Complex Variables Preliminary Examination

January 12, 2006

1. Suppose the exponential function is defined for complex arguments z by assuming a) e^z is an entire function; b) when $z = x$ is real, all of the normal properties of real functions hold; and c) for complex arguments, the product of two exponentials is the exponential of the sum of the exponents, i.e.,
$$e^{z_1} e^{z_2} = e^{z_1+z_2}\,.$$
Then we can write
$$e^z = e^{x+iy} = e^x e^{iy} \equiv e^x\left[C(y) + iS(y)\right]$$
for some functions $C(y)$ and $S(y)$. Show that the requirement that e^z be analytic, and that $e^0 = 1$, means that $C(y)$ and $S(y)$ must be $\cos y$ and $\sin y$.

2. Write the two Laurent series in powers of z that represent the function
$$f(z) = \frac{1}{z(1+z^2)}$$
in certain domains, and specify those domains.

3. Evaluate the following integrals, explaining carefully each step of the method used for the evaluation.
$$\int_0^\infty \frac{x^2\,dx}{1+x^6}$$
$$\int_{-\infty}^\infty \frac{\sin x}{x(x^2+a^2)}\,dx$$
$$\int_0^\infty \frac{x^{1/2}}{(1+x)^2}\,dx$$
$$\int_0^{2\pi} \frac{d\varphi}{1+2a\sin\varphi+a^2}\,, \qquad |a|<1.$$

4. Suppose that:
   (a) $F(z)$ is analytic in the cut plane $|\arg z| < \pi$
   (b) $F(z) \to 0$ as $z \to \infty$ uniformly in $|\arg z| < \pi$
   (c) the jump across the cut is known, i.e.,
$$F(re^{\pi i-}) - F(re^{-\pi i+}) = -2i\rho(r)\,.$$
Show that
$$F(z) = \frac{1}{\pi}\int_0^\infty \frac{\rho(r)\,dr}{r+z}\,.$$