Advanced Calculus Preliminary Examination
January 2001

1. (20 pts.) Find the critical points and the values of the maximum and minimum of the function $f = x + z$ on the sphere $x^2 + y^2 + z^2 = 1$.

2. (20 pts.) (a) Fick's law of thermal conduction states that heat flows in the direction of decreasing temperature T at a rate proportional to the temperature gradient, i.e.,

$$\mathbf{u} = -k\nabla T,$$

where k is a positive constant and $\int\int_S \mathbf{u}\cdot\mathbf{n}\, dS$ represents the heat flux, i.e. the number of calories crossing the surface S, in the direction normal to S. Let S be a closed surface forming the boundary of a region D. Consider the total amount of heat entering D through S and the rate $\int\int\int_D c\rho \frac{\partial T}{\partial t} dV$ at which heat is absorbed by the material in the region D (where the constants c and ρ are its specific heat and density, respectively), to derive the heat conduction equation in D

$$c\rho\frac{\partial T}{\partial t} = k\nabla^2 T.$$

(b) The magnetic field $\mathbf{B}$ generated by a steady current $\mathbf{j}$ satisfies

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j},$$

where the constant c is the speed of light. Consider an axisymmetric current density $\mathbf{j} = j_z(r)\,\mathbf{e}_z$ with $\mathbf{e}_z$ being the unit vector in the z-direction and r the radial coordinate of a cylindrical coordinate system. Using symmetry arguments, it can be shown that in this case the magnetic field has only an azimuthal component, $\mathbf{B} = B_\phi \mathbf{e}_\phi$ with B_ϕ constant.

i) Derive an integral expression for the magnetic field in terms of the current density.

ii) Conclude that for no current density that corresponds to a finite total current one can use this geometry, i.e. a current distribution of the form $\mathbf{j} = j_z(r)\,\mathbf{e}_z$, to obtain a magnetic field that is homogeneous everywhere within a cylindrical domain of radius R_0.

3. (20 pts.) Consider a one dimensional fluid motion, e.g., the flow is along the x axis. Let $v(x,t)$ be the velocity of the fluid at position x at time t, so that $\frac{dx}{dt} = v$. Let $f(x,t)$ denote the density of the fluid at position x at time t. A piece of fluid of density f initially occupies the interval $a_0 \le x \le b_0$. Then, at time t, it occupies the interval $a(t) \le x \le b(t)$, where $\frac{da}{dt} = v(a,t)$ and $\frac{db}{dt} = v(b,t)$. The mass of the piece of fluid is given by $F(t) = \int_{a(t)}^{b(t)} f(x,t)dx$. Show that $\frac{dF}{dt} = \int_{a(t)}^{b(t)} (\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(fv))dx = \int_{a(t)}^{b(t)} (\frac{Df}{Dt} + f\frac{\partial v}{\partial x})dx$. Here, $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}$ is the material derivative.

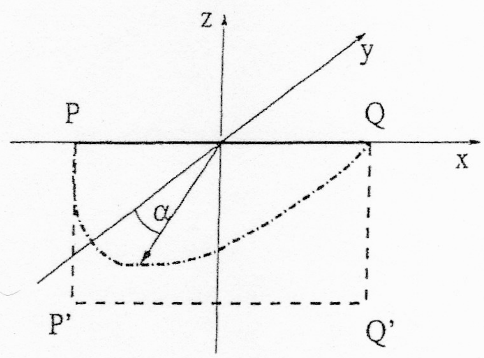

Figure 1: Sketch of the three integration paths.

4. (20 pts.) Consider the force field given in cartesian coordinates by

$$\mathbf{F}(x,y,z) = (-ay, bx, -cz). \tag{1}$$

The work W performed by the force when moving a particle along a path C is given by

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{2}$$

Calculate the work performed by the force for the following three paths C_i as indicated in fig.1:
i) in a straight line from $P = (-1,0,0)$ to $Q = (1,0,0)$ (solid line),
ii) in straight lines from P to $P' = (-1,0,-1)$ to $Q' = (1,0,-1)$ to Q (dashed line),
iii) along a circular semi-arc with radius 1 that goes from P to Q and that is rotated around the x-axis by an angle $0 \leq \alpha \leq \pi/2$ with respect to the $x-y$-plane.
Compare the amounts of work performed in each case.
For which values of a, b, and c can the force field be derived from a potential U, i.e. $\mathbf{F} = -\nabla U$? For that case determine $U(x,y,z)$.

5. (20 pts.) Consider a scalar function $F(r,\phi)$ and a vector field $\mathbf{v}(r,\phi,\theta) = v_r(r,\phi)\mathbf{e}_r + v_\phi(r,\phi)\mathbf{e}_\phi$ in spherical coordinates (r,ϕ,θ) with $0 \leq r < \infty$, $0 \leq \phi < 2\pi$, and $0 \leq \theta \leq \pi/2$.
i) Starting from the gradient operator $\nabla \equiv (\partial_x, \partial_y, \partial_z)$ in cartesian coordinates derive the r- and the ϕ-components of the gradient operator in spherical coordinates, as they are needed to determine $\nabla F(r,\phi)$ in spherical coordinates. Use them to give an explicit expression for $\nabla F(r,\phi)$. (Note, that for this F the θ-component of the gradient vanishes.)
ii) Use results from i) to determine $\nabla \cdot \mathbf{v}(r,\phi)$, i.e. determine **div** $\mathbf{v}(r,\phi)$ (with $\mathbf{v}$ independent of θ).