Linear Algebra Preliminary Exam 2012

1. Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

   Find the eigenvalues and eigenvectors of A.

2. Consider the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

   Find the values of a, b, c and d such that

   a) $(AB)^T = A^T B^T$

   b) $(AB)^T = B^T A^T$

   c) $\left(B^T A\right)^T = BA^T$

3. Consider the vector space Pol($\mathbb{R}$) of polynomials defined on the interval $[-1,1] \subset \mathbb{R}$. We define a scalar product by

$$(f,g) = \int_{-1}^{1} dx\, f(x) g(x)$$

   where $f, g \in \text{Pol}(\mathbb{R})$.

   a) Let $f_1(x) = 1$, $f_2(x) = x$ and $f_3(x) = x^2$. Construct an orthonormal set from these three vectors.

   b) Show that the polynomial functions

$$P_0(x) = \gamma_0, \quad P_n(x) = \gamma_n \frac{d^n}{dx^n}(x^2-1)^n, \quad n = 1, \ldots.$$

   with suitable constants γ_n form an orthonormal basis in Pol($\mathbb{R}$).

4. Consider three complex 2×2 matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   a) Show that

$$\sigma_n \sigma_m = I\delta_{nm} + i \sum_l \varepsilon_{nml} \sigma_l \tag{1}$$

   where δ_{nm} is the Kronecker- and ε_{nml} the Levi-Civita symbol, i.e.

$$\varepsilon_{nml} = \begin{cases} 1 & \text{if} \quad (n,m,l) = (x,y,z), (y,z,x), (z,x,y) \\ -1 & \text{if} \quad (n,m,l) = (x,z,y), (y,x,z), (z,y,x) \\ 0 & \text{otherwise} \end{cases}$$

   b) The commutator of two matrices A and B is defined as $[A,B] = AB - BA$. When $[A,B] = 0$, the operators are said to commute. Show for the matrices above that

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x \quad \text{and} \quad [\sigma_z, \sigma_x] = 2i\sigma_y. \tag{2}$$

   c) Use Eq. (1) to show that

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) I + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$$

   where

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

   and arbitrary vectors **a** and **b** that commute with $\boldsymbol{\sigma}$. Note that generally each component of the vectors **a** and **b** is a 2×2 matrix.

5. Suppose

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{pmatrix}$$

   a) Find a vector **x** orthogonal to the row space of A.

   b) Find a vector **y** orthogonal to the column space of A.

   c) What condition on **b** ensures that there exists a solution **w** to $A\mathbf{w} = \mathbf{b}$?

6. Let $A^2 = I$.

   a) What are the possible eigenvalues of A?

   b) If this A is 2 by 2, and not I or $-I$, find its trace and determinant.

   c) If the first row of A is $(3,-1)$, then what is the second row?