Differential Equations
ESAM Preliminary Examination
January 9, 2006, 9-11 a.m.

1. Solve the following differential equations for $y(x)$:

   (a) $x^3y'' - 2x^2y' + 2xy = 1,$

   (b) $\dfrac{d^4y}{dx^4} - y = e^x \sin x,$

   (c) $\dfrac{dy}{dx} = xy(y-2) + x,$

   (d) $2\dfrac{dy}{dx} = \dfrac{x+6y-2}{x-2y-2},$

   (e) $y\dfrac{dy}{dx} = x(e^{x^2} - y^2).$

2. The functions $y_1(x) = 2x+1$ and $y_2(x) = -1$ satisfy the equation

$$x\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + y = -1.$$

   Solve the initial value problem

$$x\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + y = -1, \quad y(0) = 0, \ y'(0) = 0.$$

3. Find a particular solution of the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

4. Consider the equation

$$y' = -y + \sin\left(\frac{1}{t}\right),$$

   with the initial condition $y(1) = 0$. Show that $|\ y(t)\ | \leq 1$ for $t \geq 1$. It is not necessary to find the solution explicitly.