Advanced Calculus Exam — January 6, 1993

1. (25 pts.) Consider the curve, $C$, that is the intersection of the cylinder

$$x^2 + y^2 = 1$$

with the surface

$$z = g(x, y)\,,$$

and the vector field

$$\vec{F} = (y^3 + 3x^2 y)\mathbf{i} + 3x\mathbf{j} + z^2\mathbf{k}\,.$$

Compute the work, W, done along $C$ by $\vec{F}$ (traversed in a right handed sense relative to the z-axis),

$$W = \oint_C \vec{F} \cdot d\vec{r}$$

and show it is independent of the choice of $g(x, y)$. (Hint: Use Stokes' theorem.)

2. (25 pts.) In cylindrical coordinates $(r, \theta, z)$, a torus has equation

$$(r - a)^2 + z^2 = b^2\,.$$

(a) Write and evaluate an integral for the volume of the torus in cylindrical coordinates.

(b) Change the coordinate system from cylindrical coordinates $(r, \theta, z)$ to toroidal coordinates $(\rho, \theta, \phi)$ where

$$r = a + \rho\cos(\phi) \qquad \theta = \theta \qquad z = \rho\sin(\phi)\,.$$

Write and evaluate an integral for the volume of the torus in toroidal coordinates.

3. (25 pts). Consider the surface defined by

$$F(x, y, z) = \frac{1}{2}(x+y)^2 + (y+z)^2 + (z+x)^2 = 9 .$$

(a) Evaluate $\nabla F$.

(b) Find the highest and lowest points on the surface (i.e. the points where $z$ obtains a minimum or maximum).

(c) The surface is illuminated from far above by light rays that are directed parallel to the z-axis. Find the shape of its shadow in a plane below the surface and parallel to the x-y coordinate plane.

4. (25 pts). For this problem you will need to use Maxwell's equations for the magnetic ($\vec{B}$) and electric ($\vec{E}$) fields:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad , \quad \nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} .$$

Prove that for any closed surface, $S$, bounding a volume, $V$,

$$\int\int_S \vec{P} \cdot \hat{n}\, dS = -\frac{\partial}{\partial t}\int\int\int_V U dV ,$$

where $\hat{n}$ is an outward pointing normal and $\vec{P}$ is the Poynting vector,

$$\vec{P} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) ,$$

and $U$ is the energy density,

$$U = \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2)$$