Advanced Calculus Preliminary Exam January 15, 2003

Please choose five of the following six problems to solve. Clearly indicate on the cover of your exam, which problems you have selected. Please show all work. To get full credit for a problem, you need to CLEARLY show the details of your calculations.

Problem 1 (20 pts) Find the point on the paraboloid $z = x^2 + y^2 + 1$ nearest the point $(1, 1, 3/2)$.

Problem 2 (20 pts) Consider the force $\mathbf{F} = (x^2 - y^2)\hat{\imath} + 3xy\hat{\jmath}$. Compute the work done by this force as it moves counter-clockwise around the closed path that bounds the region between the curves $y = x^2$ and $y = x^3$, $0 \leq x \leq 1$.

Problem 3 (20 pts) For $x = \xi + \eta$, $t = \xi - \eta$, convert the PDE

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0$$

into one in terms of ξ, η.

Problem 4 (20 pts) A shark that detects the presence of blood will respond by moving continually in the direction of strongest scent. In a test conducted at the surface of the ocean, the concentration C of blood in parts per million of water was given approximately by $C = e^{-(x^2+2y^2)/10^4}$. Find the shark's approach path from any point (x_0, y_0).

Problem 5 (20 pts) Let $\mathbf{v} = y^2\hat{\imath} + xy\hat{\jmath} + xz\hat{k}$ and let S be the open hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, with unit outer normal $\mathbf{n}$. Evaluate

$$\iint_S \nabla \times \mathbf{v} \cdot \mathbf{n}\, dS.$$

Problem 6 (20 pts) Let $\mathbf{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and let S be the closed surface consisting of the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and the disk $x^2 + y^2 \leq 9$ in the xy-plane. Let $\mathbf{n}$ be the unit outer normal to S. Evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dS.$$