Applied Mathematics Differential Equations Preliminary Exam

11 January 2012

1. Solve the following differential equations.

   (a) $\dfrac{dy}{dx} = -\dfrac{2xy}{1+x^2} + 1, \quad y(0) = 1.$

   (b) $(x + e^y)\dfrac{dy}{dx} = -y.$

2. Determine the general solution $y(x)$ of the differential equation

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0.$$

   given that $y = x$ is a solution.

3. Consider the damped, forced spring system described by the equation

$$y'' + 2\epsilon y' + y = \cos(t)$$

   where y is the displacement of the block from equilibrium. Find the general solution and describe the limiting amplitude of the motion. What happens as $\epsilon \to 0$?

4. Find the solution of the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

   with $x(0) = 1$, $y(0) = 0$ and $z(0) = 0$.

5. Consider the differential equation

$$\frac{dy}{dt} = \epsilon y - y^3, \quad \epsilon > 0.$$

   Let $y(t, y_0)$ denote the solution which takes the value y_0 at $t = 0$.

   (a) Find and determine the stability of all critical points.

   (b) For every y_0 determine the limiting value of $y(t, y_0)$ as $t \to \infty$ or as t approaches the limiting time for which the solution exists. It is not necessary to solve the equation explicitly.

   (c) Explain what happens as $\epsilon \to 0$.

6. Consider the nonlinear oscillator equation

$$\ddot{x} = x - 4x^3.$$

   (a) Rewrite this second order differential equation as a system of two first order equations.

   (b) Find and classify all critical points (e.g., center, saddle, (un)stable node, etc.).

   (c) Show that the energy

$$E = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + x^4$$

   is conserved along trajectories.

   (d) Find a potential energy $V(x)$ such that we can write the oscillator equation as

$$\ddot{x} = -\frac{dV}{dx}$$

   (e) Sketch the phase portrait in the $(x, \dot{x})$-phase plane.

   (f) If we now introduce some damping to the system so that

$$\ddot{x} = -\epsilon\dot{x} + x - 4x^3, \quad \epsilon > 0,$$

   what can you say about the long-term behavior of trajectories? Here we are looking for qualitative information. Explain your answer.

7. The boundary value problem

$$y'' + y = f(x), \quad 0 < x < \pi/2, \quad y'(0) = 0, \; y(\pi/2) = 0$$

may or may not have a solution, depending on the function $f(x)$.

   (a) Determine condition(s) on $f(x)$ which, if satisfied, guarantee the existence of a solution.

   (b) If $f(x)$ satisfies the condition(s), then how many solutions exist?

   (c) If $f(x) = \sin(3x - \mu)$, where μ is a real parameter, solve, if possible, the problem. Comment on uniqueness and solvability of the problem depending on μ.

You may find the trig identities

$$\begin{aligned} \sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\ \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b\,. \end{aligned}$$