problems 1998 #1, #4, #6

PRELIMINARY EXAM
ORDINARY DIFFERENTIAL EQUATIONS 1/99

Instructions: Please reduce your answers to as simple a form as is possible.

1. Find the solution of the differential equation

PC

$$\frac{dy}{dx} + y\tan x - \sin x = 0 \quad \text{with} \quad y(0) = 0\,.$$

2. The velocity v of a skydiver falling from rest through air can be described by the equation

$$m\frac{dv}{dt} = mg - C_D v^2, \quad \text{with} \quad v(0) = 0,$$

where m is the skydiver's mass, g is the gravitational acceleration, and C_D is the skydiver's *drag coefficient*.

(a) Nondimensionalize this problem.

(b) Find its solution.

(c) Determine the behavior of the velocity as $t \to \infty$.

3. Consider the system of equations

PC

$$\frac{dx}{dt} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} x,$$

where x is a 3 dimensional column vector. Determine all solutions $x(t)$ such that $e^{-t}x(t)$ is bounded as $t \to \infty$.

4. Find the general solution of the differential equation

DC

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \cos x\,.$$

5. Consider the equation for an unforced oscillator

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0.$$

DC

Determine one possible set of real values for a, b and c for which

$$x = e^{-t}\cos t$$

is a solution. For this case, give another linearly independent solution.

6. Find and classify (i.e., identify the type and stability of) the critical points of the system

DC

$$\begin{aligned}\frac{dx}{dt} &= x^2 - xy - x\\ \frac{dy}{dt} &= y^2 + xy - 3y\end{aligned}$$

PRELIM EXAM: DIFFERENTIAL EQNS 1/4Y

1. $y' + (\tan x) y = \sin x$

$$p(x) = \exp\left\{\int \frac{\sin x}{\cos x}\, dx\right\} = \exp(-\ln(\cos x)) = \frac{1}{\cos x}$$

$$\sec x\, y' + (\sec x \tan x) y = \tan x$$

$$(\sec x\, y)' = \tan x$$

$$(\sec x) y = -\ln(\cos x) + C \qquad y(0) = 0$$

$$0 = 0 + C \Rightarrow C = 0$$

$$y = \frac{\ln(\sec x)}{\sec x} \Rightarrow \boxed{y = \cos x(\ln(\sec x))}$$

2. $m\frac{dv}{dt} = mg - C_L v^2 \qquad v(0) = 0$

$m \sim M$

$g \sim \frac{L}{T^2}$

$C_D \sim \frac{M}{L}$

$$\frac{ML}{T^2} = \frac{ML}{T^2} - \frac{M}{L}\frac{L^2}{T^2}$$

$t = \alpha \hat{t}$

$$\sqrt{\underbrace{\frac{T^2}{L}}_{\frac{1}{g}} \cdot \underbrace{\frac{L}{M}}_{\frac{1}{C_D}} \cdot \frac{M}{m}} \rightarrow \alpha = \sqrt{\frac{m}{gC_D}}$$

$v = \beta \hat{v}$

$$\frac{L}{T} \sim \sqrt{\frac{L}{T^2}\frac{L}{M} M} \rightarrow \beta = \sqrt{\frac{gm}{C_D}}$$

$$\cancel{m}\sqrt{\frac{gm}{\cancel{C_D}}}\sqrt{\frac{g\cancel{C_D}}{\cancel{m}}}\frac{d\hat{v}}{d\hat{t}} = \cancel{mg} - \cancel{C_D}\frac{gm}{\cancel{C_D}}\hat{v}^2$$

$$\boxed{\frac{d\hat{v}}{d\hat{t}} = 1 - \hat{v}^2}$$

3. $\frac{dx}{dt} = \underbrace{\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}}_{A} x$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$ $\quad \det(A - \lambda I) = 0$

$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2(2-\lambda-1) + 1(2-(3-\lambda)) = 0$

$(2-\lambda)[6-5\lambda+\lambda^2-2] - 2(1-\lambda) + (-1+\lambda) = 0$

$(2-\lambda)[\lambda^2-5\lambda+4] - 2 + 2\lambda - 1 + \lambda = 0$

$(2-\lambda)(\lambda-4)(\lambda-1) - 3(-\lambda+1) = 0$

$(\lambda-2)(\lambda-4)(\lambda-1) - 3(\lambda-1) = 0$

$(\lambda-1)[\lambda^2 - 4\lambda - 2\lambda + 8 - 3] = 0$

$(\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$

$(\lambda-1)(\lambda-5)(\lambda-1) = 0$

$\lambda = 1, \ \lambda = 5$

twice

$\underline{\lambda = 1}$

$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$x_1 + 2x_2 + x_3 = 0$

$v_1 = \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix}, \quad v_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

$\underline{\lambda = 5}$

$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$v_3 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

4. $y'' - 2y' + y = \underbrace{xe^x \cos x}_{g(x)}$

$y_H'' - 2y_H' + y_H = 0$

$r^2 - 2r + r = 0$

$(r-1)(r-1) = 0$

$y_H(x) = c_1 e^x + c_2 x e^x$

$y_1 = e^x \qquad y_2 = xe^x$

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

$$y_P = -y_2 \int \frac{y_1(x)\, g(x)}{a_0 W}\, dx + y_1 \int \frac{y_2(x)\, g(x)}{a_0 W}\, dx$$

$$= -xe^x \int \frac{e^{2x} x \cos x}{e^{2x}}\, dx + e^x \int \frac{x^2 e^{2x} \cos x}{e^{2x}}\, dx$$

$$= -xe^x \int \underbrace{x}_{u} \underbrace{\cos x \; dx}_{dv} + e^x \int \underbrace{x^2}_{u} \underbrace{\cos x\, dx}_{dv} \qquad \int u\, dv = uv - \int v\, du$$

$$= -xe^x \left[x \sin x - \int \sin x\, dx \right] + e^x \left[x^2 \sin x - \int \underbrace{2x}_{u} \underbrace{\sin x\, dx}_{dv} \right]$$

$$= -xe^x \left[\cancel{x \sin x} + \cos x \right] + e^x \left[\cancel{x^2 \sin x} + 2x \cos x - 2\sin x \right]$$

$$= xe^x \cos x - 2e^x \sin x$$

check:

$y_P' = xe^x \cos x + e^x \cos x - xe^x \sin x - 2e^x \sin x - 2e^x \cos x$

$= xe^x(\cos x - \sin x) - e^x(\cos x - 2\sin x)$

$y_P'' = e^x(\cancel{\cos x} - \cancel{\sin x}) + xe^x(\cos x - \sin x) - xe^x(\sin x - \cos x)$

$\qquad - e^x(\cancel{\cos x} - 2\sin x) + e^x(\cancel{\sin x} + 2\cos x)$

$= 2xe^x(\cos x - \sin x) + 2e^x(\sin x + \cos x)$

$2xe^x(\cos x - \sin x) + 2e^x(\sin x + \cos x) - 2xe^x(\cos x - \sin x) - 2e^x(\cos x - 2\sin x)$

$\qquad + xe^x \cos x - 2e^x \sin x \neq xe^x \cos x$

MISTAKE

6. $\frac{dx}{dt} = x^2 - xy - x$

$\frac{dy}{dt} = y^2 + xy - 3y$

critical points when $\dot{x}=0$, $\dot{y}=0$

$\dot{x} = x(x-y-1) = 0$ $\qquad x=0$, $x-y-1=0$

$\dot{y} = y(y+x-3)=0$ $\qquad y=0$, $y+x-3=0$

① $x=0$

$y=0$ or $y-3=0 \Rightarrow y=3$, $x-y-1=0 \Rightarrow y=-1$

$(0,0)$, $(0,3)$, $(0,-1)$

② $y=0$

$x-1=0 \Rightarrow x=1$, $x-3=0 \Rightarrow x=3$

$(1,0)$, $(3,0)$

$$A = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 2x-y-1 & -x \\ y & 2y+x-3 \end{pmatrix}$$

$(0,0)$ $\quad A = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$

$\det(A - \lambda I) = \begin{pmatrix} -1-\lambda & 0 \\ 0 & -3-\lambda \end{pmatrix} = 0 \Rightarrow (1+\lambda)(3+\lambda)=0$

$\lambda = -1, -3$

asymptotically stable improper node

and so on...

$$x(t) = c_1 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} e^{t} + c_2 \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix} e^{t} + c_3 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} e^{5t}$$

$$e^{-t} x(t) = c_1 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} + c_2 \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix} + c_3 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} e^{4t}$$

$$\left| \lim_{t \to \infty} e^{-t} x(t) \right| = \left| \lim_{t \to \infty} c_1 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} + c_2 \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix} + c_3 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} e^{4t} \right| < \infty$$

provided that $c_3 = 0$

Then

$$e^{-t} x(t) = c_1 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} + c_2 \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix}$$ is bounded as long as c_1 and c_2 are finite

$$x(t) = c_1 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} e^{t} + c_2 \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix} e^{t}$$