Please choose five of the following six problems to solve. Clearly indicate on the cover of your exam, which problems you have selected. Please show all work. To get full credit for a problem, you need to CLEARLY show the details of your calculations.

Problem 1 (20 pts) Use the change of variable $y + x = u$, $y - x = v$ to evaluate the following integral:

$$\iint_R e^{\frac{y-x}{y+x}}\, dx\, dy$$

where R is the triangular region in the x-y plane whose vertices are $(0,0)$, $(0,1)$, $(1,0)$. Sketch the region R in both the x-y and u-v planes.

Problem 2 (20 pts) Compute the maximum and minimum values of $f(x,y) = x^2y^2 - x^2 - 4y^2$ on the circle $x^2 + y^2 = 5$.

Problem 3 (20 pts) Consider a surface S in $\mathbb{R}^3$ defined by $g(\mathrm{x}) = 0$, and let $T(\mathrm{x})$ be a temperature field defined in $\mathbb{R}^3$. At a point $\mathrm{a} \in S$, compute (1) the direction of steepest ascent of the value of T on S, and (2) the rate of increase of T in that direction.

Problem 4 (20 pts) Evaluate the flux of the vector field $\mathbf{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$ over the closed surface described by

$$z = 1 - (x^2 + y^2), \quad z \geq 0$$

where the flux is given by $\int_S \mathbf{v} \cdot \mathbf{n}\, dS$, and $\mathbf{n}$ is the unit outward normal. What happens if the unit normal $\mathbf{n}$ is chosen as the inward normal?

Problem 5 (20 pts) (a) Show that $\iint_S f\mathbf{n}\,dS = \iiint_V \nabla f\, dV$ for any piecewise smooth surface S enclosing the volume V and any differentiable scalar function $f(x,y,z)$. [Hint: consider $\iint_S \mathbf{a}f \cdot \mathbf{n}\, dS$ for fixed, but arbitrary vector $\mathbf{a}$.]

(b) Confirm your result for the function $f = x^2$ and the volume V corresponding to the box $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Problem 6 (20 pts) Let $\mathbf{F} = \dfrac{-y}{x^2+y^2}\hat{\imath} + \dfrac{x}{x^2+y^2}\hat{\jmath}$.

(a) Compute directly (i.e. without using Green's theorem): $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle, $x^2 + y^2 = 1$, $z = 0$, taken in a counter-clockwise direction.

(b) Compute the integral in part (a) by applying the formula from Green's theorem.

(c) Compare your answers for parts (a), (b). Do they support or contradict Green's theorem? Explain.