Differential Equations Preliminary Exam
January, 5, 1993

Problem 1. The boundary value problem

$$y'' + y = f(x), \qquad 0 < x < \pi$$

$$y(0) - y'(\pi) = 0, \quad y(\pi) - y'(0) = 0$$

may or may not have a solution, depending on the function $f(x)$. Determine condition(s) on $f(x)$ which, if satisfied, guarantee the existence of a solution. If $f(x)$ satisfies the condition(s), then how many solutions exist?

Problem 2. (a). Find Green's function for the operator

$$Ly \equiv -y'', \quad 0 < x < 1$$

defined for functions $y(x)$ which are twice differentiable and satisfy the boundary conditions

$$y(0) = y(1) = 0.$$

(b). Find the solution $u(x)$ of

$$-u'' = f(x), \quad 0 < x < 1$$

$$u(0) = \alpha, \quad u(1) = \beta.$$

Problem 3. Consider the boundary value problem

$$Ly \equiv (p(x)y')' - q(x)y + \lambda r(x)y = 0, \quad 0 < x < 1$$

$$y'(0) = 0, \quad y'(1) = 0,$$

where $p(x)$ is differentiable with $p(x) > 0$ and $r(x) > 0$.

(a). Show that all the eigenvalues λ are real.

(b). Show that if $q(x) \geq 0$ then all the eigenvalues $\lambda \geq 0$, and there is an eigenvalue $\lambda = 0$ if and only if $q(x) \equiv 0$.

Problem 4. Determine whether the following boundary value problems are self adjoint.

(a) $y'' + y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0$

(b) $y'' + y' + 2y = 0, \quad y(0) = y(1) = 0$

Problem 5. The population densities of two species, $u(t)$ and $v(t)$, which compete for the same food supply satisfy the system

$$\frac{du}{dt} = \frac{u}{2}(2 - 2u + v)$$

$$\frac{dv}{dt} = \frac{v}{4}(10 - u - 6v).$$

(a). Determine all possible equilibrium ($\frac{d}{dt} = 0$) solutions and their stability;

(b). Sketch a portrait of trajectories in the phase plane (u, v).

(c). On a long time basis, will the two species coexist or will one population become extinct?

Problem 6. Find the solution of the initial value problem

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$