

Preliminary Exam: Elementary Differential Equations January 2004

1. Find the general solution of the following equation:

$$\frac{d^2y}{dx^2} + y = \cos(x).$$

2. Find the general solution to the differential equation

$$xy'' - 2y' + \frac{2}{x}y = 1.$$

3. According to *Torricelli's law*, water in an open tank will flow out through a small hole in the bottom with the speed it would acquire in falling freely from the water's surface to the hole. A cylindrical tank of radius R and height H is initially full of water, and a small circular hole of radius r is punched in the bottom at time $t = 0$. How long will it take for the tank to empty itself?

4. Consider the system

$$\begin{aligned}\dot{x} &= x + 3y + e^{2t}, \\ \dot{y} &= x - y.\end{aligned}$$

- a) Find the general solution of the associated homogeneous problem.
b) Find a particular solution of the inhomogeneous problem.

5. Find the solution for the system of equations

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with $x(0) = 3$ and $y(0) = 0$.

6. Find and classify the critical points of the system of equations by their linear stability properties

$$\begin{aligned}\frac{dx}{dt} &= a - bx + x^2y - y \\ \frac{dy}{dt} &= bx - x^2y.\end{aligned}$$

for all positive values of a and b .

7. Solve the partial differential equation

$$u_t = u_{xx} \quad 0 < x < \pi \quad 0 < t$$

with boundary conditions $u(t, 0) = 0 = u(t, \pi)$ and initial condition $u(0, x) = \sin x + \sin 2x$.