Complex Variables Preliminary Examination

March 2001

1. (10 points) Find all complex z which simultaneously satisfy the following equations:
$$|z-1| = |z+i|, \qquad |z-1+i| = \sqrt{2}.$$

2. (10 points) Determine where $f'(z)$ exists and provide an expression for it ($z = x + iy = re^{i\vartheta}$):

   (a) $f(z) = 2x(1-y) + i(x^2 - y^2 + 2y)$

   (b) $f(z) = \vartheta^2 - 2ir$

3. (10 points) Suppose
$$f(z) = \frac{g(z)}{(z-2)(z-4)^2},$$
where $g(z)$ is an entire function. Determine the value of
$$\oint_C f(z)dz,$$
where
$$(a) \quad C: \; z = 5 + 2e^{i\theta}, \quad 0 \le \theta \le 2\pi;$$
$$(b) \quad C: \; z = 5 + 4e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

4. (15 points) (i) Find a linear fractional transformation $w(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ that maps the points $z_1 = -i$, $z_2 = 0$, $z_3 = i$ onto the points $w_1 = -1$, $w_2 = i$, $w_3 = 1$, respectively. (ii) Into what curve is the imaginary axis transformed? (iii) For which values of z does the transformation fail to be conformal?

5. (20 points) In the indicated region, determine a Laurent series (of the specified form) for

$$f(z) = \frac{z}{(z-1)(2-z)}$$

(Hint: Partial Fractions)

(a) $1 < |z| < 2$; Laurent series in powers of z.

(b) $|z-1| > 1$; Laurent series in powers of $(z-1)$.

6. (20 points) Use contour integration to evaluate the following integrals. Please explain carefully each step of the method used for the evaluation.

(a) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$