PRELIMINARY EXAM – LINEAR ALGEBRA 1/09

Note - all problems count equally

1. Consider the matrix

$$A = -\begin{pmatrix} u_0 & \rho_0 & 0 \\ 0 & u_0 & \frac{1}{\rho_0} \\ 0 & \gamma p_0 & u_0 \end{pmatrix},$$

where $u_0, \rho_0, p_0, \gamma > 0$. Determine the eigenvalues and left eigenvectors. Hint: It might simplify your computations to define $c_0 = \sqrt{\gamma p_0/\rho_0}$.

2. Consider the system of equations

$$\begin{aligned} x + y + z &= 4 \\ 2x + y + z &= 5 \\ 3x + 2y + 2z &= \alpha. \end{aligned}$$

Determine all values of α for which a solution exists and find the general solution when it exists.

3. Let $t_1, t_2, \ldots t_{n+1}$ be $n+1$ distinct real numbers and let $b_1, b_2, \ldots b_{n+1}$ be $n+1$ arbitrary real numbers. Show that there exists one and only one n^{th} degree polynomial, $p_n(t)$, such that $p_n(t_j) = b_j$ for $j = 1, 2 \ldots n+1$. Hint: Write this interpolation property as a system of $n+1 \times n+1$ linear equations and use what you know about polynomials to show that the system is nonsingular.

4. Let A be an $n \times n$ matrix that is diagonally dominant, i.e., for $i = 1, 2 \ldots n$ we have

$$| a_{i,i} | > \sum_{j \neq i} | a_{i,j} | .$$

Show that A is nonsingular.

5. (a) Consider the matrix

$$A = \begin{pmatrix} 0 & -i & 0 & 0 & i & 0 \\ i & 1 & i & -i & 0 & -i \\ 0 & -i & 0 & 0 & i & 0 \\ 0 & i & 0 & 0 & -i & 0 \\ -i & 0 & -i & i & 1 & i \\ 0 & i & 0 & 0 & -i & 0 \end{pmatrix}.$$

What properties of the eigenvalues and eigenvectors are known from the structure of this matrix?

(b) Compute the eigenvalues and eigenvectors for the matrix below:

$$A = \begin{pmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{pmatrix}.$$

6. Show that for any set of n real numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and any set of n linearly independent column vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ there is an $n \times n$ matrix A such that $A\vec{v_i} = \lambda_i \vec{v_i}$ (i.e., the λ's are the eigenvalues and the $\vec{v}$'s are the eigenvectors). Use your argument to construct a 2×2 matrix A such that its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, with corresponding eigenvectors $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

7. Consider the space of quadratic polynomials defined on the interval $[-1, 1]$ with the standard inner product,

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

Compute an orthonormal basis for this space.