Elementary Differential Equations Preliminary Exam, ES/AM
April 26, 1994

1. (30 points.) Solve the following differential equations:

$$yy'' + (y')^2 = -\frac{1}{2x},$$
$$y'' - 4y' + 4y = (12x^2 - 6x)e^{2x},$$
$$y' = xy(y-2).$$

2. (15 points.) One homogeneous solution of the equation

$$xy'' - y' + 4x^3 y = x^3$$

for $x > 0$ is $y_1(x) = \sin(x^2)$. Find the general solution.

3. (15 points.) Find the solution of the initial value problem

$$\dot{x} = 2x - y,$$
$$\dot{y} = x + 2y - 1,$$
$$x(0) = 1, \quad y(0) = 1.$$

4. (10 points.) Determine whether the given five functions are linearly independent on the interval $[0, 2\pi]$

$$\sin x, \quad e^x, \quad \cos(x+2), \quad \tan(x+3), \quad e^{x+1}.$$

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$ae^x + be^{x+1} = 0$
for $a = e$, $b = -1$.

5. (10 points.) Determine all real critical points of the system

$$\frac{dx}{dt} = x(x - y - 1), \quad \frac{dy}{dt} = y(y + x - 2)$$

for $x > 0$ and $y > 0$ and discuss their type and stability.

6. (10 points.) Consider the function $f(x) = 1 - \frac{1}{2}x$, $0 < x < 1$. Suppose this function was expanded in a Fourier cosine series on $0 < x < 1$. Sketch the series for $-3 \le x \le 3$.

7. (10 points.) Consider the equation

$$y'' + (5 + \sin x)y' + y = 0.$$

Let $g(x)$ be the solution such that $g(0) = 1$, $g'(0) = 0$.
Let $h(x)$ be the solution such that $h(0) = 0$, $h'(0) = 1$.
Define

$$E(x) = g(x)h'(x) - h(x)g'(x).$$

Show that

$$\lim_{x \to +\infty} E(x) = 0.$$