PRELIMINARY EXAM
ORDINARY DIFFERENTIAL EQUATIONS 1/98
DO PROBLEMS 1-4 AND EITHER PROBLEM 5 OR 6

1. Consider a spring and a linear damper as a model of a doorstop. The door of mass m swings and hits the doorstop with a given velocity, so that from the point of contact on it satisfies
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \qquad \text{with} \qquad x(0) = 0,\ x'(0) = v_0\,.$$
Find the solution. You may nodimensionalize if you wish.
   (a) Use the solution to determine the distance that it takes for the door to stop.
   (b) After the door rebounds, it loses contact with the spring-damper system (assuming there is no mechanism to keep it attached). Find the time at which the door loses contact, and the velocity at which it is moving when it loses contact.

2. For the equation
$$\frac{d^2y}{dx^2} + \left(\frac{1}{2} - \frac{1}{4}x^2\right) y = 0\,,$$
one solution is $y_1(x) = e^{-x^2/4}$. Use reduction of order to find a second solution.

3. Find the solution of the system of equations
$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$
that satisfies the initial condition $\vec{x}(0) = [\ 1\ 0\ 0\ ]^{\mathrm{T}}$.

4. Consider the differential equation
$$\frac{d\vec{w}}{dt} = A\vec{w} + \vec{f}(t),$$
where $\vec{w}$ and $\vec{f}$ are vector functions of size n and A is a constant matrix of size $n \times n$. Suppose that $\vec{f}(t)$ is periodic of period 2π. Find the most general condition on A so that there is always one and only one solution of period 2π.

5. Consider the system of differential equations

$$\frac{d\vec{w}}{dt} = A\vec{w}$$

where A is the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Determine the smallest exponent r so that

$$\lim_{t \to \infty} t^{-r} w_i < \infty, \quad i = 1, 2, 3$$

for all solutions

$$\vec{w} = \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix}$$

6. Consider the differential equation

$$\frac{dy}{dt} = \frac{y \sin(\epsilon y)}{\epsilon}, \quad \epsilon > 0$$

where $\epsilon > 0$.

(a) Find all critical points.

(b) Determine the stability of all critical points.

(c) Explain what happens as $\epsilon \to 0$. In particular,

i. Describe what happens to all of the critical points,

ii. Determine the limiting equation for $\epsilon = 0$

iii. Determine all critical points and their stability for the limiting equation.

iv. Describe how these critical points relate to the critical points for the original equation.