Date maybe wrong on Test and this could be spring 2005

Advanced Calculus Preliminary Exam **January 5, 2005**

Please do all six problems. Please show all work. To get full credit for a problem, you need to **CLEARLY** show the details of your calculations.

Problem 1 (20 pts) Use the change of variable $y + x = u$, $y - x = v$ to evaluate the following integral:

$$\iint_R e^{\frac{y-x}{y+x}}\, dx\, dy$$

where R is the triangular region in the x-y plane whose vertices are $(0,0)$, $(0,1)$, $(1,0)$. Sketch the region R in both the x-y and u-v planes.

Problem 2 (20 pts) Compute the maximum and minimum values of $f(x,y) = x^2y^2 - x^2 - 4y^2$ on the circle $x^2 + y^2 = 5$.

Problem 3 (20 pts) Consider a surface S in $\mathbb{R}^3$ defined by $g(\mathbf{x}) = 0$, and let $T(\mathbf{x})$ be a temperature field defined in $\mathbb{R}^3$. At a point $\mathbf{a} \in S$, compute (1) the direction of steepest ascent of the value of T on S, and (2) the rate of increase of T in that direction.

Problem 4 (20 pts) Evaluate the flux, $\iint_S \mathbf{v} \cdot \mathbf{n}\, dS$, of the vector field $\mathbf{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$ through the paraboloid surface described by

$$z = 1 - (x^2 + y^2), \quad z \geq 0$$

where $\mathbf{n}$ is the unit outward normal. What happens if the unit normal $\mathbf{n}$ is chosen as the inward normal?

Problem 5 (20 pts) (a) Show that $\iint_S f\mathbf{n}\, dS = \iiint_V \nabla f\, dV$ for any piecewise smooth surface S enclosing the volume V and any differentiable scalar function $f(x,y,z)$. [Hint: consider $\iint_S \mathbf{a}f \cdot \mathbf{n}\, dS$ for fixed, but arbitrary vector $\mathbf{a}$.]

Problem 6 (20 pts) Consider a three-dimensional rotational shear fluid flow where the fluid velocity is given by $\mathbf{v} = yz\hat{\imath} - xz\hat{\jmath}$.

(a) Compute the fluid vorticity, ω, where $\omega = \nabla \times \mathbf{v}$.

(b) The circulation, Γ, is defined as

$$\Gamma = \iint_S \omega \cdot \mathbf{n}\, dS.$$

where S is the surface $z = 2 - x^2 - y^2$, with $x^2 + y^2 \leq 1$, and $\mathbf{n}$ is the normal pointing in the positive $\hat{\mathbf{k}}$ direction. Use Stokes' theorem to convert the circulation formula into a line integral, and evaluate that line integral.