \leftarrow K$$

$$= K\cos\Omega t + (1-K)\cos\Omega_0 t$$

$O(\varepsilon)$: $\ddot{x}_1 + \Omega_0^2 x_0 = -\beta\dot{x}_0 - x_0^3$

$$= +\beta(K\Omega\sin\Omega t + (1-K)\Omega_0\sin\Omega_0 t)$$
$$-(K\cos\Omega t + (1-K)\cos\Omega_0 t)^3$$

[$\cos^2\Omega_0 t\cos\Omega t = \frac{1}{4}\cos(2\Omega_0 - \Omega)t + \frac{1}{4}\cos(2\Omega_0 + \Omega)t + \frac{1}{2}\cos\Omega t$

$\cos^3\Omega t = \frac{3}{4}\cos\Omega t + \frac{1}{4}\cos 3\Omega t$]

particular sol'n: $A\cos(2\Omega_0 - \Omega t)$

$\rightarrow A\left[-(2\Omega_0 - \Omega)^2 + \Omega_0^2\right] = RHS$

$= A(\Omega - \Omega_0)(3\Omega_0 - \Omega)$

$$\rightarrow A = \frac{RHS}{(\Omega - \Omega_0)(3\Omega_0 - \Omega)}$$

if $\Omega = 3\Omega_0 + O(\varepsilon)$ (close to $3\Omega_0$),

we will get an ε in the denominator

$\rightarrow$ x_1 will be $O(1)$

$\rightarrow$ problem (resonance!)

(There is also a problem near $\frac{1}{3}\Omega_0$)

"secondary resonance"

only occurs with nonlinear oscillators

$x' = x(y+1) = 0$

$y' \neq$

$\ddot{x} + \varepsilon\beta\dot{x} + x + \varepsilon x^3 = f\cos\Omega t \;,\; x(0) = \delta,\; \dot{x}(0) = 0$

$\Omega = 3(1+\varepsilon\omega)$

$x(t) = y(t_0, t_1),\; t_0 = t,\; t_1 = \varepsilon t$

$\Omega t = 3(1+\varepsilon\omega)t = 3(t_0 + \omega t_1)$

RHS: $f\cos 3(t_0 + \omega t_1)$

$y = y_0 + \varepsilon y_1$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = f\cos 3(t_0 + \omega t_1) \;,\; y_0(0,0) = \delta,\; \dfrac{\partial y}{\partial t_0}(0,0) = 0$

$y_0(t_0, t_1) = R(t_1)\cos(\underbrace{t_0 + \omega t_1 - \varphi(t_1)}_{\psi})$

$+ k\cos 3(t_0 + \omega t_1)$

$\hookrightarrow k = -f/8$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + y_1 = f_1 = -2\dfrac{\partial^2 y_0}{\partial t_0 \partial t_1} - \beta\dfrac{\partial y_0}{\partial t_0} - y_0^3$

$\rightarrow \quad 2R' + \beta R + \frac{3}{4}R^2 k\sin 3\psi = 0$

$2R(\omega - \varphi') = 3/4\, R^3 + 3/2\, Rk^2 + 3/4\, R^2 k\cos 3\psi$

Critical points

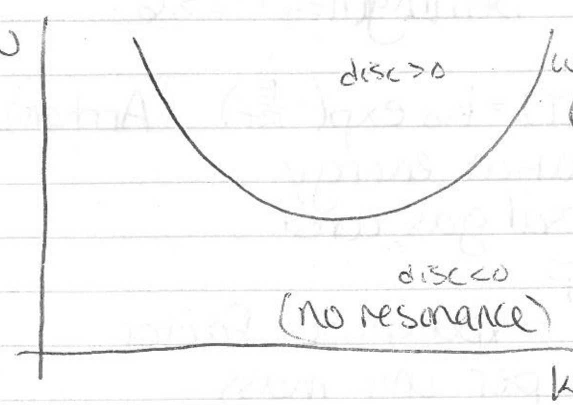
set deriv's $= 0$, square eqns + add to eliminate sine + cosine

$\beta^2 R_\infty^2 + (3/4\, R_\infty^3 + 3/2\, R_\infty k^2 - 2\omega R_\infty)^2 = \frac{9}{16}R_\infty^4 k^2$

$R_\infty = 0$ is a crit. pt.

$\beta^2 + (3/4\, R_\infty^2 + 3/2\, k^2 - 2\omega)^2 = \frac{9}{16}R_\infty^2 k^2$

$R_\infty^4 + R_\infty^2\left(3k^2 - \frac{16}{3}\omega\right) + \frac{16}{9}\beta^2 + \frac{16}{9}\left(\frac{3}{2}k^2 - 2\omega\right)^2 = 0$



$\omega = \frac{21}{32}k^2 + \frac{2}{3}\frac{\beta^2}{k^2}$

(discriminant $= 0$)

so for disc < 0, $R_\infty = 0$ is the only critical pt.

$\rightarrow$ no resonance

2-17 Asymptotic Expansions of Integrals

$I(x) = \int_a^b f(x,t)\,dt$

$f(x,t)$ is a given function

Want asymptotics of $I(x)$ as $x \to x_0$

Examples:

$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\,d\theta$

as $x \to \infty$, $\sim \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\pi}{4})$

$J_0(x)$ is a sol'n of

$xy'' + y' + xy = 0$, $y(0) = 1$, $y'(0) = 0$

let $y = \frac{u}{\sqrt{x}} \rightarrow u'' + \left(1 + \frac{1}{4x^2}\right)u = 0$

for $x \gg 1 \rightarrow u'' + u = 0$

$u = A\cos(x - \varphi)$

$y = Ax^{-1/2}\cos(x - \varphi)$

But we can't find constants:

we are looking at $x \gg 1$ and ICare@0

$(n-1)! = \Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\,dt$

$\sim \sqrt{2\pi}\, n^n e^{-n} n^{-1/2}$ $n \gg 1$

Stirling formula

$4! = 24$, Stirling$(n=5) = 23.6$

$A \xrightarrow[Q]{k(T)} B$, $k(T) = k_0 \exp(-\frac{E}{RT})$ Arrhenius

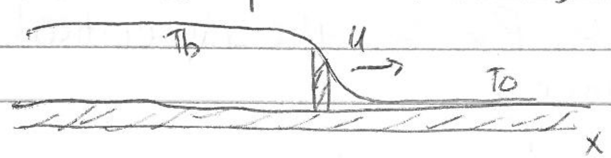
E = activation energy

R = universal gas const.

T = temp

k_0 = pre-exponential factor

Q = heat per unit mass



u: propogation speed, c = heat capacity

$$u^2 = \text{æ} \frac{c}{Q} \int_{T_0}^{T_b} k(T)\,dT$$

↑ Russian kappa

$$\int_{T_0}^{T_b} k_0 \exp\left(-\frac{E}{RT}\right) dt \sim \frac{RT_b^2}{E} k(T_b)$$

assuming $E/R \gg T_b$

Elementary Methods

Taylor Expansions & Integration by Parts

Laplace Method *

Method of Stationary Phase

Method of Steepest Descent

Ex: $I(x) = \int_0^1 \cos(xt)\,dt$ as $x \to 0$

$\sim \int_0^1 \cos(0)\,dt = 1$

Ex: $I(x) = \int_0^1 \frac{\sin(xt)}{t}\,dt$ as $x \to 0$.

Taylor expand:

$$\sim \int_0^1 \frac{xt}{t}\,dt = \int_0^1 x\,dt = x$$

OR $$\sim \int_0^1 \frac{1}{t}\left(xt - \frac{1}{3!}x^3t^3 + \frac{1}{5!}\frac{x^5}{t^5}\right)dt$$

$$= x - \frac{1}{3!}x^3\frac{1}{3} + \frac{1}{5!}x^5\frac{1}{5}$$

$I(x) = \int_a^b f(x,t)\,dt$, $x \to x_0$

If $f(x,t) \sim \sum_{n=0}^{\infty} f_n(t)(x-x_0)^{\alpha n}$, uniform in $a \le t \le b$,

then $I(x) \sim \sum_{n=0}^{\infty} \int_a^b f_n(t)\,dt\,(x-x_0)^{\alpha n}$

provided that the integrals converge.

Here $\alpha > 0$.

Ex: $I(x) = \int_0^x t^{-\frac{1}{2}} e^{-t} dt$ as $x \to 0$, let $t = xs$

$= \int_0^1 s^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-xs} x \, ds$

$= x^{\frac{1}{2}} \int_0^1 s^{-\frac{1}{2}} e^{-xs} ds$

$s^{-\frac{1}{2}} e^{-xs} = s^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n x^n s^n$

$= \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n x^n s^{n-\frac{1}{2}}$

$$I(x) = x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \int_0^1 s^{n-\frac{1}{2}} ds$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+\frac{1}{2}}}{(n+\frac{1}{2})}$$

OR:

$$I(x) = \int_0^x t^{-\frac{1}{2}} e^{-t} dt$$

$$\sim \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} t^{n-\frac{1}{2}} dt$$

$$\sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{n+\frac{1}{2}}}{(n+\frac{1}{2})}$$

Ex: $I(x) = \int_x^{\infty} e^{-t^4} dt$, $x \to 0$

$$= \underbrace{\int_0^{\infty} e^{-t^4} dt}_{\text{some const.}} - \int_0^x \underbrace{e^{-t^4}}_{\text{can expand}} dt$$

$$= \underbrace{\int_0^{\infty} e^{-s} \tfrac{1}{4} s^{-3/4} ds}_{= \frac{1}{4}\Gamma(\frac{1}{4})} - \int_0^x \sum_{n=0}^{\infty} ((-1)^n/n!) t^{4n} dt$$

$$\sim \tfrac{1}{4}\Gamma(\tfrac{1}{4}) - \sum_{n=0}^{\infty} ((-1)^n/n!) \frac{x^{4n}}{4n+1}$$

Ex: Incomplete Gamma Function

$\Gamma(a,x) = \int_x^{\infty} t^{a-1} e^{-t} dt$, $x > 0$, $x \to 0$

$a > 0$: $\int_x^{\infty} t^{a-1} e^{-t} dt$

$$= \int_0^{\infty} t^{a-1} e^{-t} dt - \int_0^x t^{a-1} e^{-t} dt$$

$$= \Gamma(a) - \int_0^x \sum_{n=0}^{\infty} ((-1)^n/n!) t^{n+a-1} dt$$

$$\sim \Gamma(a) - \sum_{n=0}^{\infty} (-1)^n/n! \, \frac{x^{n+a}}{n+a}$$

$a < 0$, $a \notin \mathbb{Z}$

say $a = -\frac{1}{2}$:

$\Gamma(-\frac{1}{2}, x) = \int_x^{\infty} t^{-3/2} e^{-t} dt$

If use same method, both integrals diverge

$= \int_0^\infty t^{-3/2}\,dt + \int_x^\infty t^{-3/2}(e^{-t}-1)\,dt$

and $e^{-t}-1 \sim t \to 2^{nd}$ integrand $\sim t^{-\frac{1}{2}}$,

which is integrable.

Say $a = -3/2$:

$t^{-5/2}e^{-t} = t^{-5/2} + t^{-5/2}(e^{-t}-1)$

$= t^{-5/2} - t^{-3/2} + t^{-5/2}(e^{-t}-1) + t^{-3/2}t^{-1/2}$

Write $e^{-t} = 1 - t + \frac{1}{2}t^2 + \underbrace{\left[e^{-t} - (1 - t + \frac{1}{2}t^2)\right]}_{=O(t^3)}$

So can multiply by $t^{-5/2} \to$ integrable

$\int_x^\infty t^{-3/2}\,dt + \int_x^\infty t^{-3/2}(e^{-t}-1)\,dt$

$\sim 2x^{-\frac{1}{2}} + \int_0^\infty t^{-3/2}(e^{-t}-1)\,dt - \int_0^x \sum_{n=1}^\infty \frac{(-1)^n}{n!} t^{n-3/2}\,dt$

$- \sum_{n=1}^\infty \frac{(-1)^n}{n!} \frac{x^{n-\frac{1}{2}}}{n-\frac{1}{2}}$

Euler's constant $\approx .577$

$\lim_{n\to\infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) - \ln n = -C$

$a = 0$ or $a = -1, -2, \ldots$

say $a = 0$

$\int_x^\infty t^{-1}e^{-t}\,dt$

previous technique: $t^{-1} + t^{-1}(e^{-t}-1)$

$\to \int_x^\infty t^{-1}\,dt + \int_x^\infty t^{-1}(e^{-t}-1)\,dt$

$= -\ln x + \int_0^\infty - \int_0^x$

problem at ∞

for $\frac{1}{t^n}$, need $n > 1$ for convergence at ∞

instead:

$\underbrace{\int_x^1 t^{-1}e^{-t}\,dt}_{\text{use prev. technique}} + \underbrace{\int_1^\infty t^{-1}e^{-t}\,dt}_{\text{some const}}$

$-\ln x + \int_x^1 \sum_{n=1}^\infty ((-1)^n/n!)\, t^{n-1} + \text{const.}$

$= -\ln x + \sum_{n=1}^\infty ((-1)^n/n!)\, t^n \Big|_{t=x}^{t=1} + \text{const.}$

2-19 Ex: $I(x) = \int_x^\infty e^{-t^4} dt$, $x \to \infty$

can't just expand

$\int_0^\infty$ converges fine but $\int_0^x$ isn't convergent

Use integration by parts

if $u = e^{-t^4}$, $dv = dt$:

$-4\int_x^\infty t \cdot t^3 e^{-t^4} dt \to$ not good

instead:

$\int_x^\infty (de^{-t^4}) \frac{1}{-4t^3} = -\frac{1}{4} t^{-3} e^{-t^4} \Big|_{t=x}^{t=\infty} - \frac{3}{4}\int_x^\infty t^{-4} e^{-t^4} dt$

$= \frac{1}{4} x^{-3} e^{-x^4} - \frac{3}{4}\int_x^\infty t^{-4} e^{-t^4} dt$

2^{nd} term is asymptotically small

$\int_x^\infty t^{-4} e^{-t^4} dt < x^{-4}\int_x^\infty e^{-t^4} dt = x^{-4} I(x) \ll I(x)$

$\to I(x) \sim \frac{1}{4} x^{-3} e^{-x^4}$

$I_n(x) = \int_x^\infty t^{-4n} e^{-t^4} dt \quad n = 0, 1, 2, \ldots$

$I_0(x) = \frac{1}{4} x^{-3} e^{-x^4} - \frac{3}{4} I_1(x)$

$I_n(x) = \int_x^\infty t^{-4n} (de^{-t^4}) \frac{1}{-4t^3} = -\frac{1}{4}\int_x^\infty t^{-(4n+3)} de^{-t^4}$

$= -\frac{1}{4} x^{-(4n+3)} e^{-x^4} - \frac{4n+3}{4} \underbrace{\int_x^\infty t^{-(4n+4)} e^{-t^4} dt}_{I_{n+1}(x)}$

$I_n(x) = \frac{1}{4}(x)^{-4n-3} e^{-x^4} - \frac{4n+3}{4} I_{n+1}(x)$

So $I(x) = \frac{1}{4} x^{-3} e^{-x^4} - \frac{3}{4} I_1(x)$

$= \frac{1}{4} x^{-3} e^{-x^4} - \frac{3}{4}\left[\frac{1}{4} x^{-7} e^{-x^4} - \frac{7}{4} I_2(x)\right]$

$= \frac{1}{4} x^{-3} e^{-x^4}\left(1 - \frac{3}{4} x^{-4}\right) + \frac{3 \cdot 7}{4^2}\left(\frac{1}{4} x^{-11} e^{-x^4} - \frac{11}{4} I_3(x)\right)$

$= \frac{1}{4} x^{-3} e^{-x^4}\left(1 - \frac{3}{4} x^{-4} + \frac{3 \cdot 7}{4^2} x^{-8}\right) - \frac{3 \cdot 7 \cdot 11}{4^3}\left[\frac{1}{4} x^{-15} e^{-x^4} - \frac{15}{4} I_4(x)\right]$

$= \frac{1}{4} x^{-3} e^{-x^4}\left(1 - \frac{3}{4} x^{-4} + \frac{3 \cdot 7}{4^2} x^{-8} - \frac{3 \cdot 7 \cdot 11}{4^3} x^{-12}\right) + \ldots$

$\to I(x) = \frac{1}{4x^3 e^{x^4}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1^n)\ 3 \cdot 7 \cdot 11 \cdot (4n-1)}{(4x^4)^n}\right]$

does this converge?

ratio test: $a_{n+1} = a_n(4n+3)$

$\to \frac{a_n}{a_{n+1}} = \frac{1}{4n+3} \to \lim_{n \to \infty} = 0$

No. radius of convergence $= 0$

	9.16
24	22.0
	21 6
	40
	24
	160
	144
	16

Ex: $I(x) = \int_0^x t^{-\frac{1}{2}} e^{-t}\,dt$, $x \to \infty$

can't use prev. method because t^{-1} will diverge at 0

$$I(x) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt - \int_x^\infty t^{-\frac{1}{2}} e^{-t}\,dt$$
$$= \sqrt{\pi} + \int_x^\infty t^{-\frac{1}{2}} e^{-t}\,dt$$
$$= \sqrt{\pi} - x^{-\frac{1}{2}} e^{-x} + \tfrac{1}{2}\int_x^\infty \underbrace{t^{-3/2} e^{-t}}_{\text{smaller}}$$

$\to I(x) \sim \sqrt{\pi} - x^{-\frac{1}{2}} e^{-x}$

$$\to \sqrt{\pi} - x^{-\frac{1}{2}} e^{-x} + \frac{1}{2}\int_x^\infty t^{-3/2}\,de^{-t}$$
$$= \sqrt{\pi} - x^{-\frac{1}{2}} e^{-x} + \frac{1}{2} x^{-3/2x} e^{-x} - \frac{(1\cdot 3)}{2^2}\int_x^\infty t^{-5/2} e^{-t}\,dt$$
$$= \sqrt{\pi} - x^{-\frac{1}{2}} e^{-x}\left(1 - \frac{1}{2}x^{-1} + \frac{1\cdot 3}{2^2 x^2} - \frac{1\cdot 3\cdot 5}{(2x)^3} + \dots\right)$$

Ex: $I(x) = \int_0^x e^{t^2}\,dt$ $x \to \infty$

prev. method won't work because $\int_0^\infty$ diverges

$$= \underbrace{\int_0^a e^{t^2}\,dt}_{=O(1)} + \int_a^x e^{t^2}\,dt$$

$$\int_a^x e^{t^2}\,dt = \int_a^x \frac{1}{2t}\,de^{t^2} = \frac{1}{2x} e^{x^2} - \underbrace{\frac{1}{2a} e^{a^2}}_{=O(1)} + \frac{1}{2}\int_a^x \frac{1}{t^2} e^{t^2}\,dt$$

$$= O(1) + \frac{1}{2x} e^{x^2} + \frac{1}{2}\int_a^x \frac{1}{2t^3}\,de^{t^2}$$
$$= O(1) + \frac{1}{2x} e^{x^2} + \frac{1}{2^2 x^3} e^{x^2} + \frac{3}{2^2}\int_a^x \frac{1}{2t^5}\,de^{t^2}$$
$$= O(1) + \frac{1}{2x} e^{x^2} + \frac{1}{2^2 x^3} e^{x^2} + \frac{3}{2^3 x^5} e^{x^2} + \dots$$
$$= O(1) + \frac{1}{2x} e^{x^2}\left(1 + \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} + \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \dots\right)$$

Ex: $I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt, \quad x \to \infty$

$= \int_0^\infty e^{-t}\left(d\ln(1+xt)\right)\frac{1}{x}$

$= \frac{1}{x}\int_0^\infty \ln(1+xt)\,e^{-t}\,dt$

$= \frac{1}{x^2}\int_0^\infty e^{-t}\,d\left[(1+xt)\ln(1+xt) - xt\right]$

→ no boundary terms

→ integration by parts method fails

instead:

let $xt = s$

$\to \int_0^\infty \frac{e^{-s/x}}{1+s}\,\frac{ds}{x}$

x large → $\frac{1}{x}$ small. say $\frac{1}{x} = \varepsilon$

$= \varepsilon\int_0^\infty \frac{e^{-\varepsilon s}}{1+s}\,ds$

$= \varepsilon\int_1^\infty \frac{e^{-\varepsilon s}}{1+s}\,ds + \underbrace{\varepsilon\int_0^1 \frac{e^{-\varepsilon s}}{1+s}\,ds}_{=O(\varepsilon)}$

$= \varepsilon\int_1^\infty \frac{e^{-\varepsilon s}}{s}\,ds - \underbrace{\varepsilon\int_1^\infty \frac{e^{-\varepsilon s}}{s(1+s)}\,ds}_{=O(\varepsilon)} + O(\varepsilon)$

let $\varepsilon s = \tau$

$= \underbrace{\varepsilon\int_\varepsilon^\infty \frac{e^{-\tau}}{\tau}\,d\tau}_{\text{large}} + O(\varepsilon)$ $\boxed{\text{subtract off part with the singularity}}$

$= \varepsilon\int_\varepsilon^1 \frac{e^{-\tau}}{\tau}\,d\tau + \underbrace{\varepsilon\int_1^\infty \frac{e^{-\tau}}{\tau}\,d\tau}_{=O(\varepsilon)} + O(\varepsilon)$

$= \varepsilon\int_\varepsilon^1 \frac{1}{\tau}\,d\tau + \underbrace{\varepsilon\int_\varepsilon^1 \frac{e^{-\tau}-1}{\tau}\,d\tau}_{O(\varepsilon)} + O(\varepsilon)$

$= -\varepsilon\ln\varepsilon = \varepsilon\ln\frac{1}{\varepsilon} = \frac{1}{x}\ln x$

↳ $I(x) \sim \frac{\ln x}{x}$

Ex: $I(x) = \int_a^b f(t)e^{x\varphi(t)}dt$, $x \to \infty$ 2-24

(Laplace integral)

$f(t), \varphi(t)$ given functions

Integration by parts

$$I(x) = \int_a^b \frac{f(t)}{x\varphi'(t)}\, de^{x\varphi(t)}, \quad \varphi'(t) \neq 0, \ a \le t \le b$$

$$= \frac{f(t)}{x\varphi'(t)} e^{x\varphi(t)}\Big|_{t=a}^{t=b} - \frac{1}{x}\int_a^b \left[\frac{f(t)}{\varphi'(t)}\right]' e^{x\varphi(t)}dt$$

$$= \frac{f(b)}{x\varphi'(b)} e^{x\varphi(b)} - \frac{f(a)}{x\varphi'(a)} e^{x\varphi(a)} - \underbrace{\frac{1}{x}\int_a^b \left[\frac{f(t)}{\varphi'(t)}\right]' e^{x\varphi(t)}dt}$$

If both $f(a)$ & $f(b) \neq 0$, then is much smaller than $I(x)$.

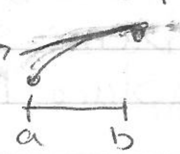
$$\left|\frac{1}{x}\int_a^b \left[\frac{f(t)}{\varphi'(t)}\right]' e^{x\varphi(t)}dt\right| \le \frac{c}{x}\int_a^b e^{x\varphi(t)}dt \qquad \text{since } \varphi'(t) \neq 0$$

So $\varphi'(t) > 0$ on $[a,b]$ or $\varphi'(t) < 0$ on $(a,b]$

Assume $\varphi'(t) > 0$

$\to \varphi(t) \le k(t-b) + \varphi(b)$

$k > 0$



So $\le \frac{c}{x}\int_a^b e^{x(k(t-b)+\varphi(b))}dt$

$= (c/x)\, e^{x\varphi(b)} \frac{1}{kx}\left[1 - e^{kx(a-b)}\right]$

$\le \frac{c_1}{x_2} e^{x\varphi(b)}$

Compare with first two terms from int. by parts

$\le \frac{1}{x}\frac{f(b)}{\varphi'(b)} e^{x\varphi(b)}$ (due to $\frac{1}{x}$)

If $\varphi(b) > \varphi(a)$, then $I(x) \sim \frac{1}{x}\frac{f(b)}{\varphi'(b)} e^{x\varphi(b)}$

If $\varphi(a) > \varphi(b)$, then $I(x) \sim -\frac{1}{x}\frac{f(a)}{\varphi'(a)} e^{x\varphi(a)}$

Ex: $I(x) = \int_a^\infty e^{-x(\cosh t - 1)}\, dt, \quad x \to \infty, \quad a > 0$

$(\cosh t = \frac{1}{2}(e^t + e^{-t}))$

$$= \int_a^\infty \frac{1}{-x\sinh t}\, de^{-x(\cosh t - 1)}$$

$$= \frac{-1}{x\sinh t} e^{-x(\cosh t - 1)}\Big|_{t=a}^{t=\infty} + \frac{1}{x}\int_a^\infty \left(\frac{1}{\sinh t}\right)' e^{-x(\cosh t - 1)}\, dt$$

$$= \frac{1}{x\sinh a} e^{-x(\cosh a - 1)} + \frac{1}{x}\int_a^\infty \left(\frac{1}{\sinh t}\right)' e^{-x(\cosh t - 1)}\, dt$$

Claim: 2^{nd} term $\ll I(x)$

as $t \to \infty$, $\sinh t \to \infty$, so $\frac{1}{\sinh t} \to 0$, so $\left(\frac{1}{\sinh t}\right)' \approx$ const (it's bounded as $t \to \infty$)

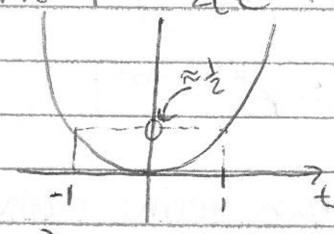
plus this term has $\frac{1}{x}$ → so small.

Ex: $I(x) = \int_{-\infty}^{\infty} e^{-x(\cosh t - 1)}\, dt$

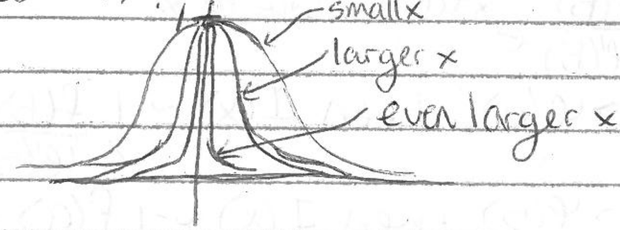
problem: $\sinh t = 0$ at $t = 0$

→ Laplace Method

$-\varphi(t) = \cosh t - 1 = \frac{1}{2}(e^t + e^{-t}) - 1$



$e^{-x(\cosh t - 1)}$:



It looks like the main contribution to the integral comes from $[-\varepsilon(x), \varepsilon(x)]$

So $I(x) \sim \int_{-\varepsilon}^{\varepsilon} e^{-x(\cosh t - 1)}\, dt$

→ can Taylor expand exponent

$\sim \int_{-\varepsilon}^{\varepsilon} e^{-x\left(\frac{1}{2}t^2+\frac{1}{24}t^4\right)} \sim \int_{-\varepsilon}^{\varepsilon} e^{-x\frac{1}{2}t^2}\,dt$

let $t\sqrt{\frac{x}{2}} = s$

$\to \sqrt{\frac{2}{x}} \int_{-\varepsilon\sqrt{x/2}}^{\varepsilon\sqrt{x/2}} e^{-s^2}\,ds \sim \sqrt{\frac{2}{x}} \int_{-\infty}^{\infty} e^{-s^2}\,ds = \sqrt{\frac{2\pi}{x}}$

Need to choose $\varepsilon(x)$ so that $\varepsilon(x) \to 0$ as $x \to \infty$

but $\varepsilon\sqrt{x} \to \infty$ as $x \to \infty$

Ex: let $\varepsilon = x^{-\frac{1}{3}}$

The main contribution to the integral, in general, comes from the neighborhood of the maximum of $\varphi(t)$.

Say max $\varphi(t)$ occurs at $t=c$

$$e^{x\varphi(t)} = e^{x\varphi(c)} \underbrace{e^{x(\varphi(t)-\varphi(c))}}$$

max $=1$, otherwise exponential is <0

$$I(x) = \int_{-\infty}^{\infty} e^{-x(\cosh t - 1)}\,dt = \int_{-\varepsilon}^{\varepsilon} e^{-x(\cosh t-1)}\,dt + \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}$$

$$= \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}t^2}\,dt + \int_{-\varepsilon}^{\varepsilon}\left(e^{-x(\cosh t-1)} - e^{-\frac{1}{2}xt^2}\right)dt$$

$$+ 2\int_{\varepsilon}^{\infty} e^{-x(\cosh t-1)}\,dt \equiv I_1 + I_2 + 2I_3$$

need to show that 2nd 2 terms are small.

let $\varepsilon = x^{-\frac{1}{3}}$

$I_1 = \int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}xt^2}\,dt$ let $s = t\sqrt{\frac{x}{2}}$

$$= \sqrt{\frac{2}{x}} \int_{-\varepsilon\sqrt{\frac{x}{2}}}^{\varepsilon\sqrt{\frac{x}{2}}} e^{-s}\,ds$$

$$\sim \sqrt{2/x} \int_{-\infty}^{\infty} e^{-s}\,ds = \sqrt{\frac{2\pi}{x}}$$

$$I_3 = \int_{\varepsilon}^{\infty} \frac{1}{-x\sinh t}\, d e^{-x(\cosh t-1)}$$

$$\sim \frac{1}{x\sinh\varepsilon} e^{-x(\cosh\varepsilon - 1)}$$

$\sinh\varepsilon \sim \varepsilon \sim x^{-\frac{1}{3}}$

$\cosh\varepsilon \sim 1 + \frac{1}{2}\varepsilon^2 \sim 1 + \frac{1}{2}x^{-\frac{2}{3}}$

$$\sim \frac{1}{x\cdot x^{-\frac{1}{3}}} e^{-x\left(\frac{1}{2}x^{-2/3}\right)} = \frac{1}{x^{2/3}} e^{-\frac{1}{2}x^{\frac{1}{3}}} \ll \sqrt{\frac{2\pi}{x}} \checkmark$$

now need to consider I_2

I_2 : $\left|e^{-x(\cosh t-1)} - e^{-\frac{1}{2}xt^2}\right|$

$= e^{-x(\cosh t-1)}\left|1 - e^{x(\cosh t-1-\frac{1}{2}t^2)}\right|$

$= e^{-x(\cosh t-1)}\left(e^{x(\cosh t-1-\frac{1}{2}t^2)}-1\right)$ ↖ exponent >0

$\le e^{-x(\cosh t-1)}\left(e^{x(\cosh x^{-1/3}-1-\frac{1}{2}x^{-2/3})}-1\right)$

since $(\;)$ is increasing → max at endpt. of $[a,b]$

$\le e^{-x(\cosh t-1)}\left(e^{xkx^{-4/3}}-1\right)$

$\le k_1 x^{-1/3} e^{-x(\cosh t-1)}$

$\ll I_1 + I_2$

$\rightarrow I_1 + I_2 \sim I_1$

$\rightarrow I(x) \sim I_1$

Ex: $I(x) = \int_0^{10} \frac{1}{1+t} e^{-xt}\, dt \qquad x \to \infty$

$\sim \int_0^{10} (1-t+t^2-t^3+\cdots)e^{-xt}\,dt$ ← good only for $|t|<1$

$\sim \int_0^{10} \sum_{n=0}^{\infty} (-1)^n t^n e^{-xt}\,dt$

$\sim \sum_{n=0}^{\infty} (-1)^n \int_0^{10} t^n e^{-xt}\,dt \qquad xt = s$

$\sim \sum_{n=0}^{\infty} (-1)^n \frac{1}{x^{n+1}} \int_0^{10x} s^n e^{-s}\,ds$

$\sim \sum_{n=0}^{\infty} (-1)^n \frac{1}{x^{n+1}} \int_0^{\infty} s^n e^{-s}\,ds$

$\sim \sum_{n=0}^{\infty} (-1)^n n!/x^{n+1}$

but the answer is correct

because main contribution comes from near $t=0$

should have replaced 10 by ε

Watson's Lemma

$I(x) = \int_0^b f(t) e^{-xt}\,dt$

Suppose $f(t)$ is continuous and has asymptotic expansion $f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}$, $\beta > 0$, $\alpha > -1$, $t \to 0$

Then the following calculation is correct.

$I(x) = \int_0^b f(t) e^{-xt}\,dt \sim \int_0^b \sum_{n=0}^{\infty} a_n t^{\beta n+\alpha} e^{-xt}\,dt$

$\sim \sum_{n=0}^{\infty} a_n \int_0^b t^{\beta n+\alpha} e^{-xt}\,dt \qquad$ let $xt = s$

$\sim \sum_{n=0}^{\infty} a_n \frac{1}{x^{\beta n+\alpha+1}} \int_0^{\infty} s^{\beta n+\alpha} e^{-s}\,ds$

$$\sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\beta n+\alpha+1)}{x^{\beta n+\alpha+1}}$$

So $I(x) = \int_0^\infty e^{-x(\cosh t-1)}\,dt$ let $s=\cosh t-1$

$ds = \sinh t\,dt$

$= \sqrt{\cosh^2 t-1}\,dt$

$= \sqrt{(s+1)^2-1}\,dt$

$= \sqrt{s^2+2s}\,dt$

$$I(x) = \int_0^\infty \frac{1}{\sqrt{s^2+2s}} e^{-xs}\,ds$$

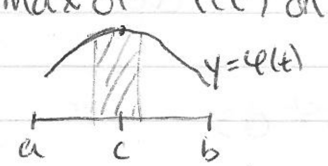
And $\frac{1}{\sqrt{s^2+2s}} = \frac{1}{\sqrt{s}\sqrt{2}\sqrt{1+s/2}}$ expand in Taylor Series

& apply Watson's Lemma.

Laplace Method 2-26

$I(x) = \int_a^b f(t) e^{x\varphi(t)}\,dt$, $x\to\infty$

max of $\varphi(t)$ on $[a,b]$ is attained at $t=c$



expand $\varphi(t) \sim \varphi(c) + \frac{1}{2}\varphi''(c)(t-c)^2$ since $\varphi'(c)=0$

if $\varphi''(c)=0$, $\varphi(t) \sim \varphi(c) + \frac{1}{24}\varphi^{(4)}(c)(t-c)^4$

expand $f(t) \sim f(c)$ unless $f(c)=0$

① Suppose $c=a \to \varphi$ is max at left end pt

Suppose $\varphi'(a)<0$, $f(a)\neq 0$

$\to I(x) \sim \int_a^{a+\varepsilon} f(a) \exp(x(\varphi(a)+\varphi'(a)(t-a)))\,dt$

let $x(t-a)=s$: $f(a)e^{x\varphi(a)}\frac{1}{x}\int_0^{\varepsilon x} e^{\varphi'(a)s}\,ds$

$\sim f(a)e^{x\varphi(a)}\frac{1}{x}\int_0^\infty e^{\varphi'(a)s}\,ds$

$= f(a)e^{x\varphi(a)}\frac{1}{x}\frac{1}{|\varphi'(a)|}$

② Suppose $c=b \to \varphi$ is max at right endpt
Suppose $\varphi'(b)>0$, $f(b)\neq 0$

$$\to I(x) \sim f(b)\, e^{x\varphi(b)} \frac{1}{x\varphi'(b)}$$

③ Suppose $c=a$ but $f(a)=0$, $\varphi'(a)<0$, $f'(a)\neq 0$

$$\to I(x) \sim \int_a^{a+\varepsilon} f'(a)(t-a)\exp[x(\varphi(a)+\varphi'(a)(t-a))]\,dt$$

$$= f'(a)\, e^{x\varphi(a)} \int_a^{a+\varepsilon} (t-a)\, e^{x\varphi'(a)(t-a)}\,dt$$

let $x(t-a)=s \to f'(a)\, e^{x\varphi(a)} \frac{1}{x}\int_0^{\infty} \frac{s}{x} e^{\varphi'(a)s}\,ds$

$$\sim f'(a)\, e^{x\varphi(a)} \frac{1}{x^2} \frac{1}{[\varphi'(a)]^2}$$

for all derivatives of f up to n^{th} $=0$:

$$f(t) \sim \frac{1}{n!} f^n(a)(t-a)^n$$

$\to \frac{1}{x^{n+1}}$ behavior

④ $f(c)\neq 0$, $\varphi''(c)<0$, $a<c<b$

$$\varphi(t) \sim \varphi(c) + \tfrac{1}{2}\varphi''(c)(t-c)$$

$$f(t) \sim f(c)$$

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(c)\exp[x(\varphi(c)+\tfrac{1}{2}\varphi''(c)(t-c)^2)]\,dt$$

$$\sim f(c)\, e^{x\varphi(c)} \int_{c-\varepsilon}^{c+\varepsilon} e^{\frac{1}{2}x\varphi''(c)(t-c)^2}\,dt$$

let $s=\sqrt{\tfrac{1}{2}x|\varphi''(c)|}\,(t-c)$

$$\to \frac{f(c)\, e^{x\varphi(c)}}{\sqrt{\frac{1}{2}x|\varphi''(c)|}} \int_{-\varepsilon\sqrt{\frac{1}{2}x|\varphi''(c)|}}^{\varepsilon\sqrt{\frac{1}{2}x|\varphi''(c)|}} e^{-s^2}\,ds$$

$$\sim \frac{f(c)\, e^{x\varphi(c)}}{\sqrt{\frac{1}{2}x|\varphi''(c)|\frac{1}{\pi}}}$$

N even, $\varphi^{(N)}(c)<0$

⑤ $a<c<b$, $f(c)\neq 0$, $\varphi(t)\sim\varphi(c)+\frac{1}{N!}\varphi^{(N)}(c)(t-c)^N$

$$I(x) \sim f(c)\, e^{x\varphi(c)} \int_{c-\varepsilon}^{c+\varepsilon} \exp[-x\tfrac{1}{N!}|\varphi^{(N)}(c)|(t-c)^N]\,dt$$

$$s = \tfrac{x}{N!}|\varphi^{(N)}(c)|(t-c)^N$$

$$\frac{s^{1/N}}{[\frac{x}{N!}|\varphi^{(N)}(c)|]^{1/N}} = t-c \to dt = \frac{\frac{1}{N}s^{\frac{1}{N}-1}}{[\ \cdots\ ]}$$

$$\to I(x) \sim \frac{f(c)\, e^{x\varphi(c)}}{[\frac{x}{N!}|\varphi^{(N)}(c)|]^{1/N}} \frac{2}{N}\int_0^{\infty} s^{\frac{1}{N}-1} e^{-s}\,ds$$

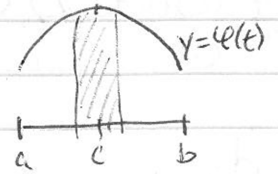
$$\sim \frac{f(c)\,e^{x\varphi(c)}}{\left[\frac{x}{N!}\left|\varphi^{(N)}(c)\right|\right]^{1/N}} \frac{2}{N}\,\Gamma\left(\frac{1}{N}\right)$$

Laplace Method

2-26

$I(x) = \int_a^b f(t) e^{x\varphi(t)} dt, \quad x \to \infty$

max of $\varphi(t)$ on $[a,b]$ is attained at $t=c$



expand $\varphi(t) \sim \varphi(c) + \frac{1}{2}\varphi''(c)(t-c)^2$ since $\varphi'(c)=0$

if $\varphi''(c)=0$, $\varphi(t) \sim \varphi(c) + \frac{1}{24}\varphi^{(4)}(c)(t-c)^4$

expand $f(t) \sim f(c)$ unless $f(c)=0$

① Suppose $c=a \to \varphi$ is max at left endpt

Suppose $\varphi'(a)<0$, $f(a) \neq 0$

$\to I(x) \sim \int_a^{a+\varepsilon} f(a) \exp(x(\varphi(a) + \varphi'(a)(t-a)))dt$

let $x(t-a)=s \to f(a) e^{x\varphi(a)} \frac{1}{x}\int_0^{\varepsilon x} \exp(\varphi'(a)s)\, ds$

$\sim f(a) e^{x\varphi(a)} \frac{1}{x}\int_0^{\infty} e^{\varphi'(a)s} ds$

$= f(a) e^{x\varphi(a)} \frac{1}{x} \frac{1}{|\varphi'(a)|}$

② Suppose $c=b \to \varphi$ is max at right endpt

Suppose $\varphi'(b)>0$, $f(b)\neq 0$

$\to I(x) \sim f(b) e^{x\varphi(b)} \frac{1}{x\varphi'(b)}$

③ Suppose $c=a$ but $f(a)=0$ $\varphi'(a)<0$, $f'(a)\neq 0$

$\to I(x) \sim \int_a^{a+\varepsilon} f'(a)(t-a) e^{x[\varphi(a) + \varphi'(a)(t-a)]} dt$

$= f'(a) e^{x\varphi(a)} \int_a^{a+\varepsilon} (t-a) e^{x\varphi'(a)(t-a)} dt$

let $x(t-a)=s \to f'(a) e^{x\varphi(a)} \frac{1}{x}\int_0^{\infty} \frac{s}{x} e^{\varphi'(a)s} ds$

$\sim f'(a) e^{x\varphi(a)} \frac{1}{x^2} \frac{1}{[\varphi'(a)]^2}$

for all derivatives of f up to $n^{th}=0$:

$f(t) \sim \frac{1}{n!} f^{(n)}(a)(t-a)^n$

$\to \frac{1}{x^{n+1}}$ behavior

④ $f(c) \neq 0, \ \varphi''(c) < 0, \quad a < c < b$

$\varphi(t) \sim \varphi(c) + \frac{1}{2}\varphi''(c)(t-c)^2$

$f(t) \sim f(c)$

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(c)\exp\left(x\left(\varphi(c) + \tfrac{1}{2}\varphi''(c)(t-c)^2\right)\right)dt$$

$$\sim f(c)e^{x\varphi(c)}\int_{c-\varepsilon}^{c+\varepsilon} e^{\frac{1}{2}x\varphi''(c)(t-c)^2}dt$$

let $s = \sqrt{\tfrac{1}{2}x|\varphi''(c)|}\,(t-c)$

$$\rightarrow \frac{f(c)e^{x\varphi(c)}}{\sqrt{\frac{1}{2}x|\varphi''(c)|}}\int_{-\varepsilon\sqrt{\frac{1}{2}x|\varphi''(c)|}}^{\varepsilon\sqrt{\frac{1}{2}x|\varphi''(c)|}} e^{-s^2}ds$$

$$\sim \frac{f(c)e^{x\varphi(c)}}{\sqrt{\frac{1}{2}x|\varphi''(c)|\frac{1}{\pi}}}$$

(N even), $\varphi^{(N)}(c) < 0$

⑤ $a < c < b, \ f(c) \neq 0, \ \varphi(t) \sim \varphi(c) + \frac{1}{N!}\varphi^{(N)}(c)(t-c)^N$

$$I(x) \sim f(c)e^{x\varphi(c)}\int_{c-\varepsilon}^{c+\varepsilon} e^{-x\frac{1}{N!}|\varphi^{(N)}(c)|(t-c)^N}dt$$

$s = \frac{x}{N!}|\varphi^{(N)}(c)|(t-c)^N$

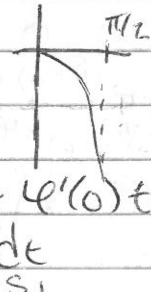
$$\frac{s^{\frac{1}{N}}}{\left[\frac{x}{N!}|\varphi^{(N)}(c)|\right]^{\frac{1}{N}}} = t-c \ \rightarrow dt = \frac{\frac{1}{N}s^{\frac{1}{N}-1}}{[\quad]^{\frac{1}{N}}}$$

$$I(x) \sim \frac{f(c)e^{x\varphi(c)}}{\left[\frac{x}{N!}|\varphi^{(N)}(c)|\right]^{\frac{1}{N}}}\frac{2}{N}\int_0^\infty s^{\frac{1}{N}-1}e^{-s}ds$$

$$\sim \frac{f(c)e^{x\varphi(c)}}{\left[\frac{x}{N!}|\varphi^{(N)}(c)|\right]^{\frac{1}{N}}}\frac{2}{N}\Gamma\left(\frac{1}{N}\right)$$

⇒ Ex: $I(x) = \int_0^{\pi/2} e^{-x\tan t}dt, \quad x \to \infty$

$\varphi(t) = -\tan t$



has a max at $t=0$

$\varphi(t) \sim \varphi(0) + \varphi'(0)t = -t$

$\rightarrow I(x) \sim \int_0^\varepsilon e^{-xt}dt \qquad s = xt \quad dt = ds\frac{1}{x}$

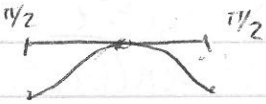
$\sim \frac{1}{x}\int_0^\infty e^{-s}ds$

$= \frac{1}{x}$

Ex: $I(x) = \int_{-\pi/2}^{\pi/2} e^{x(\cos t - 1)} (t^2+2)\,dt \quad , \ x \to \infty$

$\varphi(t) = \cos t - 1$

$f(t) = t^2 + 2$



$\varphi(t) \sim -\frac{1}{2}t^2 \ , \ f(t) \sim 2 \ , \ t \to 0$

$I(x) \sim \int_{-\varepsilon}^{\varepsilon} 2 e^{-\frac{1}{2}xt^2}\,dt \quad , \ \text{let } \sqrt{\frac{x}{2}}\,t = s$

$\sim 2\sqrt{\frac{2}{x}} \int_{-\infty}^{\infty} e^{-s^2}\,ds$

$= 2\sqrt{\frac{2\pi}{x}}$

Ex: $I(x) = \int_{-1}^{1} e^{-x\sin^4 t}\,dt$

$\varphi(t) = -\sin^4 t \sim -t^4 \ , \ t \to 0$

$I(x) \sim \int_{-\varepsilon}^{\varepsilon} e^{-xt^4}\,dt$

$= 2\int_0^{\varepsilon} e^{-xt^4}\,dt \qquad s = xt^4 \quad t = \left(\frac{s}{x}\right)^{\frac{1}{4}} \quad dt = \frac{1}{4}\frac{s^{\frac{1}{4}-1}}{x^{\frac{1}{4}}}$

$\sim 2\frac{1}{x^{1/4}}\frac{1}{4}\int_0^{\infty} s^{\frac{1}{4}-1}e^{-s}\,ds$

$= \frac{\Gamma\left(\frac{1}{4}\right)}{2x^{\frac{1}{4}}}$

Ex: $I(x) = \int_0^{\pi/2} e^{x\cos t}\ln(a+\sin t)\,dt \quad , \ a > 0$

$\varphi(t) = \cos t \sim 1 - \frac{1}{2}t^2 \ , \ t \to 0$

$f(t) = \ln(a+\sin t) \sim \ln a \ , \ t \to 0$

$I(x) \sim \ln a\, e^x \int_0^{\varepsilon} e^{-\frac{1}{2}xt^2}\,dt \qquad \text{let } s = \sqrt{\frac{x}{2}}\,t$

$\sim \ln a\, e^x \sqrt{\frac{2}{x}} \int_0^{\infty} e^{-s^2}\,ds$

$= \ln a\, e^x \sqrt{\frac{\pi}{2x}}$

→ problem for $a=1$ (expansion of f is bad)

need $f(t) = \ln(1+\sin t) \sim t$

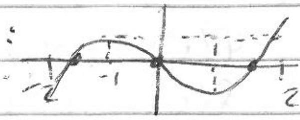
$I(x) \sim \int_0^{\varepsilon} t e^{x(1-\frac{1}{2}t^2)}\,dt$

$\sim e^x \int_0^{\infty} t e^{-\frac{1}{2}t^2 x}\,dt$

$= e^x/x$

Ex: $I(x) = \int_{-2}^{2} e^{x(\frac{1}{3}t^3 - t)} \ln(1+t^2)\,dt$

$\varphi(t) = \frac{1}{3}t^3 - t$ zeros: $t = 0, \pm\sqrt{3}$:



$f(t) = \ln(1+t^2)$ $\varphi'(t) = t^2 - 1$

$t = \pm 1$ $\varphi(-1) = 2/3,\ \varphi(2) = 2/3$

where does largest contribution come from?

t near -1:

$$\varphi(t) \sim \varphi(-1) + \tfrac{1}{2}\varphi''(-1)(t+1)^2$$
$$= 2/3 - (t+1)^2$$

$$f(t) \sim \ln 2$$

$$I(x) \sim \int_{-1-\varepsilon}^{-1+\varepsilon} \ln 2\, e^{x(2/3 - (t+1)^2)}\,dt \qquad \text{let } \sqrt{x}(t+1) = s$$
$$\sim \ln 2\, e^{2/3 x} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-s^2}\,ds$$
$$= \ln 2 \sqrt{\frac{\pi}{x}}\, e^{2/3 x}$$

t near 2:

$$\varphi(t) \sim \varphi(2) + \varphi'(2)(t-2)$$
$$\sim 2/3 + 3(t-2)$$

$$f(t) \sim \ln 5$$

$$I(x) \overset{?}{\sim} \ln 5\, e^{2/3 x} \int_{2-\varepsilon}^{2} e^{3x(t-2)}\,dt$$
$$\sim \ln 5\, e^{2/3 x} \frac{1}{3x} \int_{-\infty}^{0} e^{s}\,ds$$
$$= \ln 5\, e^{2/3 x} \frac{1}{3x}$$

$\ll$ prev. result.

if instead, $f = t+1$:

$$I(x) = \int_{-2}^{2} e^{x(\frac{1}{3}t^3 - t)}(1+t)\,dt$$

now t near -1 is different: is it still larger than the contribution near $t = 2$

Ex: $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x \to \infty$

$t^{x-1} e^{-t} = \frac{1}{t} e^{-t} e^{x \ln t}$

try $f(t) = \frac{1}{t} e^{-t}$, $\varphi(t) = \ln t$?

problem: $\varphi(t) \to \infty$ as $t \to \infty$ (no maximum)

OR $\int_0^\infty \frac{1}{t} e^{-t + x \ln t} dt \quad$ let $t = xs$

$\to \int_0^\infty \frac{1}{xs} e^{-xs + x(\ln x + \ln s)} x\, ds$

$= e^{x \ln x} \int_0^\infty \frac{1}{s} e^{x(\ln s - s)} ds$

Then $f(s) = \frac{1}{s}$, $\varphi(s) = \ln s - s$

$\varphi'(s) = \frac{1}{s} - 1 = 0 \to s = 1$

$\varphi(s) \sim \varphi(1) + \frac{1}{2} \varphi''(1)(s-1)^2$

$= -1 - \frac{1}{2}(s-1)^2$

$f(s) \sim 1$

$\Gamma(x) \sim x^x \int_{1-\varepsilon}^{1+\varepsilon} 1 \cdot e^{x(-1-\frac{1}{2}(s-1)^2)} ds \quad$ let $\frac{1}{2} x (s-1)^2 = \tau^2$

$\sim x^x e^{-x} \int_{-\varepsilon\sqrt{x/2}}^{\varepsilon\sqrt{x/2}} e^{-\tau^2} \sqrt{\frac{2}{x}}\, d\tau \quad \to s - 1 = \sqrt{\frac{2}{x}}\,\tau$

[Recall: $\varepsilon = \varepsilon(x)$, $\varepsilon\sqrt{x} \to \infty$]

$\sim x^x e^{-x} \sqrt{\frac{2\pi}{x}}$

Going further:

$\Gamma(x) \sim x^x e^{-x} \sqrt{\frac{2\pi}{x}} \left(1 + \frac{1}{12x} + \frac{1}{288x^3}\right)$

$x = 3: \; 2! = 2$, appx $= 2.00$ something else $\neq 0$

Different method:

$g = -t + x \ln t$

$dg/dt = -1 + x/t = 0 \to t = x$ is max

"moving maximum": not desirable.

let $s = t/x$, then max at $s = 1$

(same)

Higher Order Terms

$I(x) = \int_a^b f(t) e^{x\varphi(t)} dt$

$\varphi(t)$ attains maximum at $t=c$

Say $a<c<b$. Assume $\varphi''(c)<0$

$I(x) = \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{x\varphi(t)} dt + \underbrace{\int_a^{c-\varepsilon} + \int_{c+\varepsilon}^b}$

recall these terms ↗ are exp. smaller

next order terms won't be that much smaller

need to take more terms in expansions of f & φ

Recall $\varepsilon = \varepsilon(x) \to 0$ as $x \to \infty$

and $\varepsilon\sqrt{x} \to \infty$ as $x \to \infty$

say $\varepsilon = x^{-\alpha} \to \varepsilon\sqrt{x} = x^{-\alpha} x^{\frac{1}{2}} = x^{\frac{1}{2}-\alpha}$

need $0<\alpha<\frac{1}{2}$ to get correct limits

For this proof, will need $\varepsilon^3 x \to 0 \to \alpha > \frac{1}{3}$

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} \left(f(c) + f'(c)(t-c) + \tfrac{1}{2} f''(c)(t-c)^2\right) \cdot \exp\left[x\left(\varphi(c) + \tfrac{1}{2}\varphi''(c)(t-c)^2 + \tfrac{1}{6}\varphi'''(c)(t-c)^3 + \tfrac{1}{24}\varphi^{(4)}(c)(t-c)^4\right)\right] dt$$

(expand enough to keep $\frac{1}{x}$ terms)

let $s = \sqrt{x}(t-c)$, $t-c = s/\sqrt{x}$

$$\sim \frac{1}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \left[f(c) + f'(c)\frac{s}{\sqrt{x}} + \tfrac{1}{2} f''(c)\frac{s^2}{x}\right] \cdot \exp[x\varphi(c)] \exp\left[\tfrac{1}{2}\varphi''(c)s^2\right] \exp\left[\tfrac{1}{6}\varphi'''(c)\frac{s^3}{\sqrt{x}} + \tfrac{1}{24}\varphi^{(4)}(c)\frac{s^4}{x}\right] ds$$

$$\sim \frac{1}{\sqrt{x}} e^{x\varphi(c)} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \left[f(c) + f'(c)\frac{s}{\sqrt{x}} + \tfrac{1}{2} f''(c)\frac{s^2}{x}\right] \cdot e^{\frac{1}{2}\varphi''(c)s^2}\left[1 + \tfrac{1}{6}\varphi'''(c)\frac{s^3}{\sqrt{x}} + \tfrac{1}{24}\varphi^{(4)}(c)\frac{s^4}{x} + \tfrac{1}{72}\varphi'''(c)^2\frac{s^6}{x}\right] ds$$

can expand exponential due to choice of $\alpha > \frac{1}{3}$

$$\sim \frac{1}{\sqrt{x}} e^{x\varphi(c)} \int_{-\infty}^{\infty} e^{\frac{1}{2}\varphi''(c)s^2}\left[f(c) + \frac{1}{x}\left[\tfrac{1}{2} f''(c)s^2 + \tfrac{1}{6} f'(c)\varphi'''(c)s^4 + \tfrac{1}{24} f(c)\varphi^{(4)}(c)s^4 + \tfrac{1}{72} f(c)[\varphi'''(c)]^2 s^6\right]\right] ds$$

terms of order $\frac{1}{\sqrt{x}}$ are odd wrt $s \to \int_{-\infty}^{\infty} = 0 \to$ ignore

and $\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} s^{2n} ds = \sqrt{2\pi}(2n-1)!!$, so we get

$$I(x) \sim \sqrt{\frac{2\pi}{|\varphi''(c)|x}}\, e^{x\varphi(c)}\left[f(c) + \frac{1}{x}\left(\frac{\frac{1}{2}f''(c)}{|\varphi''(c)|} + \frac{1}{2}\frac{f'(c)\varphi'''(c)}{|\varphi''(c)|^2} + \frac{1}{8}\frac{f(c)\varphi^{(4)}(c)}{|\varphi''(c)|^2} + \frac{5}{24}\frac{f(c)[\varphi'''(c)]^2}{|\varphi''(c)|^3}\right)\right]$$

What about:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} s^{2n} ds \qquad \text{let } s^2 = 2\tau \to s = \sqrt{2\tau} \quad ds = \frac{\sqrt{2}}{2}\tau^{-\frac{1}{2}} d\tau$$

$$= 2\int_0^{\infty} e^{-\tau}(2\tau)^n \frac{\sqrt{2}}{2}\tau^{-\frac{1}{2}} d\tau$$

$$= 2^n\sqrt{2}\int_0^{\infty} \tau^{n-\frac{1}{2}} e^{-\tau} d\tau$$

$$= 2^n\sqrt{2}\,\Gamma(n+\tfrac{1}{2}) \qquad [\Gamma(x+1) = x\Gamma(x)]$$

$$= 2^n\sqrt{2}\,(n-\tfrac{1}{2})\Gamma(n-\tfrac{1}{2})$$

$$= 2^n\sqrt{2}\,(n-\tfrac{1}{2})(n-\tfrac{3}{2})\Gamma(n-\tfrac{3}{2})$$

$$= 2^n\sqrt{2}\,\underbrace{(n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2})\cdots\tfrac{1}{2}}_{n \text{ factors}}\Gamma(\tfrac{1}{2})$$

$$= \sqrt{2}\,(2n-1)(2n-3)(2n-5)\cdots 1\cdot\underbrace{\Gamma(\tfrac{1}{2})}_{=\sqrt{\pi}}$$

$$= \sqrt{2}\,(2n-1)!!\,\sqrt{\pi}$$

$$= \sqrt{2\pi}\,(2n-1)!!$$

$$I(x) = \int_a^b f(t)e^{ix\varphi(t)}dt \qquad (\varphi(t) \text{ real function})$$

$f(t)$ can be assumed real WLOG

Ex: Fourier Integral:

$$I(x) = \int_a^b f(t)\, e^{ixt} dt \qquad x \to \infty$$

Integration by parts:

$$I(x) = \frac{1}{ix}\int_a^b f(t)\, de^{ixt}$$

$$= \frac{1}{ix} f(b)e^{ixb} - \frac{1}{ix} f(a)e^{ixa} - \frac{1}{ix}\int_a^b f'(t)e^{ixt}dt$$

want to say last term $\to 0$ as $x \to \infty$

Distributions:

$\frac{\sin xt}{x} \to 0$ as $x \to \infty$

in the sense of distributions

because it does in the normal sense too

then we can say $\cos xt \to 0$ as $x \to 0$

in the sense of distributions

because $\int_{-\infty}^{\infty} \cos xt\ \varphi(t)dt \to 0$ as $x \to \infty$

Thus our last term integral must $\to 0$

since $e^{ixt} = \cos xt + i\sin xt$

Riemann-Lebesgue lemma:

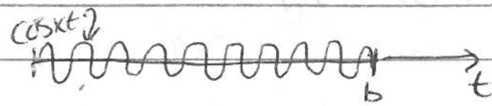
$$\lim_{x\to\infty}\int_a^b f(t)\cos xt\,dt = \lim_{x\to\infty}\int_a^b f(t)\sin xt\,dt = 0$$

provided that $f(t)$ is absolutely integrable, i.e. $\int_a^b |f(t)|dt < \infty$

(a and b can be infinite)

Intuition:

$$\int_a^b \cos xt\,dt = \frac{\sin xt}{x}\Big|_a^b = O(\tfrac{1}{x}) \quad \text{why?}$$



for each period, the contribution is zero. (only ends count)

period $T = \frac{2\pi}{x}$

the contributions at the end are $O(\frac{1}{x})$

$$\int_a^b f(t)\cos xt\,dt = O(\tfrac{1}{x}). \quad \text{Why?}$$

Assume $f(t)$ smooth

As $x\to\infty$, $f(t)$ doesn't change much over one period

$$\int_\alpha^{\alpha+\frac{2\pi}{x}} [f(\alpha)+f'(\alpha)(t-\alpha)]\cos xt\,dt \quad \leftarrow \text{over one period}$$

integral of 1st term $= 0$ 2nd term is at most $O(\frac{1}{x})$, interval $= O(\frac{1}{x}) \rightarrow$ integral $= O(\frac{1}{x^2})$

Add up all the periods from $a \to b$

length of one period $= \frac{2\pi}{x}$

# of periods $= x\frac{b-a}{2\pi} = O(x)$

$\rightarrow$ total contribution can be as large as $O(\frac{1}{x})$

$I(x) = \int_a^b f(t) e^{ix\varphi(t)} dt, \quad x \to \infty$ 3-5

$\varphi(t)$: real function, $f(t)$ real too.

$I(x) = \int_a^b f(t) e^{ixt} dt$

$= \frac{1}{ix} \int_a^b f(t) d e^{ixt}$

$= \frac{1}{ix} f(b) e^{ixb} - \frac{1}{ix} f(a) e^{ixa} - \underbrace{\frac{1}{ix} \int_a^b f'(t) e^{ixt} dt}_{}$,

$\lim_{x\to\infty} \int_a^b g(t) \cos xt\, dt = \lim_{x\to\infty} \int_a^b g(t) \sin xt\, dt = 0$

if $\int_a^b |g(t)| dt < \infty$

<u>Ex</u>: $I(x) = \int_0^1 \ln(1+t) \cos xt\, dt, \quad x \to \infty$

$= \frac{1}{x} \int_0^1 \ln(1+t)\, d \sin xt$

$= \frac{1}{x} \ln(1+t) \sin xt \Big|_{t=0}^{1} - \underbrace{\frac{1}{x} \int_0^1 \frac{1}{1+t} \sin xt\, dt}_{= o(\frac{1}{x})}$,

$\sim \frac{1}{x} \ln 2 \sin x$

<u>Ex</u>: $I(x) = \int_0^1 \frac{1}{\sqrt{t}} \cos xt\, dt, \quad x \to \infty$

$= \frac{1}{x} \int_0^1 \frac{1}{\sqrt{t}} d \sin xt$

$= \frac{1}{x} \sin x - \underbrace{\frac{1}{x}(-\frac{1}{2}) \int_0^1 \frac{1}{t^{3/2}} \sin xt\, dt}_{}$,

can't apply Riemann-Lebesgue here

the integral converges, but $\int_0^1 \frac{1}{t^{3/2}} dt$ isn't finite

as required.

Different method:

$I(x) = \int_0^\infty \frac{1}{\sqrt{t}} \cos xt\, dt - \int_1^\infty \frac{1}{\sqrt{t}} \cos xt\, dt$

(choose upper limit ∞ b/c know how to compute integral there)

$I_1(x) = \int_0^\infty \frac{1}{\sqrt{t}} \cos xt\, dt$ let $\tau = xt$

$= \int_0^\infty \sqrt{\frac{x}{\tau}} \cos\tau \frac{1}{x} d\tau$

$= \frac{1}{\sqrt{x}} \int_0^\infty \frac{1}{\sqrt{\tau}} \cos\tau\, d\tau$ let $\tau = s^2$

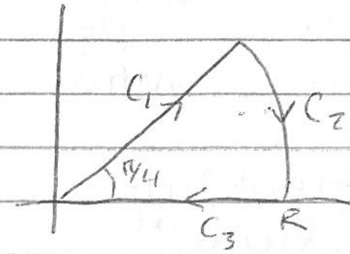
$= \frac{1}{\sqrt{x}} \int_0^\infty \frac{1}{s} \cos(s^2)\, 2s\, ds$

$= \frac{2}{\sqrt{x}} \int_0^\infty \cos(s^2)\, ds$

$= \frac{2}{\sqrt{x}} \mathrm{Re} \int_0^\infty e^{is^2} ds$

Compute $\int_0^\infty e^{is^2} ds$

let $C = C_1 \cup C_2 \cup C_3$



$\oint_C e^{iz^2} dz = 0$

$C_1: z = re^{i\pi/4}, \ r: 0 \to R$

$C_2: z = Re^{i\theta}, \ \theta: \pi/4 \to 0$

$C_3: z = r, \ r: R \to 0$

$0 = \int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz$

$\int_{-C_3} = \int_{C_1} + \int_{C_2}$

$\int_0^R e^{ir^2} dr = \int_0^R \exp(-r^2) e^{i\pi/4} dr + \underbrace{\int_{C_2} e^{iz^2} dz}_{\to 0}$

$\int_0^\infty e^{ir^2} dr = e^{i\pi/4} \int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2} e^{i\pi/4}$

So $I_1(x) = \frac{2}{\sqrt{x}} \frac{\sqrt{\pi}}{2} Re(e^{i\pi/4}) = \sqrt{\frac{\pi}{2x}}$

$I_2(x)$ is smaller:

$I_2(x) = \int_1^\infty \frac{1}{\sqrt{t}} \cos xt \, dt$ let $\tau = xt$

$= \frac{1}{\sqrt{x}} \underbrace{\int_x^\infty \frac{1}{\sqrt{\tau}} \cos \tau \, d\tau}_{\to 0 \text{ as } x \to \infty}$

Another option:

$I(x) = \int_0^1 \frac{1}{\sqrt{t}} \cos xt \, dt$ $\tau = xt$

$= \frac{1}{\sqrt{x}} \int_0^x \frac{1}{\sqrt{\tau}} \cos \tau \, d\tau$

and as $x \to \infty$, we get the same limits.

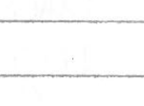
Back to our cplx integral:

$\int_{C2} e^{iz^2} dz = \int_{\pi/4}^0 \exp(iR^2 e^{2i\theta}) iRe^{i\theta} d\theta$

$= -iR \int_0^{\pi/4} \exp(iR^2(\cos 2\theta + i \sin 2\theta)) e^{i\theta} d\theta$

$\left| \int_{C2} e^{iz^2} dz \right| \le R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta$

laplace integral with $\varphi(\theta) = -\sin 2\theta$



$\int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta \sim \int_0^\varepsilon e^{-R^2 2\theta} d\theta$ let $2R^2\theta = s$

$\sim \frac{1}{2R^2} \int_0^\infty e^{-s^2} ds = \frac{1}{2R^2} \to \frac{1}{2R} \to 0$

$I(x) = \int_a^b f(t) e^{ix\psi(t)} dt \quad x \to \infty$

$= \int_a^b \frac{f(t)}{ix\psi'(t)} \, d e^{ix\psi(t)}$

$= \frac{1}{ix} \frac{f(b)}{\psi'(b)} e^{ix\psi(b)} - \frac{1}{ix} \frac{f(a)}{\psi'(a)} e^{ix\psi(a)} - \frac{1}{ix} \int_a^b \left(\frac{f(t)}{\psi'(t)} \right)' e^{ix\psi(t)} dt$

(won't work if derivative has zeros in $[a,b]$)

If $\psi'(t) \neq 0$, $a \le t \le b$, then boundary terms give asymptotics:

$I(x) \sim \frac{1}{ix} \frac{f(b)}{\psi'(b)} e^{ix\psi(b)} - \frac{1}{ix} \frac{f(a)}{\psi(a)} e^{ix\psi(a)}$

because Riemann-Lebesgue lemma states that the remaining integral $\to 0$ as $x \to \infty$

$\int_a^b g(t) e^{ix\psi(t)} dt \to 0$ as $x \to \infty$ provided that $\int_a^b |g(t)| dt < \infty$ and $\psi(t)$ is continuously differentiable and not = const for any sub-interval of $[a,b]$

What happens if $\psi'(t) = 0$ for $a \le t \le b$?

Such a "t" is called the point of stationary phase

$\to$ Can't use integration by parts

Similar to Laplace, a small sub-interval near the zero contributes the most

look at $e^{ix\psi(t)}$ near pt. of stationary phase $t=0$

$\to e^{ix[\psi(0) + \frac{1}{2}\psi''(0)t^2]}$

$\Rightarrow \cos(xt^2)$ is real part

$\cos(xt)$ oscillates rapidly: pos + neg. parts cancel

$\cos(xt^2)$ won't get this cancellation near $t=0$ (not as much, at least)

$t=a$: pt of stationary phase

$\rightarrow \psi'(a)=0 \quad \psi'(t) \neq 0$ for $t \in (a, b]$

$$I(x) = \int_a^{a+\varepsilon} f(t) e^{ix\psi(t)} dt + \int_{a+\varepsilon}^{b} f(t) e^{ix\psi(t)} dt$$

$\varepsilon = \varepsilon(x) \rightarrow 0$, but will need $\varepsilon\sqrt{x} \rightarrow \infty$ ↳ let $\varepsilon = x^{-\frac{1}{3}}$

wts: 2^{nd} integral << 1^{st} integral

$$I(x) \sim \int_a^{a+\varepsilon} f(a) e^{ix[\psi(a) + \frac{1}{2}\psi''(a)(t-a)^2]} dt + \int_{a+\varepsilon}^{b} f(t) e^{ix\psi(t)} dt$$

(let $t-a=\tau$ in 1^{st} int. 2^{nd} integral: use parts)

$$\sim f(a) e^{ix\psi(a)} \int_0^{\varepsilon} e^{ix\frac{1}{2}\psi''(a)\tau^2} d\tau + \frac{f(b)}{ix\psi'(b)} e^{ix\psi(b)} - \frac{f(a+\varepsilon)}{ix\psi'(a+\varepsilon)} e^{ix\psi(a+\varepsilon)}$$

let $s = \sqrt{\frac{x}{2}|\psi''(a)|}\,\tau$, $d\tau = \frac{1}{\sqrt{\cdots}} ds$

$$\sim \frac{f(a) e^{ix\psi(a)} \sqrt{2}}{\sqrt{x|\psi''(a)|}} \int_0^{\varepsilon\sqrt{\frac{x}{2}|\psi''(a)|}} e^{is^2 \operatorname{sgn}(\psi''(a))} ds + \text{boundary terms}$$

1^{st} boundary term $= O(\frac{1}{x})$, this $= O(\frac{1}{\sqrt{x}})$ → disregard

2^{nd} boundary term: $\psi'(a+\varepsilon) \sim \psi'(a) + \psi''(a)\varepsilon$, with $\psi'(a) \to 0$

So it's $O(\frac{1}{x\varepsilon}) = O(x^{-2/3}) << O(\frac{1}{\sqrt{x}})$ → disregard

$$\text{So } I(x) \sim \frac{f(a) e^{ix\psi(a)}}{\sqrt{\frac{x}{2}|\psi''(a)|}} e^{i\frac{\pi}{4}\operatorname{sgn}(\psi''(a))} \cdot \frac{\sqrt{\pi}}{2}$$

(Recall $\int_0^\infty e^{\pm is^2} ds = e^{\pm i\pi/4} \frac{\sqrt{\pi}}{2}$)

$I(x) = \int_a^b f(t) e^{ix\psi(t)} dt, \quad x \to \infty$ 3-10

if $\psi'(t) \neq 0$ for $a \le t \le b$ $\to$ int. by parts

$$I(x) \sim \frac{f(b)}{ix\psi'(b)} e^{ix\psi(b)} - \frac{f(a)}{ix\psi'(a)} e^{ix\psi(a)}$$

if $\psi'(a) = 0$, $\psi'(t) \neq 0$ for $a < t \le b$

$$I(x) = \int_a^{a+\varepsilon} f(t) e^{ix\psi(t)} dt + \int_{a+\varepsilon}^{b} f(t) e^{ix\psi(t)} dt$$

Taylor expand 1st integral, use parts on 2nd

$$\sim \int_a^{a+\varepsilon} f(a) e^{ix(\psi(a) + \frac{1}{2}\psi''(a)(t-a)^2)} dt$$

$$- \frac{f(a+\varepsilon)}{ix\psi'(a+\varepsilon)} e^{ix\psi(a+\varepsilon)}$$

let $s = \sqrt{\tfrac{1}{2}x|\psi''(a)|}\,(t-s)$

$$\sim \frac{f(a) e^{ix\psi(a)}}{\sqrt{\frac{1}{2}x|\psi''(a)|}} \int_0^{\infty} e^{is^2 \operatorname{sgn}(\psi''(a))} ds \sim \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}\operatorname{sgn}(\psi''(a))}$$

$$- \frac{f(a)}{ix\psi''(a)\varepsilon} e^{ix\psi(a)}$$

$$\sim \frac{f(a) \exp(ix\psi(a) + i\frac{\pi}{4}\operatorname{sgn}\psi''(a))\sqrt{\pi}}{\sqrt{2x|\psi''(a)|}}$$

if $\psi(t) \sim \psi(a) + \frac{1}{N!}\psi^{(N)}(a)(t-a)$

$$I(x) \sim \frac{f(a)\, e^{ix\psi(a) + i\frac{\pi}{2N}\operatorname{sgn}(\psi^{(N)}(a))} (N!)^{1/N}\, \Gamma(\frac{1}{N})}{|x\psi^{(N)}(a)|\, N}$$

if $\psi'(c) = 0$, $a < c < b$,

split the integral into two parts: $\int_a^c + \int_c^b$

the two sides will be equal

Ex: $I(x) = \int_0^{\pi/4} e^{ix\cos^2 t} dt, \quad x \to \infty$

$\psi(t) = \cos^2 t \quad \psi'(t) = -2\cos t \sin t = -\sin 2t$

$\to t = 0$ is a stationary point

$I(x) \sim \int_0^{\varepsilon} e^{ix(1-t^2)} dt$ let $\sqrt{x}\,t = s$

$$\sim e^{ix} \frac{1}{\sqrt{x}} \int_0^{\varepsilon\sqrt{x}} e^{-is^2} ds$$

$$\sim e^{ix} \frac{1}{\sqrt{x}} \int_0^{\infty} e^{-is^2} ds$$

$$\sim e^{ix} \frac{1}{\sqrt{x}} \frac{\sqrt{\pi}}{2} e^{-i\pi/4} = \frac{1}{2}\sqrt{\frac{\pi}{x}}\, e^{ix - i\pi/4}$$

Ex: $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin t - nt)\,dt$

$= \frac{1}{\pi} \mathrm{Re} \int_0^\pi e^{ix\sin t - int}\,dt$

prove this is $J_n(x)$:

$J_n(x)$ solves $x^2 y'' + xy' + (x^2-n^2)y = 0$

$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{n+2k} \frac{1}{k!(n+k)!}$

$x \to 0$: $\sim \left(\frac{x}{2}\right)^n \frac{1}{n!}$

let $y(x) = \int_0^\pi \cos(x\sin t - nt)\,dt$

$y'(x) = \int_0^\pi -\sin t \cdot \sin(x\sin t - nt)\,dt$ $\quad$ $u = \sin(x\sin t - nt)$, $du = (x\cos t - n)\cos(x\sin t - nt)$ $\quad$ $v = \cos t$, $dv = -\sin t$

$= 0 - \int_0^\pi \cos(x\sin t - nt)(x\cos t - n)\cos t\,dt$

$y''(x) = -\int_0^\pi \sin^2 t \cos(x\sin t - nt)\,dt$

plug into diff eq:

$x^2 y'' + xy' + (x^2-n^2)y$

$= \int_0^\pi \cos(x\sin t - nt)\left[-x^2\sin^2 t - x(x\cos^2 t - n\cos t) + (x^2-n^2)\right]dt$

$= \int_0^\pi \cos(x\sin t - nt)\left[nx\cos t - n^2\right]dt$

$= n\int_0^\pi \cos(x\sin t - nt)\left[x\cos t - n\right]dt$

let $s = x\sin t - nt$ $\quad ds = (x\cos t - n)\,dt$

$= n\int_0^{-n\pi} \cos(s)\,ds$

$= n\sin(s)\Big|_0^{n\pi}$

$= 0$ ✓

Now check behavior as $x \to 0$

$\frac{1}{\pi}\mathrm{Re}\int_0^\pi e^{-int} \sum_{k=0}^{\infty} \frac{1}{k!}(ix\sin t)^k\,dt$

$= \frac{1}{\pi}\mathrm{Re}\int_0^\pi e^{-int}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}x(e^{it}-e^{-it})\right)^k dt$

(and $\int_0^\pi e^{-int} e^{imt}\,dt = 0$ for $m \neq n$)

So leading order comes from $k=n$:

$\sim \frac{1}{\pi}\mathrm{Re}\int_0^\pi e^{-int}\frac{1}{n!}x^n\left(\frac{e^{it}}{2}\right)^n dt$

$\sim \frac{x^n}{n!\,2^n}$ ✓

As $x \to \infty$:

$\frac{1}{\pi} \mathrm{Re} \int_0^\pi e^{-int} e^{ix\sin t}$

$f(t) = e^{-int}$, $\psi(t) = \sin t$ $\psi'(t) = \cos t \to t = \pi/2$ stat ph.

$\sim \frac{1}{\pi} \mathrm{Re} \int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} e^{-in\pi/2} e^{ix(1-\frac{1}{2}(t-\pi/2)^2)} dt$

$\sim \frac{1}{\pi} \mathrm{Re}\left[e^{-in\pi/2} e^{ix} \left(\int_{-\infty}^{\infty} e^{-is^2} ds \right) \sqrt{\frac{2}{x}} \right.$

$s = \sqrt{x/2}\,(t - \pi/2)$

$\sim \frac{1}{\pi} \mathrm{Re}\left[\sqrt{\frac{2}{x}}\, e^{-in\pi/2} e^{ix} \sqrt{\pi}\, e^{-i\pi/4} \right]$

$= \sqrt{\frac{2}{\pi x}}\, \mathrm{Re}\, e^{i(x - \frac{\pi n}{2} - \pi/4)}$

$= \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right)$

$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x\sin t - nt)\, dt$, $x \to \infty$, $n \to \infty$ 3-12

let $n = x$, so they $\to \infty$ at the same rate

$J_x(x) = \frac{1}{\pi} \int_0^\pi \cos(x(\sin t - t))\, dt$

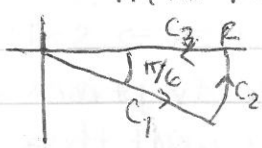
$= \frac{1}{\pi} \mathrm{Re} \int_0^\pi e^{ix(\sin t - t)} dt$, $x \to \infty$

$\psi(t) = \sin t - t$, $\psi'(t) = \cos t - 1 = 0 \to t = 0$

$\to \psi(t) \sim -\frac{1}{6} t^3$, $t \to 0$

$J_x(x) \sim \frac{1}{\pi} \mathrm{Re} \int_0^\varepsilon e^{-ix\frac{1}{6}t^3} dt$ let $s = \left(\frac{x}{6}\right)^{1/3} t$

$= \frac{1}{\pi} \left(\frac{6}{x}\right)^{1/3} \mathrm{Re} \int_0^\infty e^{-is^3} ds$



$C_1 : e^{-i\pi/6} \xi$, $0 < \xi < R$

$\int_{C_2}$ will $\to 0$

$\sim \frac{1}{\pi} \left(\frac{6}{x}\right)^{1/3} \mathrm{Re}\, e^{-i\pi/6} \int_0^\infty \exp(-i \xi^3 (-i))\, d\xi$

$= \frac{1}{\pi} \left(\frac{6}{x}\right)^{1/3} \frac{\sqrt{3}}{2} \int_0^\infty e^{-\xi^3} d\xi$ let $\tau = \xi^3$

$= \frac{1}{\pi} \left(\frac{6}{x}\right)^{1/3} \frac{1}{2}\sqrt{3}\, \frac{1}{3} \Gamma\left(\frac{1}{3}\right)$

Ex: $I(x)=\int_{-\infty}^{\infty} e^{i(3xt^2-2t^3)}dt,\ x\to\infty$

$f(t)=e^{-2it^3},\ \psi(t)=3t^2,\ \psi'(t)=6t=0\to t=0$

$f(t)\sim 1$

$I(x)\sim\int_{-\varepsilon}^{\varepsilon} e^{3ixt^2}dt \qquad \sqrt{3x}\,t=s$

$=\frac{1}{\sqrt{3x}}\int_{-\infty}^{\infty} e^{is^2}ds$

$=\sqrt{\frac{\pi}{3x}}\,e^{i\pi/4}$

This is <u>not</u> <u>correct</u>!

$\int_{\varepsilon}^{\infty} e^{3ixt^2}e^{-2it^3}dt$ integrate by parts

$=\int_{\varepsilon}^{\infty} e^{-2it^3}\frac{1}{6ixt}\,de^{3ixt^2}$

$=-e^{-2i\varepsilon^3}\frac{1}{6ix\varepsilon}e^{3ix\varepsilon^2}-\frac{1}{6ix}\int_{\varepsilon}^{\infty}\left(\frac{e^{-2it^3}}{t}\right)'e^{3ix^2}dt$

need 2^{nd} term $\to 0$ as $x\to\infty$

To use Riemann-Lebesgue, would need

$\int_{-\infty}^{\infty}\left|\left(e^{-2it^3}/t\right)'\right|dt<\infty$. Not true.

Actually it doesn't even $\to 0$.

It's the infinite limits that cause the problem.

Try substitution $t=xs \quad dt=x\,ds$

$I(x)=x\int_{-\infty}^{\infty} e^{ix^3(3s^2-2s^3)}ds$

how'd we get this sub?

$g=3xt^2-2t^3$

$dg/dt=6xt-6t^2=6t(x-t)=0\to t=0,x$

$\to t=xs$ b/c $t=x\to s=1$

$t=x$ was a stationary pt moving with x. We don't want that with infinite limits.

$\psi(s)=3s^2-2s^3 \quad \psi'(s)=6s-6s^2=6s(1-s)$

$s=0,\ s=1$ stat. points.

$I(x)\sim x\left[\int_{-\varepsilon}^{\varepsilon} e^{3ix^3s^2}ds+\int_{1-\varepsilon}^{1+\varepsilon} e^{ix^3[1-3(s-1)^2]}ds\right]$

$\psi(s)\sim\psi(1)+\frac{1}{2}\psi''(1)(s-1)^2$ near $s=1$

$=1-3(s-1)^2$

let $\tau=\sqrt{3x^3}\,s$ for 1^{st}, $\tau=\sqrt{3x^3}(s-1)$ for 2^{nd}

$\sim x\left[\frac{1}{\sqrt{3x^3}}\sqrt{\pi}e^{i\pi/4}+e^{ix^3}\frac{1}{\sqrt{3x^3}}\sqrt{\pi}e^{-i\pi/4}\right]$

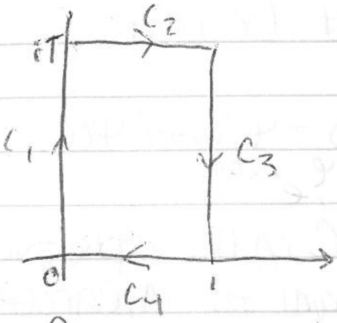
$=\sqrt{\frac{\pi}{3x}}\left(e^{i\pi/4}+e^{ix^3-i\pi/4}\right)=2\sqrt{\frac{\pi}{3x}}e^{\frac{1}{2}ix^3}\cos\left(\frac{x^3}{2}-\frac{\pi}{4}\right)$

Ex: $I(x) = \int_0^1 \ln t \, e^{ixt} \, dt$, $x \to \infty$

int. by parts won't work

neither will method of stat. phase.

consider:



$C_1: t = is$, $s: 0 \to T$

$C_3: t = 1+is$, $s: T \to 0$

$C_2: t = s+iT$, $s: 0 \to 1$

as $T \to \infty$, $\int_{C_2} \to 0$

$$\to I(x) = \int_{C_1} + \int_{C_3}$$

$$= \int_0^\infty \ln(is) \, e^{ixis} \, i\,ds - \int_0^\infty \ln(1+is) \, e^{ix(1+is)} \, i\,ds$$

$$= i\int_0^\infty \left(\ln s + i\tfrac{\pi}{2}\right) e^{-xs} \, ds - ie^{ix}\int_0^\infty \left(\ln\sqrt{1+s^2} + i\arctan s\right) e^{-xs} \, ds$$

So, to convert from complex exponential to real exponential, integrate around a different contour in the complex plane.

1st integral: 2nd term $= O(\frac{1}{x}) \to$ disregard

2nd integral: use Watson's Lemma

$\int_0^\infty f(s) e^{-xs}$: just expand $f(s)$

$\ln\sqrt{1+s^2} = \frac{1}{2}\ln(1+s^2) \sim \frac{1}{2}s^2$

$\arctan s \sim s$

leading order: $\int_0^\infty s e^{-xs} \, ds = O(\frac{1}{x^2})$

So leading order: $i\int_0^\infty \ln s \, e^{-xs} \, ds$ let $t = xs$

$$= \frac{i}{x}\int_0^\infty \ln\left(\frac{t}{x}\right) e^{-t} \, dt$$

$$= \underbrace{\frac{i}{x}\int_0^\infty \ln t \, e^{-t} \, dt}_{O(\frac{1}{x})} - \underbrace{\frac{i}{x}\int_0^\infty \ln x \, e^{-t} \, dt}_{O(\frac{1}{x}\ln x)}$$

$$\sim -i\frac{\ln x}{x}$$

Laplace integrals: $\int f(t)\, e^{x\varphi(t)}\, dt$
Fourier integrals: $\int f(t) e^{ix\psi(t)}\, dt$

Method of Steepest Descent:

$\int f(t) e^{x\rho(t)}\, dt$

where $\rho(t) = \varphi(t) + i\psi(t)$

$e^{x\rho} = e^{x\varphi} e^{ix\psi}$

integrate on C in the cplx plane
instead we want to integrate on a contour where $\psi(t)$ will be constant
Then we can just do Laplace method
(could instead make φ const, but Laplace method is more accurate than Fourier
since Laplace $\to$ neglected terms are exp smaller
Fourier $\to$ neglected terms are algebraically smaller)

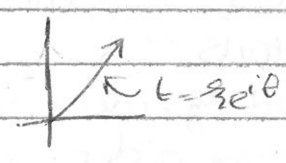
constant phase contours

$|e^{x\rho(t)}| = e^{x\varphi(t)}$

The contour where ψ is const is where φ decays the most rapidly.

From before

e^{ixt}



$$e^{ixt} = e^{ix\xi e^{i\theta}}$$
$$= e^{ix\xi(\cos\theta + i\sin\theta)}$$
$$|e^{ixt}| = e^{-x\xi\sin\theta}$$

maximize $\sin\theta$: $\theta = \pi/2$

from Calculus:

steepest descent/ascent in direction of gradient

$\nabla(e^{x\varphi(t)}) = e^{x\varphi(t)} \cdot \nabla\varphi$

$\nabla\psi$ is normal to the contour

$\nabla\varphi \cdot \nabla\psi = 0$ from Cauchy-Riemann

$\to \nabla\varphi$ tangent to contour

421-3 4 2 0 - 3

Asymptotics and Perturbation Methods

Method of Characteristics 3-31

for Linear first order PDEs
with two variables

$a(x,t)u_x + b(x,t)u_t = c(x,t)u + d(x,t)$

a,b,c,d given coefficients

$u = u(x,t)$

Recall:

$au_x + bu_t = (a,b)\cdot\nabla u$

→ directional derivative

(a,b) is like a vector field

$\left.\begin{array}{l}\frac{dx}{ds} = a(x,t)\\ \frac{dt}{ds} = b(x,t)\end{array}\right\}$ will give curve w/ (a,b) the velocity, curve described parametrically in s

$\rightarrow \frac{dx}{ds}u_x + \frac{dt}{ds}u_t = \frac{du}{ds}$

derivative of sol'n along this curve

$\rightarrow \frac{dx}{ds} = a,\ \frac{dt}{ds} = b,\ \frac{du}{ds} = cu + d$

initial condition: initial curve in x-u plane

$x = \tau,\ t = 0,\ u = f(\tau)$

$\rightarrow x = x(s,\tau),\ t = t(s,\tau),\ u = u(s,\tau)$

if we can get $s = s(x,t),\ \tau = \tau(x,t)$

can plug into u → get $u = u(s(x,t),\tau(x,t))$

→ solve first two: get curves in x-t plane

these are "characteristic curves"

then use third to get sol'n above those

Ex: $xu_x + tu_t = cu$, $c = \text{const}$, $-\infty < x < \infty$, $t > 1$

$u(x,1) = f(x)$

convert IC into parametric curve:

$x = \tau,\ t = 1,\ u = f(\tau)$

$\frac{dx}{ds} = x,\ x|_{s=0} = \tau \rightarrow x = \tau e^s$

$\frac{dt}{ds} = t,\ t|_{s=0} = 1 \rightarrow t = e^s$

$\frac{du}{ds} = cu,\ u|_{s=0} = f(\tau) \rightarrow u = f(\tau)e^{cs}$

solve for $s+\tau$ in first two:

$\tau = \frac{x}{t},\ s = \ln t$

plug into u: $u = f(\frac{x}{t})t^c$

check: $u(x,1) = f(x)$ ✓

$$x f'(\tfrac{x}{t})\tfrac{1}{t}t^c + t\left[f'(\tfrac{x}{t})(-\tfrac{x}{t^2})t^c + cf(\tfrac{x}{t})t^{c-1}\right]$$

$$= cf(\tfrac{x}{t})t^c \quad ✓$$

Ex: $a(x,t)u_x + b(x,t)u_t = 0$

$\frac{dx}{ds} = a$

$\frac{dt}{ds} = b$

$\frac{dx}{dt} = \frac{a}{b} \rightarrow f(x,t) = c$ if it can be solved.

initial curve: $x = \tau,\ t = 0,\ u = f(\tau, 0)$

sol'n: $u = f(x,t)$

Ex: $u_x + u_t = 0$

$$\left.\begin{array}{l}\frac{dx}{ds} = 1 \\ \frac{dt}{ds} = 1\end{array}\right\} \rightarrow \frac{dx}{dt} = 1 \rightarrow \begin{array}{l} x = t + c \\ x - t = c = f(x,t)\end{array}$$

Then $u = f(x,t) = x - t$

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + F(x,y,u,u_x,u_y) = 0$$

"quasi-linear"

$$\left.\begin{array}{l}\xi = \xi(x,y) \\ \eta = \eta(x,y)\end{array}\right\} \text{use these to simplify the eqn to a standard form}$$

(a) if $B^2 - AC > 0$, then the eqn reduces to

$u_{\xi\xi} - u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$

(hyperbolic)

(b) if $B^2 - AC = 0$, then

$u_{\xi\xi} = \Phi(\xi, \eta, u, u_\xi, u_\xi)$

(parabolic)

(c) if $B^2 - AC < 0$, then

$u_{\xi\xi} + u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$

(elliptic)

need change of variables to be 1-1:

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

change of variables: $\xi = \xi(x,y)$, $\eta = \eta(x,y)$

to get $\tilde{A} u_{\xi\xi} + 2\tilde{B} u_{\xi\eta} + \tilde{C} u_{\eta\eta} + \tilde{F}(\xi, \eta, u, u_\xi, u_\eta) = 0$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_{xx} = u_{\xi\xi} \xi_x^2 + u_\xi \xi_{xx} + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\eta \eta_{xx}$

$u_{yy} = u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$

$u_{xy} = u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} \xi_x \eta_y + u_\xi \xi_{xy} + u_{\xi\eta} \xi_y \eta_x$
$\quad + u_{\eta\eta} \eta_x \eta_y + u_\eta \eta_{xy}$

plug in:

$\tilde{A} = A\xi_x^2 + 2B\xi_x \xi_y + C\xi_y^2$

$\tilde{C} = A\eta_x^2 + 2B\eta_x \eta_y + C\eta_y^2$

$\tilde{B} = A\xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + C\xi_y \eta_y$

note $\tilde{B}^2 - \tilde{A}\tilde{C} = (B^2 - AC) J^2$, $J = \frac{\partial(\xi,\eta)}{\partial(x,y)}$

ⓐ $B^2 - AC > 0$

$A\varphi_x^2 + 2B\varphi_x \varphi_y + C\varphi_y^2 = 0$

$A\left(\frac{\varphi_x}{\varphi_y}\right)^2 + 2B\frac{\varphi_x}{\varphi_y} + C = 0$

$\frac{\varphi_x}{\varphi_y} = \frac{1}{A}\left(-B \pm \sqrt{B^2 - AC}\right)$

$\to A\varphi_x + (B \pm \sqrt{B^2 - AC})\varphi_y = 0$

we just learned to solve this using characteristics

$\frac{dx}{ds} = A$, $\frac{dy}{ds} = B \pm \sqrt{B^2 - AC}$

$\frac{dy}{dx} = (B \pm \sqrt{B^2 - AC})\frac{1}{A}$

$f_\pm(x,y) = \text{const}$

So, take $\xi = f_+(x,y)$, $\eta = f_-(x,y)$

$\to \tilde{A} = \tilde{C} = 0$

and $\tilde{B} \neq 0$ since $B^2 - AC > 0 \to \tilde{B}^2 - \tilde{A}\tilde{C} > 0 \to \tilde{B}^2 > 0$

can get from $u_{\xi\eta} = 0$ to $u_{\xi\xi} - u_{\eta\eta} = 0$ by letting $\xi = \alpha + \beta$, $\eta = \alpha - \beta$.

ⓑ $B^2 - AC = 0$

$\to A\varphi_x + B\varphi_y = 0$

only one sol'n: can only kill $\tilde{A}$ or $\tilde{C}$, not both.

choose either $\varphi = \xi$ or $\varphi = \eta$.

$\frac{B}{A} = \frac{C}{B}$: $B\varphi_x + C\varphi_y = 0$

say we choose η to let $\tilde{C} = 0$

Then $A\eta_x + B\eta_y = 0$

and $B\eta_x + C\eta_y = 0$

$\tilde{B} = \xi_x \underbrace{(A\eta_x + B\eta_y)}_{=0} + \xi_y \underbrace{(B\eta_x + C\eta_y)}_{=0}$

So $\tilde{B} = 0$

$\tilde{A} \neq 0$:

$\tilde{A}$ is perfect square: $\frac{1}{A}(A\xi_x + B\xi_y)^2$

if $\tilde{A} = 0$ then $A\xi_x + B\xi_y = 0$

but $A\eta_x + B\eta_y = 0$ then that would give $J = 0$ $\rightarrow\leftarrow$

(c) $B^2 - AC < 0$

$A\varphi_x + (B \pm i\sqrt{AC - B^2})\varphi_y = 0$

$\frac{dx}{ds} = A$, $\frac{dy}{ds} = B \pm i\sqrt{AC - B^2}$

$\frac{dy}{dx} = (B \pm i\sqrt{AC - B^2})\frac{1}{A}$

$f(x,y) = \text{const}$

$\varphi(x,y) = f(x,y)$, complex

choose $\xi = \operatorname{Re} f$, $\eta = \operatorname{Im} f$

then $\varphi = \xi + i\eta$

$A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2 = 0$

$A(\xi_x + i\eta_x)^2 + 2B(\xi_x + i\eta_x)(\xi_y + i\eta_y) + C(\xi_y + i\eta_y)^2 = 0$

$\rightarrow \tilde{A} - \tilde{C} + 2i\tilde{B} = 0$

$\rightarrow \tilde{A} = \tilde{C}$, $\tilde{B} = 0$

$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + F(x,y,u,u_x,u_y) = 0$ 4-2

$B^2 - AC > 0$: hyperbolic

$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$

$B^2 - AC = 0$: parabolic

$u_{\xi\xi} = \Phi$

$B^2 - AC < 0$: elliptic

$u_{\xi\xi} + u_{\eta\eta} = \Phi$

for $\xi = \xi(x,y)$, $\eta = \eta(x,y)$

$A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2 = 0$

Ex: $u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0$

$A = 1$, $B = -\sin x$, $C = -\cos^2 x$

$B^2 - AC = \sin^2 x + \cos^2 x = 1 > 0$

$\rightarrow$ hyperbolic

$\varphi_x^2 - 2\sin x\, \varphi_x\varphi_y - \cos^2 x\, \varphi_y^2 = 0$

$(\varphi_x/\varphi_y)^2 - 2\sin x(\varphi_x/\varphi_y) - \cos^2 x = 0$

$\varphi_x/\varphi_y = \sin x \pm \sqrt{\sin^2 x + \cos^2 x} = \sin x \pm 1$

$\varphi_x + (-\sin x \pm 1)\varphi_y = 0$

$dx/ds = 1$

$dy/ds = -\sin x \pm 1$

$dy/dx = -\sin x \pm 1$

$y = \cos x \pm x + \text{const.}$

$\underline{y - \cos x \pm x} = \text{const}$

"1st integral" $= \varphi$

$\xi = x + y - \cos x$

$\eta = -x + y - \cos x$

$u_x = u_\xi(1 + \sin x) + u_\eta(-1 + \sin x)$

$u_{xx} = u_{\xi\xi}(1 + \sin x)^2 + 2u_{\xi\eta}(1 + \sin x)(-1 + \sin x) + u_\xi \cos x$

$u_{\eta\eta}(-1 + \sin x)^2 + u_\eta \cos x$

$u_y = u_\xi + u_\eta$, $u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = u_{\xi\xi}(1 + \sin x) + u_{\xi\eta}(1 + \sin x) + u_{\xi\eta}(-1 + \sin x) + u_{\eta\eta}(-1 + \sin x)$

plug in:

$u_{\xi\xi}(1+\sin x)^2 - 2u_{\xi\eta}\cos^2 x + u_{\eta\eta}(-1+\sin x)^2$
$+(u_\xi + u_\eta)\cos x - 2\sin x\, u_{\xi\xi}(1+\sin x) - 2\sin x\, u_{\xi\eta}\, 2\sin x$
$-2\sin x\, u_{\eta\eta}(-1+\sin x) - \cos^2 x\, u_{\xi\xi} - 2\cos^2 x\, u_{\xi\eta}$
$-2\cos^2 x\, u_{\eta\eta} - \cos x(u_\xi + u_\eta)$

$= u_{\xi\xi}(1 + 2\sin x + \sin^2 x - 2\sin x - 2\sin^2 x - \cos^2 x)$
$+ u_{\eta\eta}(\ 0\ )$
$+ u_{\xi\eta}(-2\cos^2 x - 4\sin^2 x - 2\cos^2 x)$
$= -4u_{\xi\eta} = 0$
$\to \quad u_{\xi\eta} = 0$

Ex: $u_{xx} + x^2 u_{yy} = 0$

$A = 1, \quad B = 0, \quad C = x^2$

$B^2 - AC = -x^2 < 0$ for $x \neq 0$

$\to$ elliptic

$\varphi_x^2 + x^2\varphi_y^2 = 0$

$\varphi_x \pm i x \varphi_y = 0$

$\frac{dy}{dx} = \pm i x$

$y = \pm \frac{1}{2} i x^2 + \text{const}$

$y \pm \frac{1}{2} i x^2 = \text{const} = \varphi$

$\xi = \operatorname{Re}\varphi = y$

$\eta = \operatorname{Im}\varphi = \frac{1}{2}x^2$ $\leftarrow$ could choose + or -

$u_x = u_\eta x \qquad\qquad u_y = u_\xi$

$u_{xx} = u_{\eta\eta} x^2 + u_\eta \qquad u_{yy} = u_{\xi\xi}$

$\to \quad u_{\eta\eta} x^2 + u_\eta + x^2 u_{\xi\xi} = 0$

$2\eta(u_{\xi\xi} + u_{\eta\eta}) + u_\eta = 0$

$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{2\eta} u_\eta = 0$ okay since $x \neq 0$

$P(u,\varepsilon)=0$

Reduced problem: $P(u_0, 0)=0$

Solve both & get $u(\varepsilon)$ and u_0

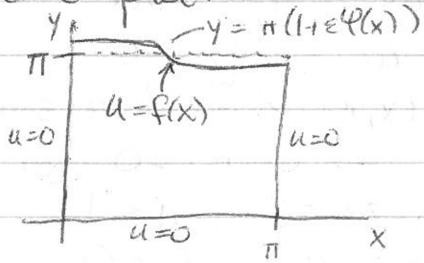
P is a regular problem if $\lim_{\varepsilon\to 0} u(\varepsilon) = u_0$

otherwise it's singular

Regular Perturbation Problems for PDEs

Ex: Steady-state temperature distribution in a nearly square plate

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$\varphi(x) = \cos x$

$f(x) = \sin x$

$u(x,0)=0,\ 0<x<\pi\ ;\ u(x, \pi(1+\varepsilon\varphi(x))) = f(x),\ 0<x<\pi$

$u(0,y)=0,\ 0<y<\pi(1+\varepsilon\varphi(0)),\ u(\pi,y)=0,\ 0<y<\pi(1+\varepsilon\varphi(\pi))$

transform domain into a perfect square:

$x=x,\ z=\frac{y}{1+\varepsilon\varphi(x)},\ u(x,y)=v(x,z)\quad \left(z=\frac{y}{1+\varepsilon\cos x}\right)$

$$u_x = v_x + v_z\frac{(\varepsilon y \sin x)}{(1+\varepsilon\cos x)^2}$$

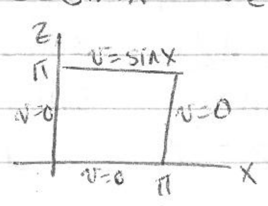
$$u_{xx} = v_{xx} + 2v_{xz}\frac{\varepsilon y\sin x}{(1+\varepsilon\cos x)^2} + v_{zz}\frac{\varepsilon^2 y^2\sin^2 x}{(1+\varepsilon\cos x)^4} + v_z\frac{\varepsilon y\cos x}{(1+\varepsilon\cos x)^2} + v_z\frac{2\varepsilon^2 y\sin^2 x}{(1+\varepsilon\cos x)^3}$$

$$u_y = \frac{v_z}{1+\varepsilon\cos x},\quad u_{yy} = \frac{v_{zz}}{(1+\varepsilon\cos x)^2}$$

$(1+\varepsilon\cos x)^p = 1+\varepsilon p\cos x + O(\varepsilon^2)$

plug in & substitute $y=(1+\varepsilon\cos x)z$:

$$v_{xx} + 2v_{xz}\,\varepsilon z\sin x + v_z\,\varepsilon z\cos x + v_{zz}(1-2\varepsilon\cos x) + O(\varepsilon^2) = 0$$



$$v_{xx}+v_{zz}+\varepsilon(2v_{xz}z\sin x+v_z z\cos x-2v_{zz}\cos x)+O(\varepsilon^2)=0$$

$$v\sim v_0(x,z)+\varepsilon v_1(x,z)$$

$$O(1):\begin{cases}\dfrac{\partial^2 v_0}{\partial x^2}+\dfrac{\partial^2 v_0}{\partial z^2}=0 & v_0(x,0)=0,\ v_0(x,\pi)=\sin x\\ & v_0(0,z)=v_0(\pi,z)=0\end{cases}$$

$$O(\varepsilon):\begin{cases}\dfrac{\partial^2 v_1}{\partial x^2}+\dfrac{\partial^2 v_1}{\partial z^2}=-\left[2z\sin x\dfrac{\partial^2 v_0}{\partial x\partial z}+z\cos x\dfrac{\partial v_0}{\partial z}-2\cos x\dfrac{\partial^2 v_0}{\partial z^2}\right]\\ v_1(x,0)=v_1(x,\pi)=v_1(0,z)=v_1(\pi,z)=0\end{cases}$$

$O(1)$: partial eigenfunction expansion

$$v_0(x,z)=\sum_{n=1}^{\infty}a_n(z)\sin nx$$

$$\sum_{n=1}^{\infty}(a_n''-n^2a_n)\sin nx=0$$

$$\rightarrow a_n''-n^2a_n=0,\quad 0<z<\pi,\ a_n(0)=0$$

$$v_0(x,\pi)=\sum_{n=1}^{\infty}a_n(\pi)\sin nx=\sin x$$

$$\rightarrow a_n(\pi)=0 \text{ for } n\neq 1,\ a_1(\pi)=1$$

$$\rightarrow a_n(z)\equiv 0,\quad n\neq 1$$

$$n=1:\ a_1''-a_1=0,\ a_1(0)=0,\ a_1(\pi)=1$$

$$a_1(z)=\sinh z/\sinh\pi$$

$$\rightarrow v_0(x,z)=\frac{\sinh z}{\sinh\pi}\sin x$$

$$O(\varepsilon):\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial z^2}=-F$$

$$F=2z\sin x\frac{\cosh z}{\sinh\pi}\cos x+z\cos x\frac{\cosh z}{\sinh\pi}\sin x$$

$$-2\cos x\frac{\sinh z}{\sinh\pi}\sin x$$

$$=\frac{\sin 2x}{\sinh\pi}\left[z\cosh z+\tfrac{1}{2}z\cosh z-\sinh z\right]$$

$$=\frac{\sin 2x}{\sinh\pi}\left[-\sinh z+\tfrac{3}{2}z\cosh z\right]$$

$v_1=\sum_{n=1}^{\infty}b_n(z)\sin nx$ all b_n will $\equiv 0$ except $n=2$

$$v_1=b(z)\sin 2x/\sinh\pi$$

$$b''\frac{\sin 2x}{\sinh\pi}-4b\frac{\sin 2x}{\sinh\pi}=\frac{\sin 2x}{\sinh\pi}\left[\sinh z-\tfrac{3}{2}z\cosh z\right)$$

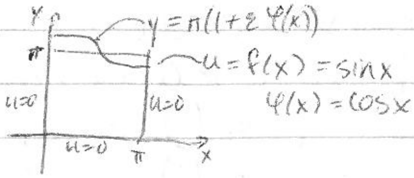
$$b''-4b=\sinh z-\tfrac{3}{2}z\cosh z$$

solve ODE, apply BC: $b(0) = b(\pi) = 0$

$b(z) = C_1 \cosh 2z + C_2 \sinh 2z + \frac{1}{2} z \cosh z$

(after method of undetermined coefficients)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ 4-7



$u(x,y) = v(x,z)$, $z = \frac{y}{1+\varepsilon\psi(x)}$

$v(x,z) \sim v_0(x,z) + \varepsilon v_1(x,z)$

$O(1)$: $\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial z^2} = 0$, $0 \le x, z \le \pi$ $\quad v(x,\pi) = \sin x$

$v(x,0) = v(0,y) = v(\pi,y) = 0$

$v_0(x,z) = \frac{\sinh z}{\sinh \pi} \cdot \sin x$

$O(\varepsilon)$: $\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = -F$ $\quad v_1 = 0$ at boundary

$-F = \frac{\sin 2x}{\sinh \pi}\left[\sinh z - \frac{3}{2} z \cosh z\right]$

$v_1(x,z) = \frac{\sin 2x}{\sinh \pi}\left[\frac{1}{2} z \cosh z - \frac{\pi}{4 \sinh \pi} \sinh 2z\right]$

$\rightarrow v(x,z) \sim \frac{\sinh z}{\sinh \pi} \sin x + \varepsilon \frac{\sin 2x}{\sinh \pi}\left[\frac{z \cosh z}{2} - \frac{\pi \sinh 2z}{4 \sinh \pi}\right]$

$\rightarrow u(x,y) = \frac{\sinh\left(\frac{y}{1+\varepsilon \cos x}\right)}{\sinh \pi} \sin x + \frac{\varepsilon \sin 2x}{\sinh \pi}\left[\frac{y \cosh y}{2} - \frac{\pi \sinh 2y}{4 \sinh \pi}\right] + O(\varepsilon^2)$

$= \frac{\sinh y \sin x}{\sinh \pi} - \frac{\pi \varepsilon}{4 \sinh^2 \pi} \sinh 2y \sin 2x + O(\varepsilon^2)$

we expanded $\sinh\left(\frac{y}{1+\varepsilon \cos x}\right)$ + it canceled w/ cosh term

general perturbed region:

solve on $0 \rightarrow 1$, change BC

$u(1+\varepsilon) = 0 \rightarrow u(1) + \varepsilon u'(1) + \frac{1}{2}\varepsilon^2 u''(1) + \dots = 0$

Taylor expand about 0

Ex: Flow past a nearly circular cylinder

2-D potential flow

potential φ

$$\nabla^2 \varphi \equiv \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0$$

for a perfect circle: $r > a, \quad 0 \le \theta < 2\pi$

perturbed: $r > a(1+\varepsilon\cos\theta), \quad 0 \le \theta < 2\pi$

as $r \to \infty$: $\varphi = U r\cos\theta + o(1)$

$= Ux + o(1)$

$r = a(1+\varepsilon\cos\theta)$: $\frac{\partial\varphi}{\partial n} = 0$

$\varepsilon = 0$: $\frac{\partial^2\varphi_0}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi_0}{\partial\theta^2} = 0, \quad r > a, \quad 0 \le \theta < 2\pi$

$\left.\frac{\partial\varphi_0}{\partial r}\right|_{r=a} = 0, \quad r \to \infty: \varphi_0 = U r\cos\theta + o(1)$

$\varphi_0(r,\theta) = R(r)\Theta(\theta)$

$\frac{r^2 R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = 0$

$\Theta_n(\theta) = a_n\cos n\theta + b_n \sin n\theta, \quad n = 1, 2, \ldots$

$\Theta_0(\theta) = a_0$

$r^2R'' + rR' - n^2 R = 0$

$R(r) = r^{\pm n}$ or for $n=0$: d_0/r

$\to \varphi_0(r,\theta) = \sum_{n=1}^{\infty}(c_n r^n + d_n r^{-n})(a_n\cos n\theta + b_n \sin n\theta)$

$+ (c_0 + d_0 \ln r)a_0$

condition as $r \to \infty$ requires:

$c_n = 0$ for $n = 2, 3, \ldots$

$c_0 = d_0 = 0, \quad b_0 = 0$

$\to \varphi_0(r,\theta) = (a_1\cos\theta + b_1\sin\theta)(c_1 r + d_1 \frac{1}{r})$

$+ \sum_{n=2}^{\infty} d_n r^{-n}(a_n\cos n\theta + b_n \sin n\theta)$

$\left.\frac{\partial\varphi_0}{\partial r}\right|_{r=a} = 0$: $(c_1 - d_1\frac{1}{a^2})(a_1\cos\theta + b_1\sin\theta)$

$+ \sum_{n=2}^{\infty} -n d_n a^{-n-1}(a_n\cos n\theta + b_n\sin n\theta) = 0$

$\to d_n = 0, \quad c_1 = d_1/a^2$

$\to \varphi_0(r,\theta) = (d_1 r/a^2 + d_1\frac{1}{r})(a_1\cos\theta)$

as $r \to \infty \sim U r\cos\theta$

$\to \varphi_0(r,\theta) = U(r + a^2/r)\cos\theta$

$$\frac{\partial \varphi}{\partial n}\Big|_{r=a(1+\varepsilon\cos\theta)} = 0$$

$$\hat{n}\cdot\nabla\varphi\,\Big|_{r=a(1+\varepsilon\cos\theta)} = 0$$

$$\hat{n} = \frac{\nabla[r-a(1+\varepsilon\cos\theta)]}{|\nabla[r-a(1+\varepsilon\cos\theta)]|}\Bigg|_{r=a(1+\varepsilon\cos\theta)}$$

and $\nabla = \hat{e}_r\,\partial/\partial r + \hat{e}_\theta \frac{1}{r}\partial/\partial\theta$

$\hat{e}_r = (\cos\theta, \sin\theta)$, $\hat{e}_\theta = (-\sin\theta, \cos\theta)$

$$\hat{n} = \frac{\hat{e}_r + \hat{e}_\theta \frac{1}{r} a\varepsilon\sin\theta}{|\hat{e}_r + \hat{e}_\theta\frac{1}{r}a\varepsilon\sin\theta|}\Bigg|_{r=a(1+\varepsilon\cos\theta)}$$

$$= \frac{\hat{e}_r + \frac{a\varepsilon\sin\theta}{a(1+\varepsilon\cos\theta)}\hat{e}_\theta}{|\hat{e}_r + \cdots \quad \hat{e}_\theta|}$$

$$= \frac{\hat{e}_r + \varepsilon\sin\theta\,\hat{e}_\theta + O(\varepsilon^2)}{\sqrt{1+\varepsilon^2\sin^2\theta/(1+\varepsilon\cos\theta)^2}}$$

$$= \hat{e}_r + \varepsilon\sin\theta\,\hat{e}_\theta + O(\varepsilon^2)$$

$$\hat{n}\cdot\nabla\varphi\,|_{r=a(1+\varepsilon\cos\theta)}$$

$$= (\hat{e}_r + \varepsilon\sin\theta\,\hat{e}_\theta)\cdot\left(\hat{e}_r\frac{\partial\varphi}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)\Big|_{r=a(1+\varepsilon\cos\theta)}$$

$$= \frac{\partial\varphi}{\partial r} + \frac{\varepsilon\sin\theta}{a(1+\varepsilon\cos\theta)}\frac{\partial\varphi}{\partial\theta} + O(\varepsilon^2)\ \Big|_{r=a(1+\varepsilon\cos\theta)}$$

$$= \frac{\partial\varphi}{\partial r}\Big|_{r=a(1+\varepsilon\cos\theta)} + \frac{\varepsilon}{a}\sin\theta\frac{\partial\varphi}{\partial\theta}\Big|_{r=a(1+\varepsilon\cos\theta)}$$

$$= \frac{\partial\varphi}{\partial r}\Big|_{r=a} + \frac{\partial^2\varphi}{\partial r^2}\Big|_{r=a} a\varepsilon\cos\theta + \frac{\varepsilon}{a}\sin\theta\frac{\partial\varphi}{\partial\theta}\Big|_{r=a} + O(\varepsilon^2)$$

$$\rightarrow \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0, \quad r>a,\ 0\le\theta<2\pi$$

$r=a$: $\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2}a\varepsilon\cos\theta + \frac{\varepsilon}{a}\sin\theta\frac{\partial\varphi}{\partial\theta} + O(\varepsilon^2) = 0$

$r\to\infty$: $\varphi \sim Ur\cos\theta + o(1)$

$\varphi \sim \varphi_0 + \varepsilon\varphi_1$

$O(1)$: $\nabla^2\varphi_0 = 0$, $r>a$, $0\le\theta<2\pi$, $\frac{\partial\varphi_0}{\partial r}\big|_{r=a} = 0$, $r\to\infty$: $\varphi_0 \sim Ur\cos\theta + o(1)$

$\varphi_0 = U(r + a^2/r)\cos\theta$ (from before)

$O(\varepsilon)$:
$$\begin{cases} \nabla^2\varphi_1 = 0, \quad r>a,\ 0\le\theta<2\pi \\ \frac{\partial\varphi_1}{\partial r}\Big|_{r=a} = -a\cos\theta\frac{\partial^2\varphi_0}{\partial r^2}\Big|_{r=a} - \frac{1}{a}\sin\theta\frac{\partial\varphi_0}{\partial\theta}\Big|_{r=a} \\ r\to\infty: \varphi_1\to 0 \end{cases}$$

$$\rightarrow \frac{\partial\varphi_1}{\partial r}\Big|_{r=a} = -a\cos\theta\, U\cos\theta\,\frac{2}{a} + \frac{\sin\theta}{a}U\sin\theta\, 2a$$

$$= -2U\cos^2\theta + 2U\sin^2\theta$$

$$= -4U\cos 2\theta$$

$$\rightarrow \varphi_1 = (c_2 r^2 + d_2\tfrac{1}{r^2})\cos 2\theta$$

need $c_2 = 0$ since as $r\to\infty$, $\varphi_1\to 0$

$\rightarrow \varphi_1 = d_2 \frac{1}{r^2} \cos 2\theta$

$\frac{\partial \varphi_1}{\partial r}\Big|_{r=a} = -2 d_2 \frac{1}{r^3} \cos 2\theta \Big|_{r=a}$

$= -2 d_2 \frac{1}{a^3} \cos 2\theta = -4U \cos 2\theta$

$d_2 = 2Ua^3$

$\rightarrow \varphi_1 = 2Ua^3 \frac{1}{r^2} \cos 2\theta$

$\rightarrow \varphi \sim U(r + a^2/r)\cos\theta + \varepsilon\, 2Ua^3 \frac{1}{r^2} \cos 2\theta$

4-9 $\nabla^2 \varphi = 0$, $r > a(1+\varepsilon\cos\theta)$, $0 \le \theta < 2\pi$

$\partial\varphi/\partial n \big|_{r=a(1+\varepsilon\cos\theta)} = 0$, $r=\infty$: $\varphi \sim Ur\cos\theta + o(1)$

reformulated: $\partial\varphi/\partial r\big|_{r=a} + \varepsilon\left[a\cos\theta \frac{\partial^2\varphi}{\partial r^2}\Big|_{r=a} + \sin\theta \frac{1}{a} \frac{\partial\varphi}{\partial\theta}\Big|_{r=a}\right] + O(\varepsilon^2) = 0$

$\varphi \sim \varphi_0 + \varepsilon\varphi_1$

$\rightarrow \varphi = U(r + \frac{a^2}{r})\cos\theta + \varepsilon U \frac{a^3}{r^2} \cos 2\theta + O(\varepsilon^2)$

could have found this without perturbation analysis

$r = \frac{a}{1-\varepsilon\cos\theta}$

$r - \varepsilon r\cos\theta = a$

$x^2 + y^2 = (a + \varepsilon x)^2$

$x^2(1-\varepsilon^2) - 2a\varepsilon x + y^2 = a^2$

$(1-\varepsilon^2)\left[x^2 - \frac{2a\varepsilon}{1-\varepsilon^2}x + \frac{a^2\varepsilon^2}{(1-\varepsilon^2)^2}\right] - \frac{a^2\varepsilon^2}{1-\varepsilon^2} + y^2 = a^2$

$(1-\varepsilon^2)(x - \frac{a\varepsilon}{1-\varepsilon^2})^2 + y^2 = a^2 \frac{1}{1-\varepsilon^2}$

$\frac{(1-\varepsilon^2)^2}{a^2}(x - \frac{a\varepsilon}{1-\varepsilon^2})^2 + \frac{1-\varepsilon^2}{a^2} y^2 = 1$

$\left(\frac{x - \frac{a\varepsilon}{1-\varepsilon^2}}{\frac{a}{1-\varepsilon^2}}\right)^2 + \left(\frac{y}{\frac{a}{\sqrt{1-\varepsilon^2}}}\right)^2 = 1 \rightarrow$ ellipse

and $\frac{1}{1-\varepsilon\cos\theta} = 1 + \varepsilon\cos\theta + O(\varepsilon^2)$, so our shape differs from an ellipse only by $O(\varepsilon^2)$

But if you look a the ellipse eqn + neglect $O(\varepsilon^2)$, it's just a circle translated to the right by $a\varepsilon$.

Could we have just used the circle (unperturbed) solution to find this?

$\varphi = U(r+\frac{a^2}{r})\cos\theta = U(x+\frac{a^2x}{x^2+y^2})$

Translated solution:

$$U\left(x-a\varepsilon + \frac{a^2(x-a\varepsilon)}{(x-a\varepsilon)^2+y^2}\right)$$

$$= U\left(x-a\varepsilon+\frac{a^2x(1-\frac{a\varepsilon}{x})}{(x^2+y^2)\left[1-\frac{2a\varepsilon x}{x^2+y^2}+O(\varepsilon^2)\right]}\right)$$

$$= U\left(x-a\varepsilon+\frac{a^2x}{x^2+y^2}\left[1-\frac{a\varepsilon}{x}+\frac{2a\varepsilon x}{x^2+y^2}\right]\right)+O(\varepsilon^2)$$

$$\sim U\left(x-a\varepsilon+\frac{a^2x}{x^2+y^2}+\left(\frac{a^2x}{x^2+y^2}\right)\left(\frac{a\varepsilon}{x}\right)\left(-1+\frac{2x^2}{x^2+y^2}\right)\right)$$

$$\sim U\left(x-a\varepsilon+\frac{a^2x}{x^2+y^2}+\frac{a^3\varepsilon}{x^2+y^2}\left(\frac{x^2-y^2}{x^2+y^2}\right)\right)$$

$$\sim U\left(x-\frac{a^2x}{x^2+y^2}-a\varepsilon+\frac{a^3\varepsilon}{r^2}\cos 2\theta\right)$$

This differs by the $-a\varepsilon$ term from our prev. sol'n
But it's just a constant, and since we were working with a potential, it doesn't matter.

Ex: $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$, $r<1$, $0\le\theta<2\pi$

$u(1,\theta)=f(\theta)$

$$\rightarrow u(r,\theta)=\int_0^{2\pi}\frac{(1-r^2)f(\theta_0)\,d\theta_0}{1+r^2-2r\cos(\theta-\theta_0)}$$

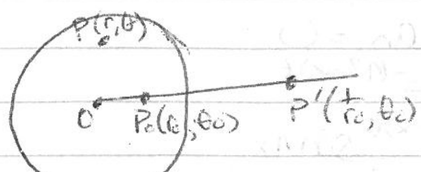
$\nabla^2 g(r,\theta;r_0,\theta_0) = -\delta(r,\theta;r_0,\theta_0)$, $g|_{r=1}=0$

$$\rightarrow u(r,\theta) = \frac{-1}{2\pi}\int_0^{2\pi}\frac{\partial g}{\partial r_0}(r,\theta;r_0,\theta_0)\Big|_{r_0=1}f(\theta_0)\,d\theta_0$$

$\nabla^2 g_0(r,\theta;r_0,\theta_0) = -\delta(r,\theta;r_0,\theta_0)$ "free space green's function"

$g_0 = -\frac{1}{2\pi}\ln|\overrightarrow{PP_0}|$

distance between $P(r,\theta)$ + $P_0(r_0,\theta_0)$



$$g(r,\theta;r_0,\theta_0) = -\frac{1}{2\pi}\ln|\overrightarrow{PP_0}| + \frac{1}{2\pi}\ln|\overrightarrow{PP'}| + \frac{1}{2\pi}\ln|\overrightarrow{OP_0}|$$

and $|\overrightarrow{PP_0}| = \sqrt{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}$

will give same result

$$\nabla^2 g(r,\theta;r_0,\theta_0) = -\delta(r,\theta;r_0,\theta_0)$$

$r < 1 + \varepsilon\varphi(\theta)$ ($\varphi(\theta)$ 2π-periodic)

$0 \le \theta < 2\pi$, $g|_{r=1+\varepsilon\varphi(\theta)} = 0$

$\rightarrow g|_{r=1} + \partial g/\partial r|_{r=1}\varepsilon\varphi(\theta) + O(\varepsilon^2) = 0$ ← Taylor expand BC

$\nabla^2 g(r,\theta;r_0,\theta_0) = -\delta(r,\theta;r_0,\theta_0)$, $r<1$

$g|_{r=1} + \partial g/\partial r|_{r=1}\varepsilon\varphi(\theta) = 0$

$g = g^{(0)} + \varepsilon g^{(1)}$

$O(1)$: $\nabla^2 g^{(0)} = -\delta(r,\theta;r_0,\theta_0)$, $g^{(0)}|_{r=1} = 0$

we already solved this.

$O(\varepsilon)$: $\nabla^2 g^{(1)} = 0$, $g^{(1)}|_{r=1} + \frac{\partial g^{(0)}}{\partial r}|_{r=1}\varphi(\theta) = 0$

Recall $u(r,\theta) = \int_0^{2\pi} \frac{(1-r^2)f(\theta')}{1+r^2-2r\cos(\theta-\theta')}d\theta$.

So sub in $f = -\varphi(\theta)\frac{\partial g^{(0)}}{\partial r}|_{r=1}$

$u_t = u_{xx} + \alpha u$, $\alpha = \text{const} > 0$

$0 < x < \pi$, $t > 0$

BC: $u(0,t) = u(\pi,t) = 0$

IC: $u(x,0) = f(x)$

which will win, the heat source (strength given by α) or the cold ends (BC)?

$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin nx$

$\sum_{n=1}^{\infty} a_n' \sin nx = \sum_{n=1}^{\infty} a_n(t)(-n^2)\sin nx + \alpha\sum_{n=1}^{\infty} a_n \sin nx$

$\rightarrow \sum_{n=1}^{\infty}[a_n' + (n^2-\alpha)a_n]\sin nx = 0$

$a_n' + (n^2-\alpha)a_n = 0$

$a_n(t) = c_n e^{-(n^2-\alpha)t}$

$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n^2-\alpha)t}\sin nx$

$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

as $t\to\infty$, the behaviour of u depends on α

$u = c_1 e^{(\alpha-1)t}\sin x + c_2 e^{(\alpha-4)t}\sin 2x + \dots$

$\alpha < 1$: $u(x,t)\to 0$ $\alpha > 1$: $u(x,t)\to\infty$

$\alpha = 1$: $u(x,t) = c_1 \sin x$

$u_t = u_{xx} + u + \varepsilon F(u)$

$0 < x < \pi, \; t > 0$

$u(0,t) = u(\pi,t) = 0$

$u(x,0) = \sin x$

Assume $F(u) > 0, \; \varepsilon > 0$

Does sol'n $\to \infty$ as we would expect?

$O(1)$: $\frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial x^2} + u_0$ BC: $u_0(0,t) = u_0(\pi,t) = 0$

IC: $u_0(x,0) = \sin x$

$u_0(x,t) = \sin x$

$O(\varepsilon)$: $\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} + F(u_0)$ BC: $u_1(0,t) = u_1(\pi,t) = 0$

IC: $u_1(x,0) = 0$

$u_1(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin nx$

$\Rightarrow \sum_{n=1}^{\infty} a_n' + (n^2-1) a_n \sin nx = F(u_0)$

$F(\sin x) = \sum_{n=1}^{\infty} f_n \sin nx$

$f_n = \frac{2}{\pi} \int_0^{\pi} F(\sin x) \sin nx \, dx$

$a_n' + (n^2-1) a_n = f_n, \quad a_n(0) = 0$

$n \neq 1$: $a_n(t) = c_n e^{-(n^2-1)t} + \frac{f_n}{n^2-1}$

$= \frac{f_n}{n^2-1} \left[1 - e^{-(n^2-1)t} \right]$

$n = 1$: $a_1(t) = f_1 t$

which $\to \infty$ as $t \to \infty$. (for $f_1 = \frac{2}{\pi} \int_0^{\pi} f(\sin x) \sin x \, dx > 0$)

(always)

★★★
★ 4-14 $\varepsilon y'' + a(x)y' + b(x)y = 0$, $\quad 0 < x < 1$, $y(0) = A$, $y(1) = B$
★★★ $\quad 0 < \varepsilon \ll 1$.

$\varepsilon = 0$: reduced problem

$$a(x)y' + b(x)y = 0$$

will not be valid for all of $[0,1]$

the sign of $a(x)$ determines where reduced problem is valid.

Ex: $\varepsilon(u_t + u_x) + u = \sin t$, $\quad -\infty < x < \infty$, $t > 0$

$u(x,0) = f(x)$ (given)

reduced: $u = \sin t$

doesn't satisfy IC (unless $f(x) \equiv 0$)

$\to$ leading order outer sol'n

exact sol'n: method of characteristics

write IC parametrically:

$x = \tau$, $t = 0$, $u = f(\tau)$ $\leftarrow$ for $s = 0$

$\frac{dx}{ds} = \varepsilon$, $\quad x|_{s=0} = \tau \quad \to \quad x = \varepsilon s + \tau$

$dt/ds = \varepsilon$, $\quad t|_{s=0} = 0 \quad \to \quad t = \varepsilon s$

$du/ds + u = \sin t = \sin \varepsilon s$, $\quad u|_{s=0} = f(\tau)$

$u_p = A \sin \varepsilon s + B \cos \varepsilon s$

$A\varepsilon \cos \varepsilon s - B\varepsilon \sin \varepsilon s + A \sin \varepsilon s + B \cos \varepsilon s = \sin \varepsilon s$

$\to A\varepsilon + B = 0$, $\quad A - B\varepsilon = 1$

$\to A = \frac{1}{1+\varepsilon^2}$, $\quad B = -\frac{\varepsilon}{1+\varepsilon^2}$

$$u = \frac{\sin \varepsilon s - \varepsilon \cos \varepsilon s}{1+\varepsilon^2} + c e^{-s}$$

$u|_{s=0} = -\frac{\varepsilon}{1+\varepsilon^2} + c = f(\tau)$

$$u = \frac{\sin \varepsilon s - \varepsilon \cos \varepsilon s}{1+\varepsilon^2} + \left(f(\tau) + \frac{\varepsilon}{1+\varepsilon^2}\right) e^{-s}$$

and $\tau = x - t$, $s = t/\varepsilon$

$$u = \frac{\sin t - \varepsilon \cos t}{1+\varepsilon^2} + \left(f(x-t) + \frac{\varepsilon}{1+\varepsilon^2}\right) e^{-t/\varepsilon}$$

approx. sol'n:

outer: $u \sim u_0(x,t) + \varepsilon u_1(x,t)$

$u_0 = \sin t$

inner: let $u(x,t) = U(x,\tau)$, $t = \delta\tau$, $\delta = \delta(\varepsilon) \ll 1$

$u_t = U_\tau \frac{1}{\delta}$, $u_x = U_x$

$\rightarrow \varepsilon(\underbrace{U_\tau \tfrac{1}{\delta}}_{\text{small}} + U_x) + \underbrace{U}_{O(1)} = \underbrace{\sin \delta\tau}_{\text{small}}$

$\rightarrow \varepsilon/\delta = O(1) \rightarrow \varepsilon = \delta$

$U_\tau + \varepsilon U_x + U = \sin \varepsilon\tau$, $U(x,0) = f(x)$

$U \sim U_0 + \varepsilon U_1$

$U_{0\tau} + U_0 = 0$, $U_0(x,0) = f(x)$

$\rightarrow U_0 = f(x)e^{-\tau}$

matching: should be automatic.

$U|_{\tau\to\infty} = u|_{t\to 0}$ yes (they are both zero)

$u(x,t) \sim \sin t + f(x)e^{-t/\varepsilon}$

$O(\varepsilon)$:

outer: $u_{0t} + u_{0x} + u_1 = 0$

$\cos t + u_1 = 0 \rightarrow u_1 = -\cos t$

$u \sim \sin t - \varepsilon \cos t$

inner: $U_{1\tau} + U_1 = -U_{0x} + \tau$, $U_1(x,0) = 0$

$= -f'(x)e^{-\tau} + \tau$

$U_1(x,\tau) = \tau - 1 - \tau f'(x)e^{-\tau} + ce^{-\tau} \rightarrow c = 1$

$U(x,\tau) \sim f(x)e^{-\tau} + \varepsilon(\tau - 1 + e^{-\tau}(1 - \tau f'(x)))$

matching: Van Dyke's 2-2

write outer sol'n in terms of inner variable, then expand in ε, keeping $O(\varepsilon)$ terms.

$u \rightarrow \sin\varepsilon\tau - \varepsilon\cos\varepsilon\tau \sim \varepsilon\tau - \varepsilon + O(\varepsilon^2)$

write inner sol'n in terms of outer variable, then expand in ε, keeping $O(\varepsilon)$ terms

$U \rightarrow f(x)e^{-t/\varepsilon} + \varepsilon[t/\varepsilon - 1 + (1 - t/\varepsilon f'(x))e^{-t/\varepsilon}]$

$t - \varepsilon + O(e^{-t/\varepsilon})$

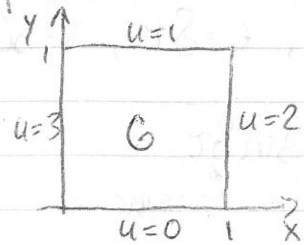
compare: equivalent. ✓ "common part" $= t - \varepsilon$

$u(x,t) \sim \sin t - \varepsilon\cos t + [\underbrace{f(x) - tf'(x)}_{\text{Taylor expansion of } f(x-t) \text{ for small } t} + \varepsilon]e^{-t/\varepsilon}$

$$\varepsilon L u = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}, \quad x, y \in G$$

$$Lu \equiv A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}$$

Suppose $B^2 - AC < 0$ (elliptic)



(Dirichlet problem)

reduced problem: $\varepsilon = 0$

$\xi = bx - ay = \text{const}$

from $\frac{dx}{ds} = a \quad \frac{dy}{ds} = b \to \frac{dy}{dx} = \frac{b}{a} \quad y = \frac{b}{a}x + c$, etc.

characteristics where sol'n is const.

can't possibly satisfy all the BC.

different example:

$$\frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y}$$

back to first example:

$$\varepsilon\left[A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}\right] = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$$

$B^2 - AC < 0$, $\quad u|_{\partial G} = u_B$, $\quad \{x, y\} \in G$

Assume $A, C > 0$ (they must have the same sign)

$\varepsilon = 0$: $a\frac{\partial u_0}{\partial x} + b\frac{\partial u_0}{\partial y} = 0$

$\xi = bx - ay = \text{const}$ "sub-characteristics"

rewrite eqn with ξ + $\eta = ax + by \leftarrow \perp$ to ξ

(ξ is $\perp$ to characteristics, η is $\parallel$ to characteristics)

$u_x = bu_\xi + au_\eta, \quad u_y = -au_\xi + bu_\eta$

$u_{xx} = b^2 u_{\xi\xi} + 2ab u_{\xi\eta} + a^2 u_{\eta\eta}$

$u_{xy} = -ab u_{\xi\xi} + b^2 u_{\xi\eta} - a^2 u_{\xi\eta} + ab u_{\eta\eta}$

$u_{yy} = a^2 u_{\xi\xi} - 2ab u_{\xi\eta} + b^2 u_{\eta\eta}$

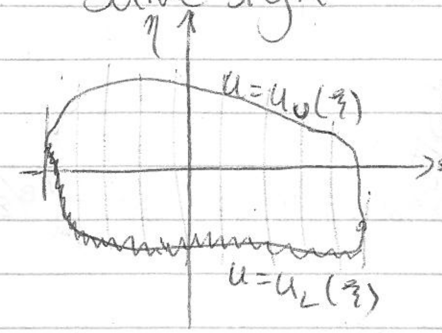
$\to \varepsilon\left[A_1 u_{\xi\xi} + 2B_1 u_{\xi\eta} + C_1 u_{\eta\eta}\right] = u_\eta$

$A_1 = \frac{1}{a^2+b^2}\left[b^2 A - 2ab B + a^2 C\right]$

$B_1 = \frac{1}{a^2+b^2}\left[ab A + (b^2 - a^2) B - ab C\right]$

$C_1 = \frac{1}{a^2+b^2}\left[a^2 A + 2ab B + b^2 C\right]$

$B_1^2 - A_1C_1 < 0$ (since change of variables was 1-1)
check sign of $A_1 + C_1$:
$A_1 + C_1 = A + C$ + $A, C > 0$, $A_1 + C_1$ have
same sign $\rightarrow$ $A_1, C_1 > 0$



(characteristics are vertical)

η derivatives will be large
BL will be near upper or lower
boundaries

$\varepsilon[Au_{xx} + 2Bu_{xy} + Cu_{yy}] = au_x + bu_y$ 4-16
in a region G, Dirichlet conditions
$B^2 - AC < 0$, $A, C > 0$
Reduced eqn: $\varepsilon = 0$
$a\frac{\partial u_0}{\partial x} + b\frac{\partial u_0}{\partial y} = 0$
constant along $\xi = bx - ay$
let $\eta = ax + by$
$\rightarrow \varepsilon[A_1 u_{\xi\xi} + 2B_1 u_{\xi\eta} + C_1 u_{\eta\eta}] = u_\eta$
then $B_1^2 - A_1C_1 < 0$, $A_1, C_1 > 0$
in ξ-η plane, characteristics are vertical.
divide boundary into upper + lower,
separated by pts where tangent lines
to boundary are vertical.
let $s = \dfrac{\eta - \eta_B^+(\xi)}{\delta(\varepsilon)}$ ← BL at top or $s = \dfrac{\eta - \eta_B^-(\xi)}{\delta(\varepsilon)}$ ← BL at bottom
in general: $s = \dfrac{\eta - \eta_B(\xi)}{\delta(\varepsilon)}$

plug into eqn:

$$\varepsilon\{O(1) + O(\tfrac{1}{\delta}) + O(\tfrac{1}{\delta^2})\} = O(\tfrac{1}{\delta})$$

$$\rightarrow \quad \varepsilon/\delta^2 = \tfrac{1}{\delta} \quad \rightarrow \delta = \varepsilon$$

So $s = \dfrac{\eta - \eta_B(\xi)}{\varepsilon}$, $u(\xi,\eta) = U(\xi,s)$

$$\partial u/\partial\xi = \partial U/\partial\xi + \partial U/\partial s\ \partial s/\partial\xi$$
$$= \partial U/\partial\xi - \eta_B'/\varepsilon\ \partial U/\partial s$$
$$\partial^2 u/\partial\xi^2 = \partial^2 U/\partial\xi^2 - 2\eta_B'/\varepsilon\ \frac{\partial^2 U}{\partial\xi\partial s} + \eta_B'^2/\varepsilon^2\ \partial^2 U/\partial s^2 - \eta_B''/\varepsilon\ \partial U/\partial s$$
$$\partial u/\partial\eta = \tfrac{1}{\varepsilon}\partial U/\partial s$$
$$\partial^2 u/\partial\eta^2 = \tfrac{1}{\varepsilon^2}\partial^2 U/\partial s^2$$
$$\partial^2 u/\partial\xi\partial\eta = \frac{\partial^2 U}{\partial\xi\partial s}\frac{1}{\varepsilon} - \frac{\eta_B'}{\varepsilon^2}\frac{\partial^2 U}{\partial s^2}$$

plug in (ignore all but leading orders)

$$A_1\eta_B'^2\ \partial^2 U/\partial s^2 - 2B_1\eta_B'\ \partial^2 U/\partial s^2 + C_1\ \partial^2 U/\partial s^2 + O(\varepsilon) = \partial U/\partial s$$

$$K(\xi)\ \partial^2 U/\partial s^2 + O(\varepsilon) = \partial U/\partial s$$

$$K(\xi) = A_1\eta_B'^2 - 2B_1\eta_B' + C_1 > 0$$

$U \sim U_0(\xi,s)$

$$K(\xi)\ \partial^2 U_0/\partial s^2 = \partial U_0/\partial s$$

$$U_0 = e^{s/k} + \text{const}$$

to avoid exp. growth, need $s<0$

$\rightarrow \eta - \eta_B < 0 \rightarrow \eta < \eta_B$

$\rightarrow$ BL at top.

$\eta_B(\xi)$ should be $\eta_B^+(\xi)$ (upper bdry)

$\rightarrow K(\xi)\ \partial^2 U_0/\partial s^2 = \partial U_0/\partial s \qquad -\infty < s < 0$

on $\xi = \eta_B^-(\xi)$, $u = u_L(\xi)$

on $\xi = \eta_B^+(\xi)$, $u = u_U(\xi)$

$U|_{s\to-\infty} = u_L(\xi)$

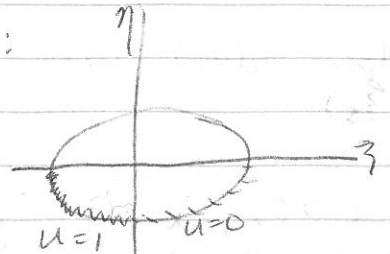
$U|_{s\to 0} = u_U(\xi)$

$U_0(\xi,s) = c_1 + c_2 e^{s/k} \rightarrow c_1 = u_L$

$U_0(\xi,s) = u_L(\xi) + (u_U(\xi) - u_L(\xi))\,e^{s/k}$

$u(\xi,\eta) \sim u_L(\xi) + (u_U(\xi))\,e^{\frac{\eta-\eta_B}{\varepsilon K(\xi)}}$

problem:



need a BL at $\xi = 0$ or $\xi = \xi_0$
large derivatives in ξ

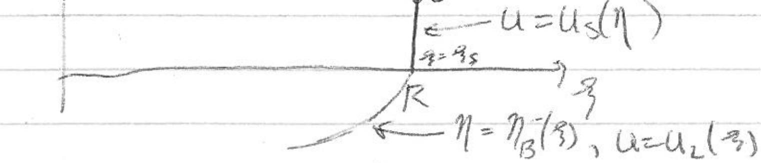
let $\xi^* = \dfrac{\xi - \xi_0}{\delta(\varepsilon)}$

$\rightarrow \varepsilon[O(\frac{1}{\delta^2}) + O(\frac{1}{\delta}) + O(1)] = O(1)$

$\delta = \sqrt{\varepsilon}$

at leading order $\rightarrow$ heat eqn
"parabolic layers" / "diffusive layers"

problem: $\eta_B' \rightarrow \infty$ at endpts of $\eta_B(\xi)$
Subcharacteristic Boundary
boundary coincides with characteristics
$\eta_B' = \infty$
typically value along characteristic curves is prescribed by BC
but BC could be different at subchar. boundary
$\rightarrow$ more BL necessary! (parabolic)



$u(\xi,\eta) = u_{PL}(\xi_*, \eta)$

$\xi_* = (\sqrt{\varepsilon}\,\xi - \xi_S)\frac{1}{\sqrt{3}}$

$u_{PL} \sim u_{PL,0}$

$\rightarrow A_1 \, \partial^2 u_{PL}/\partial \xi_*^2 = \partial u_{PL}/\partial \eta$, $\eta > 0$, $-\infty < \xi_* < \infty$

$u_{PL}(\xi, 0) = u_L(\xi_S)$ $\leftarrow$ like an IC

$u_{PL}(0,\eta) = u_S(\eta)$, $u_{PL}(-\infty,\eta) = u_L(\xi_S)$
$\uparrow$ BC $\nearrow$

$\eta \to t, \quad \xi \to -x, \quad u_{plo} - u_L(\xi_s) = u$

$\varphi(t) = u_s(\eta) - u_L(\xi_s)$

$A_1 = a^2$

$\rightarrow \frac{du}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \; x > 0 \quad u(\infty, t) = 0$

$u(x,0) = 0, \; u(0,t) = \varphi(t)$

Fourier Sine Transform

$\tilde{u}(\xi, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin(x\xi)\, dx$

$\rightarrow \partial\tilde{u}/\partial t = -a^2 \xi^2 \tilde{u}(x,t) + \sqrt{\frac{2}{\pi}} a^2 \xi \varphi(t)$

$\partial/\partial t(\tilde{u} e^{a^2\xi^2 t}) = \sqrt{\frac{2}{\pi}} a^2 \xi \varphi(t) e^{a^2\xi^2 t}$

$\tilde{u}(\xi, t) = e^{-a^2\xi^2 t} \sqrt{\frac{2}{\pi}} a^2 \xi \int_0^t \varphi(\tau) e^{a^2\xi^2\tau} d\tau$

$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{u}(\xi, t) \sin(x\xi)\, d\xi$

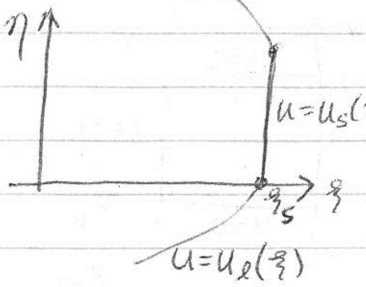
$= \frac{2}{\pi} \int_0^\infty \sin(x\xi) a^2 \xi e^{-a^2\xi^2 t} \int_0^t \varphi(\tau) e^{a^2\xi^2\tau} d\tau\, d\xi$

$= \frac{2}{\pi} a^2 \int_0^t \varphi(\tau) \underbrace{\int_0^\infty \xi \sin(x\xi) e^{-a^2\xi^2(t-\tau)} d\xi}\, d\tau$

$\mathrm{Im}\left[\frac{1}{2} \int_{-\infty}^\infty \xi e^{ix\xi} e^{-a^2\xi^2(t-\tau)} d\xi\right]$

$= \frac{x\sqrt{\pi}}{4a^3(t-\tau)^{3/2}} e^{-\frac{x^2}{4a^2(t-\tau)}}$

$$\varepsilon\left[A_1 u_{\xi\xi} + 2B_1 u_{\xi\eta} + C_1 u_{\eta\eta}\right] = u_\eta$$ 4-21



$u_\ell(\xi_s)$ does not have to be equal to $u_s(0)$

$$\xi_* = \frac{\xi-\xi_s}{\sqrt{\varepsilon}}, \quad \eta=\eta, \quad u(\xi,\eta) = u_{p\ell}(\xi_*,\eta), \quad a^2 = A_1$$

$$\partial u_{p\ell}/\partial\eta = A_1 \frac{\partial^2 u_{p\ell}}{\partial \xi_*^2}, \quad -\infty < \xi_* < 0, \quad \eta > 0$$

$$u(0,\eta) = u_s(\eta), \quad u(-\infty,\eta) = u_\ell(\xi_s)$$

$$u(\xi_*,0) = u_\ell(\xi_s)$$

let $x = -\xi_*$, $t=\eta$, $u(x,t) = u_{p\ell}(\xi_*,\eta) - u_\ell(\xi_s)$

$$u(0,t) = u_s(\eta) - u_\ell(\xi_s) = \varphi(t)$$

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad 0<x<\infty, \quad t>0$$

$$u(x,0)=0, \quad u(0,t)=\varphi(t), \quad u(\infty,t)=0$$

$$u(x,t) = \frac{x}{2a\sqrt{\pi}}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{3/2}} e^{\frac{-x^2}{4a^2(t-\tau)}}\,d\tau$$

$$= -\int_0^t \varphi(t)\frac{d}{dt}\,\mathrm{erfc}\left(\frac{x}{2a\sqrt{t-\tau}}\right)d\tau$$

check: $$\frac{d}{d\tau}\mathrm{erfc}\left(\frac{x}{2a\sqrt{t-\tau}}\right) = \frac{d}{d\tau}\left(1-\mathrm{erf}\left(\frac{x}{2a\sqrt{t-\tau}}\right)\right)$$

$$= -\frac{d}{d\tau}\left[\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t-\tau}}} e^{-s^2}\,ds\right]$$

$$= -\frac{2}{\sqrt{\pi}} e^{\frac{-x^2}{4a^2(t-\tau)}} \frac{x\left(-\frac{1}{2}\right)(-1)}{2a(t-\tau)^{-3/2}}$$

$$= \frac{-x}{2a\sqrt{\pi}}\frac{1}{(t-\tau)^{3/2}} e^{-\frac{x^2}{4a^2(t-\tau)}} \quad \checkmark$$

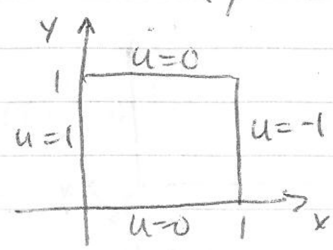
$$u(x,t) = -\varphi(\tau)\,\mathrm{erfc}\left(\frac{x}{2a\sqrt{t-\tau}}\right)\Big|_{\tau=0}^{\tau=t} + \int_0^t \varphi'(\tau)\,\mathrm{erfc}\left(\frac{x}{2a\sqrt{t-\tau}}\right)d\tau$$

$$= \varphi(0)\,\mathrm{erfc}\left(\frac{x}{2a\sqrt{t}}\right) + \int_0^t \varphi'(\tau)\,\mathrm{erfc}\left(\frac{x}{2a\sqrt{t-\tau}}\right)d\tau$$

$$u_{p\ell}(\xi_*,\eta) = u_\ell(\xi_s) + \left[u_s(0) - u_\ell(\xi_s)\right]\mathrm{erfc}\left(\frac{-\xi_*}{2\sqrt{A_1\eta}}\right)$$

$$+\int_0^\eta u_s'(\eta_0)\,\mathrm{erfc}\left(\frac{-\xi_*}{2\sqrt{A_1(\eta-\eta_0)}}\right)d\eta_0$$

Ex: $\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial x}$ $\quad 0<x,y<1$



Reduced eqn:

$$\frac{\partial u_0}{\partial x}=0 \rightarrow u_0=u_0(y)$$

sub characteristics are horiz.

sol'n is 1 everywhere except boundaries (R, U, L)

look at layer near right boundary:

$$u_R(\xi,y)=u(x,y) \qquad \xi=\frac{x-1}{\varepsilon}$$

leading order sol'n:

$$\frac{\partial^2 u_R}{\partial \xi^2}=\frac{\partial u_R}{\partial \xi}, \quad u_R|_{\xi=0}=-1$$
$$u_R|_{\xi=-\infty}=1$$

$$u_R=1-2e^{\xi}$$

bottom boundary layer: (parabolic!)

$$u_B(x,\eta)=u(x,y), \quad \eta=\frac{y}{\sqrt{\varepsilon}}$$

$$\frac{\partial u_B}{\partial x}=\frac{\partial^2 u_B}{\partial \eta^2}, \quad x>0,\ 0<\eta<\infty$$

$$u_B(x,0)=0, \quad u_B(x,\infty)=1, \quad u_B(0,\eta)=1$$

$$u_B(x,\eta)=\operatorname{erf}\left(\frac{\eta}{2\sqrt{x}}\right)$$

But, this doesn't satisfy $u(1,y)=-1$.

need another boundary layer!

Also, doesn't behave well near $(x,y)=(0,0)$

$$u(x,y)=\operatorname{erf}\left(\frac{y}{2\sqrt{\varepsilon x}}\right)$$

$$\partial u/\partial x=\frac{2}{\sqrt{\pi}}e^{-\frac{y^2}{4\varepsilon x}}\frac{y}{2\sqrt{\varepsilon}}\left(-\tfrac{1}{2}\right)x^{-3/2}$$

$$=\frac{-y}{2\sqrt{\varepsilon\pi}}x^{-3/2}e^{\frac{-y^2}{4\varepsilon x}}$$

$$\partial^2 u/\partial x^2=\frac{-y}{2\sqrt{\pi\varepsilon}}\left(-\frac{3}{2}\right)x^{-5/2}e^{\frac{-y^2}{4\varepsilon x}}-\frac{y}{2\sqrt{\pi\varepsilon}}x^{-3/2}e^{\frac{-y^2}{4\varepsilon x}}\left(\frac{y}{4\varepsilon x^2}\right)$$

$$=\frac{-y}{2\sqrt{\pi\varepsilon}}x^{-3/2}e^{-\frac{y^2}{4\varepsilon x}}\left[-\frac{3}{2}\frac{1}{x}+\frac{y}{4\varepsilon x^2}\right]$$

$$\frac{\varepsilon\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial x}\underbrace{\left[-\frac{3}{2}\frac{\varepsilon}{x}+\frac{y^2}{4x^2}\right]}_{\text{should be small in parabolic layer}}$$

if exponent is $O(1)$, there's a problem:

$$4\varepsilon x = y^2 \rightarrow x = y^2/4\varepsilon$$

$$-\frac{3}{2}\frac{\varepsilon}{x} + \frac{y^2}{4x^2} = -\frac{3}{2}\frac{\varepsilon}{y^2}\cdot 4\varepsilon + \frac{y^2\, 16\varepsilon^2}{4y^4} = -6\frac{\varepsilon^2}{y^2} + 4\varepsilon^2/y^2$$

$$= -2\varepsilon^2/y^2$$

if $y = O(\varepsilon)$, this $\uparrow$ is $O(1)$ $\rightarrow x = O(\varepsilon)$

"corner region" need to match

Left corner layer:

$$u_{cL}(\xi, \eta_*) = u(x,y), \quad \xi = \frac{x}{\varepsilon}, \ \eta_* = \frac{y}{\varepsilon} = \frac{\eta}{\sqrt{\varepsilon}}$$

$$\frac{\partial^2 u_{cL}}{\partial \xi^2} + \frac{\partial^2 u_{cL}}{\partial \eta_*^2} = \frac{\partial u_{cL}}{\partial \xi} \quad 0 < \xi, \eta_* < \infty$$

$$u_{cL}(0, \eta_*) = 1, \ u_{cL}(\xi, 0) = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}, \quad 0 < x, y < \infty, \ u(0,y) = 1, \ u(x,0) = 0$$

$$u(x,y) = e^{\frac{x}{2}} v(x,y)$$

$$\rightarrow \frac{1}{4}v + 2\frac{1}{2}\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{2}v + \frac{\partial v}{\partial x}$$

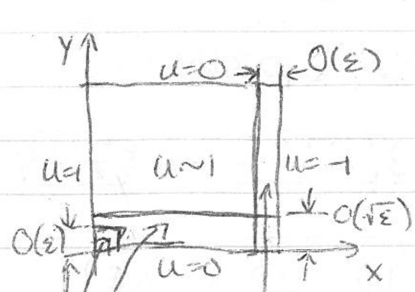
$$\rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{1}{4}v = 0, \quad x, y > 0$$

$$v(x,0) = 0, \ v(0,y) = 1$$

Green's Function

1. g_0 (free space / fundamental sol'n)
2. Method of images
3. Green's formula

4-23



$\varepsilon \nabla^2 u = \frac{\partial u}{\partial x}$

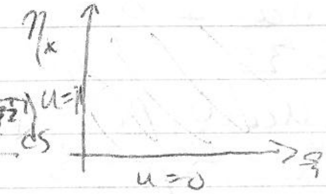
$u_R(\xi, y) = 1 - 2e^{\xi}, \quad \xi = \frac{x-1}{\varepsilon}$

$u_B(x, \eta) = \operatorname{erf}\left(\frac{\eta}{2\sqrt{x}}\right), \quad \eta = \frac{y}{\sqrt{\varepsilon}}$

$u_{CL}(\xi, \eta_*), \quad \xi = \frac{x}{\varepsilon}, \quad \eta_* = \frac{y}{\varepsilon} = \frac{\eta}{\sqrt{\varepsilon}}$

$$\frac{\partial^2 u_{CL}}{\partial \xi^2} + \frac{\partial^2 u_{CL}}{\partial \eta_*^2} = \frac{\partial u_{CL}}{\partial \xi}$$

$$u_{CL}(\xi, \eta_*) = \frac{\xi}{\pi} e^{\xi/2} \int_0^{\eta_*} \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\xi^2}\right)}{\sqrt{s^2+\xi^2}} ds$$



$\rightarrow u_{CL}(\xi, \eta_*) = e^{\xi/2} u(\xi, \eta_*)$

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta_*^2} - \frac{1}{4} u = 0 \qquad \xi \to x, \quad \eta_* \to y$$

$u_{xx} + u_{yy} - \frac{1}{4} u = 0, \quad x, y > 0$

$u(x, 0) = 0, \quad u(0, y) = 1$

Green's Function:

$\nabla^2 g(x, y \mid x_0, y_0) - \frac{1}{4} g = -\delta(x, y \mid x_0, y_0)$

$x, x_0, y, y_0 > 0, \quad g|_{y=0} = 0, \quad g|_{x=0} = 0$

method of images:

free space green's function

$\nabla^2 g_0(x, y \mid x_0, y_0) - \frac{1}{4} g_0 = -\delta(x, y \mid x_0, y_0)$

for $-\infty < x, y, x_0, y_0 < \infty$

new coordinates: center at (x_0, y_0), polar, Assume radially symmetric

$$\frac{\partial^2 g_0}{\partial r^2} + \frac{1}{r}\frac{\partial g_0}{\partial r} - \frac{1}{4} g_0 = -\delta(x_0, y_0)$$

at $r \neq 0$, RHS $= 0$

$r^2 g_{0rr} + r g_{0r} - \frac{1}{4} r^2 g_0 = 0, \quad r > 0$

$g_0 = c K_0\left(\frac{1}{2} r\right)$

$\iint_{D_\varepsilon} \left(\nabla^2 g_0 - \frac{1}{4} g_0\right) dA = -1$

$\int_{\partial D_\varepsilon} \frac{\partial g_0}{\partial n} dS = -1$

$\int_0^{2\pi} \partial g_0/\partial r \big|_{r=\varepsilon} \varepsilon \, d\theta = -1$

As $r \to 0$, $K_0(\tfrac{1}{2}r) \sim -\ln(\tfrac{r}{2})$

$\to \int_0^{2\pi} -2\pi C \frac{1}{r}\Big|_{r=\epsilon} \epsilon\, d\theta = -1$

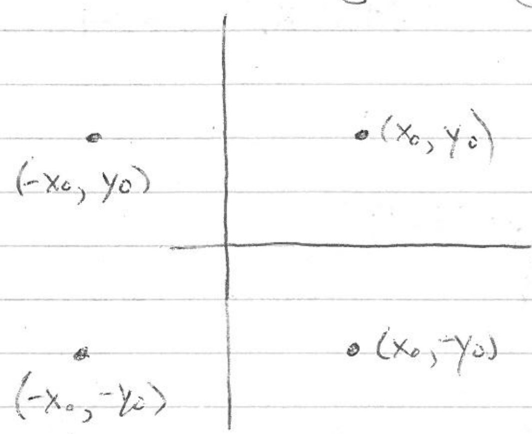
$-2\pi C = -1$

$C = \frac{1}{2\pi}$

$g_0 = \frac{1}{2\pi} K_0(\tfrac{1}{2} r)$

$= \frac{1}{2\pi} K_0\left(\tfrac{1}{2}\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$

Construct g using g_0:



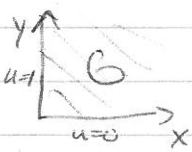
$g(x,y|x_0,y_0)$
$= g_0(x,y|x_0,y_0) - g_0(x,y|-x_0,y_0)$

Because need $g|_{x=0} = 0$

Also need $g|_{y=0} = 0$:

$$g(x,y|x_0,y_0) = g_0(x,y|x_0,y_0) - g_0(x,y|-x_0,y_0) - g_0(x,y|x_0,-y_0) + g_0(x,y|-x_0,-y_0)$$

$\int_G [(\nabla^2 u - \frac{1}{4}u)g - (\nabla^2 g - \frac{1}{4}g)u]\,dA$



$= u(x_0,y_0)$

$= \int_G g\nabla^2 u - u\nabla^2 g\, dA = u(x_0,y_0)$

$= \int_{\partial G}\left(g\frac{\partial u}{\partial n} - u\frac{\partial g}{\partial n}\right) dS$

$g=0$ on bdry. u is given on bdry

$= \int_0^\infty \frac{\partial g}{\partial x}\Big|_{x=0} dy = u(x_0,y_0)$

rename $(x,y) \longleftrightarrow (x_0,y_0)$

$u(x,y) = \int_0^\infty \frac{\partial g}{\partial x_0}(x_0,y_0|x,y)\Big|_{x_0=0} dy_0$

$= \int_0^\infty \frac{\partial g}{\partial x_0}(x,y|x_0,y_0)\Big|_{x_0=0} dy$

$\partial g/\partial x_0|_{x_0=0} = \partial/\partial x_0\left[\frac{1}{2\pi}K_0\left(\tfrac{1}{2}\sqrt{(x-x_0)^2+(y-y_0)^2}\right)\right]_{x_0=0}$

$= \frac{1}{2\pi}K_0'\left(\tfrac{1}{2}\sqrt{x^2+(y-y_0)^2}\right)\frac{1}{2}\frac{1}{\sqrt{x^2+(y-y_0)^2}}(-x)$

$-\frac{1}{2\pi}K_0'\left(\tfrac{1}{2}\sqrt{x^2+(y-y_0)^2}\right)\frac{1}{2}\frac{1}{\sqrt{x^2+(y-y_0)^2}}(x)$

$-\frac{1}{2\pi}K_0'\left(\tfrac{1}{2}\sqrt{x^2+(y+y_0)^2}\right)\frac{1}{2}\frac{1}{\sqrt{x^2+(y+y_0)^2}}(-x)$

$+\frac{1}{2\pi}K_0'\left(\tfrac{1}{2}\sqrt{x^2+(y+y_0)^2}\right)\frac{1}{2}\frac{1}{\sqrt{x^2+(y+y_0)^2}}(x)$

$$\rightarrow u(x,y) = \frac{x}{2\pi}\int_0^\infty \frac{-K_0'\left(\frac{1}{2}\sqrt{x^2+(y-y_0)^2}\right)}{\underbrace{\sqrt{x^2+(y-y_0)^2}}} + \frac{K_0'\left(\frac{1}{2}\sqrt{x^2+(y+y_0)^2}\right)}{\underbrace{\sqrt{x^2+(y+y_0)^2}}}\,dy_0$$

let $y_0 - y = s$ let $y+y_0 = s$

$$\rightarrow \frac{x}{2\pi}\int_{y}^{\infty}\frac{K_0'\left(\frac{1}{2}\sqrt{x^2+s^2}\right)}{\sqrt{x^2+s^2}}\,ds - \frac{x}{2\pi}\int_{-y}^{\infty}\frac{K_0'\left(\frac{1}{2}\sqrt{x^2+s^2}\right)}{\sqrt{x^2+s^2}}\,ds$$

$$= \frac{-x}{2\pi}\int_{-y}^{y}\frac{K_0'\left(\frac{1}{2}\sqrt{x^2+s^2}\right)}{\sqrt{x^2+s^2}}\,ds$$

$$= \frac{-x}{\pi}\int_{0}^{y}\frac{K_0'\left(\frac{1}{2}\sqrt{x^2+s^2}\right)}{\sqrt{x^2+s^2}}\,ds \qquad \text{and } K_0'(z) = -K_1(z)$$

$$= \frac{x}{\pi}\int_{0}^{y}\frac{K_1\left(\frac{1}{2}\sqrt{x^2+s^2}\right)}{\sqrt{x^2+s^2}}\,ds$$

$$\rightarrow u_{cL} = \frac{\xi}{\pi}e^{\xi/2}\int_0^{\eta_*}\frac{K_1\left(\frac{1}{2}\sqrt{s^2+\xi^2}\right)}{\sqrt{s^2+\xi^2}}\,ds$$

Check matching:

write u_{cL} in terms of parabolic layer variables

$$(\xi,\eta_*) \rightarrow (x,\eta),\quad \xi = \frac{x}{\varepsilon},\quad \eta_* = \frac{\eta}{\sqrt{\varepsilon}}$$

$$u_{cL} = \frac{x}{\varepsilon\pi}e^{\frac{x}{2\varepsilon}}\int_0^{\frac{\eta}{\sqrt{\varepsilon}}}\frac{K_1\left(\frac{1}{2}\sqrt{s^2+x^2/\varepsilon^2}\right)}{\sqrt{s^2+x^2/\varepsilon^2}}\,ds$$

$$= \frac{x}{\varepsilon\pi}e^{\frac{x}{2\varepsilon}}\int_0^{\frac{\eta}{\sqrt{\varepsilon}}}\frac{K_1\left(\frac{x}{2\varepsilon}\sqrt{1+\frac{\varepsilon^2 s^2}{x^2}}\right)}{x/\varepsilon\sqrt{1+\frac{\varepsilon^2 s^2}{x^2}}}\,ds$$

$K_1(z) \sim \sqrt{\frac{\pi}{2z}}\,e^{-z}$ as $z\rightarrow\infty$

$K_1(z) \sim \frac{1}{z}$ as $z \rightarrow 0$

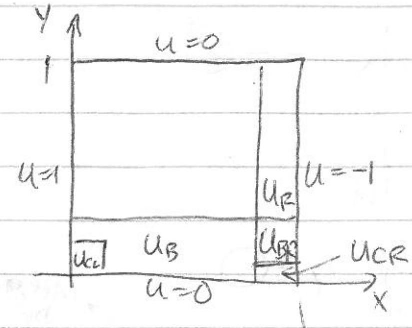
$$u_{cL} \sim \frac{1}{\pi}e^{\frac{x}{2\varepsilon}}\int_0^{\frac{\eta}{\sqrt{\varepsilon}}}\sqrt{\frac{\pi}{2\frac{x}{2\varepsilon}\sqrt{1+\frac{\varepsilon^2 s^2}{x^2}}}}\;e^{-\frac{x}{2\varepsilon}\sqrt{1+\frac{\varepsilon^2 s^2}{x^2}}}$$

$$\sim \frac{1}{\pi}e^{\frac{x}{2\varepsilon}}\int_0^{\frac{\eta}{\sqrt{\varepsilon}}}\sqrt{\frac{\pi}{2\frac{x}{2\varepsilon}}}\;e^{-\frac{x}{2\varepsilon}\left(1+\frac{\varepsilon^2 s^2}{2x^2}\right)}\,ds$$

$$= \sqrt{\frac{\varepsilon}{\pi x}}\,e^{\frac{x}{2\varepsilon}}\,e^{-\frac{x}{2\varepsilon}}\int_0^{\frac{\eta}{\sqrt{\varepsilon}}}e^{-\frac{\varepsilon s^2}{4x}}\,ds \qquad \text{let } t = \sqrt{\frac{\varepsilon}{4x}}\,s$$

$$= \sqrt{\frac{\varepsilon}{\pi x}}\sqrt{\frac{4x}{\varepsilon}}\int_0^{\frac{\eta}{2\sqrt{x}}}e^{-t^2}\,dt$$

$$= \operatorname{erf}\left(\frac{\eta}{2\sqrt{x}}\right)$$



$$\varepsilon(u_{xx}+u_{yy}) = u_x$$

Leading order solutions:

$u_{outer} = 1$

$u_R(\xi, y) = 1-2e^{\xi}, \quad \xi = \frac{x-1}{\varepsilon}$

$u_B(x,\eta) = \operatorname{erf}\left(\frac{\eta}{2\sqrt{x}}\right), \quad \eta = \frac{y}{\sqrt{\varepsilon}}$

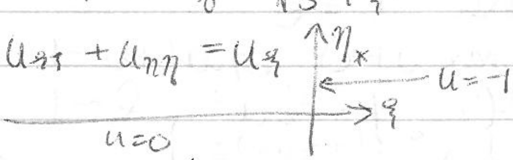
$u_{CL}(\xi,\eta^*) = \frac{\xi}{\pi} e^{\xi/2}\int_0^{\eta_*} \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\xi^2}\right)}{\sqrt{s^2+\xi^2}}\,ds, \quad \xi = \frac{x}{\varepsilon}, \ \eta_* = \frac{y}{\varepsilon} = \frac{\eta}{\sqrt{\varepsilon}}$

$u_{BR}(\xi,\eta) = \operatorname{erf}\left(\frac{\eta}{2}\right) - \left(1+\operatorname{erf}\left(\frac{\eta}{2}\right)\right)e^{\xi}, \quad \xi = \frac{x-1}{\varepsilon}, \ \eta = \frac{y}{\sqrt{\varepsilon}}$

u_{CR}: want it to vary significantly in y:
need u_{yy} term to be significant: $O(\varepsilon)\times O(\varepsilon)$

$$u_{CR}(\xi,\eta_*) = \frac{\xi}{\pi} e^{\xi/2}\int_0^{\eta_*} \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\xi^2}\right)}{\sqrt{s^2+\xi^2}}\,ds, \quad \xi = \frac{1-x}{\varepsilon}, \ \eta_* = \frac{y}{\varepsilon}$$

from: $u_{\xi\xi} + u_{\eta\eta} = u_\xi$



$$u(\xi,\eta) = -e^{\xi/2}v(x,y), \quad x=-\xi, \ \eta = y$$

gives same problem as for u_{CL}

matching:

u_{CR} & u_B:

$u_{CR} \to 0$ as $\xi \to -\infty$ (exp. decay both in coefficient & in decay of K_1)

$u_B \to 0$ b/c must replace $\eta = \sqrt{\varepsilon}\,\eta_*$, then leading order of argument of erf $\to 0$, so erf $\to 0$.

u_{CR} + u_{BR}:

u_{CR}: let $\eta_* \to \infty$: $\frac{\xi}{\pi}\int_0^{\infty} \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\xi^2}\right)}{\sqrt{s^2+\xi^2}}\,ds$

u_{BR}: let $\eta \to 0$: $-e^{\xi}$

Want $\frac{\xi}{\pi} e^{\xi/2} \int_0^\infty \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\xi^2}\right)}{\sqrt{s^2+\xi^2}}\, ds = -e^{\xi}$

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh u - \nu u}\, du$$

$K_\nu(z)$ solves $z^2 y'' + z y' - (z^2+\nu^2) y = 0$

$$y = \int_{-\infty}^{\infty} e^{-z\cosh u - \nu u}\, du$$

$$y' = \int_{-\infty}^{\infty} -\cosh u\, e^{-z\cosh u - \nu u}\, du = -\int_{-\infty}^{\infty} \sinh u (z\sinh u + \nu) e^{-z\cosh u - \nu u}\, du$$

integration by parts

$$y'' = \int_{-\infty}^{\infty} \cosh^2 u\, e^{-z\cosh u - \nu u}\, du$$

Plug in:

$$\int_{-\infty}^{\infty} \left[z^2\cosh^2 u - z\sinh u (z\sinh u + \nu) - (z^2+\nu^2)\right] e^{-z\cosh u - \nu u}\, du$$

$$\int_{-\infty}^{\infty} (-z\nu \sinh u - \nu^2) e^{-z\cosh u - \nu u}\, du$$

$$= \nu e^{-z\cosh u - \nu u}\Big|_{-\infty}^{\infty} = 0 \quad \checkmark$$

So the integral is a sol'n of the eqn

$\rightarrow \; = c_1 I_\nu(z) + c_2 K_\nu(z)$

which is it?

for $z \to \infty$: Laplace Method

$\varphi = -\cosh u, \quad f = e^{-\nu u}$

$\hookrightarrow$ max at 0, equal to -1

$\varphi \sim -1 - \frac{1}{2}u^2, \quad f \sim 1$

integral $\sim \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1+\frac{1}{2}u^2)z}\, du \qquad s = \sqrt{\frac{z}{2}}\, u$

$= \frac{1}{2} e^{-z} \sqrt{\frac{2}{z}} \sqrt{\pi}$

$= \sqrt{\frac{\pi}{2z}}\, e^{-z}$

matches Asymptotics of $K_\nu(z)$

doesn't blow up as $z \to \infty$ like $I_\nu(z)$ does

Alternate expression:

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu} \int_0^\infty e^{-t - \frac{z^2}{4t}}\, t^{-\nu-1}\, dt$$

can get this by substituting into prev. integral

$e^u = \frac{2t}{z}$

Look at a more general integral:

$$\int_0^\infty \frac{K_\mu(a\sqrt{x^2+y^2})}{(x^2+y^2)^{\mu/2}}\,dx$$ replace K_μ with 2^{nd} integral representation

$$= \int_0^\infty \frac{1}{(x^2+y^2)^{\mu/2}} \frac{1}{2}\left(\frac{a\sqrt{x^2+y^2}}{2}\right)^\mu \left[\int_0^\infty e^{-t-\frac{a^2(x^2+y^2)}{4t}}\, t^{-\mu-1}\,dt\right] dx$$

$$= \frac{1}{2}\left(\frac{a}{2}\right)^\mu \int_0^\infty e^{-t}\, t^{-\mu-1}\, e^{-\frac{a^2y^2}{4t}} \left[\int_0^\infty e^{-\frac{a^2x^2}{4t}}\,dx\right] dt$$

$$= \frac{\sqrt{\pi}}{2a}\left(\frac{a}{2}\right)^\mu \int_0^\infty t^{-\mu-1+\frac{1}{2}}\, e^{-t-\frac{a^2y^2}{4t}}\,dt$$

integral is equiv. to 2^{nd} integral rep. of $K_\nu(z)$, with $z=ay$, $\mu-\frac{1}{2}=\nu$

$$= \frac{\sqrt{\pi}}{2a}\left(\frac{a}{2}\right)^\mu K_{\mu-\frac{1}{2}}(ay)\, 2\left(\frac{2}{ay}\right)^{\mu-\frac{1}{2}}$$

$$= \sqrt{\frac{a}{2}}\, y^{\frac{1}{2}-\mu}\, \frac{\sqrt{\pi}}{a}\, K_{\mu-\frac{1}{2}}(ay) = \sqrt{\frac{\pi}{2a}}\, y^{\frac{1}{2}-\mu}\, K_{\mu-\frac{1}{2}}(ay)$$

now let $\mu=1$, $a=\frac{1}{2}$, $y=\xi$:

$$\int_0^\infty \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\xi^2}\right)}{\sqrt{s^2+\xi^2}}\,ds = \sqrt{\frac{\pi}{\xi}}\, K_{\frac{1}{2}}\left(\frac{\xi}{2}\right) = \sqrt{\frac{\pi}{\xi}}\sqrt{\frac{\pi}{2(\xi/2)}}\, e^{-\xi/2}$$

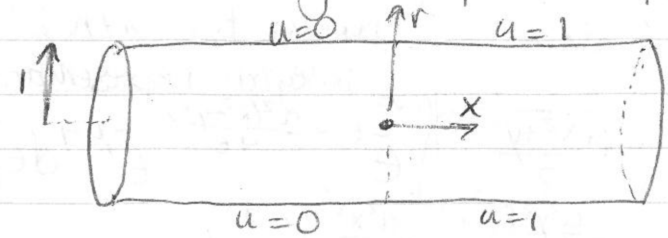
$$= \frac{\pi}{\xi} e^{-\xi/2}$$ really we should have $|\xi|$

our sol'n was premultiplied by $\xi/\pi\, e^{\xi/2}$

and our $\xi<0$, so replace $|\xi|$ by $-\xi$

$\rightarrow -e^{\xi}$

Fluid flow through a cylindrical pipe with heated walls



steady state
infinitely long

cylindrical coordinates (x, r) (doesn't depend on θ)
velocity : $(1-r^2)$
temp : $u = u(x,r)$
Heat eqn : $\varepsilon(u_{rr} + \frac{1}{r}u_r + u_{xx}) = (1-r^2)u_x$
small thermal diffusivity / heat conduction
$-\infty < x < \infty, \quad 0 \le r < 1$

$$u(x,1) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$u(-\infty, r) = 0$

Outer sol'n : $\varepsilon = 0 \to u_x = 0 \to u = \text{const} = 0$
expect large derivatives in r : will need to keep r-derivative(s)
$\rho = \frac{1-r}{\sqrt{\varepsilon}}$ won't work because it makes the x-deriv. small.
Try $\rho = \frac{1-r}{\varepsilon^\alpha}$

4-30 $\varepsilon(u_{rr} + \frac{1}{r}u_r + u_{xx}) = (1+r^2)u_x$
reduced eqn : $u_x = 0 \to u = \text{const} = 0$
$u(x,r) = U(x, \rho), \quad \rho = \frac{1-r}{\varepsilon^\alpha}$
$u_r = -\frac{1}{\varepsilon^\alpha}U_\rho$
$\to \varepsilon\left[\frac{1}{\varepsilon^{2\alpha}}U_{\rho\rho} + \frac{1}{1-\varepsilon^\alpha\rho}\left(\frac{-1}{\varepsilon^\alpha}\right)U_\rho + U_{xx}\right. = \left[1-(1-\varepsilon^\alpha\rho)^2\right]U_x$
$\sim \varepsilon^{1-2\alpha}U_{\rho\rho} + \varepsilon^{1-\alpha}\frac{1}{\rho}U_{\rho\rho} + \varepsilon U_{xx} = \left[2\varepsilon^\alpha\rho - \varepsilon^{2\alpha}\rho^2\right]U_x$
$\to 1 - 2\alpha = \alpha \to \alpha = \frac{1}{3}$
Leading order inner eqn: $U(x,\infty) = 0$
$U_{\rho\rho} = 2\rho U_x, \quad x > 0, \ \rho > 0, \ U(x,0) = 1, \ U(0,\rho) = 0$

Similarity solutions

$U(x,\rho) = v(z), \quad z = \frac{\rho}{x^{1/3}}$

$\rho \to \alpha\rho, \ x \to \beta x:$

$\to \frac{1}{\alpha^2} U_{\rho\rho} = 2\alpha\rho \frac{1}{\beta} U_x$

$\frac{1}{\alpha^3} U_{\rho\rho} = 2\rho \frac{1}{\beta} U_x$

$\to$ let $\beta = \alpha^3$

Then eqn + BC's are invariant under this transformation.

$\to U(x,\rho) = U(\alpha^3 x, \alpha\rho)$ since sol'n is unique

let $\alpha = x^{-1/3}$: $U(1, \rho/x^{1/3})$

$U_x = \frac{dv}{dz}\rho(-\frac{1}{3})x^{-4/3}$

$U_\rho = \frac{dv}{dz}\frac{1}{x^{1/3}}$

$U_{\rho\rho} = \frac{d^2v}{dz^2}\frac{1}{x^{2/3}}$

$\to \frac{1}{x^{2/3}} v'' = 2\rho^2 v' \frac{1}{x^{4/3}}(-\frac{1}{3})$

$v'' + \frac{2}{3} v' \rho^2 \frac{1}{x^{2/3}} = v'' + \frac{2}{3} v' z^2 = 0$

$z > 0, \ v(0) = 1, \ v(\infty) = 0$

integrating factor: $\exp(\frac{2}{9}z^3)$

$(v' \exp(\frac{2}{9}z^3))' = 0$

$v' \exp(\frac{2}{9}z^3) = c_1$

$v = c_2 + \int_0^z c_1 e^{-\frac{2}{9}s^3} ds$

$v(0) = c_2 = 1$

$v(\infty) = 0 = 1 + c_1 \int_0^\infty e^{-\frac{2}{9}s^3} ds$

$\int_0^\infty e^{-\frac{2}{9}s^3} ds$ let $\frac{2}{9}s^3 = \tau \to s = (\frac{9}{2})^{\frac{1}{3}} \tau^{1/3}$

$\int_0^\infty e^{-\tau} (\frac{9}{2})^{\frac{1}{3}} \frac{1}{3} \tau^{-2/3} d\tau$

$= \frac{1}{6^{1/3}} \Gamma(\frac{1}{3})$

$\to v(z) = 1 - \frac{6^{1/3}}{\Gamma(\frac{1}{3})} \int_0^z e^{-\frac{2}{9}s^3} ds$

$\to U(x,\rho) = 1 - \frac{6^{1/3}}{\Gamma(\frac{1}{3})} \int_0^{\rho/x^{1/3}} e^{-\frac{2}{9}s^3} ds$

$\to u(x,r) = 1 - \frac{6^{1/3}}{\Gamma(\frac{1}{3})} \int_0^{\frac{1-r}{(\varepsilon x)^{1/3}}} e^{-\frac{2}{9}s^3} ds$ for $x > 0$

width of BL: $(\varepsilon x)^{\frac{1}{3}}$

(get this by setting $u =$ some # + solve for eqn in u + r)

approximation no longer valid as $x \to \infty$ (no longer BL)

Parabolic Equations

Burgers' Eqn

$u_t + uu_x = \varepsilon u_{xx}$ quasilinear

Conservation Laws

$u_t + [q(u)]_x = 0$

say ρ is the conserved quantity

integral conservation law

$\underbrace{\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx}_{\text{rate of change of chemical in } (x_1,x_2)} = \underbrace{q(\rho(x_1,t),x_1,t) - q(\rho(x_2,t),x_2,t)}_{\text{flux}}$

if $\rho(x,t)$ is smooth, (let $\Delta x = x_2 - x_1$

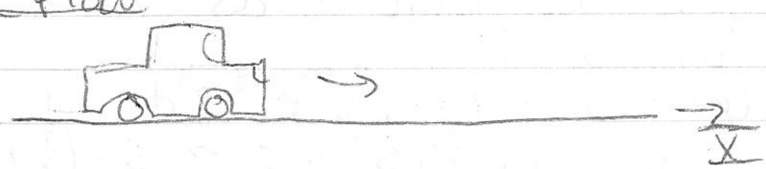
$\rightarrow \frac{1}{\Delta x}\int_{x_1}^{x_2} \frac{\partial \rho}{\partial t}\,dx = -\left[\frac{q(\rho(x_2,t),x_2,t) - q(\rho(x_1,t),x_1,t)}{\Delta x}\right]$

as $\Delta x \to 0$: $\rho_t + q(\rho,x,t)_x = 0$

$\rho_t + c(\rho,x,t)\rho_x = -q_x$ for $c = q_\rho$

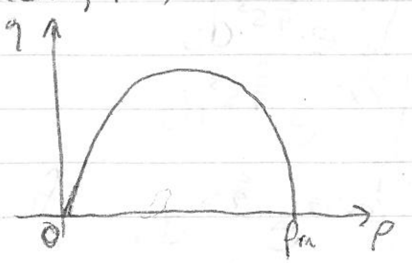
use method of characteristics

Traffic Flow



no onramps or offramps

$\rho(X,T)$: car density = # of cars / unit length



q doesn't depend on T, X

$q = K\rho(\rho_m - \rho) - k\frac{\partial \rho}{\partial X}$

$\rho_T + [K\rho(\rho_m - \rho) - k\frac{\partial \rho}{\partial X}]_X = 0$

nondimensionalize: (should be $(X,T) = (x,t)$)

$\rho = \frac{1}{2}\rho_m (1-u)$

$\to -\frac{1}{2}\rho_m u_T + \left[K\frac{1}{2}\rho_m(1-u^2)\frac{1}{2}\rho_m + k\frac{1}{2}\rho_m \frac{\partial u}{\partial X}\right]_X = 0$

$-u_T + K\rho_m\, \partial/\partial X\left[\frac{1}{2}(1-u^2) + \frac{k}{K\rho_m} u_X\right] = 0$

let $X = xL$, $T = \frac{L}{K\rho_m}$

$\to -u_t + \partial/\partial x\left[\frac{1}{2}(1-u^2) + \varepsilon\frac{\partial u}{\partial x}\right] = 0$

$\varepsilon = \frac{k}{K\rho_m L}$

$-u_t + (-uu_x) + \varepsilon u_{xx} = 0$

$u_t + uu_x = \varepsilon u_{xx}$ → Burgers' eqn

Cole-Hopf Transformation

let $u(x,t) = -2\varepsilon \dfrac{v_x(x,t)}{v(x,t)}$

$u_t = -2\varepsilon\dfrac{v_{xt}}{v} + 2\varepsilon\dfrac{v_x v_t}{v^2}$

$u_x = -2\varepsilon\dfrac{v_{xx}}{v} + 2\varepsilon\dfrac{v_x^2}{v^2}$

$u_{xx} = -2\varepsilon\dfrac{v_{xxx}}{v} + 2\varepsilon\dfrac{v_{xx}v_x}{v^2} + 4\varepsilon\dfrac{v_x v_{xx}}{v^2} - 4\varepsilon\dfrac{v_x^2 v_x}{v^3}$

$\to -2\varepsilon\dfrac{v_{xt}}{v} + 2\varepsilon\dfrac{v_x v_t}{v^2} + \cancel{4\varepsilon^2\dfrac{v_{xx}v_x}{v^2}} - \cancel{4\varepsilon^2\dfrac{v_x^3}{v^3}} = -2\varepsilon^2\dfrac{v_{xxx}}{v} + \cancel{6}^{\,2}\varepsilon^2\dfrac{v_{xx}v_x}{v^2} - \cancel{4\varepsilon^2\dfrac{v_x^3}{v^3}}$

$\to \frac{1}{v}(-v_{xt} + \varepsilon v_{xxx}) + \frac{v_x}{v^2}(v_t - \varepsilon v_{xx}) = 0$

$-\frac{1}{v}(v_t - \varepsilon v_{xx})_x + \frac{v_x}{v^2}(v_t - \varepsilon v_{xx}) = 0$

$\partial/\partial x\left[-\frac{1}{v}(v_t - \varepsilon v_{xx})\right] = 0$

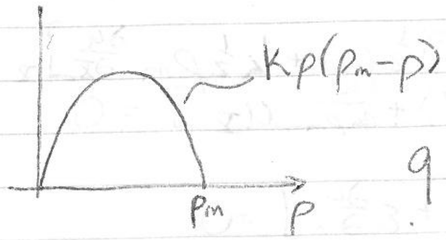
$\frac{1}{v}(v_t - \varepsilon v_{xx}) = a(t)$

$v_t - \varepsilon v_{xx} = a(t)\, v$

let $v = \exp\left(\int a(t)\,dt\right) w$

$\to w_t - \varepsilon w_{xx} = 0$

5-5 $\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = q|_{x=x_1} - q|_{x=x_2}$



$q = K\rho(\rho_m-\rho) - k\frac{\partial\rho}{\partial x}$

$\rho_t + q_x = 0$, $u = \frac{\frac{1}{2}\rho_m - \rho}{\frac{1}{2}\rho_m}$

$\rightarrow u_t + uu_x = \varepsilon u_{xx}$

$u = \frac{-2\varepsilon v_x}{v}$

$\rightarrow v_t = \varepsilon v_{xx}$

$u_t + uu_x = \varepsilon u_{xx}, \quad -\infty < x < \infty, \quad t > 0$

$u(x,0) = f(x), \quad -\infty < x < \infty$

$\rightarrow v_t = \varepsilon v_{xx}, \quad -\infty < x < \infty, \quad t > 0$

$v(x,0) = h(x)$

$f(x) = 2\varepsilon \frac{h'(x)}{h(x)}$

$\ln h = -\frac{1}{2\varepsilon}\int_0^x f(s)\,ds$

$h(x) = \exp\left(-\frac{1}{2\varepsilon}\int_0^x f(s)\,ds\right)$

Say $f_1(x) = \begin{cases} 1 & x<0 \\ -1 & x>0 \end{cases}$

$f_2(x) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$

$\rightarrow$ IC of piecewise constant function can always be reduced to one of these two forms by a change of variable.

$\bar{x} = \frac{x - x_0 - \frac{1}{2}t(u_1+u_2)}{2/|u_1-u_2|}$

$\bar{t} = \frac{t}{4/(u_1-u_2)^2}$

$\bar{u} = \frac{2u - (u_1+u_2)}{|u_1-u_2|}$

x_0 is pt of discontinuity, u_1 is value to left of x_0, u_2 is value to right of x_0

$$v(x,t)=\frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{\infty}h(s)\,e^{\frac{-(x-s)^2}{4\varepsilon t}}\,ds$$

$$u(x,t)=\frac{\int_{-\infty}^{\infty}\frac{(x-s)}{t}h(s)\,e^{\frac{-(x-s)^2}{4\varepsilon t}}\,ds}{\int_{-\infty}^{\infty}h(s)\,e^{\frac{-(x-s)^2}{4\varepsilon t}}\,ds}$$

$$f_1(x)=\begin{cases}1 & x<0\\ -1 & x>0\end{cases}\rightarrow h_1=e^{-\frac{1}{2\varepsilon}\int_0^x f_1(s)ds}=\begin{cases}e^{-\frac{x}{2\varepsilon}} & x<0\\ e^{\frac{x}{2\varepsilon}} & x>0\end{cases}$$

$$u_1(x,t)=\frac{\int_{-\infty}^{0}\frac{x-s}{t}e^{-\frac{s}{2\varepsilon}-\frac{(x-s)^2}{4\varepsilon t}}ds+\int_0^{\infty}\frac{x-s}{t}e^{\frac{s}{2\varepsilon}-\frac{(x-s)^2}{4\varepsilon t}}ds}{\int_{-\infty}^{0}e^{-\frac{s}{2\varepsilon}-\frac{(x-s)^2}{4\varepsilon t}}ds+\int_0^{\infty}e^{\frac{s}{2\varepsilon}-\frac{(x-s)^2}{4\varepsilon t}}ds}$$

$$=\frac{I_1+I_2}{I_3+I_4}$$

$$I_4=\int_0^{\infty}e^{-\frac{1}{4\varepsilon t}\left(s^2-2s(x+t)+(x+t)^2-(x+t)^2+x^2\right)}ds$$

$$=\int_0^{\infty}e^{-\frac{1}{4\varepsilon t}\left[(s-(x+t))^2\right.}ds\cdot e^{\frac{1}{4\varepsilon t}((x+t)^2-x^2)}$$

let $\tau=\frac{1}{\sqrt{4\varepsilon t}}(s-x+t)$

$$\rightarrow e^{\frac{t^2+2xt}{4\varepsilon t}}\sqrt{4\varepsilon t}\int_{-\frac{x+t}{\sqrt{4\varepsilon t}}}^{\infty}e^{-\tau^2}d\tau$$

$$=\sqrt{\varepsilon t\pi}\,e^{\frac{t+2x}{4\varepsilon}}\operatorname{erfc}\left[\frac{-(x+t)}{\sqrt{4\varepsilon t}}\right]$$

Similarly, $I_3=e^{\frac{t-2x}{4\varepsilon}}\sqrt{\pi\varepsilon t}\operatorname{erfc}\left[\frac{x-t}{\sqrt{4\varepsilon t}}\right]$

$$I_2=-e^{\frac{t^2+2xt}{4\varepsilon t}}\sqrt{\pi\varepsilon t}\,\frac{2}{\sqrt{\pi}}\int_{\frac{-(x+t)}{\sqrt{4\varepsilon t}}}^{\infty}\left(1+\tau\sqrt{\frac{4\varepsilon}{t}}\right)e^{-\tau^2}d\tau$$

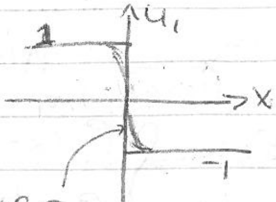
$$I_1=e^{\frac{t^2-2xt}{4\varepsilon t}}\sqrt{\pi\varepsilon t}\,\frac{2}{\sqrt{\pi}}\int_{\frac{x-t}{\sqrt{4\varepsilon t}}}^{\infty}\left(1+\tau\sqrt{\frac{4\varepsilon}{t}}\right)e^{-\tau^2}d\tau$$

$$\rightarrow u_1(x,t)=\frac{e^{-x/\varepsilon}\operatorname{erfc}\left(\frac{x-t}{2\sqrt{\varepsilon t}}\right)-\operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\varepsilon t}}\right)}{e^{-x/\varepsilon}\operatorname{erfc}\left(\frac{x-t}{2\sqrt{\varepsilon t}}\right)+\operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\varepsilon t}}\right)}$$

As $\varepsilon\to 0$: $u\to\frac{0\;\;-2}{0\;\;+2}\to -1$

$x>0$

as $\varepsilon \to 0$, $x<0$: $u \to 1$



want to see a transition layer

Let $\xi = \frac{x}{\varepsilon} = O(1)$

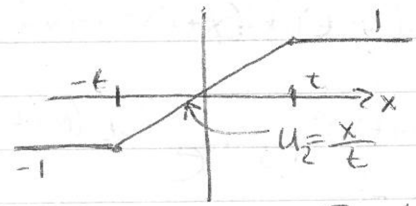
$$u_1 = \frac{e^{-\xi}\operatorname{erfc}\left(\frac{\varepsilon\xi - t}{2\sqrt{\varepsilon t}}\right) - \operatorname{erfc}\left(-\frac{\varepsilon\xi + t}{2\sqrt{\varepsilon t}}\right)}{e^{-\xi}\operatorname{erfc}\left(\frac{\varepsilon\xi - t}{2\sqrt{\varepsilon t}}\right) + \operatorname{erfc}\left(-\frac{\varepsilon\xi + t}{2\sqrt{\varepsilon t}}\right)}$$

$$\sim \frac{2e^{-\xi} - 2}{2e^{-\xi} + 2} = \frac{e^{-\xi/2} - e^{\xi/2}}{e^{-\xi/2} + e^{\xi/2}} = \frac{-2\sinh \xi/2}{2\cosh \xi/2} = -\tanh(\xi/2)$$

$$f_2(x) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$$

$$u_2(x,t) = \frac{e^{-x/\varepsilon}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right) - \operatorname{erfc}\left(\frac{x+t}{2\sqrt{\varepsilon t}}\right)}{e^{-x/\varepsilon}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right) + \operatorname{erfc}\left(\frac{x+t}{2\sqrt{\varepsilon t}}\right)}$$

$\to$



$\leftarrow$ for $\varepsilon = 0$

$\varepsilon \to 0$, $x > t$: $u_2 \to ?$ (should be 1)

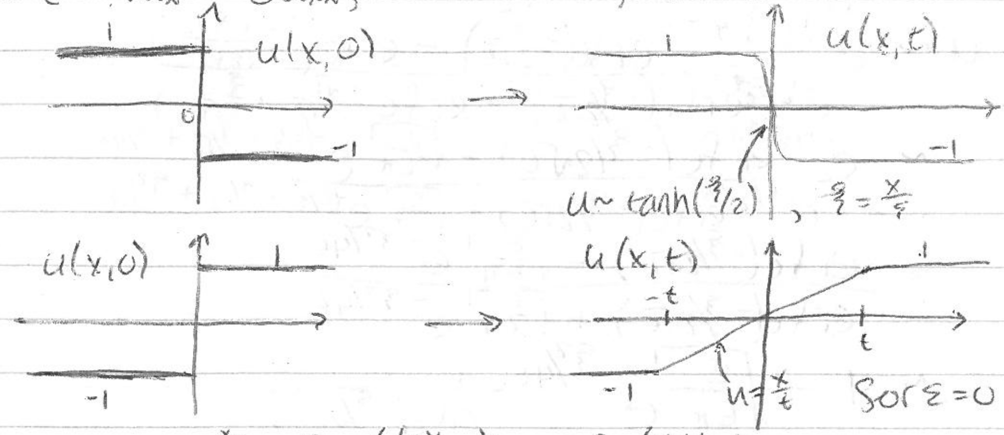
As $z \to \infty$, $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-s^2}\,ds \sim \frac{1}{z\sqrt{\pi}} e^{-z^2}$

So as $\varepsilon \to 0$, $u_2 \to \dfrac{e^{-x/\varepsilon}\cdot 2 + \frac{1}{\sqrt{\pi}}\left(\frac{2\sqrt{\varepsilon t}}{x+t}\right)e^{-\frac{(x+t)^2}{4\varepsilon t}}}{e^{-x/\varepsilon}\cdot 2 - \frac{1}{\sqrt{\pi}}\left(\frac{2\sqrt{\varepsilon t}}{x+t}\right)e^{-\frac{(x+t)^2}{4\varepsilon t}}}$

the $e^{-x/\varepsilon}$ win since

$$x < \frac{(x+t)^2}{4t} \to 4xt < x^2 + 2xt + t^2$$
$$0 < x^2 - 2xt + t^2$$
$$0 < (x-t)^2 \checkmark$$

$u_t + u u_x = \varepsilon u_{xx}, \quad -\infty < x < \infty, \quad t > 0$



$$u = \frac{e^{-x/\varepsilon}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right) - \operatorname{erfc}\left(\frac{x+t}{2\sqrt{\varepsilon t}}\right)}{e^{-x/\varepsilon}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right) + \operatorname{erfc}\left(\frac{x+t}{2\sqrt{\varepsilon t}}\right)}$$

as $z \to \infty$, $\operatorname{erfc}(z) \to \frac{1}{z\sqrt{\pi}} e^{-z^2}$ $\quad = 1 - \operatorname{erf}(z)$

so for $x > t$, $$u \sim \frac{e^{-x/\varepsilon}\, 2 - \frac{2\sqrt{\varepsilon t}}{(x+t)\sqrt{\pi}} e^{-\frac{(x+t)^2}{4\varepsilon t}}}{e^{-x/\varepsilon}\, 2 + \frac{2\sqrt{\varepsilon t}}{(x+t)\sqrt{\pi}} e^{-\frac{(x+t)^2}{4\varepsilon t}}}$$

$$= \frac{1 - \frac{\sqrt{\varepsilon t}}{(x+t)\sqrt{\pi}} e^{\frac{x}{\varepsilon} - \frac{(x+t)^2}{4\varepsilon t}}}{1 + (\text{same})}$$

$$\frac{x}{\varepsilon} - \frac{(x+t)^2}{4\varepsilon t} = \frac{1}{4\varepsilon t}\left(4tx - (x+t)^2\right) = \frac{-(x-t)^2}{4\varepsilon t}$$

so as $x \to 0$, exponent $\to -\infty$

$\to u \to 1$

$x < -t$: similar.

for $-t < x < t$:

$$u \sim \frac{e^{-x/\varepsilon} \frac{2\sqrt{\varepsilon t}}{(t-x)\sqrt{\pi}} e^{-\frac{(t-x)^2}{4\varepsilon t}} - \frac{2\sqrt{\varepsilon t}}{(x+t)\sqrt{\pi}} e^{-\frac{(x+t)^2}{4\varepsilon t}}}{(\ldots) + (\ldots)}$$

$$= \frac{e^{\left(-x/\varepsilon - \frac{(t-x)^2}{4\varepsilon t} + \frac{(x+t)^2}{4\varepsilon t}\right)} \frac{1}{t-x} - \frac{1}{x+t}}{[\ldots] + [\ldots]}$$

$-4xt - (t-x)^2 + (x+t)^2 = 0$ in exponent.

$$\to \frac{\frac{1}{t-x} - \frac{1}{x+t}}{\frac{1}{t-x} + \frac{1}{x+t}} = \frac{x+t-(t-x)}{x+t+t-x} = \frac{2x}{2t} = \frac{x}{t} \checkmark$$

sharp corners near $x = \pm t$

$\to$ corner layers.

let $\xi = \frac{x-t}{\sqrt{\varepsilon}}$, $x = t + \xi\sqrt{\varepsilon}$

$$u \sim \frac{e^{-\frac{1}{\varepsilon}(t+\xi\sqrt{\varepsilon})}\,\mathrm{erfc}\left(-\frac{\xi}{2\sqrt{t}}\right) - \mathrm{erfc}\left(\frac{2t+\xi\sqrt{\varepsilon}}{2\sqrt{\varepsilon t}}\right)}{e^{-\frac{1}{\varepsilon}t-\frac{\xi}{\sqrt{\varepsilon}}}\,\mathrm{erfc}\left(-\xi/2\sqrt{t}\right) + \mathrm{erfc}\left(\frac{\sqrt{t}}{\sqrt{\varepsilon}} + \frac{\xi}{2\sqrt{t}}\right)}$$

$$\sim \frac{e^{-t/\varepsilon-\xi/\sqrt{\varepsilon}}\,\mathrm{erfc}\left(-\xi/(2\sqrt{t})\right) - \sqrt{\frac{\varepsilon}{t\pi}}\, e^{-(t/\varepsilon + \xi/\sqrt{\varepsilon} + \xi^2/4t)}}{e^{-t/\varepsilon-\xi/\sqrt{\varepsilon}}\,\mathrm{erfc}\left(-\xi/2\sqrt{t}\right) + \sqrt{\frac{\varepsilon}{t\pi}}\, e^{-(t/\varepsilon + \xi/\sqrt{\varepsilon} + \xi^2/4t)}}$$

$$= \frac{\mathrm{erfc}\left(-\xi/2\sqrt{t}\right) - \sqrt{\frac{\varepsilon}{t\pi}}\, e^{-\xi^2/4t}}{\mathrm{erfc}\left(-\xi/2\sqrt{t}\right) + \sqrt{\frac{\varepsilon}{t\pi}}\, e^{-\xi^2/4t}}$$

$$\sim \frac{1 - \sqrt{\frac{\varepsilon}{t\pi}}\, e^{-\xi^2/4t}\,\frac{1}{\mathrm{erfc}\left(-\xi/2\sqrt{t}\right)}}{1 + \boxed{\wedge}}$$

$$\sim \left(1 - \sqrt{\frac{\varepsilon}{t\pi}}\, e^{-\xi^2/4t}\,\frac{1}{\mathrm{erfc}\left(-\xi/2\sqrt{t}\right)}\right)^2$$

$$\sim 1 - \frac{2\sqrt{\frac{\varepsilon}{\pi t}}\, e^{-\xi^2/4t}}{\mathrm{erfc}\left(-\xi/2\sqrt{t}\right)}$$

as $\xi \to \infty$, should $\to 1$ ✓

$\xi \to -\infty$ should $\to \frac{x}{t}$

$$1 - 2\sqrt{\varepsilon/\pi t}\,\frac{e^{-\xi^2/4t}}{-\frac{2\sqrt{t}}{\xi\sqrt{\pi}}e^{-\xi^2/4t}} = 1 - 2\sqrt{\frac{\varepsilon}{\pi t}}\left(\frac{\xi\sqrt{\pi}}{-2\sqrt{t}}\right)$$

$$= 1 + \sqrt{\varepsilon}\,\xi/t = 1 + \sqrt{\varepsilon}\,(x-t)/t\sqrt{\varepsilon}$$

$$= 1 + \frac{x-t}{t} = x/t \quad ✓$$

Now, solve using perturbation methods.

$u_t + uu_x = 0$ (outer sol'n), $t > 0$, $-\infty < x < \infty$

$u(x,0) = f(x)$

$x = \tau$, $t = 0$, $u = f(\tau)$

$dt/ds = 1$, $t|_{s=0} = 0$

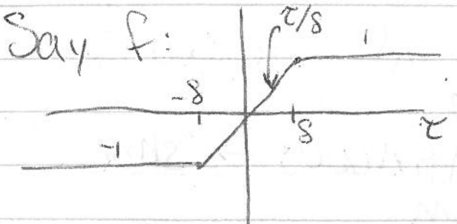
$dx/ds = u$ $\quad x|_{s=0} = \tau$

$du/ds = 0$ $\quad u|_{s=0} = f(\tau)$

$\to t = s$, $u = f(\tau)$, $x = us + \tau = f(\tau)s + \tau$

$\to x = f(\tau)t + \tau$, $u = f(\tau)$

solve x for τ, plug into eqn for u

Say f: 

$\tau > \delta$: $f(\tau) = 1$
$\rightarrow x = t + \tau$, $u = 1$
(for $x > t + \delta$, $u = 1$)

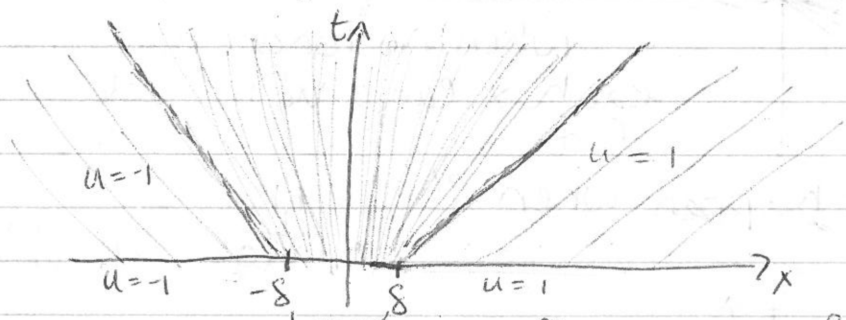
$\tau < -\delta$: $f(\tau) = -1$, $x = -t + \tau$, $u = -1$
(for $x < -t - \delta$, $u = -1$)

$-\delta < \tau < \delta$: $f(\tau) = \tau/\delta \rightarrow x = \tau/\delta\, t + \tau = \tau(t/\delta + 1)$

$\tau = \frac{x\delta}{t+\delta}$

$u = f(\tau) = \tau/\delta = \frac{x}{t+\delta}$ for $-t-\delta < x < t+\delta$

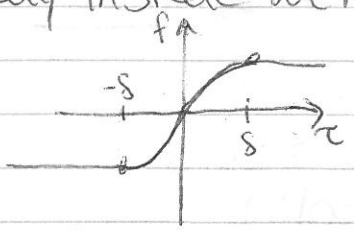
in limit $\delta \to 0$, get same as before.
"rarefaction wave"



$\rightarrow x = \frac{\tau}{\delta} t + \tau \rightarrow t = \frac{\delta}{\tau} x - \delta$

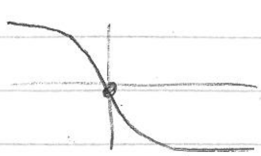
$\rightarrow$ "expansion fan"

Say instead we had $f = \begin{cases} 1 & \tau > \delta \\ \varphi(\tau) & |\tau| < \delta \\ -1 & \tau < -\delta \end{cases}$

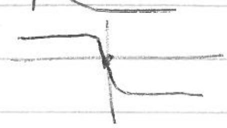
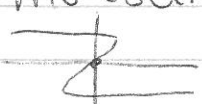


$-\delta < \tau < \delta \rightarrow x = \varphi(\tau) t + \tau$, $u = \varphi(\tau)$
$\rightarrow \frac{x}{t} = \varphi(\tau) + \tau/t$
$\rightarrow u = \frac{x}{t} - \tau/t$ linear for $\delta \ll 1$ or $t \gg 1$.

What if f was decreasing? ⌐_ ?

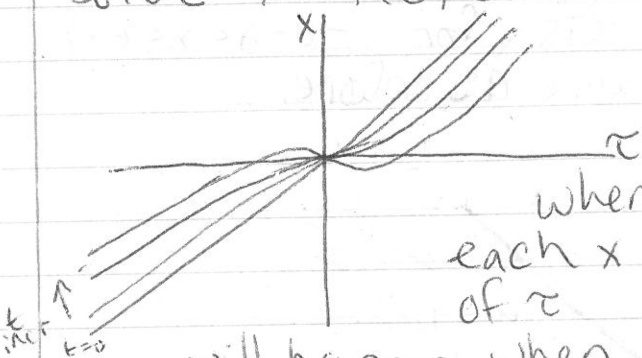


$\tau = x - ut$ as time increases, sol'n moves at speed u

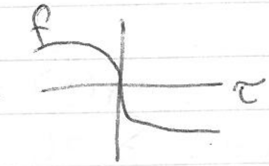
↳  $\rightarrow$  multi-valued → bad. shock formation.

where derivative becomes infinite, can't use diff eq. any more
→ sol'n becomes discontinuous → "shock"
need to derive eqn from integral conservation law instead for region near shock.

Solve $x = f(\tau)t + \tau$



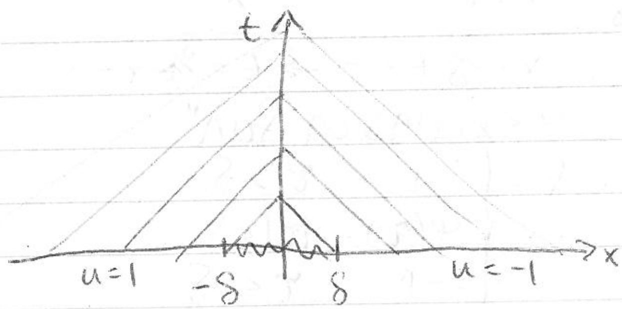
for



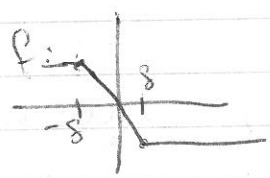
when no longer monotonic, each x can give multiple values of τ

will happen when $f'(\tau)t + 1 = 0$

$\rightarrow t = \frac{-1}{f'(0)}$ since symmetric about $\tau = 0$

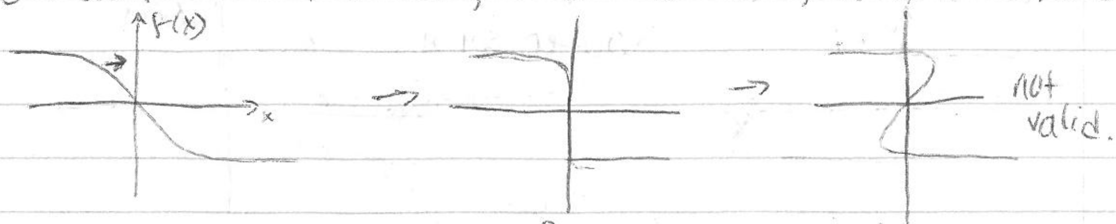


for f:



intersecting characteristics → bad
→ multiple values of sol'n.
stationary shock at $x = 0$

$u_t + uu_x = 0$, $t > 0$, $-\infty < x < \infty$, $u(x,0) = f(x)$ 5-12



solve to left + right of shock as usual
discontinuity where the meet : shock
need to determine how shock propogates.

Rankine Hugoniot Condition

$\frac{d}{dt} \int_{x_1}^{x_2} \rho(x,t)\,dx = q(\rho(x_1,t)) - q(\rho(x_2,t))$

suppose shock is at $x_c(t)$, $x_1 < x_c < x_2$.

$\rightarrow \frac{d}{dt}\left[\int_{x_1}^{x_c(t)} \rho(x,t)\,dx + \int_{x_c(t)}^{x_2} \rho(x,t)\,dx \right] = q|_{x=x_1} - q|_{x=x_2}$

$\rightarrow \frac{d}{dt}\left[\rho(x_c^-,t)(x_c(t)-x_1) + \rho(x_c^+,t)(x_2 - x_c(t)) + O((x_2-x_1)^2) \right]$

$= q(\rho(x_c^-,t)) + q_x(\rho(x_c^-,t))(x_1 - x_c)$

$- q(\rho(x_c^+,t)) - q_x(\rho(x_c^+,t))(x_2 - x_c) + \cdots$

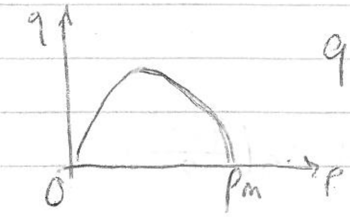
$\rightarrow \rho(x_c^-,t)\frac{dx_c}{dt} - \rho(x_c^+,t)\frac{dx_c}{dt} = q(\rho(x_c^-,t)) - q(\rho(x_c^+,t))$

Σ in $\lim (x_2 \to x_1)$

now $[f] = f|_{x=x_c^+} - f|_{x=x_c^-}$

$\rightarrow \frac{dx_c}{dt} = \frac{[q]}{[\rho]}$

Ex:



$q = \rho(\rho_m - \rho)$

$\rightarrow \frac{dx_c}{dt} = \frac{\rho_+(\rho_m - \rho_+) - \rho_-(\rho_m - \rho_-)}{\rho_+ - \rho_-}$

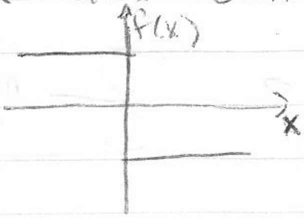
$= \frac{\rho_m(\rho_+ - \rho_-) - (\rho_+ + \rho_-)(\rho_+ - \rho_-)}{\rho_+ - \rho_-}$

$= \rho_m - \rho_+ + \rho_-$

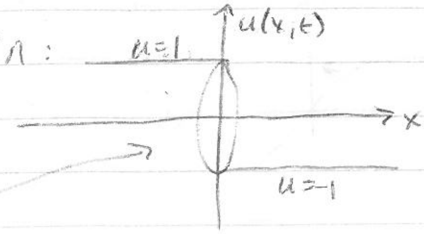
$q = \frac{1}{2}(1 - u^2)$

$\rightarrow \frac{dx_c}{dt} = \frac{\frac{1}{2}(1-u_+^2) - \frac{1}{2}(1-u_-^2)}{u_+ - u_-} = -\frac{1}{2}(u_+ + u_-)$

$u_t + uu_x = \varepsilon u_{xx}$



outer sol'n :



inner : $u \sim -\tanh\left(\frac{x}{2\varepsilon}\right)$

let $\xi = \frac{x}{\varepsilon}$, $u(x,t) = U(\xi, t)$

$$U_t + U U_\xi \frac{1}{\varepsilon} = \varepsilon \frac{1}{\varepsilon^2} U_{\xi\xi}$$

Leading order:

$$U U_\xi = U_{\xi\xi}$$

$$\frac{1}{2}\frac{d}{d\xi}(U^2) = U_{\xi\xi}$$

$$\to \frac{1}{2}U^2 = U_\xi + c$$ matching: $U(\pm\infty) = \mp 1$

$$\to c = \frac{1}{2}$$

$$\int \frac{2\,dU}{U^2-1} = \int d\xi$$

$$\int \frac{1}{U-1} - \frac{1}{U+1}\, dU = \int d\xi$$

$$\ln\left|\frac{U-1}{U+1}\right| = \xi - \xi_0$$

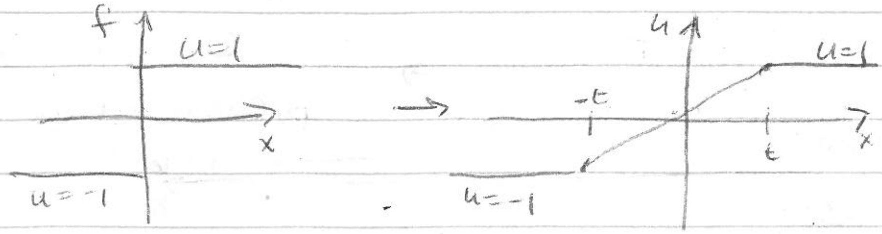
$$\left|\frac{U-1}{U+1}\right| = e^{\xi - \xi_0}$$

$$\frac{U-1}{U+1} = -e^{\xi - \xi_0}$$

$$U(\xi, t) = \frac{1 - e^{\xi-\xi_0}}{1 + e^{\xi - \xi_0}} = -\tanh\left(\frac{\xi - \xi_0}{2}\right)$$

by symmetry of problem (odd IC), we can say $\xi_0 = 0$. otherwise it's undetermined.

new IC:



near $x = t$:

$\underline{u(x,t) = U(\xi, t)}$, $\xi = \frac{x-t}{\varepsilon^{\alpha}}$

not right, need $u(x,t) = 1 + \varepsilon^{\beta} U(\xi, t)$

$$\varepsilon^{\beta} U_t + \varepsilon^{\beta} U_{\xi}\left(-\frac{1}{\varepsilon^{\alpha}}\right) + (1+\varepsilon^{\beta} U)\,\varepsilon^{\beta} U_{\xi} \frac{1}{\varepsilon^{\alpha}}$$
$$= \varepsilon\, \varepsilon^{\beta} \frac{1}{\varepsilon^{2\alpha}} \frac{\partial^2 U}{\partial \xi^2}$$

$$\varepsilon^{\beta} U_t + \varepsilon^{2\beta-\alpha} U U_{\xi} = \varepsilon^{1-\beta-2\alpha} U_{\xi\xi}$$

balancing: $\alpha = \beta = \frac{1}{2}$

$\rightarrow U_t + U U_{\xi} = U_{\xi\xi}$

$\rightarrow$ original equation
(solve using Cole-Hopf as before)

Combustion Theory (Reaction-Diffusion Models)

$$\frac{\partial T}{\partial t} = \varkappa \frac{\partial^2 T}{\partial x^2} + q\, C k(T)$$

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} - C k(T)$$

$C \rightarrow$ product (exothermic)

Reaction rate $= Ck(T)$

Diffusion coefficient: D

(no flow $\rightarrow$ "diffusion")

T: temperature

$k(T) = k_0 e^{-E/RT}$

E: activation energy
R: gas constant
$\Big\}$ $\frac{E}{R}$: activation temperature $\sim 30{,}000$ K

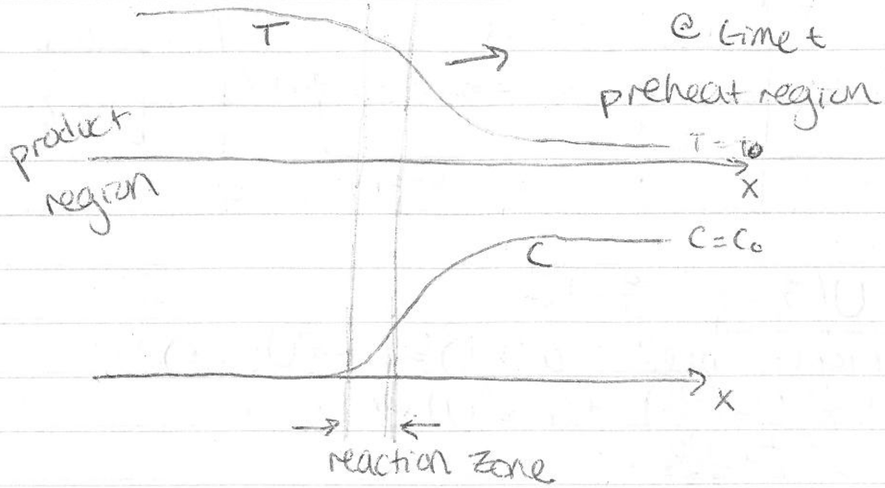
$T_{max} \sim 3{,}000$ K

$k(T_{max}) \sim k_0 e^{-3\cdot 10^4/3\cdot 10^3} = k_0 e^{-10}$

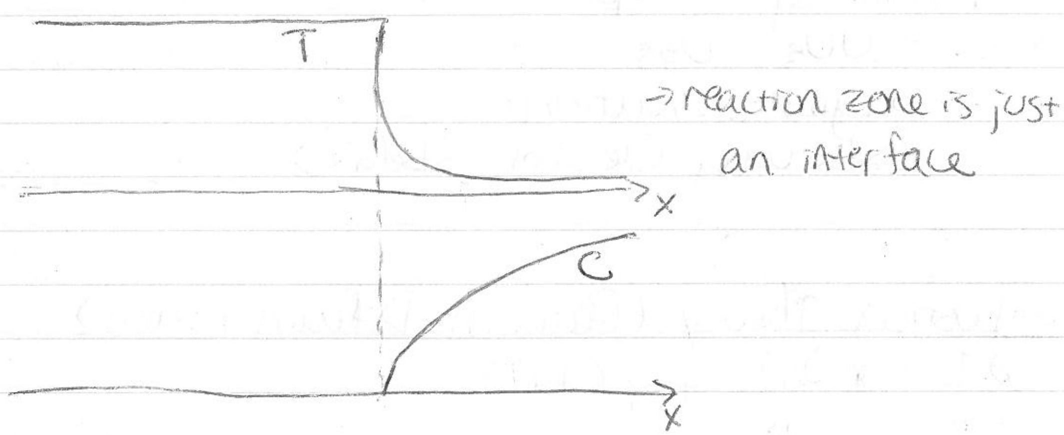
$k(T_0 = 300K) = k_0 e^{-3\cdot 10^4/3\cdot 10^2} = k_0 e^{-100}$

$k(T_{max})/k(T_0) = e^{90} \sim 20^{30} = 2^{30}\cdot 10^{30} \sim 10^9 \cdot 10^{30} = 10^{39}$

(if a reaction occurs in 1s @ T_{max}, it would take 10^{39} s @ T_0.)



simplified:



Nondimensional problem:

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} + Z^2 c \cdot e^{\frac{Z\theta}{1+\delta\theta}}$$

$$\frac{\partial c}{\partial \tau} = Le \frac{\partial^2 c}{\partial \xi^2} - Z^2 c \, e^{\frac{Z\theta}{1+\delta\theta}}$$

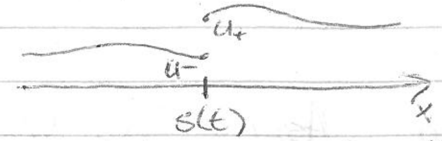
for $c = C/C_0$, $\theta = \frac{T - T_{ad}}{T_{ad} - T_0}$, $T_{ad} \equiv T_0 + qC_0$

$Le = D/\varkappa$, $\tau = t/t_*$, $\xi = x/x_*$

$Z = \frac{E(T_{ad} - T_0)}{R T_{ad}^2} >> 1$, Zeldovich number

$u_t + (q(u))_x = 0$ conservation law

$u_t + (q(u))_x = \varepsilon u_{xx} \longleftarrow$ "artificial viscosity"



$x = s(t)$: location of shock

internal layer: $\xi = \frac{x - s(t)}{\varepsilon}$, $u(x,t) = U(\xi, t)$

$$\rightarrow \quad U_t + U_\xi\left(-\frac{s'(t)}{\varepsilon}\right) + \frac{1}{\varepsilon}\left((q(U))_\xi = \varepsilon \frac{1}{\varepsilon^2} U_{\xi\xi}\right.$$

$$\rightarrow \quad -s'(t)\, U_\xi + (q(U))_\xi = U_{\xi\xi}$$

$$U(-\infty, t) = u_-, \quad U(+\infty, t) = u_+$$

$$-s'(t)\int_{-\infty}^{\infty} U_\xi \, d\xi + \int_{-\infty}^{\infty} (q(U))_\xi \, d\xi = \int_{-\infty}^{\infty} U_{\xi\xi} \, d\xi$$

$$-s'(t)(u_+ - u_-) + q(u_+) - q(u_-) = 0$$

since $U = \text{const}$ @ $\xi \to \pm\infty$

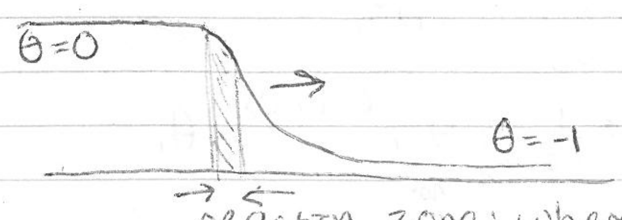
$$\rightarrow \quad s'(t) = \frac{[q]}{[u]}$$

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} + Z^2 C e^{Z\theta/(1+\delta\theta)}$$

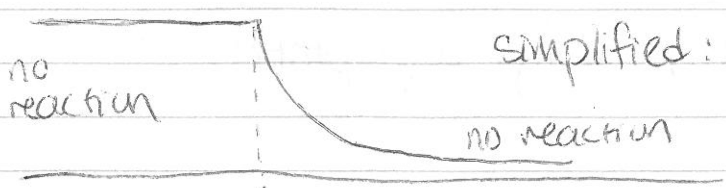
$$\frac{\partial C}{\partial \tau} = Le \frac{\partial^2 C}{\partial \xi^2} - Z^2 C e^{Z\theta/(1+\delta\theta)}$$

$$\theta = \frac{T - T_{ad}}{T_{ad} - T_0}, \quad T_{ad} = T_0 + qC_0$$

$$Le = \frac{D}{\kappa}, \quad Z = \frac{E(T_a - T_0)}{RT_a^2} \gg 1$$



reaction zone: where exponent is significant



simplified: reaction is like a shock: solve before + after

instantaneous reaction

$\xi = \varphi(t)$

$\xi_1 = \frac{\xi - \varphi(t)}{\varepsilon}$, $\varepsilon = \frac{1}{Z}$ $\left(\xi_1 = (\xi - \varphi(t))Z\right)$

$\theta_1 = \frac{\theta - \theta_r(t)}{\varepsilon} = Z(\theta - \theta_r)$, $\theta_r = \theta$ at reaction

$C_1 = C/\varepsilon = ZC$,

$$\rightarrow \frac{\partial^2 \theta_1}{\partial \xi_1^2} + C_1 e^{\theta_1 + Z\theta_r} = 0 \quad (*)$$

Assume $Z\theta_r = O(1)$

$$Le \frac{\partial^2 C_1}{\partial \xi_1^2} - C_1 e^{\theta_1 + Z\theta_r} = 0$$

$$\rightarrow \frac{\partial^2 \theta_1}{\partial \xi_1^2} + Le \frac{\partial^2 C_1}{\partial \xi_1^2} = 0$$

$$\rightarrow \frac{\partial \theta_1}{\partial \xi_1}\Big|_{-\infty}^{\infty} + Le \frac{\partial C_1}{\partial \xi_1}\Big|_{-\infty}^{\infty} = 0$$

and $\frac{\partial \theta_1}{\partial \xi_1} = \frac{\partial \theta}{\partial \xi} \rightarrow \left[\frac{\partial \theta}{\partial \xi}\right] = \frac{\partial \theta_1}{\partial \xi_1}\Big|_{-\infty}^{\infty}$

$$\rightarrow \left[\frac{\partial \theta}{\partial \xi}\right] + Le\left[\frac{\partial C}{\partial \xi}\right] = 0$$

$$\varkappa\left[\partial\theta/\partial\xi\right] + D\left[\partial C/\partial \xi\right] = 0$$

$$\frac{\partial \theta_1}{\partial \xi_1} + Le \frac{\partial C_1}{\partial \xi_1} = 0$$

integrate from $-\infty$ to current time instead of $-\infty \rightarrow \infty$.

Assuming $\frac{\partial \theta_1}{\partial \xi_1} \rightarrow \frac{\partial \theta}{\partial \xi} = 0$, $\frac{\partial C_1}{\partial \xi_1} \rightarrow \frac{\partial C}{\partial \xi} = 0$ actually $\neq 0$, but small

$$\theta_1 + Le\, C_1 = 0$$

$$\rightarrow C_1 = -\frac{1}{Le}\theta_1$$

$$\rightarrow \frac{\partial^2 \theta_1}{\partial \xi_1^2} - \frac{\theta_1}{Le} e^{\theta_1 + Z\theta_r} = 0$$

(from (*) above)

$$\frac{\partial^2 \theta_1}{\partial \xi_1^2}\frac{\partial \theta_1}{\partial \xi_1} - \frac{\theta_1}{Le} e^{\theta_1 + Z\theta_r}\frac{\partial \theta_1}{\partial \xi_1} = 0$$

integrate $-\infty \rightarrow \infty$:

$$\frac{1}{2}\left(\frac{\partial \theta_1}{\partial \xi_1}\right)^2\Big|_{-\infty}^{\infty} + \frac{1}{Le}\int_{(-\infty)}^{0} \theta_1 e^{\theta_1 + Z\theta_r}\, d\theta_1$$

$(-\infty)$ not important

$$\frac{1}{2}\left[\left(\frac{\partial \theta}{\partial \xi}\right)^2\right] = \frac{1}{Le} e^{Z\theta_r}$$

"large activation energy asymptotics"

$\varepsilon(u_{xx} - u_{tt}) = a u_x + b u_t \quad , \quad -\infty < x < \infty, \ t > 0$

$u(x,0) = F(x)$ } (given)
$u_t(x,0) = G(x)$ }

Initial Layer

$u(x,t) = U(x,\tau) \ , \ \tau = t/\varepsilon$

$\to U_{\tau\tau} + b U_\tau = \varepsilon^2 U_{xx} - \varepsilon a U_x$

$U(x,0) = F(x), \ U_\tau(x,0) = \varepsilon G(x)$

$U(x,\tau) \sim U_0(x,\tau) + \varepsilon U_1(x,\tau)$

$O(1): \ \frac{\partial^2 U_0}{\partial \tau^2} + b \frac{\partial U_0}{\partial \tau} = 0 \ , \ \tau > 0, \ U_0(x,0) = F(x)$
$\partial U_0/\partial\tau \, (x,0) = 0$

$U_0(x,\tau) = c_1(x) + c_2(x) e^{-b\tau}$

$U_0(x,0) = c_1(x) + c_2(x) = F(x)$

$\partial U_0/\partial\tau(x,0) = -b c_2(x) = 0 \quad \to c_2 \equiv 0$

$U_0(x,\tau) = F(x)$

$O(\varepsilon): \ \frac{\partial^2 U_1}{\partial \tau^2} + \frac{b \partial U_1}{\partial \tau} = -a \frac{\partial U_0}{\partial x} = -a F'(x)$

$U_1(x,0) = 0 \ , \ \partial U_1/\partial\tau(x,0) = G(x)$

$U_1 = c_2(x) + c_3(x) e^{-b\tau} - \frac{a}{b}\tau F'(x)$

$U_1(x,0) = c_2 + c_3 = 0$

$\partial U_1/\partial\tau(x,0) = -b c_3 - a/b \, F'(x) = G(x)$

$U(x,\tau) = F(x) + \varepsilon\left[-\frac{a}{b}\tau F'(x) + (1 - e^{-b\tau})\frac{1}{b}\left(\frac{a}{b}F'(x) + G(x)\right)\right]$

Outer Solution

$u \sim u_0 + \varepsilon u_1$

$O(1): \ a\frac{\partial u_0}{\partial x} + b\frac{\partial u_0}{\partial t} = 0 \ , \ u_0(x,t) = f(x - \frac{a}{b}t)$
$= f(\xi)$

$O(\varepsilon): \ a\frac{\partial u_1}{\partial x} + b\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial t^2} = f''(\xi) - \frac{a^2}{b^2} f''(\xi) = (1 - \frac{a^2}{b^2}) f''(x - \frac{a}{b}t)$

$u_1(x,t) = v(\xi, t)$

$\to a v_\xi + b(v_t - \frac{a}{b} v_\xi) = (1 - a^2/b^2) f''(\xi)$

$b v_t = (1 - a^2/b^2) f''(\xi)$

$v(\xi,t) = t/b \, (1 - a^2/b^2) f''(\xi) + f_1(\xi)$

$$u_1(x,t) = v(x - a/b,\ t)$$
$$= \tfrac{t}{b}(1 - a^2/b^2) f''(x - a/b) + f_1(x - a/b)$$
$$u(x,t) \sim f(x - a/b\, t) + \varepsilon\left[\tfrac{t}{b}(1 - a^2/b^2) f''(x - a/b) + f_1(x - a/b)\right]$$

Matching:

$$U \sim F(x) - a/b\, t F'(x) + \tfrac{\varepsilon}{b}\left(a/b\, F'(x) + G(x)\right)$$
$$u \sim f(x) - f'(x)\tfrac{a}{b}\varepsilon\tau + \varepsilon f_1(x)$$
$$\rightarrow f(x) - f'(x)\, a/b\, t + \varepsilon f_1(x)$$
$$\rightarrow \begin{cases} f(x) = F(x) \\ f_1(x) = \tfrac{1}{b}\left(\tfrac{a}{b} F'(x) + G(x)\right) \end{cases}$$

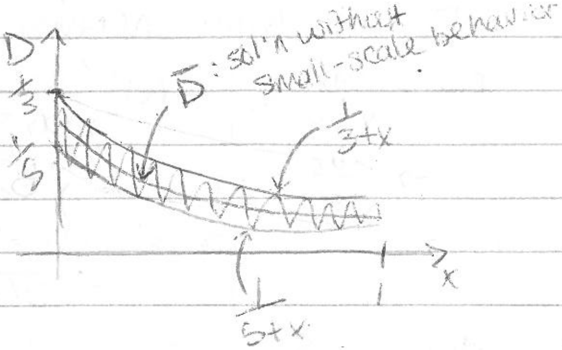
Common part $= F(x) - a/b\, t F'(x) + \varepsilon \tfrac{1}{b}\left(\tfrac{a}{b} F'(x) + G(x)\right)$

$$u \sim -\varepsilon e^{-bt/\varepsilon} \tfrac{1}{b}\left(\tfrac{a}{b} F'(x) + G(x)\right)$$
$$+ F(x - a/b\, t) + \varepsilon\Big[\tfrac{t}{b}(1 - a^2/b^2) F''(x - a/b\, t)$$
$$+ \tfrac{1}{b}\left(a/b\, F'(x - a/b\, t) + G(x - a/b\, t)\right)\Big]$$

Homogenization 5-21

$\frac{d}{dx}\left(D\frac{du}{dx}\right)=f(x), \quad 0<x<1, \quad u(0)=a, \; u(1)=b$

Ex: $D=\dfrac{1}{4+x+\cos(\frac{x}{\varepsilon})}$



$\frac{d}{dx}\left(\bar{D}\frac{du}{dx}\right)=f(x), \; 0<x<1$

$u(0)=a, \; u(1)=b$

guess: $\bar{D}=\frac{1}{4+x}$

let $y=\frac{x}{\varepsilon}$, $D=D(x,y)$

$\bar{D}=\lim_{y\to\infty}\frac{1}{y}\int_0^y D(x,s)\,ds$

$\lim_{y\to\infty}\frac{1}{y}\int_0^y \frac{1}{4+x+\cos s}\,ds \neq \frac{1}{4+x}$

Instead, use multiple scales.

$y=\frac{x}{\varepsilon}, \; u=u(x,y)$

$\frac{d}{dx}\to\partial/\partial x+\partial/\partial y\,\frac{1}{\varepsilon}$

$\to\left(\partial/\partial y+\varepsilon\,\partial/\partial x\right)\left(D(x,y)\left(\frac{\partial u}{\partial y}+\varepsilon\frac{\partial u}{\partial x}\right)\right)=\varepsilon^2 f(x)$

$u(x,y)\sim u_0+\varepsilon u_1+\varepsilon^2 u_2$

$u(x,y)$ bdd as $y\to\infty$

$O(1)$: $\partial/\partial y\left[D(x,y)\frac{\partial u_0}{\partial y}\right]=0$

$D(x,y)\frac{\partial u_0}{\partial y}=c_0(x)$

$u_0=\int_0^y\frac{c_0(x)}{D(x,s)}\,ds+c_1(x)$

Assume $0<D_m(x)\le D(x,y)\le D_M(x)$

$\int_0^y\frac{c_0(x)}{D(x,s)}\,ds\ge\int_0^y\frac{c_0(x)}{D_M(x)}=\frac{c_0(x)}{D_M(x)}\,y$

As $y\to\infty$, integral $\to\infty$

$\to c_0(x)\equiv 0$

$\to u_0=c_1(x)$

$O(\varepsilon)$: $\partial/\partial y\left(D(x,y)\frac{\partial u_1}{\partial y}\right) = -\partial/\partial y\left(D(x,y)\frac{\partial u_0}{\partial x}\right)$

$$D(x,y)\frac{\partial u_1}{\partial y} = -D(x,y)\frac{\partial u_0}{\partial x} + b_0(x)$$

$$u_1 = -\frac{\partial u_0}{\partial x} y + b_0(x)\int_0^y \frac{1}{D(x,s)} ds + b_1(x)$$

1st two terms increase linearly in y.

need $\lim_{y\to\infty} \frac{1}{y}\left[b_0(x)\int_0^y \frac{ds}{D(x,s)} - \frac{\partial u_0}{\partial x} y\right] = 0$

$$\rightarrow \frac{\partial u_0}{\partial x} = \lim_{y\to\infty} \frac{1}{y} b_0(x)\int_0^y \frac{ds}{D(x,s)}$$

$$= b_0(x) \lim_{y\to\infty} \frac{1}{y}\int_0^y \frac{ds}{D(x,s)}$$

$$= b_0(x) \left\langle \frac{1}{D(x,y)} \right\rangle_\infty$$

due to $\langle v(y)\rangle_\infty = \lim_{y\to\infty} \frac{1}{y}\int_0^y v(s) ds$

$O(\varepsilon^2)$: $\partial/\partial y\left(D(x,y)\frac{\partial u_2}{\partial y}\right) = f(x) - \partial/\partial y\left(D(x,y)\frac{\partial u_1}{\partial x}\right)$

$$-\partial/\partial x\left(D(x,y)\frac{\partial u_1}{\partial y}\right) - \partial/\partial x\left(D(x,y)\frac{\partial u_0}{\partial x}\right)$$

$$= f(x) - \partial/\partial y\left(D(x,y)\frac{\partial u_1}{\partial x}\right) - \partial/\partial x\, b_0(x)$$

$$D\frac{\partial u_2}{\partial y} = (f(x) - b_0'(x))y - D\frac{\partial u_1}{\partial x} + k_0(x)$$

$$u_2 = (f(x) - b_0'(x))\int_0^y \frac{s}{D(x,s)} ds - \frac{\partial u_1}{\partial x} y + \int_0^y \frac{k_0(x)}{D(x,s)} ds + k_1(x)$$

1st term: quadratic in y : need $f(x) = b_0'(x)$

and from $O(\varepsilon)$, we have

$$\frac{1}{\langle D^{-1}(x,y)\rangle_\infty} \frac{\partial u_0}{\partial x} = b_0(x)$$

$$\rightarrow \partial/\partial x\left[\frac{1}{\langle D^{-1}(x,y)\rangle_\infty}\frac{\partial u_0}{\partial x}\right] = b_0'(x) = f(x)$$

$$\frac{d}{dx}\left[\bar{D}\frac{du_0}{dx}\right] = f(x), \quad \bar{D} = \frac{1}{\left\langle \frac{1}{D(x,y)}\right\rangle_\infty} \quad \text{"harmonic mean"}$$

So for $D(x,y) = \frac{1}{4+x+\cos y}$

$$D^{-1}(x,y) = 4 + x + \cos y$$

$$\langle D^{-1}(x,y)\rangle_\infty = \lim_{y\to\infty} \frac{1}{y}\int_0^y 4 + x + \cos s\, ds$$

$$= 4 + x$$

$\rightarrow \bar{D} = \frac{1}{4+x}$, as suspected.

$\nabla_x \cdot (D \nabla_x u) = f(x)$ in Ω, $u = g$ on $\partial\Omega$

$D = D(\vec{x}, \frac{\vec{x}}{\varepsilon})$, let $\vec{x} = (x_1, x_2)$, $\vec{y} = \frac{\vec{x}}{\varepsilon}$

$= D(\vec{x}, \vec{y})$

Assume $D(y_1, y_2) = D(y_1 + p_1, y_2) = D(y_1, y_2 + p_2)$

for period $\vec{p} = (p_1, p_2)$

Also $D'(y_1, y_2) = D'(y_1 + p_1, y_2) = D'(y_1, y_2 + p_2)$

$\nabla_x \to \nabla_x + \frac{1}{\varepsilon}\nabla_y$

$(\nabla_y + \varepsilon\nabla_x) \cdot (D(x,y)(\nabla_y u + \varepsilon \nabla_x u)) = \varepsilon^2 f(x)$

$u(x,y) \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2$

require $\vec{p}$-periodic in $\vec{y}$.

$O(1)$: $\nabla_y \cdot (D(x,y) \nabla_y u_0) = 0$

$u_0 = u_0(x)$ (doesn't depend on y)

Why? Assume u has a maximum

Consider $\int_C D(x,y) \nabla_y u_0 \cdot \hat{n}\, dS = 0$, where C is a level curve surrounding the max

but on the level curve, $\frac{\partial u}{\partial \hat{n}} < 0$, so the integral $\neq 0$.

$\to$ no max/min

$\to$ $u_0 = u_0(x)$

$O(\varepsilon)$: $\nabla_y \cdot (D(x,y) \nabla_y u_1) = -\nabla_y \cdot (D(x,y) \nabla_x u_0)$

$= -(\nabla_y D) \cdot (\nabla_x u_0)$

$u_1(x,y) = \vec{a}(x,y) \cdot \nabla_x u_0$ $\leftarrow$ Assume + plug in

$\to \nabla_y \cdot \left(D\left(\frac{\partial a_1}{\partial y_1}\frac{\partial u_0}{\partial x_1} + \frac{\partial a_1}{\partial y_1}\frac{\partial u_0}{\partial x_2}, \frac{\partial a_1}{\partial y_2}\frac{\partial u_0}{\partial x_1} + \frac{\partial a_1}{\partial y_2}\frac{\partial u_0}{\partial x_2}\right)\right)$

$= -\left(\frac{\partial D}{\partial y_1}\frac{\partial u_0}{\partial x_1} + \frac{\partial D}{\partial y_2}\frac{\partial u_0}{\partial x_2}\right)$

$\to \frac{\partial}{\partial y_1}\left(D\frac{\partial a_1}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(D\frac{\partial a_1}{\partial y_2}\right) = \frac{-\partial D}{\partial y_1}$

$\to \nabla_y \cdot (D \nabla_y a_1) = -\frac{\partial D}{\partial y_1}$

Similarly, $\nabla_y \cdot (D \nabla_y a_2) = -\frac{\partial D}{\partial y_2}$

5-26 Ex: (recap)

$\nabla \cdot (D \nabla u) = f(\vec{x}), \quad \vec{x} \in \Omega$

$D = D(\vec{x}, \frac{\vec{x}}{\varepsilon}), \quad \vec{x} = (x_1, x_2)$

$= D(\vec{x}, \vec{y}), \quad \vec{y} = \frac{\vec{x}}{\varepsilon} = (y_1, y_2)$

Assume D periodic in y with period $\vec{p} = (p_1, p_2)$

$\rightarrow D(\vec{x}, y_1 + p_1, y_2) = D(\vec{x}, y_1, y_2)$

$D(\vec{x}, y_1, y_2 + p_2) = D(\vec{x}, y_1, y_2)$

+ Derivatives

$u(\vec{x}, \vec{y})$: $\vec{p}$-periodic in $\vec{y}$

$u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2$

$u_j = u_j(\vec{x}, \vec{y})$

$\nabla \rightarrow \nabla_x + \frac{1}{\varepsilon} \nabla_y$

$\rightarrow (\nabla_y + \varepsilon \nabla_x) \cdot [D(\nabla_y + \varepsilon \nabla_x) u] = \varepsilon^2 f$

$O(1)$: $\nabla_y \cdot [D \nabla_y u_0] = 0$

$\rightarrow u_0 = u_0(\vec{x})$

$O(\varepsilon)$: $\nabla_y \cdot [D \nabla_y u_1] = -\nabla_y \cdot [D \nabla_x u_0]$

$\rightarrow u_1(\vec{x}, \vec{y}) = \underbrace{b(x)}_{\text{homog. sol'n}} + \underbrace{\vec{a}(x,y) \cdot \nabla_x u_0}_{\text{particular sol'n}}$

$\nabla_y \cdot [D(\vec{x}, \vec{y}) \nabla_y a_j] = -\frac{\partial D}{\partial y_j}, \quad j = 1, 2$

$O(\varepsilon^2)$: $\nabla_y \cdot [D \nabla_y u_2] = f(x) - \nabla_y \cdot [D \nabla_x u_1] - \nabla_x \cdot [D \nabla_y u_1] - \nabla_x \cdot [D \nabla_x u_0]$

$\langle v \rangle_p = \frac{\iint_{\Omega_0} v(s)\, dA}{|\Omega_0|}$

$\langle \nabla_y \cdot [D \nabla_y u_2] \rangle_p = \frac{1}{|\Omega_0|} \iint_{\Omega_0} \nabla_y \cdot [D \nabla_y u_2]\, dA_y$

$= \frac{1}{|\Omega_0|} \int_{\partial \Omega_0} D \nabla_y u_2 \cdot \hat{n}\, dS_y$

$= 0$ due to periodicity of deriv.s

2^{nd} term on RHS $= 0$ similarly.

$\langle \nabla_x \cdot [D \nabla_x u_0] \rangle_p = \frac{1}{|\Omega_0|} \iint_{\Omega_0} \nabla_x \cdot [D \nabla_x u_0]\, dA_y$

$= \frac{1}{|\Omega_0|} \nabla_x \cdot \iint_{\Omega_0} D\, dA_y\, \nabla_x u_0$

$= \nabla_x \cdot \langle D \rangle_p \nabla_x u_0$

$\langle \nabla_x \cdot [D \nabla_y u_1] \rangle = \frac{1}{|\Omega_0|} \nabla_x \cdot \iint_{\Omega_0} D \nabla_y (\vec{a} \cdot \nabla_x u_0)\, dA_y$

$= \partial/\partial x_1 \langle D \frac{\partial a_1}{\partial y_1} \frac{\partial u_0}{\partial x_1} + D \frac{\partial a_2}{\partial y_1} \frac{\partial u_0}{\partial x_2} \rangle, \; \partial/\partial x_2 \langle D \frac{\partial a_1}{\partial y_2} \frac{\partial u_0}{\partial x_1} + D \frac{\partial a_2}{\partial y_2} \partial$

$$= \partial/\partial x_1 \left(\left\langle D \frac{\partial a_1}{\partial y_1} \right\rangle \frac{\partial u_0}{\partial x_1} \right) + \partial/\partial x_1 \left(\left\langle D \frac{\partial a_2}{\partial y_1} \right\rangle \frac{\partial u_0}{\partial x_2} \right)$$
$$+ \partial/\partial x_2 \left(\left\langle D \frac{\partial a_1}{\partial y_2} \right\rangle \frac{\partial u_0}{\partial x_1} \right) + \partial/\partial x_2 \left(\left\langle D \frac{\partial a_2}{\partial y_2} \right\rangle \frac{\partial u_0}{\partial x_2} \right)$$
$\rightarrow \nabla_x \cdot (\underbrace{\bar{D} \nabla_x u_0}) = f(x)$ after averaging entire $O(\varepsilon^2)$ eqn

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x_1} \\ \frac{\partial u_0}{\partial x_2} \end{pmatrix} = \begin{matrix} D_{11} \frac{\partial u_0}{\partial x_1} + D_{12} \frac{\partial u_0}{\partial x_2}, \\ D_{21} \frac{\partial u_0}{\partial x_1} + D_{22} \frac{\partial u_0}{\partial x_2} \end{matrix}$$

$$\bar{D} = \begin{pmatrix} \langle D \frac{\partial a_1}{\partial y_1} \rangle_p + \langle D \rangle_p & \langle D \frac{\partial a_2}{\partial y_1} \rangle_p \\ \langle D \frac{\partial a_1}{\partial y_2} \rangle_p & \langle D \frac{\partial a_2}{\partial y_2} \rangle_p + \langle D \rangle_p \end{pmatrix}$$

Ex: $D = D_0(x_1, x_2) e^{\alpha(y_1) + \beta(y_2)}$

$\alpha(y_1)$: a-periodic, $\beta(y_2)$: b-periodic

$\rightarrow a_1 = a_1(y_1) \quad a_2 = a_2(y_2)$

$\rightarrow \partial/\partial y_1 \left[D_0 e^{\alpha(y_1) + \beta(y_2)} \frac{\partial a_1}{\partial y_1} \right] = -D_0 \, \partial/\partial y_1 \, e^{\alpha(y_1) + \beta(y_2)}$

$$\rightarrow \begin{cases} \partial/\partial y_1 \left[e^{\alpha(y_1)} \frac{\partial a_1}{\partial y_1} \right] = -\partial/\partial y_1 \left[e^{\alpha(y_1)} \right] \\ \partial/\partial y_2 \left[e^{\beta(y_2)} \frac{\partial a_2}{\partial y_2} \right] = -\partial/\partial y_2 \left[e^{\beta(y_2)} \right] \end{cases}$$ similarly

$\partial a_1/\partial y_1 = -1 + c_1 e^{-\alpha(y_1)}$

$a_1(y_1) = -y_1 + c_1 \int_0^{y_1} e^{-\alpha(s)} ds$

$a_1(a) = a_1(0): \quad -a + c_1 \int_0^a e^{-\alpha(s)} ds = 0$

$\rightarrow c_1 = \frac{a}{\int_0^a e^{-\alpha(s)} ds}$

$\partial a_2/\partial y_2 = -1 + c_2 e^{-\beta(y_2)}, \; c_2 = b / \int_0^b e^{-\beta(s)} ds$

Off diagonal entries of $\bar{D}$ are zero.

$\langle D \, \partial a_1/\partial y_1 \rangle_p = \langle -D + c_1 D e^{-\alpha(y_1)} \rangle_p$

$= -\langle D \rangle_p + c_1 \langle D_0 e^{\beta(y_2)} \rangle$

$= -\langle D \rangle_p + c_1 D_0 \langle e^{\beta(y_2)} \rangle$

$= -\langle D \rangle_p + \frac{a D_0}{\int_0^a e^{-\alpha(s)} ds} \frac{a}{ab} \int_0^b e^{\beta(y_2)} dy_2$

$= -\langle D \rangle_p + \frac{a \int_0^b e^{\beta(y_2)} dy_2}{b \int_0^a e^{-\alpha(y_1)} dy_1} D_0$

$$\bar{D} = \begin{pmatrix} \lambda_1 D_0(x_1, x_2) & 0 \\ 0 & \lambda_2 D_0(x_1, x_2) \end{pmatrix}$$

$$\lambda_1 = \frac{a \int_0^b e^{\beta(s)} ds}{b \int_0^a e^{-\alpha(s)} ds}, \quad \lambda_2 = \frac{b \int_0^a e^{\alpha(s)} ds}{a \int_0^b e^{-\beta(s)} ds}$$

5-28 Multiple Scales in PDEs

Ex: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon \frac{\partial u}{\partial t}$, $0<x<1$, $t>0$

$u(x,0)=g(x)$, $u_t(x,0)=0$; $u(0,t)=u(1,t)=0$

$u(x,t) \sim u_0(x,t) + \varepsilon u_1(x,t)$

$O(1)$: $\frac{\partial^2 u_0}{\partial x^2} = \frac{\partial^2 u_0}{\partial t^2}$, $0<x<L$, $t>0$

$u_0(0,t) = u_0(1,t) = 0$; $u_0(x,0) = g(x)$, $u_{0t}(x,0) = 0$

$u_0(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin n\pi x$

$\rightarrow \sum_{n=1}^{\infty} (b_n'' + n^2\pi^2 b_n) \sin n\pi x = 0$

$u_0(x,0) = \sum_{n=1}^{\infty} b_n(0) \sin n\pi x = g(x)$

$u_{0t}(x,0) = \sum_{n=1}^{\infty} b_n'(0) \sin n\pi x = 0$

$\rightarrow b_n'' + n^2\pi^2 b_n = 0$, $b_n(0) = g_n = 2\int_0^1 g(x)\sin n\pi x\,dx$
$b_n'(0) = 0$

$b_n(t) = g_n \cos n\pi t$

$\rightarrow u_0(x,t) = \sum_{n=1}^{\infty} g_n \cos n\pi t \sin n\pi x$

$O(\varepsilon)$: $\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial u_0}{\partial t}$ $\quad u_1(0,t) = u_1(1,t) = 0$
$u_1(x,0) = u_{1t}(x,0) = 0$

$= \sum_{n=1}^{\infty} g_n(-n\pi \sin n\pi t) \sin n\pi x$

$u_1(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x$

$\rightarrow -n^2\pi^2 c_n - c_n'' = -n\pi g_n \sin n\pi t$

$c_n'' + n^2\pi^2 c_n = n\pi g_n \sin n\pi t$

$c_n(t) = -\frac{1}{2} g_n t \cos n\pi t + \frac{1}{2n\pi} g_n \sin n\pi t$

$u_1(x,t) = \sum_{n=1}^{\infty} \frac{1}{2} g_n \left(-t\cos n\pi t + \frac{1}{\pi n} \sin n\pi t\right) \sin n\pi x$

$\Rightarrow u(x,t) \sim \sum_{n=1}^{\infty} g_n \left[\cos n\pi t\left(1 - \frac{\varepsilon}{2} t\right) + \frac{\varepsilon}{2n\pi} \sin n\pi t\right] \sin n\pi x$

if $t = O(\frac{1}{\varepsilon})$, this term isn't a small correction

Exact solution:

$u(x,t) = \sum_{n=1}^{\infty} d_n(t) \sin n\pi x$

$\rightarrow d_n'' + \varepsilon d_n' + n^2\pi^2 d_n = 0$, $d_n(0) = g_n$, $d_n'(0) = 0$

$$a_n(t) = g_n e^{-\frac{1}{2}\varepsilon t}\left[\cos(\sqrt{n^2\pi^2 - \varepsilon^2/4}\, t) + \frac{\varepsilon}{2\sqrt{n^2\pi^2-\varepsilon^2/4}} \sin(\sqrt{n^2\pi^2-\varepsilon^2/4}\, t)\right]$$

$$\sim g_n (1 - \tfrac{1}{2}\varepsilon t)\left[\cos n\pi t + \frac{\varepsilon}{2n\pi} \sin n\pi t\right]$$

$$= g_n\left[(1-\tfrac{1}{2}\varepsilon t)\cos n\pi t + \frac{\varepsilon}{2n\pi}\sin n\pi t\right]$$

agrees with previous expansion

But for large t, we can't expand the exponential.

Multiple scales:

$t_0 = t,\ t_1 = \varepsilon t \ \to u(x, t_0, t_1)$

?

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 u}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 u}{\partial t_1^2} + \varepsilon \frac{\partial u}{\partial t_0} + \varepsilon^2 \frac{\partial u}{\partial t_1}$$

$u(x,0,0) = g(x),\ (u_{t_0} + \varepsilon u_{t_1})|_{t_0=t_1=0} = 0$

$u \sim u_0(x,t_0,t_1) + \varepsilon u_1(x,t_0,t_1)$

$O(1)$: $\dfrac{\partial^2 u_0}{\partial x^2} = \dfrac{\partial^2 u_0}{\partial t_0^2},\ u_0(x,0,0) = g(x),\ u_{t_0}(x,0,0) = 0$

$u_0 = \sum_{n=1}^{\infty} a_n(t_0,t_1)\sin n\pi x$

$a_{nt_0t_0} + n^2\pi^2 a_n = 0$

$a_n = c_{1n}(t_1)\cos n\pi t_0 + c_{2n}(t_1)\sin n\pi t_0$

$c_{1n}(0) = g_n,\ c_{2n}(0) = 0$

$O(\varepsilon)$: $\dfrac{\partial^2 u_1}{\partial x^2} - \dfrac{\partial^2 u_1}{\partial t_0^2} = 2\dfrac{\partial^2 u_0}{\partial t_0 \partial t_1} + \dfrac{\partial u_0}{\partial t_0}$

$= \sum_{n=1}^{\infty}\big[2c_{1n}'(-n\pi)\sin n\pi t_0 + 2c_{2n}'(n\pi)\cos n\pi t_0$
$\quad + c_{1n}(-n\pi)\sin n\pi t_0 + c_{2n}(n\pi)\cos n\pi t_0\big]$

$u_1(x,t_0,t_1) = \sum_{n=1}^{\infty} v_n(t_0,t_1)\sin n\pi x$

$-v_{nt_0t_0} - n^2\pi^2 v_n = -n\pi \sin n\pi t_0(2c_{1n}' + c_{1n}) + n\pi\cos n\pi t_0(2c_{2n}' + c_{2n})$

$\to\ 2c_{1n}' + c_{1n} = 0,\ 2c_{2n}' + c_{2n} = 0$

$c_{1n}(0) = g_n \quad c_{2n}(0) = 0$

$\to\ c_{1n}(t_1) = g_n e^{-\frac{1}{2}t_1},\ c_{2n}(t_1) = 0$

$\to\ u_0 = \sum_{n=1}^{\infty} g_n e^{-\frac{1}{2}t_1}\cos n\pi t_0 \sin n\pi x$

$\to\ u \sim \sum_{n=1}^{\infty} g_n e^{-\frac{1}{2}\varepsilon t}\cos n\pi t \sin n\pi x$

$c(\varepsilon x) \sim c(0) + \varepsilon x c'(0) + \cdots$

Ex: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} + c(\varepsilon x) u$, $0 < x < \pi$ $\quad u(x,0) = f(x)$

$u(0,t) = u(\pi,t) = 0$

Try regular expansion:

$u \sim u_0(x,t) + \varepsilon u_1(x,t)$

$O(1)$: $\dfrac{\partial^2 u_0}{\partial x^2} = \dfrac{\partial u_0}{\partial t} + c(0)u_0$ $\quad u(0,t) = u(\pi,t) = 0$

$u(x,0) = f(x)$

$u_0(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx$

$b_n' + (n^2 + c(0)) b_n = 0$, $\quad b_n(0) = f_n \equiv \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$b_n(t) = f_n e^{-(n^2 + c(0))t}$

$O(\varepsilon)$: $\dfrac{\partial^2 u_1}{\partial x^2} - \dfrac{\partial u_1}{\partial t} - c(0) u_1 = x c'(0) u_0$

$u_1(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin nx$

$-n^2 w_n - w_n' - c(0) w_n = x c'(0) f_n e^{(-n^2 - c(0))t}$

will give εt terms

6-2 Ex: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} + \varepsilon \dfrac{\partial u}{\partial t}$, $t > 0$, $-\infty < x < \infty$

$u(x,0) = F(x)$, $u_t(x,0) = 0$

Regular perturbation Expansion

$u(x,t) \sim u_0(x,t) + \varepsilon u_1(x,t)$

$O(1)$: $\dfrac{\partial^2 u_0}{\partial x^2} = \dfrac{\partial^2 u_0}{\partial t^2}$, $u_0(x,0) = F(x)$

$\partial u_0 / \partial t (x,0) = 0$

$\xi_1 = x - t$, $\xi_2 = x + t$

$\rightarrow \partial^2/\partial x^2 - \partial^2/\partial t^2 = (\partial/\partial\xi_1 + \partial/\partial\xi_2)^2 - (-\partial/\partial\xi_1 + \partial/\partial\xi_2)^2$

$= 4 \partial^2 / \partial\xi_1 \partial\xi_2$

$\rightarrow \dfrac{\partial^2 u_0}{\partial \xi_1 \partial \xi_2} = 0 \rightarrow \dfrac{\partial u_0}{\partial \xi_2} = f_0(\xi_2)$

$\rightarrow u_0 = \underbrace{\int f_0(\xi_2) d\xi_2}_{f(\xi_2)} + g(\xi_1)$

$f(x) + g(x) = F(x)$, $f'(x) = g'(x)$

$\rightarrow f(x) = g(x) + C$

$\rightarrow g(x) = \frac{1}{2}F(x) - \frac{1}{2}C \quad \rightarrow g(\xi_1) = \frac{1}{2}F(\xi_1) - \frac{1}{2}C$

$f(x) = \frac{1}{2}F(x) + \frac{1}{2}C \quad f(\xi_2) = \frac{1}{2}F(\xi_2) + \frac{1}{2}C$

$\rightarrow u_0 = \frac{1}{2}(F(x-t) + F(x+t))$

"D'Alembert Solution"

$O(\varepsilon)$: $\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial u_0}{\partial t}$

$$\rightarrow 4\frac{\partial^2 u_1}{\partial\xi_1\partial\xi_2} = -\frac{\partial u_0}{\partial\xi_1} + \frac{\partial u_0}{\partial\xi_2}$$

$$= \frac{1}{2}(-F'(\xi_1) + F'(\xi_2))$$

$$\rightarrow \frac{\partial u_1}{\partial\xi_2} = \frac{1}{8}\left[-F(\xi_1) + F'(\xi_2)\xi_1\right] + f_1(\xi_2)$$

$$u_1 = \frac{1}{8}\left[-F(\xi_1)\xi_2 + F(\xi_2)\xi_1\right] + f_2(\xi_2) + g_2(\xi_1)$$

$$u \sim \frac{1}{2}(F(x-t) + F(x+t)) + \varepsilon\Big[\frac{1}{8}(-(x+t)F(x-t) + (x-t)F(x+t)) + f_2(x+t) + g_2(x-t)\Big]$$

need scales εx & εt

$u = u(\xi_1, \xi_2, x_1, t_1) \quad \xi_1 = x - t,\ \xi_2 = x + t,\ x_1 = \varepsilon x,\ t_1 = \varepsilon t$

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2} + \varepsilon\frac{\partial}{\partial x_1}\right)^2 - \left(\frac{-\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2} + \varepsilon\frac{\partial}{\partial t_1}\right)^2$$

$$= \frac{4\partial^2}{\partial\xi_1\partial\xi_2} + 2\varepsilon\left(\frac{\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2}\right)\frac{\partial}{\partial x_1} - 2\varepsilon\left(\frac{-\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2}\right)\frac{\partial}{\partial t_1} + O(\varepsilon^2)$$

$$\rightarrow \frac{4\partial^2 u}{\partial\xi_1\partial\xi_2} = -2\varepsilon\left(\frac{\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2}\right)\frac{\partial u}{\partial x_1} + 2\varepsilon\left(\frac{-\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2}\right)\frac{\partial u}{\partial t_1} + \varepsilon\left(\frac{-\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2}\right)u + O(\varepsilon^2)$$

IC: at $\xi_1 = \xi_2,\ t_1 = 0$: $u(x, x, x_1, 0) = F(x)$

$$\left(-\partial u/\partial\xi_1 + \partial u/\partial\xi_2 + \varepsilon\frac{\partial u}{\partial t_1}\right)\Big|_{\substack{t_1=0\\ \xi_1=\xi_2=x}} = 0$$

$u \sim u_0 + \varepsilon u_1$

$O(1)$: $\frac{4\partial^2 u_0}{\partial\xi_1\partial\xi_2} = 0$, $\quad u_0(x, x, x_1, 0) = F(x)$

$\partial u_0/\partial\xi_1 = \partial u_0/\partial\xi_2$ at $\xi_1 = \xi_2 = x,\ t_1 = 0$

$\rightarrow u_0(\xi_1, \xi_2, x_1, t_1) = f(\xi_1, x_1, t_1) + g(\xi_2, x_1, t_1)$

$$\begin{cases} f(x, x_1, 0) + g(x, x_1, 0) = F(x) \\ \partial f/\partial x\ (x, x_1, 0) = \partial g/\partial x\ (x, x_1, 0) \end{cases}$$

$f(x, x_1, 0) = g(x, x_1, 0) + c(x_1)$

$O(\varepsilon)$: $\frac{4\partial^2 u_1}{\partial \xi_1 \partial \xi_2} = +2\left(\frac{-\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2}\right)\frac{\partial u_0}{\partial t_1} + \left(\frac{-\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2}\right) u_0$

$= \frac{\partial}{\partial \xi_1}\left(-2\frac{\partial f}{\partial t_1} - f\right) + \frac{\partial}{\partial \xi_2}\left(2\frac{\partial g}{\partial t_1} + g\right)$

$\rightarrow \begin{cases} 2\partial f/\partial t_1 + f = 0 & f(x,0) = \frac{1}{2}F(x) \\ 2\partial g/\partial t_1 + g = 0 & g(x,0) = \frac{1}{2}F(x) \end{cases}$

$\rightarrow \begin{cases} f(x,t) = \frac{1}{2}F(x)e^{-\frac{1}{2}t_1} \\ g(x,t) = \frac{1}{2}F(x)e^{-\frac{1}{2}t_1} \end{cases}$

$\rightarrow \begin{cases} f(\xi_1, t_1) = \frac{1}{2}F(\xi_1)e^{-\frac{1}{2}t_1} \\ g(\xi_2, t_1) = \frac{1}{2}F(\xi_2)e^{-\frac{1}{2}t_1} \end{cases}$

$\rightarrow u \sim \frac{1}{2}(F(x-t) + F(x+t))e^{-\frac{1}{2}\varepsilon t}$

Klein-Gordon Eqn

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + u + \varepsilon u^3$, $u(x,0) = F(x)$, $\frac{\partial u}{\partial t}(x,0) = G(x)$

$-\infty < x < \infty$, $t > 0$

$\varepsilon = 0$: $u_{xx} = u_{tt} + u$

$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx$

$\rightarrow \tilde{u}_{tt} + (k^2+1)\tilde{u} = 0$

$\tilde{u}(k,t) = A_1(k)e^{-i\omega t} + B_1(k)e^{+i\omega t}$ $\quad \omega = k^2+1$

$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)} + B(k)e^{i(kx+\omega t)}dk$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + u + \varepsilon u^3$

want $u_0(x,t) = \alpha\cos(kx - \omega t)$

$\Rightarrow \alpha\cos kx = F(x) \rightarrow u(x,0) = \alpha\cos kx$

and $u_t(x,0) = \alpha\omega\sin kx$

Regular perturbation method gives secular terms from the cubic term

let $\xi = kx - \omega t$, $u = u(\xi, x_1, t_1)$, $x_1 = \varepsilon x$, $t_1 = \varepsilon t$

$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} = \left(k\frac{\partial}{\partial \xi} + \varepsilon\frac{\partial}{\partial x_1}\right)^2 - \left(-\omega\frac{\partial}{\partial \xi} + \varepsilon\frac{\partial}{\partial t_1}\right)^2$ $\omega = k^2+1$

$\rightarrow \frac{\partial^2 u}{\partial \xi^2} - 2\varepsilon\left(k\frac{\partial}{\partial x_1} + \omega\frac{\partial}{\partial t_1}\right)\frac{\partial u}{\partial \xi} + u + \varepsilon u^3 = 0$

$+ O(\varepsilon^2)$

$u(kx, x_1, 0) = \alpha \cos kx$

$\left(-\omega \partial u/\partial \xi + \varepsilon \partial u/\partial t_1\right)\Big|_{\xi = kx,\, t_1 = 0} = \alpha \omega \sin kx$

$u \sim u_0 + \varepsilon u_1$

$O(1)$: $\dfrac{\partial^2 u_0}{\partial \xi^2} + u_0 = 0$, $u_0(kx, x_1, 0) = \alpha \cos kx$

$-\omega \partial u/\partial \xi\,(kx, x_1, 0) = \alpha \omega \sin kx$

$\rightarrow \partial u/\partial \xi\,(kx, x_1, 0) = -\alpha \sin kx$

$u_0(\xi, x_1, t) = A(x_1, t_1) \cos(\xi + \varphi(x_1, t_1))$

$u_0(\xi, x_1, 0) = A(x_1, 0) \cos(kx + \varphi(x_1, 0)) = \alpha \cos kx$

$A(x_1, 0) \sin(kx + \varphi(x_1, 0)) = \alpha \sin kx$

$\rightarrow A(x_1, 0) = \alpha$, $\varphi(x_1, 0) = 0$

$O(\varepsilon)$: $\dfrac{\partial^2 u_1}{\partial \xi^2} + u_1 = 2\left(k \dfrac{\partial}{\partial x_1} + \omega \dfrac{\partial}{\partial t_1}\right) \dfrac{\partial u_0}{\partial \xi} - u_0^3$

plug u_0 into RHS:

$= -2A\left(k \dfrac{\partial \varphi}{\partial x_1} + \omega \dfrac{\partial \varphi}{\partial t_1} + 3/8\, A^2\right) \cos(\xi + \varphi)$

$-2\left(k\, \partial A/\partial x_1 + \omega\, \partial A/\partial t_1\right) \sin(\xi + \varphi) - \frac{1}{4} A^3 \cos 3(\xi + \varphi)$

$\rightarrow \begin{cases} k\, \partial \varphi/\partial x_1 + \omega\, \partial \varphi/\partial t_1 + 3/8\, A^2 = 0 \\ k\, \partial A/\partial x_1 + \omega\, \partial A/\partial t_1 = 0 \end{cases}$

Solve using characteristics

let $r = \omega x_1 + k t_1$, $s = \omega x_1 - k t_1$

$\rightarrow \begin{cases} \partial A/\partial r = 0 \rightarrow A = A(s) \equiv \alpha \text{ from } A(x, 0) = \alpha \\ \partial \varphi/\partial r + \dfrac{3}{16 \omega k} A^2 = 0 \end{cases}$

$\partial \varphi/\partial r + \dfrac{3}{16 \omega k} \alpha^2 = 0$ solve for φ

$\rightarrow u \sim \alpha \cos\left(kx - \omega t\left(1 + \frac{3 \alpha^2 \varepsilon}{16 \omega^2}\right)\right)$

phase velocity increases due to nonlinearity

1) $u_{xx} = u_{tt} + \varepsilon u_t + \varepsilon f(x)\sin(\omega t)$, $0<x<1$, $t>0$
$u(0,t)=u(1,t)=0$, $u(x,0)=0$, $u_t(x,0)=0$

Assume $u \sim u_0 + \varepsilon u_1$

$O(1)$: $u_{0xx} = u_{0tt}$, $u_0(0,t)=u_0(1,t)=u_0(x,0)=u_{0t}(x,0)=0$

let $\xi_1 = x-t$, $\xi_2 = x+t$

$\rightarrow \partial^2/\partial x^2 - \partial^2/\partial t^2 = \left(\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2}\right)^2 - \left(-\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2}\right)^2$

$\rightarrow \frac{\partial^2 u_0}{\partial\xi_1\partial\xi_2} = 0$

$\rightarrow \frac{\partial u_0}{\partial\xi_2} = f_0(\xi_2)$

$u_0 = f(\xi_2) + g(\xi_1)$

$u_0(x,0) = f(x) + g(x) = 0 \rightarrow f(x) = -g(x)$

$u_{0t}(x,0) = -f'(x) + g'(x) = 0 \rightarrow f'(x) = g'(x) =$

$\rightarrow f(x) = g(x) + c \quad = -g(x)$

$\rightarrow 2g(x) = c \rightarrow g(x) = c_1$

$\rightarrow f(x) = -c_1$

$\rightarrow f = g = 0 \rightarrow u_0 \equiv 0$

$O(\varepsilon)$: $u_{1xx} - u_{1tt} = -u_{0t} + f(x)\sin(\omega t)$
$= f(x)\sin(\omega t)$

$\rightarrow 4u_{1\xi_1\xi_2} = f(\xi_1+\xi_2)\sin\frac{\omega}{2}(\xi_2-\xi_1)$

$u_{1\xi_1} = \frac{1}{4}\int_0^{\xi_2} f(\xi_1+s)\sin\frac{\omega}{2}(s-\xi_1)\,ds$

$\left(\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2}\right)\left(-\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2}\right)$

let $\eta_1 = x_1 - t_1$, $\eta_2 = x_1 + t_1$

$\to \partial/\partial t_1 = (-\partial/\partial\eta_1 + \partial/\partial\eta_2)$

$\partial/\partial x_1 = (\partial/\partial\eta_1 + \partial/\partial\eta_2)$

$\to \partial/\partial t_1 + \partial/\partial x_1 = \partial/\partial\eta_2$

$-\partial/\partial t_1 + \partial/\partial x_1 = \partial/\partial\eta_1$

$\to 2 f_{\eta_2} + \frac{A'(0)}{A(0)} f = 0$

$f = c_1 \exp(-\frac{1}{4}\frac{A'}{A}\eta_2)$

$g = c_2 \exp(-\frac{1}{4}\frac{A'}{A}\eta_1)$

$p = f + g$

$p|_{t=0} = h(x_0) = c_1 \exp(-\frac{1}{4}\frac{A'}{A}x_1) + c_2 \exp(-\frac{1}{4}\frac{A'}{A}x_1)$

$h(x_0) = (c_1 + c_2)\exp(-\frac{1}{4}\frac{A'}{A}x_1)$

$\to c_1 + c_2 = \dfrac{h(x_0)}{\exp(-\frac{1}{4}\frac{A'}{A}x_1)}$

$p_{t_0}|_{t=0} = 0 \to$

ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Homework 1 (DUE TUESDAY, 9/30/08)

Problem 1 (Holmes, # 1.3.1a, page 6).
What values of α, if any, yield $f = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$, $\varepsilon > 0$, where
(i) $f = \sqrt{1+\varepsilon^2}$
(ii) $f = \varepsilon \sin \varepsilon$ want $|f(\varepsilon)| \le K|\varepsilon^\alpha|$
(iii) $f = (1 - e^\varepsilon)^{-1}$
(iv) $f = \ln(1+\varepsilon)$
(v) $f = \varepsilon \ln \varepsilon$
(vi) $f = \sin(1/\varepsilon)$.
Find all such α in each case.

Problem 2 (Holmes, # 1.3.2c, page 6).
Give an example to show that $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$, $\varepsilon > 0$, does not necessarily mean that $e^f = O(e^g)$ as $\varepsilon \to 0$.

Problem 3 (Holmes, # 1.5.1a,d page 23).
Find a two-term asymptotic expansion in ε, $0 < \varepsilon \ll 1$, of each solution of the equations

(a) $x^2 + (1 - \varepsilon - \varepsilon^2)x + \varepsilon - 2e^{\varepsilon^2} = 0.$ Taylor expand

(d) $x^2 - 2x + (1 - \varepsilon^2)^{25} = 0.$ expand

Problem 4.
Determine the first four terms of the Taylor expansion of the function $f(x) = \sqrt{1 + \sin(x)}$ about $x = 0$, i.e., determine the coefficients of the expansion

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + O(x^4).$$

Haley Lepo
9/30/08
ESAM 420-1
Hwk #1

1) What values of α, if any, yield $f = O(\varepsilon^{\alpha})$ as $\varepsilon \to 0$, $\varepsilon > 0$?

i) $f = \sqrt{1+\varepsilon^2}$

find Taylor series: $f = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$

$f'(\varepsilon) = \varepsilon(1+\varepsilon^2)^{-1/2}$ $\quad f'(0) = 0$

$f''(\varepsilon) = (1+\varepsilon^2)^{-1/2} + \varepsilon(2\varepsilon)(-\tfrac{1}{2})(1+\varepsilon^2)^{-3/2}$

$= (1+\varepsilon^2)^{-1/2} - \varepsilon^2(1+\varepsilon^2)^{-3/2}$

$f''(0) = 1$

$f'''(\varepsilon) = -\varepsilon(1+\varepsilon^2)^{-3/2} - 2\varepsilon(1+\varepsilon^2)^{-3/2} + 3\varepsilon^3(1+\varepsilon^2)^{-5/2}$

$= -3\varepsilon(1+\varepsilon^2)^{-3/2} + 3\varepsilon^3(1+\varepsilon^2)^{-5/2}$

$f'''(0) = 0$

$f^{(4)}(\varepsilon) = -3(1-\varepsilon^2)^{-3/2} + 9\varepsilon^2(1+\varepsilon^2)^{-5/2}$
$+ 9\varepsilon^2(1+\varepsilon^2)^{-5/2} - 15\varepsilon^4(1+\varepsilon^2)^{-7/2}$

$= -3(1-\varepsilon^2)^{-3/2} + 18\varepsilon^2(1+\varepsilon^2)^{-5/2} - 15\varepsilon^4(1+\varepsilon^2)^{-7/2}$

$f^{(4)}(0) = -3$

$f = 1 + \frac{1}{2}(1)\varepsilon^2 + \frac{1}{24}(-3)\varepsilon^4 + \cdots$

$= 1 + \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \cdots$

Now test: $f(\varepsilon) = O(\phi(\varepsilon)) \leftrightarrow \lim_{\varepsilon\to 0} f(\varepsilon)/\phi(\varepsilon)$ finite

$$\lim_{\varepsilon \to 0} \frac{1 + \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \cdots}{\varepsilon^{\alpha}}$$

$= \lim_{\varepsilon\to 0} \varepsilon^{-\alpha} + \frac{1}{2}\varepsilon^{2-\alpha} - \frac{1}{8}\varepsilon^{4-\alpha}$

This is finite when the exponents ≤ 0

The smallest is $-\alpha$, so we need

$-\alpha \leq 0 \rightarrow \alpha \geq 0$

$\boxed{f = O(\varepsilon^{\alpha}) \text{ for } \alpha \geq 0.}$ +

1) (continued)

ii) $f = \varepsilon \sin\varepsilon$, $f(0) = 0$

$f'(\varepsilon) = \sin\varepsilon + \varepsilon\cos\varepsilon$, $f'(0) = 0$

$f''(\varepsilon) = 2\cos\varepsilon - \varepsilon\sin\varepsilon$, $f''(0) = 2$

$f'''(\varepsilon) = -2\sin\varepsilon - \sin\varepsilon - \varepsilon\cos\varepsilon$, $f'''(0) = 0$

$f^{(4)}(\varepsilon) = -4\cos\varepsilon + \varepsilon\sin\varepsilon$, $f^{(4)}(0) = -4$

$f = \varepsilon^2 - \frac{1}{6}\varepsilon^4 + \cdots$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^2 - \frac{1}{6}\varepsilon^4 + \cdots}{\varepsilon^\alpha} = \lim_{\varepsilon\to 0} \varepsilon^{2-\alpha} - \frac{1}{6}\varepsilon^{4-\alpha} + \cdots$$

?

For the limit to be finite, we need $2-\alpha \geq 0$

$\rightarrow \alpha \geq 2$

$\boxed{f = O(\varepsilon^\alpha) \text{ for } \alpha \geq 2.}$ $\pm$

iii) $f = (1-e^\varepsilon)^{-1}$

$e^\varepsilon = 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + O(\varepsilon^4)$

So $f = (1 - (1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + O(\varepsilon^4)))^{-1}$

$= (-(\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + O(\varepsilon^4)))^{-1}$

$$\lim_{\varepsilon\to 0} \frac{-1}{\varepsilon^\alpha(\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + O(\varepsilon^4))}$$

$$= \lim_{\varepsilon\to 0} \frac{\varepsilon^{-\alpha}}{\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + O(\varepsilon^4)}$$

For this limit to be finite, we need a non-zero term in the denominator.

So we need $-\alpha \geq 1$

$\rightarrow \alpha \leq -1$

$\boxed{f = O(\varepsilon^\alpha) \text{ for } \alpha \leq -1}$ $+$

1) (continued)

(iv) $f = \ln(1+\varepsilon)$, $f(0) = 0$

$f'(\varepsilon) = \dfrac{1}{1+\varepsilon}$, $f'(0) = 1$

$f''(\varepsilon) = \dfrac{-1}{(1+\varepsilon)^2}$, $f''(0) = -1$

$f'''(\varepsilon) = \dfrac{2}{(1+\varepsilon)^3}$, $f'''(0) = 2$

$f^{(4)}(\varepsilon) = \dfrac{-6}{(1+\varepsilon)^4}$, $f^{(4)}(0) = -6$

$f = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \frac{1}{4}\varepsilon^4 + \cdots$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \frac{1}{4}\varepsilon^4 + \cdots}{\varepsilon^\alpha} = \lim_{\varepsilon\to 0} \varepsilon^{1-\alpha} - \tfrac{1}{2}\varepsilon^{2-\alpha} + \tfrac{1}{3}\varepsilon^{3-\alpha} - \cdots$$

For the limit to be finite, we need $1-\alpha \leq 0$

$\rightarrow \alpha \geq 1$

$\boxed{f = O(\varepsilon^\alpha) \text{ for } \alpha \geq 1.}$

v) $f = \varepsilon \ln \varepsilon$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon \ln \varepsilon}{\varepsilon^\alpha} = \lim_{\varepsilon\to 0} \frac{\ln \varepsilon}{\varepsilon^{\alpha-1}}$$

Since the natural log grows more slowly than any polynomial of degree ≥ 1, we need $\alpha - 1 \geq 1 \rightarrow \alpha \geq 2$.

$\boxed{f = O(\varepsilon^\alpha) \text{ for } \alpha \geq 2.}$

1)(continued)

vi) $f = \sin\frac{1}{\varepsilon}$

Claim: $f = O(\varepsilon^{\alpha})$ for $\alpha \leq 0$

Must Show: $|\sin\frac{1}{\varepsilon}| \leq K\varepsilon^{\alpha}$ where K is some number not dependent on ε.

Consider $\alpha \leq 0$: let $\beta = -\alpha$, $\beta > 0$

Since $\varepsilon < 1$, we have

$\varepsilon^{\beta}|\sin\frac{1}{\varepsilon}| \leq |\sin\frac{1}{\varepsilon}|$

And since |sine| is bounded by 1,

$|\sin\frac{1}{\varepsilon}| \leq 1$

So $\varepsilon^{\beta}|\sin\frac{1}{\varepsilon}| \leq 1$

$\rightarrow |\sin\frac{1}{\varepsilon}| \leq \varepsilon^{-\beta} = \varepsilon^{\alpha}$

So we have shown there exists a K, namely K=1, for $\alpha \leq 0$.

Consider $\alpha > 0$:

We would need $|\sin\frac{1}{\varepsilon}| \leq K\varepsilon^{\alpha}$.

But, however small ε gets, $|\sin\frac{1}{\varepsilon}|$ will always take on a value of 1 at an even smaller value of ε.

That is, for any ε_0 near zero, there exists $0 < \varepsilon_1 < \varepsilon_0$ such that $|\sin\frac{1}{\varepsilon_1}| = 1$.

Thus there exists no K to make the inequality true.

$\boxed{f = O(\varepsilon^{\alpha}) \text{ for } \alpha \leq 0}$ +

4/6

Lepo 3

2) Give an example to show that $f(\varepsilon)=O(g(\varepsilon))$ as $\varepsilon\to 0$, $\varepsilon>0$ does not necessarily mean that $e^f=O(e^g)$ as $\varepsilon\to 0$

Let $f=\ln\varepsilon$, $g=2\ln\varepsilon$ ✓

Then since $\lim_{\varepsilon\to 0}\frac{\ln\varepsilon}{2\ln\varepsilon}=\frac{1}{2}$, $f(\varepsilon)=O(g(\varepsilon))$.

However, consider $e^f=e^{\ln\varepsilon}=\varepsilon$, $e^g=e^{2\ln\varepsilon}=\varepsilon^2$

Then $\lim_{\varepsilon\to 0}\frac{e^f}{e^g}=\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon^2}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\to\infty$

Since this limit is infinite, it is not the case that $e^f=O(e^g)$.

2/2

3) Find a two-term asymptotic expansion in ε, $0<\varepsilon\ll 1$

A) $x^2+(1-\varepsilon-\varepsilon^2)x+\varepsilon-2e^{\varepsilon^2}=0$

Find Taylor expansion for $-2e^{\varepsilon^2}=f$, $f(0)=-2$

$f'=-4\varepsilon e^{\varepsilon^2}$, $f'(0)=0$

$f''=-4e^{\varepsilon^2}-8\varepsilon^2 e^{\varepsilon^2}$, $f''(0)=-4$

$f'''=-8\varepsilon e^{\varepsilon^2}-16\varepsilon e^{\varepsilon^2}-16\varepsilon^3 e^{\varepsilon^2}$

$=-24\varepsilon e^{\varepsilon^2}-16\varepsilon^3 e^{\varepsilon^2}$, $f'''(0)=0$

$\to x^2+(1-\varepsilon-\varepsilon^2)x+\varepsilon-2-2\varepsilon^2+O(\varepsilon^4)=0$

Assume $x=x_0+x_1\varepsilon+x_2\varepsilon^2+O(\varepsilon^3)$

Plug in: $(x_0+x_1\varepsilon+x_2\varepsilon^2+O(\varepsilon^3))^2$

$+(1-\varepsilon-\varepsilon^2)(x_0+x_1\varepsilon+x_2\varepsilon^2+O(\varepsilon^3))+\varepsilon-2-2\varepsilon^2+O(\varepsilon^4)=0$

?

$\to x_0^2+2x_0x_1\varepsilon+2x_0x_2\varepsilon^2+O(\varepsilon^3)+x_0+x_1\varepsilon+x_2\varepsilon^2$ $x_1^2\varepsilon^2$

$-x_0\varepsilon-x_1\varepsilon^2-x_0\varepsilon^2+\varepsilon-2-2\varepsilon^2=0$

$\varepsilon^2(x_0x_2+x_2-x_1-x_0-2)+\varepsilon(x_0x_1+x_1-x_0+1)$

$+(x_0^2+x_0-2)+O(\varepsilon^3)=0$

$\to\begin{cases}x_0x_2+x_2-x_1-x_0-2=0\\ x_0x_1+x_1-x_0+1=0\\ x_0^2+x_0-2=0\end{cases}$

continued on reverse

3) (continued)

A) (continued)

$x_0^2 + x_0 - 2 = 0 \rightarrow (x_0+2)(x_0-1) = 0$

$\rightarrow x_0 = -2, 1$

$x_0 x_1 + x_1 - x_0 + 1 = 0$

$\rightarrow x_0 = -2: -2x_1 + x_1 + 2 + 1 = 0 \rightarrow -x_1 + 3 = 0 \rightarrow x_1 = 3$

$x_0 = 1: x_1 + x_1 - 1 + 1 = 0 \rightarrow x_1 = 0$

$x_0 x_2 + x_2 - x_1 - x_0 - 2 = 0$

$x_0 = -2$ & $x_1 = 3$: $\rightarrow -2x_2 + x_2 - 3 + 2 - 2 = 0$

$\rightarrow -x_2 - 3 = 0$

$\rightarrow x_2 = -3$

$x_0 = 1$ & $x_1 = 0$:

$x_2 + x_2 - 1 - 2 = 0$

$2x_2 = 3$

$x_2 = 3/2$

So $x_0 = -2, x_1 = 3\ x_2 = -3$ OR $x_0 = 1, x_1 = 0, x_2 = -3$

$$\boxed{\begin{array}{l} ① \ x = -2 + 3\varepsilon + O(\varepsilon^2) \\ ② \ x = 1 - 3\varepsilon^2 + O(\varepsilon^3) \end{array}}$$

2/3

D) $x^2 - 2x + (1-\varepsilon^2)^{25} = 0$

Assume $x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3)$

Plugin: $(x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3))^2$

$-2(x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3)) + (1-\varepsilon^2)^{25} = 0$

$\rightarrow +x_0^2 + 2x_0x_1\varepsilon + 2x_0x_2\varepsilon^2 + O(\varepsilon^3) - 2x_0 - 2x_1\varepsilon$

$-2x_2\varepsilon^2 + 1 - 25\varepsilon^2 = 0$

$\rightarrow (x_0^2 - 2x_0 + 1) + \varepsilon(x_0x_1 - 2x_1)$

$+ \varepsilon^2(x_0x_2 - 2x_2 - 25) + O(\varepsilon^3) = 0$

$$\rightarrow \begin{cases} x_0^2 - 2x_0 + 1 = 0 \\ x_0x_1 - 2x_1 = 0 \\ x_0x_2 - 2x_2 - 25 = 0 \end{cases}$$

$\rightarrow (x_0 - 1)^2 = 0 \rightarrow x_0 = 1$ (double root)

$\rightarrow x_1 - 2x_1 = 0 \rightarrow x_1 = 0$

? $\rightarrow x_2 - 2x_2 - 25 = 0 \rightarrow -x_2 = 25 \rightarrow x_2 = -25$

$\rightarrow \boxed{x = 1 - 25\varepsilon^2 + O(\varepsilon^3)}$

2/3

Lepo 4

4) Determine the first four terms of the Taylor expansion of the function $f(x) = \sqrt{1+\sin(x)}$ about $x = 0$.

Taylor series def'n: $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$

$f(x) = (1+\sin x)^{\frac{1}{2}} \quad f(0) = 1$

$f'(x) = \frac{1}{2}\cos x(1+\sin x)^{-\frac{1}{2}}$

$f'(0) = \frac{1}{2}\cdot 1 \cdot 1 = \frac{1}{2}$

$f''(x) = -\frac{1}{2}\sin x(1+\sin x)^{-\frac{1}{2}} - \frac{1}{4}\cos^2 x(1+\sin x)^{-3/2}$

$f''(0) = 0 - \frac{1}{4}\cdot 1 \cdot 1 = -\frac{1}{4}$

$f'''(x) = -\frac{1}{2}\cos x(1+\sin x)^{-\frac{1}{2}} + \frac{1}{4}\sin x\cos x(1+\sin x)^{-3/2}$
$+ \frac{1}{2}\sin x\cos x(1+\sin x)^{-3/2} + \frac{3}{8}\cos^3 x(1+\sin x)^{-5/2}$

$f'''(0) = -\frac{1}{2}\cdot 1 \cdot 1 + 0 + 0 + \frac{3}{8}\cdot 1 \cdot 1$
$= -\frac{4}{8} + \frac{3}{8} = -\frac{1}{8}$

$f(x) = 1\cdot 1 + \frac{1}{1!}\cdot\frac{1}{2}x + \frac{1}{2!}\cdot\left(-\frac{1}{4}\right)x^2 + \frac{1}{3!}\cdot\left(-\frac{1}{8}\right)x^3 + O(x^4)$
$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 + O(x^4)$

$$\boxed{\begin{array}{l} f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 + O(x^4) \checkmark \\ \text{i.e. } c_0 = 1,\ c_1 = \frac{1}{2},\ c_2 = -\frac{1}{8},\ c_3 = -\frac{1}{48} \end{array}}$$

2/2

1	2	3	4	
4	2	4	2	12
6	2	6	2	16

ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Homework 2 (DUE TUESDAY, 10/7/2008)

Problem 1.
Find a two-term asymptotic expansion in ε, $0 < \varepsilon \ll 1$, of **each** real solution of the equations

(a) $x^3 - 3x^2 + (3-\varepsilon)x + \varepsilon - 1 = 0,$

(b) $\varepsilon^2 x^3 + 2\varepsilon x^2 + x + \varepsilon = 0,$

(c) $\varepsilon e^x = 1 + \dfrac{\varepsilon}{1+x},$

(d) $x^2 = \varepsilon \sin x,$

(e) $x^3 - 3x^2 + (3-\varepsilon)x + \varepsilon = \sin \dfrac{\pi(x+\varepsilon)}{2}.$

Haley Lepo
10-7-08
ESAM 420
Hwk #2

1) Find a two-term asymptotic expansion in ε, $0<\varepsilon\ll 1$ of each real solution of the equations

A) $x^3-3x^2+(3-\varepsilon)x+\varepsilon-1=0$

Assume $x=x_0+x_1\varepsilon+x_2\varepsilon^2+x_3\varepsilon^3+O(\varepsilon^4)$

Plug in:

$$(x_0+x_1\varepsilon+x_2\varepsilon^2+x_3\varepsilon^3+O(\varepsilon^4))^3$$
$$-3(x_0+x_1\varepsilon+x_2\varepsilon^2+x_3\varepsilon^3+O(\varepsilon^4))^2$$
$$+(3-\varepsilon)(x_0+x_1\varepsilon+x_2\varepsilon^2+x_3\varepsilon^3+O(\varepsilon^4))$$
$$+\varepsilon-1=0$$

$$\to x_0^3+x_1^3\varepsilon^3+3x_0^2x_1\varepsilon+3x_0^2x_2\varepsilon^2+3x_0^2x_3\varepsilon^3$$
$$+3x_1^2x_0\varepsilon^2+3x_0x_1x_2\varepsilon^3+O(\varepsilon^4)$$
$$-3x_0^2-3x_1^2\varepsilon^2-6x_0x_1\varepsilon-6x_0x_2\varepsilon^2-6x_0x_3\varepsilon^3$$
$$-6x_1x_2\varepsilon^3+3x_0+3x_1\varepsilon+3x_2\varepsilon^2+3x_3\varepsilon^3-x_0\varepsilon$$
$$-x_1\varepsilon^2-x_2\varepsilon^3+\varepsilon-1=0$$

$$\to O(\varepsilon^4)+\varepsilon^3(x_1^3+3x_0^2x_3+3x_0x_1x_2-6x_0x_3$$
$$-6x_1x_2+3x_3-x_2)+\varepsilon^2(3x_0^2x_2+3x_1^2x_0$$
$$-3x_1^2-6x_0x_2+3x_2-x_1)+\varepsilon(3x_0^2x_1-6x_0x_1$$
$$+3x_1-x_0+1)+(x_0^3-3x_0^2+3x_0-1)=0$$

$\to \varepsilon^0$: $x_0^3-3x_0^2+3x_0-1=0\to(x_0-1)^3=0\to x_0=1$

ε^1: $3x_1-6x_1+3x_1=0\to 0=0$

ε^2: $\cancel{3x_2}+\cancel{3x_1^2}-\cancel{3x_1^2}-\cancel{6x_2}+\cancel{3x_2}-x_1=0$

$\to -x_1=0\to x_1=0$

ε^3: $\cancel{3x_3}-\cancel{6x_3}+\cancel{3x_3}-x_2=0$

$\to x_2=0$

$\to x=1+O(\varepsilon^4)$

(there should be three solutions...
but I don't know the other two)

A) (continued)

Try iterations:

$$x^3 - 3x^2 + 3x - x\varepsilon + \varepsilon - 1 = 0$$

$$x_n = \frac{1 - \varepsilon + x_{n-1}\varepsilon + 3x_{n-1}^2 - x_{n-1}^3}{3}$$

Guess $x_0 = 0$

$$x_1 = (1-\varepsilon)/3 = \tfrac{1}{3} - \tfrac{1}{3}\varepsilon \qquad (1-\varepsilon)$$

$$x_2 = \frac{1 - \varepsilon + (\varepsilon - \varepsilon^2)/3 + \frac{1}{3}(1 + \varepsilon^2 - 2\varepsilon) - (1 + \varepsilon^2 - 2\varepsilon - \varepsilon - \varepsilon^4 + 2\varepsilon^2)}{3}$$

$$= \tfrac{1}{3}\left[\cancel{1} - \varepsilon + \tfrac{1}{3}\varepsilon - \tfrac{1}{3}\varepsilon^2 + \tfrac{1}{3} + \tfrac{1}{3}\varepsilon^2 - \tfrac{2}{3}\varepsilon - \cancel{1} - 3\varepsilon^2 + 3\varepsilon + \varepsilon^4\right.$$

$$= \tfrac{1}{9}$$

not good...

1/3

B) $\varepsilon^2 x^3 + 2\varepsilon x^2 + x + \varepsilon = 0$

Reduced problem: $x = 0$ $\rightarrow$ singular!

Try iterations:

$$x = -\varepsilon^2 x^3 - 2\varepsilon x^2 - \varepsilon$$

$$x_n = -\varepsilon^2 x_{n-1}{}^3 - 2\varepsilon x_{n-1}{}^2 - \varepsilon$$

Let $x_0 = 0$

$$x_1 = -\varepsilon$$

$$x_2 = -\varepsilon^5 - 2\varepsilon^3 - \varepsilon$$

$$x_3 = -\varepsilon^2(-\varepsilon^5 - 2\varepsilon^3 - \varepsilon)^3 - 2\varepsilon(-\varepsilon^5 - 2\varepsilon^3 - \varepsilon)^2 - \varepsilon$$

$$= -\varepsilon^2(-\varepsilon^3 - 4\varepsilon^5 + O(\varepsilon^6)) - 2\varepsilon(\varepsilon^2 + 4\varepsilon^4 + O(\varepsilon^6)) - \varepsilon$$

$$= \varepsilon^5 + O(\varepsilon^6) - 2\varepsilon^3 - 8\varepsilon^5 - \varepsilon$$

$$= -\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6)$$

$$x_4 = -\varepsilon^2(-\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6))^3$$

$$- 2\varepsilon(-\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6))^2 - \varepsilon$$

$$= -\varepsilon^2(-\varepsilon^3 - 6\varepsilon^5 + O(\varepsilon^6)) - 2\varepsilon(\varepsilon^2 + 4\varepsilon^4 + O(\varepsilon^6)) - \varepsilon$$

$$= \varepsilon^5 + O(\varepsilon^6) - 2\varepsilon^3 - 8\varepsilon^5 - \varepsilon$$

$$= -\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6)$$

$$x = -\varepsilon - 2\varepsilon^3 + O(\varepsilon^5)$$

There should be three solutions

(continued next page)

A) (continued)

Try iterations:

$$x^3 - 3x^2 + 3x - x\varepsilon + \varepsilon - 1 = 0$$

$$x_n = \frac{1 - \varepsilon + x_{n-1}\varepsilon + 3x_{n-1}^2 - x_{n-1}^3}{3}$$

Guess $x_0 = 0$

$$x_1 = (1-\varepsilon)/3 = \tfrac{1}{3} - \tfrac{1}{3}\varepsilon \qquad (1-\varepsilon)$$

$$x_2 = \frac{1 - \varepsilon + (\varepsilon - \varepsilon^2)/3 + \frac{1}{3}(1 + \varepsilon^2 - 2\varepsilon) - (1 + \varepsilon^2 - 2\varepsilon - \varepsilon - \varepsilon^4 + 2\varepsilon^2)}{3}$$

$$= \tfrac{1}{3}\left[\cancel{1} - \varepsilon + \tfrac{1}{3}\varepsilon - \tfrac{1}{3}\varepsilon^2 + \tfrac{1}{3} + \tfrac{1}{3}\varepsilon^2 - \tfrac{2}{3}\varepsilon - \cancel{1} - 3\varepsilon^2 + 3\varepsilon + \varepsilon^4\right.$$

$$= \tfrac{1}{9}$$

not good...

$1/3$

B) $\varepsilon^2 x^3 + 2\varepsilon x^2 + x + \varepsilon = 0$

Reduced problem: $x = 0$ $\rightarrow$ singular!

Try iterations:

$$x = -\varepsilon^2 x^3 - 2\varepsilon x^2 - \varepsilon$$

$$x_n = -\varepsilon^2 x_{n-1}^3 - 2\varepsilon x_{n-1}^2 - \varepsilon$$

Let $x_0 = 0$

$$x_1 = -\varepsilon$$

$$x_2 = -\varepsilon^5 - 2\varepsilon^3 - \varepsilon$$

$$x_3 = -\varepsilon^2(-\varepsilon^5 - 2\varepsilon^3 - \varepsilon)^3 - 2\varepsilon(-\varepsilon^5 - 2\varepsilon^3 - \varepsilon)^2 - \varepsilon$$

$$= -\varepsilon^2(-\varepsilon^3 - 4\varepsilon^5 + O(\varepsilon^6)) - 2\varepsilon(\varepsilon^2 + 4\varepsilon^4 + O(\varepsilon^6)) - \varepsilon$$

$$= \varepsilon^5 + O(\varepsilon^6) - 2\varepsilon^3 - 8\varepsilon^5 - \varepsilon$$

$$= -\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6)$$

$$x_4 = -\varepsilon^2(-\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6))^3 - 2\varepsilon(-\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6))^2 - \varepsilon$$

$$= -\varepsilon^2(-\varepsilon^3 - 6\varepsilon^5 + O(\varepsilon^6)) - 2\varepsilon(\varepsilon^2 + 4\varepsilon^4 + O(\varepsilon^6)) - \varepsilon$$

$$= \varepsilon^5 + O(\varepsilon^6) - 2\varepsilon^3 - 8\varepsilon^5 - \varepsilon$$

$$= -\varepsilon - 2\varepsilon^3 - 7\varepsilon^5 + O(\varepsilon^6)$$

$$x = -\varepsilon - 2\varepsilon^3 + O(\varepsilon^5)$$

There should be three solutions

(continued next page)

B) (continued)

Large terms: x for sure

Then either $\varepsilon^2 x^3$ or $2\varepsilon x^2$

$x = \varepsilon^2 x^3$	$x = 2\varepsilon x^2$
$1 = \varepsilon^2 x^2$	$1 = 2\varepsilon x$
$1/\varepsilon^2 = x^2$	$x = 1/2\varepsilon$
$x = 1/\varepsilon$	

In either case $x \approx a/\varepsilon$

Use the substitution $x = a/\varepsilon$

$\rightarrow \varepsilon^2 (a/\varepsilon)^3 + 2\varepsilon (a/\varepsilon)^2 + a/\varepsilon + \varepsilon = 0$

$\rightarrow \varepsilon^2 a^3/\varepsilon^3 + 2\varepsilon a^2/\varepsilon^2 + a/\varepsilon + \varepsilon = 0$

$\rightarrow a^3/\varepsilon + 2a^2/\varepsilon + a/\varepsilon + \varepsilon = 0$

$\rightarrow a^3 + 2a^2 + a + \varepsilon^2 = 0$

Assume $a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3)$

Plugin: $(a_0 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3))^3$

$+ 2(a_0 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3))^2 + (a_0 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3)) + \varepsilon^2 = 0$

$\rightarrow a_0^3 + 3a_0^2 a_1 \varepsilon + 3a_0 a_1^2 \varepsilon^2 + a_0^2 a_2 \varepsilon^2 + O(\varepsilon^3)$

$+ 2a_0^2 + 2a_1\varepsilon^2 + 4a_0 a_1 \varepsilon + 4a_0 a_2 \varepsilon^2$

$+ a_0 + a_1\varepsilon + a_2\varepsilon^2 + \varepsilon^2 = 0$

$\rightarrow O(\varepsilon^3) + \varepsilon^2(3a_0 a_1^2 + a_0^2 a_2 + 2a_1 + 4a_0 a_2 + a_2 + 1)$

$+ \varepsilon(3a_0^2 a_1 + 4a_0 a_1 + a_1) + (a_0^3 + 2a_0^2 + a_0) = 0$

$\rightarrow a_0^3 + 2a_0^2 + a_0 = 0$

$\rightarrow a_0(a_0^2 + 2a_0 + 1) = 0 \rightarrow a_0(a_0+1)^2 = 0$

$\rightarrow a_0 = 0, -1, -1$

$3a_0^2 a_1 + 4a_0 a_1 + a_1 = 0$

$a_0 = 0 \rightarrow a_1 = 0$

$a_0 = -1 \rightarrow 3a_1 - 4a_1 + a_1 = 0 \rightarrow 0 = 0$

$3a_0 a_1^2 + a_0^2 a_2 + 2a_1 + 4a_0 a_2 + a_2 + 1 = 0$

$a_0 = 0, a_1 = 0 \rightarrow a_2 + 1 = 0 \rightarrow a_2 = -1$

$a_0 = -1 \rightarrow -3a_1 + a_2 + 2a_1 - 4a_2 + a_2 + 1 = 0$

$-a_1 - 2a_2 + 1 = 0 \rightarrow a_2 = \frac{1-a_1}{2}$

So from this we have

① $a = -\varepsilon^2 + O(\varepsilon^3)$

② $a = -1 + a_1\varepsilon + \frac{1-a_1}{2}\varepsilon^2 + O(\varepsilon^3)$

Giving

① $x = -\varepsilon + O(\varepsilon^2)$

(This agrees with our previous solution

② $x = -\frac{1}{\varepsilon} + a_1 + \frac{1-a_1}{2}\varepsilon + O(\varepsilon^2)$

$x = -\frac{1}{\varepsilon} + a_1 + O(\varepsilon)$

Let's try to find a_1:

Plug $x = a_1 - \frac{1}{\varepsilon}$ into the original equation

$\varepsilon^2(a_1 - \frac{1}{\varepsilon})^3 + 2\varepsilon(a_1 - \frac{1}{\varepsilon})^2 + a_1 - \frac{1}{\varepsilon} + \varepsilon = 0$

$\to a_1^3\varepsilon^2 - \frac{1}{\varepsilon} - 3a_1^2\varepsilon + 3a_1 + 2\varepsilon a_1^2 - 2a_1 + \frac{2}{\varepsilon} + a_1 - \frac{1}{\varepsilon} + \varepsilon = 0$

equating like powers we find

$\varepsilon: -3a_1^2 + 2a_1^2 + 1 = 0$

$-a_1^2 + 1 = 0$

$a_1 = \pm 1$

So $x = 1 - \frac{1}{\varepsilon}$, $x = -1 - \frac{1}{\varepsilon}$

Three solutions:

$$\boxed{\begin{array}{l} ①\ x = -\varepsilon - 2\varepsilon^3 + O(\varepsilon^5) \\ ②\ x = 1 - \frac{1}{\varepsilon} + O(\varepsilon) \\ ③\ x = -1 - \frac{1}{\varepsilon} + O(\varepsilon) \end{array}}$$

✓

3/3

c) $\varepsilon e^x = 1 + \dfrac{\varepsilon}{1+x}$

Graphically, we see there are two solutions, one of which $\to -1$ and the other $\to \infty$ as $\varepsilon \to 0$

Let one of the solutions be $x = -1 - \delta$

Plug in to rewritten form

$$x = -1 + \varepsilon e^x(1+x)$$
$$-1-\delta = -1 + \varepsilon e^{-1-\delta}(1-1-\delta)$$
$$-\delta = -\varepsilon \delta e^{-1-\delta}$$
$$1 = \varepsilon e^{-1-\delta}$$
$$\frac{1}{\varepsilon} = e^{-1-\delta}$$
$$\ln\frac{1}{\varepsilon} = -1-\delta$$
$$\delta = -1 - \ln\frac{1}{\varepsilon}$$

So $x = -1 - (-1 - \ln\frac{1}{\varepsilon})$

$$= -1 + 1 + \ln\frac{1}{\varepsilon}$$
$$\boxed{x = \ln\frac{1}{\varepsilon}}$$ 2-term expansion?

Let another solution be $x = \delta_0 + \delta_1 + \delta_2 + \cdots$ where $\delta_0 > \delta_1 > \delta_2 \cdots$

Plug into $x = -1 + \varepsilon e^x(1+x)$

$$\to x = -1 + \varepsilon(1+x)(1 + x + \tfrac{1}{2}x^2 + \cdots)$$
$$\delta_0 + \delta_1 + \delta_2 + \cdots = -1 + \varepsilon(1 + \delta_0 + \delta_1 + \delta_2 + \cdots)\big(1 + \delta_0 + \delta_1 + \delta_2 + \cdots + \tfrac{1}{2}(\delta_0 + \delta_1 + \delta_2 + \cdots)^2 + \cdots\big)$$
$$\to \delta_0 + \delta_1 + \delta_2 + \cdots = -1 + \varepsilon + \varepsilon\delta_0 + \varepsilon\delta_1 + \varepsilon\delta_0^2 + 2\delta_0\delta_1 + \varepsilon\delta_1^2 + \varepsilon\delta_1\delta_2 + \cdots$$

Find big terms:

$$\delta_0 = \varepsilon\delta_0 + \varepsilon\delta_0^2$$
$$\delta_0(1-\varepsilon) = \varepsilon\delta_0^2$$
$$1 - \varepsilon = \varepsilon\delta_0$$
$$\delta_0 = \frac{1}{\varepsilon} - 1 \quad ?$$

1/3

$$\boxed{x = -1 + \frac{1}{\varepsilon}}$$

D) $x^2 = \varepsilon \sin x$

Graphically, we see that there are two solutions, one at $x=0$ and one approaching $x=0$ as $\varepsilon \to 0$

$x^2 = \varepsilon\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)$

$x = \varepsilon\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots\right)$

Iterations:

$x_0 = 0$

$x_1 = \varepsilon$

$x_2 = \varepsilon - \frac{1}{6}\varepsilon^3 + \frac{1}{120}\varepsilon^5 + \cdots$

$x_3 = \varepsilon\left(1 - \frac{1}{6}\left(\varepsilon - \frac{1}{6}\varepsilon^3 + \frac{1}{120}\varepsilon^5\right)^2 + \frac{1}{120}\left(\varepsilon - \frac{1}{6}\varepsilon^3 + \frac{1}{120}\varepsilon^5\right)^4 + \cdots\right.$

$= \varepsilon\left(1 - \frac{1}{6}\left(\varepsilon^2 - \frac{1}{3}\varepsilon^4\right) + \frac{1}{120}\left(\varepsilon^4\right) + O\left(\varepsilon^6\right)\right)$

$= \varepsilon\left(1 - \frac{1}{6}\varepsilon^2 + \frac{1}{18}\varepsilon^4 + \frac{1}{120}\varepsilon^4 + O\left(\varepsilon^6\right)\right)$

$= \varepsilon - \frac{1}{6}\varepsilon^3 + \frac{23}{360}\varepsilon^5 + O\left(\varepsilon^7\right)$

So $\boxed{x = \varepsilon - \frac{1}{6}\varepsilon^3 + O\left(\varepsilon^5\right)}$ ✓

And there is an exact solution at $\boxed{x=0}$. ✓

E) $x^3-3x^2+(3-\varepsilon)x+\varepsilon = \sin\frac{\pi(x+\varepsilon)}{2}$

Reduced problem: $\varepsilon = 0$

$\rightarrow x^3-3x^2+3x = \sin\frac{\pi}{2}x$

$\rightarrow$ solutions at $x=0$, $x=1$

Let $x = x_0 + x_1\varepsilon + O(\varepsilon^2)$

Plug in:

$$(x_0+x_1\varepsilon)^3 - 3(x_0+x_1\varepsilon)^2 + (3-\varepsilon)(x_0+x_1\varepsilon)+\varepsilon = \frac{\pi(x_0+x_1\varepsilon+\varepsilon)}{2} - \frac{1}{6}\left[\frac{\pi(x_0+x_1\varepsilon+\varepsilon)}{2}\right]^3$$

$$x_0^3 + x_1^3\varepsilon^3 + 3x_0^2x_1\varepsilon + 3x_0x_1^2\varepsilon^2 - 3x_0^2 - 3x_1^2\varepsilon^2 - 6x_0x_1\varepsilon + 3x_0 + 3x_1\varepsilon - x_0\varepsilon - \cancel{x_1\varepsilon^2} + \varepsilon$$
$$= \frac{\pi}{2}x_0 + \frac{\pi}{2}x_1\varepsilon + \frac{\pi}{2}\varepsilon - \frac{\pi^3}{48}(x_0^3 + 3x_0x_1\varepsilon^2 + 3x_0x_1^2\varepsilon^2 + 3x_0\varepsilon^2)$$

ε: $3x_0^2x_1 - 6x_0x_1 + 3x_1 - x_0 + 1 = \frac{\pi}{2}x_1 + \frac{\pi}{2} - (3x_0^2 + 3x_0^2x_1)\frac{\pi^3}{48}$

ε^0: $x_0^3 - 3x_0^2 + 3x_0 = \frac{\pi}{2}x_0 - \frac{\pi^3}{48}x_0^3$

$(1+\pi^3/48)x_0^3 - 3x_0^2 + (3-\pi/2)x_0 = 0$

$x_0\left[(1+\pi^3/48)x_0^2 - 3x_0 + 3 - \pi/2\right] = 0$

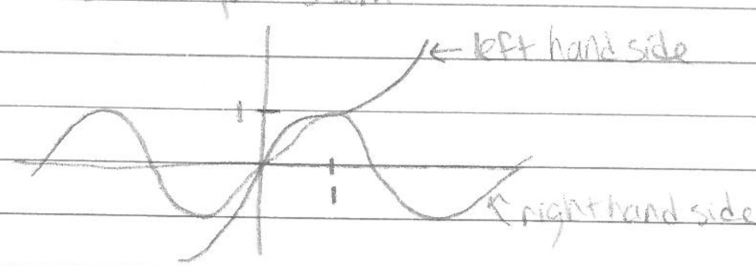
$x_0 = 0$ and

$x_0 = \dfrac{3 \pm \sqrt{9-4(1+\pi^3/48)(3-\pi/2)}}{2(1+\pi^3/48)}$ (a complex number)

I'm pretty sure this isn't right - I'm not sure it would be useful to continue.

2/3

Graphically we can see the solutions from the reduced problem:



ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Homework 3 (DUE ~~WEDNESDAY~~, 10/14/08)

Tuesday

Problem 1.
Consider the incomplete exponential integral

$$I(\varepsilon) = \int_{1/\varepsilon}^{\infty} x^{-1} e^{-x} dx.$$

(a). Use repeated integration by parts to show that

$$I(\varepsilon) = \varepsilon e^{-1/\varepsilon}(1 - 1!\varepsilon + 2!\varepsilon^2 + \ldots + (-1)^n n!\varepsilon^n) + R_n(\varepsilon),$$

and determine the form of $R_n(\varepsilon)$.

(b). Prove that

$$I(\varepsilon) \sim \varepsilon e^{-1/\varepsilon}(1 - 1!\varepsilon + 2!\varepsilon^2 + \ldots + (-1)^n n!\varepsilon^n) \equiv S_n(\varepsilon)$$

(i.e., that the sum is an asymptotic expansion of the integral) by showing that $R_n(\varepsilon) = o(\varepsilon^{n+1} e^{-1/\varepsilon})$.

Problem 2.
An asymptotic expansion of the Bessel function $J_\nu(x)$ for large x has the form

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \frac{4\nu^2 - 1}{8x} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \right], \qquad x \gg 1.$$

Let x_n be the n-th zero of the function $J_\nu(x)$, i.e., the n-th solution of the equation $J_\nu(x) = 0$. Show that for n sufficiently large

$$x_n = \left(n + \frac{3}{4} + \frac{\nu}{2}\right)\pi - \frac{4\nu^2 - 1}{2\pi(4n + 3 + 2\nu)} + \ldots.$$

Holley Lepo
10-14-08
ESAM 420
Hwk #3

1) Consider the incomplete exponential integral
$I(\varepsilon) = \int_{1/\varepsilon}^{\infty} x^{-1} e^{-x} dx.$

A) Use repeated integration by parts to show
$I(\varepsilon) = \varepsilon e^{-\frac{1}{\varepsilon}} (1 - 1!\varepsilon + 2!\varepsilon^2 + \cdots + (-1)^n n! \varepsilon^n) + R_n(\varepsilon)$,
and determine the form of $R_n(\varepsilon)$.

$I(\varepsilon) = \int_{1/\varepsilon}^{\infty} x^{-1} e^{-x} dx \qquad \begin{pmatrix} u = x^{-1} & v = -e^{-x} \\ du = -x^{-2}dx & dv = e^{-x}dx \end{pmatrix}$

$= -x^{-1}e^{-x}\Big|_{1/\varepsilon}^{\infty} - \int_{1/\varepsilon}^{\infty} x^{-2} e^{-x} dx = \varepsilon e^{-1/\varepsilon} - \int_{1/\varepsilon}^{\infty} x^{-2} e^{-x} dx$

Consider $I_n(\varepsilon) = \int_{1/\varepsilon}^{\infty} x^{-n} e^{-x} dx \qquad \begin{pmatrix} u = x^{-n} & v = -e^{-x} \\ du = -nx^{-(n+1)}dx & dv = e^{-x} \end{pmatrix}$

$= -x^{-n}e^{-x}\Big|_{x=1/\varepsilon}^{\infty} - \int_{1/\varepsilon}^{\infty} n x^{-(n+1)} e^{-x} dx$

$= \varepsilon^n e^{-1/\varepsilon} - n I_{n+1}(\varepsilon)$

So $I(\varepsilon) = \varepsilon e^{-1/\varepsilon} - I_2$

$= \varepsilon e^{-1/\varepsilon} - \varepsilon^2 e^{-1/\varepsilon} + 2 I_3(\varepsilon)$

$= \varepsilon e^{-1/\varepsilon} - \varepsilon^2 e^{-1/\varepsilon} + 2\varepsilon^3 e^{-1/\varepsilon} - 2\cdot 3 I_4(\varepsilon)$

$= \varepsilon e^{-1/\varepsilon} - \varepsilon^2 e^{-1/\varepsilon} + 2\varepsilon^3 e^{-1/\varepsilon} - 2\cdot 3\varepsilon^4 e^{-1/\varepsilon} + 2\cdot 3\cdot 4 I_{n+1}(\varepsilon)$

$= \varepsilon e^{-1/\varepsilon}(1 - \varepsilon + 2\varepsilon^3 - 2\cdot 3\varepsilon^4 + 2\cdot 3\cdot 4\varepsilon^5 \ldots)$

$= \varepsilon e^{-1/\varepsilon}(1 - 1!\varepsilon + 2!\varepsilon^3 - 3!\varepsilon^4 + 4!\varepsilon^5 - \ldots)$

Comparing this to the desired form of solution
$I(\varepsilon) = \varepsilon e^{-1/\varepsilon}(1 - 1!\varepsilon + 2!\varepsilon^2 + \cdots + (-1)^n n! \varepsilon^n) + R_n(\varepsilon)$,
we see that $R_n(\varepsilon) = -n I_{n+1}(\varepsilon)$

$\rightarrow \boxed{R_n(\varepsilon) = -n \int_{1/\varepsilon}^{\infty} x^{-(n+1)} e^{-x} dx}$

or, alternatively,

$\boxed{R_n(\varepsilon) = \varepsilon e^{-1/\varepsilon} \sum_{k=n+1}^{\infty} (-1)^k k! \varepsilon^k}$

1) (continued)

B) Prove that

$I(\varepsilon) \sim \varepsilon e^{-1/\varepsilon}(1 - 1!\,\varepsilon + 2!\,\varepsilon^2 + \cdots + (-1)^n n!\,\varepsilon^n) \equiv S_n(\varepsilon)$

by showing that $R_n(\varepsilon) = o(\varepsilon^{n+1} e^{-1/\varepsilon})$.

We must show:

$$\lim_{\varepsilon \to 0} \left| \frac{R_n(\varepsilon)}{\varepsilon^{n+1} e^{-1/\varepsilon}} \right| = 0$$

$$\lim_{\varepsilon \to 0} \left| \frac{R_n(\varepsilon)}{\varepsilon^{n+1} e^{-1/\varepsilon}} \right| = \lim_{\varepsilon \to 0} \left| \frac{\varepsilon e^{-1/\varepsilon} \sum_{k=n+1}^{\infty} [(-1)^k k!\,\varepsilon^k]}{\varepsilon^{n+1} e^{-1/\varepsilon}} \right|$$

$$= \lim_{\varepsilon \to 0} \left| \frac{\sum_{k=n+1}^{\infty} [(-1)^k k!\,\varepsilon^k]}{\varepsilon^n} \right|$$

$$= \lim_{\varepsilon \to 0} \left| \frac{(-1)^{n+1}(n+1)!\,\varepsilon^{n+1} + (-1)^{n+2}(n+2)!\,\varepsilon^{n+2} + \cdots}{\varepsilon^n} \right|$$

$$= \lim_{\varepsilon \to 0} \left| (n+1)!\,\varepsilon^n + (n+2)!\,\varepsilon^{n+1} + (n+3)!\,\varepsilon^{n+2} + \cdots \right|$$

$$= 0$$

Thus $S_n(\varepsilon)$ is an asymptotic expansion of $I(\varepsilon)$

The series for $R_n(\varepsilon)$ diverges for any ε.
You cannot replace R_n by the series in your $\lim_{\varepsilon \to 0}$

3/4

Lepo2

2) An asymptotic expansion of the Bessel function $J_\nu(x)$ for large x has the form

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)-\frac{4\nu^2-1}{8x}\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right],\ x \gg 1$$

Let x_n be the n^{th} zero of the function $J_\nu(x)$, i.e. the n^{th} solution of the equation $J_\nu(x)=0$.
Show that for n sufficiently large,

$$x_n = \left(n+\frac{3}{4}+\frac{\nu}{2}\right)\pi - \frac{4\nu^2-1}{2\pi(4n+3+2\nu)} + \cdots$$

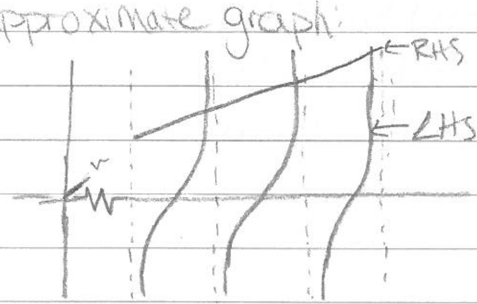
$J_\nu(x)=0$

$$\rightarrow \sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)-\frac{4\nu^2-1}{8x}\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right]=0$$

$$\rightarrow \cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)=\frac{4\nu^2-1}{8x}\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)$$

$$\rightarrow \tan\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)=\frac{8x}{4\nu^2-1}$$

approximate graph:



so as $x \to \infty$, the intersections occur very close to where the LHS is infinite

$\tan\alpha$ is infinite where $\alpha = \pi/2 + \pi n$, n integer
So $\tan(x - \frac{\nu\pi}{2} - \pi/4)$ is infinite where

$$x - \frac{\nu\pi}{2} - \pi/4 = \pi/2 + \pi n$$

$$\rightarrow x = 3\pi/4 + \nu\pi/2 + \pi n$$

Then the intersections (the solutions to $J_\nu(x)=0$) occur at $x-\delta$ where δ is small:

$$x_n = 3\pi/4 + \nu\pi/2 + \pi n - \delta_n$$

(continued on reverse)

2) (continued)

So we have, for x_n the n^{th} sol'n to $J_\nu(x)=0$:

$$\tan\left(x_n - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \frac{8x_n}{4\nu^2-1}$$

$$\rightarrow \tan\left(\frac{\pi}{2} + \pi n - \delta_n\right) = \frac{8\left(\frac{3\pi}{4} + \frac{\nu\pi}{2} + \pi n - \delta_n\right)}{4\nu^2-1}$$

and since $\tan\alpha = \tan(\alpha + \pi n)$

$$\rightarrow \tan\left(\frac{\pi}{2} - \delta_n\right) = \frac{6\pi + 4\nu\pi + 8\pi n - 8\delta_n}{4\nu^2 - 1}$$

And $\tan(\frac{\pi}{2} - \delta_n) = \frac{\sin(\frac{\pi}{2}-\delta_n)}{\cos(\frac{\pi}{2}-\delta_n)} = \frac{\sin\frac{\pi}{2}\cos\delta_n - \cos\frac{\pi}{2}\sin\delta_n}{\cos\frac{\pi}{2}\cos\delta_n + \sin\frac{\pi}{2}\sin\delta_n}$

$$= \frac{\cos\delta_n}{\sin\delta_n} = \frac{-\left(1 - \frac{1}{2}\delta_n^2 + O(\delta_n^4)\right)}{\delta_n - \frac{1}{6}\delta_n^3 + O(\delta_n^5)}$$

So we have

$$-\left[1-\tfrac{1}{2}\delta_n^2 + O(\delta_n^4)\right](4\nu^2-1) = \left[\delta_n - \tfrac{1}{6}\delta_n^3 + O(\delta_n^5)\right](6\pi + 4\nu\pi + 8\pi n - 8\delta_n)$$

$$\rightarrow -(4\nu^2-1) + \tfrac{1}{2}(4\nu^2-1)\delta_n^2 + O(\delta_n^4) =$$

$$(6\pi + 4\nu\pi + 8\pi n)\delta_n - 8\delta_n^2$$

$$\frac{1}{6}(6\pi + 4\nu\pi + 8\pi n)\delta_n^3 + O(\delta_n^4)$$

Keeping only large terms, we have

$$-(4\nu^2-1) = 2\pi(3 + 2\nu + 4n)\delta_0$$

$$\rightarrow \delta_0 = \frac{-(4\nu^2-1)}{2\pi(3+2\nu+4n)}$$ this is the first term in an expansion of δ_n

[if we would have substituted $\delta_n = \delta_0 + \delta_1\varepsilon + \delta_2\varepsilon^2 + \ldots$ we would have gotten the same answer.] see next page for δ_n found by iteration

Thus

$$x_n = \pi\left(\frac{3}{4} + \frac{\nu}{2} + n\right) - \frac{4\nu^2-1}{2\pi(3+2\nu+4n)} + \ldots$$

Lepo 3

2) (continued)

Iterations to find δ_n

$$\delta_{n+1} = \frac{-(4\nu^2-1) + (2\nu^2+\frac{15}{2})\delta_n^2 - \frac{1}{3}\pi(3+2\nu+4n)\delta_n^3 + O(\delta_n^4)}{2\pi(3+2\nu+4n)}$$

$\delta_0 = 0$

$$\delta_1 = \frac{-(4\nu^2-1)}{2\pi(3+2\nu+4n)}$$

$$\delta_2 = \frac{-(4\nu^2-1)}{2\pi(3+2\nu+4n)} + \frac{(2\nu^2+\frac{15}{2})(4\nu^2-1)^2}{(2\pi(3+2\nu+4n))^3} - \frac{\pi(3+2\nu+4n)(4\nu^2-1)^3}{3(2\pi(3+2\nu+4n))^4} + \cdots$$

Obviously, the first term will remain unchanged.

Thus $\delta_n^0 = -\dfrac{(4\nu^2-1)}{2\pi(3+2\nu+4n)}$

So $x_n = \pi\left(\frac{3}{4} + \frac{\nu}{2} + n\right) - \dfrac{4\nu^2-1}{2\pi(3+2\nu+4n)} + \cdots$ ✓

2/2

Haley Lepo
10-21-08
ESAM 420
Hwk #4

1) Consider the eigenvalue problem

$$x^2u'' + xu' + (\lambda x^2 - 1)u = 0, \quad 1<x<2, \quad u(1)=u(2)=0$$

Determine a two-term expansion of the n^{th} positive eigenvalue λ_n for large n.

Let $u = y\exp\left(-\frac{1}{2}\int \frac{x}{x^2}dx\right) = y\exp\left(-\frac{1}{2}\int\frac{1}{x}dx\right)$

$= y\exp\left(-\frac{1}{2}\ln x\right) = x^{-\frac{1}{2}}y$

Then $u' = -\frac{1}{2}x^{-3/2}y + x^{-\frac{1}{2}}y'$

$u'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-\frac{1}{2}}y''$

Plug in:

$$x^2\left[\tfrac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-\frac{1}{2}}y''\right] + x\left[-\tfrac{1}{2}x^{-3/2}y + x^{-\frac{1}{2}}y'\right] + (\lambda x^2 - 1)x^{-\frac{1}{2}}y = 0$$

$$\rightarrow \tfrac{3}{4}x^{-\frac{1}{2}}y - x^{\frac{1}{2}}y' + x^{3/2}y'' - \tfrac{1}{2}x^{-\frac{1}{2}}y + x^{\frac{1}{2}}y' + (\lambda x^{3/2} - x^{-\frac{1}{2}})y = 0$$

$$\rightarrow x^{3/2}y'' + (\lambda x^{3/2} - \tfrac{3}{4}x^{-\frac{1}{2}})y = 0$$

$$y'' + (\lambda - \tfrac{3}{4}x^{-2})y = 0$$

let $-\frac{3}{4}x^{-2} = \varepsilon q(x)$, with $\varepsilon = \frac{1}{4}$, $q(x) = -3/x^2$

Then, using the result we found in class,

$$\lambda_n = \pi^2n^2 - \tfrac{1}{2}\int_1^2 -\tfrac{3}{x^2}\sin^2(\pi n(x-1))\,dx \checkmark \quad 3/3$$

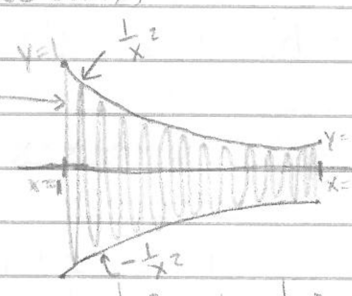
$$\lambda_n = \pi^2n^2 + \tfrac{3}{2}\int_1^2 \tfrac{1}{x^2}\sin^2(\pi n x)\,dx$$

$$= \pi^2n^2 + \tfrac{3}{4}\int_1^2 \tfrac{1}{x^2}(1-\cos(2\pi n x))dx$$

$$= \pi^2n^2 - \tfrac{3}{4}\tfrac{1}{x}\Big|_1^2 - \tfrac{3}{4}\int_1^2\tfrac{1}{x^2}\cos(2\pi n x)$$

$$= \pi^2n^2 + 3/8 - \tfrac{3}{4}\int_1^2\tfrac{1}{x^2}\cos(2\pi n x)$$

$\frac{1}{x^2}\cos(2\pi n x)$ →



As $n\to\infty$, the integral will → the area between $\frac{1}{x^2}$ and $-\frac{1}{x^2}$, which is zero since it is symmetric to the y-axis

$$\boxed{\lambda_n = \pi^2n^2 + 3/8}$$

2) Find a two-term expansion in ε, $0<\varepsilon<<1$, of the solution of the following problem

$$y'' + \varepsilon y' - y = 1 \quad , \quad y(0) = y(1) = 1$$

Let $y \sim y_0 + \varepsilon y_1$

$$y_0'' + \varepsilon y_1'' + \varepsilon y_0 - y_0 - \varepsilon y_1 + O(\varepsilon^2) = 1$$

$$y_0(0) + \varepsilon y_1(0) = 1$$

$$y_0(1) + \varepsilon y_1(1) = 1$$

ε^0 terms: $y_0'' - y_0 = 1 \quad , \quad y_0(0) = y_0(1) = 1$

$$y_0 = C_1 \sinh x + C_2 \cosh x - 1$$

$$y_0(0) = C_2 - 1 = 1 \rightarrow C_2 = 2$$

$$y_0(1) = C_1 \sinh(1) + 2\cosh(1) - 1 = 1$$

$$C_1 = \frac{2 - 2\cosh(1)}{\sinh(1)} \approx -.924$$

$$y_0 = \left(\frac{2-2\cosh 1}{\sinh 1}\right) \sinh x + 2\cosh x - 1$$

? $\quad y_0' = y_0 + 1$, not $y_0 - 1$

ε^1 terms: $y_1'' - y_1 = 1 - y_0 \quad , \quad y_1(0) = 0, \; y_1(1) = 0$

$$y_1 = C_3 \sinh x + C_4 \cosh x + y_p$$

Let $y_p = Ax\sinh x + Bx\cosh x + C$

$$\rightarrow y_p' = A\sinh x + Ax\cosh x + B\cosh x + Bx\sinh x$$

$$\rightarrow y_p'' = 2A\cosh x + Ax\sinh x + 2B\sinh x + Bx\cosh x$$

So $y_p'' - y_p = 2A\cosh x + 2B\sinh x - C = -\left(\frac{2-2\cosh 1}{\sinh 1}\right)\sinh x - 2\cosh x + 2$

So $A = -1$, $B = \frac{-1 + \cosh 1}{\sinh 1}$, $C = -2$

$$y_1 = C_3 \sinh x + C_4 \cosh x - x\sinh x + \left(\frac{-1+\cosh 1}{\sinh 1}\right) x\cosh x - 2$$

$$y_1(0) = 1 = C_4 - 2 \rightarrow C_4 = 3$$

$$y_1(1) = 1 = C_3 \sinh 1 + 3\cosh 1 - \sinh 1 + \left(\frac{-1+\cosh 1}{\sinh 1}\right)\cosh 1 - 2$$

$$\rightarrow C_3 = \frac{\left(3 - 3\cosh 1 + \sinh 1 + (1-\cosh 1)\frac{\cosh 1}{\sinh 1}\right)}{\sinh 1} \approx -.993$$

Lepo 2

2) (continued)

$$y_1 = \left[3 - 3\cosh 1 + \sinh 1 + \frac{(1-\cosh 1)\frac{\cosh 1}{\sinh 1}}{\sinh 1}\right]\sinh x + 3\cosh x - x\sinh x + \left(\frac{-1+\cosh 1}{\sinh 1}\right)x\cosh x - 2$$

$$y \sim \left(\frac{2-2\cosh 1}{\sinh 1}\right)\sinh x + 2\cosh x - 1$$
$$+ \varepsilon\left[\left(3 - 3\cosh 1 + \sinh 1 + \frac{(1-\cosh 1)}{(\sinh 1)^2}\right)\sinh x + 3\cosh x - x\sinh x + \left(\frac{\cosh 1 - 1}{\sinh 1}\right)x\cosh x - 2\right]$$

or, with numbers: ← don't have to

$$y \approx -0.924\sinh x + 2\cosh x - 1$$
$$+ \varepsilon(-0.993\sinh x + 3\cosh x - x\sinh x - 0.391\,x\cosh x - 2)$$

3/4

3) Find a two-term expansion in ε, $0<\varepsilon\ll 1$, of the solution of the following problem

$$y''-y=0, \quad y(0)=0, \quad y(1+\varepsilon)=1$$

$$y(x)''-y(x)=0, \quad y(x=0)=0, \quad y(x=1+\varepsilon)=1$$

Let $t=\frac{x}{1+\varepsilon}$ $\quad y(t)=y\left(\frac{x}{1+\varepsilon}\right)$ $\quad y'(t)=\frac{1}{1+\varepsilon}\,y'\left(\frac{x}{1+\varepsilon}\right)$

$$y''(t)=\frac{1}{(1+\varepsilon)^2}\,y''\left(\frac{x}{1+\varepsilon}\right)$$

$$y(x=0)=y(t=0)=0$$

$$y(x=1+\varepsilon)=y(t=1)=1$$

Plug in: $(1+\varepsilon)^2 y''(t)-y(t)=0$

$$(1+2\varepsilon+\varepsilon^2)\,y''(t)-y(t)=0$$

let $y(t)\sim y_0(t)+\varepsilon y_1(t)$

$$(1+2\varepsilon+\varepsilon^2)\left(y_0''(t)+\varepsilon y_1''(t)\right)-\left(y_0(t)+\varepsilon y_1(t)\right)$$

$$y_0(0)+\varepsilon y_1(0)=0, \quad y_0(1)+\varepsilon y_1(1)=1$$

ε^0: $y_0''(t)-y_0(t)=0$, $\quad y_0(0)=0$, $\quad y_0(1)=1$

$y_0=c_1\sinh x+c_2\cosh x$ $\qquad$ Is $\cancel{x}$ t now x?

$y_0(0)=0=c_2$

$y_0(1)=1=c_1\sinh(1)$

$c_1=1/\sinh(1)$

$$y_0=\frac{\sinh x}{\sinh(1)}$$

ε^1: $y_1''(t)-y_1(t)=-2y_0''(t)$

$y_1(0)=0$, $\quad y_1(1)=0$

$y_1''(t)-y_1(t)=-2\sinh x/\sinh(1)$

$y_1=c_3\sinh x+c_4\cosh x+y_p$

Let $y_p=Ax\sinh x+Bx\cosh x$

$y_p'=A\sinh x+Ax\cosh x+B\cosh x+Bx\sinh x$

$y_p''=2A\cosh x+Ax\sinh x+2B\sinh x+Bx\cosh x$

So $y_p''-y_p=2A\cosh x+2B\sinh x=-\frac{2}{\sinh 1}\sinh x$

$\rightarrow A=0$, $\quad B=-1/\sinh 1$

Lepo 3

3) (continued)

$\rightarrow y_p = \frac{-x}{\sinh(1)} \cosh(x)$

$y_1 = C_3 \sinh x + C_4 \cosh x - \frac{x}{\sinh(1)} \cosh(x)$

$y_1(0) = 0 = C_4$

$y_1(1) = 0 = C_3 \sinh(1) - \frac{1}{\sinh(1)} \cosh(1)$

$C_3 = \cosh(1) / [\sinh(1)]^2$

$$y_1 = \frac{\cosh(1)}{[\sinh(1)]^2} \sinh(x) - \frac{x}{\sinh(1)} \cosh(x)$$

Algebra?

$y = y_0 + \varepsilon y_1$

$$\boxed{y(t) = \frac{\sinh(x)}{\sinh(1)} + \varepsilon \left[\frac{\cosh(1)}{[\sinh(1)]^2} \cdot \sinh(x) - \frac{x}{\sinh(1)} \cdot \cosh(x) \right]}$$

3/4

ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Homework 5 (DUE TUESDAY, 10/28/08)

Problem 1.
Consider the boundary value problem

$$\varepsilon y'' + (1 + 2\varepsilon)y' + y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 1,$$

where ε is a small positive parameter. Determine a two-term asymptotic expansion of the solution in $\varepsilon \ll 1$ that is uniformly valid for $0 < x < 1$ (i.e., determine a two-term outer solution, a two-term inner solution, match them and form a composite solution).

Problem 2.
Consider the boundary value problem

$$y'' + (1 + \varepsilon)y = 1, \quad 0 < x < \pi, \quad y(0) = y(\pi) = 0,$$

where ε is a small positive parameter. In class we have determined the leading order solution of the problem as

$$y(x) \sim \frac{4}{\pi\varepsilon} \sin x.$$

Determine a *two*-term expansion of the solution in powers of ε.

Haley Lepo
10/28/08
ESAM 4120
Hwk #5

1) Consider the boundary value problem

$\varepsilon y'' + (1+2\varepsilon)y' + y = 0, \quad 0<x<1, \quad y(0)=0, \quad y(1)=1$

where ε is a small positive parameter. Determine a two-term asymptotic expansion of the solution in $\varepsilon \ll 1$ that is uniformly valid for $0<x<1$.

Outer region:

$y(x) \sim y_0(x) + \varepsilon y_1(x)$

$\varepsilon^0: \quad y_0' + y_0 = 0, \qquad y_0(1)=1$

$y_0 = c_1 e^{-x}$

$y_0(1) = c_1/e = 1 \rightarrow c_1 = e$

$\rightarrow y_0 = e^{1-x}$

$\varepsilon^1: \quad y_1' + y_1 = -y_0'' - 2 \cdot y_0' \qquad y_1(1) = 0$

$y_1' + y_1 = -e^{1-x} + 2e^{1-x} = e^{1-x}$

$y_h = e^{-x}, \quad y_p = Axe^{-x}, \quad y_p' = Ae^{-x} - Axe^{-x}$

$Ae^{-x} - Axe^{-x} + Axe^{-x} = ee^{-x} \rightarrow A = e$

$y_1 = c_2 e^{-x} + xe^{1-x}$

$y_1(1) = c_2/e + 1 = 0 \rightarrow c_2 = -e$

$y_1 = -e^{1-x} + xe^{1-x}$

$y_1 = e^{1-x}(x-1)$

$y(x) = e^{1-x}(1 + \varepsilon(x-1))$

Inner region:

Let $z = \frac{x}{\varepsilon} \quad y(x) = Y(z), \quad \frac{dy}{dx} = \frac{dY}{dz}\frac{dz}{dx} = \frac{1}{\varepsilon}\frac{dY}{dx}$

$\frac{d^2y}{dx^2} = \frac{d^2Y}{dz^2}\frac{d^2z}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2Y}{dz^2}$

$\frac{1}{\varepsilon}Y'' + (1+2\varepsilon)\frac{1}{\varepsilon}Y' + Y = 0, \quad 0<z<\varepsilon, \quad Y(0)=0$

$Y'' + (1+2\varepsilon)Y' + \varepsilon Y = 0$

$Y(z) \sim Y_0(z) + \varepsilon Y_1(z)$

$\varepsilon^0: \quad Y_0'' + Y_0' = 0, \qquad Y_0(0) = 0$

$Y_0 = c_3 + c_4 e^{-z}$

$Y_0(0) = c_3 + c_4 = 0 \rightarrow c_4 = -c_3$

$Y_0 = c_3(1 - e^{-z})$

$\varepsilon^1: \quad Y_1'' + Y_1' = -2Y_0' - Y_0, \qquad Y_1(0) = 0$

$Y_1'' + Y_1' = -2c_3 e^{-z} - c_3(1 - e^{-z}) = c_3(-1 - e^{-z})$

$Y_h = c_5 + c_6 e^{-z}$, $Y_p = Az + Bze^{-z}$

$Y_p' = A + Be^{-z} - Bze^{-z}$, $Y_p'' = -2Be^{-z} + Bze^{-z}$

$A + Be^{-z} - Bze^{-z} - 2Be^{-z} + Bze^{-z} = c_3(-1 - e^{-z})$

$A = -c_3$, $B = c_3$

$Y_1 = c_5 + c_6 e^{-z} + c_3 z(-1 + e^{-z})$

$Y_1(0) = c_5 + c_6 = 0 \rightarrow c_6 = -c_5$

$Y_1 = c_5(1 - e^{-z}) - c_3 z(1 - e^{-z})$

$Y_1 = (1 - e^{-z})(c_5 - c_3 z)$

$Y = c_3(1 - e^{-z}) + \varepsilon[(1 - e^{-z})(c_5 - c_3 z)]$

$Y = (1 - e^{-z})[c_3 + \varepsilon(c_5 - c_3 z)]$

Match:

Let $\eta = \frac{x}{\varepsilon^\alpha}$, $0 < \alpha < 1$ $\rightarrow x = \eta\varepsilon^\alpha$, $z = \eta\varepsilon^{\alpha-1}$

$y_1(x) = e^{1-x}(1 + \varepsilon(x-1)) = e^{1-\eta\varepsilon^\alpha}(1 + \varepsilon(\eta\varepsilon^\alpha - 1))$

$\sim e(1 - \eta\varepsilon^\alpha)(1 + \eta\varepsilon^{\alpha+1} - \varepsilon)$

$= e + e\eta\varepsilon^{\alpha+1} - e\varepsilon - e\eta\varepsilon^\alpha - e\eta^2\varepsilon^{2\alpha+1} + e\eta\varepsilon^{\alpha+1}$

$= e - e\varepsilon - e\eta\varepsilon^\alpha + \cancel{2e\eta\varepsilon^{\alpha+1}} - \cancel{e\eta^2\varepsilon^{2\alpha+1}}$

$Y(z) = (1 - \underbrace{e^{-\eta\varepsilon^{\alpha-1}}})[c_3 + \varepsilon(c_5 - c_3\eta\varepsilon^{\alpha-1})]$

since $\alpha - 1 < 0$, $\eta\varepsilon^{\alpha-1}$ is large, making $e^{-\eta\varepsilon^{\alpha-1}}$ small

$\approx c_3 + \varepsilon c_5 - \varepsilon c_3 \eta\varepsilon^\alpha$

Compare

$e - e\varepsilon - e\eta\varepsilon^\alpha$ and $c_3 + \varepsilon c_5 - \varepsilon c_3\eta\varepsilon^\alpha$

$\rightarrow c_5 = -e$ and $c_3 = e$

(continued next page)

1) (continued)

$$y_{composite} = y_{outer} + y_{inner} - \text{common part}$$

$$= e^{1-x}(1+\varepsilon(x-1))$$
$$+(1-e^{-\frac{x}{\varepsilon}})[e+\varepsilon(-e-e^{\frac{x}{\varepsilon}})]$$
$$-(e-e\varepsilon-ex)$$

$$= e^{1-x}(1+\varepsilon x-\varepsilon)$$
$$+(1-e^{-\frac{x}{\varepsilon}})e[1-\varepsilon-x]$$
$$-e(1-\varepsilon-x)$$

$$= e^{1-x}(1+\varepsilon x-\varepsilon)+(1-\varepsilon-x)e(-1+1-e^{-\frac{x}{\varepsilon}})$$

$$= e^{1-x}(1+\varepsilon x-\varepsilon)-e^{-\frac{x}{\varepsilon}+1}(1-\varepsilon-x)$$

$$\boxed{y(x)=e^{1-x}(1-\varepsilon+\varepsilon x)-e^{-\frac{x}{\varepsilon}+1}(1-\varepsilon-x)}$$ ✓

6/6

2) Consider the boundary value problem

$y'' + (1+\varepsilon)y = 1, \quad 0 < x < \pi, \quad y(0) = y(\pi) = 0$

where ε is a small positive parameter. In class we have determined the leading order solution of the problem as

$y(x) \sim \frac{4}{\pi\varepsilon} \sin x.$

Determine a two-term expansion of the solution in powers of ε.

Let $y(x) = z(x)/\varepsilon$

$\to z'' + (1+\varepsilon)z = \varepsilon \quad , \quad z(0) = z(\pi) = 0$

$z \sim \frac{4}{\pi} \sin x + \varepsilon z_1 + u(x) \quad , \quad u = o(z_1)$

$O(\varepsilon)$: $z_1'' + z_1 = 1 - \frac{4}{\pi}\sin x \quad , \quad z_1(0) = z_1(\pi) = 0$

$z_{1H} = c_1 \sin x + c_2 \cos x$

$z_{1P} = A + Bx\sin x + Cx\cos x$

$z_{1P}' = B\sin x + Bx\cos x + C\cos x - Cx\sin x$

$z_{1P}'' = 2B\cos x - Bx\sin x - 2C\sin x - Cx\cos x$

$A + 2B\cos x - 2C\sin x = 1 - \frac{4}{\pi}\sin x$

$A = 1, \quad -2C = -4/\pi \to C = \frac{2}{\pi}$

$z_1 = c_1\sin x + c_2\cos x + 1 + \frac{2}{\pi} x\cos x$

$z_1(0) = c_2 + 1 = 0 \to c_2 = -1$

$z_1(\pi) = 1 + 1 - \frac{2}{\pi}\pi = 0$

$z_1 = c_1 \sin x - \cos x + \frac{2}{\pi} x\cos x + 1$

$o(\varepsilon)$: $u'' + u = -z_1 \quad , \quad u(0) = u(\pi) = 0$

$u'' + u = -c_1\sin x + \cos x - \frac{2}{\pi}x\cos x - 1$

$u_H'' + u_H = 0 \quad , \quad u_H(0) = u_H(\pi) = 0$

$u_H = c_3 \sin x$

need $\int_0^\pi c_3 \sin x\left[-c_1\sin x + \cos x - \frac{2}{\pi}x\cos x - 1\right] dx = 0$

$\to -c_1\int_0^\pi \sin^2 x\,dx + \int_0^\pi \sin x\cos x\,dx - \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx - \int_0^\pi \sin x\,dx = 0$

$\to -\frac{c_1}{2}\int_0^\pi 1 - \cos 2x\,dx + \frac{1}{2}\sin^2 x\Big|_0^\pi - \frac{2}{\pi}\left[\frac{1}{2}x\sin^2 x\Big|_0^\pi - \frac{1}{2}\int_0^\pi \sin^2 x\,dx\right] + \cos x\Big|_0^\pi = 0$

$\to -\frac{c_1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi + 0 + \frac{1}{\pi}\cdot\frac{1}{2}\int_0^\pi 1 - \cos 2x\,dx + (-1 - 1) = 0$

$-\frac{c_1}{2}[\pi] + \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi - 2 = 0$

$-\frac{c_1\pi}{2} + \frac{1}{2\pi}[\pi] - 2 = 0 \qquad -\frac{c_1\pi}{2} + \frac{1}{2} - 2 = 0$

$-c_1\pi = 3 \quad \to c_1 = \frac{-3}{\pi}$

answer on reverse

$$z_1 = -\frac{3}{\pi}\sin x - \cos x + \frac{2}{\pi}x\cos x + 1$$

So $z \sim \frac{4}{\pi}\sin x + \varepsilon\left[-\frac{3}{\pi}\sin x - \cos x + \frac{2}{\pi}x\cos x + 1\right]$

$$\boxed{y \sim \frac{1}{\varepsilon}\frac{4}{\pi}\sin x - \frac{3}{\pi}\sin x - \cos x + \frac{2}{\pi}x\cos x + 1}$$

6	4
6	4

10

ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Midterm Examination (DUE THURSDAY, 11/6/08, IN CLASS)

Problem 1. Find a two-term asymptotic expansion in ε, $0 < \varepsilon \ll 1$, of each real solution of the equation

$$x - \frac{\varepsilon}{3x^2} - \frac{3\varepsilon^2}{10x^4} = 0.$$

Problem 2. Find a two-term asymptotic expansion in ε, $0 < \varepsilon \ll 1$, of each *large* ($x \gg 1$) solution of the equation

$$xe^x \tan x = 1.$$

Problem 3. Find a two-term asymptotic expansion in ε, $0 < \varepsilon \ll 1$, of $x(\varepsilon)$, the solution *near zero* of

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{x^2}{2} = -\frac{\varepsilon}{6}.$$

matching?

Problem 4. Find a two-term asymptotic expansion in small ε, $0 < \varepsilon \ll 1$, of the solution of the boundary value problem

$$\varepsilon y'' - y' + \frac{1}{4-x}y = 0, \quad y(1) = 2, \ y(2) = 1$$

that is uniform in x, $1 \le x \le 2$ (i.e., find a two-term inner solution, a two-term outer solution, match them, and form a composite solution).

Problem 5. Consider the following boundary value problems and determine a leading order asymptotic expansion of the solution in $\varepsilon \ll 1$ that is uniformly valid for $0 < x < 1$ (i.e., determine the leading order outer solution, the leading order inner solution, match them and form a composite solution):

(a). $\varepsilon y'' + 2y' + e^y = 0, \quad 0 < x < 1, \ y(0) = 0, \ y(1) = 0,$

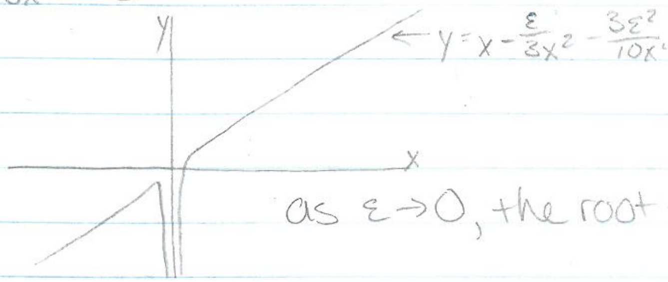
(b). $\varepsilon y'' + (1 + x^2)y' - x^3 y = 0, \quad 0 < x < 1, \ y(0) = 1, \ y(1) = 1.$

Please simplify your answers and clearly write them.

Haley Lepo
11-6-08
ESAM 420
Mid-term

1) Find a two-term asymptotic expansion in ε, $0<\varepsilon\ll 1$, of each real solution of the equation

$$x - \frac{\varepsilon}{3x^2} - \frac{3\varepsilon^2}{10x^4} = 0$$



as $\varepsilon \to 0$, the root $\to 0$ ✓

So for $x \sim x_0 + \delta$, $x_0 = 0$ $(\delta = o(1))$

$\to x \sim \delta$ plug in:

$$\delta - \frac{1}{3}\varepsilon/\delta^2 - \frac{3}{10}\varepsilon^2/\delta^4 = 0$$

$$\varepsilon/\delta^2 = \varepsilon^2/\delta^4 \to \delta^2 = \varepsilon \to \delta = \sqrt{\varepsilon}$$

wrong balance the terms are of the same sign and cannot cancel each other

check: first term: $\sqrt{\varepsilon}$

second + third terms: $\varepsilon/\delta^2 = \varepsilon/\varepsilon = 1$

$\to$ first term is small in comparison ✓

Let $x \sim x_1\varepsilon^{1/2} + x_2\varepsilon + x_3\varepsilon^{3/2}$

$$\to x_1\varepsilon^{1/2} + x_2\varepsilon + x_3\varepsilon^{3/2} - \frac{\varepsilon}{3(x_1\varepsilon^{1/2}+x_2\varepsilon+x_3\varepsilon^{3/2})^2} - \frac{3\varepsilon^2}{10(x_1\varepsilon^{1/2}+x_2\varepsilon+x_3\varepsilon^{3/2})^4} = 0$$

$$\to x_1\varepsilon^{1/2} + x_2\varepsilon + x_3\varepsilon^{3/2} - \frac{\varepsilon}{3x_1\varepsilon^{1/2}}\left(\frac{1}{1+\frac{x_2\varepsilon + x_3\varepsilon^{3/2}}{x_1\varepsilon^{1/2}}}\right)^2 - \frac{3\varepsilon^2}{10x_1\varepsilon^{1/2}}\left(\frac{1}{1+\frac{x_2\varepsilon+x_3\varepsilon^{3/2}}{x_1\varepsilon^{1/2}}}\right)^4 = 0$$

$$\to x_1\varepsilon^{1/2} + x_2\varepsilon + x_3\varepsilon^{3/2} - \frac{\varepsilon}{3x_1\varepsilon^{1/2}}\left(1 - \frac{x_2}{x_1}\varepsilon^{1/2} - \frac{x_3}{x_1}\varepsilon\right)^2$$
$$- \frac{3\varepsilon^2}{10x_1\varepsilon^{1/2}}\left(1 - \frac{x_1}{x_2}\varepsilon^{1/2} - \frac{x_3}{x_1}\varepsilon\right)^4 = 0$$

$$\to x_1\varepsilon^{1/2} + x_2\varepsilon + x_3\varepsilon^{3/2} - \frac{1}{3x_1}\varepsilon^{1/2}\left(1 - 2\frac{x_2}{x_1}\varepsilon^{1/2} + \left(\frac{x_2^2}{x_1^2} - \frac{2x_3}{x_1}\right)\varepsilon\right) - \frac{3}{10x_1}\varepsilon^{3/2} + O(\varepsilon^2) = 0$$

$$\to x_1\varepsilon^{1/2} + x_2\varepsilon + x_3\varepsilon^{3/2} - \frac{1}{3x_1}\varepsilon^{1/2} + \frac{2x_2}{3x_1}\varepsilon + \left(-\frac{x_2^2}{3x_1^3} + \frac{2x_3}{3x_1^2}\right)\varepsilon^{3/2} - \frac{3}{10x_1}\varepsilon^{3/2} = 0$$

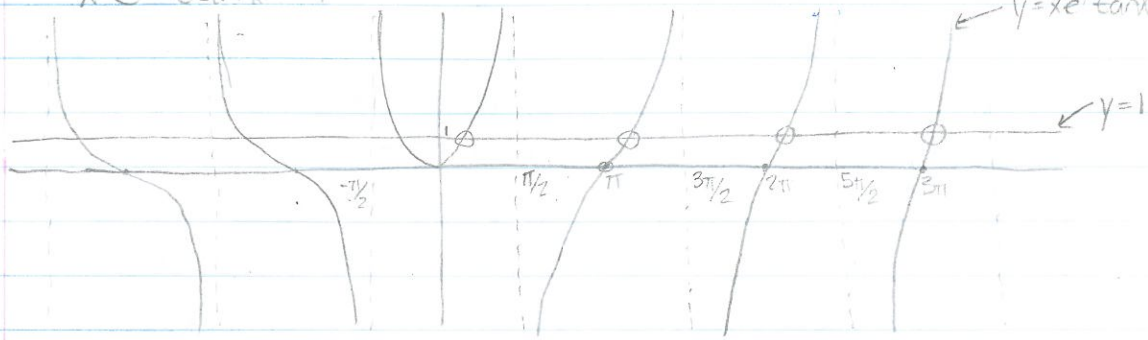
$$\to \left(x_1 - \frac{1}{3x_1}\right)\varepsilon^{1/2} + \left(x_2 + \frac{2x_2}{3x_1}\right)\varepsilon + \left(x_3 - \frac{x_2^2}{3x_1^3} + \frac{2x_3}{3x_1^2} - \frac{3}{10x_1}\right)\varepsilon^{3/2} = 0$$

$$\to \begin{cases} x_1 - \frac{1}{3x_1} = 0 \to x_1^2 = \frac{1}{3} \to x_1 = \frac{1}{\sqrt{3}} \\ x_2 + \frac{2x_2}{3x_1} = 0 \to x_2\left(1 + \frac{2\sqrt{3}}{3}\right) = 0 \to x_2 = 0 \\ x_3 + \frac{2x_3}{3x_1^2} - \frac{3}{10x_1} = 0 \quad x_3(1+2) = +\frac{3\sqrt{3}}{10} \to x_3 = \frac{\sqrt{3}}{10} \end{cases}$$

$$\boxed{x \sim \frac{1}{\sqrt{3}}\varepsilon^{1/2} + \frac{\sqrt{3}}{10}\varepsilon^{3/2}}$$

Lepo 2

2) Find a two-term asymptotic expansion in ε, $0<\varepsilon \ll 1$, of each large ($x \gg 1$) solution of $xe^x \tan x = 1$



Graphically, we see that as $x \to \infty$, the intersections of $y=xe^x\tan x$ and $y=1$ become closer + closer to the roots of $y=xe^x\tan x$.

The roots are $x=0$ and $x=n\pi$. ($n \in \mathbb{Z}$)

So let the intersections occur a distance δ_n from the n^{th} root.

$\rightarrow x_n \sim n\pi + \delta_n$, $\delta_n \ll 1$

Plug into equation:

$$(n\pi+\delta_n)e^{n\pi+\delta_n}\tan(n\pi+\delta_n) = 1$$

$$\rightarrow (n\pi+\delta_n)e^{n\pi}e^{\delta_n}\tan\delta_n = 1 \qquad \text{since } \tan(n\pi+\theta)=\tan\theta$$

Taylor expand:

$$(n\pi+\delta_n)e^{n\pi}(1+\delta_n+O(\delta_n^2))(\delta_n+O(\delta_n^3)) = 1$$

$$e^{n\pi}\delta_n(n\pi+\delta_n+n\pi\delta_n) + O(\delta_n^2) = 1$$

$$e^{n\pi}\delta_n n\pi + O(\delta_n^2) = 1$$

$$\delta_n = \frac{1}{n\pi}e^{-n\pi}$$

So $x_n \sim n\pi + \frac{1}{n\pi}e^{-n\pi}$

$$\boxed{x_n \sim n\pi + \frac{1}{n\pi}e^{-n\pi}}$$ ✓

This isn't in terms of ε... I'm not sure how to change it so that it is...

Lepo 3

3) Find a two-term asymptotic expansion in ε, $0 < \varepsilon \ll 1$, of $x(\varepsilon)$, the solution near zero of
$\sqrt{2}\sin(x + \frac{\pi}{4}) - 1 - x + \frac{x^2}{2} = -\varepsilon/6$

[note: $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\sqrt{2}(\sin x \cos \pi/4 + \cos x \sin \pi/4) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\varepsilon$

$\sin x + \cos x - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\varepsilon$

Reduced problem: let $\varepsilon = 0$

Then we find $x_0 = 0$ is a solution

Taylor expand:

$(\cancel{x} - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots) + (\cancel{1} - \cancel{\frac{1}{2}x^2} + \frac{1}{24}x^4 + \cdots) \cancel{-1} \cancel{-x} + \cancel{\frac{1}{2}x^2} = -\frac{1}{6}\varepsilon$

$-\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + O(x^6) = -\frac{1}{6}\varepsilon$

Let $x \sim x_0 + \delta = \delta$, $\delta = o(1)$

$-\frac{1}{6}\delta^3 + \frac{1}{24}\delta^4 + \frac{1}{120}\delta^5 + O(\delta^6) = -\frac{1}{6}\varepsilon$

$\rightarrow \delta^3 = \varepsilon \rightarrow \delta = \varepsilon^{1/3}$

Let $x \sim x_1\varepsilon^{1/3} + x_2\varepsilon^{2/3}$

$-\frac{1}{6}(x_1\varepsilon^{1/3} + x_2\varepsilon^{2/3})^3 + \frac{1}{24}(x_1\varepsilon^{1/3} + x_2\varepsilon^{2/3})^4 + O(\varepsilon^{5/3}) = -\frac{1}{6}\varepsilon$

$-\frac{1}{6}x_1\varepsilon - \frac{1}{2}x_1^2x_2\varepsilon^{4/3} + \frac{1}{24}x_1^4\varepsilon^{4/3} + O(\varepsilon^{5/3}) = -\frac{1}{6}\varepsilon$

$-\frac{1}{6}x_1 = -\frac{1}{6} \rightarrow x_1 = 1$

$-\frac{1}{2}x_1^2x_2 + \frac{1}{24}x_1^4 = 0$

$-\frac{1}{2}x_2 = -\frac{1}{24}$

$x_2 = \frac{1}{12}$

$$\boxed{x \sim \varepsilon^{1/3} + \frac{1}{12}\varepsilon^{2/3}}$$ ✓

Lepu4

4) Find a two-term asymptotic expansion in small ε, $0<\varepsilon<<1$, of the solution of the BVP $\varepsilon y'' - y' + \frac{1}{4-x} y = 0$, $y(1)=2$, $y(2)=1$ that is uniform in x, $1 \le x \le 2$.

Note that the coefficient of y' is negative, meaning that there can be a boundary layer near $x=2$.

Outer:

let $y(x) \sim y_0 + y_1 \varepsilon$

$O(1)$: $-y_0' + \frac{1}{4-x} y_0 = 0$, $y_0(1) = 2$

$\rightarrow \quad y_0' + \frac{1}{x-4} y_0 = 0$

$dy_0/dx = -\frac{1}{x-4} y_0$

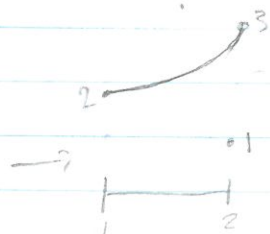
$\int dy_0/y_0 = \int -\frac{1}{x-4} dx$

$\ln y_0 = -\ln(x-4) + c$

$y_0 = c \cdot \frac{1}{x-4}$

$y_0(1) = 2 = c \cdot \frac{1}{-3} \rightarrow c = -6$

$y_0 = -6/(x-4)$ ✓



$O(\varepsilon)$: $-y_1' + \frac{1}{4-x} y_1 = -\ y_0''$, $y_1(1) = 0$

$y_1' + \frac{1}{x-4} y_1 = y_0'' = \frac{12}{(x-4)^3}$

$y_{1p} = \frac{A}{(x-4)^2}$, $y_{1p}' = \frac{-2A}{(x-4)^3}$

$\frac{-2A}{(x-4)^3} + \frac{1}{(x-4)} \cdot \frac{A}{(x-4)^2} = \frac{12}{(x-4)^3}$

$-2A + A = 12 \rightarrow A = -12$

$y_1 = \frac{c}{x-4} - \frac{12}{(x-4)^2}$

$y_1(1) = 0 = \frac{c}{-3} - \frac{12}{9} \rightarrow c = -4$

$y_1 = \frac{-4}{x-4} - \frac{12}{(x-4)^2}$

$y(x) = \frac{-6}{x-4} + \varepsilon \left[\frac{-4}{x-4} - \frac{12}{(x-4)^2} \right]$

Signs?

Boundary layer near $x=2$:

let $y(x) = Y(\xi)$, $\xi = \frac{x-2}{\delta}$, $\delta = o(1)$, $x = \xi\delta + 2$, $\xi < 0$

$\rightarrow \varepsilon/\delta^2 Y'' - 1/\delta\, Y' + \frac{1}{4-\xi\delta-2} Y = 0$

coefficient of Y:

$\frac{1}{2-\xi\delta} = \frac{1}{2}\cdot\frac{1}{1-(\xi\delta/2)} = \frac{1}{2}\left(1 + \frac{1}{2}\xi\delta + O(\delta^2)\right)$

$\rightarrow \varepsilon/\delta^2\, Y'' - \frac{1}{\delta} Y' + \left(\frac{1}{2} + \frac{1}{4}\xi\delta\right)Y = 0$

$\varepsilon/\delta^2 = \frac{1}{\delta} \rightarrow \delta = \varepsilon \rightarrow \xi = \frac{x-2}{\varepsilon}$, $x = \xi\varepsilon + 2$

$\rightarrow \frac{1}{\varepsilon} Y'' - \frac{1}{\varepsilon} Y' + \left(\frac{1}{2} + \frac{1}{4}\xi\varepsilon\right) Y = 0$

$\rightarrow Y'' - Y' + \left(\frac{1}{2}\varepsilon + \frac{1}{4}\xi\varepsilon^2\right) Y = 0$

$\rightarrow Y'' - Y' + \frac{1}{2}\varepsilon Y = 0$

let $Y(\xi) \sim Y_0 + Y_1\varepsilon$

$O(1)$: $Y_0'' - Y_0' = 0$, $Y_0(x=2) = Y_0(\xi=0) = 1$

$Y_0 = c_1 + c_2 e^{\xi}$

$Y_0(0) = 1 = c_1 + c_2 e^0 \rightarrow c_1 = 1 - c_2$

$Y_0 = 1 - c_2 + c_2 e^{\xi}$

$Y_0 = 1 + c_2(e^{\xi} - 1)$ ✓

$O(\varepsilon)$: $Y_1'' - Y_1' = -\frac{1}{2} Y_0$, $Y_1(2) = 0$

$Y_1'' - Y_1' = -\frac{1}{2}(1 + c_2(e^{\xi} - 1))$

$Y_{1p} = A\xi + B\xi e^{\xi}$

$Y_{1p}' = A + Be^{\xi} + B\xi e^{\xi}$

$Y_{1p}'' = 2Be^{\xi} + B\xi e^{\xi}$

$\rightarrow \quad Be^{\xi} - A = -\frac{1}{2} + \frac{1}{2}c_2 \quad -\frac{1}{2}c_2 e^{\xi}$

$B = -c_2$, $A = \frac{1}{2} - \frac{1}{2}c_2$

$Y_1 = c_3 + c_4 e^{\xi} + \left(\frac{1}{2} - \frac{1}{2}c_2\right)\xi - c_2\xi e^{\xi}$

$Y_1(0) = 0 = c_3 + c_4$

$c_3 = -c_4$

$Y_1 = c_4(e^{\xi} - 1) + \frac{1}{2}\xi(1 - c_2) - c_2\xi e^{\xi}$

$Y(\xi) = 1 + c_2(e^{\xi} - 1) + \varepsilon\left[c_4(e^{\xi} - 1) + \frac{1}{2}\xi(1 - c_2) - c_2\xi e^{\xi}\right]$

Lepo 5

4) (continued)

Matching:

O(1) terms:

$$y_0 = \frac{-6}{x-4} \qquad y_0(2) = \frac{-6}{2-4} = \frac{-6}{-2} = 3$$

$$Y_0 = 1 + c_2(e^{\xi} - 1) \;, \; Y(-\infty) = 1 - c_2$$

$$\to \quad 3 = 1 - c_2$$

$$c_2 = -2$$

Now compare $y(\xi)$ to $Y(x)$:

$$y(x) = \frac{-6}{x-4} + \varepsilon\left[\frac{-4}{x-4} - \frac{12}{(x-4)^2}\right]$$

$$y(\xi) = \frac{-6}{\xi\varepsilon - 2} + \varepsilon\left[\frac{-4}{\xi\varepsilon-2} - \frac{12}{(\xi\varepsilon-2)^2}\right]$$

$$= 3\left[\frac{1}{1-\xi\varepsilon/2}\right] + \varepsilon\left(2\left[\frac{1}{1-\xi\varepsilon/2}\right] - 3\left[\frac{1}{1-\xi\varepsilon/2}\right]^2\right.$$

$$= 3(1 + \xi\varepsilon/2) + \varepsilon\left[2(1+\xi\varepsilon/2) - 3(1+\xi\varepsilon/2)^2\right]$$

$$= 3 + \tfrac{3}{2}\xi\varepsilon + \varepsilon\left[2 + \xi\varepsilon - 3 - 3\xi\varepsilon - \tfrac{3}{4}\xi^2\varepsilon^2\right]$$

$$= 3 + \tfrac{3}{2}\xi\varepsilon + 2\varepsilon - 3\varepsilon + \xi\varepsilon^2 - 3\xi\varepsilon^2 + O(\varepsilon^3)$$

$$= 3 + \varepsilon\left(\tfrac{3}{2}\xi - 1\right) + \varepsilon^2(-2\xi)$$

$$Y(\xi) = 1 - \underline{2(e^{\xi}} - 1) + \varepsilon\left[c_4(\underline{e^{\xi}} - 1) + \tfrac{3}{2}\xi + \underline{2\xi e^{\xi}}\right]$$

these terms are very small - neglect them

$$\sim 1 + 2 + \varepsilon\left[-c_4 + \tfrac{3}{2}\xi\right] = 3 + \varepsilon\left(\tfrac{3}{2}\xi - c_4\right)$$

Comparing with $y(\xi)$, we see $c_4 = 1$

$$\text{common part} = 3 + \varepsilon\left(\tfrac{3}{2}\left(\tfrac{x-2}{\varepsilon}\right) - 1\right) = 3 + \tfrac{3}{2}x - 3 - \varepsilon$$

$$= \tfrac{3}{2}x - \varepsilon$$

$$y_c(x) = \frac{-6}{x-4} + \varepsilon\left[\frac{-4}{x-4} - \frac{12}{(x-4)^2}\right] - \frac{3}{2}x + \cancel{\varepsilon}$$

$$+ \cancel{1} - 2e^{\frac{x-2}{\varepsilon}} + \cancel{2} + \varepsilon\left[e^{\frac{x-2}{\varepsilon}} - \cancel{1} + \cancel{\tfrac{3}{2}\tfrac{x-2}{\varepsilon}} + 2\left(\tfrac{x-2}{\varepsilon}\right)e^{\frac{x-2}{\varepsilon}}\right]$$

$$= \frac{-6}{x-4} + \varepsilon\left[\frac{-4}{x-4} - \frac{12}{(x-4)^2}\right] \quad -6e^{\frac{x-2}{\varepsilon}} + 2xe^{\frac{x-2}{\varepsilon}} + \varepsilon e^{\frac{x-2}{\varepsilon}}$$

$$\boxed{y_c(x) = \frac{-6}{x+4} + e^{\frac{x-2}{\varepsilon}}(2x-6) + \varepsilon\left[\frac{-4}{x-4} - \frac{12}{(x-4)^2} + e^{\frac{x-2}{\varepsilon}}\right]}$$

Lepo6

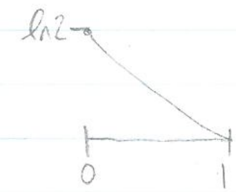
5) Consider the following boundary value problems and determine a leading order asymptotic expansion of the solution in $\varepsilon \ll 1$ that is uniformly valid for $0<x<1$.

A) $\varepsilon y'' + 2y' + e^{y} = 0,\ 0<x<1,\ y(0)=y(1)=0$

$2>0 \to$ Boundary layer possible at $x=0$

Outer solution: let $y \sim y_0$

$$2y_0' + e^{y_0} = 0,\quad y(1)=0$$
$$2\, dy_0/dx = -e^{y_0}$$
$$\int 2\, dy_0\, e^{-y_0} = \int -dx$$
$$-2e^{-y_0} = -x + C$$
$$e^{-y_0} = \frac{x+c}{2}$$
$$-y_0 = \ln \frac{x+c}{2}$$
$$y_0 = -\ln \frac{x+c}{2}$$
$$y_0(1) = -\ln \frac{1+c}{2} = 0$$
$$\ln \frac{2}{1+c} = 0$$
$$\frac{2}{1+c} = 1$$
$$2 = c+1 \to c = 1$$



$$y_0 = \ln \frac{2}{x+1}$$

Boundary layer near $x=0$:

Let $Y(\xi) = y(x),\ \xi = \frac{x}{\varepsilon},\ \delta = o(1)$

$$Y'' + 2Y' + \varepsilon e^{Y} = 0$$

Let $Y \sim Y_0$

$$Y_0'' + 2Y_0' = 0,\quad Y_0(0) = 0$$
$$Y_0 = c_1 + c_2 e^{-2\xi}$$
$$Y_0(0) = c_1 + c_2 = 0 \to c_1 = -c_2$$
$$Y_0 = c_2(e^{-2\xi} - 1)$$

Matching: $y_0(0) = \ln 2$

$$Y_0(\infty) = -c_2 \to c_2 = \ln 2$$

$$y_c = \ln\frac{2}{x+1} - \ln 2\, e^{-2\frac{x}{\varepsilon}} - \ln 2 + \ln 2$$

$$\boxed{y_c = \ln\frac{2}{x+1} - \ln 2 \cdot e^{-\frac{2x}{\varepsilon}}}$$

✓

Lepo 7

5) (continued)

B) $\varepsilon y'' + (1+x^2)y' - x^3 y = 0$, $0<x<1$, $y(0)=y(1)=1$

$1+x^2>0 \rightarrow$ Boundary layer possible at $x=0$

Outer solution: let $y \sim y_0$

$$(1+x^2)y_0' - x^3 y_0 = 0, \quad y_0(1) = 1$$

$$y_0' - \frac{x^3}{1+x^2} y_0 = 0$$

$$\frac{d}{dx}\left[y_0 \exp\left(\int -\frac{x^3}{1+x^2}dx\right)\right] = 0 \qquad \begin{matrix} u=x^2 \\ du = 2x \end{matrix} \qquad \begin{matrix} v = \frac{1}{2}\ln(1+x^2) \\ dv = \frac{x}{1+x^2} \end{matrix}$$

$$\frac{d}{dx}\left[y_0 \exp\left(-\tfrac{1}{2}x^2 \ln(1+x^2) + \int x \ln(1+x^2)dx\right)\right]$$

$$\frac{d}{dx}\left[y_0 \exp\left(-\tfrac{1}{2}x^2 + \tfrac{1}{2}\ln(1+x^2)\right)\right]$$

$$\frac{d}{dx}\left[y_0 e^{-\frac{1}{2}x^2}\sqrt{1+x^2}\right] = 0$$

$$y_0 e^{-\frac{1}{2}x^2}\sqrt{1+x^2} = c$$

$$y_0 = c\, e^{\frac{1}{2}x^2}(1+x^2)^{-\frac{1}{2}}$$

$$y_0(1) = c\, e^{\frac{1}{2}}\, 2^{-\frac{1}{2}} = 1$$

$$c = \sqrt{2}/e^{\frac{1}{2}}$$

$$y_0 = \sqrt{2}\, e^{\frac{1}{2}(x^2-1)}(1+x^2)^{-\frac{1}{2}}$$

Inner solution:

Let $Y(\xi) = y(x)$, $\xi = \frac{x}{\delta}$, $\delta = o(1)$

$$\rightarrow \varepsilon \frac{1}{\delta^2} Y'' + (1+\xi^2\delta^2)\frac{1}{\delta}Y' - \xi^3\delta^3 Y = 0$$

$$\varepsilon/\delta^2\, Y'' + \frac{1}{\delta}Y' + \underbrace{\xi^2 \delta Y'}_{\text{small}} + \underbrace{\xi^3\delta^3 Y}_{\text{small}} = 0$$

$$\varepsilon/\delta^2 = \frac{1}{\delta} \rightarrow \varepsilon = \delta \rightarrow \xi = \frac{x}{\varepsilon},\ x = \varepsilon\xi$$

Let $Y \sim Y_0$

$$\rightarrow Y_0'' + (1+\xi^2\varepsilon^2)Y' + \xi^3\varepsilon^4 Y = 0$$

$$Y_0'' + Y' = 0, \quad Y(0) = 1$$

$$Y_0 = c_1 + c_2 e^{-\xi}$$

$$Y_0(0) = c_1 + c_2 = 1 \rightarrow c_1 = 1 - c_2$$

$$Y_0 = 1 + c_2(e^{-\xi} - 1)$$

Matching: $y_0(0) = \sqrt{2}\, e^{-\frac{1}{2}}$

$Y_0(\infty) = 1 - c_2 \rightarrow 1 - c_2 = \sqrt{2/e} \rightarrow c_2 = 1 - \sqrt{2/e}$

$$y_c = \sqrt{2}\, e^{\frac{1}{2}(x^2-1)}(1+x^2)^{-\frac{1}{2}} + 1 + (1-\sqrt{2/e})(e^{-\frac{x}{\varepsilon}} - 1) - \sqrt{2/e}$$

$$\boxed{y_c = \sqrt{2}\, e^{\frac{1}{2}(x^2-1)}(1+x^2)^{-\frac{1}{2}} + (1-\sqrt{2/e})\, e^{-\frac{x}{\varepsilon}}}$$

✓

	1	2	3	4	5a	5b	
	3	3	4	7	5	5	27
max	5	3	4	9	5	5	31

Haley Lepo
11/11/08
ESAM 420
Hwk #6

A) $4\varepsilon y'' + 6\sqrt{x}\,y' - 3y = -3, \quad 0<x<1, \quad y(0)=0, \quad y(1)=3$

$6\sqrt{x} > 0 \rightarrow$ BL possible near $x=0$

Outer: $6\sqrt{x}\,y' - 3y = -3 \quad y(1)=3$

$$6\sqrt{x}\,y_h' = 3y_h$$

$$\int dy_h/y_h = \int \tfrac{1}{2}x^{-1/2}\,dx$$

$$\ln y_h = x^{1/2} + C$$

$$y_h = c_1 e^{\sqrt{x}}$$

$$y_p = A, \quad -3A = -3 \rightarrow A = 1$$

$$y = c_1 e^{\sqrt{x}} + 1$$

$$y(1) = c_1 e + 1 = 3$$

$$c_1 = 2/e$$

$$y = 2e^{\sqrt{x}-1} + 1 \checkmark$$

Boundary layer near $x=0$:

let $y(x) = Y(\xi), \ \xi = \frac{x}{\delta}, \ \delta = o(1)$

$$4(\varepsilon/\delta^2)Y'' + 6\xi^{1/2}\delta^{1/2}/\delta)Y' - 3Y = -3$$

$$\varepsilon/\delta^2 = \delta^{-1/2} \rightarrow \varepsilon = \delta^{3/2} \rightarrow \delta = \varepsilon^{2/3}$$

$$\xi = \frac{x}{\varepsilon^{2/3}}, \quad x = \xi\varepsilon^{2/3}$$

$$4Y'' + 6\xi^{1/2}Y' - 3\varepsilon^{1/2}Y = -3\varepsilon^{1/3}$$

$$4Y_0'' + 6\xi^{1/2}Y_0' = -3, \quad Y_0(0)=0$$

$$Y_0'' + \tfrac{3}{2}\xi^{1/2}Y_0' = \cancel{-3/4}$$

$$\left[\int \tfrac{3}{2}\xi^{1/2}d\xi = \xi^{3/2}\right]$$

$$d/d\xi\left[Y_0' e^{\xi^{3/2}}\right] = -3/4\, e^{\xi^{3/2}}$$

$$Y_0' = e^{-\xi^{3/2}}\left[(-3/4)\int e^{\xi^{3/2}}d\xi + c_1\right]$$

$$Y_0 = -\tfrac{3}{4}\int e^{-\xi^{3/2}}\left[\int_0^{\xi} e^{s^{3/2}}ds + c_1\right]d\xi + c_2$$

$$Y_0 = -\tfrac{3}{4}\int_0^{\xi} e^{-r^{3/2}}\left[\int_0^r e^{s^{3/2}}ds + c_1\right]dr + c_2$$

$$Y_0(0) = c_2 = 0$$

$$Y_0(\infty) = y_0(0) = 2/e + 1$$

$$2/e + 1 = -3/4\int_0^{\infty} e^{-r^{3/2}}\left[\int_0^r e^{s^{3/2}}ds + c_1\right]dr$$

$$= -3/4\left[\int_0^{\infty} e^{-r^{3/2}}\left[\int_0^r e^{s^{3/2}}ds\right] + c_1\int_0^{\infty} e^{-r^{3/2}}dr\right]$$

$$c_1 = \frac{-4/3(2/e+1) - \int_0^{\infty} e^{-r^{3/2}}\left[\int_0^r e^{s^{3/2}}ds\right]dr}{\int_0^{\infty} e^{-r^{3/2}}dr}$$

$$y_c = 2e^{\sqrt{x}-1} - 2/e - 3/4\int_0^{x\varepsilon^{-2/3}} e^{-r^{3/2}}\left[\int_0^r e^{s^{3/2}}ds + c_1\right]dr$$

?

B) $\varepsilon y'' + y(y'+3) = 0, \quad 0 < x < 1, \quad y(0) = y(1) = 1$ BL near $x=1$

Outer: $y_0(y_0'+3) = 0, \quad y(0) = 1$

Either $y_0 = 0$ or $y_0' + 3 = 0 \rightarrow y_0 = -3x + C$

doesn't work since $y(0)=1$

$y_0 = -3x + C$

$y_0(0) = 1 = C \rightarrow y_0 = -3x + 1$

BL near $x=1$:

$y(x) = Y(\xi), \quad \xi = \frac{x-1}{\delta}, \quad \delta = o(1)$

$\varepsilon/\delta^2 Y'' + Y(\frac{1}{\delta} Y' + 3) = 0$

$\varepsilon/\delta^2 Y'' + \frac{1}{\delta} YY' + 3Y = 0$

$\varepsilon/\delta^2 = \frac{1}{\delta} \rightarrow \varepsilon = \delta, \quad \xi = \frac{x-1}{\varepsilon}$

$\rightarrow Y'' + YY' + 3\varepsilon Y = 0$

$Y \sim Y_0$

$Y_0'' + Y_0 Y_0' = 0, \quad Y_0(x=1) = Y_0(0) = 1 \quad Y_0(-\infty) = -2.$

$Y_0' + \frac{1}{2} Y_0^2 = C_1$

$\frac{1}{2}(-2)^2 = C_1 = 2$

$dY_0/d\xi = 2 - \frac{1}{2} Y_0^2 = \frac{1}{2}(4 - Y_0^2)$

$\int \frac{dY_0}{4 - Y_0^2} = \int \frac{1}{2} d\xi$

$\frac{1}{4 - Y_0^2} = \frac{1/4}{2 + Y_0} + \frac{1/4}{2 - Y_0}$

$\int \frac{1/4}{2+Y_0} + \frac{1/4}{2-Y_0} \, dY_0 = \int \frac{1}{2} d\xi$

$\frac{1}{4}[\ln(2+Y_0) - \ln(2-Y_0)] = \frac{1}{2}\xi + C_1$

$\ln\left(\frac{2+Y_0}{2-Y_0}\right) = 2\xi + C_2$

$Y_0(0) = 1 \rightarrow \ln 3 = C_2$

$\ln\left(\frac{2+Y_0}{2-Y_0}\right) = 2\xi + \ln 3$

$\ln\left(\left(\frac{2+Y_0}{2-Y_0}\right) \cdot \frac{1}{3}\right) = 2\xi$

$\frac{1}{3}\left(\frac{2+Y_0}{2-Y_0}\right) = e^{2\xi}$

$$2 + Y_0 = 3e^{2\xi}(2 - Y_0)$$

$$Y_0(1 + 3e^{2\xi}) = 6e^{2\xi} - 2$$

$$Y_0 = \frac{6e^{2\xi} - 2}{3e^{2\xi} + 1} = \frac{2(3e^{2\xi}+1) - 4}{3e^{2\xi}+1} = 2 - \frac{4}{3e^{2\xi}+1}$$

$$y_c = -3x + 1 + 2 - \frac{4}{3e^{2\left(\frac{x-1}{\varepsilon}\right)} + 1} + 2$$

$$y_c = 5 - 3x - \frac{4}{3e^{2\left(\frac{x-1}{\varepsilon}\right)} + 1}$$

✓

c) $\varepsilon y'' + \underbrace{(x^{1/2}+x)}_{>0,\text{ so BL near } x=0} y' - (1+x^{1/2})y = 0, \quad 0<x<1, \quad y(0)=0, \quad y(1)=e^2$

Outer:

$(x^{1/2}+x)y_0' - (1+x^{1/2})y_0 = 0, \quad y_0(1)=e^2$

$$\int \frac{dy_0}{y_0} = \int \frac{1+x^{1/2}}{x+x^{1/2}}\,dx$$

$$\begin{array}{r|l} & \frac{1}{\sqrt{x}} \\ \hline x+\sqrt{x} & \sqrt{x}+1 \\ & \sqrt{x}+1 \\ \hline & 0 \end{array}$$

$$\ln y_0 = \int x^{-1/2}\,dx = 2\sqrt{x} + C$$

$$y_0 = c_1 e^{2\sqrt{x}}$$

$$y_0(1) = e^2 = c_1 e^2 \to c_1 = 1$$

$$y_0 = e^{2\sqrt{x}}$$

BL near $x=0$:

$$y(x) = Y(\xi), \quad \xi = \frac{x}{\delta}, \quad \delta = o(1)$$

$$\varepsilon/\delta^2\, Y'' + (\xi^{1/2}\delta^{1/2} + \xi\delta)\tfrac{1}{\delta} Y' - (1+\xi^{1/2}\delta^{1/2})Y = 0$$

$$\varepsilon/\delta^2\, Y'' + (\xi^{1/2}\delta^{-1/2} + \xi)Y' - (1+\xi^{1/2}\delta^{1/2})Y = 0$$

$$\varepsilon/\delta^2 = \delta^{-1/2} \to \varepsilon = \delta^{3/2}, \quad \delta = \varepsilon^{2/3}$$

$$Y'' + (\xi^{1/2} + \xi\varepsilon^{1/3})Y' - (\varepsilon^{1/3} + \xi\varepsilon^{2/3})Y = 0$$

$$Y_0'' + \xi^{1/2} Y_0' = 0, \quad Y(0) = 0$$

$$\int dY_0'/Y_0' = \int -\xi^{1/2}\,d\xi$$

$$\ln Y_0' = -2/3\,\xi^{3/2} + C$$

$$Y_0' = c_1 e^{-2/3\,\xi^{3/2}}$$

$$Y_0 = \int_0^{\xi} c_1 e^{-2/3\,s^{3/2}}\,ds + c_2$$

$$Y_0(0) = 0 = c_2$$

$$Y_0(\xi=\infty) = y_0(x=0) = 1 \to c_1 \int_0^\infty e^{-2/3\,s^{3/2}}\,ds = 1$$

$$c_1 = 1/\int_0^\infty e^{-2/3\,s^{3/2}}\,ds$$

$$Y_0 = \frac{\int_0^{\xi} e^{-2/3\,s^{3/2}}\,ds}{\int_0^\infty e^{-2/3\,s^{3/2}}\,ds}$$

$$y_c = e^{2\sqrt{x}} + \frac{\int_0^{x\varepsilon^{-2/3}} e^{-2/3\,s^{3/2}}\,ds}{\int_0^\infty e^{-2/3\,s^{3/2}}\,ds} - 1$$

✓

c) $\varepsilon y'' + \underbrace{(x^{1/2}+x)}_{>0,\text{ so BL near } x=0} y' - (1+x^{1/2})y = 0, \quad 0<x<1, \quad y(0)=0, \quad y(1)=e^2$

Outer:

$(x^{1/2}+x)\,y_0' - (1+x^{1/2})y_0 = 0, \quad y_0(1)=e^2$

$$\int \frac{dy_0}{y_0} = \int \frac{1+x^{1/2}}{x+x^{1/2}}\,dx$$

$$\begin{array}{r|l} & \frac{1}{\sqrt{x}} \\ \hline x+\sqrt{x} & \sqrt{x}+1 \\ & \underline{\sqrt{x}+1} \\ & \quad 0 \end{array}$$

$\ln y_0 = \int x^{-1/2}\,dx = 2\sqrt{x} + C$

$y_0 = c_1 e^{2\sqrt{x}}$

$y_0(1) = e^2 = c_1 e^2 \rightarrow c_1 = 1$

$y_0 = e^{2\sqrt{x}}$

BL near $x=0$:

$y(x) = Y(\xi), \quad \xi = \frac{x}{\delta}, \quad \delta = o(1)$

$\varepsilon/\delta^2\, Y'' + (\xi^{1/2}\delta^{1/2} + \xi\delta)\frac{1}{\delta}Y' - (1+\xi^{1/2}\delta^{1/2})Y = 0$

$\varepsilon/\delta^2\, Y'' + (\xi^{1/2}\delta^{-1/2} + \xi)Y' - (1+\xi^{1/2}\delta^{1/2})Y = 0$

$\varepsilon/\delta^2 = \delta^{-1/2} \rightarrow \varepsilon = \delta^{3/2}, \quad \delta = \varepsilon^{2/3}$

$Y'' + (\xi^{1/2} + \xi\varepsilon^{1/3})Y' - (\varepsilon^{1/3} + \xi\varepsilon^{2/3})Y = 0$

$Y_0'' + \xi^{1/2} Y_0' = 0, \quad Y(0) = 0$

$\int dY_0'/Y_0' = \int -\xi^{1/2}\,d\xi$

$\ln Y_0' = -\frac{2}{3}\xi^{3/2} + C$

$Y_0' = c_1 e^{-\frac{2}{3}\xi^{3/2}}$

$Y_0 = \int_0^{\xi} c_1 e^{-\frac{2}{3}s^{3/2}}\,ds + c_2$

$Y_0(0) = 0 = c_2$

$Y_0(\xi=\infty) = y_0(x=0) = 1 \rightarrow c_1 \int_0^\infty e^{-\frac{2}{3}s^{3/2}}\,ds = 1$

$c_1 = 1 \Big/ \int_0^\infty e^{-\frac{2}{3}s^{3/2}}\,ds$

$$Y_0 = \frac{\int_0^{\xi} e^{-\frac{2}{3}s^{3/2}}\,ds}{\int_0^\infty e^{-\frac{2}{3}s^{3/2}}\,ds}$$

$$y_c = e^{2\sqrt{x}} + \frac{\int_0^{x\varepsilon^{-2/3}} e^{-\frac{2}{3}s^{3/2}}\,ds}{\int_0^\infty e^{-\frac{2}{3}s^{3/2}}\,ds} - 1 \quad \checkmark$$

D) $\varepsilon y'' + \underbrace{\varepsilon(x+1)^2}_{>0} y' - y = x-1$, $0<x<1$, $y(0)=0$, $y(1)=-1$

$>0 \to$ BL possible near $x=0$

Outer: $-y = x-1 \to y = 1-x$

doesn't match either BC.

$\to$ 2 BL

BL near $x=0$:

$y(x) = Y(\xi)$ $\xi = \frac{x}{\delta}$, $\delta = o(1)$

$\varepsilon/\delta^2 Y'' + \varepsilon(\xi\delta+1)^2 \frac{1}{\delta} Y' - Y = \xi\delta - 1$

$\varepsilon/\delta^2 Y'' + \varepsilon/\delta(\xi^2\delta^2 + 2\xi\delta + 1) Y' - Y = \xi\delta - 1$

$\varepsilon/\delta^2 Y'' + (\xi^2 \varepsilon\delta + 2\xi\varepsilon + \frac{\varepsilon}{\delta}) Y' - Y = \xi\delta - 1$

(1) (2) (3) (4)

(3)+(4): $\varepsilon/\delta = \delta \to \delta^2 = \varepsilon \to \delta = \sqrt{\varepsilon}$: $1, \varepsilon^{3/2}, \varepsilon, \varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}}$ ✗

(1)+(4): $\varepsilon/\delta^2 = \delta \to \delta^3 = \varepsilon \to \delta = \varepsilon^{\frac{1}{3}}$: $\varepsilon^{\frac{1}{3}}, \varepsilon^{4/3}, \varepsilon, \varepsilon^{2/3}, \varepsilon^{\frac{1}{3}}$ ✓

$\varepsilon^{\frac{1}{3}} Y'' + (\xi^2 \varepsilon^{4/3} + 2\xi\varepsilon + \varepsilon^{2/3}) Y' - Y = \xi\varepsilon^{\frac{1}{3}} - 1$

will not work ($Y(0) \neq 0$)

Try $\delta = \sqrt{\varepsilon}$

$Y'' + (\varepsilon^{3/2}\xi^2 + 2\varepsilon^{\frac{1}{2}}\xi + \varepsilon^{\frac{1}{2}}) Y' - Y = \varepsilon^{\frac{1}{2}}\xi - 1$

$Y_0'' - Y_0 = -1$ $r^2 - 1 = 0$ $r = \pm 1$

$Y_0 = c_1 e^{\xi} + c_2 e^{-\xi} + 1 \to c_1 = 0$

$Y_0(0) = c_2 + 1 = 0 \to c_2 = -1$

$Y_0 = 1 - e^{-\xi}$ Need $Y_0(\xi = \infty) = y_0(x=0) = 1$ ✓

BL near $x=1$:

$y(x) = Z(\eta)$, $\eta = \frac{x-1}{\delta}$, $\delta = o(1)$ $x = \eta\delta + 1$

$\varepsilon/\delta^2 Z'' + \varepsilon(\eta\delta+2)^2 \frac{1}{\delta} Z' - Z = \eta\delta$

$\varepsilon/\delta^2 Z'' + (\eta^2\varepsilon\delta + 4\eta\varepsilon + 2^{\varepsilon}/\delta) Z' - Z = \eta\delta$

$\delta^2 = \varepsilon \to \delta = \varepsilon^{\frac{1}{2}}$

$Z'' + (\eta^2\varepsilon^{3/2} + 4\eta\varepsilon + 2\varepsilon^{\frac{1}{2}}) Z' - Z = \eta\varepsilon^{\frac{1}{2}}$

$Z_0'' - Z_0 = 0$, $Z(\eta=0) = -1$

$Z_0 = c_3 e^{-\eta} + c_4 e^{\eta} \to c_3 = 0$

$Z_0(0) = c_4 = -1$

$Z_0(\eta) = -e^{\eta}$ need $Z_0(\eta = -\infty) = y_0(x=1) = 0$ ✓

$y_c = 1 - x + 1 - e^{-x/\varepsilon^{\frac{1}{2}}} - e^{\frac{x-1}{\varepsilon^{\frac{1}{2}}}} - 1$

$y_c = 1 - x - e^{-\frac{x}{\sqrt{\varepsilon}}} - e^{\frac{x-1}{\sqrt{\varepsilon}}}$ ✓

	a	b	c	d
	3	4	4	4
max	4	4	4	4

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ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Homework 6 (DUE TUESDAY, 11/11/08)

Problem 1.

Consider the following boundary value problems and determine a leading order asymptotic expansion of the solution in $\varepsilon \ll 1$ that is uniformly valid for $0 < x < 1$ (i.e., determine the leading order outer solution, the leading order inner solution, match them and form a composite solution):

(a). $4\varepsilon y'' + 6\sqrt{x}y' - 3y = -3, \quad 0 < x < 1, \;\; y(0) = 0, \;\; y(1) = 3,$

(b). $\varepsilon y'' + y(y' + 3) = 0, \quad 0 < x < 1, \;\; y(0) = 1, \;\; y(1) = 1,$

(c). $\varepsilon y'' + (x^{1/2} + x)y' - (1 + x^{1/2})y = 0, \quad 0 < x < 1, \;\; y(0) = 0, \;\; y(1) = e^2,$

(d). $\varepsilon y'' + \varepsilon(x + 1)^2 y' - y = x - 1, \quad 0 < x < 1, \;\; y(0) = 0, \;\; y(1) = -1.$

(In problem (b) find the solution with a boundary layer near $x = 1$; the solution with a boundary layer near $x = 0$ has been discussed in class.)

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ESAM 420
11-18-08
Hwk #7

A) $\varepsilon y'' + \underbrace{(x^2 - \tfrac{1}{4})}\, y' = 0, \quad 0 < x < 1, \quad y(0) = 1, \quad y(1) = -1$

< 0 at $x = 0$, > 0 at $x = 1$, 0 at $x = \frac{1}{2}$

can't have BL at $x = 0$ or $x = 1$

Outer: $(x - \frac{1}{4}) y_0' = 0$

$y_0 = C \rightarrow y_{left} = 1, \quad y_{right} = -1$

Innerlayer at $x = \frac{1}{2}$:

let $y(x) = \varepsilon^{\gamma} Y(\xi), \quad \xi = \frac{x - \frac{1}{2}}{\delta}, \quad \delta = o(1), \quad x = \xi\delta + \frac{1}{2}$

$\rightarrow \varepsilon/\delta^2\, Y'' + \left[(\xi\delta + \tfrac{1}{2})^2 - \tfrac{1}{4}\right] \frac{1}{\delta} Y' = 0$

$\varepsilon/\delta^2\, Y'' + (\xi^2\delta^2 + \xi\delta) \frac{1}{\delta} Y' = 0$

$\varepsilon/\delta^2\, Y'' + (\xi^2\delta + \xi) Y' = 0$

$\varepsilon/\delta^2 = 1 \rightarrow \varepsilon = \delta^2 \rightarrow \delta = \sqrt{\varepsilon}$

$\rightarrow Y'' + (\xi^2\sqrt{\varepsilon} + \xi) Y' = 0$

$Y_0'' + \xi Y_0' = 0$

$d/dx\left[Y_0' e^{\frac{1}{2}\xi^2}\right] = 0$

$Y_0' = c_1 e^{-\frac{1}{2}\xi^2}$

$Y_0 = \int_0^{\xi} c_1 e^{-\frac{1}{2}s^2} ds + c_2$

BC: $\varepsilon^{\gamma} Y_0(\xi = -\infty) = 1$:

$\varepsilon^{\gamma}\left(\int_0^{-\infty} c_1 e^{-\frac{1}{2}s^2} ds + c_2\right) = 1$

$\varepsilon^{-\gamma}\left(-c_1 \tfrac{1}{2}\sqrt{2\pi} + c_2\right) = 1$

$\varepsilon^{\gamma} Y_0(\xi = \infty) = -1$:

$\varepsilon^{\gamma}\left(\int_0^{\infty} c_1 e^{-\frac{1}{2}s^2} ds + c_2\right) = -1$

$\varepsilon^{\gamma}\left(c_1 \tfrac{1}{2}\sqrt{2\pi} + c_2\right) = -1$

$\gamma = 0: \quad 2c_2 = 0 \rightarrow c_2 = 0$

$c_1\sqrt{2\pi} = -2 \rightarrow c_1 = -2/\sqrt{2\pi}$

$Y_0 = -2/\sqrt{2\pi} \int_0^{\xi} e^{-\frac{1}{2}s^2} ds$

$$y_c = \begin{cases} 1 - 2/\sqrt{2\pi} \int_0^{\xi} e^{-\frac{1}{2}s^2} ds - 1 & 0 < x < \frac{1}{2} \\ -1 - 2/\sqrt{2\pi} \int_0^{\xi} e^{-\frac{1}{2}s^2} ds + 1 & \frac{1}{2} < x < 1 \end{cases}$$

$$\boxed{y_c = -2/\sqrt{2\pi} \int_0^{\frac{x - \frac{1}{2}}{\varepsilon}} e^{-\frac{1}{2}s^2} ds}$$ ✓

B) $\varepsilon y'' + \underbrace{e^x(xy'-y)} = x^2$, $-1<x<1$, $y(-1)=1$, $y(1)=-1$

<0 for $x=-1$, >0 for $x=1$, $=0$ for $x=0$

$\rightarrow$ can't have BC at $x=-1$ or $x=1$

Outer: $e^x(xy_0'-y_0)=x^2$

$$xy_0'-y_0=e^{-x}x^2$$

$$y_0'-\frac{1}{x}y_0=e^{-x}x$$

$$\left[\exp\int -\frac{1}{x}\,dx=\exp(-\ln x)=\frac{1}{x}\right]$$

$$\rightarrow \frac{d}{dx}\left[\frac{1}{x}y_0\right]=e^{-x}$$

$$\frac{1}{x}y_0=-e^{-x}+c_1$$

$$y_0=c_1x-xe^{-x}$$

left: $y_0(-1)=-c_1+e=1$

$$\rightarrow c_1=e-1$$

right: $y_0(1)=c_1-e^{-1}=-1$

$$\rightarrow c_1=e^{-1}-1$$

Inner: let $\varepsilon^{\gamma}Y(\xi)=y(x)$, $\xi=\frac{x}{\delta}$, $\delta=o(1)$

$$\varepsilon/\delta^2\, Y''+e^{\xi\delta}(\xi Y'-Y)=\xi^2\delta^2$$

$$\varepsilon/\delta^2\, Y''+(1+\xi\delta+\xi^2\delta^2)(\xi Y'-Y)=\xi^2\delta^2$$

$$\underline{\varepsilon/\delta^2\, Y''}+\underline{\xi Y'}\,\underline{-Y}+\xi^2\delta Y'-\xi\delta Y+\xi^3\delta^2 Y'-\xi^2\delta^2 Y=\xi^2\delta^2$$

$$\varepsilon/\delta^2=1 \rightarrow \delta^2=\varepsilon \rightarrow \delta=\sqrt{\varepsilon}$$

$$\rightarrow Y_0''+\xi Y_0'-Y_0=0$$

$Y_0=\xi$ is a solution

reduction of order: $Y_0=\xi\cdot u(\xi)$

$$Y_0'=u+\xi u',\quad Y_0''=2u'+\xi u''$$

$$\rightarrow \xi u''+u'+u'+\xi u+\xi^2u'-\xi u=0$$

$$\xi u''+(\xi^2+2)u'=0$$

$$u''+(\xi+2/\xi)u'=0$$

$$\left[\exp\int \xi+2/\xi\right)=\exp\left(\tfrac{1}{2}\xi^2+2\ln\xi\right)=e^{\frac{1}{2}\xi^2}\cdot\xi^2$$

$$\frac{d}{d\xi}\left[u'\xi^2e^{\frac{1}{2}\xi^2}\right]=0$$

$$u'=c_1\xi^{-2}e^{-\frac{1}{2}\xi^2}\qquad u=e^{-\frac{1}{2}\xi^2}\quad du=-\xi e^{-\frac{1}{2}\xi^2}d\xi\qquad v=-\xi^{-1}\quad dv=\xi^{-2}$$

$$u=c_1\left(-\xi^{-1}e^{-\frac{1}{2}\xi^2}-\int e^{-\frac{1}{2}\xi^2}d\xi\right)+c_2$$

$$\rightarrow Y_0=c_1\left(-e^{-\frac{1}{2}\xi^2}-\xi\int e^{-\frac{1}{2}\xi^2}d\xi\right)+c_2\xi$$

$$=c_3e^{-\frac{1}{2}\xi^2}+c_3\xi\int_0^{\xi}e^{-\frac{1}{2}s^2}ds+c_2\xi$$

$$\psi_0(\xi) = \begin{cases} \xi \to -\infty: \left(-c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2\right)\cdot \xi \\ \xi \to \infty: \left(c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2\right)\cdot \xi \end{cases}$$

$$= \begin{cases} \xi \to -\infty: \left(-c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2\right) \frac{x}{\sqrt{\varepsilon}} \\ \xi \to \infty: \left(c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2\right) \frac{x}{\sqrt{\varepsilon}} \end{cases}$$

$$\varepsilon^{\gamma}\psi_0(\xi) = \begin{cases} \xi \to -\infty: \left(-c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2\right) x\,\varepsilon^{\gamma-\frac{1}{2}} \\ \xi \to \infty: \left(c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2\right) x\,\varepsilon^{\gamma-\frac{1}{2}} \end{cases}$$

$$y_0(x) = \begin{cases} (e-1)x - xe^{-x} & -1 < x < 0 \\ (e^{-1}-1)x - xe^{-x} & 0 < x < 1 \end{cases}$$

$$= \begin{cases} (e-1-e^{-x})x & -1 < x < 0 \\ (e^{-1}-1-e^{-x})x & 0 < x < 1 \end{cases}$$

$$y_0(\xi) = \begin{cases} (e-1-e^{\xi\sqrt{\varepsilon}})\,\xi\sqrt{\varepsilon} & \xi < 0 \\ (e^{-1}-1-e^{\xi\sqrt{\varepsilon}})\,\xi\sqrt{\varepsilon} & \xi > 0 \end{cases}$$

$\to \gamma = 1/2$

$$e-1 = -c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2$$
$$e^{-1}-1 = c_3 \tfrac{1}{2}\sqrt{2\pi} + c_2$$
$$\to e + e^{-1} - 2 = 2c_2 \to c_2 = \tfrac{1}{2}(e + e^{-1} - 1)$$
$$e^{-1} - e = c_3\sqrt{2\pi} \to c_3 = (e^{-1}-e)/\sqrt{2\pi}$$

$$y_c = -xe^{-x} + \left(\frac{(e^{-1}-e)}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\frac{x^2}{\varepsilon}} + \frac{x}{\sqrt{\varepsilon}}\int_0^{\frac{x}{\sqrt{\varepsilon}}} e^{-\frac{1}{2}s^2}\,ds\right] + \frac{(e+e^{-1}-2)}{2}\frac{x}{\sqrt{\varepsilon}}\right)\sqrt{\varepsilon}$$

$$y_c = -xe^{-x} + \frac{(e^{-1}-e)}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\frac{x^2}{\varepsilon}}\sqrt{\varepsilon} + x\int_0^{\frac{x}{\sqrt{\varepsilon}}} e^{-\frac{1}{2}s^2}\,ds\right] + \frac{(e+e^{-1}-2)x}{2}$$

O.K.

a	b
4	5
4	5

ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Homework 8 (DUE TUESDAY, 11/25/08)

Problem 1. Consider the equation

$$\varepsilon^2 y'' - q(x,\varepsilon)y = 0.$$

Assuming $q(x,\varepsilon) \sim q_0(x) + \varepsilon q_1(x)$, $q_0(x) \neq 0$, show that

$$y \sim q_0(x)^{-1/4}\left[a_0 \exp\left(-\frac{1}{\varepsilon}\kappa(x,\varepsilon)\right) + b_0 \exp\left(\frac{1}{\varepsilon}\kappa(x,\varepsilon)\right)\right],$$

where

$$\kappa(x,\varepsilon) = \int^x \sqrt{q_0(s)}\left[1 + \frac{\varepsilon}{2}\frac{q_1(s)}{q_0(s)}\right] ds.$$

Problem 2. Consider the equation

$$\varepsilon^2 y'' + p(x)y' + q(x)y = 0.$$

Assuming $p(x) \neq 0$, show that the WKB approximation of the general solution is

$$y \sim a_0 \exp\left(-\int^x \frac{q(s)}{p(s)}\, ds\right) + \frac{b_0}{p(x)} \exp\left(\int^x \frac{q(s)}{p(s)}\, ds - \frac{1}{\varepsilon^2}\int^x p(s)ds\right).$$

Hint: seek solutions in the form

$$y(x) = \exp\left(\frac{\theta(x)}{\epsilon^\alpha}\right)[y_0(x) + \epsilon^\alpha y_1(x) + \ldots],$$

where α has to be determined.

Haley Lepo
11-25-08
ESAM 420
Hwk #8

1) Consider $\varepsilon^2 y'' - q(x,\varepsilon) y = 0$

Assume $q(x,\varepsilon) \sim q_0(x) + \varepsilon q_1(x)$, $q_0(x) \neq 0$

Assume $y(x) \sim e^{\frac{1}{\varepsilon}\theta(x)}\left[y_0(x) + \varepsilon y_1(x)\right]$

$O\left(\frac{1}{\varepsilon^2}\right)$: $\left[\theta'^2(x) - q(x,\varepsilon)\right] y_0(x) = 0$

$y_0(x) \neq 0 \rightarrow \theta'^2(x) = q(x,\varepsilon)$ — needs to be expanded

$\rightarrow \theta'(x) = \pm\sqrt{q(x,\varepsilon)}$

$\rightarrow \theta(x) = \pm\int^x \sqrt{q(s,\varepsilon)}\, ds$

$O\left(\frac{1}{\varepsilon}\right)$: $\underbrace{\left[\theta'^2(x) - q(x,\varepsilon)\right] y_1}_{=0} + 2\theta' y_0' + \theta'' y_0 = 0$

$$\rightarrow \int \frac{dy_0}{y_0} = \int \frac{-\theta''}{2\theta'}\, dx$$

$$\ln y_0 = -\tfrac{1}{2} \ln|\theta'| + c_0$$

$$y_0 = \frac{c_1}{\sqrt{|\theta'|}} = \frac{c_1}{q(x,\varepsilon)^{1/4}}$$

$$y(x) \sim \frac{1}{q(x,\varepsilon)^{1/4}}\left[a_0 e^{-\frac{1}{\varepsilon}\int^x \sqrt{q(s,\varepsilon)}\, ds} + b_0 e^{\frac{1}{\varepsilon}\int^x \sqrt{q(s,\varepsilon)}\, ds}\right]$$

$$\sqrt{q(x,\varepsilon)} \sim \left(q_0(x) + \varepsilon q_1(x)\right)^{\frac{1}{2}}$$
$$\sim q_0(x)^{\frac{1}{2}} + \tfrac{1}{2} q_0(x)^{-\frac{1}{2}} \varepsilon q_1(x)$$
$$= \sqrt{q_0(x)}\left[1 + \frac{\varepsilon q_1(x)}{2 q_0(x)}\right]$$

So $y(x) \sim \dfrac{1}{q_0(x)^{1/4}}\left[a_0 e^{-\frac{1}{\varepsilon}K} + b_0 e^{\frac{1}{\varepsilon}K}\right]$

$$K = \int^x \sqrt{q_0(s)}\left[1 + \frac{\varepsilon q_1(s)}{2 q_0(s)}\right] ds$$

2) Consider $\varepsilon^2 y'' + p(x) y' + q(x) y = 0$
Assume $p(x) \neq 0$.

Assume $y(x) = \exp\left(\frac{\theta(x)}{\varepsilon^\alpha}\right)\left(y_0(x) + \varepsilon^\alpha y_1(x) + \cdots\right)$

$$y'(x) = \theta'(x) \frac{1}{\varepsilon^\alpha} \exp\left(\frac{\theta(x)}{\varepsilon^\alpha}\right)\left(y_0(x) + \varepsilon^\alpha y_1(x) + \cdots\right) + \exp\left(\frac{\theta(x)}{\varepsilon^\alpha}\right)\left(y_0'(x) + \varepsilon^\alpha y_1'(x) + \cdots\right)$$

$$y''(x) = \theta''(x) \frac{1}{\varepsilon^\alpha} \exp\left(\frac{\theta(x)}{\varepsilon^\alpha}\right)\left(y_0(x) + \varepsilon^\alpha y_1(x) + \cdots\right) + \theta'(x)^2 \frac{1}{\varepsilon^{2\alpha}} \exp\left(\frac{\theta(x)}{\varepsilon^\alpha}\right)\left(y_0(x) + \varepsilon^\alpha y_1(x) + \cdots\right) + 2\theta'(x) \frac{1}{\varepsilon^\alpha} \exp\left(\frac{\theta(x)}{\varepsilon^\alpha}\right)\left(y_0'(x) + \varepsilon^\alpha y_1'(x) + \cdots\right) + \exp\left(\frac{\theta(x)}{\varepsilon^\alpha}\right)\left(y_0''(x) + \varepsilon^\alpha y_1''(x) + \cdots\right)$$

Plug into $\varepsilon^2 y'' + p(x) y' + q(x) y = 0$
(exponentials cancel)

$$\varepsilon^2\left[\theta'' \varepsilon^{-\alpha}(y_0 + \varepsilon^\alpha y_1) + \theta'^2 \varepsilon^{-2\alpha}(y_0 + \varepsilon^\alpha y_1) + 2\theta' \varepsilon^{-\alpha}(y_0' + \varepsilon^\alpha y_1') + (y_0'' + \varepsilon^\alpha y_1'')\right] + p\left[\theta' \varepsilon^{-\alpha}(y_0 + \varepsilon^\alpha y_1) + (y_0' + \varepsilon^\alpha y_1')\right] + q\left[y_0 + \varepsilon^\alpha y_1\right] = 0$$

$$\rightarrow \theta'' \varepsilon^{2-\alpha} y_0 + \theta'' \varepsilon^2 y_1 + \theta'^2 \varepsilon^{2-2\alpha} y_0 + \theta'^2 \varepsilon^{2-\alpha} y_1 + 2\theta' \varepsilon^{2-\alpha} y_0' + 2\theta' \varepsilon^2 y_1' + \varepsilon^2 y_0'' + \varepsilon^{2+\alpha} y_1'' + p\theta' \varepsilon^{-\alpha} y_0 + p\theta' y_1 + p y_0' + p\varepsilon^\alpha y_1' + q y_0 + q\varepsilon^\alpha y_1 = 0$$

worrying terms: $\varepsilon^{2-2\alpha}$, $\varepsilon^{-\alpha}$
let $2 - 2\alpha = -\alpha \rightarrow 2 = \alpha$ ✓

$$\rightarrow \theta'' y_0 + \theta'' \varepsilon^2 y_1 + \theta'^2 \varepsilon^{-2} y_0 + \theta'^2 y_1 + 2\theta' y_0' + 2\theta' \varepsilon^2 y_1' + \varepsilon^2 y_0'' + \varepsilon^4 y_1'' + p\theta' \varepsilon^{-2} y_0 + p\theta' y_1 + p y_0' + p\varepsilon^2 y_1' + q y_0 + q\varepsilon^2 y_1 = 0$$

$O(\varepsilon^{-2})$: $y_0 \theta'(\theta' + p) = 0$ ✓

$O(1)$: $\theta'' y_0 + \theta'^2 y_1 + 2\theta' y_0' + p\theta' y_1 + p y_0' + q y_0 = 0$ ✓

$O(\varepsilon^2)$: $\theta'' y_1 + 2\theta' y_1' + y_0'' + p y_1' + q y_1 = 0$

	1	2
	2	5
max	4	5

7

Assume $y_0 \neq 0$, $\theta' \neq 0$

$\rightarrow \theta' = -p(x)$ ✓

$\rightarrow \theta = -\int^x p(s)\,ds$

$O(1)$: $y_1 \theta'(\theta' + p) + y_0(\theta'' + q) + y_0'(2\theta' + p) = 0$

$\rightarrow y_0(\theta'' + q) - \theta' y_0' = 0$

$$\int \frac{dy_0}{y_0} = +\int \frac{\theta'' + q(x)}{\theta'} dx$$

$$\ln|y_0| = +\int \frac{-p'(x) + q(x)}{-p(x)} dx$$

$$= \int \frac{p'(x)}{p(x)} dx - \int \frac{q(x)}{p(x)}$$

$$y_0 = e^{\ln \frac{1}{p(x)}} e^{+\int \frac{q(x)}{p(x)} dx}$$

$$y_0 = \frac{1}{p(x)} \exp^{+}\int^x \frac{q(s)}{p(s)} ds$$

Now Assume $\theta' = 0$ ✓

$\rightarrow \theta = c$, $\theta'' = 0$

$O(1)$: $y_0(\theta'' + q) + y_0' p = 0$

$$\int \frac{dy_0}{y_0} = \int \frac{-p(x)}{q(x)} dx$$

$$\ln|y_0| = -\int \frac{p(x)}{q(x)} dx$$

$$y_0 = \exp - \int^x \frac{p(s)}{q(s)} ds$$

$$y(x) \sim a_1 \exp(\varepsilon^{-2}) \exp\left(-\int^x \frac{p(s)}{q(s)} ds\right) + b_0 \exp\left(\varepsilon^{-2}\left(-\int^x p(s)\,ds\right)\right) \frac{1}{p(x)} \exp^{+}\int^x \frac{q(s)}{p(s)} ds)$$

$$y(x) \sim a_1 \exp\left(\varepsilon^{-2}\left(-\int^x \frac{p(s)}{q(s)} ds\right)\right) + \frac{b_0}{p(x)} \exp\left(-\varepsilon^{-2}\int^x p(s)\,ds + \int^x \frac{q(s)}{p(s)} ds\right)$$

let $a_0 = a_1 e^{\frac{1}{\varepsilon^2}}$

$$\boxed{y(x) \sim a_0 e^{-\int^x \frac{p(s)}{q(s)} ds} + \frac{b_0}{p(x)} e^{\int^x \frac{q(s)}{p(s)} ds - \frac{1}{\varepsilon^2}\int^x p(s)\,ds}}$$ ✓

ES_APPM 420-1 "Asymptotic and Perturbation Methods"

Final Examination DUE TUESDAY, 12/09/08 BY 2 PM.

Use the method of matched asymptotic expansions to find leading-order uniform approximations to the solutions of the following problems ($0 < \varepsilon \ll 1$).

1. $\varepsilon^2 y'' + \underbrace{(2\varepsilon^2 + x^4)}_{>0} y' + (x^4 - \varepsilon)y = x^5, \quad 0 < x < 1, \quad y(0) = 1, \; y(1) = 2e^{-1};$

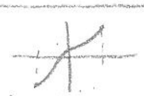
→ may have BL near x = 0

2. $6\varepsilon(x+1)y'' + \underline{(x^2 - 3x + 2)}y' + (x-2)y = 0, \quad 0 < x < 2, \quad y(0) = 1, \; y(2) = 3;$

may have BL near x = 0

could be problems at x = 1 + 2

3. $\varepsilon y'' + \underline{(\sqrt{\varepsilon}e^x + \tan x)}y' + (\sqrt{\varepsilon}e^x - \tan x)y = 0, \quad -1 < x < 1, \quad y(-1) = y(1) = 1;$



no BL at endpts

could be problem near x = 0

4. $\varepsilon y'' + \underline{x}y' - (1 + x\cos x)y = 0, \quad -1 < x < \pi, \quad y'(-1) = 0, \; y(\pi) = \pi;$

no BL at endpts

could be problem at x = 0

Please simplify your answers and clearly write them.

Haley Lepo
12-9-08
ESAM 420
Final Exam

1) $\varepsilon^2 y'' + (2\varepsilon^2 + x^4)y' + (x^4 - \varepsilon)y = x^5$, $0<x<1$, $y(0)=1$, $y(1)=2e^{-1}$

Since the coefficient of y', $2\varepsilon^2+x^4$, is greater than zero, there is the possibility of a boundary layer near $x=0$.

Outer solution:

Assume $y(t) \sim y_0(t) + \varepsilon y_1(t)$

$O(1)$: $x^4 y_0' + x^4 y_0 = x^5$, $y(1) = 2e^{-1}$

$y_0' + y_0 = x$ ← this is valid because $x \neq 0$

$y_{oh} = c_1 e^{-x}$

$y_{op} = A + Bx$, $y_{op}' = B$

$B + A + Bx = x$

$\rightarrow B = 1 \rightarrow A = -1$

$y_0 = c_1 e^{-x} + x - 1$

$y_0(1) = c_1 e^{-1} + 1 - 1 = 2e^{-1}$

$\rightarrow c_1 = 2$

$y_0 = 2e^{-x} + x - 1$

Check $y_0(0) = 2e^0 + 0 - 1 = 2 - 1 = 1$

So y_0 fits both boundary conditions.
It has no worrying areas otherwise (it is smooth and bounded).
thus there is no reason for a boundary layer.

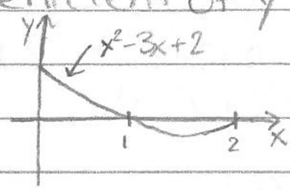
$\boxed{y(x) \sim 2e^{-x} + x - 1}$ ✓

Lepo 2

2) $\varepsilon(x+1)y'' + (x^2-3x+2)y' + (x-2)y = 0,$

$0 < x < 2, \quad y(0) = 1, \quad y(2) = 3$

Coefficient of y': Since this is positive at $x=0$ there may be a BL there.
Since it is zero at $x=1$ and $x=2$, there may be problems there too.



Outer solution:

Assume $y(t) \sim y_0(t) + \varepsilon y_1(t)$

$O(1)$: $(x^2-3x+2)y_0' + (x-2)y_0 = 0$

$$(x-2)(x-1)y_0' = -(x-2)y_0$$

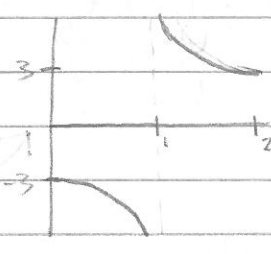
$$\int \frac{dy}{y_0} = \int \frac{-(x-2)\,dx}{(x-2)(x-1)}$$

$$\ln|y_0| = -\ln|x-1|$$

$$y_0 = \frac{c_1}{x-1}$$

$$y_0(2) = \frac{c_1}{2-1} = c_1 = 3 \rightarrow c_1 = 3$$

$$y_0 = \frac{3}{x-1}$$

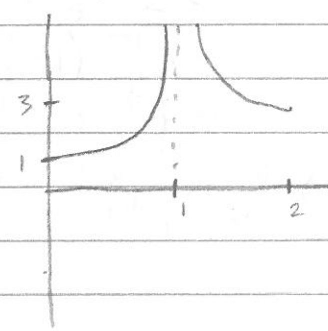


There must be an inner layer at $x=1$.

So we can separate y_0 into right and left solutions, matching the left BC as well:

$$y_0(0) = \frac{c_1}{0-1} = -c_1 = 1 \rightarrow c_1 = -1$$

$$y_0 = \begin{cases} -1/x-1 & \text{left} \\ 3/x-1 & \text{right} \end{cases}$$



Inner Layer:

let $\varepsilon^{-\gamma} Y(\xi) = y(x)$ with $\xi = \frac{x-1}{\delta}$, $\delta = o(1)$, $x = \xi\delta + 1$

Plug in:

$$6\varepsilon(\xi\delta+1+1)\frac{1}{\delta^2}Y'' + ((\xi\delta+1)^2 - 3(\xi\delta+1)+2)\frac{1}{\delta}Y' + (\xi\delta+1-2)Y = 0$$

$$6\varepsilon(\xi\delta+2)Y'' + (\xi^2\delta^2+2\xi\delta+1-3\xi\delta-3+2)\delta Y'$$
$$+(\xi\delta-1)\delta^2 Y = 0$$

$$(6\xi\varepsilon\delta+12\varepsilon)Y'' + (\xi^2\delta^3-\xi\delta^2)Y' + (\xi\delta^3-\delta^2)Y = 0$$

$$(6\xi\varepsilon\delta+12\varepsilon)Y'' + (\xi\delta^3-\delta^2)(\xi Y'+Y) = 0$$

$$\rightarrow \varepsilon = \delta^2 \rightarrow \delta = \sqrt{\varepsilon}$$

$$(6\xi\varepsilon^3+12\varepsilon)Y'' + (\xi\varepsilon^{3/2}-\varepsilon)(\xi Y'+Y) = 0$$

Assume $Y(\xi) \sim Y_0(\xi) + \varepsilon Y_1(\xi)$

$O(\varepsilon)$: $12Y_0'' - \xi Y_0' - Y_0 = 0$

$$12Y_0'' - d/d\xi[\xi Y_0] = 0$$

$$d/d\xi[12Y_0'] = d/d\xi[\xi Y_0]$$

$$12Y_0' = \xi Y_0 + C_2$$

$$12Y_0' - \xi Y_0 = C_2$$

integrating factor: $e^{\int -\frac{\xi}{12}d\xi} = e^{-\frac{1}{24}\xi^2}$

$$d/d\xi[Y_0 e^{-\frac{1}{24}\xi^2}] = C_2 e^{-\frac{1}{24}\xi^2}$$

$$Y_0 e^{-\frac{1}{24}\xi^2} = C_2\int_0^\xi e^{-\frac{1}{24}s^2}ds + C_3$$

$$Y_0 = e^{+\frac{1}{24}\xi^2}\left(C_2\int_0^\xi e^{-\frac{1}{24}s^2}ds + C_3\right)$$

this function $\rightarrow \infty$ so fast as $\xi \rightarrow \pm\infty$ that you cannot match it to anything.

Matching:

left: [let $\gamma = 0$]

$$Y_0(\xi) \sim e^{\frac{1}{24}\xi^2}\left(C_2\tfrac{1}{2}\sqrt{24\pi} + C_3\right)$$
$$= e^{\frac{1}{24}\frac{(x-1)^2}{\varepsilon}}\left(C_2\tfrac{1}{2}\sqrt{24\pi} + C_3\right)$$
$$= e^{\frac{1}{24\varepsilon}(x^2-2x+1)}\left(C_2\tfrac{1}{2}\sqrt{24\pi} + C_3\right)$$
$$= e^{\frac{1}{24\varepsilon}} e^{x^2/24\varepsilon} e^{-x/12\varepsilon}\left(C_2\tfrac{1}{2}\sqrt{24\pi} + C_3\right)$$
$$\sim e^{\frac{1}{24\varepsilon}}\left(1+\frac{x^2}{24\varepsilon}\right)\left(1-\frac{x}{12\varepsilon}\right)\left(C_2\tfrac{1}{2}\sqrt{24\pi} + C_3\right)$$
$$\sim e^{\frac{1}{24\varepsilon}}\left(C_2\tfrac{1}{2}\sqrt{24\pi} + C_3\right)$$

$$y_0\text{left} = \frac{-1}{x-1} = \frac{1}{1-x} \sim 1+x+x^2 \sim 1$$

$$\rightarrow e^{\frac{1}{24\varepsilon}}\left(C_2\tfrac{1}{2}\sqrt{24\pi} + C_3\right) = 1$$

right:

$$Y_0(\xi) \sim e^{\frac{1}{24}\xi^2}\left(-c_2\tfrac{1}{2}\sqrt{24\pi}+c_3\right)$$

$$\sim e^{\frac{1}{24\varepsilon}}\left(-c_2\tfrac{1}{2}\sqrt{24\pi}+c_3\right)$$

$$y_{right} = 3/x-1 = -3/1-x = -3(1+x+x^2\ldots) \sim -3$$

$$\rightarrow e^{\frac{1}{24\varepsilon}}\left(-c_2\tfrac{1}{2}\sqrt{24\pi}+c_3\right) = -3$$

$$\rightarrow 2c_3 e^{\frac{1}{24\varepsilon}} = -2$$

$$c_3 = -e^{-1/24\varepsilon}$$

$$c_2\sqrt{24\pi}\, e^{\frac{1}{24\varepsilon}} = 4$$

$$c_2 = \frac{e^{-1/24\varepsilon}\,4}{\sqrt{24\pi}} = \frac{e^{-1/24\varepsilon}\,2}{\sqrt{6\pi}}$$

$$y(x) = \begin{cases} -\frac{1}{x-1} + e^{\frac{(x-1)^2}{24\varepsilon}}\left(\left(e^{-\frac{1}{24\varepsilon}}\cdot 2/\sqrt{6\pi}\right)\int_0^{\frac{x-1}{\sqrt{\varepsilon}}} e^{-\frac{1}{24}s^2}ds - e^{-\frac{1}{24\varepsilon}}\right) - 1 \\ 3/x-1 + e^{\frac{(x-1)^2}{24\varepsilon}}\left(\left(e^{-\frac{1}{24\varepsilon}}\cdot 2/\sqrt{6\pi}\right)\int_0^{\frac{x-1}{\sqrt{\varepsilon}}} e^{-\frac{1}{24}s^2}ds - e^{-\frac{1}{24\varepsilon}}\right) + 3 \end{cases}$$

$$= \begin{cases} -\frac{1}{x-1} + e^{(x^2-2x)/24\varepsilon}\left[\left(2/\sqrt{6\pi}\right)\int_0^{\frac{x-1}{\sqrt{\varepsilon}}} e^{-\frac{1}{24}s^2}ds - 1\right] - 1 \\ 3/x-1 + e^{(x^2-2x)/24\varepsilon}\left[\left(2/\sqrt{6\pi}\right)\int_0^{\frac{x-1}{\sqrt{\varepsilon}}} e^{-\frac{1}{24}s^2}ds - 1\right] + 3 \end{cases}$$

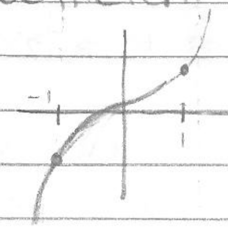
$$\boxed{y(x) = \begin{cases} -\frac{x}{x-1} + e^{(x^2-2x)/24\varepsilon}\left[\left(2/\sqrt{6\pi}\right)\int_0^{\frac{x-1}{\sqrt{\varepsilon}}} e^{-\frac{1}{24}s^2}ds - 1\right], & x<1 \\ \frac{3x}{x-1} + e^{(x^2-2x)/24\varepsilon}\left[\left(2/\sqrt{6\pi}\right)\int_0^{\frac{x-1}{\sqrt{\varepsilon}}} e^{-\frac{1}{24}s^2}ds - 1\right], & x>1 \end{cases}}$$

Lepo'4

3) $\varepsilon y'' + (\sqrt{\varepsilon}\, e^x + \tan x)\, y' + (\sqrt{\varepsilon}\, e^x - \tan x)\, y = 0$,

$-1 < x < 1, \quad y(-1) = y(1) = 1$

Coefficient of y':



Since this is less than zero at $x=-1$ and greater than zero at $x=1$, there are no BL at the endpts. Since it equals zero near $x=0$, there may be a problem there.

Outer solution

Assume $y(x) \sim y_0(x) + \varepsilon y_1(x)$

$O(1)$: $\tan x\, y_0' - \tan x\, y_0 = 0$

$y_0' - y_0 = 0$ for $\tan x \neq 0 \to x \neq 0$

$y_0 = c_1 e^x$

$y_0(-1) = c_1 e^{-1} = 1 \to c_1 = e$ (left)

$y_0(1) = c_1 e^{1} = 1 \to c_1 = e^{-1}$ (right)

$y_0(x) = \begin{cases} e^{x+1} & \text{left} \\ e^{x-1} & \text{right} \end{cases}$

Inner layer near $x=0$:

Let $Y(\xi) = y(x)$, $\xi = \frac{x}{\delta}$, $\delta = o(1)$

Plug in:

$\varepsilon/\delta^2\, Y'' + (\sqrt{\varepsilon}\, e^{\xi\delta} + \tan \xi\delta)\frac{1}{\delta} Y' + (\sqrt{\varepsilon}\, e^{\xi\delta} - \tan(\xi\delta))Y = 0$

$\varepsilon Y'' + (\sqrt{\varepsilon}(1 + \xi\delta + \frac{1}{2}\xi^2\delta^2) + (\xi\delta + \frac{1}{3}\xi^3\delta^3))\delta Y'$
$+ (\sqrt{\varepsilon}(1 + \xi\delta + \frac{1}{2}\xi^2\delta^2) - (\xi\delta + \frac{1}{3}\xi^3\delta^3))\delta^2 Y = 0$

$\varepsilon = \sqrt{\varepsilon}\,\delta \to \delta = \sqrt{\varepsilon}$

$\varepsilon Y'' + (\varepsilon + \xi\varepsilon^{3/2} + \xi\varepsilon)Y' + (\varepsilon^{3/2} + \xi\varepsilon^2 - \xi\varepsilon^{3/2})Y = 0$

Assume $Y(\xi) \sim Y_0(\xi) + \varepsilon Y_1(\xi)$

$O(\varepsilon)$: $Y_0'' + (1+\xi)Y_0' = 0$

$\int \frac{dY_0'}{Y_0'} = -\int (1+\xi)\, d\xi$

$Y_0' = c_1 e^{-(\xi + \frac{1}{2}\xi^2)}$

$Y_0 = c_1 \int_0^{\xi} e^{-(s+\frac{1}{2}s^2)} ds + c_2$

$$Y_0 = c_2 + c_1 \int_0^{\frac{x}{\varepsilon}} e^{-(\frac{1}{2}s^2+s)}\,ds$$

complete the square: $\frac{1}{2}s^2 + s = \frac{1}{2}(s^2+2s+1-1)$

$$= \frac{1}{2}((s+1)^2-1) = \left(\frac{s+1}{\sqrt{2}}\right)^2 - \frac{1}{2}$$

$$Y_0 = c_2 + c_1 e^{-\frac{1}{2}} \int_0^{\frac{x}{\varepsilon}} e^{-\left(\frac{s+1}{\sqrt{2}}\right)^2} ds \qquad \text{let } \frac{s+1}{\sqrt{2}} = t \to \frac{1}{\sqrt{2}}ds = dt$$

$$Y_0 = c_2 + c_3 \int_0^{\frac{x}{\varepsilon}} e^{-t}\,dt$$

?

Matching:

$Y_0(-\infty) = y_0\text{left}(0)$:

$$c_2 - c_3 \tfrac{1}{2}\sqrt{\pi} = e$$

$Y_0(\infty) = y_0\text{right}(0)$:

$$c_2 + c_3 \tfrac{1}{2}\sqrt{\pi} = e^{-1}$$

$$\to 2c_2 = e + e^{-1}$$

$$c_2 = (e+e^{-1})\tfrac{1}{2}$$

$$\sqrt{\pi}\, c_3 = e^{-1} - e$$

$$c_3 = \frac{e^{-1}-e}{\sqrt{\pi}}$$

$$y(x) = \begin{cases} e^{x+1} + \frac{1}{2}(e+e^{-1}) + \frac{1}{\sqrt{\pi}}(e^{-1}-e)\int_0^{x/\sqrt{\varepsilon}} e^{-t}dt \quad - e \\ e^{x-1} + \frac{1}{2}(e+e^{-1}) + \frac{1}{\sqrt{\pi}}(e^{-1}-e)\int_0^{x/\sqrt{\varepsilon}} e^{-t}dt \quad - e^{-1} \end{cases}$$

$$= \begin{cases} ee^x - \frac{1}{2}e + \frac{1}{2}e^{-1} + \frac{1}{\sqrt{\pi}}(e^{-1}-e)\int_0^{x/\sqrt{\varepsilon}} e^{-t}dt \\ e^{-1}e^x - \frac{1}{2}e^{-1} + \frac{1}{2}e + \frac{1}{\sqrt{\pi}}(e^{-1}-e)\int_0^{x/\sqrt{\varepsilon}} e^{-t}dt \end{cases}$$

$$\boxed{y(x) = \begin{cases} e(e^x - \frac{1}{2}) + \frac{1}{2}e^{-1} + \frac{1}{2}(e^{-1}-e)\,\mathrm{erf}\left(\frac{x}{\sqrt{\varepsilon}}\right), & x<0 \\ e^{-1}(e^x-\frac{1}{2}) + \frac{1}{2}e + \frac{1}{2}(e^{-1}-e)\,\mathrm{erf}\left(\frac{x}{\sqrt{\varepsilon}}\right), & x>0 \end{cases}}$$

lepos

4) $\varepsilon y'' + xy' - (1 + x\cos x)y = 0, \quad -1 < x < \pi, \quad y'(-1) = 0, \quad y(\pi) = \pi$

Since the coefficient of y' is less than zero at $x=-1$ and greater than zero at $x=\pi$, there are no BL at the endpoints. Since it is zero at $x=0$, there may be a problem there.

Outer solution:

Assume $y(x) \sim y_0(x) + \varepsilon y_1(x)$

$O(1)$: $xy_0' - (1 + x\cos x)y_0 = 0$

$$\int \frac{dy_0}{y_0} = \int \frac{1 + x\cos x}{x}\,dx = \int \frac{dx}{x} + \int \cos x\,dx \quad \text{if } x \neq 0$$

$$\ln y_0 = \ln x + \sin x + c$$

$$y_0 = e^{\ln x} e^{\sin x} e^{c}$$

$$y_0 = c_1 x e^{\sin x}$$

$$y_0(\pi) = c_1 \pi e^{\sin\pi} = c_1\pi = \pi \rightarrow c_1 = 1 \quad \text{(right)}$$

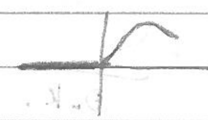
$$y_0' = c_1\left(e^{\sin x} + x\cos x\, e^{\sin x}\right)$$

$$y_0'(-1) = c_1\left(e^{\sin(-1)} - \cos(-1)e^{\sin(-1)}\right) = 0$$

$$c_1 e^{\sin(-1)}\left(1 - \cos(-1)\right) = 0$$

$$\rightarrow c_1 = 0 \quad \text{(left)}$$

$$y_0(x) = \begin{cases} 0 & x < 0 \\ xe^{\sin x} & x > 0 \end{cases}$$



Inner layer near $x=0$: ("corner layer")

let $\varepsilon^{\alpha} Y(\xi) = y(x), \quad \xi = x/\delta, \quad \delta = o(1)$

Plug in:

$$\varepsilon/\delta^2\, Y'' + \xi\delta \tfrac{1}{\delta} Y' - (1 + \xi\delta\cos\xi\delta)Y = 0$$

$$\varepsilon Y'' + \xi\delta^2 Y' - (1 + \xi\delta(1 + \tfrac{1}{2}\xi^2\delta^2))\delta^2 Y = 0$$

$$\varepsilon Y'' + \xi\delta^2 Y' - (\delta^2 + \xi\delta^3 + \tfrac{1}{2}\xi^3\delta^5) = 0$$

$$\rightarrow \varepsilon = \delta^2, \quad \delta = \sqrt{\varepsilon}$$

$$\varepsilon Y'' + \xi\varepsilon Y' - (\varepsilon + \xi\varepsilon^{3/2} + \tfrac{1}{2}\xi^3\varepsilon^{5/2})Y = 0$$

Assume $Y(\xi) \sim Y_0(\xi) + \varepsilon Y_1(\xi)$

$O(\varepsilon)$: $Y_0'' + \xi Y_0' - Y_0 = 0$

$y_0 = c_1 \xi$ is a solution.

Reduction of order: Assume ξu is a sol'n, $u = u(\xi)$

$[\xi u]' = u + \xi u'$

$[\xi u]'' = 2u' + \xi u''$

$\rightarrow 2u' + xu'' + u + \xi^2 u' - u = 0$

$\xi u'' + (2+\xi^2) u' = 0$

$\int \frac{du'}{u'} = \int \frac{-(2+\xi^2)}{\xi} \, d\xi$

$\ln u' = -2 \ln \xi - \frac{1}{2}\xi^2 + c$

$u' = c_2 \frac{1}{\xi^2} e^{-\frac{1}{2}\xi^2}$

$u = c_2 \int \frac{1}{\xi^2} e^{-\frac{1}{2}\xi^2} d\xi + c_3$

$= c_2 [\frac{1}{\xi} e^{-\frac{1}{2}\xi^2} - \int_0^\xi e^{-\frac{1}{2}s^2} ds$

$y_0 = c_1 \xi + c_2 [-e^{-\frac{1}{2}\xi^2} - \xi \int_0^\xi e^{-\frac{1}{2}s^2} ds$

Matching:

left: $(x<0)$

As $\xi \to -\infty$, $y_0(\xi) \to c_1 \xi - c_2 \frac{1}{2}\sqrt{2\pi}\, \xi$

$\varepsilon^{-\gamma} y_0(\frac{x}{\sqrt{\varepsilon}}) \sim c_1 x \varepsilon^{-\gamma-\frac{1}{2}} - c_2 \frac{1}{2}\sqrt{2\pi}\, x \varepsilon^{-\gamma-\frac{1}{2}}$

$\underset{\text{o.k.}}{\underline{\text{let } \gamma = -\tfrac{1}{2}}}$: $\sim c_1 x - \frac{1}{2}\sqrt{2\pi}\, c_2 x$

$= x(c_1 - c_2 \frac{1}{2}\sqrt{2\pi})$

y_0 left $= 0$

So $c_1 - \frac{1}{2}\sqrt{2\pi}\, c_2 = 0$

right: $(x>0)$

As $\xi \to \infty$, $y_0(\xi) \to c_1 \xi + c_2 \frac{1}{2}\sqrt{2\pi}\, \xi$

$\varepsilon^{\gamma} y_0(\frac{x}{\sqrt{\varepsilon}}) \sim c_1 x \varepsilon^{-\gamma-\frac{1}{2}} + c_2 \frac{1}{2}\sqrt{2\pi}\, x \varepsilon^{-\gamma-\frac{1}{2}}$

let $\gamma = -\frac{1}{2}$: $\sim c_1 x + \frac{1}{2}\sqrt{2\pi}\, c_2 x$

$= x(c_1 + \frac{1}{2}\sqrt{2\pi}\, c_2$

$y_{right} = x e^{\sin x}$

$\sim x(1 + \sin x + \frac{1}{2}\sin^2 x) \sim x(1 + x + \frac{1}{2}x^2) \sim x$

So $c_1 + \frac{1}{2}\sqrt{2\pi}\, c_2 = 1$

Lep06

$$c_1 - \tfrac{1}{2}\sqrt{2\pi}\,c_2 = 0$$
$$c_1 + \tfrac{1}{2}\sqrt{2\pi}\,c_2 = 1$$
$$2c_1 = 1 \rightarrow c_1 = \tfrac{1}{2}$$
$$\sqrt{2\pi}\,c_2 = 1$$
$$c_2 = \tfrac{1}{\sqrt{2\pi}}$$

$$Y_0 = \tfrac{1}{2}\xi + \tfrac{1}{\sqrt{2\pi}}\left[-e^{-\frac{1}{2}\xi^2} - \xi\int_0^{\xi} e^{-\frac{1}{2}s^2}ds\right] \qquad \varepsilon^{\gamma}\,?$$

$$\boxed{y(x) = \begin{cases} \frac{1}{2}\frac{x}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{1}{2}x^2/\varepsilon} - \frac{x}{\sqrt{\varepsilon}}\int_0^{x/\sqrt{\varepsilon}} e^{-\frac{1}{2}s^2}ds\right], & x<0 \\ xe^{\sin x} + \frac{1}{2}\frac{x}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{1}{2}x^2/\varepsilon} - \frac{x}{\sqrt{\varepsilon}}\int_0^{x/\sqrt{\varepsilon}} e^{-\frac{1}{2}s^2}ds\right] - 1, & x>0 \end{cases}}$$

common part is not 1, it is x.

	1	2	3	4	
	10	3	9	8	30
max	10	10	10	10	

ES_APPM 420-2 "Asymptotic and Perturbation Methods"

Homework 1 (DUE TUESDAY, 1/27/08)

Problem 1. Find the leading term of the expansion of the solution of the following problem in small ε that is valid for large time, $t = O(1/\varepsilon^2)$,

$$\phi'' + \varepsilon(1 + \cos\phi)\phi' + \sin\phi = \varepsilon\alpha, \quad \phi(0) = \phi'(0) = 0,$$

where α is a real parameter (an $O(1)$ quantity).

Hints:

1. Since $\phi(t) \equiv 0$ for $\varepsilon = 0$, assume $|\phi(t)| \ll 1$, specifically, assume that $\phi(t) = O(\varepsilon)$.
2. Introduce two times $\tau = \omega t$ and $t_1 = \varepsilon t$ to solve the problem (as we did in class).

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$$\phi'' + \varepsilon(1+\cos\phi)\phi' + \sin\phi = \varepsilon\alpha, \quad \phi(0)=\phi'(0)=0$$

Let $\phi(t) = \varepsilon y(\tau, t_1)$, where $y = O(1)$

$\tau = \omega t, \quad t_1 = \varepsilon t$

$$\rightarrow \omega^2\varepsilon\frac{\partial^2 y}{\partial\tau^2} + 2\omega\varepsilon^2\frac{\partial^2 y}{\partial\tau\partial t_1} + \varepsilon^3\frac{\partial^2 y}{\partial t_1^2}$$

$$+ \varepsilon(1+\cos(\varepsilon y))\left(\omega\varepsilon\frac{\partial y}{\partial\tau} + \varepsilon^2\frac{\partial y}{\partial t_1}\right) + \sin\varepsilon y = \varepsilon\alpha$$

$$\rightarrow \omega^2\varepsilon\frac{\partial^2 y}{\partial\tau^2} + 2\omega\varepsilon^2\frac{\partial^2 y}{\partial\tau\partial t_1} + \varepsilon^3\frac{\partial^2 y}{\partial t_1^2}$$

$$+ \varepsilon\left(2 - \frac{1}{2}\varepsilon^2 y^2\right)\left(\omega\varepsilon\frac{\partial y}{\partial\tau} + \varepsilon^2\frac{\partial y}{\partial t_1}\right) + \varepsilon y - \frac{\varepsilon^3 y^3}{6} = \varepsilon\alpha$$

$$\rightarrow \Big[\omega^2\frac{\partial^2 y}{\partial\tau^2} + 2\omega\varepsilon\frac{\partial^2 y}{\partial\tau\partial t_1} + \varepsilon^2\frac{\partial^2 y}{\partial t_1^2}$$

$$+ 2\omega\varepsilon\frac{\partial y}{\partial\tau} + 2\varepsilon^2\frac{\partial y}{\partial t_1} - \frac{\omega\varepsilon^3}{2}y^2\frac{\partial y}{\partial\tau} + y - \frac{\varepsilon^2 y^3}{6} = \alpha + O(\varepsilon^4)$$

let $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2, \quad \omega \sim \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2$

$O(1)$: $\omega_0^2\frac{\partial^2 y_0}{\partial\tau^2} + y_0 = \alpha, \quad y_0(0,0) = \frac{\partial y_0}{\partial\tau}(0,0) = 0$

$\omega_0^2 r^2 + 1 = 0$

$r^2 = -\frac{1}{\omega_0^2} = i\omega_0$

$y_0 = A_0(t_1)\cos\omega_0\tau + B_0(t_1)\sin\omega_0\tau + \alpha$

$y_0(0,0) = A_0(0) + \alpha = 0$

$A_0(0) = -\alpha$

$\frac{\partial y_0}{\partial\tau}(0,0) = B_0(0) = 0$

$O(\varepsilon)$: $\omega_0^2\frac{\partial y_1}{\partial\tau^2} + y_1 = -\left[2\omega_1\frac{\partial^2 y_0}{\partial\tau^2} + 2\omega_0\frac{\partial^2 y_0}{\partial\tau\partial t_1} + 2\omega_0\frac{\partial y_0}{\partial\tau}\right]$

$= 2\omega_1\omega_0^2(A_0(t_1)\cos\omega_0\tau + B_0(t_1)\sin\omega_0\tau)$

$-2\omega_0^2(B_0'(t_1)\cos\omega_0\tau - A_0'(t_1)\sin\omega_0\tau)$

$-2\omega_0^2(B_0(t_1)\cos\omega_0\tau - A_0(t_1)\sin\omega_0\tau)$

$$\rightarrow \begin{cases} 2\omega_1 \omega_0^2 A_0(t_1) - 2\omega_0^2 B_0'(t_1) - 2\omega_0^2 B_0(t_1) = 0 \\ 2\omega_1 \omega_0^2 B_0(t_1) + 2\omega_0^2 A_0'(t_1) + 2\omega_0^2 A_0(t_1) = 0 \end{cases}$$

$$\rightarrow \begin{cases} 2\omega_0^2 A_0'(t_1) = -2\omega_0^2 A_0(t_1) - 2\omega_1\omega_0^2 B_0(t_1) \\ 2\omega_0^2 B_0'(t_1) = 2\omega_1\omega_0^2 A_0(t_1) - 2\omega_0^2 B_0(t_1) \end{cases}$$

$$\rightarrow \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}' = \begin{pmatrix} -1 & -\omega_1 \\ \omega_1 & -1 \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

$$\lambda^2 + 2\lambda + 1 + \omega^2 = 0$$

$$\lambda = -1 \pm i\omega$$

$$\lambda = -1 - i\omega: \begin{pmatrix} i\omega_1 & -\omega_1 \\ \omega_1 & i\omega_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$ai - b = 0 \rightarrow b = ai$$

$$a + bi = 0$$

$$\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = e^{-t_1}(\cos\omega_1 t_1 - i\sin\omega_1 t_1)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

Real: $e^{-t_1}\begin{pmatrix} \cos\omega_1 t_1 \\ \sin\omega_1 t_1 \end{pmatrix}$

Imag: $e^{-t_1}\begin{pmatrix} -\sin\omega_1 t_1 \\ \cos\omega_1 t_1 \end{pmatrix}$

$$A_0 = e^{-t_1}(c_1 \cos\omega_1 t_1 - c_2 \sin\omega_1 t_1)$$

$$B_0 = e^{-t_1}(c_1 \sin\omega_1 t_1 + c_2 \cos\omega_1 t_1)$$

$$A_0(0) = c_1 = -\alpha$$

$$B_0(0) = c_2 = 0$$

$$A_0 = -\alpha e^{-t_1} \cos\omega_1 t_1$$

$$B_0 = -\alpha e^{-t_1} \sin\omega_1 t_1$$

$$\rightarrow y_0 = -\alpha e^{-t_1}\left[\cos\omega_1 t_1 \cos\omega_0 \tau + \sin\omega_1 t_1 \sin\omega_0 \tau\right] + \alpha$$

$$= -\alpha e^{-t_1}\left[\cos(\omega_1 t_1 - \omega_0 \tau)\right] + \alpha$$

$$\omega_0^2 \frac{\partial y_1}{\partial \tau^2} + y_1 = 0 \quad , \; y_1(0,0) = 0, \; \left[\omega_0 \frac{\partial y_1}{\partial \tau} + \omega_1 \frac{\partial y_0}{\partial \tau} + \frac{\partial y_0}{\partial t_1}\right]_{(0,0)} = 0$$

$$y_1 = A_1(t_1)\cos\omega_0\tau + B_1(t_1)\sin\omega_0\tau$$

$$y_1(0,0) = A_1(0) = 0$$

$$[\omega_0(\omega_0 A_1(t_1)(-\sin\omega_0\tau) + \omega_0 B_1(t_1)\cos\omega_0\tau$$
$$+\omega_1(-\alpha e^{-t}\omega_0\sin(\omega_1 t_1 - \omega_0\tau))$$
$$+\alpha e^{-t_1}\cos(\omega_1 t_1 - \omega_0\tau) + \alpha e^{-t_1}\omega_1\sin(\omega_1 t_1 - \omega_0\tau)]_{(0,0)}$$
$$= \omega_0^2 B_1(0) + \alpha = 0$$
$$B_1(0) = -\alpha/\omega_0^2$$

$$O(\varepsilon^2): \omega_0^2 \frac{\partial y_2}{\partial \tau^2} + y_2 = -\left[(\omega_1^2 + 2\omega_0\omega_2)\frac{\partial^2 y_0}{\partial \tau^2} + 2\omega_0\omega_1 \frac{\partial^2 y_1}{\partial \tau^2}\right.$$
$$+ 2\omega_1 \frac{\partial^2 y_0}{\partial \tau \partial t_1} + 2\omega_0 \frac{\partial^2 y_1}{\partial \tau \partial t_1} + \frac{\partial^2 y_0}{\partial t_1^2}$$
$$\left. + 2\omega_1 \frac{\partial y_0}{\partial \tau} + 2\omega_0 \frac{\partial y_1}{\partial \tau} + 2\frac{\partial y_0}{\partial t_1} - \frac{y_0^3}{6}\right]$$

via Mathematica:

$$= \tfrac{1}{6}\alpha^3 + 2\omega_0^3\omega_1 A_1\cos(\tau\omega_0) - 2\omega_0^2 B_1\cos(\tau\omega_0)$$
$$-e^{-t_1}\alpha\cos(\tau\omega_0)\cos(t_1\omega_1) - \tfrac{1}{2}e^{-t_1}\alpha^3\cos(\tau\omega_0)\cos(t_1\omega_1)$$
$$-e^{-t_1}\alpha\omega_1^2\cos(\tau\omega_0)\cos(t_1\omega_1) + 2e^{-t_1}\alpha\omega_0\omega_1^2\cos(\tau\omega_0)\cos(t_1\omega_1)$$
$$-e^{-t_1}\alpha\omega_0^2\omega_1^2\cos(\tau\omega_0)\cos(t_1\omega_1) - 2e^{-t_1}\alpha\omega_0^3\omega_2\cos(\tau\omega_0)\cos(t_1\omega_1)$$
$$+\tfrac{1}{2}e^{-2t_1}\alpha^3\cos(t\omega_0)^2\cos(t_1\omega_1)^2 - \tfrac{1}{6}e^{-3t_1}\alpha^3\cos(t\omega_0)^3\cos(t_1\omega_1)^3$$
$$+2\omega_0^2 A_1\sin(\tau\omega_0) + 2\omega_0^3\omega_1 B_1\sin(\tau\omega_0)$$
$$-e^{-t_1}\alpha\sin(\tau\omega_0)\sin(t_1\omega_1) - \tfrac{1}{2}e^{-t_1}\alpha^3\sin(\tau\omega_0)\sin(t_1\omega_1)$$
$$-e^{-t_1}\alpha\omega_1^2\sin(\tau\omega_0)\sin(t_1\omega_1) + 2e^{-t_1}\alpha\omega_0\omega_1^2\sin(\tau\omega_0)\sin(t_1\omega_1)$$
$$-e^{-t_1}\alpha\omega_0^2\omega_1^2\sin(t\omega_0)\sin(t_1\omega_1) - 2e^{-t_1}\alpha\omega_0^3\omega_2\sin(\tau\omega_0)\sin(t_1\omega_1)$$
$$+e^{-2t_1}\alpha^3\sin(t\omega_0)^2\sin(t_1\omega_1)^2 - \tfrac{1}{2}e^{-3t_1}\alpha^3\cos(\tau\omega_0)\cos(t_1\omega_1)$$
$$\cdot\sin(t\omega_0)^2\sin(t_1\omega_1)^2 - \tfrac{1}{6}e^{-3t_1}\alpha^3\sin(\tau\omega_0)^3\sin(t_1\omega_1)^3$$
$$+2\omega_0^2\sin(\tau\omega_0)A_1' - 2\omega_0^2\cos(\tau\omega_0)B_1'$$

In this mess, I would find the terms that would produce secular terms.
Anything with a factor of $\cos \omega_0 \tau$ or $\sin \omega_0 \tau$ needs to disappear, so group them together and set the coefficient equal to zero.
This will give A_1 + B_1, and hopefully also tell us something about ω.

So far all I have is:

$$y_0 = -\alpha e^{-t_1}[\cos(\omega_1 t_1 - \omega_0 \tau)] + \alpha$$

$\omega_2 = ?$ 5/8

ES_APPM 420-2 "Asymptotic and Perturbation Methods"

Homework 2 (DUE TUESDAY, 2/3/09)

Problem 1. Using multiple scales find the leading order term of the expansion in small ε of the solution of

$$y'' + \varepsilon(y')^3 + y = 0, \quad y(0) = 0, \ \ y'(0) = 1$$

that is valid for large time, $t = O(1/\varepsilon)$,

$$(a+b)^3 = (a+b)(a^2+2ab+b^2)$$
$$= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b+c)^3 = (a+b+c)(a+b+c)(a+b+c)$$
$$= (a+b+c)(a^2+ab+ac+ab+b^2+bc+ac+bc+c^2)$$
$$= (a+b+c)(a^2+2ab+2bc+2ac+b^2+c^2)$$
$$= \cancel{a^3} + \cancel{2a^2b} + \cancel{2abc} + \cancel{2a^2c} + \cancel{ab^2} + \cancel{ac^2}$$
$$+ \cancel{a^2b} + \cancel{2ab^2} + \cancel{2b^2c} + \cancel{2abc} + \cancel{b^3} + \cancel{bc^2}$$
$$+ \cancel{a^2c} + \cancel{2abc} + \cancel{2bc^2} + \cancel{2ac^2} + \cancel{b^2c} + \cancel{c^3}$$
$$= a^3 + b^3 + c^3 + 6abc + 3a^2b + 3a^2c + 3ab^2 + 3ac^2$$
$$+ 3b^2c + 3bc^2$$

$$\sin^3 t = \cancel{\sin t (} \frac{3\sin t - \sin 3t}{4} \qquad \cos^3 t = \frac{3\cos t + \cos 3t}{4}$$

$$\sin t \cos^2 t = \tfrac{1}{2}\sin t(1+\cos 2t)$$
$$= \tfrac{1}{2}\sin t + \tfrac{1}{2}\sin t \cos 2t$$
$$= \tfrac{1}{2}\sin t + \tfrac{1}{4}\sin 3t \cancel{+} \tfrac{1}{4}\sin t$$
$$= \tfrac{1}{4}\sin t + \tfrac{1}{4}\sin 3t$$

$$\sin^2 t \cos t = \tfrac{1}{2}\cos t(1-\cos 2t)$$
$$= \tfrac{1}{2}\cos t - \tfrac{1}{2}\cos t \cos 2t$$
$$= \tfrac{1}{2}\cos t - \tfrac{1}{4}\cos 3t - \tfrac{1}{4}\cos t$$
$$= \cancel{\tfrac{1}{4}\cos 3t}$$
$$= \tfrac{1}{4}\cos t - \tfrac{1}{4}\cos 3t$$

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$$y'' + \varepsilon (y')^3 + y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Let $\tau = \omega t, \quad t_1 = \varepsilon t$

$$x(\tau, t_1) = y(t)$$

$$\omega^2 \frac{\partial^2 x}{\partial \tau^2} + 2\varepsilon\omega \frac{\partial^2 x}{\partial\tau \partial t_1} + \varepsilon^2 \frac{\partial^2 x}{\partial t_1^2} + \varepsilon\left[\omega\frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial t_1}\right]^3 + x = 0$$

$$x(0,0) = 0, \quad \omega \frac{\partial x}{\partial \tau}(0,0) + \varepsilon \frac{\partial x}{\partial t_1}(0,0) = 1$$

we know $\omega_0 = 1$ since sol'n with $\varepsilon = 0$ is 2π-periodic

$$\rightarrow \omega \sim 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2, \quad x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2$$

$$\rightarrow \omega^2 \frac{\partial^2 x}{\partial \tau^2} + 2\varepsilon\omega\frac{\partial^2 x}{\partial \tau \partial t_1} + \varepsilon^2 \frac{\partial^2 x}{\partial t_1^2} + x + \varepsilon\omega^3\left(\frac{\partial x}{\partial \tau}\right)^3 + 3\varepsilon^2\omega^2\left(\frac{\partial x}{\partial \tau}\right)^2 \frac{\partial x}{\partial t_1} + 3\varepsilon^3\omega\left(\frac{\partial x}{\partial \tau}\right)\left(\frac{\partial x}{\partial t_1}\right)^2 + \varepsilon^4\left(\frac{\partial x}{\partial t_1}\right)^3 = 0$$

$O(1)$: $\dfrac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0, \quad x_0(0,0) = 0, \quad \dfrac{\partial x_0}{\partial \tau}(0,0) = 1$

$$x_0 = A_0(t_1)\cos\tau + B_0(t_1)\sin\tau$$

$$x_0(0,0) = A_0(0) = 0$$

$$\frac{\partial x_0}{\partial \tau}(0,0) = B_0(0) = 1$$

$O(\varepsilon)$: $\dfrac{\partial^2 x_1}{\partial \tau^2} + x_1 = f_1, \quad x_1(0,0) = 0, \quad \left.\dfrac{\partial x_1}{\partial \tau} + \omega_1 \dfrac{\partial x_0}{\partial \tau} + \varepsilon \dfrac{\partial x_1}{\partial t_1}\right|_{(0,0)} = 0$

$$-f_1 = 2\omega_1 \frac{\partial^2 x_0}{\partial \tau^2} + 2\frac{\partial^2 x_0}{\partial \tau \partial t_1} + \left(\frac{\partial x_0}{\partial \tau}\right)^3$$

$$= -2\omega_1(A_0(t_1)\cos\tau + B_0(t_1)\sin\tau)$$
$$+ 2(-A_0'(t_1)\sin\tau + B_0'(t_1)\cos\tau)$$
$$- [A_0(t_1)]^3 \sin^3 t + [B_0(t_1)]^3 \cos^3\tau$$
$$+ 3[A_0(t_1)]^2 \sin^2 t\, B_0(t_1)\cos\tau$$
$$- 3A_0(t_1)\sin\tau\,[B_0(t_1)]^2\cos^2\tau$$

$$= \cos\tau(-2\omega_1 A_0 + 2B_0') + \sin\tau(-2\omega_1 B_0 - 2A_0')$$
$$- A_0^3 \tfrac{3}{4}(\sin\tau - \sin 3\tau) + B_0^3 \tfrac{3}{4}(\cos\tau + \cos 3\tau)$$
$$+ 3A_0^2 B_0 \tfrac{1}{4}(\cos\tau + \cos 3\tau) - 3A_0 B_0^2 \tfrac{1}{4}(\sin\tau + \sin 3\tau).$$

Secular producing terms:

$\cos\tau$: $-2\omega_1 A_0 + 2B_0' + \frac{3}{4}B_0^3 + \frac{3}{4}A_0^2 B_0 = 0$

$\sin\tau$: $-2\omega_1 B_0 - 2A_0' - \frac{3}{4}A_0^3 - \frac{3}{4}A_0 B_0^2 = 0$

But, because time of $O(\frac{1}{\varepsilon})$ is taken care of with $t_1 = \varepsilon t$, we know $\omega_1 = 0$

$$\rightarrow \begin{cases} 2B_0' + \frac{3}{4}B_0^3 + \frac{3}{4}A_0^2 B_0 = 0 \\ 2A_0' + \frac{3}{4}A_0^3 + \frac{3}{4}A_0 B_0^2 = 0 \end{cases}$$

$$\rightarrow \begin{cases} 2B_0' + \frac{3}{4}B_0(B_0^2 + A_0^2) = 0 \\ 2A_0' + \frac{3}{4}A_0(A_0^2 + B_0^2) = 0 \end{cases}$$

$$\rightarrow \begin{cases} (B_0^2)' + \frac{3}{4}B_0^2(B_0^2 + A_0^2) = 0 \\ (A_0^2)' + \frac{3}{4}A_0^2(B_0^2 + A_0^2) = 0 \end{cases}$$

$$\rightarrow (A_0^2 + B_0^2)' + (B_0^2 + A_0^2)(A_0^2 + B_0^2)\tfrac{3}{4} = 0$$

let $z = A_0^2 + B_0^2$

$$z' + \tfrac{3}{4}z^2 = 0, \quad z(0) = A_0(0)^2 + B_0(0)^2 = 1$$

$$\int \frac{dz}{z^2} = \int -\frac{3}{4}\,dt_1$$

$$-\frac{1}{z} = -\tfrac{3}{4}t_1 + C$$

$$z = \frac{1}{\frac{3}{4}t_1 + C_0}, \quad z(0) = \frac{1}{C_0} = 1 \rightarrow C_0 = 1$$

$$2B_0' + \tfrac{3}{4}B_0 z = 0$$

$$\int \frac{2\,dB_0}{B_0} = \frac{-\frac{3}{4}\,dt_1}{\frac{3}{4}t_1 + 1}$$

$$2\ln B_0 = -\ln(\tfrac{3}{4}t_1 + 1)$$

$$B_0^2 = \frac{1}{\frac{3}{4}t_1 + 1}$$

$$B_0 = \frac{1}{\sqrt{\frac{3}{4}t_1 + 1}}$$

$\rightarrow A_0(t_1) \equiv 0$ (since $B_0^2 = z$

$$x_0 = \frac{\sin\tau}{\frac{3}{4}t_1 + 1}$$

$$y(t) = x(\tau, t_1) \sim \frac{\sin t}{\sqrt{\frac{3}{4}\varepsilon t + 1}}$$

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ES_APPM 420-2 "Asymptotic and Perturbation Methods"

Homework 3 (DUE TUESDAY, 2/10/09)

Problem 1. Consider the problem

$$\frac{d}{dt}\left(D(\varepsilon t)\frac{dy}{dt}\right) + y = 0, \quad y(0) = \alpha, \ y'(0) = \beta.$$

The coefficient D is a smooth positive function with $D' > 0$. Find a one-term expansion of the solution that is valid for large time.

Problem 2. Consider the boundary value problem

$$\varepsilon y'' + (1+x)y' + y = 0, \quad y(0) = 1, \ y(1) = 1.$$

Determine the leading order uniform solution of the problem using the **method of multiple scales**.

1) $\frac{d}{dt}\left(D(\varepsilon t)\frac{dy}{dt}\right) + y = 0$, $y(0)=\alpha$, $y'(0)=\beta$

$D(\varepsilon t)\frac{d^2y}{dt^2} + \varepsilon D'(\varepsilon t)\frac{dy}{dt} + y = 0$

Try $t_0 = t$, $t_1 = \varepsilon t$ let $x(t_0, t_1) = y(t)$

$\rightarrow D(t_1)\left[\frac{\partial^2 x}{\partial t_0^2} + \varepsilon^2\frac{\partial^2 x}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 x}{\partial t_0 \partial t_1}\right] + \varepsilon D'(t_1)\left[\frac{\partial x}{\partial t_0} + \varepsilon\frac{\partial x}{\partial t_1}\right] + x = 0$

$O(1)$: $D(t_1)\frac{\partial^2 x}{\partial t_0^2} + \cancel{D'(t_1)\frac{\partial x}{\partial t_0}} + x = 0$ $r = \pm i\sqrt{1/D}$

$Dr^2 + \cancel{D'r} + 1 = 0$ $r = \cancel{\frac{1}{2D}\left(-D' \pm \sqrt{(D')^2 - 4D}\right)}$

$x = A_0(t_1)e^{r_1(t_1)t_0} + B_0(t_1)e^{r(t_1)t_0}$

not good → period should depend on fast time!

instead: $t_0 = f(t, \varepsilon)$, $t_1 = \varepsilon t$

$\rightarrow \frac{dy}{dt} = f_t\frac{\partial x}{\partial t_0} + \varepsilon\frac{\partial x}{\partial t_1}$, $\frac{d^2y}{dt^2} = f_t^2\frac{\partial^2 x}{\partial t_0^2} + f_{tt}\frac{\partial x}{\partial t_0} + 2\varepsilon f_t\frac{\partial^2 x}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 x}{\partial t_1^2}$

Haleylepo
2-10-09
ESAM 420-2
Hwk #3

$$D\frac{d}{dt}\left(D(\varepsilon t)\frac{dy}{dt}\right)+y=0\ ,\quad y(0)=\alpha,\ y'(0)=\beta$$

let $t_0=f(t,\varepsilon),\ t_1=\varepsilon t,\ x(t_0,t_1)=y(t)$

$$D(\varepsilon t)\frac{d^2y}{dt^2}+D'(\varepsilon t)\frac{dy}{dt}+y=0$$

Plug in: $D(t_1)\left[f_t^2\frac{\partial^2 x}{\partial t_0^2}+f_{tt}\frac{\partial x}{\partial t_0}+2\varepsilon f_t\frac{\partial^2 x}{\partial t_0\partial t_1}+\varepsilon^2\frac{\partial^2 x}{\partial t_1^2}\right]$

$$+\varepsilon D'(t_1)\left[f_t\frac{\partial x}{\partial t_0}+\varepsilon\frac{\partial x}{\partial t_1}\right]+x=0$$

$$x(0,0)=\alpha,\ f_t\frac{\partial x}{\partial t_0}(0,0)+\varepsilon\frac{\partial x}{\partial t_1}(0,0)=\beta$$

let $D(\varepsilon t)f_t^2=1$

$$\rightarrow f_t=\frac{1}{\sqrt{D(\varepsilon t)}}$$

$$f(t,\varepsilon)=\int_0^t\frac{1}{\sqrt{D(\varepsilon s)}}ds=t_0$$

$$f_{tt}=\frac{-\varepsilon D'(\varepsilon t)}{2(D(\varepsilon t))^{3/2}}$$

$$\rightarrow D\left[\frac{1}{D}\frac{\partial^2 x}{\partial t_0^2}-\frac{D'\varepsilon}{2D^{3/2}}\frac{\partial x}{\partial t_0}+\frac{2\varepsilon}{D^{1/2}}\frac{\partial^2 x}{\partial t_0\partial t_1}+\varepsilon^2\frac{\partial^2 x}{\partial t_1^2}\right]$$

$$+\varepsilon D'\left[\frac{1}{D^{1/2}}\frac{\partial x}{\partial t_0}+\varepsilon\frac{\partial x}{\partial t_1}\right]+x=0$$

$$\rightarrow \frac{\partial^2 x}{\partial t_0^2}-\frac{\varepsilon D'}{2D^{1/2}}\frac{\partial x}{\partial t_0}+2\varepsilon D^{1/2}\frac{\partial^2 x}{\partial t_0\partial t_1}+\varepsilon^2 D\frac{\partial^2 x}{\partial t_1^2}$$

$$+\varepsilon\frac{D'}{D^{1/2}}\frac{\partial x}{\partial t_0}+\varepsilon^2 D'\frac{\partial x}{\partial t_1}+x=0$$

$$x(0,0)=\alpha,\ \frac{1}{D^{1/2}}\frac{\partial x}{\partial t_0}(0,0)+\varepsilon\frac{\partial x}{\partial t_1}(0,0)=\beta$$

$O(1): \frac{\partial^2 x_0}{\partial t_0^2} + x_0 = 0, \quad x_0(0,0) = \alpha, \quad \frac{\partial x_0}{\partial t_0}(0,0) = \beta\sqrt{D(0)}$

$x_0 = A_0(t_1)\cos t_0 + B_0(t_1)\sin t_0$

$x_0(0,0) = A_0(0) = \alpha$

$\frac{\partial x_0}{\partial t_0}(0,0) = B_0(0) = \beta\sqrt{D(0)}$

$O(\varepsilon): \frac{\partial^2 x}{\partial t_0^2} + x_1 = \frac{D'}{2D^{\frac{1}{2}}}\frac{\partial x_0}{\partial t_0} - \frac{2D}{D^{\frac{1}{2}}}\frac{\partial^2 x_0}{\partial t_0 \partial t_1} - \frac{D'}{D^{\frac{1}{2}}}\frac{\partial x_0}{\partial t_0}$

$= \frac{-D'}{2D^{\frac{1}{2}}}\frac{\partial x_0}{\partial t_0} - \frac{2D}{D^{\frac{1}{2}}}\frac{\partial^2 x_0}{\partial t_0 \partial t_1}$

$= \frac{-D'}{2D^{\frac{1}{2}}}\left(-A_0(t_1)\sin t_0 + B_0(t_1)\cos t_0\right)$

$- \frac{2D}{D^{\frac{1}{2}}}\left(-A_0'(t_1)\sin t_0 + B_0'(t_1)\cos t_0\right)$

$\sin t_0: \; 2DA_0' + \frac{1}{2}D'A_0 = 0$

$\cos t_0: \; 2DB_0' + \frac{1}{2}D'B_0 = 0$

$A_0' = -\frac{\frac{1}{2}D'A_0}{2D} = -\frac{1}{4}\frac{D'}{D}A_0$

$\int \frac{dA_0}{A_0} = \int -\frac{1}{4}\frac{D'}{D}\,dt$

$\ln A_0 = -\frac{1}{4}\ln D + c$

$\begin{cases} A_0 = c_1 D^{-\frac{1}{4}} \\ B_0 = c_2 D^{-\frac{1}{4}} \end{cases}$

$\begin{cases} A_0(0) = c_1 D(0)^{-1/4} = \alpha \to c_1 = \alpha D(0)^{\frac{1}{4}} \\ B_0(0) = c_2 D(0)^{-1/4} = \beta D(0)^{1/2} \to c_2 = \beta D(0)^{3/4} \end{cases}$

$\to \begin{cases} A_0(t_1) = \alpha D(0)^{1/4} D(t_1)^{-1/4} \\ B_0(t_1) = \beta D(0)^{3/4} D(t_1)^{-1/4} \end{cases}$

$\to x_0 = D(t_1)^{-1/4}\left[\alpha D(0)^{1/4}\cos t_0 + \beta D(0)^{3/4}\sin t_0\right]$

$y(t) = D(\varepsilon t)^{-1/4}\left[\alpha D(0)^{1/4}\cos\int_0^t \frac{ds}{\sqrt{D(\varepsilon s)}} + \beta D(0)^{3/4}\sin\int_0^t \frac{ds}{\sqrt{D(\varepsilon s)}}\right]$

✓

2) $\varepsilon y'' + (1+x)y' + y = 0$, $y(0)=1$, $y(1)=1$

first, solve with boundary layer method:
since $(1+x) > 0$ on $0<x<1$, there
may be a boundary layer near $x=0$
outer solution:

$$(1+x)y' + y = 0, \quad y(1)=1$$

$$\int \frac{dy}{y} = \int \frac{-dx}{1+x}$$

$$\ln y = \ln \frac{1}{1+x} + C$$

$$y = \frac{C_1}{1+x}$$

$$y(1) = C_1/2 = 1 \quad \rightarrow \quad C_1 = 2$$

$$y(x) = \frac{2}{1+x}$$

boundary layer near $x=0$:

let $Y(\xi) = y(x)$, $\xi = \frac{x}{\delta}$, $\delta \ll 1$

$$\varepsilon/\delta^2 Y'' + \tfrac{1}{\delta}(1+\xi\delta)Y' + Y = 0$$

$$\varepsilon Y'' + (\delta + \xi\delta^2)Y' + \delta^2 Y = 0$$

$$\rightarrow \varepsilon = \delta \quad \rightarrow \quad \xi = \frac{x}{\varepsilon}, \quad x = \varepsilon\xi$$

$$\rightarrow Y'' + (1+\varepsilon\xi)Y' + \varepsilon Y = 0$$

$$\int \frac{dY'}{Y'} = \int -d\xi$$

$$\ln Y' = -\xi + C$$

$$Y' = C_1 e^{-\xi}$$

$$Y = -C_1 e^{-\xi} + C_2$$

$$Y(0) = -C_1 + C_2 = 1 \quad \rightarrow \quad C_1 = C_2 - 1$$

$$Y(\xi) = (1-C_2)e^{-\xi} + C_2$$

matching:

$$Y(\xi \rightarrow \infty) = C_2 = y(0) = 2$$

$$\rightarrow Y = -e^{-\xi} + 2$$

$$y_c = \frac{2}{1+x} - e^{-\frac{x}{\varepsilon}}$$

Now, solve using method of multiple scales:

let $t_1 = \frac{x}{\varepsilon}$, $t_2 = x$

$$y(x) = z(t_1, t_2), \quad 0<t_1<\infty, \; 0<t_2<1$$

Plug in: $\varepsilon\left[\frac{1}{\varepsilon^2}\frac{\partial^2 z}{\partial t_1^2} + \frac{2}{\varepsilon}\frac{\partial^2 z}{\partial t_1 \partial t_2} + \frac{\partial^2 z}{\partial t_2^2}\right]$

$$+(1+t_2)\left[\frac{1}{\varepsilon}\frac{\partial z}{\partial t_1} + \frac{\partial z}{\partial t_2}\right] + z = 0$$

$$z(0,0)=1, \quad z(\infty,1)=1$$

$$\frac{\partial^2 z}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 z}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2 z}{\partial t_2^2} + (1+t_2)\left(\frac{\partial z}{\partial t_1} + \varepsilon\frac{\partial z}{\partial t_2}\right) + \varepsilon z = 0$$

$O(1)$: $\frac{\partial^2 z_0}{\partial t_1^2} + (1+t_2)\frac{\partial z_0}{\partial t_1} = 0$, $z_0(0,0)=1$, $z_0(\infty,1)=1$

$$z_0 = A_0(t_2)e^{-(1+t_2)t_1} + B_0(t_2)$$

$$z_0(0,0) = A_0(0) + B_0(0) = 1$$

$$z_0(\infty,1) = B_0(1) = 1$$

t_2 derivative results in t_1 as a factor. ~~Cannot solve like this!~~

$O(\varepsilon)$: $\frac{\partial^2 z_1}{\partial t_1^2} + (1+t_2)\frac{\partial z_1}{\partial t_1} = -2\frac{\partial^2 z_0}{\partial t_1 \partial t_2} - (1+t_2)\frac{\partial z_0}{\partial t_2} - z_0$

$$= 2A_0'(1+t_2)e^{-(1+t_2)t_1} - (1+t_2)(A_0'e^{-(1+t_2)t_1} + B_0')$$
$$- A_0 e^{-(1+t_2)t_1} - B_0$$

exp. terms: $2A_0'(1+t_2) - A_0'(1+t_2) - A_0 = 0$

const. terms: $-(1+t_2)B_0' - B_0 = 0$

$$\rightarrow \begin{cases} A_0'(1+t_2) - A_0 = 0 \\ B_0'(1+t_2) + B_0 = 0 \end{cases}$$

$$\int\frac{dA_0}{A_0} = \int\frac{dt_2}{1+t_2} \rightarrow \ln A_0 = \ln(1+t_2) + C$$

$$\rightarrow A_0 = 1 + t_2 + C_1$$

$$\int\frac{dB_0}{B_0} = \int\frac{-dt_2}{1+t_2} \rightarrow \ln B_0 = -\ln(1+t_2) + C$$

$$\rightarrow B_0 = \frac{C_2}{1+t_2}$$

Hnleuleno

$B_0(1) = C_2/2 = 1 \rightarrow C_2 = 2$

$A_0(0) + B_0(0) = 1 + C_1 + 2 = 1 \rightarrow C_1 = -2$

$$\rightarrow \begin{cases} A_0 = t_2 - 1 \\ B_0 = 2/1+t_2 \end{cases}$$

$$\rightarrow z_0 = (t_2 - 1)e^{-(1+t_2)t_1} + \frac{2}{1+t_2}$$

$$y \sim (x-1)e^{-(1+x)\frac{x}{\varepsilon}} + \frac{2}{1+x}$$

1	2
7	2
7	7

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ES_APPM 420-2 "Asymptotic and Perturbation Methods"

Midterm Examination (DUE TUESDAY, 2/17/09)

Problem 1. Consider van der Pol's equation

$$x'' - \varepsilon(1 - x^2)x' + x = 0$$

with initial conditions

$$x(0) = a, \ x'(0) = 0.$$

Determine the leading order solution of the problem that is valid for large t (i.e., $t = O(1/\varepsilon)$). Comment on the behavior of the solution depending on the initial condition a.

Problem 2. Consider the initial value problem

$$\frac{d^2x}{dt^2} + 2\varepsilon\frac{dx}{dt} + e^{-2\varepsilon t}x = 0, \quad x(0) = 0, \ x'(0) = 1.$$

Determine the leading order solution of the problem that is valid for large t (i.e., $t = O(1/\varepsilon)$).

Problem 3. Consider the equation

$$\ddot{x} + \varepsilon\mu\dot{x} + x + \varepsilon x^2 = F\cos\Omega t.$$

Show that there is a secondary resonance for $\Omega \approx 1/2$ by using the regular, one-time, perturbation approach to solving the equation. Study the behavior of the solution for $\Omega = (1+\varepsilon\omega)/2$ by introducing multiple scales.

2 continued)

$O(\varepsilon)$: $\Phi_{1rr} + \frac{1}{r}\Phi_{1r} + \frac{1}{r^2}\Phi_{1\theta\theta} = 0$

$\Phi_1(b,\theta) = 0$, $\Phi_1(a,\theta) = -\Phi_0'(a,\theta)\cdot a\sin n\theta$

$\Phi_1 = c_5 + c_6 \ln r + \sum_{k=1}^{\infty}(c_{3k}r^k + c_{4k}r^{-k})(c_{1k}\cos k\theta + c_{2k}\sin k\theta)$

$\Phi_1(b,\theta) = c_5 + c_6\ln b + \sum_{k=1}^{\infty}(c_{3k}b^k + c_{4k}b^{-k})(c_{1k}\cos k\theta + c_{2k}\sin k\theta) = 0$

$\rightarrow c_5 = -c_6 \ln b \quad c_{3k} = -c_{4k}b^{-2k}$

$\Phi_1(r,\theta) = c_6\ln(\frac{r}{b}) + \sum_{k=1}^{\infty}(r^{-k} - r^k b^{-2k})(c_{1k}\cos k\theta + c_{2k}\sin k\theta)$

$\Phi_1(a,\theta) = -\Phi_0'(a,\theta)\cdot a\sin n\theta$:

$c_6\ln(\frac{a}{b}) + \sum_{k=1}^{\infty}(a^{-k} - a^k b^{-2k})(c_1\cos k\theta + c_2\sin k\theta) = \frac{-Va}{\ln b/a}\sin n\theta$

$\rightarrow c_6 = 0$, $c_{1k} = 0\ \forall k$, $c_{2k} = 0$ for $k \neq n$,

$(a^{-n} - a^n b^{-2n})c_{2n} = -Va/\ln(\frac{b}{a})$

$\rightarrow c_{2n} = \dfrac{-Va}{\ln\frac{b}{a}(a^{-n} - a^n b^{-2n})}$

$$\rightarrow \Phi_1(r,\theta) = \frac{(r^n b^{-2n} - r^{-n})Va\sin n\theta}{\ln\frac{b}{a}(a^{-n} - a^n b^{-2n})}$$

$$\rightarrow \boxed{\Phi(r,\theta) \sim \frac{V\ln\frac{r}{a}}{\ln\frac{b}{a}} + \varepsilon\frac{(r^n b^{-2n} - r^{-n})V\overset{?}{a}\sin n\theta}{\ln\frac{b}{a}(a^{-n} - a^n b^{-2n})}}$$

lep04

3) $u_t + \alpha u_x + \beta u = \varepsilon u_{xx}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ $\quad (\alpha, \beta > 0,\ 0 < \varepsilon \ll 1)$

Outer: let $\varepsilon = 0$

$u_t + \alpha u_x + \beta u = 0$

method of characteristics:

$\frac{dx}{ds} = \alpha$, $\frac{dt}{ds} = 1$, $\frac{du}{ds} = -\beta u$

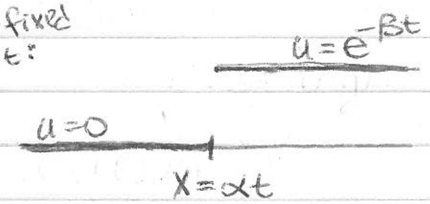
$x(s=0) = \tau$, $t(s=0) = 0$, $u(s=0) = H(\tau)$

$\rightarrow x = \alpha s + \tau$, $t = s$, $u = e^{-\beta s} H(\tau)$

$\rightarrow \tau = x - \alpha t$, $s = t$

$\rightarrow u = e^{-\beta t} H(x - \alpha t)$

for fixed t:



need inner layer at $x = \alpha t$

let $\xi = \frac{x - \alpha t}{\varepsilon^\gamma} \rightarrow \partial/\partial t = \frac{\partial \xi}{\partial t}\partial/\partial \xi + \frac{\partial t}{\partial t}\partial/\partial t = -\frac{\alpha}{\varepsilon^\gamma}\partial/\partial \xi + \partial/\partial t$

$\partial/\partial x = \frac{\partial \xi}{\partial x}\partial/\partial \xi + \frac{\partial t}{\partial x}\partial/\partial t = \frac{1}{\varepsilon^\gamma}\partial/\partial \xi$, $\partial^2/\partial x^2 = \frac{1}{\varepsilon^{2\gamma}}\partial^2/\partial \xi^2$

$\rightarrow -\frac{\alpha}{\varepsilon^\gamma} u_\xi + u_t + \frac{\alpha}{\varepsilon^\gamma} u_\xi + \beta u = \varepsilon^{1-2\gamma} u_{\xi\xi}$

$\rightarrow u_t + \beta u = \varepsilon^{1-2\gamma} u_{\xi\xi} \rightarrow 1 - 2\gamma = 0 \rightarrow \gamma = \frac{1}{2}$

$\rightarrow u_t + \beta u = u_{\xi\xi}$, $-\infty < \xi < \infty$, $t > 0$

$u(\xi, 0) = H(\sqrt{\varepsilon}\,\xi) = H(\xi)$

let $u = e^{-\beta t} v(\xi, t)$

$\rightarrow -\beta e^{-\beta t} v + e^{-\beta t} v_t + \beta e^{-\beta t} v = e^{-\beta t} v_{\xi\xi}$

$\rightarrow v_t = v_{\xi\xi}$

The solution to this is known:

$v = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} H(s) e^{\frac{-(\xi - s)^2}{4t}} ds$

$= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{\frac{-(s-\xi)^2}{4t}} ds$ $\quad$ let $\frac{s - \xi}{\sqrt{4t}} = \tau$

$\rightarrow ds = \sqrt{4t}\, d\tau$

$\rightarrow \frac{1}{\sqrt{\pi}} \int_{-\frac{\xi}{\sqrt{4t}}}^{\infty} e^{-\tau^2} d\tau$

$= \frac{1}{2} \operatorname{erfc}(-\xi/\sqrt{4t})$

$\rightarrow u = \frac{1}{2} e^{-\beta t} \operatorname{erfc}\left(\frac{-\xi}{\sqrt{4t}}\right)$

(check matching: as $\xi \rightarrow \infty$,

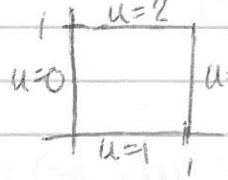
$u \rightarrow \frac{1}{2} e^{-\beta t} \operatorname{erfc}(-\infty) = e^{-\beta t}$ ✓

as $\xi \rightarrow -\infty$, $u \rightarrow \frac{1}{2} e^{-\beta t} \operatorname{erfc}(\infty) = 0$ ✓)

$$\boxed{u(x,y) = \frac{1}{2} e^{-\beta t} \operatorname{erfc}\left(\frac{\alpha t - x}{\sqrt{4\varepsilon t}}\right)}$$ ✓

4) $u_{xx} + \varepsilon u_{yy} - u_y = 0$, $0<x<1$, $0<y<1$

$u(0,y) = u(1,y) = 0$, $u(x,0) = 1$, $u(x,1) = 2$



Note that if we hold x constant, the coefficient of the first derivative in y is <0, so there can't be a boundary layer near y=0.

Outer problem: let $\varepsilon = 0$

$\rightarrow u_{xx} - u_y = 0$

Assume $u = X(x)Y(y)$

$\rightarrow \frac{X''}{X} = \frac{Y'}{Y} = \lambda$

$\rightarrow X'' - \lambda X = 0$

want two zeros in X: need $\lambda < 0$. let $\lambda = -\alpha^2$

$X = c \sin \alpha x$

$X(1) = 0 \rightarrow c \sin \alpha = 0 \rightarrow \alpha = n\pi$, $n = 0,1,2,\ldots$

$\rightarrow X = c \sin n\pi x$

$Y' - \lambda Y = 0$

$\rightarrow Y = e^{\lambda y} = e^{-n^2\pi^2 y}$

$\rightarrow u = \sum_{n=0}^{\infty} c_n e^{-n^2\pi^2 y} \sin n\pi x$

$u(x,0) = \sum_{n=0}^{\infty} c_n \sin n\pi x = 1$

$\rightarrow \int_0^1 c_m \sin^2 m\pi x \, dx = \int_0^1 \sin m\pi x \, dx$

$\rightarrow c_m \frac{1}{2} \int_0^1 1 - \cos 2m\pi x \, dx = \frac{1}{m\pi} \int_0^1 \cos m\pi x \, dx$

$\frac{1}{2} c_m = \frac{2}{m\pi}$

$\rightarrow c_m = \frac{4}{m\pi}$

$$\rightarrow \boxed{u = \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2 y} \sin n\pi x \quad \text{Outer solution}}$$

As $y \rightarrow 1$, this $\rightarrow \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x$, $\neq 2$ $\rightarrow$ need BL near $y=1$

BL near $y=1$:

let $\eta = \frac{1-y}{\varepsilon^{\alpha}}$

$\rightarrow u_{xx} + \varepsilon^{1-2\alpha} u_{\eta\eta} + \varepsilon^{-\alpha} u_{\eta} = 0$

$\rightarrow 1-2\alpha = -\alpha \rightarrow \alpha = 1$

$\rightarrow u_{xx} + \frac{1}{\varepsilon} u_{\eta\eta} + \frac{1}{\varepsilon} u_{\eta} = 0$

$\rightarrow u_{\eta\eta} = -u_{\eta}$, $u(x,0) = 2$, $u(x,\infty) = \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x$

$\rightarrow u = c_1(x) e^{-\eta} + c_2(x)$

$u(x,0) = c_1(x) + c_2(x) = 2$

$\rightarrow c_1(x) = 2 - c_2(x)$

$u(x,\infty) = c_2(x) = \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x$

$\rightarrow u(x,\eta) = e^{-\eta}\left(2 - \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x\right) + \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x$

$= (1-e^{-\eta}) \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x + 2e^{-\eta}$

$\boxed{u(x,y) = (1-e^{\frac{y-1}{\varepsilon}}) \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x + 2e^{\frac{y-1}{\varepsilon}}}$

solution in boundary layer near $y=1$ of width ε

Check behavior near $x=0$: $u_{BL} \approx 2e^{\frac{y-1}{\varepsilon}}$

should be 0, according to boundary conditions

$\rightarrow$ need a corner layer near $x=0$.

Same near $x=1$.

Corner Layer near $x=0,\ y=1$:

let $\xi = \frac{x}{\varepsilon^\alpha},\ \eta = \frac{1-y}{\varepsilon^\beta}$

$\rightarrow \varepsilon^{-2\alpha} u_{\xi\xi} + \varepsilon^{1-2\beta} u_{\eta\eta} - \varepsilon^{-\beta} u_\eta = 0$

must match all terms if possible:

$1-2\beta = -\beta \rightarrow \beta = 1$

$-2\alpha = -\beta \rightarrow \alpha = \frac{1}{2}\beta = \frac{1}{2}$

$\rightarrow u_{\xi\xi} + u_{\eta\eta} - u_\eta = 0,\ u(\xi,0)=2,\ u(0,\eta)=0$

let $u = e^{\eta/2} v$

$\rightarrow e^{\eta/2} v_{\xi\xi} - [\frac{1}{2} e^{\eta/2} v + e^{\eta/2} v_\eta] + \frac{1}{4} e^{\eta/2} v + e^{\eta/2} v_\eta + e^{\eta/2} v_{\eta\eta} = 0$

$\rightarrow v_{\xi\xi} + v_{\eta\eta} - \frac{1}{4} v = 0,\quad \xi>0,\ \eta>0$

$v(\xi,0) = 2,\ v(0,\eta) = 0$

following sol'n in notes: (let $\xi = x,\ \eta = y$)

Green's function:

$\nabla^2 g(x,y\,|\,x_0,y_0) - \frac{1}{4} g = -\delta(x,y\,|\,x_0,y_0)$

$x, x_0, y, y_0 > 0,\ g|_{y=0} = 0,\ g|_{x=0} = 0$

free space green's function, polar, radially symmetric:

$\frac{\partial^2 g_0}{\partial r^2} + \frac{1}{r}\frac{\partial g_0}{\partial r} - \frac{1}{4} g_0 = -\delta(x_0,y_0)$

$r \neq 0$: $r^2 g_{0rr} + r g_{0r} - \frac{1}{4} r^2 g_0 = 0,\quad r>0$

$\rightarrow g_0 = c K_0(\frac{1}{2} r)$

And $\iint_{D_\varepsilon} (\nabla^2 g_0 - \frac{1}{4} g_0)\, dA = -1$

$\rightarrow \int_{\partial D_\varepsilon} \partial g_0/\partial n \, dS = -1$

$\rightarrow \int_0^{2\pi} \partial g_0/\partial r|_{r=\varepsilon}\, \varepsilon \, d\theta = -1$

As $r \to 0,\ K_0(\frac{1}{2} r) \to -\ln(\frac{r}{2})$

$\rightarrow \int_0^{2\pi} -2\pi c \frac{1}{r}|_{r=\varepsilon}\, \varepsilon \, d\theta = -1$

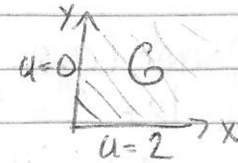
$\rightarrow -2\pi c = -1$

$\rightarrow c = \frac{1}{2\pi}$

$g_0 = \frac{1}{2\pi} K_0(\frac{1}{2} r) = \frac{1}{2\pi} K_0(\frac{1}{2}\sqrt{(x-x_0)^2 + (y-y_0)^2})$

Construct g using g_0:

$g = g_0(x,y\,|\,x_0,y_0) - g_0(x,y\,|\,-x_0,y_0)$
$\quad - g_0(x,y\,|\,x_0,-y_0) + g_0(x,y\,|\,-x_0,-y_0)$



$$\int_G [(\nabla^2 v - \tfrac{1}{4} v) g - (\nabla^2 g - \tfrac{1}{4} g) v]\, dA = v(x_0, y_0)$$

$$= \int_G g \nabla^2 v - v \nabla^2 g \, dA = v(x_0, y_0)$$

$$= \int_{\partial G} \left(g \frac{\partial v}{\partial n} - v \frac{\partial g}{\partial n}\right) ds$$

$g = 0$ on boundary, $v = 0$ on y-axis

$$= \int_0^\infty -2 \left.\frac{\partial g}{\partial y}\right|_{y=0} dx = v(x_0, y_0)$$

rename $(x_0, y_0) \to (x, y)$

$$\Rightarrow v(x,y) = -2 \int_0^\infty \left.\frac{\partial g}{\partial y_0}\right|_{y_0=0} dx_0$$

$$= -2 \int_0^\infty \frac{\partial}{\partial y_0}\left[\frac{1}{2\pi} K_0\left(\tfrac{1}{2}\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)\right]_{y_0=0} dx$$

$$= \frac{-y}{\pi} \int_0^\infty \frac{-K_0'\left(\frac{1}{2}\sqrt{(x-x_0)^2 + y^2}\right)}{\sqrt{(x-x_0)^2 + y^2}} + \frac{K_0'\left(\frac{1}{2}\sqrt{(x-x_0)^2 + y^2}\right)}{\sqrt{(x-x_0)^2 + y^2}}\, dx_0$$

let $x_0 - x = s$ let $x + x_0 = s$

$$= \frac{-y}{\pi} \int_x^\infty \frac{K_0'\left(\frac{1}{2}\sqrt{s^2 + y^2}\right)}{\sqrt{s^2 + y^2}}\, ds + \frac{y}{\pi} \int_{-x}^\infty \frac{K_0'\left(\frac{1}{2}\sqrt{s^2 + y^2}\right)}{\sqrt{s^2 + y^2}}\, ds$$

$$= \frac{y}{\pi} \int_{-x}^{x} \frac{K_0'\left(\frac{1}{2}\sqrt{s^2 + y^2}\right)}{\sqrt{s^2 + y^2}}\, ds$$

$$= \frac{-2y}{\pi} \int_0^x \frac{K_1\left(\frac{1}{2}\sqrt{s^2 + y^2}\right)}{\sqrt{s^2 + y^2}}\, ds$$

$$\Rightarrow u = \frac{-2\eta}{\pi} e^{\eta/2} \int_0^{\xi} \frac{K_1\left(\frac{1}{2}\sqrt{s^2 + \eta^2}\right)}{\sqrt{s^2 + \eta^2}}\, ds$$

$$\boxed{u(x,y) = \frac{-2\left(\frac{1-y}{\varepsilon}\right)}{\pi} e^{\frac{1-y}{2\varepsilon}} \int_0^{\frac{x}{\sqrt{\varepsilon}}} \frac{K_1\left(\frac{1}{2}\sqrt{s^2 + \left(\frac{1-y}{\varepsilon}\right)^2}\right)}{\sqrt{s^2 + \left(\frac{1-y}{\varepsilon}\right)^2}}\, ds}$$

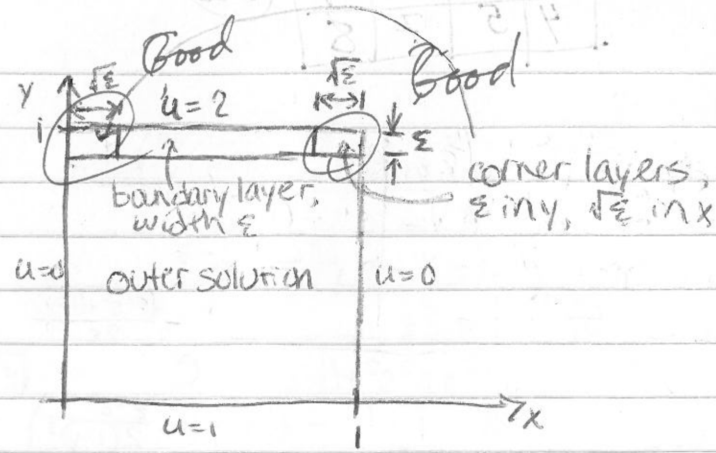
solution in corner layer near $(x,y) = (0,1)$ of width ε in y, $\sqrt{\varepsilon}$ in x.

Corner layer near $(x,y) = (1,1)$: same as above, except now $\xi = \frac{1-x}{\sqrt{\varepsilon}}$:

$$\boxed{u(x,y) = \frac{-2(1-y)}{\varepsilon\pi} e^{\frac{1-y}{2\varepsilon}} \int_0^{\frac{1-x}{\sqrt{\varepsilon}}} \frac{K_1\left(\frac{1}{2}\sqrt{s^2 + \left(\frac{1-y}{\varepsilon}\right)^2}\right)}{\sqrt{s^2 + \left(\frac{1-y}{\varepsilon}\right)^2}}\, ds}$$

solution in corner layer near $(x,y) = (1,1)$ of width ε in y, $\sqrt{\varepsilon}$ in x

Lepo 7



Uniform solution:

Outer + boundary layer − common part

$$= \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2 y} \sin n\pi x - e^{\frac{y-1}{\varepsilon}} \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x + 2e^{\frac{y-1}{\varepsilon}}$$

add corner layers − common part:

$$\boxed{u = \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2 y} \sin n\pi x - e^{\frac{y-1}{\varepsilon}} \sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x - \frac{2(1-y)}{\varepsilon\pi} e^{\frac{1-y}{2\varepsilon}} \left[\int_0^{\frac{x}{\sqrt{\varepsilon}}} \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\left(\frac{1-y}{\varepsilon}\right)^2}\right) ds}{\sqrt{s^2+\left(\frac{1-y}{\varepsilon}\right)^2}} + \int_0^{\frac{1-x}{\sqrt{\varepsilon}}} \frac{K_1\left(\frac{1}{2}\sqrt{s^2+\left(\frac{1-y}{\varepsilon}\right)^2}\right) ds}{\sqrt{s^2+\left(\frac{1-y}{\varepsilon}\right)^2}} \right]}$$

I don't think my corner layer solution is correct. See reverse side of this page for a failed attempt at matching.

1	2	3	4
4	4	7	7
4	5	7	8

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check matching for corner layers

$$\frac{2(y-1)}{\varepsilon\pi} e^{\frac{1-y}{2\varepsilon}} \int_0^{\frac{x}{\sqrt{\varepsilon}}} \frac{K_1\left(\frac{1}{2\varepsilon}\sqrt{\varepsilon^2 s^2+(1-y)^2}\right)}{\frac{1}{\varepsilon}\sqrt{\varepsilon^2 s^2+(1-y)^2}} ds$$

As $z \to \infty$, $K_1(z) \to \sqrt{\frac{\pi}{2z}}\, e^{-z}$;

$$\to \frac{2(y-1)}{\pi} e^{\frac{1-y}{2\varepsilon}} \int_0^{\frac{x}{\sqrt{\varepsilon}}} \sqrt{\frac{\pi\, 2\varepsilon}{2\sqrt{\varepsilon^2 s^2+(1-y)^2}}}\; e^{-\frac{\sqrt{\varepsilon^2 s^2+(1-y)^2}}{2\varepsilon}} \frac{ds}{\sqrt{\varepsilon^2 s^2+(1-y)^2}}$$

$$\sim -2\sqrt{\frac{\varepsilon}{\pi}}\, e^{\frac{1-y}{2\varepsilon}} \int_0^{\frac{x}{\sqrt{\varepsilon}}} \sqrt{\frac{1}{1-y}}\; e^{-\frac{(1-y)}{2\varepsilon}\left(1+\frac{\varepsilon^2 s^2}{2(1-y)^2}\right)} ds$$

$$= \frac{-2\sqrt{\varepsilon}}{\sqrt{(1-y)\pi}} \int_0^{\frac{x}{\sqrt{\varepsilon}}} e^{-\frac{\varepsilon s^2}{4(1-y)}} ds \qquad \text{let } \tau = \sqrt{\frac{\varepsilon}{4(1-y)}}\, s$$

$$= \frac{-2\sqrt{\varepsilon}}{\sqrt{\pi}\sqrt{(1-y)}} \sqrt{\frac{4(1-y)}{\varepsilon}} \int_0^{\frac{x}{2\sqrt{1-y}}} e^{-\tau^2} d\tau$$

$$= -2\, \mathrm{erf}\left(\frac{x}{2\sqrt{1-y}}\right)$$

for $x=0$, $\mathrm{erf}(0)=0$ ✓

for $y \to 1$, $\mathrm{erf}(\infty) = 1 \to -2$

(should be $+2$?)

As $z \to \infty$, $\mathrm{erf}(z) \to 1 - \frac{1}{z\sqrt{\pi}} e^{-z^2}$

So as we move away from $(x,y)=(0,1)$ in the x-direction,

$$\sim -2\left(1 - \frac{1}{\sqrt{\pi}} \frac{2\sqrt{1-y}}{x} e^{\frac{x^2}{4(1-y)}}\right)$$

$$= -2 + \frac{4}{x}\sqrt{\frac{1-y}{\pi}}\, e^{\frac{x^2}{4(1-y)}}$$

(should match $2e^{\frac{y-1}{\varepsilon}}$?)

And as we move away in the y-direction,

$$\sim 2 - \frac{4}{x}\sqrt{\frac{1-y}{\pi}}\, e^{\frac{x^2}{4(1-y)}}$$

(should match $\sum_{n=0}^{\infty} \frac{4}{n\pi} e^{-n^2\pi^2} \sin n\pi x$?)

This doesn't seem right at all.

ES_APPM 420-3 "Asymptotic and Perturbation Methods"

Homework 4 (DUE TUESDAY, 6/2/2009, IN CLASS)

Problem 1. Consider the two-dimensional problem

$$\nabla \cdot (D \nabla u) = f(x), \quad x \in \Omega,$$

$$u = g(x), \quad x \in \partial\Omega.$$

Here $x = (x_1, x_2)$, the coefficient D is positive and smooth and $D = D(x_1, x_2, y_1)$, where $y_1 = x_1/\varepsilon$ (unlike the problem we considered in class D does not depend on $y_2 = x_2/\varepsilon$). The coefficient D is periodic in y_1 with period p. What do the homogenized equations derived in class reduce to in this case? (You can use the results derived in class, i.e., you do not have to re-derive everything.)

Problem 2. Consider the following problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon u, \quad 0 < x < 1, \quad t > 0,$$

where

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = g(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

Using multiple scales, find a first-term approximation of the solution that is valid for large t, i.e., for $t = O(1/\varepsilon)$.

Haley Lepo
6-2-09
ESAM 420-3
Hwk #4

1) $\nabla\cdot(D\nabla u)=f(x)$, $x\in\Omega$; $u=g(x)$ for $x\in\partial\Omega$
$x=(x_1,x_2)$, $D=D(x_1,x_2,y_1)>0$, $y_1=x_1/\varepsilon$
D periodic in y_1 with period p
Find $\bar{D}$ to get the form $\nabla\cdot(\bar{D}\nabla u)=f(x)$.

$O(1)$: $u_0=u_0(x_1,x_2)$
$O(\varepsilon)$: $\nabla_y[D\nabla_y u_1]=-\nabla_y\cdot[D\nabla_x u_0]$
$u_1(\vec{x},\vec{y})=b(\vec{x})+\vec{a}(\vec{x},\vec{y})\cdot\nabla_x u_0$

find particular sol'n

$\vec{a}=(a_1(x_1,x_2,y_1),\ a_2(x_1,x_2,y_1))$

$$\begin{cases}\nabla_y\cdot[D(x_1,x_2,y_1)\nabla_y a_1]=-\frac{\partial D}{\partial y_1}\\ \nabla_y\cdot[D(x_1,x_2,y_1)\nabla_y a_2]=-\frac{\partial D}{\partial y_2}\end{cases}$$

$$\begin{cases}\nabla_y\cdot[D(x_1,x_2,y_1)(\frac{\partial a_1}{\partial y_1},\frac{\partial a_1}{\partial y_2})]=-\frac{\partial D}{\partial y_1}\\ \nabla_y\cdot[D(x_1,x_2,y_1)(\frac{\partial a_2}{\partial y_1},\frac{\partial a_2}{\partial y_2})]=0\end{cases}$$

$$\begin{cases}\partial/\partial y_1[D\frac{\partial a_1}{\partial y_1}]+D\frac{\partial^2 a_1}{\partial y_2^2}=-\frac{\partial D}{\partial y_1}\\ \partial/\partial y_1[D\frac{\partial a_2}{\partial y_1}]+D\frac{\partial^2 a_2}{\partial y_2^2}=0\end{cases}$$

$$\begin{cases}\partial/\partial y_1[D\,\partial a_1/\partial y_1]=-\partial D/\partial y_1\\ \partial/\partial y_1[D\,\partial a_2/\partial y_1]=0\end{cases}$$

$$\begin{cases}D\,\partial a_1/\partial y_1=-D+c_1\\ D\,\partial a_2/\partial y_1=c_2\end{cases}$$

$$\begin{cases}\partial a_1/\partial y_1=-1+c_1/D\\ \partial a_2/\partial y_1=c_2/D\end{cases}$$

$$\begin{cases}a_1=-y_1+\int_0^{y_1}c_1/D(\vec{x},s)\,ds+c_3\\ a_2=\int_0^{y_1}c_2/D(\vec{x},s)\,ds+c_4\end{cases}$$

a_1 + a_2 periodic in p:

$$\begin{cases}0=-p+\int_0^p c_1/D(\vec{x},s)\,ds\\ 0=\int_0^p c_2/D(\vec{x},s)\,ds\end{cases}$$

$$\begin{cases}c_1=p\left[\int_0^p\frac{ds}{D(\vec{x},s)}\right]^{-1}\\ c_2=0\end{cases}$$

$$\rightarrow\frac{\partial a_1}{\partial y_1}=-1+\frac{p}{\int_0^p ds/D(\vec{x},s)}$$

$$\frac{\partial a_2}{\partial y_1}=0$$

continued on reverse

$$\rightarrow \bar{D} = \begin{pmatrix} \langle D \frac{\partial a_1}{\partial y_1} \rangle_p + \langle D_p \rangle & \cancel{\langle D \frac{\partial a_2}{\partial y_1} \rangle_p} \to 0 \\ \cancel{\langle D \frac{\partial a_1}{\partial y_2} \rangle_p} \to 0 & \cancel{\langle D \frac{\partial a_2}{\partial y_2} \rangle_p} \to 0 + \langle D \rangle_p \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{p} \int_0^p \left(-D + \frac{p}{\int_0^{y_1} ds/D} + D \right) & 0 \\ 0 & \frac{1}{p} \int_0^p D \, dy_1 \end{pmatrix}$$

$$= \begin{pmatrix} \int_0^p \frac{1}{\int_0^{y_1} ds / D(\vec{x}, s)} dy_1 & 0 \\ 0 & \frac{1}{p} \int_0^p D \, dy_1 \end{pmatrix}$$

$$\bar{D} = \begin{pmatrix} \frac{p}{\int_0^{y_1} \frac{ds}{D(\vec{x}, s)}} & 0 \\ 0 & \frac{1}{p} \int_0^p D(x_1, x_2, y_1) \, dy_1 \end{pmatrix}$$

$$\nabla \cdot (\bar{D} \nabla u) = f(x)$$

$$\boxed{\nabla \cdot \left[\left(\frac{p}{\int_0^{y_1} \frac{ds}{D(x_1, x_2, s)}} \partial u / \partial x_1 , \frac{1}{p} \int_0^p D(x_1, x_2, y_1) dy_1 \frac{\partial u}{\partial x_2} \right) \right] = f(x)}$$

✓

2) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} + \varepsilon u \quad , \; 0<x<1, \; t>0$

$u(0,t) = u(1,t) = 0, \; u(x,0) = g(x), \; u_t(x,0) = 0$

Determine the correct scaling: try without.

$O(1)$: $u_{0xx} = u_{0tt}$, $u_0(0,t) = u_0(1,t) = 0$; $u_0(x,0) = g(x)$, $u_{0t}(x,0) = 0$

$$\frac{X''}{X} = \frac{T''}{T} = \lambda$$

$$X'' - \lambda X = 0$$

$$X = \sin n\pi x \quad , \quad \lambda = -n^2\pi^2 \quad , \quad n \in \mathbb{N}$$

$$T'' + n^2\pi^2 T = 0$$

$$T = c_1 \cos n\pi t$$

$$u = \textstyle\sum_{n=1}^{\infty} c_n \cos(n\pi t)\sin(n\pi x)$$

$$u_0(x,0) = \textstyle\sum_{n=1}^{\infty} c_n \sin n\pi x = g(x)$$

$$\int_0^1 c_m \sin^2 m\pi x \, dx = \int_0^1 g(x) \sin m\pi x \, dx$$

$$c_m \tfrac{1}{2} \int_0^1 1 - \cos 2m\pi x \, dx = \int_0^1 g(x) \sin m\pi x \, dx$$

$$c_m = 2\int_0^1 g(x) \sin m\pi x \, dx$$

$O(\varepsilon)$: $u_{1xx} - u_{1tt} = u_0 = \sum_{n=1}^{\infty} c_n \cos(n\pi t)\sin(n\pi x)$, $u(x,0) = u_t(x,0) = 0$

$$u_1 = \textstyle\sum_{n=1}^{\infty} d_n(t) \sin n\pi x$$

$$\rightarrow \textstyle\sum_{n=1}^{\infty} (-n^2\pi^2 d_n(t) - d_n''(t)) \sin(n\pi x) = \sum_{n=1}^{\infty} c_n \cos(n\pi t)\sin(n\pi x)$$

$$d_n''(t) + n^2\pi^2 d_n(t) = -c_n \cos(n\pi t) \quad , \quad d_n(0) = d_{nt}(0) = 0$$

homog: $a\cos n\pi t + b\sin n\pi t$

particular: $ft\cos n\pi t + ht\sin n\pi t$

$$\rightarrow \partial/\partial t = f\cos n\pi t - n\pi f t \sin n\pi t + h \sin n\pi t + n\pi h t \cos n\pi t$$

$$\begin{aligned} \partial^2/\partial t^2 &= -n\pi f \sin n\pi t - n\pi f \sin n\pi t - n^2\pi^2 f t \cos n\pi t \\ &\quad + n\pi h \cos n\pi t + n\pi h \cos n\pi t - n^2\pi^2 h t \sin n\pi t \\ &= \sin n\pi t\,(-2n\pi f - n^2\pi^2 h t) \\ &\quad + \cos n\pi t\,(2n\pi h - n^2\pi^2 f t) \end{aligned}$$

$$\rightarrow -2n\pi f = 0, \quad 2n\pi h - 2n^2\pi^2 f t = -c_n$$

$$\rightarrow h = -\frac{1}{2n\pi} c_n$$

$$\rightarrow d_n = a\cos n\pi t + b\sin n\pi t - \frac{1}{2\pi n} c_n t \cos n\pi t$$

$d_n(0) = 0$: $a = 0$ $\quad d_{nt}(0) = 0$: $n\pi b - \frac{1}{2\pi n} c_n = 0 \rightarrow b = \frac{1}{2\pi^2 n^2} c_n$

$$d_n(t) = \frac{1}{2n^2\pi^2} \sin n\pi t - \frac{1}{2\pi n} c_n t \cos n\pi t$$

ES_APPM 420-3 "Asymptotic and Perturbation Methods"

Final Examination (DUE MONDAY, JUNE 8, 2009, BY NOON)

Problem 1. The problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon\frac{\partial u}{\partial t} + \varepsilon f(x)\sin(\omega t), \quad 0 < x < 1, \ t > 0,$$

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

describes oscillations of a weakly damped, forced elastic string.
(a). When $\omega \neq n\pi$, where n is an integer, find a first-term approximation of the solution that is valid for large t, i.e., for $t = O(1/\varepsilon)$.
(b). When $\omega = \pi$, find a first-term approximation of the solution that is valid for large t, i.e., for $t = O(1/\varepsilon)$.

Problem 2. The problem

$$\frac{\partial^2 p}{\partial t^2} - \frac{1}{A}\frac{\partial}{\partial x}\left(A\frac{\partial p}{\partial x}\right) = 0, \quad -\infty < x < \infty, \ t > 0,$$

$$p(x,0) = h(x), \quad \left.\frac{\partial p}{\partial t}\right|_{t=0} = 0$$

where $p(x,t)$ is the pressure, describes acoustic wave propagation in a tube with a slowly-varying cross-section $A = A(\varepsilon x)$. Find a first-term approximation of the solution that is valid for large t, i.e., for $t = O(1/\varepsilon)$.

Problem 3. Consider the advection problem with weak diffusion

$$\varepsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}, \quad -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = f(x).$$

Using multiple scales find a first-term approximation of the solution that is valid for large t, i.e., for $t = O(1/\varepsilon)$.

Haley Lepo
5-5-09
ESAM 420-3
Hwk #2

1) $\varepsilon\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = u, \quad 0<r<1, \ 0 \le \theta < 2\pi$

$u(1,\theta) = 1$

reduced problem: let $\varepsilon = 0$:

$\rightarrow \quad u = 0$

$\rightarrow$ need boundary layer near $r=1$

let $\rho = \frac{1-r}{\sqrt{\varepsilon}} \rightarrow r = 1-\rho\sqrt{\varepsilon}, \ \partial/\partial r = -\frac{1}{\sqrt{\varepsilon}}\partial/\partial\rho$

$\rightarrow \varepsilon\left(\frac{1}{\varepsilon}\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{1-\rho\sqrt{\varepsilon}}\left(\frac{-1}{\sqrt{\varepsilon}}\right)\frac{\partial u}{\partial \rho} + \frac{1}{(1-\rho\sqrt{\varepsilon})^2}\frac{\partial^2 u}{\partial\theta^2}\right) = u$

$\rightarrow (1-\rho\sqrt{\varepsilon})^2 u_{\rho\rho} - (1-\rho\sqrt{\varepsilon})\sqrt{\varepsilon}\, u_\rho + \varepsilon u_{\theta\theta} = (1-\rho\sqrt{\varepsilon})^2 u$

$\rightarrow (1-2\rho\sqrt{\varepsilon} + \rho^2\varepsilon) u_{\rho\rho} - (\sqrt{\varepsilon} - \rho\varepsilon) u_\rho + \varepsilon u_{\theta\theta} = (1-2\sqrt{\varepsilon} + \rho^2\varepsilon) u$

$O(1)$ terms: $u_{\rho\rho} = u, \ u(0) = 1, \ u(\infty) = 0$

$\rightarrow \quad u = c_1 e^{\rho} + c_2 e^{-\rho}$

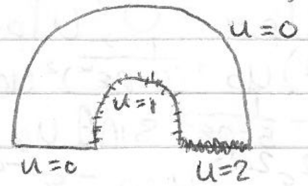
need $c_1 = 0, \ c_2 = 1$

$\rightarrow \quad u = e^{-\rho}$

$\rightarrow \quad u = e^{-\left(\frac{1-r}{\sqrt{\varepsilon}}\right)}$

$\rightarrow \quad \boxed{u = e^{\frac{r-1}{\sqrt{\varepsilon}}}}$ ✓

2) $\varepsilon \nabla^2 u = u_x$, $3<r<5$, $0<\theta<\pi$



Reduced problem: let $\varepsilon = 0$

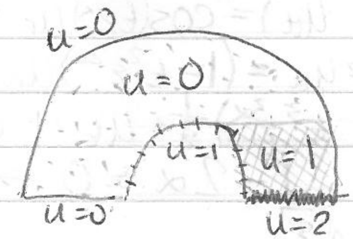
$\rightarrow \quad u_x = 0$

$\rightarrow \quad u = u(y) = \text{const.}$ since BC are const.

Thus characteristics are horizontal.

As we move along a characteristic, $u_{yy}=0$, and the eqn becomes $\varepsilon u_{xx} - u_x = 0$.

the coefficient of the first derivative is <0, so the boundary layers will be on the right.



(The solution propogates from the left boundary, along the characteristics.)

BLA

Need a boundary layer near $r=3$, $\theta: \pi/2 \rightarrow \pi$

$$\varepsilon\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = u_x = \cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta$$

Let $\rho = \frac{r-3}{\varepsilon^\alpha}$. Then $u|_{\rho=0} = 1$, $u|_{\rho\to\infty} \rightarrow 0$

$$\rightarrow \varepsilon\left(\frac{1}{\varepsilon^{2\alpha}}u_{\rho\rho} + \frac{1}{\rho\varepsilon^\alpha+3}\left(\frac{1}{\varepsilon^\alpha}\right)u_\rho + \frac{1}{(\rho\varepsilon^\alpha+3)^2}u_{\theta\theta}\right) = \cos\theta \frac{1}{\varepsilon^\alpha}u_\rho - \frac{1}{\rho\varepsilon^\alpha+3}\sin\theta\, u_\theta$$

$$\rightarrow \varepsilon^{1-2\alpha}(\rho\varepsilon^\alpha+3)^2 u_{\rho\rho} + \varepsilon^{1-\alpha}(\rho\varepsilon^\alpha+3)u_\rho + \varepsilon u_{\theta\theta} = \cos\theta\, \varepsilon^{-\alpha}(\rho\varepsilon^\alpha+3)^2 u_\rho - \sin\theta(\rho\varepsilon^\alpha+3)u_\theta$$

$$\rightarrow (\rho^2\varepsilon + 6\rho\varepsilon^{1-\alpha} + 9\varepsilon^{1-2\alpha})u_{\rho\rho} + (\rho\varepsilon + 3\varepsilon^{1-\alpha})u_\rho + \varepsilon u_{\theta\theta} = \cos\theta(\rho^2\varepsilon^\alpha + 6\rho + 9\varepsilon^{-\alpha})u_\rho - \sin\theta(\rho\varepsilon^\alpha+3)u_\theta$$

match largest terms: $1-2\alpha = -\alpha \rightarrow \alpha = 1$ $\downarrow$ BL has width ε

$$\rightarrow 9u_{\rho\rho} = 9\cos\theta\, u_\rho$$

$$u_{\rho\rho} = \cos\theta\, u_\rho$$

BLB

Need a BL near $r=5$, $\theta: 0 \to \sin^{-1}(3/5)$

let $\rho = \frac{5-r}{\varepsilon^\alpha}$ Then, $u|_{\rho=0}=0$, $u|_{\rho\to\infty} \to 1$

$\to \varepsilon\left(\frac{1}{\varepsilon^{2\alpha}} u_{\rho\rho} + \frac{1}{5-\rho\varepsilon^\alpha}\left(-\frac{1}{\varepsilon^\alpha}\right) u_\rho + \frac{1}{(5-\rho\varepsilon^\alpha)^2} u_{\theta\theta}\right)$

$= \cos\theta\left(\frac{-1}{\varepsilon^\alpha}\right) u_\rho - \frac{1}{5-\rho\varepsilon^\alpha} \sin\theta\, u_\theta$

$\to \varepsilon^{1-2\alpha}(25 - 10\rho\varepsilon^\alpha + \rho^2\varepsilon^{2\alpha}) u_{\rho\rho} - \varepsilon^{1-\alpha}(5-\rho\varepsilon^\alpha) u_\rho$

$+ \varepsilon u_{\theta\theta} = -\cos\theta\, \varepsilon^{-\alpha}(25 - 10\rho\varepsilon^\alpha + \rho^2\varepsilon^{2\alpha}) u_\rho$

$-(5-\rho\varepsilon^\alpha)\sin\theta\, u_\theta$

match largest terms: $1-2\alpha = -\alpha \to \alpha = 1$

BL has width ε

$\to 25 u_{\rho\rho} = -25\cos\theta\, u_\rho$

$u_{\rho\rho} = -\cos\theta\, u_\rho$

BLC

Need a BL near $\theta = 0$, $r: 3 \to 5$

let $t = \frac{\theta}{\varepsilon^\alpha}$, Then $u|_{t=0} = 2$, $u|_{t\to\infty} = 1$, $u|_{r=3} = 1$

$\to \varepsilon\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2}\frac{1}{\varepsilon^{2\alpha}} u_{tt}\right) = \cos(t\varepsilon^\alpha) u_r - \frac{1}{r}\sin(t\varepsilon^\alpha)\frac{1}{\varepsilon^\alpha} u_t$

$\to \varepsilon u_{rr} + \varepsilon \frac{1}{r} u_r + \varepsilon^{1-2\alpha} \frac{1}{r^2} u_{tt} = (1 - t^2\varepsilon^{2\alpha}) u_r$

$-\frac{1}{r}(t\varepsilon^\alpha - t^3\varepsilon^{3\alpha})\frac{1}{\varepsilon^\alpha} u_t + O(\varepsilon^{4\alpha})$

match largest terms: $1-2\alpha = 0 \to \alpha = \frac{1}{2}$

BL has width $\sqrt{\varepsilon}$

$\to \frac{1}{r^2} u_{tt} = u_r - \frac{1}{r} t u_t$

$u_{tt} + r t u_t = r^2 u_r$

BLD

Need a BL near $y=3$, $x>0$

$\varepsilon(u_{xx} + u_{yy}) = u_x$

let $\eta = \frac{y-3}{\varepsilon^\alpha}$, Then $u|_{\eta=0} = 1$, $u|_{\eta\to\infty} \to 0$

$u|_{x=0} = 0$

$\to \varepsilon\left(u_{xx} + \frac{1}{\varepsilon^{2\alpha}} u_{\eta\eta}\right) = u_x$

$\varepsilon u_{xx} + \varepsilon^{1-2\alpha} u_{\eta\eta} = u_{xx}$

match largest terms: $1-2\alpha = 0 \to \alpha = \frac{1}{2}$

BL has width $\sqrt{\varepsilon}$

$\to u_{\eta\eta} = u_x$

Problem near $r=3$, $\theta=\pi$: Along the width of BLA at $\theta=\pi$, the function value varies when it should be constant at $u=0$.
$\rightarrow$ need a corner layer.
Use same width in r-direction, and scale θ so that θ-derivatives remain in leading order equation:

$\boxed{\text{CLA}}$

$$\rho=\frac{r-3}{\varepsilon}, \quad t=\frac{\pi-\theta}{\varepsilon} \quad \text{(from inspection of BLA calculations)}$$

$$\rightarrow \varepsilon\left(\frac{1}{\varepsilon^2}u_{\rho\rho}+\frac{1}{\rho\varepsilon+3}\left(\frac{1}{\varepsilon}\right)u_\rho+\frac{1}{(\rho\varepsilon+3)^2}\left(\frac{1}{\varepsilon^2}\right)u_{tt}\right)$$

$$=\cos(\pi-t\varepsilon)\left(\frac{1}{\varepsilon}\right)u_\rho-\frac{1}{(\rho\varepsilon+3)}\left(-\frac{1}{\varepsilon}\right)\sin(\pi-t\varepsilon)u_t$$

$$\rightarrow \frac{1}{\varepsilon}(\rho^2\varepsilon^2+6\rho\varepsilon+9)u_{\rho\rho}+(\rho\varepsilon+3)u_\rho+\frac{1}{\varepsilon}u_{tt}$$

$$=-\frac{1}{\varepsilon}\cos(t\varepsilon)(\rho^2\varepsilon^2+6\rho\varepsilon+9)u_\rho+\frac{1}{\varepsilon}(\rho\varepsilon+3)\sin(t\varepsilon)u_t$$

$$\rightarrow 9u_{\rho\rho}+u_{tt}=-9u_\rho \quad \text{at leading order}$$

need $u|_{\rho=0}=1$, $u|_{t=0}=0$
(other BC should match automatically)

~~Problem near $r=3$, $\theta=0$: Along the height of BLC at $r=3$ the function value varies while it should be constant at $u=1$.~~
~~$\rightarrow$ need a corner layer~~
~~Use same height in θ-direction and scale r so that r-derivs stay in leading order eqn.~~

CLB

Need a corner layer near $r=3,\ \theta=0$.

Use same scaling as for previous's corner layer.

$\rho = \frac{r-3}{\varepsilon}, \quad t = \frac{\theta}{\varepsilon}$

$$\to \varepsilon\left(\frac{1}{\varepsilon^2}u_{\rho\rho} + \frac{1}{\rho\varepsilon+3}\frac{1}{\varepsilon}u_\rho + \frac{1}{(\rho\varepsilon+3)^2}\frac{1}{\varepsilon^2}u_{tt}\right) = \cos t\varepsilon \frac{1}{\varepsilon}u_\rho - \frac{1}{\rho\varepsilon+3}\sin t\varepsilon \frac{1}{\varepsilon}u_t$$

$$\to 9u_{\rho\rho} + u_{tt} = 9u_\rho$$

need $u|_{\rho=0} = 1, \quad u|_{t=0} = 2$

~~Need a corner layer near $r=5, \theta=0$~~

$\cancel{\rho = \frac{5-r}{\varepsilon}, \quad t = \theta/\varepsilon}$

$$\cancel{\to \varepsilon\left(\frac{1}{\varepsilon^2}u_{\rho\rho} + \frac{1}{5-\rho\varepsilon}\left(-\frac{1}{\varepsilon}\right)u_\rho + \frac{1}{(5-\rho\varepsilon)^2}\frac{1}{\varepsilon^2}u_{tt}\right) = \cos t\varepsilon\left(-\frac{1}{\varepsilon}\right)u_\rho - \frac{1}{5-\rho\varepsilon}\sin t\varepsilon \frac{1}{\varepsilon}u_t}$$

$$\cancel{\to 25u_{\rho\rho} + u_{tt} = -25u_\rho}$$

~~need $u|_{\rho=0} = 0, \quad u|_{t=0} = 2$~~

CLC

Problem near $r=5,\ \theta = \sin^{-1}(3/5)$:

Approaching the point along $r=5$ from below, the function is constant at $u=0$, while if you approach from $y=3$, it is constant at $u=1$. $\to$ need a corner layer.

$\rho = \frac{5-r}{\varepsilon} \quad \eta = \frac{3-y}{\varepsilon} = \frac{1}{\varepsilon}(3 - r\sin\theta), \quad \sin\theta = \frac{1}{r}(3-\eta\varepsilon) = \frac{3-\eta\varepsilon}{5-\rho\varepsilon}$

$$\partial/\partial r = \frac{\partial\rho}{\partial r}\frac{\partial}{\partial\rho} + \frac{\partial\eta}{\partial r}\frac{\partial}{\partial\eta} = -\frac{1}{\varepsilon}\partial/\partial\rho - \frac{1}{\varepsilon}\sin\theta\,\partial/\partial\eta$$

$$\partial^2/\partial r^2 = \partial/\partial r\left[-\frac{1}{\varepsilon}\left(\partial/\partial\rho + \sin\theta\,\partial/\partial\eta\right)\right]$$

$$= \frac{1}{\varepsilon^2}\left(\partial^2/\partial\rho^2 + \sin\theta\frac{\partial^2}{\partial\rho\partial\eta}\right) + \frac{1}{\varepsilon^2}\sin\theta\left(\frac{\partial^2}{\partial\rho\partial\eta} + \sin\theta\,\partial^2/\partial\eta^2\right)$$

$$= \frac{1}{\varepsilon^2}\partial^2/\partial\rho^2 + 2\frac{1}{\varepsilon^2}\sin\theta\frac{\partial^2}{\partial\rho\partial\eta} + \frac{1}{\varepsilon^2}\sin^2\theta\,\partial^2/\partial\eta^2$$

$$\partial/\partial\theta = \frac{\partial\rho}{\partial\theta}\partial/\partial\rho + \frac{\partial\eta}{\partial\theta}\partial/\partial\eta = -\frac{1}{\varepsilon}r\cos\theta\,\partial/\partial\eta$$

$$\partial^2/\partial\theta^2 = \frac{1}{\varepsilon^2}r^2\cos^2\theta\,\partial^2/\partial\eta^2$$

$$\to \varepsilon\Big(\frac{1}{\varepsilon^2}u_{\rho\rho} + \frac{2}{\varepsilon^2}\sin\theta\,u_{\rho\eta} + \frac{1}{\varepsilon^2}\sin^2\theta\,u_{\eta\eta} - \frac{1}{5-\rho\varepsilon}\frac{1}{\varepsilon}u_\rho - \frac{1}{5-\rho\varepsilon}\frac{1}{\varepsilon}\sin\theta\,u_\eta + \frac{1}{(5-\rho\varepsilon)^2}\frac{1}{\varepsilon^2}(5-\rho\varepsilon)^2\cos^2\theta\,u_{\eta\eta}\Big)$$

$$= -\cos\theta\frac{1}{\varepsilon}(u_\rho + \sin\theta\,u_\eta) + \frac{1}{(5-\rho\varepsilon)}\sin\theta\frac{1}{\varepsilon}(5-\rho\varepsilon)\cos\theta\,u_\eta$$

CLC

$$\to \frac{1}{\varepsilon} u_{\rho\rho} + \frac{2}{\varepsilon}\left(\frac{3-\eta\varepsilon}{5-\rho\varepsilon}\right) u_{\rho\eta} + \frac{1}{\varepsilon}\left(\frac{3-\eta\varepsilon}{5-\rho\varepsilon}\right)^2 u_{\eta\eta} - \frac{1}{5-\rho\varepsilon} u_\rho$$
$$- (3-\eta\varepsilon) u_\eta + \frac{1}{\varepsilon}\left(1-\left(\frac{3-\eta\varepsilon}{5-\rho\varepsilon}\right)^2\right) u_{\eta\eta}$$
$$= -\sqrt{1-\left(\frac{3-\eta\varepsilon}{5-\rho\varepsilon}\right)^2}\,\frac{1}{\varepsilon}\left(u_\rho + \frac{3-\eta\varepsilon}{5-\rho\varepsilon} u_\eta\right) + \frac{1}{\varepsilon}\,\frac{3-\eta\varepsilon}{5-\rho\varepsilon}\sqrt{1-\left(\frac{3-\eta\varepsilon}{5-\rho\varepsilon}\right)^2}\, u_\eta$$

$$\to \frac{1}{\varepsilon}(5-\rho\varepsilon)^2 u_{\rho\rho} + \frac{2}{\varepsilon}(5-\rho\varepsilon)(3-\eta\varepsilon) u_{\rho\eta} + \frac{1}{\varepsilon}(3-\eta\varepsilon) u_{\eta\eta}$$
$$-(5-\rho\varepsilon) u_\rho - (5-\rho\varepsilon)^2(3-\eta\varepsilon) u_\eta + \frac{1}{\varepsilon}\left[(5-\rho\varepsilon)^2-(3-\eta\varepsilon)^2\right] u_{\eta\eta}$$
$$= -\sqrt{(5-\rho\varepsilon)^2-(3-\eta\varepsilon)^2}\,\frac{1}{\varepsilon}\left((5-\rho\varepsilon)u_\rho + (3-\eta\varepsilon)u_\eta\right)$$
$$+ \frac{1}{\varepsilon}(3-\eta\varepsilon)\sqrt{(5-\rho\varepsilon)^2-(3-\eta\varepsilon)^2}\, u_\eta$$

gather $O(\frac{1}{\varepsilon})$ terms:

$$25 u_{\rho\rho} + 30 u_{\rho\eta} + 3 u_{\eta\eta} - 5u_\rho - 75 u_\eta$$
$$+ 16 u_{\eta\eta} = -20 u_\rho - 12 u_\eta$$
$$\to 25 u_{\rho\rho} + 30 u_{\rho\eta} + 19 u_{\eta\eta} = -15 u_\rho + 63 u_\eta$$

need $u|_{\rho=0} = 0$, $u|_{\eta=0} = 1$

CLD

Near $r=5$, $\theta=0$ we expect that BLC won't match $u=0$ at $r=5$. $\to$ need a corner layer

$\rho = \frac{5-r}{\varepsilon}$, $t = \frac{\theta}{\sqrt{\varepsilon}}$

$$\to \varepsilon\left(\frac{1}{\varepsilon^2} u_{\rho\rho} + \frac{1}{(5-\rho\varepsilon)}\left(\frac{-1}{\varepsilon}\right) u_\rho + \frac{1}{(5-\rho\varepsilon)^2}\frac{1}{\varepsilon} u_{tt}\right)$$
$$= \cos(t\sqrt{\varepsilon})\left(\frac{-1}{\varepsilon}\right) u_\rho - \frac{1}{(5-\rho\varepsilon)}\sin(t\sqrt{\varepsilon})\frac{1}{\sqrt{\varepsilon}} u_t$$
$$\to \frac{1}{\varepsilon}(5-\rho\varepsilon)^2 u_{\rho\rho} - (5-\rho\varepsilon) u_\rho + u_{tt}$$
$$= -\frac{1}{\varepsilon}(5-\rho\varepsilon)^2 \cos(t\sqrt{\varepsilon}) u_\rho - \frac{1}{\sqrt{\varepsilon}}(5-\rho\varepsilon)\sin(t\sqrt{\varepsilon}) u_t$$
$$\to 25 u_{\rho\rho} = -25 u_\rho \to u_{\rho\rho} = -u_\rho$$

need $u|_{\rho=0} = 0$, $u|_{\rho\to\infty} \to u_{BLC}|_{r=5}$

CLE

We then expect that CLD won't match $u=2$ at $\theta=0$. $\to$ need another corner.

$\rho = \frac{5-r}{\varepsilon}$, $t = \theta/\varepsilon$

$$\to \varepsilon\left(\frac{1}{\varepsilon^2} u_{\rho\rho} + \frac{1}{5-\rho\varepsilon}\left(-\frac{1}{\varepsilon}\right) u_\rho + \frac{1}{(5-\rho\varepsilon)^2}\frac{1}{\varepsilon^2} u_{tt}\right)$$
$$= \cos t\varepsilon\left(-\frac{1}{\varepsilon}\right) u_\rho - \frac{1}{5-\rho\varepsilon}\sin t\varepsilon\,\frac{1}{\varepsilon} u_t$$
$$\to 25 u_{\rho\rho} + u_{tt} = -25 u_\rho$$

need $u|_{\rho=0} = 0$, $u|_{t=0} = 2$

There also seems to be a problem near $r=3,\ \theta=\pi/2$. It is not likely that BLA, BLD, and the outer solution $u=0$ will match there. I'm not entirely sure how this should be fixed, but here is a guess:

CLF?

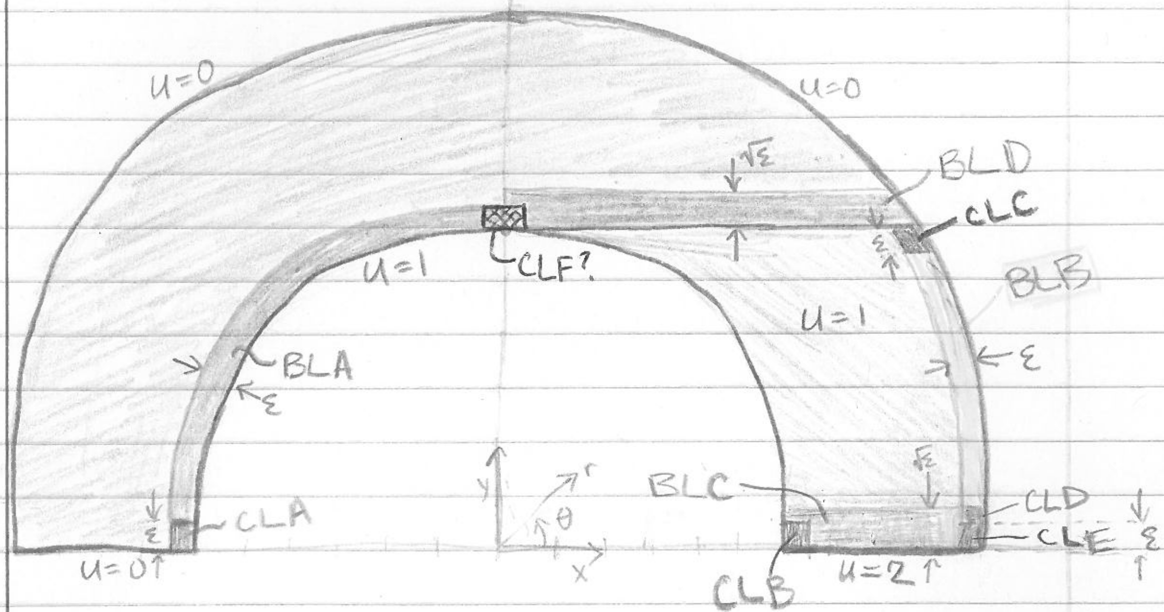
$\xi = \frac{x}{\varepsilon}$, at $\xi=0$, $u=1$. as $\xi\to\infty$, $u\to 0$

$\eta = \frac{y-3}{\varepsilon}$, as $\eta\to\infty$, $u\to \text{BLA}|_{\theta=\pi/2}$

$\eta\to-\infty$, $u\to \text{BLD}|_{x=0}$

$\varepsilon\left(\frac{1}{\varepsilon^2}u_{\xi\xi}+\frac{1}{\varepsilon^2}u_{\eta\eta}\right)=\frac{1}{\varepsilon}u_\xi$

$\to\quad u_{\xi\xi}+u_{\eta\eta}=u_\xi$



Name	Width	Leading order BVP	BC
$\boxed{\text{BLA}}$	ε	$u_{\rho\rho} = \cos u_\rho$	$u\vert_{\rho=0}=1,\ u\vert_{\rho\to\infty}\to 0$
$\boxed{\text{BLB}}$	ε	$u_{\rho\rho} = -\cos u_\rho$	$u\vert_{\rho=0}=0,\ u\vert_{\rho\to\infty}\to 1$
$\boxed{\text{BLC}}$	$\sqrt{\varepsilon}$	$u_{tt}+rtu_t=r^2u_r$	$u\vert_{t=0}=2,\ u\vert_{t\to\infty}\to 1,\ u\vert_{r=3}=1$
$\boxed{\text{BLD}}$	$\sqrt{\varepsilon}$	$u_{\eta\eta}=u_x$	$u\vert_{\eta=0}=1,\ u\vert_{\eta\to\infty}\to 0,\ u\vert_{x=0}=0$
$\boxed{\text{CLA}}$	$\varepsilon\times\varepsilon$	$9u_{\rho\rho}+u_{tt}=-9u_\rho$	$u\vert_{\rho=0}=1,\ u\vert_{t=0}=0$
$\boxed{\text{CLB}}$	$\varepsilon\times\varepsilon$	$9u_{\rho\rho}+u_{tt}=9u_\rho$	$u\vert_{\rho=0}=1,\ u\vert_{t=0}=2$
$\boxed{\text{CLC}}$	$\varepsilon\times\varepsilon$	$25u_{\rho\rho}+30u_{\rho\eta}+19u_{\eta\eta}=-15u_\rho+63u_\eta$	$u\vert_{\rho=0}=0,\ u\vert_{\eta=0}=1$
$\boxed{\text{CLD}}$	$\varepsilon\times\sqrt{\varepsilon}$	$u_{\rho\rho}=-u_\rho$	$u\vert_{\rho=0}=0,\ u\vert_{\rho\to\infty}\to u_{BLC}\vert_{r=5}$
$\boxed{\text{CLE}}$	$\varepsilon\times\varepsilon$	$25u_{\rho\rho}+u_{tt}=-25u_\rho$	$u\vert_{\rho=0}=0,\ u\vert_{t=0}=2$

CLF $\varepsilon\times\varepsilon$ $u_{\xi\xi}+u_{\eta\eta}=u_\xi$ $u|_{\xi=0}=1,\ u|_{\xi\to\infty}\to 0$

$u|_{\eta\to\infty}\to BLA|_{\theta=\pi/2}$

$u|_{\eta\to-\infty}\to BLD|_{x=0}$

ES_APPM 420-3 "Asymptotic and Perturbation Methods"

Homework 3 (DUE TUESDAY, 5/12/2009, IN CLASS)

Problem 1. Consider the initial value problem for Burgers' equation

$$u_t + uu_x = \varepsilon u_{xx}, \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}.$$

(a). Use the Cole-Hopf transformation to find the exact solution.
(b). Determine the behavior of the exact solution for small ε.

x_0: pt of disc'ty, u_1: value to left of x_0, u_2: value to right of x_0

$\bar{x} = \dfrac{x - x_0 - \frac{1}{2}t(u_1+u_2)}{2/|u_1-u_2|}$; $\bar{u} = \dfrac{2u-(u_1+u_2)}{|u_1-u_2|}$

$= \dfrac{x - \frac{1}{2}t}{2} = \frac{1}{2}x - \frac{1}{4}t$; $= 2u - 1$; $2u - 1 \to u = \frac{1}{2}\bar{u} + \frac{1}{2}$

$\frac{1}{2}x = \bar{x} + \frac{1}{4}t$

$x = 2\bar{x} + \frac{1}{2}t$

$\to \frac{1}{2}\bar{u}_t + (\frac{1}{2}\bar{u} + \frac{1}{2})\frac{1}{2}\bar{u}_x = \frac{1}{2}\varepsilon \bar{u}_{xx}$

$\bar{u}_t + \frac{1}{2}(\bar{u}+1)\bar{u}_x = \varepsilon u_{xx}$

$\bar{u}_t + \frac{1}{2}(\bar{u}+1)\bar{u}_{\bar{x}}(\frac{1}{2} - \frac{1}{4}) = \varepsilon u_{xx}$

$\frac{1}{4}\bar{u}_{\bar{t}} + \frac{1}{2}(\bar{u}+1)(2\bar{u}_{\bar{x}} + \frac{1}{8}\bar{u}_{\bar{t}}) = \varepsilon(4\bar{u}_{\bar{x}\bar{x}} + \frac{1}{4}u_{\bar{x}\bar{t}} + \frac{1}{4}u_{\bar{x}\bar{t}} + \frac{1}{64}u_{\bar{t}\bar{t}})$

$\bar{t} = \dfrac{t}{4/(u_1-u_2)^2} = \dfrac{t}{4}$

$\dfrac{\partial}{\partial t} = \dfrac{\partial \bar{t}}{\partial t}\dfrac{\partial}{\partial \bar{t}}$

$\dfrac{\partial}{\partial x} = \dfrac{\partial \bar{x}}{\partial x}\dfrac{\partial}{\partial \bar{x}}$

start with $v_t = \varepsilon v_{xx}$

$(\frac{1}{2}\bar{u} + \frac{1}{2})_t = \varepsilon(\frac{1}{2}\bar{u} + \frac{1}{2})_{xx}$

$\frac{1}{2}\bar{u}_t = \frac{1}{2}\varepsilon \bar{u}_{xx}$

$\frac{1}{8}\bar{u}_{\bar{t}} = \frac{1}{2}\varepsilon($

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$$u_t + uu_x = \varepsilon u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

First, we use the following change of variables:

$$\bar{x} = \tfrac{1}{2}x - \tfrac{1}{4}t, \quad \bar{t} = \tfrac{1}{4}t, \quad \bar{u} = 2u - 1$$

plug into equation:

$$\left(\tfrac{1}{2}\bar{u}(\bar{x},\bar{t}) + \tfrac{1}{2}\right)_t + \left(\tfrac{1}{2}\bar{u}(\bar{x},\bar{t}) + \tfrac{1}{2}\right)\left(\tfrac{1}{2}\bar{u}(\bar{x},\bar{t}) + \tfrac{1}{2}\right)_x = \varepsilon\left(\tfrac{1}{2}\bar{u}(\bar{x},\bar{t}) + \tfrac{1}{2}\right)_{xx}$$

$$\tfrac{1}{2}\left[\frac{\partial\bar{x}}{\partial t}\bar{u}_{\bar{x}} + \frac{\partial\bar{t}}{\partial t}\bar{u}_{\bar{t}}\right] + \tfrac{1}{2}(\bar{u}+1)\tfrac{1}{2}\left[\frac{\partial\bar{x}}{\partial x}u_{\bar{x}} + \frac{\partial\bar{t}}{\partial x}u_{\bar{t}}\right] = \varepsilon\tfrac{1}{2}\left(\frac{\partial\bar{x}}{\partial x}\frac{\partial}{\partial\bar{x}} + \frac{\partial\bar{t}}{\partial x}\frac{\partial}{\partial\bar{t}}\right)\left[\frac{\partial\bar{x}}{\partial x}\bar{u}_{\bar{x}} + \frac{\partial\bar{t}}{\partial x}\bar{u}_{\bar{t}}\right]$$

$$-\tfrac{1}{4}\bar{u}_{\bar{x}} + \tfrac{1}{4}\bar{u}_{\bar{t}} + (\bar{u}+1)\tfrac{1}{4}\bar{u}_{\bar{x}} = \tfrac{1}{4}\varepsilon\bar{u}_{\bar{x}\bar{x}}$$

new problem
$$\begin{cases} \bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} = \varepsilon\bar{u}_{\bar{x}\bar{x}} \\ \bar{u}(\bar{x},0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} \end{cases}$$

A) Cole-Hopf transformation

let $\bar{u}(\bar{x},\bar{t}) = -2\varepsilon\dfrac{v_x(x,t)}{v(x,t)}$

plug into new problem

$$-2\varepsilon\frac{v_{xt}}{v} + 2\varepsilon\frac{v_x v_t}{v^2} + 4\varepsilon^2\frac{v_{xx}v_x}{v^2} - 4\varepsilon^2\frac{v_x^3}{v^3} = -2\varepsilon^2\frac{v_{xxx}}{v} + 6\varepsilon^2\frac{v_{xx}v_x}{v^2} - 4\varepsilon^2\frac{v_x^3}{v^3}$$

$$\rightarrow \frac{1}{v}(-v_{xt} + \varepsilon v_{xxx}) + \frac{v_x}{v^2}(v_t - \varepsilon v_{xx}) = 0$$

$$\rightarrow \partial/\partial x\left[-\frac{1}{v}(v_t - \varepsilon v_{xx})\right] = 0$$

$$\frac{1}{v}(v_t - \varepsilon v_{xx}) = c(t)$$

$$v_t - \varepsilon v_{xx} = c(t)\,v$$

let $v = e^{\int c(t)dt}\,\omega$

$$\rightarrow \omega_t - \varepsilon\omega_{xx} = 0$$

$$\bar{u}(\bar{x},\bar{t}) = -2\varepsilon\frac{v_x(x,t)}{v(x,t)} = -2\varepsilon\frac{e^{\int c(t)dt}\,\omega_x(x,t)}{e^{\int c(t)dt}\,\omega(x,t)} = -2\varepsilon\frac{\omega_x}{\omega}$$

Initial condition: say $w(x,0) = h(x)$, $\bar{u}(\bar{x},0) = f(x)$

then $f(x) = -2\varepsilon \dfrac{h'(x)}{h(x)}$

$$\rightarrow \ln(h(x)) = -\frac{1}{2\varepsilon}\int_0^x f(s)\,ds$$

$$\rightarrow h(x) = \exp\left(-\frac{1}{2\varepsilon}\int_0^x f(s)\,ds\right)$$

The solution to this form of differential equation is known:

$$w(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{\infty} h(s)\, e^{\frac{-(x-s)^2}{4\varepsilon t}}\,ds$$

$$\rightarrow \bar{u}(\bar{x},\bar{t}) = \frac{\int_{-\infty}^{\infty}\frac{(\bar{x}-s)}{\bar{t}}\, h(s)\, e^{\frac{-(\bar{x}-s)^2}{4\varepsilon\bar{t}}}\,ds}{\int_{-\infty}^{\infty} h(s)\, e^{\frac{-(\bar{x}-s)^2}{4\varepsilon\bar{t}}}\,ds}$$

and $f(x) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases} \rightarrow h(x) = \begin{cases} e^{\frac{x}{2\varepsilon}} & x<0 \\ e^{-\frac{x}{2\varepsilon}} & x>0 \end{cases}$

$$\rightarrow \bar{u} = \frac{\int_{-\infty}^{0}\frac{(\bar{x}-s)}{\bar{t}}\, e^{-\frac{(\bar{x}-s)^2}{4\varepsilon\bar{t}}+\frac{s}{2\varepsilon}}\,ds + \int_0^{\infty}\frac{\bar{x}-s}{\bar{t}}\, e^{-\frac{(\bar{x}-s)^2}{4\varepsilon\bar{t}}-\frac{s}{2\varepsilon}}\,ds}{\int_{-\infty}^{0} e^{-\frac{(\bar{x}-s)^2}{4\varepsilon\bar{t}}+\frac{s}{2\varepsilon}}\,ds + \int_0^{\infty} e^{-\frac{(\bar{x}-s)^2}{4\varepsilon\bar{t}}-\frac{s}{2\varepsilon}}\,ds}$$

$$= (I_1 + I_2)/(I_3 + I_4)$$

$$I_3 = \int_{-\infty}^{0}\exp\left(-\frac{1}{4\varepsilon\bar{t}}\left[s^2 - 2\bar{x}s - 2s\bar{t} + \bar{x}^2 + (\bar{t}+\bar{x})^2 - (\bar{t}+\bar{x})^2\right]\right)ds$$

$$= \exp\left(\frac{(\bar{t}+\bar{x})^2 - \bar{x}^2}{4\varepsilon\bar{t}}\right)\int_{-\infty}^{0}\exp\left(-\frac{1}{4\varepsilon\bar{t}}(s-\bar{t}-\bar{x})^2\right)ds$$

$$\text{let } \tau = \frac{1}{2\sqrt{\varepsilon\bar{t}}}(s-\bar{t}-\bar{x})$$

$$= 2\sqrt{\varepsilon\bar{t}}\,\exp\left(\frac{\bar{t}^2+2\bar{t}\bar{x}}{4\varepsilon\bar{t}}\right)\int_{-\infty}^{-\frac{1}{2\sqrt{\varepsilon\bar{t}}}(\bar{t}+\bar{x})} e^{-\tau^2}\,d\tau$$

$$= \sqrt{\pi\varepsilon\bar{t}}\,\exp\left(\frac{\bar{t}^2+2\bar{t}\bar{x}}{4\varepsilon\bar{t}}\right)\operatorname{erfc}\left(\frac{(\bar{t}+\bar{x})}{2\sqrt{\varepsilon\bar{t}}}\right)$$

I_4: let $s = -s$, $\bar{x} = -\bar{x}$, then $I_3 = +I_4$

$$= \sqrt{\pi\varepsilon\bar{t}}\,\exp\left(\frac{\bar{t}^2-2\bar{t}\bar{x}}{4\varepsilon\bar{t}}\right)\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right)$$

$$I_1 = \int_{-\infty}^{0}\frac{(\bar{x}-s)}{\bar{t}}\exp\left(-\frac{1}{4\varepsilon\bar{t}}\left[s^2 + s(-2\bar{t}-2\bar{x}) + \bar{x}^2 + (\bar{t}+\bar{x})^2 - (\bar{t}+\bar{x})^2\right]\right)ds$$

$$= \exp\left(\frac{\bar{t}^2+2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)\int_{-\infty}^{0}\left(\frac{\bar{x}}{\bar{t}} - \frac{s}{\bar{t}}\right)\exp\left(-\frac{1}{4\varepsilon\bar{t}}(s-\bar{t}-\bar{x})^2\right)ds$$

$$\text{let } \tau = \frac{1}{2\sqrt{\varepsilon\bar{t}}}(s-\bar{t}-\bar{x}) \rightarrow s = 2\tau\sqrt{\varepsilon\bar{t}} + \bar{t} + \bar{x}$$

$$= 2\sqrt{\varepsilon\bar{t}}\,\exp\left(\frac{\bar{t}^2+2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)\int_{-\infty}^{-\frac{(\bar{t}+\bar{x})}{2\sqrt{\varepsilon\bar{t}}}}\left(-2\tau\sqrt{\frac{\varepsilon}{\bar{t}}} - 1\right)\exp^{-\tau^2}\,d\tau$$

$$= -\sqrt{\pi\varepsilon\bar{t}}\,\exp\left(\frac{\bar{t}^2+2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)\operatorname{erfc}\left(\frac{\bar{t}+\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) + \underbrace{2\varepsilon\exp\left(\frac{\bar{t}^2+2\bar{x}\bar{t}-\bar{t}^2-2\bar{x}\bar{t}-\bar{x}^2}{4\varepsilon\bar{t}}\right)}_{=2\varepsilon\exp(-\bar{x}^2/4\varepsilon\bar{t})}$$

I_2: let $s = -s$, $\bar{x} = -\bar{x}$ then $I_2 = I_1$

$$= +\sqrt{\pi\varepsilon\bar{t}}\exp\left(\frac{\bar{t}^2-2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) - 2\varepsilon\exp\left(-\frac{\bar{x}^2}{4\varepsilon\bar{t}}\right)$$

$$\bar{u} = \frac{-\sqrt{\pi\varepsilon\bar{t}}\, e^{\left(\frac{\bar{t}^2+2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)}\operatorname{erfc}\left(\frac{\bar{t}+\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) + 2\varepsilon e^{\left(\frac{\bar{t}^2+2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)} e^{-\frac{(t+x)^2}{4\varepsilon t}} + \sqrt{\pi\varepsilon\bar{t}}\, e^{\left(\frac{\bar{t}^2-2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)}\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) - 2\varepsilon e^{\left(\frac{\bar{t}^2-2\bar{x}\bar{t}}{4\varepsilon\bar{t}}\right)} e^{-\frac{(\bar{t}-x)^2}{4\varepsilon\bar{t}}}}{-\sqrt{\pi\varepsilon\bar{t}}\, e^{\left(\frac{\bar{t}^2+2\bar{t}\bar{x}}{4\varepsilon\bar{t}}\right)}\operatorname{erfc}\left(\frac{\bar{t}+\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) - \sqrt{\pi\varepsilon\bar{t}}\, e^{\left(\frac{\bar{t}^2-2\bar{t}\bar{x}}{4\varepsilon\bar{t}}\right)}\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right)}$$

$$= \frac{-e^{-\bar{x}/\varepsilon}\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) + \operatorname{erfc}\left(\frac{\bar{t}+\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) + \cancel{\cdots}}{-e^{-\bar{x}/\varepsilon}\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) - \operatorname{erfc}\left(\frac{\bar{t}+\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right)}$$

For small ε,

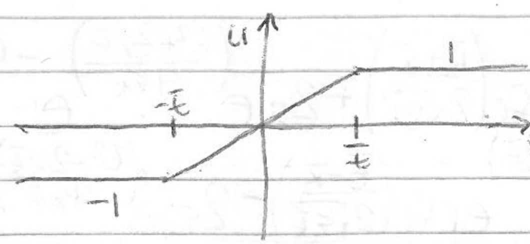
$$u \sim \frac{+e^{-\bar{x}/\varepsilon}\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) - \operatorname{erfc}\left(\frac{\bar{t}+\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right)}{+e^{-\bar{x}/\varepsilon}\operatorname{erfc}\left(\frac{\bar{t}-\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right) + \operatorname{erfc}\left(\frac{\bar{t}+\bar{x}}{2\sqrt{\varepsilon\bar{t}}}\right)}$$

for $x > t$:
$$u \sim \frac{2e^{-\bar{x}/\varepsilon} - \frac{2}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(\bar{t}+\bar{x})^2}{4\varepsilon t}}}{2e^{-\bar{x}/\varepsilon} + \frac{2}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(t+x)^2}{4\varepsilon t}}}$$
$$= \frac{1 - \frac{1}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{\frac{-t^2-2xt-x^2+4xt}{4\varepsilon t}}}{1 + \frac{1}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{(-t^2+2xt-x^2)/4\varepsilon t}}$$
$$= \frac{1 - \frac{1}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-(t-x)^2/4\varepsilon t}}{1 + \frac{1}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-(t-x)^2/4\varepsilon t}}$$
$$\sim 1 \text{ as } \varepsilon \to 0$$

for $x < t$:
$$\frac{e^{-\bar{x}/\varepsilon}\frac{2}{\bar{t}-\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(t-\bar{x})^2}{4\varepsilon\bar{t}}} - 2}{e^{-\bar{x}/\varepsilon}\frac{2}{\bar{t}-\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(\bar{t}-\bar{x})^2}{4\varepsilon\bar{t}}} + 2}$$
$$= \frac{\frac{2}{\bar{t}-\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{\frac{-\bar{t}^2+2\bar{x}\bar{t}-\bar{x}^2-4\bar{x}\bar{t}}{4\varepsilon\bar{t}}} - 2}{\frac{2}{\bar{t}-\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-(\bar{t}+\bar{x})^2/4\varepsilon t} + 2}$$
$$\sim -1 \text{ as } \varepsilon \to 0$$

for $-t < x < t$:
$$u \sim \frac{e^{-\bar{x}/\varepsilon}\frac{2}{\bar{t}-\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(t-x)^2}{4\varepsilon t}} - \frac{2}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(t+x)^2}{4\varepsilon t}}}{e^{-\bar{x}/\varepsilon}\frac{2}{\bar{t}-\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(t-x)^2}{4\varepsilon t}} + \frac{2}{\bar{t}+\bar{x}}\sqrt{\frac{\varepsilon\bar{t}}{\pi}}\, e^{-\frac{(t+x)^2}{4\varepsilon t}}}$$
$$= \frac{\exp\left(-4\bar{x}\bar{t} - \bar{t}^2 + 2\bar{t}\bar{x} - \bar{x}^2 + t^2 + 2\bar{t}\bar{x} + x^2\right)\frac{2}{\bar{t}-\bar{x}} - \frac{2}{\bar{t}+\bar{x}}}{\exp(0)\frac{2}{\bar{t}-\bar{x}} + \frac{2}{\bar{t}+\bar{x}}}$$

$$= \frac{\bar{t}+\bar{x} - \bar{t}+\bar{x}}{\bar{t}+\bar{x} + \bar{t}-\bar{x}} = \frac{\bar{x}}{\bar{t}}$$



Sharp corners near $\bar{x}=\bar{t}$: we know they should be smooth in reality.

let $\xi = \frac{\bar{x}-t}{\sqrt{\varepsilon}} \rightarrow \bar{x} = t + \xi\sqrt{\varepsilon}$

$$\bar{u} \sim \frac{e^{-\frac{1}{\varepsilon}(t+\xi\sqrt{\varepsilon})}\operatorname{erfc}\left(\frac{-\xi}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{2t+\xi\sqrt{\varepsilon}}{2\sqrt{\varepsilon t}}\right)}{e^{-\frac{1}{\varepsilon}t - \xi/\sqrt{\varepsilon}}\operatorname{erfc}\left(-\frac{\xi}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\sqrt{t}/\sqrt{\varepsilon} + \frac{\xi}{2\sqrt{t}}\right)}$$

$$= \frac{\operatorname{erfc}(-\xi/2\sqrt{t}) - \sqrt{\frac{\varepsilon}{t\pi}}\, e^{-\xi^2/4t}}{\operatorname{erfc}(-\xi/(2\sqrt{t})) + \sqrt{\varepsilon/t\pi}\, e^{-\xi^2/4t}}$$

$$\sim \frac{1 - \sqrt{\varepsilon/t\pi}\, e^{-\xi^2/4t}\, \frac{1}{\operatorname{erfc}(-\xi/2\sqrt{t})}}{1 + \sqrt{\varepsilon/t\pi}\, e^{-\xi^2/4t}\, \frac{1}{\operatorname{erfc}(-\xi/2\sqrt{t})}}$$

$$\sim \left(1 - \sqrt{\frac{\varepsilon}{t\pi}}\, \frac{e^{-\xi^2/4t}}{\operatorname{erfc}(-\xi/2\sqrt{t})}\right)^2$$

$$\sim 1 - \frac{2\sqrt{\varepsilon/\pi\bar{t}}\, e^{-\xi^2/4\bar{t}}}{\operatorname{erfc}(-\xi/2\sqrt{\bar{t}})}$$

$$= 1 - 2\sqrt{\frac{\varepsilon}{\pi\bar{t}}}\left(\frac{e^{-\frac{1}{4t}(\bar{x}-\bar{t})^2/\varepsilon}}{\operatorname{erfc}\left(\frac{-\bar{x}+\bar{t}}{2\sqrt{\varepsilon\bar{t}}}\right)}\right)$$

Another sharp corner near $\bar{x} = -\bar{t}$

let $\xi = \frac{\bar{x}+\bar{t}}{\sqrt{\varepsilon}} \rightarrow \bar{x} = \xi\sqrt{\varepsilon} - \bar{t}$

$$\tilde{u} \sim \frac{e^{-\frac{1}{\varepsilon}(\xi\sqrt{\varepsilon}-\bar{t})}\operatorname{erfc}\left(\frac{2\bar{t}-\xi\sqrt{\varepsilon}}{2\sqrt{t\varepsilon}}\right) - \operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}}\right)}{e^{-\frac{1}{\varepsilon}(\xi\sqrt{\varepsilon}-t)}\operatorname{erfc}\left(\frac{2\bar{t}-\xi\sqrt{\varepsilon}}{2\sqrt{t\varepsilon}}\right) + \operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}}\right)}$$

$$\sim \frac{\exp\left(-\frac{1}{\varepsilon}(\xi\sqrt{\varepsilon}-t) - \frac{1}{4t\varepsilon}(2t-\xi\sqrt{\varepsilon})^2\right)\sqrt{\frac{t\varepsilon}{\pi}}\left(\frac{2}{2t-\xi\sqrt{\varepsilon}}\right) - \operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}}\right)}{\exp\left(-\frac{1}{\varepsilon}(\xi\sqrt{\varepsilon}-t) - \frac{1}{4t\varepsilon}(4t^2+\xi^2\varepsilon-4t\xi\sqrt{\varepsilon})\right)\sqrt{\frac{t\varepsilon}{\pi}}\left(\frac{2}{2t-\xi\sqrt{\varepsilon}}\right) + \operatorname{erfc}\left(\xi/2\sqrt{t}\right)}$$

$$\sim \frac{\exp\left(-\frac{1}{4t\varepsilon}\left(\xi^2\varepsilon\right)\right)\sqrt{\frac{t\varepsilon}{\pi}}\left(\frac{2}{2t-\xi\sqrt{\varepsilon}}\right) - \operatorname{erfc}\left(\xi/2\sqrt{t}\right)}{\exp\left(-\xi^2/t4\right)\sqrt{\frac{\varepsilon}{\pi t}} + \operatorname{erfc}\left(\xi/2\sqrt{t}\right)}$$

$$= \frac{\frac{\exp\left(-\frac{\xi^2}{t4}\right)}{\operatorname{erfc}\left(\xi/2\sqrt{t}\right)}\sqrt{\frac{\varepsilon}{\pi t}} - 1}{\frac{\exp\left(-\xi^2/t4\right)}{\operatorname{erfc}\left(\xi/2\sqrt{t}\right)}\sqrt{\frac{\varepsilon}{\pi t}} + 1}$$

$$\sim \left(-1 + \sqrt{\frac{\varepsilon}{t\pi}}\,\frac{e^{-\xi^2/t4}}{\operatorname{erfc}\left(\xi/2\sqrt{t}\right)}\right)^2$$

$$\sim -1 + 2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{e^{-\xi^2/t4}}{\operatorname{erfc}\left(\xi/2\sqrt{t}\right)}$$

$$= -1 + 2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{e^{-\frac{1}{4\varepsilon t}(\bar{x}+\bar{t})^2}}{\operatorname{erfc}\left(\frac{\bar{x}+\bar{t}}{2\sqrt{t\varepsilon}}\right)}$$

Exact Solution in terms of original variables:

$$2u-1 = \frac{+e^{-\frac{1}{\varepsilon}\left(\frac{1}{2}x-\frac{1}{4}t\right)}\operatorname{erfc}\left(\frac{\frac{1}{4}t-\frac{1}{2}x+\frac{1}{4}t}{\sqrt{\varepsilon t}}\right)-\operatorname{erfc}\left(\frac{\frac{1}{4}t+\frac{1}{2}x-\frac{1}{4}t}{\sqrt{\varepsilon t}}\right)}{e^{-\frac{1}{\varepsilon}\left(\frac{1}{2}x-\frac{1}{4}t\right)}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right)+\operatorname{erfc}\left(\frac{x}{2\sqrt{\varepsilon t}}\right)}$$

$$\boxed{u = \frac{1}{2}+\frac{1}{2}\,\frac{e^{-\frac{1}{\varepsilon}\left(\frac{1}{2}x-\frac{1}{4}t\right)}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right)-\operatorname{erfc}\left(\frac{x}{2\sqrt{\varepsilon t}}\right)}{e^{-\frac{1}{\varepsilon}\left(\frac{1}{2}x-\frac{1}{4}t\right)}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right)+\operatorname{erfc}\left(\frac{x}{2\sqrt{\varepsilon t}}\right)}}$$

Behavior for small ε (smoothed) in terms of original variables: near $x_2 = t$

$$2u-1 \sim 1-2\sqrt{\frac{\varepsilon}{\frac{1}{4}\pi t}}\left(\frac{e^{-\frac{1}{t}\left(\frac{1}{2}x-\frac{1}{4}t-\frac{1}{4}t\right)^2/\varepsilon}}{\operatorname{erfc}\left(\frac{-\frac{1}{2}x+\frac{1}{4}t+\frac{1}{4}t}{2\sqrt{\varepsilon\frac{1}{4}t}}\right)}\right)$$

$$\boxed{u \sim 1-2\sqrt{\frac{\varepsilon}{\pi t}}\left(\frac{e^{-\frac{1}{4\varepsilon t}(x-t)^2}}{\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right)}\right)}$$

behavior different for $\bar{x} < -\bar{t} \rightarrow \frac{1}{2}x-\frac{1}{4}t < -\frac{1}{4}t \rightarrow x<0$

$\bar{x} > \bar{t} \rightarrow \frac{1}{2}x-\frac{1}{4}t > \frac{1}{4}t \rightarrow x>t$

and $0<x<t$

$x>t$: $u \sim 1-2\sqrt{\frac{\varepsilon}{\pi t}}\,\frac{e^{-\frac{1}{4\varepsilon t}(x-t)^2}}{2}$

~ 1 ✓

$x<t$: $u \sim 1-2\sqrt{\frac{\varepsilon}{\pi t}}\sqrt{\frac{\pi}{\varepsilon t}}\,\frac{(t-x)}{2}\,e^{\frac{(t-x)^2}{4\varepsilon t}-\frac{(x-t)^2}{4\varepsilon t}}$

$= 1-\frac{(t-x)}{t}\,e^{\frac{1}{4\varepsilon t}\left(t^2-2xt+x^2-x^2+2xt-t^2\right)}$

$= 1-\left(1-\frac{x}{t}\right)e^0$

$u = x/t$ ✓

Behavior for small ε near $x=0$:

$$2u-1 \sim -1+2\sqrt{\frac{\varepsilon}{\frac{1}{4}t\pi}}\,\frac{\exp\left(-\frac{1}{\varepsilon t}\left(\frac{1}{2}x-\frac{1}{4}t+\frac{1}{4}t\right)^2\right)}{\operatorname{erfc}\left(\frac{1}{\sqrt{\varepsilon t}}\left(\frac{1}{2}x-\frac{1}{4}t+\frac{1}{4}t\right)\right)}$$

$$\boxed{u \sim 1+2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{\exp(-x^2/4\varepsilon t)}{\operatorname{erfc}\left(\frac{1}{\sqrt{\varepsilon t}}\left(\frac{1}{2}x\right)\right)}}$$

$$x<0:\ 1+2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{\exp\left(-\frac{x^2}{4\varepsilon t}\right)}{2-\exp\left(\frac{1}{t\varepsilon}\left(\frac{1}{2}x\right)^2\right)\frac{2\sqrt{t\varepsilon}}{x\sqrt{\pi}}}$$

$$=1+2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{\exp\left(-\frac{1}{4\varepsilon t}(x^2+x^2)\right)}{2\exp\left(-\left(\frac{1}{4\varepsilon t}\right)x^2\right)-2\sqrt{t\varepsilon}/x\sqrt{\pi}}$$

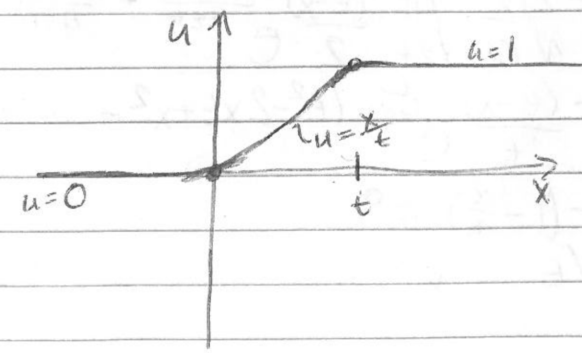
? (should $\to 0$)

$$x>0:\ 1+2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{x\sqrt{\pi}}{2\sqrt{t\varepsilon}}\,e^{-\frac{1}{4\varepsilon t}x^2+\frac{1}{4\varepsilon t}x^2}$$

$1+\frac{x}{t}$? (should $\to \frac{x}{t}$) ?

Overall:

$$u \sim \begin{cases} 2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{\exp(-x^2/4\varepsilon t)}{\operatorname{erfc}\left(\frac{x}{2\sqrt{\varepsilon t}}\right)} & x<t/2 \\ 1-2\sqrt{\frac{\varepsilon}{t\pi}}\,\frac{\exp\left(\frac{-(x-t)^2}{4\varepsilon t}\right)}{\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon t}}\right)} & x>t/2 \end{cases}$$



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ES_APPM 420-3 "Asymptotic and Perturbation Methods"

Midterm Examination (DUE ~~TUESDAY,~~ Thursday 5/21/09)

Problem 1. A flagpole oscillated at the base is described by the nondimensional equation

$$\frac{\partial^2 y}{\partial t^2} + \alpha^4 \frac{\partial^4 y}{\partial x^4} = 0, \qquad 0 < x < 1, \quad t > 0$$

for the nondimensional amplitude $y(x,t)$ of small transverse displacements. The solution is subject to the boundary conditions

$$y_{xx} = y_{xxx} = 0 \text{ at } x = 1; \qquad y = \cos t, \ y_x = 0 \text{ at } x = 0$$

and a condition of periodicity in time. Suppose that $\alpha \gg 1$, and use $\varepsilon = 1/\alpha^4$ as a small parameter. Find the solution correct to $O(\varepsilon)$ by a regular perturbation method.

Problem 2. Find the electric potential ϕ, satisfying $\nabla^2 \phi = 0$ between the two cylinders $r = a$, on which $\phi = 0$, and $r = b > a$, on which $\phi = V = const.$ Suppose that the inner cylinder is perturbed to $r = a(1 + \varepsilon \sin n\theta)$. Calculate ϕ correct to $O(\varepsilon)$.

look at old hwk

Problem 3. Consider the linear diffusion problem

similarity soln? see Diff Eq p. 315

$$u_t + \alpha u_x + \beta u = \varepsilon u_{xx}, \qquad -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = H(x) \equiv \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

Here α and β are positive constants and $0 < \varepsilon \ll 1$. Find the first terms in the inner and outer expansions of the solution and the composite solution.

Problem 4. Consider the boundary value problem

$$u_{xx} + \varepsilon u_{yy} - u_y = 0, \quad 0 < x < 1, \ 0 < y < 1,$$

$$u(0,y) = u(1,y) = 0, \ u(x,0) = 1, \ u(x,1) = 2.$$

(a). Make a sketch showing the location of all boundary layers and indicate the size of each of the boundary layer.
(b). Determine leading order in $\varepsilon \ll 1$ uniform solution.
Hint: You may want to use separation of variables when solving the outer problem.

Haley Lepo
5-21-09
ESAM 421-3
Midterm

1) $y_{tt} + \alpha^4 y_{xxxx} = 0$, $0 < x < 1$, $t > 0$

$y_{xx}(1,t) = y_{xxx}(1,t) = 0$, $y(0,t) = \cos t$, $y_x(0,t) = 0$

$\alpha \gg 1 \rightarrow \varepsilon = \frac{1}{\alpha^4} \ll 1$

$\rightarrow \varepsilon y_{tt} + y_{xxxx} = 0$

let $y \sim y_0 + y_1$

outer solution: $O(1)$:

$\rightarrow y_{0xxxx} = 0$

$\rightarrow y_0(x,t) = c_1(t) + c_2(t)x + c_3(t)x^2 + c_4(t)x^3$

$y_0(0,t) = c_1(t) = \cos t$

$y_{0x}(x,t) = c_2(t) + 2c_3(t)x + 3c_4(t)x^2$

$y_{0x}(0,t) = c_2(t) = 0$

$y_{0xx}(x,t) = 2c_3(t) + 6c_4(t)x$

$y_{0xx}(1,t) = 2c_3(t) + 6c_4(t) = 0$

$\rightarrow c_3(t) = -3c_4(t)$

$y_{0xxx}(x,t) = 6c_4(x)$

$y_{0xxx}(1,t) = 6c_4(x) = 0$

$\rightarrow y_0(x,t) = \cos t$

$O(\varepsilon)$: $y_{1xxxx} = -y_{0tt} = \cos t$

$y_{1xx}(1,t) = y_{1xxx}(1,t) = 0$, $y_1(0,t) = 0$, $y_{1x}(0,t) = 0$

$y_{1p} = Ax^4 \rightarrow y_{1pxxxx} = 24A = \cos t \rightarrow A = \frac{\cos t}{24}$

$y_1 = c_1(t) + c_2(t)x + c_3(t)x^2 + c_4(t)x^3 + \frac{1}{24}\cos t\, x^4$

$y_1(0,t) = c_1(t) = 0$

$y_{1x}(x,t) = c_2 + 2c_3 x + 3c_4 x^2 + \frac{1}{6}\cos t\, x^3$

$y_{1x}(0,t) = c_2 = 0$

$y_{1xx}(x,t) = 2c_3 + 6c_4 x + \frac{1}{2}\cos t\, x^2$

$y_{1xx}(1,t) = 2c_3 + 6c_4 + \frac{1}{2}\cos t = 0$

$\rightarrow c_3 = -3c_4 - \frac{1}{4}\cos t$

$y_{1xxx}(x,t) = 6c_4 + \cos t = 0$

$c_4 = -\frac{1}{6}\cos t$

$\rightarrow c_3 = \frac{1}{4}\cos t$

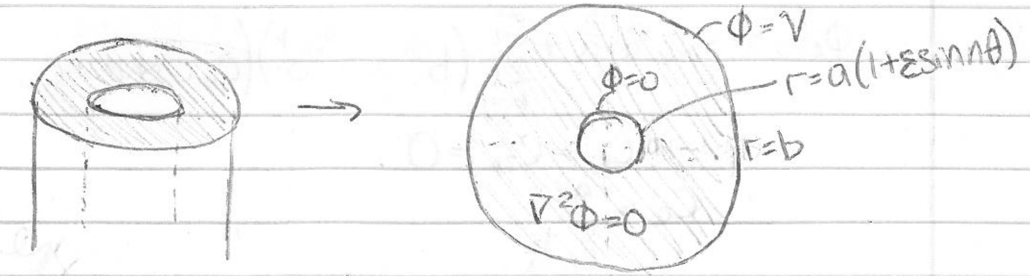
$y_1(x,t) = \cos t\left(\frac{1}{4}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4\right)$

$$y(x,t) \approx \cos t\left[1 + \varepsilon\left(\tfrac{1}{4}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4\right)\right]$$

$\checkmark$

Lepo 2

2)



$$\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0, \quad a(1+\varepsilon\sin n\theta) < r < b$$

$$\phi(b,\theta) = V$$

$$\phi(a(1+\varepsilon\sin n\theta),\theta) = 0$$

let $\phi \sim \phi_0 + \varepsilon\phi_1$

inner BC. becomes:

$$\phi(a+\varepsilon a\sin n\theta,\theta) = \phi_0(a) + \phi_0'(a)\varepsilon a\sin n\theta + \varepsilon\phi_1(a) + O(\varepsilon^2) = 0$$

$O(1)$: $\phi_{0rr} + \frac{1}{r}\phi_{0r} + \frac{1}{r^2}\phi_{0\theta\theta} = 0,$

$\phi_0(a,\theta) = 0, \quad \phi_0(b,\theta) = V$

Assume $\phi = R\Theta$

$$\rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} = \frac{-\Theta''}{\Theta} = \lambda$$

$\rightarrow \Theta'' + \lambda\Theta = 0$, 2π-periodic.

$\Theta = c_1\cos k\theta + c_2\sin k\theta, \quad \lambda = k^2, \quad = 0,1,2,\dots$

$\rightarrow r^2R'' + rR' - k^2R = 0$

$R = c_3r^k + c_4r^{-k}$ for $k \neq 0$

$= c_5 + c_6\ln r$ for $k = 0$

$$\rightarrow \phi_0 = c_5 + c_6\ln r + \sum_{k=1}^{\infty}(c_{3k}r^k + c_{4k}r^{-k})(c_{1k}\cos k\theta + c_{2k}\sin k\theta)$$

$$\phi_0(a,\theta) = c_5 + c_6\ln a + \sum_{k=1}^{\infty}(c_{3k}a^k + c_{4k}a^{-k})(c_{1k}\cos k\theta + c_{2k}\sin k\theta) = 0$$

$\rightarrow c_5 = -c_6\ln a, \quad c_{3k} = -c_{4k}a^{-2k}$ (absorb c_4 into c_1 + c_2)

$$\rightarrow \phi_0 = c_6\ln\frac{r}{a} + \sum_{k=1}^{\infty}(-a^{-2k}r^k + r^{-k})(c_{1k}\cos k\theta + c_{2k}\sin k\theta)$$

$$\phi_0(b,\theta) = c_6 \ln\frac{b}{a} + \sum_{k=1}^{\infty}\left(b^{-k} - a^{-2k} b^{k}\right)\left(c_{1k}\cos k\theta + c_{2k}\sin k\theta\right) = V$$

$$\to c_{1k} = c_{2k} = 0$$

$$c_6 = \frac{V}{\ln\frac{b}{a}}$$

$$\to \phi_0(r,\theta) = \frac{V \ln\frac{r}{a}}{\ln\frac{b}{a}}$$

sorry for the mess. Please see next page.

$$\cancel{O(\epsilon):\ \phi_{1,rr} + \frac{1}{r}\phi_{1,r} + \frac{1}{r^2}\phi_{1,\theta\theta} = 0}$$

$$\cancel{\phi_1(a,\theta) = -\phi_0'(a)\, a \sin n\theta}$$

$$\cancel{\phi_1(b,\theta) = 0}$$

$$\cancel{\phi_1 = c_5 + c_6 \ln r + \sum_{k=1}^{\infty}\left(c_{3k} r^{k} + c_{4k} r^{-k}\right)\left(c_{1k}\cos k\theta + c_{2k}\sin k\theta\right)}$$

$$\cancel{\phi_1(a,\theta) = c_5 + c_6 \ln a + \sum\left(c_{3k} a^{k} + c_{4k} a^{-k}\right)\left(c_{1k}\cos k\theta + c_{2k}\sin k\theta\right) = -\frac{V a\, a \sin n\theta}{a \ln\frac{b}{a}}}$$

$$\cancel{\to c_5 = -c_6 \ln a,\ c_{1k} = 0,\ c_{2k} = 0 \text{ for } k \neq n}$$

$$\cancel{\to \left(c_{3n} a^{n} + c_{4n} a^{-n}\right)\sin n\theta = -\frac{V a \sin n\theta}{\ln\frac{b}{a}}}$$

$$\cancel{\to c_{3n} = -\frac{V a^{1-n}}{\ln\frac{b}{a}} - c_{4n} a^{-2n}}$$

$$\cancel{\phi_1 = c_6 \ln\frac{r}{a} - \frac{V a^{1-n} r^{n}}{\ln\frac{b}{a}} + c_{4n}\left(a^{-2n} r^{n} + r^{-n}\right)\sin n\theta}$$

$$\cancel{\phi_1(b,\theta) = c_6 \ln\frac{b}{a} - \frac{V a^{1-n} b^{n}}{\ln\frac{b}{a}} + c_{4n}\left(a^{-2n} b^{n} + b^{-n}\right)\sin n\theta = 0}$$

$$\cancel{\to c_{4n} = 0}$$

$$\cancel{c_6 = \frac{V a^{1-n} b^{n}}{\left(\ln\frac{b}{a}\right)^2}}$$

$$\cancel{\phi_1 = \frac{V a^{1-n}}{\ln\frac{b}{a}}\left(\frac{\ln\frac{r}{a}}{\ln\frac{b}{a}} b^{n} - r^{n}\right)}$$

$$\cancel{\phi(r,\theta) = \frac{V}{\ln\frac{b}{a}}\left[\ln\frac{r}{a} + \epsilon a^{1-n}\left(\frac{\ln\frac{r}{a}}{\ln\frac{b}{a}} b^{n} - r^{n}\right)\right]}$$

$$\cancel{\begin{matrix} ab \sin n\theta \\ c_{3n} + c_{4n} \\ \text{for } k = n \end{matrix}}$$

$$I_3(x) = \int_0^1 \sin\left[x\left(t + \tfrac{1}{6}t^3 - \sinh t\right)\right] dt$$
$$= \operatorname{Im}\int_0^1 e^{ix(t + \frac{1}{6}t^3 - \sinh t)}\, dt$$

$\psi = t + \frac{1}{6}t^3 - \sinh t$

$\psi' = 1 + \frac{1}{2}t^2 - \cosh t = 0$

$\rightarrow \cosh t = 1 + \frac{1}{2}t^2$

$\rightarrow t = 0$

$\psi'' = t - \sinh t \qquad \psi''(0) = 0$

$\psi''' = 1 - \cosh t \qquad \psi'''(0) = 0$

$\psi^{(4)} = -\sinh t \qquad \psi^{(4)}(0) = 0$

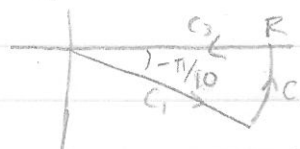
$\psi^{(5)} = -\cosh t \qquad \psi^{(5)}(0) = -1$

$I_3(x) \sim \operatorname{Im}\int_0^\varepsilon e^{-ix\frac{1}{120}t^5}\, dt \qquad$ let $s^5 = \frac{x}{120}t^5$

$= \left(\frac{120}{x}\right)^{\frac{1}{5}} \operatorname{Im}\int_0^{(\frac{x}{120})^{\frac{1}{5}}\varepsilon} e^{-is^5}\, ds \qquad s = \left(\frac{x}{120}\right)^{\frac{1}{5}} t \qquad dt = \left(\frac{120}{x}\right)^{\frac{1}{5}} ds$

$\sim \left(\frac{120}{x}\right)^{\frac{1}{5}} \operatorname{Im}\int_0^\infty e^{-is^5}\, ds \qquad$ if we let $\varepsilon x^{\frac{1}{5}} \to \infty$, so let $\varepsilon = x^{-\frac{1}{6}}$

Consider $\oint_C e^{-is^5}\, ds$ where $C = C_1 \cup C_2 \cup C_3$



as $R \to \infty$

Then $\int_0^\infty e^{-is^5} = \int_{C_1} + \int_{C_2}$

C_1: $s = e^{-i\pi/10} r \qquad ds = e^{-i\pi/10} dr, \quad r: 0 \to R$

$\int_{C_1} e^{-is^5} ds = \int_0^R \exp\left(-i\left(e^{-i\pi/10} r\right)^5\right) e^{-i\pi/10} dr$

$= e^{-i\pi/10} \int_0^R \exp\left(-i\, e^{-i\pi/2} r^5\right) dr$

$= e^{-i\pi/10} \int_0^\infty e^{-r^5} dr \qquad$ let $\tau = r^5 \quad r = \tau^{\frac{1}{5}} \quad dr = \frac{1}{5}\tau^{-4/5} d\tau$

$= e^{-i\pi/10} \frac{1}{5} \int_0^\infty \tau^{-4/5} e^{-\tau} d\tau$

$= \frac{1}{5} e^{-i\pi/10}\, \Gamma\left(\frac{1}{5}\right)$

C_2: $s = Re^{i\theta}, \quad ds = iRe^{i\theta} d\theta \quad \theta: -\pi/10 \to 0$

$\int_{C_2} e^{-is^5} ds = \int_{-\pi/10}^0 \exp\left(-i\left(Re^{i\theta}\right)^5\right) iRe^{i\theta} d\theta$

$= iR\int_{-\pi/10}^0 \exp\left(-iR^5(\cos 5\theta + i\sin 5\theta)\right) e^{i\theta} d\theta$

magnitude of this $\leq R\int_{-\pi/10}^0 e^{R^5 \sin 5\theta} d\theta$

$\varphi = \sin 5\theta \quad \varphi' = 5\cos 5\theta \rightarrow$ max at $\theta = 0 \rightarrow \varphi \sim 5\theta$

$\sim R\int_{-\varepsilon}^0 e^{R^5 5\theta} d\theta \qquad u = R^5 5\theta \quad du = 5R^5 d\theta$

$= 5R^{-4} \int_{-\varepsilon R^5 5}^0 e^u du \sim 5R^{-4} e^u \Big|_{-\infty}^0 = 5/R^4 \to 0$ as $R \to \infty$

So $\int_0^\infty e^{is^5}\,ds = \frac{1}{5} e^{-i\pi/10}\Gamma\left(\frac{1}{5}\right)$

Thus $I_3(x) \sim \left(\frac{120}{x}\right)^{\frac{1}{5}} \mathrm{Im}\left[\frac{1}{5} e^{-i\pi/10}\Gamma\left(\frac{1}{5}\right)\right]$

$$= \left(\frac{120}{x}\right)^{\frac{1}{5}} \frac{1}{5}\Gamma\left(\frac{1}{5}\right) \mathrm{Im}\left(\cos\pi/10 - i\sin\pi/10\right)$$

$$= \frac{-1}{5}\left(\frac{120}{x}\right)^{\frac{1}{5}}\Gamma\left(\frac{1}{5}\right)\sin\pi/10$$

✓

$I_4(x) = \int_0^1 \cos(xt)\tan t\, dt$

$= \mathrm{Re} \int_0^1 e^{ixt} \tan t\, dt$

$\psi = t \quad , \quad \psi'(t) = 1 \quad \rightarrow \psi' \neq 0 \text{ in } [0,1]$

$f = \tan t \quad , \quad f(0) = 0 \quad , \quad f(1) = \tan 1 \neq 0$

So we can do integration by parts.

$u = \tan t \qquad v = \frac{1}{ix} e^{ixt}$

$du = \sec^2 t \qquad dv = e^{ixt}$

$I_4(x) = \mathrm{Re}\left[\tan t \frac{1}{ix} e^{ixt} \Big|_0^1 - \frac{1}{ix} \int_0^1 \sec^2 t\, e^{ixt}\, dt \right]$

$= \mathrm{Re}\left[\tan 1 \frac{1}{ix} e^{ix} - \frac{1}{ix} \int_0^1 \sec^2 t\, e^{ixt}\, dt \right.$

If we continue doing integration by parts on the remaining integral, each time we will get another factor of $\frac{1}{x}$. Thus the terms get progressively smaller (since $x \to \infty$)

$I_4(x) \sim \mathrm{Re}\left[\tan(1) \left(-\frac{i}{x}\right)(\cos x + i \sin x) \right]$

$= \mathrm{Re}\left[\tan(1) \left(\frac{1}{x}\right)(-i\cos x + \sin x) \right]$

$= \underline{\dfrac{\tan(1) \sin x}{x}}$ ✓

$I_5(x) = \int_{-4}^{4} (t+2) e^{xt(t^2-12)} dt$

$\varphi = t(t^2-12) = t^3 - 12t$

$\varphi' = 3t^2 - 12 = 0$

$\rightarrow 3t^2 = 12$

$t^2 = 4$

$t = \pm 2$

$\varphi'' = 6t \rightarrow \varphi''(2) = 12 > 0 \rightarrow$ minimum

$\varphi''(-2) = -12 < 0 \rightarrow$ maximum

$\varphi(-4) = -16, \ \varphi(4) = 16, \ \varphi(-2) = 16$

So possible major contributions come from near $t=4$ and $t=-2$

near $t=4$:

$f \sim 6$

$\varphi \sim 16 + 36(t-4)$

$\rightarrow \sim 6 \int_{4-\varepsilon}^{4} e^{x(16+36(t-4))} dt$ let $\tau = t-4$

$\rightarrow 6e^{16x} \int_{-\varepsilon}^{0} e^{36x\tau} dt$ let $s = 36x\tau$

$\rightarrow 6e^{16x} \frac{1}{36x} \int_{-36\varepsilon s}^{0} e^s ds$ let $\varepsilon = x^{-1/2}$

$\sim \frac{1}{6x} e^{16x} \ e^s \Big|_{-\infty}^{0}$

$= \frac{1}{6x} e^{16x}$

near $t=-2$

$f \sim 0 + (t+2)$

$\varphi \sim 16 - 6(t+2)^2$

$\rightarrow \sim \int_{-2-\varepsilon}^{-2+\varepsilon} (t+2) e^{x(16-6(t+2))^2} dt$ let $\tau = t+2$

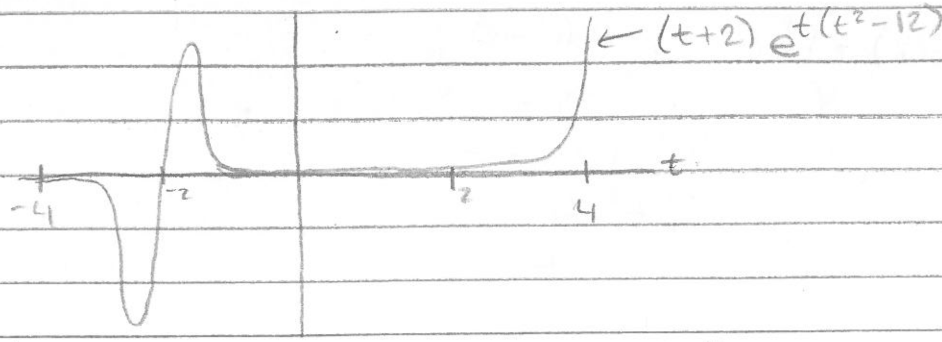
$\rightarrow e^{16x} \int_{-\varepsilon}^{\varepsilon} \tau e^{-6x\tau^2} d\tau$ let $u = -6x\tau^2$ $du = -12x\tau \, d\tau$

$\rightarrow -\frac{1}{12x} e^{16x} \int_{6x\varepsilon^2}^{6x\varepsilon^2} e^u du$

$= 0$

This region does not contribute significantly to I_5. This can be verified by examining a plot of the integrand of I_5:

(over)



You can see that the bits just before $t=-2$ and just after $t=-2$ will cancel out, and that the part near $t=4$ will contribute the most.

Thus $I_5(x) \sim \dfrac{e^{16x}}{6x}$ ✓

2) Find the leading order term of the asymptotics of the integral below as $x \to 0$ $(x>0)$.

$$I(x) = \int_0^\infty e^{-xt^2} \sinh t \, dt$$
$$= \int_0^\infty e^{-xt^2} \cdot \tfrac{1}{2}(e^{t} - e^{-t})\, dt$$
$$= \tfrac{1}{2}\int_0^\infty e^{-xt^2+t} - e^{-xt^2-t}$$

let $s = xt \to t = \frac{s}{x}$, $dt = \frac{1}{x} ds$, $t^2 = \frac{s^2}{x^2}$

$$= \tfrac{1}{2}\int_0^\infty e^{-x s^2/x^2 + s/x} - e^{-x s^2/x^2 - s/x} \; \tfrac{1}{x} ds$$
$$= \tfrac{1}{2}\tfrac{1}{x}\int_0^\infty e^{\frac{1}{x}(-s^2+s)} - e^{+\frac{1}{x}(-s^2-s)} ds$$

let $\xi = \frac{1}{x}$, then as $x \to 0$, $\xi \to \infty$
(and we can use Laplace method)

$$= \tfrac{1}{2}\xi\int_0^\infty \underbrace{e^{\xi(-s^2+s)}}_{A} \underbrace{- e^{\xi(-s^2-s)}}_{B} ds$$

B: $-\frac{1}{2}\xi\int_0^\infty e^{\xi(-s^2-s)} ds$

$\varphi = -s^2 - s$

$\varphi' = -2s - 1 = 0 \to s = -\frac{1}{2} \to \varphi' \neq 0 \in [0, \infty]$

$$= -\tfrac{1}{2}\xi\int_0^\infty \frac{(-2s-1)\xi}{(-2s-1)\xi} e^{\xi(-s^2-s)} ds$$

$u = -\frac{1}{(2s+1)\xi}$ $\quad v = e^{-\xi(s^2+s)}$

$du = \frac{2}{(2s+1)^2\xi}$ $\quad dv = -(2s+1)\xi e^{-\xi(s^2+s)}$

$$= -\tfrac{1}{2}\xi\left[-\frac{1}{(2s+1)\xi} e^{-\xi(s^2+s)}\Big|_0^\infty - \frac{2}{\xi}\int_0^\infty \frac{1}{(2s+1)^2} e^{-\xi(s^2+s)} ds\right]$$
$$= -\frac{1}{2} + \int_0^\infty \frac{1}{(2s+1)^2} e^{-\xi(s^2+s)} ds$$

Each time we integrate the remaining integral by parts, we will get another factor of $\frac{1}{\xi}$, so terms get smaller and smaller (since $\xi \to \infty$)

$\sim -\frac{1}{2}$

(continued on reverse)

A: $\frac{1}{2}\xi\int_0^\infty e^{\xi(-s^2+s)}\,ds$

$\varphi = -s^2+s$

$\varphi' = -2s+1 = 0 \quad \rightarrow s=\frac{1}{2}$

$\varphi'' = -2 < 0$, so $s=\frac{1}{2}$ is a maximum

$\varphi \sim \frac{1}{4} - (s-\frac{1}{2})^2$

$\rightarrow \sim \frac{1}{2}\xi\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} e^{\xi(\frac{1}{4}-(s-\frac{1}{2})^2)}\,ds \quad$ let $\tau = s-\frac{1}{2}$

$= \frac{1}{2}\xi e^{\frac{1}{4}\xi}\int_{-\varepsilon}^{\varepsilon} e^{-\xi\tau^2}\,ds \quad$ let $s^2=\xi\tau^2 \rightarrow s=\sqrt{\xi}\tau \quad ds=\sqrt{\xi}\,d\tau$

$= \frac{1}{2}\sqrt{\xi}\,e^{\frac{1}{4}\xi}\int_{-\varepsilon\sqrt{\xi}}^{\varepsilon\sqrt{\xi}} e^{-s^2}\,ds \quad$ let $\varepsilon = \xi^{-\frac{1}{3}}$

$\sim \frac{1}{2}\sqrt{\xi}\,e^{\frac{1}{4}\xi}\int_{-\infty}^{\infty} e^{-s^2}\,ds$

$= \frac{1}{2}\sqrt{\xi\pi}\,e^{\frac{1}{4}\xi}$

Note that this is $\gg \frac{1}{2}$ for $\xi\rightarrow\infty$.

So $I(x) \sim \frac{1}{2}\sqrt{\pi\xi}\,e^{\frac{1}{4}\xi}$

$= \frac{1}{2}\sqrt{\frac{\pi}{x}}\,e^{\frac{1}{4x}}$ ✓

1-1	1-2	1-3	1-4	1-5	2
6	6	6	6	6	6
6	6	6	6	6	6

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ES_APPM 420-3 "Asymptotic and Perturbation Methods"

Homework 1 (DUE TUESDAY, 4/21/09, IN CLASS)

Problem 1. Find a two-term asymptotic expansion, for small ε, of the solution of Laplace's equation in the unit disk

$$\nabla^2 u = 0, \qquad r < 1, \;\; 0 \le \theta < 2\pi$$

subject to the boundary condition

$$u(1,\theta) = \exp(\varepsilon \cos\theta), \quad 0 \le \theta < 2\pi.$$

Here (r,θ) are polar coordinates in the plane.

Problem 2. For plane inviscid, incompressible flows, the equations of motion reduce to solving Laplace's equation for the streamfunction ψ; the velocity field (v_r, v_θ) is then obtained by direct differentiation

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r}.$$

The flow past a body is therefore described (in polar coordinates) by

$$\nabla^2 \psi = 0,$$

$$\psi(r,\theta) \sim Ur\sin\theta \quad \text{as} \quad r \to \infty,$$

$$\psi(r,\theta)\,|_{\text{surface}} = 0,$$

where the boundary conditions reflect the facts that the flow approaching the body is unidirectional with speed U, and there is no penetration of the liquid into the body. Consider now the flow past the region $r = a(1 - \varepsilon \sin^2\theta)$, $0 < \varepsilon \ll 1$.

(a). Obtain an approximate solution for ψ correct to $O(\varepsilon)$.

(b). Determine the velocity field (v_r, v_θ).

(c). Show that along the surface of the body the speed

$$\sqrt{v_r^2 + v_\theta^2} \sim 2U\sin\theta + \varepsilon U \sin 3\theta.$$

$\nabla^2(\psi_0 + \varepsilon\psi_1)$

$(\psi_0 + \varepsilon\psi_1)(a + \varepsilon a \sin^{-2}\theta) = 0$

Haley Lepo
4-21-09
ESAM 420-3
Hwk #1

1) $\nabla^2 u = 0$, $r<1$, $0 \le \theta < 2\pi$

$u(1,\theta) = e^{\varepsilon\cos\theta}$

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$\frac{r^2 R''}{R} + \frac{r R'}{R} = \frac{-\Theta''}{\Theta} = \lambda$

$\Theta'' + \lambda\Theta = 0$, must be 2π-periodic

$\lambda = n^2$

$\Theta = c_1 \cos n\theta + c_2 \sin n\theta$, $n = 0, 1, 2, \cdots$

$r^2 R'' + rR' - n^2 = 0$

$R = c_3 r^n + c_4 r^{-n}$, $n = 1, 2, 3, \cdots$

or $c_5 + c_6 \ln r$ for $n=0$

must be finite for $r=0$: $c_4 = c_6 = 0$

$\rightarrow R = c_7 r^n$, $n = 0, 1, 2, \cdots$

$u = \sum_{n=0}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$

$u(1,\theta) = e^{\varepsilon\cos\theta} \sim 1 + \varepsilon\cos\theta$

let $u(r,\theta) \sim u_0 + \varepsilon u_1$

$u_0(1,\theta) = 1 = \sum_{n=0}^{\infty}[a_n \cos n\theta + b_n \sin n\theta]$

$\rightarrow a_n = b_n = 0$ for $n \neq 0$. $a_0 = 1$

$u_0(r,\theta) = r^0(\cos 0 + b_n \sin 0) = 1$

$u_1(1,\theta) = \cos\theta = \sum_{n=0}^{\infty}[a_n \cos n\theta + b_n \sin n\theta]$

$\rightarrow a_n = b_n = 0$ for $n \neq 1$, $b_1 = 0$

$u_1(r,\theta) = r\cos\theta$

$u(r,\theta) \sim 1 + \varepsilon r\cos\theta$

2) $\nabla^2 \psi = 0$, $\psi(r,\theta) \sim Ur\sin\theta$ as $r \to \infty$

$\psi(r,\theta)|_{surface} = 0$

region: $r = a(1-\varepsilon\sin^2\theta)$

Let $\psi \sim \psi_0 + \varepsilon\psi_1$

$O(1)$: $\nabla^2\psi_0 = 0$, $r < a$; $0 \le \theta < 2\pi$

$\psi_0(a,\theta) = 0$, $\psi_0(\infty,\theta) = Ur\sin\theta$

$\psi_0 = (c_0 + d_0 \ln r) + \sum_{n=1}^{\infty}(c_n r^n + d_n r^{-n})(a_n\cos n\theta + b_n \sin n\theta)$

$\psi_0(\infty,\theta) = Ur\sin\theta$:

$c_0 = 0, d_0 = 0, \ a_1 = 0, \ b_n = 0$ for $n > 1$

$\psi_0 = (c_1 r + d_1 \frac{1}{r})(b_1\sin\theta) + \sum_{n=2}^{\infty} r^{-n}(a_n\cos n\theta + b_n \sin n\theta)$

$c_1 b_1 = U$

$\psi_0(a,\theta) = 0$:

$a_n = b_n = 0$ for $n > 1$, $(Ua + d_1\frac{1}{a}) = 0$

$d_1 = -Ua^2$

$\psi_0(r,\theta) = U(r - a^2/r)\sin\theta$

$O(\varepsilon)$: $\nabla^2\psi_1 = 0$, $\psi(r,\theta) = 0$ as $r \to \infty$

$\psi_0(a(1-\varepsilon\sin^2\theta),\theta) + \varepsilon\psi_1(a(1-\varepsilon\sin^2\theta),\theta) = 0$

$= U\left(a(1-\varepsilon\sin^2\theta) - \frac{a}{1-\varepsilon\sin^2\theta}\right)\sin\theta + \varepsilon\psi_1(a(1-\varepsilon\sin^2\theta,\theta))$

$\sim Ua(1+\varepsilon\sin^2\theta - 1 - \varepsilon\sin^2\theta)\sin\theta = -\varepsilon\psi_1(a(1-\varepsilon\sin^2\theta,\theta)$

$= 2Ua\varepsilon\sin^3\theta = \varepsilon\psi_1(a(1-\varepsilon\sin^2\theta),\theta)$

$\to 2Ua\sin^3\theta = \psi_1(a(1-\varepsilon\sin^2\theta),\theta) \sim \psi_1(a,\theta)$

$\psi_1 = (c_0 + d_0\ln r) + \sum_{n=1}^{\infty}(c_n r^n + d_n r^{-n})(a_n\cos n\theta + b_n\sin n\theta)$

$r \to \infty$: $c_0 = d_0 = c_n = 0$

$\psi_1 = \sum_{n=1}^{\infty} r^{-n}(a_n\cos n\theta + b_n\sin n\theta)$

$\psi_1(a,\theta) = \sum_{n=1}^{\infty} a^{-n}(a_n\cos n\theta + b_n\sin n\theta) = 2Ua\sin^3\theta$

$= Ua(\frac{3}{2}\sin\theta - \frac{1}{2}\sin 3\theta)$

$\to a_n = 0$, $b_n = 0$ for $n \ne 1, 3$

$b_1\frac{1}{a}\sin\theta = Ua\,3/2\sin\theta$, $b_3\frac{1}{a^3}\sin 3\theta = -\frac{1}{2}Ua\sin 3\theta$

$b_1 = 3/2\,Ua^2$, $b_3 = -\frac{1}{2}Ua^4$

$\psi_1 = 3/2\,Ua^2\frac{1}{r}\sin\theta - \frac{1}{2}Ua^4\frac{1}{r^3}\sin 3\theta$

$$\boxed{\psi \sim U\left(r-\frac{a^2}{r}\right)\sin\theta + \varepsilon U\left(\tfrac{3}{2}\tfrac{a^2}{r}\sin\theta - \tfrac{1}{2}\tfrac{a^4}{r^3}\sin 3\theta\right)}$$ ✓

B) $(v_r, v_\theta) = \left(\frac{1}{r}\psi_\theta, -\psi_r\right)$

$$= \Big(\tfrac{1}{r}\big(U(r-a^2/r)\cos\theta + \varepsilon U(\tfrac{3}{2}\tfrac{a^2}{r}\cos\theta - \tfrac{3}{2}\tfrac{a^4}{r^3}\cos 3\theta)\big),$$
$$-\big(U(1+a^2/r^2)\sin\theta + \varepsilon U(-\tfrac{3}{2}a^2/r^2\sin\theta + \tfrac{3}{2}a^4/r^4\sin 3\theta\big)\Big)$$
$$= \Big(U(1-a^2/r^2)\cos\theta + \varepsilon U(\tfrac{3}{2}a^2/r^2\cos\theta - \tfrac{3}{2}a^4/r^4\cos 3\theta),$$
$$U(-1-a^2/r^2)\sin\theta + \varepsilon U(\tfrac{3}{2}a^2/r^2\sin\theta - \tfrac{3}{2}a^4/r^4\sin 3\theta)\Big)$$ ✓

C) Show $\sqrt{v_r^2+v_\theta^2} \sim 2U\sin\theta + \varepsilon U\sin 3\theta$

$v_r^2+v_\theta^2 \sim$

$$U^2(1-a^2/r^2)^2\cos^2\theta + 2\varepsilon U^2(1-a^2/r^2)\cos\theta\,\cdot$$
$$\cdot\,(\tfrac{3}{2}a^2/r^2\cos\theta - \tfrac{3}{2}a^4/r^4\cos 3\theta)$$
$$+U^2(1+a^2/r^2)^2\sin^2\theta - 2\varepsilon U^2(1+a^2/r^2)\sin\theta\,\cdot$$
$$\cdot\,(\tfrac{3}{2}a^2/r^2\sin\theta - \tfrac{3}{2}a^4/r^4\sin 3\theta)$$
$$= U^2\Big[(1-2a^2/r^2+a^4/r^4)\cos^2\theta + (1+2a^2/r^2+a^4/r^4)\sin^2\theta$$
$$+2\varepsilon(1-a^2/r^2)\cos\theta\,\tfrac{3}{2}(a^2/r^2)(\cos\theta - a^2/r^2\cos 3\theta)$$
$$-2\varepsilon(1+a^2/r^2)\sin\theta\,\tfrac{3}{2}(a^2/r^2)(\sin\theta - a^2/r^2\sin 3\theta)\Big]$$

1	2
4	7
4	9

max (4+2+3)

$$= U^2\Big[1 + a^4/r^4 + 2a^2/r^2(\sin^2\theta - \cos^2\theta)$$
$$+3\varepsilon\, a^2/r^2\big(\cos\theta(1-a^2/r^2)(\cos\theta - a^2/r^2\cos 3\theta)$$
$$-\sin\theta(1+a^2/r^2)(\sin\theta - a^2/r^2\sin 3\theta)\big)\Big]$$

$\sqrt{v_r^2+v_\theta^2} =$

$$= U\sqrt{f(\varepsilon)} \sim U\sqrt{f(0)} + \tfrac{1}{2}\varepsilon U\, f'(0)/\sqrt{f(0)}$$
$$= U\sqrt{1+a^4/r^4+2a^2/r^2(\sin^2\theta-\cos^2\theta)}$$
$$+\frac{\tfrac{1}{2}\varepsilon U\, 3a^2/r^2\big[\cos\theta(1-a^2/r^2)(\cos\theta - a^2/r^2\cos 3\theta) - \sin\theta(1+a^2/r^2)(\sin\theta - a^2/r^2\sin 3\theta)\big]}{\sqrt{1+a^4/r^4+2a^2/r^2(\sin^2\theta-\cos^2\theta)}}$$

not quite...

ES_APPM 420-3 "Asymptotic and Perturbation Methods"

Homework 2 (DUE TUESDAY, 5/05/2009, IN CLASS)

Problem 1. Consider the solution $u(r, \theta)$ of the problem

$$\varepsilon\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=u, \quad 0<r<1, \ 0\le\theta<2\pi$$

subject to the boundary condition $u(1, \theta) = 1$. Determine a leading order uniformly valid approximation to $u(r, \theta)$ for small ε.

Problem 2. Consider the equation

$$\varepsilon\nabla^2 u = \frac{\partial u}{\partial x}$$

in the region that is given in polar coordinates as

$$3 < r < 5, \quad 0 < \theta < \pi$$

(see the sketch below). The boundary conditions for $u(r, \theta)$ are (see also the sketch)

$$u(r, 0) = 2, \quad 3 < r < 5,$$

$$u(r, \pi) = 0, \quad 3 < r < 5,$$

$$u(3, \theta) = 1, \quad 0 < \theta < \pi,$$

$$u(5, \theta) = 0, \quad 0 < \theta < \pi.$$

(a). Determine the outer solution of the problem.
(b). Sketch all the boundary/internal layers and corner regions that are needed to produce a uniformly valid solution.
(c). Determine the appropriate scaling in each layer/ corner region.
(d). Formulate the leading order boundary value problems (including the boundary conditions and indicating the spatial domain in which the problem has to be solved) for each boundary/internal layer and corner region.

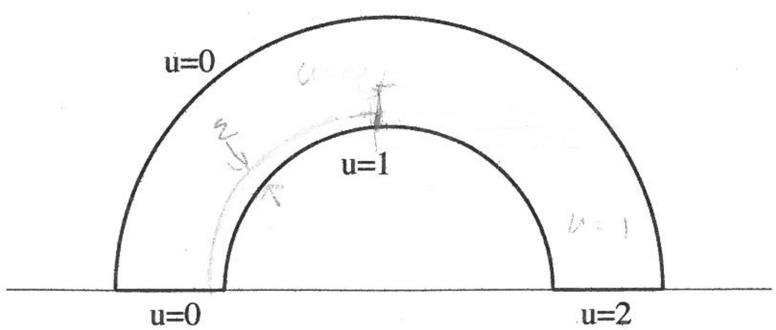

Figure 1: Sketch of the region in Problem 2.

Haleylepo
2-17-09
ESAM 420-2
Mid-term

1) $x'' - \varepsilon(1-x^2)x' + x = 0;\quad x(0)=a,\ x'(0)=0$

let $t_0 = t,\ t_1 = \varepsilon t,\ y(t_0,t_1) = x(t)$

$$\to \frac{\partial^2 y}{\partial t_0} + 2\varepsilon\frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 y}{\partial t_1^2} - \varepsilon(1-y^2)\left(\frac{\partial y}{\partial t_0} + \varepsilon\frac{\partial y}{\partial t_1}\right) + y = 0$$

$$\to \frac{\partial^2 y}{\partial t_0} + \frac{2\varepsilon\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 y}{\partial t_1^2} - \varepsilon\frac{\partial y}{\partial t_0} - \varepsilon^2\frac{\partial y}{\partial t_1} + \varepsilon y^2 \frac{\partial y}{\partial t_0} + \varepsilon^2 y^2 \frac{\partial y}{\partial t_1} + y = 0$$

let $y \sim y_0 + \varepsilon y_1$

$O(1)$: $\frac{\partial^2 y_0}{\partial t_0} + y_0 = 0,\quad y_0(0,0) = a,\quad \frac{\partial y_0}{\partial t_0}(0,0) = 0$

$y_0(t_0,t_1) = A_0(t_1)\cos t_0 + B_0(t_1)\sin t_0$

$y_0(0,0) = A_0(0) = a$

$\frac{\partial y_0}{\partial t_0}(0,0) = B_0(0) = 0$

$O(\varepsilon)$: $\frac{\partial^2 y_1}{\partial t_0} + y_1 = -\frac{2\partial^2 y_0}{\partial t_0 \partial t_1} + \frac{\partial y_0}{\partial t_0} - y_0^2 \frac{\partial y_0}{\partial t_0}$

$= 2A_0'\sin t_0 - 2B_0'\cos t_0 - A_0 \sin t_0 + B_0 \cos t_0$
$\quad -(A_0\cos t_0 + B_0 \sin t_0)^2(-A_0\sin t_0 + B_0\cos t_0)$

$= 2A_0'\sin t_0 - 2B_0'\cos t_0 - A_0\sin t_0 + B_0\cos t_0$
$\quad (-A_0^2\cos^2 t_0 - B_0^2 \sin^2 t_0 - 2A_0B_0\cos t_0 \sin t_0)$
$\quad \cdot(-A_0\sin t_0 + B_0\cos t_0)$

$= 2A_0'\sin t_0 - 2B_0'\cos t_0 - A_0\sin t_0 + B_0\cos t_0$
$\quad + A_0^3\cos^2 t_0 \sin t_0 + A_0B_0^2\sin^3 t_0 + 2A_0^2 B_0 \cos t_0 \sin^2 t_0$
$\quad - A_0^2 B_0 \cos^3 t_0 - B_0^3 \cos t_0 \sin^2 t_0 - 2A_0B_0^2\cos^2 t_0 \sin t_0$

$\cos^2 t_0 \sin t_0 = \frac{1}{2}(1+\cos 2t_0)\sin t_0$
$= \frac{1}{2}\sin t_0 + \frac{1}{2}\cos 2t_0 \sin t_0$
$= \frac{1}{2}\sin t_0 + \frac{1}{4}\sin 3t_0 - \frac{1}{4}\sin t_0$
$= \frac{1}{4}\sin t_0 + \frac{1}{4}\sin 3t_0$

$\sin^2 t_0 \cos t_0 = \frac{1}{2}(1-\cos 2t_0)\cos t_0$
$= \frac{1}{2}\cos t_0 - \frac{1}{2}\cos 2t_0 \cos t_0$
$= \frac{1}{2}\cos t_0 - \frac{1}{4}\cos t_0 - \frac{1}{4}\cos 3t_0$
$= \frac{1}{4}\cos t_0 - \frac{1}{4}\cos 3t_0$

$\sin^3 t_0 = \frac{1}{2}(1-\cos 2t_0)\sin t_0 = \frac{1}{2}\sin t_0 - \frac{1}{2}\cos 2t_0 \sin t_0$
$= \frac{1}{2}\sin t_0 - \frac{1}{4}\sin 3t_0 + \frac{1}{4}\sin t_0 = \frac{3}{4}\sin t_0 - \frac{1}{4}\sin 3t_0.$

$$\begin{cases} \cos^3 t_0 = \frac{1}{2}(1+\cos 2t_0)\cos t_0 = \frac{1}{2}\cos t_0 + \frac{1}{2}\cos 2t_0 \cos t_0 \\ \qquad = \frac{1}{2}\cos t_0 + \frac{1}{4}\cos t_0 + \frac{1}{4}\cos 3t_0 = \frac{3}{4}\cos t_0 + \frac{1}{4}\cos 3t_0 \end{cases}$$

So, secular-producing terms:

$\sin t_0$: $2A_0' - A_0 + \frac{1}{4}A_0^3 - \frac{1}{2}A_0B_0^2 + \frac{3}{4}A_0B_0^2 = 0$

$\cos t_0$: $-2B_0' + B_0 + \frac{1}{2}A_0^2B_0 - \frac{1}{4}B_0^3 - \frac{3}{4}A_0^2B_0 = 0$

$$\to \begin{cases} 2A_0' - A_0 + \frac{1}{4}A_0^3 + \frac{1}{4}A_0B_0^2 = 0 \\ 2B_0' - B_0 + \frac{1}{4}B_0^3 + \frac{1}{4}A_0^2B_0 = 0 \end{cases}$$

$$\to \begin{cases} (A_0^2)' + A_0^2\left(\frac{1}{4}A_0^2 + \frac{1}{4}B_0^2 - 1\right) = 0 \\ (B_0^2)' + B_0^2\left(\frac{1}{4}B_0^2 + \frac{1}{4}A_0^2 - 1\right) = 0 \end{cases}$$

$$\to (A_0^2+B_0^2)' + \frac{1}{4}(A_0^2+B_0^2)^2 - (A_0^2+B_0^2) = 0$$

let $z(t_1) = A_0(t_1)^2 + B_0(t_1)^2$, $z(0) = a^2$

$$\to z' + \frac{1}{4}z^2 - z = 0$$

$$z' = -\frac{1}{4}(z^2 - 4z)$$

$$\int \frac{dz}{z(z-4)} = \int -\frac{1}{4}\,dt_1 \qquad \frac{1}{z(z+4)} = \frac{\frac{1}{4}}{z} + \frac{-\frac{1}{4}}{z-4}$$

$$1 = A(z-4) + Bz$$

$$A + B = 0, \quad 1 = -4A \Rightarrow A = -\frac{1}{4}, \quad B = +\frac{1}{4}$$

$$-\frac{1}{4}\int\frac{dz}{z} + \frac{1}{4}\int\frac{dz}{z-4} = -\frac{1}{4}\int dt_1$$

$$-\ln z + \ln z-4 = -t + c$$

$$\frac{z-4}{z} = c_1 e^{-t_1}$$

$$z(0) = a^2 \to \frac{a^2-4}{a^2} = c_1$$

$$a^2(z-4) = z(a^2-4)e^{-t_1}$$

$$z = \frac{-4a^2}{(a^2-4)e^{-t_1} - a^2} = \frac{4a^2}{a^2 + (4-a^2)e^{-t_1}}$$

$$\to 2A_0' = A_0\left(1 - \frac{1}{4}z\right)$$

$$\int\frac{dA_0}{A_0} = \int \frac{1}{2}\,\frac{a^2+(4-a^2)e^{-t_1} - a^2}{a^2+(4-a^2)e^{-t_1}}\,dt_1$$

$$\ln A_0 = \ln\left(a^2+(4-a^2)e^{-t_1}\right)^{-\frac{1}{2}} + c_2$$

$$A_0 = c_3\left(a^2+(4-a^2)e^{-t_1}\right)^{-\frac{1}{2}}$$

$$A_0(0) = c_3(a^2+4-a^2) = c_3/2 = a \to c_3 = 2a$$

$$A_0(t_1) = \frac{2a}{\sqrt{a^2+(4-a^2)e^{-t_1}}}$$

Lepo 2

Notice $A^2 = z$, and since $z = A^2 + B^2$, this implies $B(t_1) \equiv 0$. (This could also be found by integrating in the same manner used to find A - then using the initial condition $B(0) = 0$, we find that the constant multiplying $(a^2 + (4-a^2)e^{-t})^{-1/2}$ must be zero.)

$$y_0(t_0, t_1) = \frac{2a\cos t_0}{\sqrt{a^2 + (4-a^2)e^{-t_1}}}$$

$$x(t) \sim \frac{2a\cos t}{\sqrt{a^2 + (4-a^2)e^{-\varepsilon t}}}$$ ✓

If $a^2 = 4$ we have $\frac{\pm 4\cos t}{\sqrt{4+0}} = \pm 2\cos t$

If $a^2 < 4$ the coefficient of $e^{-\varepsilon t}$ is > 0, meaning the $e^{-\varepsilon t}$ term decreases with time, approaching zero. Then the denominator decreases with time, approaching $|a|$, so $x(t)$ increases with time, approaching $\pm 2\cos t$

If $a^2 > 4$ the coefficient of $e^{-\varepsilon t}$ is < 0, meaning the $e^{-\varepsilon t}$ term increases with time, approaching zero. Then the denominator increases with time, approaching $|a|$, so $x(t)$ decreases with time, approaching $\pm 2\cos t$.

This type of behavior is called a limit cycle.

Lepo 3

2) $\frac{d^2x}{dt^2} + 2\varepsilon\frac{dx}{dt} + e^{-2\varepsilon t}x = 0$; $x(0)=0$, $x'(0)=1$

let $t_0 = f(\varepsilon, t)$, $t_1 = \varepsilon t$, $y(t_0, t_1) = x(t)$

$f_t^2\frac{\partial^2 y}{\partial t_0^2} + f_{tt}\frac{\partial y}{\partial t_0} + 2\varepsilon f_t\frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 y}{\partial t_1^2} + 2\varepsilon\left(f_t\frac{\partial y}{\partial t_0} + \varepsilon\frac{\partial y}{\partial t_1}\right) + e^{-2\varepsilon t}y = 0$

$$y(0,0)=0,\ \left(f_t\frac{\partial y}{\partial t_0} + \varepsilon\frac{\partial y}{\partial t_1}\right)\bigg|_{(0,0)} = 1$$

let $f_t^2 = e^{-2\varepsilon t}$

$\rightarrow f_t = e^{-\varepsilon t}$ (choose + root)

$f = \int e^{-\varepsilon t}dt = -\frac{1}{\varepsilon}e^{-\varepsilon t} = t_0$ (choose const = 0)

then at $t=0$ $\Rightarrow t_0 = -\frac{1}{\varepsilon}$

$f_{tt} = -\varepsilon e^{-\varepsilon t}$

$\rightarrow e^{-2\varepsilon t}\frac{\partial^2 y}{\partial t_0} - \varepsilon e^{-\varepsilon t}\frac{\partial y}{\partial t_0} + 2\varepsilon e^{-\varepsilon t}\frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 y}{\partial t_1^2}$

$+ 2\varepsilon e^{-\varepsilon t}\frac{\partial y}{\partial t_0} + 2\varepsilon^2\frac{\partial y}{\partial t_1} + e^{-2\varepsilon t}y = 0$

$\rightarrow \frac{\partial^2 y}{\partial t_0} - \varepsilon e^{\varepsilon t}\frac{\partial y}{\partial t_0} + 2\varepsilon e^{\varepsilon t}\frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2 e^{\varepsilon^2 t}\frac{\partial^2 y}{\partial t_1^2}$

$+ 2\varepsilon e^{\varepsilon t}\frac{\partial y}{\partial t_0} + 2\varepsilon^2 e^{\varepsilon t}\frac{\partial y}{\partial t_1} + y = 0$

$\rightarrow \frac{\partial^2 y}{\partial t_0} - \varepsilon\frac{\partial y}{\partial t_0} + 2\varepsilon\frac{\partial^2 y}{\partial t_0 \partial t_1} + 2\varepsilon\frac{\partial y}{\partial t_0} + y + O(\varepsilon^2) = 0$

$$y(0,0)=0,\ \left(e^{\varepsilon t}\frac{\partial y}{\partial t_0} + \varepsilon\frac{\partial y}{\partial t_1}\right)\bigg|_{(0,0)} = 1$$

f.k.

$y \sim y_0 + \varepsilon y_1$

$O(1)$: $\frac{\partial^2 y_0}{\partial t_0} + y_0 = 0$, $y_0(0,0) = 0$, $\frac{\partial y_0}{\partial t_0} = 1$

$y_0 = A_0(t_1)\cos t_0 + B_0(t_1)\sin t_0$

$y_0(0,0) = A_0(0) = 0$

$\frac{\partial y_0}{\partial t_1}(0,0) = B_0(0) = 1$

(continued on reverse)

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + y_1 = \dfrac{\partial y_0}{\partial t_0} - 2\dfrac{\partial^2 y_0}{\partial t_0 \partial t_1} - 2\dfrac{\partial y_0}{\partial t_0}$

$= +A_0 \sin t_0 - B_0 \cos t_0 + 2A_0' \sin t_0 - 2B_0' \cos t_0$

secular-producing terms:

$\sin t_0$: $A_0 + 2A_0' = 0$

$-B_0 - 2B_0' = 0$

$\rightarrow \begin{cases} A_0' = -\frac{1}{2}A_0 \\ B_0' = -\frac{1}{2}B_0 \end{cases}$

$\rightarrow \begin{cases} A_0 = c_1 e^{-\frac{1}{2}t_1} \\ B_0 = c_2 e^{-\frac{1}{2}t_1} \end{cases}$

$A_0(0) = c_1 = 0$

$B_0(0) = c_2 = 1$

$\rightarrow A_0 = 0, \quad B_0 = e^{-\frac{1}{2}t_1}$

$\rightarrow y_0(t_0, t_1) = e^{-\frac{1}{2}t_1} \sin t_0$

$\rightarrow \quad x(t) \sim e^{-\frac{1}{2}\varepsilon t} \sin(-\frac{1}{\varepsilon} e^{-\varepsilon t})$

Lepak

3) $\ddot{x} + \varepsilon\mu\dot{x} + x + \varepsilon x^2 = F\cos\Omega t$

First, show there is secondary resonance for $\Omega \approx \frac{1}{2}$ by using regular, one-time perturbation.

$x \sim x_0 + \varepsilon x_1$

$O(1)$: $\ddot{x}_0 + x_0 = F\cos\Omega t$

$x_{0p} = A\cos\Omega t$, $\ddot{x}_{0p} = -A\Omega^2\cos\Omega t$

$\rightarrow (-A\Omega^2 + A)\cos\Omega t = F\cos\Omega t$

$\rightarrow A(1-\Omega^2) = F \rightarrow A = F/(1-\Omega^2)$

$x_0 = a\cos t + b\sin t + \frac{F}{1-\Omega^2}\cos\Omega t$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -\mu\dot{x}_0 - x_0^2$

$$\ddot{x}_1 + x_1 = -\mu\left(-a\sin t + b\cos t - \tfrac{F\Omega}{1-\Omega^2}\sin\Omega t\right)$$
$$- a^2\cos^2 t - 2ab\cos t\sin t - 2\tfrac{aF}{1-\Omega^2}\cos t\cos\Omega t$$
$$- b^2\sin^2 t - 2\tfrac{bF}{1-\Omega^2}\sin t\cos\Omega t - \left(\tfrac{F}{1-\Omega^2}\right)^2\cos^2\Omega t$$
$$= \mu a\sin t - \mu b\cos t + \tfrac{\mu F\Omega}{1-\Omega^2}\sin\Omega t$$
$$- \tfrac{1}{2}a^2(1+\cos 2t) - ab\sin 2t - \tfrac{aF}{1-\Omega^2}\cos(1+\Omega)t$$
$$- \tfrac{aF}{1-\Omega^2}\cos(1-\Omega)t - \tfrac{1}{2}b^2(1-\cos 2t)$$
$$- \tfrac{bF}{1-\Omega^2}\sin(1+\Omega)t - \tfrac{bF}{1-\Omega^2}\sin(1-\Omega)t - \tfrac{1}{2}\left(\tfrac{F}{1-\Omega^2}\right)^2(1+\cos 2\Omega t)$$
$$= \mu a\sin t - \mu b\cos t + \tfrac{\mu F\Omega}{1-\Omega^2}\sin\Omega t$$
$$(\tfrac{1}{2}b^2 - \tfrac{1}{2}a^2)\cos 2t - ab\sin 2t - \tfrac{aF}{1-\Omega^2}\cos(1+\Omega)t$$
$$- \tfrac{aF}{1-\Omega^2}\cos(1-\Omega)t - \tfrac{bF}{1-\Omega^2}\sin(1+\Omega)t - \tfrac{bF}{1-\Omega^2}\sin(1-\Omega)t$$
$$- \tfrac{1}{2}\left(\tfrac{F}{1-\Omega^2}\right)^2\cos 2\Omega t - \tfrac{1}{2}a^2 - \tfrac{1}{2}b^2 - \tfrac{1}{2}\left(\tfrac{F}{1-\Omega^2}\right)^2$$

So $x_{1p} = Bt\sin t + Ct\cos t + Dt\sin\Omega t + Et\cos\Omega t$
$+ F\cos 2t + G\sin 2t + H\cos(1+\Omega)t + I\cos(1-\Omega)t$
$+ J\sin(1+\Omega)t + K\sin(1-\Omega)t + L\cos 2\Omega t + M$

Now say we want to determine L.

$\rightarrow -4\Omega^2 L\cos 2\Omega t + L\cos 2\Omega t = -\frac{1}{2}\left(\frac{F}{1-\Omega^2}\right)^2\cos 2\Omega t$

$\rightarrow -4\Omega^2 L + L = -\frac{1}{2}\left(\frac{F}{1-\Omega^2}\right)^2$

$$L = \frac{-1}{2}\,\frac{F^2}{(1-\Omega^2)^2(1-4\Omega^2)}$$

So we encounter a "problem" when $(1-4\Omega^2) = 0$

$\rightarrow \Omega = \pm\frac{1}{2}$. This "problem" is secondary resonance.

Note: when $\Omega = \pm\frac{1}{2}$ we are dividing by zero, which is bad, but also when Ω is very close to $\pm\frac{1}{2}$ we have a problem because then x_1 is no longer a "small correction" to x_0.

Now Study the behavior of the solution for $\Omega = \frac{1}{2}(1+\varepsilon\omega)$ using multiple scales.

$$\ddot{x} + \varepsilon\mu\dot{x} + x + \varepsilon x^2 = F\cos\left(\tfrac{1}{2}(1+\varepsilon\omega)t\right)$$

let $t_0 = t$, $t_1 = \varepsilon t$, $y(t_0, t_1) = x(t)$

$$\rightarrow \frac{\partial^2 y}{\partial t_0^2} + 2\varepsilon\frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 y}{\partial t_1^2} + \varepsilon\mu\frac{\partial y}{\partial t_0} + \varepsilon^2\mu\frac{\partial y}{\partial t_1}$$

$$+ y + \varepsilon y^2 = F\cos\left(\tfrac{1}{2}(t_0 + \omega t_1)\right)$$

1	2	3	
8	7	7	22
8	8	8	

$y \sim y_0 + \varepsilon y_1$

$O(1)$: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = F\cos\left(\tfrac{1}{2}(t_0 + \omega t_1)\right)$

$$y_0 = R(t_1)\cos(t_0 + \omega t_1 - \varphi(t_1)) + k\cos\tfrac{1}{2}(t_0 + \omega t_1)$$

$$\Leftarrow -\tfrac{1}{4}k + k = F$$

$$\rightarrow \tfrac{3}{4}k = F \quad \rightarrow k = \tfrac{4}{3}F$$

$$y_0(t_0, t_1) = R(t_1)\cos(t_0 + \omega t_1 - \varphi(t_1)) + \tfrac{4}{3}F\cos\tfrac{1}{2}(t_0 + \omega t_1)$$

$O(\varepsilon)$: $\dfrac{\partial^2 y_1}{\partial t_0^2} + y_1 = -2\dfrac{\partial^2 y_0}{\partial t_0 \partial t_1} - \mu\dfrac{\partial y_0}{\partial t_0} - y_0^2$

$$\left[\text{let } t_0 + \omega t_1 - \varphi(t_1) = \psi \rightarrow \partial/\partial t_1 \psi = \omega - \varphi'\right]$$

$$= +2\,\partial/\partial t_1\left(R\sin\psi + \tfrac{2}{3}F\sin\tfrac{1}{2}(t_0 + \omega t_1)\right)$$

$$+\mu\left(R\sin\psi + \tfrac{2}{3}F\sin\tfrac{1}{2}(t_0 + \omega t_1)\right)$$

$$-\left(R^2\cos^2\psi + \tfrac{8}{3}RF\cos\psi\cos\tfrac{1}{2}(t_0 + \omega t_1) + \tfrac{16}{9}F^2\cos^2\tfrac{1}{2}(t_0 + \omega t_1)\right)$$

$$= 2R'\sin\psi + 2R(\omega - \varphi')\cos\psi + \tfrac{2}{3}\omega F\cos\tfrac{1}{2}(t_0 + \omega t_1)$$

$$+\mu R\sin\psi + \tfrac{2}{3}\mu F\sin\tfrac{1}{2}(t_0 + \omega t_1) - \tfrac{1}{2}R^2(1 + \cos 2\psi)$$

$$-\tfrac{8}{3}RF\cos\psi\cos\tfrac{1}{2}(t_0 + \omega t_1) - \tfrac{16}{9}F^2(1 + \cos(t_0 + \omega t_1))$$

Secular-producing terms:

$\cos\psi$: $2R(\omega - \varphi') - \tfrac{8}{3}RF\cos\tfrac{1}{2}(t_0 + \omega t_1) = 0$?

$\sin\psi$: $2R' + \mu R = 0$

$$\rightarrow R' + \tfrac{1}{2}\mu R = 0$$

$$R = a\cos\sqrt{\mu/2}\,t_1 + b\sin\sqrt{\mu/2}\,t_1 = a_0 e^{i\sqrt{\frac{\mu}{2}}} + b_0 e^{-i\sqrt{\frac{\mu}{2}}}$$

So, assuming $\mu > 0$, as $t \to \infty$ the amplitude is simply oscillatory.

If $\mu < 0$, the amplitude might grow or decay as $t \to \infty$, depending on initial conditions

ES_APPM 420-2 “Asymptotic and Perturbation Methods”

Homework 4 (DUE TUESDAY, 3/3/09)

Problem 1. Find the first two terms of the asymptotics of the following integrals as $x \to 0$ ($x > 0$):

$$I_1(x) = \int_x^1 \cos(xt)\, dt, \quad I_2(x) = \int_x^1 \sin(xt)\, dt,$$

$$I_3(x) = \int_0^1 \frac{e^x - e^{xt}}{1-t}\, dt, \quad I_4(x) = \int_0^\infty \frac{\exp(-t)}{1+xt}\, dt.$$

Problem 2. Find the leading order term of the asymptotics of the integral

$$\int_x^{1/x} f(xt)\, dt$$

as $x \to 0$ ($x > 0$). Here $f(s)$ is a continuous function for $s > 0$ such that as $s \to 0$

$$f(s) = \frac{1}{s} + O(1).$$

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3-3-09
ESAM 4202
Hwk # 4

1) Find first two terms of asymptotics as $x \to 0$

$I_1(x) = \int_x^1 \cos(xt)\,dt$

As $x \to 0$, $\cos(xt) \sim 1 + \frac{1}{2}x^2t^2$

So $I_1(x) \sim \int_x^1 1 + \frac{1}{2}x^2t^2\,dt$

$= t + \frac{1}{6}x^2t^3 \Big|_x^1$

$= 1 + \frac{1}{6}x^2 - x - \frac{1}{6}x^5$

$\sim 1 - x$

$\boxed{I_1(x) \sim 1 - x}$ ✓

$I_2(x) = \int_x^1 \sin(xt)\,dt$

As $x \to 0$, $\sin(xt) \sim xt - \frac{1}{6}x^3t^3$

So $I_2(x) \sim \int_x^1 xt - \frac{1}{6}x^3t^3\,dt$

$= \frac{1}{2}xt^2 - \frac{1}{24}x^3t^4 \Big|_x^1$

$= \frac{1}{2}x - \frac{1}{24}x^3 - \frac{1}{2}x^3 + \frac{1}{24}x^7$

$\sim \frac{1}{2}x - \frac{13}{24}x^3$

$\boxed{I_2(x) \sim \frac{1}{2}x - \frac{13}{24}x^3}$ ✓

$I_3(x) = \int_0^1 \frac{e^x - e^{xt}}{1-t}\,dt$

$\sim \int_0^1 \frac{1 + x + \frac{1}{2}x^2 - 1 - xt - \frac{1}{2}x^2t^2}{1-t}\,dt$

$= \int_0^1 \frac{x(1-t) + \frac{1}{2}x^2(1-t^2)}{1-t}\,dt$

$= \int_0^1 x + \frac{1}{2}x^2(1+t)\,dt$

$= \int_0^1 x + \frac{1}{2}x^2 + \frac{1}{2}x^2t\,dt$

$= xt + \frac{1}{2}x^2t + \frac{1}{4}x^2t^2 \Big|_0^1$

$= x + \frac{3}{4}x^2$

$\boxed{I_3(x) \sim x + \frac{3}{4}x^2}$ ✓

(continued on reverse)

$$I_4(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \qquad \text{let } s = 1+xt \to t = \frac{1}{x}(s-1)$$
$$dt = \frac{1}{x}\,ds$$

$$= \int_1^\infty \frac{e^{\frac{1}{x}(1-s)}}{xs}\,ds$$

$$= \frac{e^{\frac{1}{x}}}{x}\int_1^\infty \frac{e^{-s/x}}{s}\,ds \qquad u = \frac{1}{s} \quad v = -xe^{-s/x}$$
$$du = -\frac{1}{s^2}\,ds \quad dv = e^{-s/x}\,ds$$

$$= \frac{e^{\frac{1}{x}}}{x}\left[\frac{-xe^{-s/x}}{s}\Bigg|_1^\infty - x\int_1^\infty \frac{e^{-s/x}}{s^2}\,ds\right] \qquad u = \frac{1}{s^2} \quad v = -xe^{-s/x}$$
$$du = \frac{-2}{s^3} \quad dv = e^{-s/x}$$

$$= \frac{e^{\frac{1}{x}}}{x}\left[\frac{-xe^{-s/x}}{s}\Bigg|_1^\infty + \frac{x^2e^{-s/x}}{s^2}\Bigg|_1^\infty + 2x^2\int_1^\infty \frac{e^{-s/x}}{s^3}\,ds\right]$$

$$= \frac{e^{\frac{1}{x}}}{x}\left[\left(0 + xe^{-\frac{1}{x}}\right) + \left(0 - x^2e^{-\frac{1}{x}}\right) + 2x^2\int_1^\infty \frac{e^{-s/x}}{s^3}\,ds\right]$$

$$= 1 - x + \underbrace{2xe^{\frac{1}{x}}\int_1^\infty \frac{e^{-s/x}}{s^3}\,ds}$$

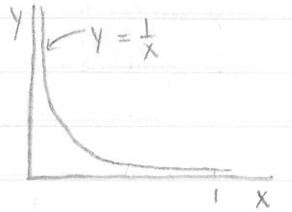
if we were to continue, we would get a term of $O(x^2)$, then $O(x^3)$, etc.

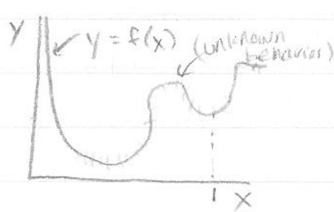
So $\boxed{I_4(x) \sim 1-x}$ ✓

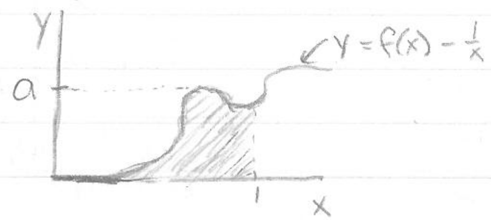
2) Find the leading order asymptotics of the integral $\int_x^{\frac{1}{x}} f(xt)\,dt$ as $x \to 0$, where as $s \to 0$, $f(s) \sim \frac{1}{s} + O(1)$

$\int_x^{\frac{1}{x}} f(xt)\,dt$ let $s = xt \to t = \frac{1}{x}s \to dt = \frac{1}{x}ds$

$$= \frac{1}{x}\int_{x^2}^{1} f(s)\,ds$$
$$= \frac{1}{x}\left[\int_{x^2}^{1} \frac{1}{s}\,ds + \int_{x^2}^{1} f(s) - \frac{1}{s}\,ds\right]$$
$$= \frac{1}{x}\left[\ln s\,\Big|_{x^2}^{1} + \int_{x^2}^{1} f(s) - \frac{1}{s}\,ds\right]$$
$$= \frac{1}{x}\left[-2\ln x + \int_{x^2}^{1} f(s) - \frac{1}{s}\,ds\right]$$







The value of the second integral is equal to the area of the shaded region above. Since f is continuous and we are looking at [0,1], a finite interval, this value must be O(1). And $O(1) \ll \ln x$ for $x \to 0$, so the leading order term is just the first term above.

$$\int_x^{\frac{1}{x}} f(xt)\,dt \sim \frac{-2\ln x}{x}$$

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3-10-09
ESAM 420-2
Hwk #5

Use the method of stationary phase to find leading order behavior as $x \to \infty$

$I_1(x) = \int_0^1 e^{ix(t-\sin t)}\,dt$

$f(t) = 1$, $\varphi(t) = t - \sin t$. $\varphi'(t) = 1-\cos t$, $\varphi''(t) = \sin t$

$\varphi'(t) = 0$ on $[0,1]$ at $t=0$ $\qquad \varphi'''(t) = \cos t$

near $t=0$, $\varphi(t) \sim \varphi(0) + \frac{1}{2}\varphi''(0)t^2 + \frac{1}{6}\varphi'''(0)t^3$

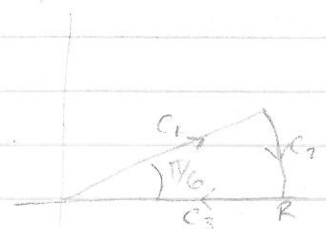
$= 0 + 0 + \frac{1}{6}t^3$

$I_1(x) \sim \int_0^\varepsilon e^{ix\frac{1}{6}t^3}\,dt$ $\quad$ let $s^3 = \frac{1}{6}xt^3$ $\quad s = \left(\frac{x}{6}\right)^{\frac{1}{3}}t$

$= \left(\frac{6}{x}\right)^{\frac{1}{3}} \int_0^{\left(\frac{x}{6}\right)^{\frac{1}{3}}\varepsilon} e^{is^3}\,ds$ $\quad t = \left(\frac{6}{x}\right)^{\frac{1}{3}}s \quad dt = \left(\frac{6}{x}\right)^{\frac{1}{3}}ds$

want $\varepsilon(x) \to 0$ and $\varepsilon(x)x^{\frac{1}{3}} \to \infty$ as $x \to \infty$: let $\varepsilon(x) = x^{-\frac{1}{6}}$

$\sim \left(\frac{6}{x}\right)^{\frac{1}{3}} \int_0^\infty e^{is^3}\,ds$



let $C = C_1 \cup C_2 \cup C_3$

Then $\oint_C e^{is^3}\,ds = 0$

$\to \int_{-C_3} = \int_{C_1} + \int_{C_2}$

$\int_{C_1} e^{is^3}\,ds$ $\quad$ let $s = re^{i\pi/6}$ $\quad ds = e^{i\pi/6}dr$, $r: 0 \to R$

$= \int_0^R \exp(i(re^{i\pi/6})^3)\,e^{i\pi/6}\,dr$

$= e^{i\pi/6}\int_0^R \exp(ir^3 e^{i\pi/2})\,dr$

$= e^{i\pi/6}\int_0^R e^{-r^3}\,dr$

$\approx e^{i\pi/6}\cdot(0.893)$

(not sure how to evaluate this)

– reduce to the gamma-function

$\int_{C_2} e^{is^3}\,ds$ $\quad$ let $s = Re^{i\theta}$ $\quad ds = Rie^{i\theta}d\theta$, $\theta: \pi/6 \to 0$

$= \int_{\pi/6}^0 \exp(i(Re^{i\theta})^3)\,Rie^{i\theta}\,d\theta$

$= -Ri\int_0^{\pi/6} \exp(iR^3e^{i3\theta})\,e^{i\theta}\,d\theta$

$= -Ri\int_0^{\pi/6} \exp(iR^3(\cos 3\theta + i\sin 3\theta))\,e^{i\theta}\,d\theta$

$= -Ri\int_0^{\pi/6} e^{iR^3\cos 3\theta}\,e^{-R^3\sin 3\theta}\,e^{i\theta}\,d\theta$

Magnitude of this $\uparrow$ $\leq R\int_0^{\pi/6} e^{-R^3\sin 3\theta}\,d\theta$

Laplace integral with $\varphi = \sin 3\theta$: max in $[0, \pi/6]$ at $\theta = \pi/6$

$\sim R\int_{\pi/6-\varepsilon}^{\pi/6} e^{-R^3 3\theta}\,d\theta$ $\quad$ let $\tau = \theta - \pi/6$

$\sim R\int_{-\varepsilon}^{0} e^{-R^2 2(\tau+\pi/6)}\,d\tau$

$= Re^{-R^2\pi/3}\int_{-\varepsilon}^{0} e^{-2R^2\tau}\,d\tau$ let $u=-2R^2\tau$ $du=-2R^2d\tau$

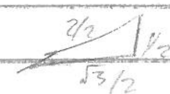
$= \frac{-1}{2R}e^{-R^2\pi/3}\int_{2R^2\varepsilon}^{0} e^{u}\,du$

$= -\frac{1}{2R}e^{-R^2\pi/3}\, e^{u}\Big|_{u=2R^2\varepsilon}^{u=0}$

$= -\frac{1}{2R}e^{-R^2\pi/3} + \frac{1}{2R}e^{R^2(2\varepsilon-\pi/3)}$ need $2\varepsilon-\pi/3<0$ as $R\to\infty$

Then this $\to 0$ as $R\to\infty$ $\varepsilon<\pi/6$ let $\varepsilon=\frac{1}{R}$ ✓

Thus $I_1(x)\sim\left(e^{i\pi/6}\int_0^\infty e^{-r^3}dr\right)\left(\frac{6}{x}\right)^{\frac{1}{3}}$



$= \left(\frac{6}{x}\right)^{\frac{1}{3}}\frac{1}{2}(\sqrt{3}+i)\int_0^\infty e^{-r^3}dr$ ✓

o.k.

$f(t) = e^{-5it}$

$\psi(t) = \cos t$

$$I_2(x) = \int_0^\pi \sin(x\cos t - 5t)\,dt$$
$$= \mathrm{Im}\left[\int_0^\pi e^{ix\cos t} e^{-5it}\,dt\right]$$
$$= \mathrm{Im}\left[\int_0^\pi \cos 5t\, e^{ix\cos t} - i\int_0^\pi \sin 5t\, e^{ix\cos t}\,dt\right]$$
$$= \mathrm{Im}\left[I_A - i\, I_B\right]$$

$\psi(t) = \cos t \quad \psi'(t) = -\sin t \quad \psi''(t) = -\cos t$

$\psi'(t) = 0$ at $t = 0$ + $t = \pi$

near $t=0$: $\psi(t) \sim \psi(0) + \frac{1}{2}\psi''(0)t^2$
$$= 1 - \tfrac{1}{2}t^2$$

$f_A(t) \sim f_A(0) = 1$

$f_B(t) \sim f_B(0) + f_B'(0)t = 5t$

near $t=\pi$: $\psi(t) \sim \psi(\pi) + \frac{1}{2}\psi''(\pi)(t-\pi)^2$
$$= -1 + \tfrac{1}{2}(t-\pi)^2$$

$f_A(t) \sim f_A(\pi) = -1$

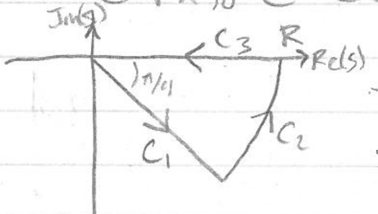
$f_B(t) \sim f_B(\pi) + f_B'(\pi)(t-\pi) = -5(t-\pi)$

$$\to I_A \sim \int_0^{\varepsilon_1} e^{ix(1-\frac{1}{2}t^2)}\,dt + \int_{\pi-\varepsilon_2}^{\pi} -e^{ix(-1+\frac{1}{2}(t-\pi)^2)}\,dt$$
$$= I_{A1} + I_{A2}$$
$$I_B \sim \int_0^{\varepsilon_3} 5t\, e^{ix(1-\frac{1}{2}t^2)}\,dt + \int_{\pi-\varepsilon_4}^{\pi} -5(t-\pi)\, e^{ix(-1+\frac{1}{2}(t-\pi)^2)}\,dt$$
$$= I_{B1} + I_{B2}$$

$$I_{A1} = e^{ix}\int_0^{\varepsilon_1} e^{-ix\frac{1}{2}t^2}\,dt \qquad \left\{ \text{let } s^2 = \tfrac{1}{2}xt^2,\ s = \sqrt{\tfrac{x}{2}}\,t \quad t = \sqrt{\tfrac{2}{x}}\,s \quad dt = \sqrt{\tfrac{2}{x}}\,ds \right.$$
$$= e^{ix}\sqrt{\tfrac{2}{x}}\int_0^{\varepsilon_1\sqrt{\frac{x}{2}}} e^{-is^2}\,ds$$

need $\varepsilon_1(x) \to 0$ + $\varepsilon_1(x)\sqrt{x} \to \infty$ as $x \to \infty$, let $\varepsilon(x) = x^{-1/3}$

$$\sim e^{ix}\sqrt{\tfrac{2}{x}}\int_0^\infty e^{-is^2}\,ds$$



let $C = C_1 \cup C_2 \cup C_3$

Then $\oint_C e^{-is^2}\,ds = 0$

$\to \int_{-C_3} = \int_{C_1} + \int_{C_2}$

$\int_{C_1} e^{-is^2}\,ds \qquad$ let $s = re^{-i\pi/4} \to ds = e^{-i\pi/4}\,dr$, $r: 0 \to R$
$$= \int_0^R \exp(-i(re^{-i\pi/4})^2)\, e^{-i\pi/4}\,dr$$
$$= \sqrt{2}(1-i)\int_0^R \exp(-ir^2 e^{-i\pi/2})\,dr$$
$$= \sqrt{2}(1-i)\int_0^R e^{-r^2}\,dr = \tfrac{1}{2}\sqrt{2}(1-i)\sqrt{\pi} \quad \text{as } R \to \infty$$

$\int_{C_2} e^{-is^2}\,ds$ let $s = Re^{i\theta}$, $ds = Rie^{i\theta}\,d\theta$, $\theta: -\pi/4 \to 0$

$\int_{-\pi/4}^{0} \exp(-i(Re^{i\theta})^2)\,Rie^{i\theta}\,d\theta$

$= Ri\int_{-\pi/4}^{0} \exp(-iR^2 e^{i2\theta})\,e^{i\theta}\,d\theta$

$= Ri\int_{-\pi/4}^{0} \exp(-iR^2(\cos 2\theta + i\sin 2\theta))\,e^{i\theta}\,d\theta$

$= Ri\int_{-\pi/4}^{0} e^{-iR^2\cos 2\theta}\, e^{R^2\sin 2\theta}\, e^{i\theta}\,d\theta$

magnitude of this ↑ $\le R\int_{-\pi/4}^{0} e^{R^2\sin 2\theta}\,d\theta$

Laplace integral with $\varphi = \sin 2\theta$: max on $[-\pi/4, 0]$ at $\theta = 0$

$\sim R\int_{-\varepsilon}^{0} e^{R^2 2\theta}\,d\theta$ let $u = 2R^2\theta$ $du = 2R^2\,d\theta$

$= R\int_{-2R^2\varepsilon}^{0} e^{u}\,du$

$= R\, e^{u}\Big|_{u=-2R^2\varepsilon}^{u=0}$ let $\varepsilon(R) = R^{-1}$ then,

$= R - Re^{-2R^2\varepsilon}$ as $R \to \infty$ this $\to 0$

Thus $I_{A1} \sim (1-i)\sqrt{\pi/2}\, e^{ix}\sqrt{\frac{2}{x}} = (1-i)e^{ix}\sqrt{\frac{\pi}{x}}$

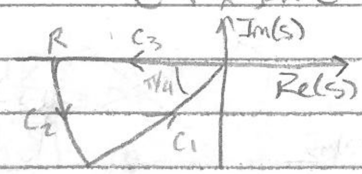
$I_{A2} = -e^{-ix}\int_{\pi-\varepsilon_2}^{\pi} e^{\frac{1}{2}ix(t-\pi)^2}\,dt$ let $\tau = t - \pi$

$= -e^{-ix}\int_{-\varepsilon_2}^{0} e^{i\frac{1}{2}x\tau^2}\,d\tau$ let $s^2 = \frac{1}{2}x\tau^2$, $s = \sqrt{\frac{x}{2}}\,\tau$

$= -e^{-ix}\sqrt{\frac{2}{x}}\int_{-\varepsilon_2\sqrt{\frac{x}{2}}}^{0} e^{is^2}\,ds$ $\tau = \sqrt{\frac{2}{x}}\,s$, $d\tau = \sqrt{\frac{2}{x}}\,ds$

need $\varepsilon_2(x) \to 0$ and $\varepsilon_2(x)\sqrt{x} \to \infty$ as $x \to \infty$: let $\varepsilon_2(x) = x^{-\frac{1}{3}}$

$\sim -e^{-ix}\sqrt{\frac{2}{x}}\int_{-\infty}^{0} e^{is^2}\,ds$



let $C = C_1 \cup C_2 \cup C_3$

Then $\oint_C e^{is^2}\,ds = 0$

$\to \int_{-C_3} = \int_{C_1} + \int_{C_2}$

$\int_{C_1} e^{is^2}\,ds$ let $s = e^{-i3\pi/4} r$, $ds = e^{-i3\pi/4}\,dr$, $r: R \to 0$

$\int_{R}^{0} \exp(i(e^{-i3\pi/4} r)^2)\,e^{-i3\pi/4}\,dr$

$= -e^{-i3\pi/4}\int_{0}^{R} \exp(i e^{-i3\pi/2} r^2)\,dr$

$= \sqrt{2}(1+i)\int_{0}^{R} e^{-r^2}\,dr$

$= \sqrt{2}(1+i)\frac{1}{2}\sqrt{\pi} = (1+i)\sqrt{\pi/2}$

$\int_{C_2} e^{is^2}\,ds$ let $s = Re^{i\theta}$, $ds = Rie^{i\theta}\,d\theta$, $\theta: \pi \to 5\pi/4$

$= \int_{\pi}^{5\pi/4} \exp(i(Re^{i\theta})^2)\,Rie^{i\theta}\,d\theta$

$= Ri\int_{\pi}^{5\pi/4} e^{iR^2\cos 2\theta}\, e^{-R^2\sin 2\theta}\, e^{i\theta}\,d\theta$

magnitude of ↑ $\le R\int_{\pi}^{5\pi/4} e^{-R^2\sin 2\theta}\,d\theta$

Laplace integral with $\varphi = \sin 2\theta$ max on $[\pi, 5\pi/4]$ at $5\pi/4$ $d\theta$

$\sim R\int_{5\pi/4-\varepsilon}^{5\pi/4} e^{-R^2 2\theta}\,d\theta$ let $\tau=\theta-5\pi/4$

$= R\int_{-\varepsilon}^{0} e^{-R^2 2(\tau+5\pi/4)}\,d\tau$

$= Re^{-\frac{5\pi}{2}R^2}\int_{-\varepsilon}^{0} e^{-2R^2\tau}$ let $u=-2R^2\tau$ $du=-2R^2\,d\tau$

$= -\frac{1}{2R} e^{-\frac{5\pi}{2}R^2}\int_{2R^2\varepsilon}^{0} e^{u}\,du$

$= -\frac{1}{2R} e^{-\frac{5\pi}{2}R^2}\left(1-e^{2R^2\varepsilon}\right)$

$= -\frac{1}{2R} e^{-\frac{5\pi}{2}R^2}+\frac{1}{2R}e^{R^2(2\varepsilon-\frac{5\pi}{2})}$ need $2\varepsilon-\frac{5\pi}{2}<0$ as $R\to\infty$

$\varepsilon<\frac{5\pi}{4}$ let $\varepsilon=\frac{1}{R}$ ✓

Then this $\to 0$ as $R\to\infty$

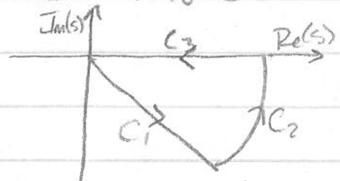
Thus $I_{A2}\sim(1+i)\sqrt{\pi/2}\left(-e^{-ix}\sqrt{\frac{2}{x}}\right) = (1+i)e^{-ix}\sqrt{\frac{\pi}{x}}$

$I_{B1} = 5e^{ix}\int_0^{\varepsilon_3} t\, e^{-\frac{1}{2}ixt^2}\,dt$ $\Big\{$ let $s^2=\frac{1}{2}xt^2 \to s=\sqrt{\frac{x}{2}}\,t$, $t=\sqrt{\frac{2}{x}}\,s \to dt=\sqrt{\frac{2}{x}}\,dt$

$= 5e^{ix}\frac{2}{x}\int_0^{\sqrt{\frac{x}{2}}\varepsilon_3} s\,e^{-is^2}\,ds$

need $\varepsilon_3(x)\to 0$, $\varepsilon_3(x)\sqrt{x}\to\infty$ as $x\to\infty$: let $\varepsilon_3(x)=x^{-\frac{1}{3}}$.

$\sim 5e^{ix}\frac{2}{x}\int_0^{\infty} s\,e^{-is^2}\,ds$



let $C=C_1\cup C_2\cup C_3$

then $\oint_C s\,e^{-is^2}=0$

$\to \int_{-C_3}=\int_{C_1}+\int_{C_2}$

$\int_{C_1} s\,e^{-is^2}\,ds$ let $s=e^{-i\pi/4}r$, $ds=e^{-i\pi/4}dr$, $r: 0\to R$

$\int_0^R e^{-i\pi/4}r\exp(-i(e^{-i\pi/4}r)^2)e^{-i\pi/4}\,dr$

$= e^{-i\pi/2}\int_0^R r\exp(-ie^{-i\pi/2}r^2)\,dr$

$= e^{-i\pi/2}\int_0^R r\,e^{-r^2}\,dr$ let $u=-r^2$ $du=-2r\,dr$

$= +\frac{1}{2}i\int_0^{-R^2} e^{u}\,du$

$= -\frac{1}{2}i\; e^{u}\Big|_{-R^2}^{0} = -\frac{1}{2}i$

$\int_{C_2} s\,e^{-is^2}\,ds$ let $s=Re^{i\theta}$, $ds=iRe^{i\theta}d\theta$, $\theta: -\pi/4\to 0$

$\int_{-\pi/4}^{0} Re^{i\theta}\exp(-i(Re^{i\theta})^2)\,iRe^{i\theta}\,d\theta$

$= iR^2\int_{-\pi/4}^{0}\exp(-iR^2e^{i2\theta})\,e^{i2\theta}\,d\theta$

$= iR^2\int_{-\pi/4}^{0}\exp(-iR^2(\cos 2\theta+i\sin 2\theta))\,e^{i2\theta}\,d\theta$

$= iR^2\int_{-\pi/4}^{0} e^{-iR^2\cos 2\theta}\,e^{R^2\sin 2\theta}\,e^{i2\theta}\,d\theta$

magnitude of this ↑ $\le R^2\int_{-\pi/4}^{0} e^{R^2\sin 2\theta}$

from calculations for I_{A1}, we know this $\to 0$ as $R\to\infty$

(this has an extra factor of R but that is overpowered by the exponential decay.)

Thus $I_{B1}\sim -\frac{1}{2}i\left(5e^{ix}\frac{2}{x}\right) = -5i\frac{1}{x}e^{ix}$

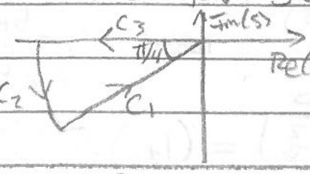
$I_{B2} = -5e^{-ix}\int_{\pi-\varepsilon_4}^{\pi}(t-\pi)e^{\frac{1}{2}ix(t-\pi)^2}dt$ let $\tau = t-\pi$

$= -5e^{-ix}\int_{-\varepsilon_4}^{0}\tau e^{x\frac{1}{2}i\tau^2}d\tau$ let $s^2=\frac{1}{2}x\tau^2$, $s=\sqrt{\frac{x}{2}}\tau$

$= -5e^{-ix}\frac{2}{x}\int_{-\varepsilon_4\sqrt{\frac{x}{2}}}^{0} se^{is^2}ds$ $\tau=\sqrt{\frac{2}{x}}s$ $d\tau=\sqrt{\frac{2}{x}}ds$

need $\varepsilon_4(x)\to 0$ and $\varepsilon_4(x)\sqrt{x}\to\infty$ as $x\to\infty$: let $\varepsilon_4(x)=x^{-\frac{1}{3}}$.

$\sim -\frac{10}{x}e^{-ix}\int_{-\infty}^{0} se^{is^2}ds$



let $C = C_1\cup C_2\cup C_3$

Then $\oint_C se^{is^2}ds = 0$

$\rightarrow \int_{-C_3} = \int_{C_1}+\int_{C_2}$

$\int_{C_1} se^{is^2}ds$ let $s=e^{-3\pi/4 i}r$ $ds=e^{-3\pi/4 i}dr$, $r: R\to 0$

$=\int_R^0 re^{-3\pi/4 i}\exp(i(e^{-3\pi/4 i}r)^2)e^{-3\pi/4 i}dr$

$=-e^{-\frac{3\pi}{2}i}\int_0^R r\exp(ie^{-\frac{3\pi}{2}i}r^2)dr = -e^{-\frac{3\pi}{2}i}\int_0^R re^{-r^2}dr$

$=\frac{1}{2}e^{-\frac{3\pi}{2}i}e^{-r^2}\Big|_0^R = -\frac{1}{2}e^{-\frac{3\pi}{2}i} = -\frac{1}{2}i$

$\int_{C_2} se^{is^2}ds$ let $s=Re^{i\theta}$ $ds=iRe^{i\theta}d\theta$, $\theta: \pi\to 5\pi/4$

$=\int_\pi^{5\pi/4}Re^{i\theta}\exp(i(Re^{i\theta})^2)iRe^{i\theta}d\theta$

$=R^2 i\int_\pi^{5\pi/4}e^{2i\theta}\exp(iR^2e^{2i\theta})d\theta$

$=R^2 i\int_\pi^{5\pi/4}e^{2i\theta}\exp(iR^2(\cos 2\theta+i\sin 2\theta))d\theta$

$=R^2 i\int_\pi^{5\pi/4}e^{2i\theta}e^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}d\theta$

magnitude of this $\uparrow$ $\le R^2\int_\pi^{5\pi/4}e^{-R^2\sin 2\theta}d\theta$

from calculations for I_{A2}, we know this $\to 0$ as $R\to\infty$

(this has an extra factor of R, but that is overpowered by the exponential decay)

Thus $I_{B2}\sim -\frac{1}{2}i\left(-\frac{10}{x}e^{-ix}\right) = -5i\frac{1}{x}e^{-ix}$

So, $I_2(x) = \mathrm{Im}\left[I_{A1}+I_{A2}-i\left[I_{B1}+I_{B2}\right]\right]$

$=\mathrm{Im}\left[(1+i)e^{-ix}\sqrt{\frac{\pi}{x}}+(1-i)\sqrt{\frac{\pi}{x}}e^{ix}-i\left(-5i\frac{1}{x}e^{ix}-5i\frac{1}{x}e^{-ix}\right)\right]$

$=\mathrm{Im}\left[\left[(1+i)(\cos x-i\sin x)+(1-i)(\cos x+i\sin x)\right]\sqrt{\frac{\pi}{x}}\right.$

$\left.-5\frac{1}{x}(\cos x+i\sin x+\cos x-i\sin x)\right]$

$=\mathrm{Im}\left[\sqrt{\frac{\pi}{x}}(2\sin x)-5\frac{1}{x}(2\cos x)\right]$ Algebra?

$=0$?

1	2
2	2
3	3

4

np. 86

ES_APPM 420-2 "Asymptotic and Perturbation Methods"

Final Examination (DUE TUESDAY, 3/17/08)

Problem 1.
Find the leading order term of the asymptotics of the following integrals as $x \to \infty$:

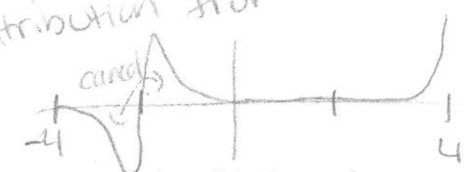
$$I_1(x) = \int_0^1 \sqrt{t(1-t)}(t+2)^{-x}\, dt,$$ maybe. $\sqrt{t(t-1)} \sim \sqrt{t}$?

$$I_2(x) = \int_0^\infty e^{-t-x/t^2}\, dt,$$ maybe.

$$I_3(x) = \int_0^1 \sin\left[x\left(t + \frac{1}{6}t^3 - \sinh t\right)\right]\, dt,$$

$$I_4(x) = \int_0^1 \cos(xt)\tan t\, dt,$$

$$I_5(x) = \int_{-4}^4 (t+2)e^{xt(t^2-12)}\, dt.$$ contribution from x=4



Problem 2. Find the leading order term of the asymptotics of the integral

$$I(x) = \int_0^\infty e^{-xt^2}\sinh t\, dt$$ maybe?

as $x \to 0$ $(x > 0)$.

do we need to let limits $\to \infty$?

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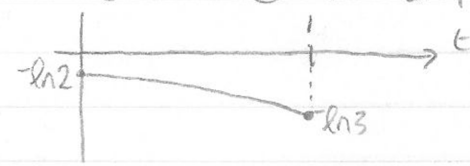
3-17-09
ESAM 420-2
Final

1) Find the leading order term of the asymptotics of the following integrals as $x \to \infty$:

$$I_1(x) = \int_0^1 \sqrt{t(1-t)}\,(t+2)^{-x}\,dt$$
$$= \int_0^1 \sqrt{t(1-t)}\, e^{-x\ln(t+2)}$$
$$\varphi = -\ln(t+2)$$
$$\varphi' = -\frac{1}{t+2} \neq 0 \text{ on } [0,1]$$



maximum at $t=0$

The major contribution to the integral comes from near $t=0$.

near $t=0$: $\varphi \sim -\ln 2 - \frac{1}{2}t$

$f \sim \sqrt{t}$

$$I_1(x) \sim \int_0^\varepsilon \sqrt{t}\, e^{-x(\ln 2 + \frac{1}{2}t)}$$

let $\frac{1}{2}xt = s^2 \to t = \frac{2}{x}s^2 \quad dt = \frac{4}{x}s\,ds$

$s = \sqrt{\frac{x}{2}}\sqrt{t} \qquad \sqrt{t} = \sqrt{\frac{2}{x}}\,s$

$$\to e^{-x\ln 2}\sqrt{\frac{2}{x}}\,\frac{4}{x}\int_0^{\sqrt{\frac{x\varepsilon}{2}}} s^2 e^{-s^2}\,ds \qquad u = s^2 \quad v = -\frac{1}{2s}e^{-s^2}$$
$$\sim e^{-x\ln 2}\sqrt{\frac{2}{x}}\,\frac{4}{x}\left[-\frac{1}{2}se^{-s^2}\Big|_0^\infty + \int_0^\infty e^{-s^2}ds\right] \qquad du = 2s\,ds \quad dv = e^{-s^2}$$
$$= e^{-x\ln 2}\sqrt{\frac{2}{x}}\,\frac{4}{x}\,\sqrt{\pi}\,\frac{1}{2}$$
$$= \frac{1}{2}e^{-x\ln 2}\left(\frac{2}{x}\right)^{3/2}\sqrt{\pi}$$

- check your algebra

$I_2(x) = \int_0^\infty e^{-t - x/t^2}\,dt$

$$\text{exponent} = g = -t - \frac{x}{t^2}$$

$$\partial g/\partial t = -1 + \frac{2x}{t^3} = 0$$

$$\frac{2x}{t^3} = 1$$

$$2x = t^3$$

$$(2x)^{\frac{1}{3}} = t$$

There is a moving maximum!

Let $s = t(2x)^{-\frac{1}{3}} \to t = s(2x)^{\frac{1}{3}} \to dt = (2x)^{\frac{1}{3}}ds$

For ease of notation, let $2^{\frac{1}{3}} = a$ and $x^{\frac{1}{3}} = \xi$

$$s = t/a\xi \qquad t = sa\xi \qquad dt = a\xi\,ds$$

$$\frac{x}{t^2} = \frac{x}{(sa\,x^{\frac{1}{3}})^2} = \xi/a^2s^2$$

$$\to I_2(\xi) = \int_0^\infty e^{-sa\xi - \xi/a^2s^2}(a\xi)\,ds$$

$$= +a\xi\int_0^\infty e^{-\xi\left(\frac{1}{a^2s^2} + sa\right)}ds$$

$$\varphi = -\frac{1}{a^2s^2} - sa$$

$$\varphi' = \frac{2}{a^2s^3} - a = 0$$

$$\frac{2}{a^2s^3} = a$$

$$\frac{2}{a^3} = s^3$$

$$\frac{2}{2} = s^3$$

$$s = 1$$

$$\varphi'' = -\frac{6}{a^2s^3}, \quad \varphi''(1) = -\frac{6}{a^2} < 0$$

So φ has a maximum at $s=1$

$$\to I_2(\xi) \sim +a\xi\int_{1-\varepsilon}^{1+\varepsilon} e^{\xi\left(-\frac{1}{a^2} - a - \frac{3}{a^2}(s-1)^2\right)}ds$$

let $\tau = s-1$, note $\frac{1}{a^2} + a = 3/a^2$

$$= +a\xi e^{-\frac{3}{a^2}\xi}\int_{-\varepsilon}^{\varepsilon} e^{-3/a^2\,\xi\tau^2}\,d\tau$$

let $s^2 = \frac{3}{a^2}\xi\tau^2$, $s = \frac{\sqrt{3\xi}}{a}\tau$, $d\tau = \frac{a}{\sqrt{3\xi}}ds$

$$\to \frac{a}{\sqrt{3\xi}}(+a\xi)e^{-3/a^2\,\xi}\int_{(-\varepsilon\sqrt{3\xi})/a}^{(\varepsilon\sqrt{3\xi})/a} e^{-s^2}ds$$

$\sim +\frac{1}{\sqrt{3}}a^2\sqrt{\xi}\,e^{-3/a^2\,\xi}\int_{-\infty}^{\infty} e^{-s^2}ds$ ← if we let $\varepsilon\sqrt{\xi} \to \infty$, so let $\varepsilon = \xi^{-\frac{1}{3}} = x^{-\frac{1}{9}}$

$$= +\sqrt{\frac{\pi\xi}{3}}\,a^2 e^{-3/a^2\,\xi}$$

$$= +\sqrt{\frac{\pi}{3}}\,2^{2/3}x^{\frac{1}{6}}e^{-3/2\,\xi}$$

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2-17-09
ESAM 420-2
Mid-term

1) $x'' - \varepsilon(1-x^2)x' + x = 0$; $x(0)=a$, $x'(0)=0$.

let $t_0 = t$, $t_1 = \varepsilon t$, $y(t_0,t_1) = x(t)$

$\rightarrow \frac{\partial^2 y}{\partial t_0} + 2\varepsilon \frac{\partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 y}{\partial t_1^2} - \varepsilon(1-y^2)\left(\frac{\partial y}{\partial t_0} + \varepsilon \frac{\partial y}{\partial t_1}\right) + y = 0$

$\rightarrow \frac{\partial^2 y}{\partial t_0} + \frac{2\varepsilon \partial^2 y}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 y}{\partial t_1^2} - \varepsilon \frac{\partial y}{\partial t_0} - \varepsilon^2 \frac{\partial y}{\partial t_1} + \varepsilon y^2 \frac{\partial y}{\partial t_0} + \varepsilon^2 y^2 \frac{\partial y}{\partial t_1} + y = 0$

let $y \sim y_0 + \varepsilon y_1$

$O(1)$: $\frac{\partial^2 y_0}{\partial t_0} + y_0 = 0$, $y_0(0,0) = a$, $\frac{\partial y_0}{\partial t_0}(0,0) = 0$

$y_0(t_0,t_1) = A_0(t_1)\cos t_0 + B_0(t_1)\sin t_0$

$y_0(0,0) = A_0(0) = a$

$\frac{\partial y_0}{\partial t_0}(0,0) = B_0(0) = 0$

$O(\varepsilon)$: $\frac{\partial^2 y_1}{\partial t_0} + y_1 = -\frac{2\partial^2 y_0}{\partial t_0 \partial t_1} + \frac{\partial y_0}{\partial t_0} - y_0^2 \frac{\partial y_0}{\partial t_0}$

$= 2A_0' \sin t_0 - 2B_0' \cos t_0 - A_0 \sin t_0 + B_0 \cos t_0$
$-(A_0 \cos t_0 + B_0 \sin t_0)^2(-A_0 \sin t_0 + B_0 \cos t_0)$

$= 2A_0' \sin t_0 - 2B_0' \cos t_0 - A_0 \sin t_0 + B_0 \cos t_0$
$(-A_0^2 \cos^2 t_0 - B_0^2 \sin^2 t_0 - 2A_0 B_0 \cos t_0 \sin t_0)$
$\cdot(-A_0 \sin t_0 + B_0 \cos t_0)$

$= 2A_0' \sin t_0 - 2B_0' \cos t_0 - A_0 \sin t_0 + B_0 \cos t_0$
$+ A_0^3 \cos^2 t_0 \sin t_0 + A_0 B_0^2 \sin^3 t_0 + 2A_0^2 B_0 \cos t_0 \sin^2 t_0$
$- A_0^2 B_0 \cos^3 t_0 - B_0^3 \cos t_0 \sin^2 t_0 - 2A_0 B_0^2 \cos^2 t_0 \sin t_0$

$\cos^2 t_0 \sin t_0 = \frac{1}{2}(1+\cos 2t_0)\sin t_0$
$= \frac{1}{2}\sin t_0 + \frac{1}{2}\cos 2t_0 \sin t_0$
$= \frac{1}{2}\sin t_0 + \frac{1}{4}\sin 3t_0 - \frac{1}{4}\sin t_0$
$= \frac{1}{4}\sin t_0 + \frac{1}{4}\sin 3t_0$

$\sin^2 t_0 \cos t_0 = \frac{1}{2}(1-\cos 2t_0)\cos t_0$
$= \frac{1}{2}\cos t_0 - \frac{1}{2}\cos 2t_0 \cos t_0$
$= \frac{1}{2}\cos t_0 - \frac{1}{4}\cos t_0 - \frac{1}{4}\cos 3t_0$
$= \frac{1}{4}\cos t_0 - \frac{1}{4}\cos 3t_0$

$\sin^3 t_0 = \frac{1}{2}(1-\cos 2t_0)\sin t_0 = \frac{1}{2}\sin t_0 - \frac{1}{2}\cos 2t_0 \sin t_0$
$= \frac{1}{2}\sin t_0 - \frac{1}{4}\sin 3t_0 + \frac{1}{4}\sin t_0 = \frac{3}{4}\sin t_0 - \frac{1}{4}\sin 3t_0$.