Preliminary Examinations: Complex Variables **January 13, 2010**

Name: ______________________

Score

1	/12
2	/12
3	/12
4	/12
5	/12
6	/13
7	/13
8	/13
Total	/99

Show all your work.

Problem 1 (12 points). Express $(2+2i)^{11}$ in the form $a+ib$.

Problem 2 (12 points). For $f(z)$ given in polar coordinates as

$$f(z) = \theta^2 - 2ir$$

find an expression for $f'(z)$ in polar coordinates and indicate where $f'(z)$ exists.

Problem 3 (12 points). Express the function

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

in a Laurent series expansion about $z = 0$, which is valid for $1 < |z| < 2$.

Problem 4 (12 points). Evaluate the integral

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}d\theta.$$

Problem 5 (12 points). Compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x + 1}\, dx \qquad 0 < \alpha < 1,$$

by considering a contour given by the rectangle with vertices at $-R$, R, $R + 2\pi i$, $-R + 2\pi i$, where $R \to \infty$.

Problem 6 (13 points). Use the Bromwich integral $f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds$ to determine the inverse Laplace transform of

$$F(s) = \frac{s^2-1}{(s^2+1)^2}.$$

Sketch the closed contour you are using.

Problem 7 (13 points). Compute the integral

$$\int_0^\infty \frac{x^{\frac{1}{2}} \log x}{(1+x)^2}\, dx.$$

Sketch the closed contour you are using. Explain which contributions to the contour integral vanish.

Problem 8 (13 points). Use a conformal mapping to solve for $u(x, y)$, which is harmonic in the semi-annulus $1 < r^2 \equiv x^2 + y^2 < 4$, $y > 0$, and satisfies the boundary conditions,

$$\frac{\partial u}{\partial r} = 0, \quad \text{for} \quad r = 1, \quad r = 2, \quad 0 < \theta < \pi$$

$$u = 1, \quad \text{for} \quad y = 0 \quad 1 < x < 2$$

$$u = 0, \quad \text{for} \quad y = 0 \quad -2 < x < -1.$$

Hint: the transformation $\zeta = \log z$ conformally maps the semi-annulus onto a rectangle.