PRELIMINARY EXAM - DIFFERENTIAL EQUATIONS 1/02

~~1~~. Determine the general solution of

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} = \cos x.$$

Hyperbolic + trig identities

~~2~~. Solve the initial value problem

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3e^{3x},$$

where $y(0) = 1$ and $\frac{dy}{dx}(0) = 1$.

~~3~~. Determine the general solution of the differential equation

$$x^3\frac{d^2y}{dx^2} - 4x^2\frac{dy}{dx} + 6xy = x^3.$$

~~4~~. Find all the critical points of the system of equations

$$\begin{aligned} \frac{dx}{dt} &= (y-2)(x^2+x) \\ \frac{dy}{dt} &= (x-1)(y-1) \end{aligned}$$

Determine the type and stability of each of the critical points.

~~5~~. Find the general solution of the system of equations

$$\frac{d}{dt}\vec{X} = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} \vec{X}$$

6. Find the eigenvalues (show graphically if necessary) and eigenfunctions that satisfy

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad 0 < x < L$$

where $y(x)$ satisfies the boundary conditions

$$y(0) = 0$$
$$y(L) + L\frac{dy}{dx}(L) = 0.$$

7. Determine the solution $u(x,t)$ of the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $0 < x < L$ for $t > 0$. Assume that u satisfies the no-flux boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.$$

Also assume that initial value of u is given by

$$u(x,0) = \begin{cases} 1 & 0 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}$$

What is the limiting value of u as $t \to \infty$?

PRELIM : DIFFERENTIAL EQNS 1/02

1. $y'' + \tan x\, y' = \cos x$

$y_H'' + \tan x\, y_H' = 0$

$\left(\frac{1}{\cos x} y_H'\right)' = 0$

$y_H' = A\cos x$

$y_H = A \sin x + B$

Check: $y_H'' = -A\sin x$

$-A\sin x + \frac{\sin x}{\cos x}(A\cos x) = 0$

$y_p(x) = -y_1(x)\int' \frac{y_2(x)g(x)}{w(x)}\,dx + y_2(x)\int' \frac{y_1(x)g(x)}{w(x)}$

$y_1(x) = \sin x, \quad y_2(x) = 1$

$w(x) = \begin{vmatrix} \sin x & 1 \\ \cos x & 0 \end{vmatrix} = -\cos x \qquad g(x) = \cos x$

$y_p(x) = -\sin x \int -dx + \int \frac{\sin x \cos x}{-\cos x}\,dx$

$= x\sin x + \cos x$

Check:

$y_p' = x\cos x + \sin x - \sin x = x\cos x$

$y_p'' = -x\sin x + \cos x$

$-x\sin x + \cos x + \frac{\sin x}{\cos x}(x\cos x) = \cos x$

$y(x) = y_H + y_p$

$y(x) = A\sin x + B + x\sin x + \cos x$

2. $y'' + y' - 2y = 3e^{3x}$ $\quad y(0)=1 \quad y'(0)=1$

$y_H'' + y_H' - 2y_H = 0$

$r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0$

$r = 1, -2$

$y_H = C_1 e^{-2x} + C_2 e^{x}$

$y_p'' + y_p' - 2y_p = \underbrace{3e^{3x}}_{g(x)}$

Try $y_p(x) = Ae^{3x}$

$y_p' = 3Ae^{3x}$; $y_p'' = 9Ae^{3x}$

$9Ae^{3x} + 3Ae^{3x} - 2Ae^{3x} = 3e^{3x}$

$9A + 3A - 2A = 3$

$10A = 3 \Rightarrow A = \frac{3}{10}$

$y_p(x) = \frac{3}{10}e^{3x}$

$y = y_H + y_p = C_1 e^{-2x} + C_2 e^{x} + \frac{3}{10}e^{3x}$

$y' = -2C_1 e^{-2x} + C_2 e^{x} + \frac{9}{10}e^{3x}$

$y(0) = C_1 + C_2 + \frac{3}{10} = 1 \Rightarrow C_2 = \frac{7}{10} - C_1$

$y'(0) = -2C_1 + C_2 + \frac{9}{10} = 1$

$-2C_1 + \frac{7}{10} - C_1 + \frac{9}{10} = 1 \Rightarrow -3C_1 = 1 - \frac{16}{10} = -$

$C_1 = 1/5$

$C_2 = 1/2$

$y = \frac{1}{5}e^{-2x} + \frac{1}{2}e^{x} + \frac{3}{10}e^{3x}$

3. $x^3y'' - 4x^2y' + 6xy = x^3$

$x^2y'' - 4xy' + 6y = x^2$

$y = x^K \qquad y' = Kx^{K-1} \qquad K(K-1)x^{K-2}$

$K(K-1)x^K - 4Kx^K + 6x^K = 0$

$K(K-1) - 4K + 6 = 0$

$K^2 - K - 4K + 6 = 0$

$K^2 - 5K + 6 = 0 \implies (K-3)(K-2) = 0$

$K = 3, 2$

$y_H = C_1x^2 + C_2x^3$

$y_P = -y_1(x)\int \frac{y_2(x)g(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)g(x)}{W(x)}dx \qquad g(x) = x^2$

$W(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$

$y_P(x) = -x^2\int x\,dx + x^3\int dx = -x^2\frac{x^2}{2} + x^3x = \frac{x^4}{2}$

$y(x) = C_1x^2 + C_2x^3 + \frac{x^4}{2}$

Check: $y_H = C_1x^2 + C_2x^3$

$y_H' = 2C_1x + 3C_2x^2$

$y_H'' = 2C_1 + 6C_2x$

$x^2(2C_1 + 6C_2x) - 4x(2C_1x + 3C_2x^2) + 6(C_1x^2 + C_2x^3) = 0$

ind a particular solution

$x^2 y_p'' - 4x y_p' + 6 y_p = x^2$

$y_p'' - \frac{4}{x} y_p' + \frac{6}{x^2} y_p = 1$

$y_1(x) = x^2 \qquad y_2(x) = x^3$

$w(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4 \qquad g(x) = 1$

$y_p(x) = -y_1(x) \int \frac{y_2(x) g(x)}{w(x)} dx + y_2(x) \int \frac{y_1(x) g(x)}{w(x)} dx$

$= -x^2 \int \frac{x^3}{x^4} dx + x^3 \int \frac{x^2}{x^4} dx = -x^2 \int \frac{dx}{x} + x^3 \int x^{-2} dx$

$= -x^2 \ln x + x^3 \left(\frac{-1}{x}\right) = -x^2 \ln x - x^2$

check: $y_p = -x^2 \ln x - x^2$

$y_p' = -2x \ln x - x - 2x = -2x \ln x - 3x$

$y_p'' = -2 \ln x - 5$

$x^2(-2\ln x - 5) - 4x(-2x \ln x - 3x) + 6(-x^2 \ln x - x^2)$

$= -\cancel{2x^2 \ln x} - 5x^2 + \cancel{8x^2 \ln x} + 12x^2 - \cancel{6x^2 \ln x} - 6x^2 = \underline{\underline{x^2}}$

$\boxed{y = C_1 x^2 + C_2 x^3 - x^2 \ln x - x^2}$

4. $\dot{x} = (y-2)(x^2+x)$

$\dot{y} = (x-1)(y-1)$

critical pts: $(1,2)$, $(0,1)$, $(-1,1)$

$$A = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 2xy + y - 4x - 2 & x^2 + x \\ y-1 & x-1 \end{pmatrix}$$

$$A_{(1,2)} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} -\lambda & 2 \\ 1 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 2 = 0 \Rightarrow \lambda^2 = 2 \Rightarrow \lambda = \pm\sqrt{2}$$

$\Rightarrow$ unstable saddle

$$A_{(0,1)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\det\begin{pmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = 0 \Rightarrow (-1-\lambda)^2 = 0 \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$\lambda = -1$$

$\Rightarrow$ asymptotically stable proper node

$$A_{(-1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & -2-\lambda \end{pmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda) = 0$$

$$\lambda = 1, \quad \lambda = -2$$

$\Rightarrow$ unstable saddle

5. $\frac{d}{dt}\vec{x} = \underbrace{\begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}}_{A} \vec{x}$

$\det(A-\lambda I) = \det \begin{bmatrix} -2-\lambda & -9 & 0 \\ 1 & 4-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (-2-\lambda)(4-\lambda)(1-\lambda)$
$+9(1-\lambda) = 0$
$\Rightarrow \lambda = 1$ threetimes

$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$a_1 + 3a_2 = 0 \Rightarrow a_1 = -3a_2$

$v_1 = \begin{Bmatrix} -3 \\ 1 \\ 0 \end{Bmatrix} \quad v_2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} -3 \\ 1 \\ 0 \end{Bmatrix}$

this won't work
$b_1 + 3b_2 = 1$
$b_1 + 3b_2 = 0$ $\quad \phi$

$= \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

won't work either
$b_1 + 3b_2 = 0$
$b_1 + 3b_2 = 1$ $\quad \phi$

$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \underset{\uparrow_1}{c_1} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \underset{\uparrow_1}{c_2} \begin{Bmatrix} -3 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -3 \\ 1 \\ 1 \end{Bmatrix}$

$-3b_1 - 9b_2 = -3 \Rightarrow b_1 + 3b_2 = 1$

$b_1 + 3b_2 = 1$
$b_1 + 3b_2 = 1$

$v_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

$\vec{x}(t) = c_1 \begin{Bmatrix} -3 \\ 1 \\ 0 \end{Bmatrix} e^t + c_2 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} e^t + c_3 \left[\begin{Bmatrix} -3 \\ 1 \\ 1 \end{Bmatrix} te^t + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} e^t \right]$

6. $y'' + \lambda y = 0 \qquad 0<x<L \qquad y(0)=0$

$$y(L) + L\,y'(L) = 0$$

$\underline{\lambda > 0}$

$y'' + \lambda y = 0 \Rightarrow y = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$

$y' = \cancel{-A\sqrt{\lambda}\sin(\sqrt{\lambda}\,x)} + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$

$y(0)=0 \Rightarrow A = 0$

$y(L) + L\,y'(L) = 0$

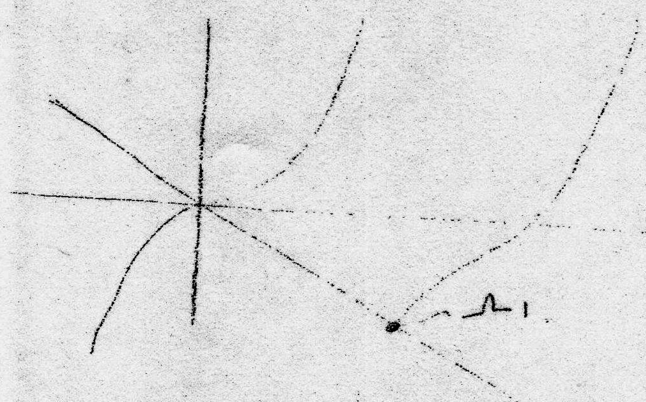
$B\sin(\sqrt{\lambda}\,L) + L B\sqrt{\lambda}\cos(\sqrt{\lambda}\,L) = 0$

$B\left(\sin(\sqrt{\lambda}\,L) + L\sqrt{\lambda}\cos(\sqrt{\lambda}\,L)\right) = 0$

want $\sin(\sqrt{\lambda}\,L) + L\sqrt{\lambda}\cos(\sqrt{\lambda}\,L) = 0$

$-\dfrac{\sin(\sqrt{\lambda}\,L)}{\cos(\sqrt{\lambda}\,L)} = \sqrt{\lambda}\,L \Rightarrow \tan(\sqrt{\lambda}\,L) = -\sqrt{\lambda}\,L$

$\Omega = \sqrt{\lambda}\,L \Rightarrow \dfrac{\Omega}{L} = \sqrt{\lambda} \Rightarrow \dfrac{\Omega^2}{L^2} = \lambda$



$\underline{\lambda = 0}$

$y'' = 0$ $\qquad\qquad$ $\lambda=0$ is not an eigenvalue

$y' = A$

$y = Ax + B$

$y(0)=0 \Rightarrow 0 = B$

$y = Ax$

$y(L) + L\,y'(L) = 0$

$AL + LA = 0$

$2LA = 0$

$A = 0$

$\underline{\lambda < 0}$

$y'' - |\lambda| y = 0$

$y = A\cosh(\sqrt{|\lambda|}\,x) + B\sinh(\sqrt{|\lambda|}\,x)$

$y(0) = 0 = A$

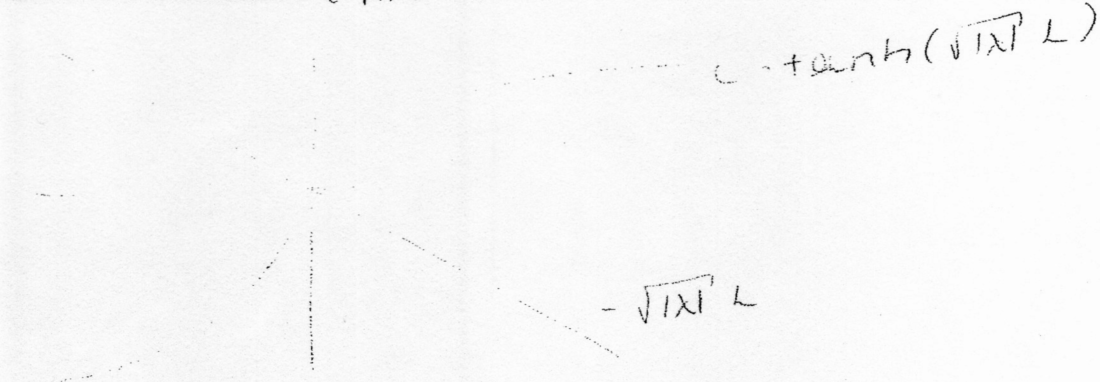
$y = B\sinh(\sqrt{|\lambda|}\,x) \qquad y' = B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}\,x)$

$y(L) + L y'(L) = 0$

$B\sinh(\sqrt{|\lambda|}\,L) + L B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}\,L) = 0$

Dont want $B=0$. assume $B \neq 0$

$\tanh(\sqrt{|\lambda|}\,L) = -\sqrt{|\lambda|}\,L$



no eigenvalues for $\lambda < 0$

t. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < L \quad , t > 0$

BC

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

IC

$$u(x,0) = \begin{cases} 1 & 0 \leq x \leq L/2 \\ 0 & L/2 \leq x \leq L \end{cases}$$

assume a solution of the form $u(x,t) = X(x)T(t)$

$$X(x)T'(t) = X''(x)T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

$$T'(t) + \lambda^2 T(t) = 0 \implies \frac{T'}{T} = -\lambda^2 \implies T = Ae^{-\lambda^2 t}$$

$$X''(x) + \lambda^2 X(x) = 0 \implies X = B\cos(\lambda x) + C\sin(\lambda x)$$

$$u(x,t) = e^{-\lambda^2 t}(C_1 \cos(\lambda x) + C_2 \sin(\lambda x))$$

BCs

$$\frac{\partial u}{\partial x} = e^{-\lambda^2 t}(-\lambda C_1 \sin(\lambda x) + \lambda C_2 \cos(\lambda x))$$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = e^{-\lambda^2 t}(\lambda C_2) = 0 \implies C_2 = 0$$

$$\frac{\partial u}{\partial x}\Big|_{x=L} = e^{-\lambda^2 t}(-\lambda C_1 \sin(\lambda L)) = 0 \implies \sin(\lambda L) = 0$$

$$\implies \lambda L = \pm n\pi \implies \lambda = \frac{\pm n\pi}{L}$$

$$\lambda^2 = \frac{n^2\pi^2}{L^2}$$

$$u_n(x,t) = C_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L} x\right)$$

$$u_n(x,0) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L} x\right) = \begin{cases} 1 & 0 \leq x \leq L/2 \quad \phi_1(x) \\ 0 & L/2 \leq x \leq L \quad \phi_2(x) \end{cases}$$

$$\phi(x) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L} x\right)$$

$$\cos\left(\frac{m\pi x}{L}\right)\phi(x) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$$\int_0^1 \cos\left(\frac{m\pi x}{L}\right)\phi(x)\,dx = \int_0^1 C_n \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\int_0^1 \cos\left(\frac{m\pi x}{L}\right)\phi(x)\,dx = \frac{C_n}{2}$$

$$\Rightarrow C_n = 2\int_0^1 \cos\left(\frac{m\pi x}{L}\right)\phi(x)\,dx$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$$

where $C_n = 2\int_0^1 \cos\left(\frac{n\pi x}{L}\right)\phi(x)\,dx$

$$\phi(x)=1 \Rightarrow C_n = 2\int_0^{1} \cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_0^1 = \frac{2L}{n\pi}\sin\left(\frac{n\pi}{L}\right)$$

$$\phi(x)=0 \Rightarrow C_n = 2\int_0^{1} 0\,dx = 0$$

$$u(x,t) = \begin{cases} \sum_{n=1}^{\infty} \frac{2L}{n\pi} e^{-\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}\right) & 0 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}$$

$$\lim_{t\to\infty} u(x,t) = \begin{cases} \sum_{n=1}^{\infty} \frac{2L}{n\pi} \cos\left(\frac{n\pi}{L}\right) & 0 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}$$