$u = -3v$

$\begin{matrix} 3 & 9 \\ -1 & -3 \end{matrix}$

$y = ax^2 + bx + c$

$y' = 2ax + b$

$y'' = 2a$

$2a - 2ax^2 - bx + 2ax^2 + 2bx + 2c = 0$

$2a + bx + 2bx + 2c = 0$

DIFFERENTIAL EQUATIONS
ESAM Preliminary Examination
April 2, 2010, 9:30-11:30am

1. Find the solution of the initial value problem

$$y' = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$y = ax + b$

$y' = a$

$y'' = 0$

2. Find the solution of the initial value problem

$$y'' - xy' + 2y = x, \qquad y(0) = 1, \;\; y'(0) = 1$$

(Hint: the homogeneous equation has a polynomial solution).

$y = x^2 - 1$

$y' = 2x$

$y'' = 2$

$2 - 2x^2 + 2x^2 + 2 = 0$

$-ax + 2(ax + b) = 0$

3. Determine all real critical points of the system

$$\frac{dx}{dt} = x - x^2 - xy, \quad \frac{dy}{dt} = 3y - xy - 2y^2$$

and discuss their type and stability.

$1 - 2x - y$

4. The boundary value problem

$$y'' + y = f(x), \quad 0 < x < \pi$$

$$y(0) - y'(\pi) = 0, \quad y(\pi) - y'(0) = 0$$

may or may not have a solution, depending on the function $f(x)$. Determine condition(s) on $f(x)$ which, if satisfied, guarantee the existence of a solution. If $f(x)$ satisfies the condition(s), then how many solutions exist?

5. Can the function $f(x) = \cos x$ be expanded in a Fourier sine series over the interval $(0, \pi)$? Explain. Over the interval $(-\pi, \pi)$? Explain.

6. Consider the boundary value problem

$(-\tfrac{1}{2} - \lambda)(-3 - \lambda) = 0$

(y_n, Ly_n)

$$Ly \equiv (p(x)y')' - q(x)y + \lambda r(x)y = 0, \quad 0 < x < 1$$

$$y'(0) = 0, \quad y'(1) = 0,$$

$-y^3 + x + 4x$

$5x - x^3$

where $p(x)$ is differentiable with $p(x) > 0$ and $r(x) > 0$.

(a). Show that all the eigenvalues λ are real.

(b). Show that if $q(x) \geq 0$ then all the eigenvalues $\lambda \geq 0$, and there is an eigenvalue $\lambda = 0$ if and only if $q(x) \equiv 0$.

$-3(-3) - 4(2)$

$9 - 8 = 1$

-1

$\left[\begin{array}{cc|c} -3 & -4 & 1 \\ 1 & 1 & -1 \end{array}\right]$ $\qquad$ $\left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 1 & -1 \end{array}\right]$

$[-3, 2]$

$c_2 = 2$

$c_1 = -1 - 2 = -3$

$\frac{1}{2} - \frac{1}{3}$ $\quad \frac{1}{4} - \frac{1}{6}$ $\quad \frac{6}{24} - \frac{4}{24} = \frac{1}{12}$

$\frac{1}{3} - \frac{1}{2} + \frac{1}{4}$ $\quad \frac{3}{6} - \frac{2}{6}$

$\frac{8}{24} - \frac{12}{24} + \frac{6}{24} = \frac{1}{12}$

PRELIMINARY EXAM - LINEAR ALGEBRA - SPRING, 2010

Note - all problems count equally

1. Consider the matrix

$$A = \begin{pmatrix} -1.5 & -.5 & -1.5 \\ 1 & .5 & 1 \\ 1 & .5 & 1 \end{pmatrix}.$$

Show that $A^n \to 0$ as $n \to \infty$.

2. Let U be the matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(a) Determine all values of θ in the interval $0 \leq \theta < 2\pi$ so that $U(\theta)$ has real eigenvalues.

(b) Show that $U(\theta + \phi) = U(\theta)U(\phi) = U(\phi)U(\theta)$ for any θ and ϕ.

(c) Determine $U^{-1}(\theta)$

3. Let $f(p)$ be a **linear** map of the space of cubic polynomials onto itself such that $f(1) = x^3$, $f(x) = 2x$, $f(x^2) = x^2(1+x)$ and $f(x^3) = 1$.

$= x^2 + x^3$

(a) Show that the transformation is one-to-one, so that for every cubic polynomial p there is a unique cubic polynomial q so that $f(q) = p$.

(b) Find q when $p = 1 + x + x^2 + x^3$.

4. Consider the space of quadratic polynomials on the interval $[0,1]$ with inner product defined as

$$(p,q) = \int_0^1 p(x)q(x)dx.$$

Determine an orthonormal basis for this space.

$(x^2 - x + \frac{1}{6})(x^2 - x + \frac{1}{6})$

$x^4 - x^3 + \frac{1}{6}x^2 - x^3 + x^2 - \frac{1}{6}x + \frac{x^2}{6} - \frac{x}{6} + \frac{1}{36}$

$x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}$

$\frac{3}{12} - \frac{4}{12} + \frac{1}{12} - \frac{2}{12} + \frac{3}{12} - \frac{1}{12}$

5. Differentiation is a linear operator that maps the space of quadratic polynomials onto itself. Write the matrix representation of this operator with respect to the basis you computed in problem 4.

6. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

Let $\vec{x}$ be a column vector of size 3 and define

$$H(\vec{x}) = (\vec{x}, A\vec{x}).$$

Determine the global minimum and global maximum of $H(\vec{x})$ over all vectors of unit length.

$\sqrt{2} = 1$

$\frac{\sqrt{2}}{2}$

$3a = -v$

$\frac{1}{4} - \frac{1}{2}, \frac{16}{36} - \frac{6}{36} + \frac{1}{36}$

$a\sqrt{1+9} =$

$\frac{1}{\sqrt{10}}$ $\frac{1}{10}$

$\frac{9}{10}$