PRELIMINARY EXAM - LINEAR ALGEBRA 4/98

1. Find all solutions of the system of equations

$$\begin{aligned} x + y + z &= 3 \\ 2x + y + 3z &= 2 \\ 3x + y + 5z &= 1 \end{aligned}$$

2. Compute $(Q^t(A^tQ^{-1})^{-1}A^t)^{-1}Q^{-1}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(Hint: What kind of matrices are A and Q?)

3. Let V be the space of smooth functions $(f(x), g(x))^t$ defined on the interval $[0,1]$ such that

$$f(0) = f(1) = g(0) = g(1) = 0, \quad f'(0) = f'(1), \quad g'(0) = g'(1).$$

Define an inner product on V by

$$< (f_1, g_1), (f_2, g_2) >= \int_0^1 f_1(x)f_2(x) + g_1(x)g_2(x)\, dx$$

Find the adjoint of the operator A given by

$$A = \begin{bmatrix} a\partial_x & b\partial_x \\ c\partial_x^2 & d\partial_x^2 \end{bmatrix}$$

4. Compute

$$\lim_{n\to\infty} \frac{A^n x}{\|A^n x\|_2}$$

where $A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ and (a) $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and (b) $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \end{bmatrix}$. Solve the problem

$$Au = \lambda u$$

for the λ's and eigenvectors u.

6. Let P be the space of quadratic polynomials of the form $a + bx + cx^2$.

   (a) Prove that the elements 1, x, $2x^2 - 1$ form a basis for the space P.

   (b) Consider the linear transformation $L : P \to P$ where

   $$L(p(x)) = \frac{d}{dx}((x-1)p(x)).$$

   Express L as a matrix with respect to the basis elements 1, x, $2x^2 - 1$.

   (c) Prove that the transformation L is invertible.

   (d) Find a polynomial $p(x)$ such that $L(p(x)) = 3p(x)$.

   (e) Prove that there is no polynomial $q(x)$ such that $L(q(x)) = 4q(x)$.