PRELIMINARY EXAM – DIFFERENTIAL EQUATIONS 1/07

Note - all problems count equally

1. Consider the differential equation

$$x(2-x)\frac{dy}{dx}+y=0. \tag{1}$$

   (a) Solve (1) for initial condition $y(1)=1$.

   (b) Solve (1) for initial condition $y(3)=1$.

2. Find all values of λ for which the boundary value problem

$$\frac{d^2u}{dt^2}+2\frac{du}{dt}+\lambda u=0, \quad u(0)=0, \quad u(1)=0,$$

   has a non-trivial solution, and write the corresponding solution(s).

3. Find the general solution of the equation

$$\frac{1}{2}x^2(x-1)\frac{d^2y}{dx^2}-x(x-2)\frac{dy}{dx}+(x-3)y=0,$$

   given that $y=x^2$ is a solution.

4. Find the general solution of the following system of equations:

$$\begin{aligned}\frac{dx}{dt}&=-7x-10y+4e^t,\\ \frac{dy}{dt}&=4x+5y-2e^t.\end{aligned}$$

5. Consider the initial value problem

$$y' = -\alpha y + f(t) \quad y(t_0) = y_0, \tag{2}$$

where $\alpha > 0$ and

$$f(t) = \frac{\sin t}{t}, \quad t \neq 0, \quad f(0) = 1.$$

(a) Show that $f(t)$ is continuous at $t = 0$.

(b) Determine the solution to (2) in terms of integrals of known functions. It is not necessary to actually perform the integration.

(c) Show that $\mid f(t) \mid \leq 1$ for all t.

(d) Show that for $t \geq t_0$

$$\mid y(t) \mid \leq \mid y_0 \mid + \frac{1}{\alpha}.$$

(e) Show that for a fixed value of α, (2) can have at most one solution that is bounded for $-\infty < t < \infty$.

(f) For a given fixed value of α, determine the unique bounded solution to (2) in terms of integrals of known functions. It is not necessary to perform the integration.

6. Consider the initial value problem

$$y'' + y = \sin \omega t, \quad y(0) = y'(0) = 0. \tag{3}$$

(a) Determine the solution when $\omega = 2$.

(b) Determine the solution when $\omega = 1$.

7. Consider the system

$$\begin{aligned} x' &= y, \\ y' &= xy, \quad x(0) = x_0, \ y(0) = y_0. \end{aligned}$$

(a) Determine all critical points.

(b) Sketch the trajectories in the phase plane. Make certain you show the direction of motion on the trajectories (use arrows).

(c) Consider the solution with initial conditions $x(0) = 0, y(0) = -4$. Determine $\lim_{t\to\infty} x(t), y(t)$.