1995

Complex Variables Preliminary Examination

1. Evaluate:

   (a) $(i)^{1/2}$ (b) $(-i)^{1/3}$

2. Determine where $f'(z)$ exists and provide an expression for it:

   (a) $f(z) = |z|^2$

   (b) $f(z) = (r^2 \cos 2\theta - 2r \sin\theta) + i(r^2 \sin 2\theta + 2r \cos\theta)$

3. For the function
$$f(z) = (z^2+1)^{-1/2},$$

   (a) Find the Taylor series expansion about $z = 0$, determine its radius of convergence, and explain why it has this value.

   (b) Find the Laurent expansion of $f(z)$ about $z = 0$ valid for $|z| > 1$. Why is there no branch cut in this Laurent series?

4. Evaluate the following integrals:

   (a) $$\int_0^{\infty} \frac{dx}{x^2+3x+2}$$
   (Hint: introduce a factor $\log z$ into the numerator)

   (b) $$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$$
   (Hint: convert it to an integral around the circle $|z| = 1$)

5. Evaluate the integral:
$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{4x^2-1}\, dx$$

6. Use the conformal mapping $z = e^w$ to solve the boundary value problem for the harmonic function $u(x, y)$ which satisfies the boundary conditions

$$u = 0 \quad \text{on} \quad x^2+y^2 = 1, \qquad u = 1 \quad \text{on} \quad x^2+y^2 = 4, \qquad \text{both for } y > 0;$$

$$u_y = 0 \quad \text{on} \quad y = 0, \quad 1 \le |x| \le 2.$$