Advanced Calculus Preliminary Exam

Tuesday, January 8, 2008 (9 – 11:00 am)

CHOOSE 5 PROBLEMS OUT OF 6

Please show all work. To get full credit you must **clearly** describe the details of all your calculations.

Problem 1. Consider the vector field

$$\vec{F} = (y^3 + 3x^2y)\mathbf{i} + 3x\mathbf{j} + z^2\mathbf{k}$$

and the curve, C, that is the intersection of the cylinder $x^2 + y^2 = 1$ with the surface $z = g(x, y)$. Compute the work, W, done along C by $\vec{F}$,

$$W = \oint_C \vec{F} \cdot d\vec{r}$$

(C is oriented clockwise when viewed from above). Does W depend on the choice of $g(x, y)$?

Problem 2. In cylindrical coordinates (r, θ, z), a torus has equation

$$(r - a)^2 + z^2 = b^2, \quad (a > b > 0).$$

Write and evaluate an integral for the volume of the torus in cylindrical coordinates.

Problem 3. Evaluate

$$I = \int\int_D (x + y)^2 \, dx \, dy$$

where D is the region

$$(x - y)^2 + \frac{(x + y)^2}{4} \leq 1 \; .$$

Problem 4. Consider the surface defined by

$$F(x, y, z) = \frac{1}{2}(x + y)^2 + (y + z)^2 + (x + z)^2 = 9.$$

(a) Evaluate ∇F.

(b) Find the highest and lowest points on the surface (i.e. the points where z attains a maximum or minimum).

(c) The surface is illuminated from far above by light rays that are directed parallel to the z-axis. Find the shape of its shadow in a plane below the surface and parallel to the (x, y) coordinate plane.

Problem 5. Compute directly the flux,

$$\int\int \vec{F} \cdot \hat{n} \, dS$$

out of the cone

$$x^2 + y^2 = z^2, \qquad 0 \leq z \leq 1$$

for the vector field

$$\vec{F} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}.$$

Verify your answer with the divergence theorem.

Problem 6. Find the area of the plane region bounded by the curve

$$(x^2 + y^2)^3 = x^4 + y^4.$$