COMPLEX VARIABLES REVIEW PROBLEMS

1. Compute all possible values of the following

$$(-2+2i)^{1/4}, \quad 1^i, \quad \log(3), \quad \sin(\pi), \quad \ln(i^3), \quad i^{\sqrt{3}}, \quad \frac{1}{(1+i)^{1/2}}.$$

2. Find all complex z which simultaneously satisfy the following equations:

$$|z+1| = |z-i|, \qquad |z-1+i| = \sqrt{2}.$$

ANSWER: $z_1 = 0$, $z_2 = 2(1-i)$.

3. Determine where $f'(z)$ exists and provide an expression for it:

(a) $f(z) = (x^3 - 3xy^2 - 3x) + i(-y^3 + 3x^2y - 3y)$
(b) $f(z) = e^{ix}e^{-|y|}$
(c) $f(z) = |z|^2$
(d) $f(z) = r^2\cos 2\theta - 2r\sin\theta + i(r^2\sin 2\theta + 2r\cos\theta)$
(e) $f(z) = \theta^2 - 2ir$

ANSWERS: (a) everywhere, $f'(z) = 3z^2 - 3$; (b) $y \geq 0$, $f'(z) = ie^{iz}$; (c) $z = 0$, $f'(0) = 0$; (d) everywhere, $f'(z) = 2(z+i)$; (e) $r = \theta$, $f'(z) = -2ie^{-i\theta}$

4. Verify that the given function u is harmonic everywhere in the plane and find all analytic functions $f(z)$ such that $u = \mathrm{Re} f(z)$:

(a) $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$
(b) $u = r\sin\theta$

ANSWERS: (a) $f(z) = (1-2i)(\sin z + z^2) + ci$; (b) $f(z) = -iz + ic$; c – real constant

5. Construct an entire function (i.e. analytic everywhere in the plane) $f(z) = u(x,y) + iv(x,y)$ such that $f(0) = i$ and

$$u(x,y) = 2v(x,y) + 5x - 2.$$

ANSWER: $f(z) = (1-2i)z + i$.

6. Prove that if $v(x,y)$ is a harmonic conjugate of $u(x,y)$ in a domain D, then $w(x,y) = -u(x,y)$ is a harmonic conjugate of $v(x,y)$ in D.

7. Use complex variables to evaluate the following integrals more on -cess!

$$\cancel{I_1} = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} \quad (|a| < 1), \qquad \cancel{I_2} = \int_0^{2\pi} \frac{d\theta}{(5 + 4\sin\theta)^2},$$

$$\cancel{I_3} = \int_0^{\infty} \frac{dx}{x^4 + 1}, \qquad I_4 = \int_0^{\infty} \frac{dx}{x^7 + 1}, \qquad \cancel{I_5} = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^4},$$

$$\cancel{I_6} = \int_0^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} \quad (a > 0,\ b > 0), \qquad I_7 = \int_{-\infty}^{\infty} \frac{\sin x}{(x + a)^2 + b^2} dx \quad (a > 0,\ b > 0),$$

$$I_8 = \int_0^{\infty} \frac{x^{-\alpha}}{x + 1}\, dx \quad (0 < \alpha < 1), \qquad I_9 = \int_0^{\infty} \frac{\sqrt{x}}{x^2 + 1}\, dx,$$

$$I_{10} = \int_0^{\infty} \frac{x^{-1/2} \ln x}{(x + 1)^2}\, dx, \qquad I_{11} = \int_0^{\infty} \frac{\ln x}{x^2 + a^2}\, dx \quad (a > 0).$$

ANSWERS:

$$I_1 = \frac{2\pi}{1 - a^2}, \quad I_2 = \frac{10}{27}\pi, \quad I_3 = \frac{\sqrt{2}}{4}\pi, \quad I_4 = \frac{\pi/7}{\sin(\pi/7)}, \quad I_5 = \frac{5}{16}\pi,$$

$$I_6 = \frac{\pi}{4b^3}(1 + ab)e^{-ab}, \quad I_7 = -\frac{\pi}{b}e^{-b}\sin a, \quad I_8 = \frac{\pi}{\sin \alpha\pi}, \quad I_9 = \frac{\sqrt{2}}{2}\pi,$$

$$I_{10} = -\pi, \quad I_{11} = \frac{\pi}{2a}\ln a.$$

8. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D. Suppose that

$$u^2(x, y) + v^2(x, y) = 1$$

for all $z = x + iy$ in D. Show that $f(z)$ is constant in D.

9. Suppose

$$f(z) = \frac{g(z)}{z(z - 1)^2},$$

where $g(z)$ is an entire function (i.e., analytic everywhere). Determine the value of

$$\oint_C f(z)dz,$$

where

(a) $C:\ z = i + \frac{5}{4}e^{i\theta},\ 0 \le \theta \le 2\pi$

(b) $C:\ z = 3ie^{i\theta},\ 0 \le \theta \le 2\pi$.

ANSWER: (a) $2\pi i g(0)$; (b) $2\pi i[g(0) + g'(1) - g(1)]$

10. Evaluate the integral

$$I = \oint_C \frac{dz}{z^3(z^2+4)},$$

where

(a) $C\colon |z+1| = 2$
(b) $C\colon |z-1| = 2$
(c) $C\colon |z| = 3$

ANSWER: (a) $I = -\pi i/8$, (b) $I = -\pi i/8$, (c) $I = 0$.

11. Evaluate the integral

$$I = \int_C [2x(1-y) + i(x^2 - y^2 + 2y)]\, dz,$$

where

(a) $C\colon x^2 - xy + y^2 + x + y = 0$
(b) $C\colon x = \cos(t^2/2),\ y = \sin(t^2/2),\ 0 \le t \le \sqrt{\pi}$.

ANSWER: (a) $I = -\pi i/8$, (b) $I = -\pi i/8$, (c) $I = 0$.

12. Evaluate the integral

$$I = \oint_C \bar{z}\, dz, \quad C: \ x = \cos t,\ y = \sin t, \quad 0 \le t \le 2\pi.$$

ANSWER: $2\pi i$.

13. Evaluate the integral

$$I = \oint_C \frac{dz}{\sin z(e^z+1)}, \quad C: \ x = \pi + 4\cos t,\ y = \pi + 4\sin t, \quad 0 \le t \le 2\pi.$$

ANSWER:

$$-\frac{1}{e^\pi + 1} + \frac{2i}{e^\pi - e^{-\pi}}.$$

14. Evaluate the integral

$$I = \oint_C \frac{z^2+1}{z^3+3z^2+z+1},$$

where C is a closed contour that encloses all the zeros of the polynomial in the denominator.

ANSWER: $I = 2\pi i$.

15. Evaluate the integral

$$I = \oint_C \frac{dz}{z^6 - 1},$$

where contour C is shown in the figure.

ANSWER: $I = 0$.

16. Determine the coefficients c_n, $n = 0, 1, 2, \ldots$ of the expansion

$$\frac{1}{(1-z)^3} = \sum_{n=0}^{\infty} c_n z^n.$$

ANSWER: $c_n = \frac{1}{2}(n+1)(n+2)$.

17. Find two separate Laurent expansions of the function $1/(z^2+1)$ in powers of $(z-i)$: one valid for $0 < |z-i| < 1$ and the other valid for $|z-i| > 1$.

ANSWER:

$$\frac{1}{4}\sum_{n=-1}^{\infty}\left(\frac{i}{2}\right)^n (z-i)^n, \quad \sum_{n=1}^{\infty}(-2i)^n \frac{1}{(z-i)^{n+1}}.$$

18. Find the Laurent expansion of the function $\cos\frac{z}{z+1}$ in powers of $(z+1)$.

19. Calculate the inverse Laplace transform of the function $(z+8)/(z^2+z-2)$, i.e. calculate the integral

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{(z+8)e^{zt}}{z^2+z-2}\,dz,$$

for $\delta > 1$ and t real.

20. Use the Argument Principle to determine how many roots the equation

$$4e^z = \frac{1}{z}$$

has in the unit circle $|z| < 1$.

ANSWER: 1.

21. Use the conformal mapping $z = e^w$ to solve the boundary value problem

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0,$$

$$u(x,0) = H(x), \quad -\infty < x < \infty$$

22. Find the image of the quadrant $x > 0$, $y > 0$ under the transformation

$$w = \frac{z-i}{z+i}.$$