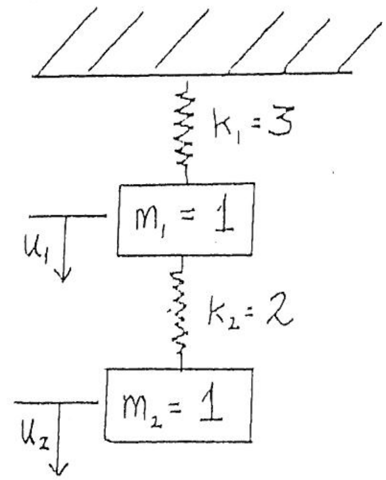
PRELIMINARY EXAM - DIFFERENTIAL EQUATIONS 1/96

20 points per problem

1. Consider the spring-mass system shown below.



Assuming linear springs and no damping, the equations of motion are

$$\begin{aligned} \ddot{u}_1 &= -2(u_1 - u_2) - 3u_1 \\ \ddot{u}_2 &= -2(u_2 - u_1) \end{aligned}$$

(a) Find the general solution $(u_1(t), u_2(t))$.

(b) Suppose that the initial conditions are

$$u_1(0) = 1\ , \quad \dot{u}_1(0) = 0\ , \quad u_2(0) = 2\ , \quad \dot{u}_2(0) = 0. \tag{1}$$

Solve for $u_1(t), u_2(t)$.

(c) Suppose that the initial conditions are

$$u_1(0) = -2\ , \quad \dot{u}_1(0) = 0\ , \quad u_2(0) = 1\ , \quad \dot{u}_2(0) = 0. \tag{2}$$

Solve for $u_1(t), u_2(t)$.

(d) Briefly describe (in words) the motion of the masses in both parts (b) and (c), *i.e.*, give the frequency of motion and describe how the masses move relative to each other.

2. The flow of heat in a one dimensional rod of length L is governed by the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $0 < x < L$ and $t > 0$. Here u is the temperature in the rod. Suppose that along the boundaries $u(0, t) = 1$ and $u(L, t) = 50$. Also suppose that initially $u(x, 0) = 1 + x$. Find the temperature in the rod for $t > 0$.

3. Consider the differential equation

$$\frac{dy}{dt} = y^2 - \epsilon, \quad \epsilon > 0.$$

Let $y(t, y_0)$ denote the solution which takes the value y_0 at $t = 0$.

(a) Find and determine the stability of all critical points.

(b) For every y_0 determine the limiting value of $y(t, y_0)$ as $t \to \infty$ or t approaches the limiting time for which the solution exists. It is not necessary to solve the equation explicitly.

(c) Explain what happens as $\epsilon \to 0$.

4. Consider the system of equations:

$$\begin{aligned} \dot{x} &= x(2 - x - y) \\ \dot{y} &= y(1 - x) \end{aligned}$$

(a) Find all equilibrium solutions (*i.e.*, critical points).

(b) For each equilibrium, find the corresponding linear system, and determine whether the equilibrium is asymptotically stable, stable, or unstable.

(c) Sketch the phase portrait in the (x, y)-phase plane.

5. Consider the differential equation

$$\frac{dy}{dt} = \lambda y + f(t),$$

where λ is a complex number. Suppose $f(t)$ is bounded for all t, i.e., $| f(t) | \leq M$ for $-\infty < t < \infty$.

(a) Find all values of λ for which a <u>unique</u> bounded solution exists. Note the solution must be bounded for <u>all</u> values of t, i.e., for $-\infty < t < \infty$.

(b) Write down this solution in terms of integrals of f.

(c) For every value of λ for which there is <u>not</u> a <u>unique</u> bounded solution, determine conditions on f so that bounded solutions exist.