Complex Variables Exam

Friday, January 17, 2003 (11:00 a.m. - 1:00 p.m.)

Please show all work. To get full credit for a problem you need to clearly describe your calculations.

1. **(10 points)** Determine where $f'(z)$ exists and provide an expression for it:

$$f(z) = e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y), \quad z = x + iy.$$

2. **(45 points)** Use complex variables to evaluate the following integrals

$$\text{(a)} \quad \int_0^\infty \frac{x^3\,dx}{x^5+1}, \qquad \text{(b)} \quad \int_0^{2\pi} \frac{d\theta}{1-2k\cos\theta+k^2} \quad (k>1),$$

$$\text{(c)} \quad \int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+2\pi x+1+\pi^2}.$$

Please give the answers in the real form (not as complex numbers).

3. **(10 points)** Suppose $g(z)$ is an entire function (i.e. analytic everywhere). Determine the value of

$$\oint_C \frac{g(z)}{(z-i)(z-2i)(z-3i)}dz,$$

where $C:\ z = 1+3e^{i\theta}$, $0 \le \theta < 2\pi$.

4. **(15 points)** Find three separate Laurent series expansions of the function

$$\frac{z}{(z-1)(z-2)}$$

in powers of z: one valid for $0 < |z| < 1$, one valid for $1 < |z| < 2$, and one valid for $|z| > 2$.

5. **(10 points)** Verify that $u(x,y) = x^2 - y^2$ is harmonic everywhere in the plane and find all analytic functions $f(z)$, $z = x+iy$, such that $u = \text{Re}f(z)$.

6. **(10 points)** Find the image of the circle $x^2 + (y-1)^2 = 1$ under the transformation

$$w = \frac{z-i}{z+i} \quad (z = x+iy).$$