PRELIMINARY EXAM – LINEAR ALGEBRA 1/10

Note - all problems count equally

1. Consider the matrix
$$A = \begin{pmatrix} 0 & -4 & -4 \\ -1 & 0 & -1 \\ 1 & 4 & 5 \end{pmatrix}.$$
Determine $\sqrt{A}$.

2. Consider the system of equations
$$A\vec{x} = \vec{b}, \quad \vec{b} = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix},$$
where A is the same matrix as in the first problem. Determine the most general condition on α and β so that this system has at least one solution.

3. Let A be a real $n \times n$ matrix. Define the function $\xi(A)$ as the number of nonzero elements of A.
(1) - Show that if $\xi(A) \le n-1$ the matrix A is singular.
(2) - Show that if $n = 2$ and $\xi(A) = n+1$ the matrix A is nonsingular.
(3) - Give a counterexample to show that this is not true if $n = 3$.

4. Let A be an $n \times n$ matrix that is diagonally dominant, i.e., for $i = 1, 2 \ldots n$ we have
$$\mid a_{i,i} \mid > \sum_{j \neq i} \mid a_{i,j} \mid.$$
Write A as
$$A = D - OD,$$
where D contains just the diagonal entries of A and OD contains all of the off-diagonal entries. Show that the iteration scheme
$$D\vec{x}_{m+1} = \vec{b} + OD\vec{x}_m,$$
converges as $m \to \infty$ to the solution of
$$A\vec{x} = \vec{b}.$$

5. Let A be the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Determine $\exp(A)$.

6. Let A be an $n \times n$ matrix and define

$$| A |^2 = \max \frac{(A\vec{x}, A\vec{x})}{(\vec{x}, \vec{x})},$$

for any nonzero vector $\vec{x}$.
(1) Show that $| A |$ is a norm on the space of $n \times n$ matrices.
(2) Show that $| AB | \leq | A || B |$ for any two $n \times n$ matrices A and B.
(3) Show that $| A | = \sqrt{3}$ for the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

7. Let $f(p)$ be a **linear** map of the space of quadratic polynomials onto itself such that $f(1) = 1 - x + x^2$, $f(x) = -2 + x + 2x^2$, and $f(x^2) = -2 - x + 4x^2$. Find the quadratic polynomial p such that $f(p) = 1 + 2x + 4x^2$.