Applied Mathematics Differential Equations Preliminary Exam

30 March 2012

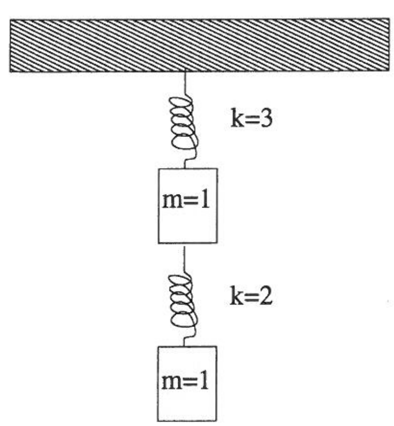
1. Determine all values of μ for which the boundary value problem
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \mu y = 0 \qquad y(0) = 0 = y(1)$$
has a non-trivial solution. Give the corresponding solution(s).

2. For the problem
$$\frac{dA}{dt} = (\alpha - 1)A + 2A^3 - A^5$$
characterize the critical points and their stability for a) $\alpha < 0$; b) $0 < \alpha < 1$; c) $\alpha > 1$.

3. Find the general solution to the differential equation
$$x^3\frac{d^2y}{dx^2} - 4x^2\frac{dy}{dx} + 6xy = x^3.$$

4. A frictionless mass-spring system consists of two masses and two springs as sketched below. The equations of motion for the masses are given by
$$\begin{aligned} \ddot{u}_1 &= -2\,(u_1 - u_2) - 3u_1 \\ \ddot{u}_2 &= -2\,(u_2 - u_1)\,. \end{aligned}$$
   (a) Determine the general solution $\{u_1(t), u_2(t)\}$.

   (b) Give the solution satisfying the initial conditions
$$u_1(0) = 1 \quad \dot{u}_1(0) = 0 \quad u_2(0) = 2 \quad \dot{u}_2(0) = 0.$$
   (c) Give the solution satisfying the initial conditions
$$u_1(0) = -2 \quad \dot{u}_1(0) = 0 \quad u_2(0) = 1 \quad \dot{u}_2(0) = 0.$$
   (d) Describe briefly in words the motion of the masses in parts (b) and (c); what is their frequency and how do the masses relatively to each other?



5. One day it started snowing at a heavy and steady rate. At noon a snowplow started to clear a road; this snowplow can clear a constant volume of snow per hour. The snowplow cleared the first mile of that road by 1:00 p.m. and the next mile by 3:00 p.m. At what time did it start snowing?

6. Find the solution of the problem

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{y} \quad \text{with} \quad y(1) = 1\,.$$

7. Find the solution of the system of equations

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^{-2t} \\ 6 \end{bmatrix}.$$

8. Consider the system

$$\frac{dx}{dt} = y^2,$$

$$\frac{dy}{dt} = x^2.$$

a) Sketch the phase portrait of the system.
b) What is the qualitative difference in behavior of the solution with $x(0) = y(0) = -1000$ and that with $x(0) = -1000$, $y(0) = -1000.000001$?