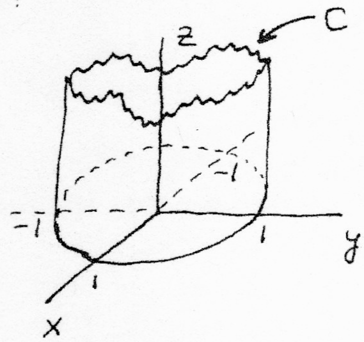
Advanced Calculus Exam

Friday. January 12, 2007 (12:00 – 2:00)

choose 5 *problems out of* 6

Please show all work. To get full credit you must **clearly** describe your calculations.

1. (**20 points**) A broken wine bottle sits on the xy plane as shown. The bottle consists of a portion of a cylinder of radius 1 along the z axis, with the unit disc in the xy plane at the bottom (see the figure below). Let **C** be the path along the broken edge oriented counterclockwise when viewed from above. If F is the vector field $\mathrm{F} = \langle -y,\ 2x,\ 10z\rangle$, what is the value of $\oint \mathrm{F} \cdot d\mathrm{r}$?



2. (**20 points**) Inconspicuously, James Bond put a drop of cyanide into a glass of whiskey that he saw the double agent X was drinking. The drop started to spread. Let $C(x, y, z, t)$ denote the concentration of cyanide in the whiskey. Fick's law states that the flux q of cyanide is negatively proportional to the gradient of C (the cyanide spreads from points of high concentration to points of low concentration), i.e. $\mathrm{q} = -k\nabla C$. Show that C satisfies the equation $\frac{\partial C}{\partial t} = k\nabla^2 C$.

3. (**20 points**)

   (a) For what values of α is the force field $\mathrm{F} = (e^x \sin y - y)\,\mathrm{i} + \alpha(e^x \cos y - x - 2)\,\mathrm{j}$ conservative? For each such value of α find a potential ϕ such that $\mathrm{F} = \nabla\phi$.

   (b) Consider a force field F derivable from a potential U, $\mathrm{F} = -\nabla U$. The work done by F on a particle moving from point A to point B is defined as $W = \int_A^B \mathrm{F} \cdot d\mathrm{r}$.

   (c) Show that $W = U(A) - U(B)$.

   (d) Let the particle move according to the Newton's Second law of motion, $\mathrm{F} = m\, d^2\mathrm{r}/dt^2$, where m is the mass of the particle and r is its position. Show that the sum of the potential and the kinetic energy, $U + mv^2/2$, where $v = |\mathrm{v}|$, $\mathrm{v} = d\mathrm{r}/dt$, is conserved (therefore such a force field is called conservative).

4. (**20 points**) Find the mass of the plane region R in the first quadrant of the (x, y)-plane bounded by the hyperbolas $xy = 1$, $xy = 2$, $x^2 - y^2 = 3$, $x^2 - y^2 = 5$. Assume the density at the point (x, y) is $\rho(x, y) = x^2 + y^2$.

5. **(20 points)** A function $y(x)$ is given in implicit form $F(x, y) = 0$. Find dy/dx and d^2y/dx^2 in terms of $\partial F/\partial x$ and $\partial F/\partial y$.

6. **(20 points)** Consider the force field $\mathbf{F} = v_0\mathbf{k}$, where $\mathbf{k}$ is the unit vector along the $+z$-direction.

   a) By direct computation find the flux of the vector field $\mathbf{F}$ across the hemispherical surface $|\mathbf{r}| = a$, $z > 0$ in the direction pointing away from the origin.

   b) Find the flux of the vector field $\mathbf{F}$ across the disk $\{x^2 + y^2 = a^2,\ z = 0\}$ in the $\mathbf{k}$ direction.

   c) Without appealing to Stokes theorem, present an argument showing why the fluxes in a) and b) must be equal.