PRELIMINARY EXAM - LINEAR ALGEBRA - SPRING, 2010

Note - all problems count equally

1. Consider the matrix

$$A = \begin{pmatrix} -1.5 & -.5 & -1.5 \\ 1 & .5 & 1 \\ 1 & .5 & 1 \end{pmatrix}.$$

   Show that $A^n \to 0$ as $n \to \infty$.

2. Let U be the matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

   (a) Determine all values of θ in the interval $0 \leq \theta < 2\pi$ so that $U(\theta)$ has real eigenvalues.

   (b) Show that $U(\theta + \phi) = U(\theta)U(\phi) = U(\phi)U(\theta)$ for any θ and ϕ.

   (c) Determine $U^{-1}(\theta)$

3. Let $f(p)$ be a **linear** map of the space of cubic polynomials onto itself such that $f(1) = x^3$, $f(x) = 2x$, $f(x^2) = x^2(1+x)$ and $f(x^3) = 1$.

   (a) Show that the transformation is one-to-one, so that for every cubic polynomial p there is a unique cubic polynomial q so that $f(q) = p$.

   (b) Find q when $p = 1 + x + x^2 + x^3$.

4. Consider the space of quadratic polynomials on the interval $[0, 1]$ with inner product defined as

$$(p, q) = \int_0^1 p(x)q(x)dx.$$

   Determine an orthonormal basis for this space.

5. Differentiation is a linear operator that maps the space of quadratic polynomials onto itself. Write the matrix representation of this operator with respect to the basis you computed in problem 4.

6. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

Let $\vec{x}$ be a column vector of size 3 and define

$$H(\vec{x}) = (\vec{x}, A\vec{x}).$$

Determine the global minimum and global maximum of $H(\vec{x})$ over all vectors of unit length.