Advanced Calculus Exam

January 11, 2002

Please show all work. To get full credit for a problem you need to **CLEARLY** describe your calculations.

Problem 1 (25 pts) Compute the integrals

$$\int_0^{\pi}\left(\int_y^{\pi}\frac{\sin x}{x}\,dx\right)dy, \qquad \iint_{\Omega}\sqrt{x^2+y^2+1}\,dx\,dy,$$

where $\Omega = \{(x,y):\ x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 4\}$.

Problem 2 (20 pts) The cylinder $x^2+y^2=4$ and the plane $2x+2y+z=2$ intersect in an ellipse. Find the points on the ellipse that are nearest to and farthest from the origin.

Problem 3 (10 pts) Let $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$ and $r = |\vec{r}|$ (i.e., r is the length of the vector $\vec{r}$). Compute $\mathbf{curl}(r^3\vec{r})$. Simplify your answer.

Problem 4 (15 pts) Find the work done by the force

$$\vec{F} = x\vec{i} - y\vec{j} + (x+y+z)\vec{k}$$

on a particle that moves along the parabola $y = 3x^2$, $z = 0$ from the origin to the point $(2, 12, 0)$.

Problem 5 (15 pts) Find the center of mass of a plate shaped like the region between $y = x^2$ and $y = 2x$, where the density $\rho(x,y)$ varies as $\rho(x,y) = x$.

Problem 6 (15 pts) Suppose that f is such that for any closed, oriented surface S

$$\iint_S \frac{\partial f}{\partial n}\,dS = 0$$

(here $\partial f/\partial n$ is the normal derivative). Show that then

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} = 0.$$