FLOW OF A NEWTONIAN FLUID
THE CASE OF BLOOD IN LARGE ARTERIES

by

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I dedicate this dissertation to the memory of my father,

SAMUEL OUKOUOMI.

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Contents

Chapter 1

Introduction

1.1 Background

The physiology of the cardiovascular system has been elucidated only gradually, over many centuries. Among the major actors in the lengthy process have been some of the central characters in human history.
Aristote (384-322 B.C.), for example, identified the role of blood vessels in transferring " animal heat" from the heart to the periphery of the body (although he ignored blood circulation).
In the third century B.C., Praxagoras realized that arteries and veins have different roles (believing that arteries transported air while veins transported blood). Galen was the first to observe the presence of blood in the arteries. Much later, in the 17th century, Sir William Harvey inaugurated modern cardiovascular research with his De Motu Cardis and Sanguinis Animalibus, in which he wrote, " When I turned to vivisection I found the task so hard I was about to think that only God could understand the heart motion". Harvey observed that the morphology of veins is such that they are effective only if blood is flowing toward the heart. His conclusion: " I began privately to consider if it (the blood) had a movement, as it is, it would be in a circle".

In the 18th century, the Reverend S. Hales introduced quantitative studies of blood pressure (Hemostatics, 1773). Later, Euler and D. Bernoulli both made great contributions to fluid dynamic research. Bernoulli in particular as a Professor of anatomy at the University of Basel in Switzerland, investigated the laws governing blood and formulated his famous law relating pressure, density, and velocity: $p + \frac{1}{2}\rho| u |^2 =$ const (vis viva equation, 1730).

In the 19th century, J.P. Poiseuille, medical doctor and physicist, was studying the flow of blood in arteries when he derived the first simplified mathematical model of flow in a cylindrical pipe, a model that still bears his name today.

T. Young later made fundamental contributions to research on elastic properties of arterial tissues and on the propagation of pressure: " The inquiry in what manner and at what degree the circulation of the blood depends on the muscular and the elastic power of the heart and the arteries, supposing the nature of these powers to be known, must become simply a question belonging to the most renowned department of the theory of hydraulics" (From a lesson given by Young at the Royal Society of London in 1809).

At the beginning of the 20th century, O. Franck introduced the idea that the circulatory system is analogous to an electric network. In 1955, J. Womersley, studying vascular flows, found the analytical counterpart of Poiseuille flow in pressure gradients that vary periodically in time, a situation that more closely describes actual pressure variations during the cardiac cycle.

In the second half of the 20th century, developments in mathematical modeling were limited to basic paradigms, such as flow in morphologically simple regions (e.g., Poiseuille or Womersley solutions), or to models based on electric network analogies. Exact solutions are very difficult to obtain in more general situations, because of the strong nonlinear interactions among different parts of the system and the geometric complexities of individual vascular morphologies.
The historical details in this section is extracted from *Modeling the Cardiovascular System-A Mathematical Adventure: Part I* by Alfio Quarteroni [13].

1.2 Outline of the dissertation

The first requirement to study blood flow is to gain a general understanding of the physiology involved. In this regard, a brief outline of the circulatory system and the description of some cardiovascular equations are provided in chapter 2 to give the reader a better understanding of how blood is circulated and regulated in the human body. The similarities between the cardiovascular system and electrical circuits are also considered at the end of this chapter.

Chapter 3 presents a theoretical model of blood flow in the large arteries. The hypothesis of a Newtonian rheology is considered and blood flow is described by the Navier-Stokes equations, with specific initial and boundary conditions. The well-posedness of the Navier-Stokes problem is addressed, having provided boundary conditions which can be considered as a generalization of the mean pressure drop problem investigated in [7]. The properties of the Stokes operator are introduced and the eigenfunctions of its inverse operator are basis functions for the Galerkin approximations.

In chapter 4 the conclusion is presented, followed by the appendix, as well as the bibliography.

Chapter 2

Cardiovascular Physiology and Equations

Blood is a circulating tissue composed of fluid plasma and cells (red blood cells, white blood cells, platelets). The main function of blood is to supply nutrients to the tissues and to remove waste products. Blood also enables cells (leukocytes, abnormal tumor cells) and different substances (lipids, hormones, amino acids) to be transported between tissues and organs [8].

2.1 Anatomy of blood

Blood is composed of several kinds of corpuscles; these formed elements of the blood constitute about 45% of whole blood. The other 55% is blood plasma, a yellowish fluid that is the blood's liquid medium. The normal pH of human arterial blood is approximately 7.40 (normal range is 7.35-7.45). Blood that has a pH below 7.35 is acidic, while blood pH above 7.45 is alkaline. Blood pH reading is helpful in determining the acid-base balance of the body. Blood is about 7% of the human body weight, so the average adult has a blood volume of about 5 liters, of which 2.7-3 liters is plasma. The combined surface area of all the red blood cells in the human anatomy would be roughly 2,000 times as great as the body's exterior surface. The corpuscles are:

1. Red blood cells or erythrocytes (96%). In mammals, mature red blood cells lack a nucleus and organelles. They contain the blood's hemoglobin and distribute oxygen. The red blood cells (together with endothelial vessel cells and some other cells) are also marked by proteins that define different blood types.

2. White blood cells or leukocytes (3.0%), are part of the immune system; they destroy infectious agents.

3. Platelets or thrombocytes (1.0%) are responsible for blood clotting (coagulation).

Blood plasma is essentially an aqueous solution containing 96 percent of water, 4% of proteins, and trace amounts of other materials. Some components are albumin, blood clotting factors, immunoglobulins (antibodies), hormones, various other proteins and various electrolytes (mainly sodium and chlorine) [8, 15].

2.2 Cardiovascular physiology

The problem of mathematically modelling a human function is not easy. First, a good foundation of the physiology to be modelled must be formed. In this regard, we introduce the elementary physiology of the cardiovascular system and highlight some cardiovascular equations. The cardiovascular system includes the heart and the blood vessels. The heart pumps blood and the blood vessels channel and deliver it throughout the body. Arteries carry blood filled with nutrients away from the heart to all parts of the body and blood comes back to the heart through veins. Arteries are thick-walled tubes with a circular covering of yellow, elastic fibers, which contain a filling of muscle that absorbs the tremendous pressure wave of the heartbeat and slows the blood down. This pressure can be felt in the arm and wrist: it is the pulse. Eventually, arteries divide into smaller arterioles and then into even smaller capillaries, the smallest of all blood vessels. One arteriole can serve hundred capillaries. Here, in every tissue of every organ, blood's work is done when it gives up what the cells need and take away the waste products that they don't need. Capillaries join to form venules, which flow into veins, and these deliver deoxygenated blood back to the heart. Veins, unlike arteries, have thin, slack walls, because the blood has lost the pressure which forced it out of the heart, so the dark, reddish-blue blood which flows through the veins on its way to the lungs moves along very slowly on its way to be reoxygenated . Back to the heart, the blood enters a special vessel called the pulmonary arteries, from the right side. It flows along to the lungs to collect oxygen, then to the pulmonary veins and back to the heart's left side to begin its journey around the body again.

The flow of blood through the cardiovascular system follows physical laws known from fluid mechanics (e.g., the law of conservation of matter states that mass or energy can neither be created nor destroyed). The total blood volume is approximately 5 liters in a healthy adult. The right atrium receives venous blood from the caval veins, and the left atrium receives oxygenated blood from the pulmonary veins. On average, atrial systole contributes only about 15% of the total ventricular filling. The total blood volume (5 liters) is distributed with 60-75% in veins and venules, 20% in arteries and arterioles, and only 5 percent in capillaries at rest. Of the total blood volume only 12 percent is found in the pulmonary low-pressure system [10].

2.3 Cardiovascular equations

2.3.1 A geometrical relation

The relationship between linear mean velocity $\overline{v}$ and the blood flow in one unit of time (the flux Q), is determined by the cross sectional area A:

$$Q = \overline{v}A. \tag{2.1}$$

2.3.2 Poiseuille's law

The volume rate or the flux ΔV is equal to the driving pressure ΔP divided by the resistance:

$$\Delta V = \frac{\Delta P}{resistance}.$$

In the left ventricle, the blood flow is described by the cardiac output Q (i.e. flux, blood flow per unit of time), so that the equation reads:

$$Q = \frac{\Delta P}{R}, \tag{2.2}$$

where R is the total peripheral vascular resistance. R is directly related to the blood viscosity μ and to the length L of the vascular system, and inversely related to its radius to the 4th power:

$$R = \frac{8\mu L}{\pi r^4} \quad [9]. \tag{2.3}$$

Doubling the length of the system only doubles the resistance, where halving the radius increases the resistance sixteen-fold.

2.3.3 Vascular resistance in parallel organs

In the systemic or peripheral circulation the resistance of the organs are mainly placed in parallel, and the resistance of all organs R_1 to R_n are related to the total resistance R by the following relation:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n} \quad [15]. \tag{2.4}$$

2.3.4 Vascular resistance in portal circulations

There are only a few serially connected elements (portal circulation): Spleen/liver, gut/liver, pancreas/liver and hypothalamus/pituitary. For serial arranged organs, the total resistance is given by the formula:

$$R_{total} = R_1 + R_2 + ... + R_n \quad [15]. \tag{2.5}$$

2.4 Electrical analogy

The cardiovascular system presents many similarities with electric circuits. In fact, for every closed fluid system, there is an electrical circuit whose behavior is identical (up to conversion factors). The heart develops a pressure difference which moves the blood through the system. Similarly, a battery (or other power supply) develops a potential difference which moves electrons around the circuit (to a position of least energy). Thus a fluid pressure drop (energy per unit volume) corresponds to a potential difference, or "voltage drop" (energy per unit charge). Likewise, the flow rate of a fluid is analogous to the current in a circuit, which is the rate of movement of charge.

Poiseuille's law corresponds to Ohm's law for electric circuits. If we solve Poiseuille's law for the pressure drop, and replace pressure drop by voltage and flow rate by current, we have that

$$V = I \, (\frac{8\mu l}{\pi r^4}).$$

where V is the voltage, I is the current, μ the viscosity, l is the length of the blood vessel and r its radius. The quantity in parentheses has the characteristics of a resistance: for a given voltage drop, a larger value of l results in less flow, while a larger value of r results in greater flow. In fact, the electrical analog of this fluid quantity is the resistance

$$R = \frac{\rho L}{A},$$

where ρ is the resistivity, L is the length of the conductor, and A is its radius. The fluid analog of the resistivity is essentially the ratio of viscosity to cross sectional area.

Ohm's law is in a sense more useful than Poiseuille's law, for it is not restricted to long straight wires with constant current. This is essentially due to the order of magnitude of the size of the charge carriers. The speed of the charge carriers is much less than the speed of energy movement, just as the flow rate was much smaller than the speed with which pressure changes are propagated in a fluid. In the cardiovascular system, pressure changes propagated with the speed of sound (about 1500 m/s) while the blood move at less than 1 m/s.

Fluid conservation principles are also applicable to electric circuits. In the cardiovascular case, the pressure drop from the ventricle to the atrium is equal to the pressure at the ventricle; in the electric case, the voltage drop around any closed loop is equal to the voltage of the source (battery or power supply) in that loop (or zero if there is no source in that loop). This is essentially the conservation of energy. In the cardiovascular case, the flow into a branch is equal to the sum of the flows out; in the electrical case, the same is true for currents at a junction of conductors: conservation of charge. In the electrical parlance, these principles are known as Kirchoff's laws. These laws lead to the following rules for treating voltages and currents in series and parallel configurations:

1. The voltage drop across two components in series is equal to the sum of the voltage drops across each component. We think of the pressure drops in the arteries and in the arterioles.

2. The currents flowing through two components in series are equal. If an arteriole and a venule were joined by a single capillary, the fluid flow through them all would be the same.

3. The voltage drops across two components in parallel are equal. This corresponds to a branching of an artery into different arterioles.

4. The current flowing through two components in parallel is equal to the sum of the currents flowing through each component. This implies that the resistances through alternate paths determine how much current flows through each of them: electricity tends to the path of least resistance ([9]).

Chapter 3

Blood Flow in Large Arteries

3.1 Introduction

This chapter provides a model for blood flow in the large arteries and addresses the analysis of the Navier-Stokes problem, having provided boundary conditions which can be considered as a generalization of the mean pressure drop problem investigated in [7], as they arise in bioengineering applications. Our purpose is to consider both the stationary case and the nonstationary case. In this regard, we will prove the existence of a weak solution for the stationary case based on Brouwer's fixed point theorem and afterward, we will establish a well-posedness analysis for the nonstationary case based on a suitable energy estimate that we are going to derive as well as a well-known compactness argument.

3.1.1 Mechanical properties of blood

Blood is a very complex fluid in its nature. It consists of plasma with red blood cells (erythrocytes), white blood cells (leucocytes), and platelets (thrombocytes) in suspension. Blood plasma is essentially an aqueous solution containing 96 percent of water, 4% of proteins, and trace amounts of other materials. Erythrocytes make up more than 99% of all blood cells and approximately 40 to 45% of the blood (cells plus plasma); this percentage is called the hematocrit. Blood cells are deformable, with the erythrocytes being the most deformable. Significant deformations occur when the cells are passing through capillaries. However, the cell membranes do not rupture because each cell has a cytoskeleton that supports its shape.
Therefore, the mechanical properties of blood should be studied by analyzing a liquid containing a suspension of flexible particles. A liquid is said to be Newtonian if the coefficient of viscosity is constant at all rates of shear. This condition exists in most homogeneous liquids, including blood plasma (which, since it consists mostly of water, is Newtonian). But the mechanical behavior of a liquid containing a suspension of particles can vary such that the liquid becomes non-Newtonian. These deviations become

particularly significant when the particle size becomes appreciably large in comparison to the dimension of the channel in which the fluid is flowing. This situation happens in the microcirculation (small arterioles and capillaries).
Consider a suspension in which the suspending fluid has a Newtonian behavior. If the suspended particles are spherical and have the same density as the suspending fluid, then for any motion the shear stress will be proportional to the rate of shear and the suspension will behave as a Newtonian fluid. This rule applies as long as the concentration of spheres is low, less than 30%. This rule was arrived at through experiments performed under steady state conditions with suspensions of rigid spheres. These experiments showed that the viscosity of the suspensions, was independent of the shear rate for volume concentrations of suspended spheres up to 30%. However, if the suspended particles are not spherical or are deformable in any way, then the shear stress is not proportional to the shear rate unless the concentration is much less than 30%.
The cells suspended in blood are not rigid spheres, and the volume fraction of erythrocytes is about 40 to 45%. Therefore, one should expect that the behavior of blood is non-Newtonian. But it has been shown that human blood is Newtonian at all rates of shear for hematocrits up to about 12% [11]. In general blood has a higher viscosity than plasma, and as the hematocrit rises, the viscosity of the suspension increases and non-Newtonian behavior is observed, being detectable first at very low rates of shear. Studies with human blood show that viscosity is independent of shear rate when the shear rate is high. With a reduction of shear rate the viscosity increases slowly until a shear rate less than $1\mathrm{s}^{-1}$, where it rises extremely steeply [2]. The shear stresses can be divided into two groups according to the effect of the shear rate:

1. At low shear rates, the viscosity increases markedly. The reason for this increase is that at low shear rates a tangled network of aggregated cell structures (Rouleaux) can be formed [11]. If blood is subjected to shear stress less than a critical value, these aggregated structures form. As a result, they exhibit a yield stress. This behavior is, however, only present if the hematocrit is high. If the hematocrit falls below a critical value, there are not enough cells to produce the aggregated structures and no yield stresses will be found.

2. At high shear rates, the viscosity in small vessels is lower than it is in larger vessels. The progressive diminution with the size of the vessels is detectable in vessels with an internal diameter less than 1 mm. It is even more pronounced in vessels with a diameter of 0.1 mm to 0.2 mm. The decreased viscosity with vessel size is known as the Fåhraeus and Lindqvist effect. Experiments were performed at high enough shear rates for the erythrocytes not to aggregate. It was found that the viscosity was approximately constant in vessels larger than 1 mm, but when the radius dropped below that, there was a substantial decrease in viscosity.

In the larger vessels it is reasonable to assume blood has a constant viscosity, because the vessel diameters are large compared with the individual cell diameters and because shear rates are high enough for viscosity to be independent of them. Hence in these

vessels the non-Newtonian behavior becomes insignificant and blood can be considered to be a Newtonian fluid.
In the microcirculation, it is no longer possible to think of the blood as a homogeneous fluid; it is essential to treat it as a suspension of red cells in plasma. The reason for this is that even the largest vessels of the microcirculation are only approximately 15 red cell widths in diameter.
In summary, we can conclude that blood is generally a non-Newtonian fluid, but it is reasonable to regard it as a Newtonian fluid when modelling arteries with diameters larger than 1 mm. For very small vessels it is not easy to reach conclusions as to the Newtonian nature of blood because some effects tend to decrease the viscosity and others to increase it. The latter influence on viscosity is due to a small flow, which increases the viscosity significantly, as well as to the fact that cells often become stuck at constrictions in small vessels (see [2, 11, 12]).

3.1.2 Basic notations

In this subsection, we summarize some notations that will occur throughout the dissertation. Vectors and tensors are denoted by bold-face letters.
$\mathbf{x} = (x_1, x_2, x_3):$ location of fluid particle.
$\mathbf{u}$: velocity field of the flow.
$p:$ pressure.
$\nabla p:$ the pressure gradient
$\mathbf{I}$: identity tensor.
$\mathbf{T}$: symmetric Cauchy stress tensor.
μ: dynamic viscosity.
ρ: fluid mass density.
$\mathbb{R}$: the set of real numbers.
$|\cdot|$: the absolute value of $\mathbb{R}$ and correspondingly, the norm of $\mathbb{R}^3$.
Ω denotes a bounded domain in $\mathbb{R}^3$.
Γ : denotes the boundary of Ω.
$\nabla\mathbf{u}$: the gradient of $\mathbf{u}$.
$\nabla\cdot\mathbf{u}$: the divergence of $\mathbf{u}$.
$\Delta\mathbf{u}$: the Laplacian of $\mathbf{u}$.

$$\mathbf{u} = (u_1, u_2, u_3), \quad \mathbf{v} = (v_1, v_2, v_3), \quad \mathbf{w} = (w_1, w_2, w_3).$$

$$\nabla \cdot \mathbf{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}.$$

$$(\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} = \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j.$$

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i.$$

$$\nabla\mathbf{u} : \nabla\mathbf{v} = \sum_{i,j=1}^{3}(\frac{\partial u_i}{\partial x_j})(\frac{\partial v_i}{\partial x_j}).$$

$$\nabla p = (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}).$$

$$\Delta\mathbf{u} = (\sum_{j=1}^{3}\frac{\partial^2 u_1}{\partial x_j^2}, \sum_{j=1}^{3}\frac{\partial^2 u_2}{\partial x_j^2}, \sum_{j=1}^{3}\frac{\partial^2 u_3}{\partial x_j^2}).$$

$$(\nabla\mathbf{v})\mathbf{u} = (\mathbf{u}\cdot\nabla)\mathbf{v} = (\sum_{j=1}^{3}u_j\frac{\partial v_1}{\partial x_j}, \sum_{j=1}^{3}u_j\frac{\partial v_2}{\partial x_j}, \sum_{j=1}^{3}u_j\frac{\partial v_3}{\partial x_j}).$$

The spaces $C(\Omega)$, $C^k(\Omega)$, $C_0^k(\Omega)$, $C_0^\infty(\Omega)$ and their vector-valued analogues $\mathbf{C}(\Omega)$, $\mathbf{C}^k(\Omega)$, $\mathbf{C}_0^k(\Omega)$, $\mathbf{C}_0^\infty(\Omega)$ are defined as usual, the superscript indicating continuous derivatives to a certain order and the subscript zero indicating functions with compact support.

$$C(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is continuous}\}.$$

$$C(\overline{\Omega}) = \{u \in C(\Omega) \mid u \text{ is uniformly continuous on bounded subsets of } \Omega\}.$$

$$C^k(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is k-times continuously differentiable}\}.$$

$$\begin{aligned} C^k(\overline{\Omega}) = & \{u \in C^k(\Omega) \mid D^\alpha u \text{ is uniformly continuous on bounded} \\ & \text{subsets of } \Omega, \text{ for all multiindex } |\alpha| \le k\}. \end{aligned}$$

$$C^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is infinitely differentiable}\}.$$

$C_0(\Omega)$, $C_0^k(\Omega)$ and $C_0^\infty(\Omega)$ denote these functions in $C(\Omega)$, $C^k(\Omega)$ and $C^\infty(\Omega)$ respectively with compact support.

The Hölder space $C^{k,\gamma}(\Omega)$ consists of those functions u that are k-times continuously differentiable and whose k^{th}-partial derivatives are bounded and Hölder continuous with exponent γ. The corresponding vector space $(C^{k,\gamma}(\Omega))^3$ will be denoted by $\mathbf{C}^{k,\gamma}(\Omega)$.

We denote by $L^P(\Omega)$ $(1 \le p < \infty)$ the space of real functions which are Lebesgue measurable, with

$$\int_\Omega |u|^p\, dx < \infty.$$

It is endowed with the norm denoted by $\|u\|_{L^P(\Omega)}$;

$$\|u\|_{L^P(\Omega)} = (\int_\Omega |u|^p\, dx)^{\frac{1}{p}}.$$

$L^2(\Omega)$ is a Hilbert space. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the associated inner product and norm in $L^2(\Omega)$ respectively,

$$(u, v) = \int_\Omega uv\ dx \text{ and } \|u\|^2 = \int_\Omega u^2\, dx\ .$$

The corresponding vector space $(L^p(\Omega))^3$ will be denoted by $\mathbf{L}^p(\Omega)$. We still denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $\mathbf{L}^2(\Omega)$ respectively.

Sobolev spaces $W_p^k(\Omega)$ $(1 \leq p < \infty)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belong to $L^p(\Omega)$. It is endowed with the norm denoted by $\|\cdot\|_{W_p^k(\Omega)}$;

$$\|u\|_{W_p^k(\Omega)} = (\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p dx)^{\frac{1}{p}},$$

where $D^0 u = u$. The corresponding vector space $(W_p^k(\Omega))^3$ will be denoted by $\mathbf{W}_p^k(\Omega)$. If $p = 2$, we write

$$H^k(\Omega) = W_2^k(\Omega), \quad \text{and} \quad \|\cdot\|_k = \|\cdot\|_{W_2^k(\Omega)} (k = 0, 1, ...).$$

$H^k(\Omega)$ is a Hilbert space. the associated inner product $(\cdot,\cdot)_k$ is defined by

$$(u, v)_k = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u : D^\alpha v \, dx.$$

Correspondingly, we set $\mathbf{H}^k(\Omega)$=$(H^k(\Omega))^3$ and we still denote its norm by $\|\cdot\|_k$ and its inner product by $(\cdot,\cdot)_k$.

We denote by $\mathbf{H}_0^k(\Omega)$ the completion of $\mathbf{C}_0^\infty(\Omega)$ in $\mathbf{H}^k(\Omega)$.

For functions depending on space and time, for a given space $\mathbf{V}$ of space dependent functions, we define (for some $T > 0$)

$$L^2(0, T; \mathbf{V}) = \{\mathbf{v} : (0, T) \to \mathbf{V} \mid \ \mathbf{v} \text{ is measurable and} \int_0^T \|\mathbf{v}(t)\|_V^2 dt < \infty\}$$

with norm $\|\mathbf{v}\|_{L^2(0,T;V)}$=$(\int_0^T \|\mathbf{v}(t)\|_V^2 dt)^{1/2}$ and

$$L^\infty(0, T; \mathbf{V}) = \{\mathbf{v} : (0, T) \to \mathbf{V} \mid \operatorname{ess} \sup_{t\in(0,T)} \|\mathbf{v}(t)\|_V < \infty\}$$

with norm $\|\mathbf{v}\|_{L^\infty(0,T;V)} = \operatorname{ess\,sup}_{t\in(0,T)} \|\mathbf{v}(t)\|_V$.

Let Γ_0 be an open nonempty subset of Γ. We denote by $H^{1/2}(\Gamma_0)$ the space of functions defined on Γ_0 which are traces of functions in $H^1(\Omega)$.
We recall that the trace operator $Tr : H^1(\Omega) \to H^{1/2}(\Gamma)$ is surjective and continuous.

When considering functions which depend only on time, we define the space

$$\mathbf{L}^\infty(0, T) = \{\mathbf{z} : (0, T) \to \mathbb{R}^3 \mid \operatorname{ess} \sup_{t\in(0,T)} |\mathbf{z}(t)| < \infty\}$$

endowed with the norm $\|\mathbf{z}\|_{L^\infty(0,T)} = \operatorname{ess\,sup}_{t\in(0,T)} |\mathbf{z}(t)|$.

Let $\mathbf{Y}$ be a real Banach space, with norm $\|\cdot\|_{\mathbf{Y}}$. The space

$$\mathbf{L}^p(0,T;\,\mathbf{Y})$$

consists of all strongly measurable functions $\mathbf{u}:\,[0,T]\to\mathbf{Y}$ with

$$\|\mathbf{u}\|_{\mathbf{L}^p(0,T;\,\mathbf{Y})} := \left(\int_0^T \|\mathbf{u}(t)\|_{\mathbf{Y}}^p dt\right)^{\frac{1}{p}} < \infty$$

for $1\le p<\infty$.

We recall that a function $\mathbf{u}:\,[0,T]\to\mathbf{Y}$ is strongly measurable if there exist simple functions $\mathbf{s}_k:\,[0,T]\to\mathbf{Y}$ such that

$$\mathbf{s}_k(t)\to\,\mathbf{u}(t)\quad\text{for a.e. }\ 0\le t\le T,\ \text{ see [1].}$$

The space

$$\mathbf{C}(0,T;\,\mathbf{Y})$$

comprises all continuous functions $\mathbf{u}:\,[0,T]\to\mathbf{Y}$ with

$$\|\mathbf{u}\|_{\mathbf{C}(0,T;\,\mathbf{Y})} := \max_{0\le t\le T}\|\mathbf{u}(t)\|_{\mathbf{Y}}<\infty.$$

The Sobolev space

$$\mathbf{W}_p^1(0,T;\,\mathbf{Y})$$

consists of all functions $\mathbf{u}\in L^p(0,T;\,\mathbf{Y})$ such that $\mathbf{u}'$ exists in the weak sense and belongs to $\mathbf{L}^p(0,T;\,\mathbf{Y})$. We write $\mathbf{H}^1(0,T;\,\mathbf{Y}) := \mathbf{W}_2^1(0,T;\,\mathbf{Y})$.
Furtermore,

$$\|\mathbf{u}\|_{\mathbf{W}_p^1(0,T;\,\mathbf{Y})} := \left(\int_0^T \|\mathbf{u}(t)\|_{\mathbf{Y}}^p + \|\mathbf{u}'(t)\|_{\mathbf{Y}}^p dt\right)^{\frac{1}{p}}.$$

3.1.3 Preliminaries

In this subsection, we recall some assumptions used in [7, 14], and we are going to make use of them throughout our analysis. In what follows, Ω is a bounded domain of $\mathbb{R}^3$ with a boundary Γ sufficiently smooth. Γ consists of the artery wall denoted by Γ_{wall} and some artificial sections (see Fig 3.1). The velocity is required to be zero on the artery wall. To account for homogeneous Dirichlet boundary conditions on the artery wall, we define

$$\mathbf{X}\equiv\{\boldsymbol{\varphi}\in\mathbf{H}^1(\Omega):\boldsymbol{\varphi}|_{\Gamma_{\text{wall}}}=0\}.$$

Poincaré's inequality

$$\|\boldsymbol{\varphi}\|\le C_\Omega\|\nabla\boldsymbol{\varphi}\| \tag{3.1}$$

holds for $\boldsymbol{\varphi}\in\mathbf{X}$ ([4, 7, 14]). The artificial sections consist of the upstream section on the side of the heart and the downstream sections on the side of the peripheral vessels.

Rather than giving serious thought to the artificial sections boundary conditions, in seeking a variational formulation, the test space is left free on these portions of the boundary. In this regard, we introduce

$$\mathbf{V} \equiv \{\boldsymbol{\varphi}\in\mathbf{H}^1(\Omega) : \boldsymbol{\varphi}|_{\Gamma_{\text{wall}}} = 0, \nabla \cdot \boldsymbol{\varphi} = 0\} \tag{3.2}$$

as the test space. To prove an existence theorem for a Navier-Stokes problem, either steady or non-steady, it is convenient to construct the solution as a limit of Galerkin approximations in terms of the eigenfunctions of the corresponding steady Stokes problem. This use of the Stokes eigenfunctions originated with Prodi and was further developed by Heywood [6]. To define the corresponding Stokes operator, we introduce $\mathbf{V}^\star$ as the completion of $\mathbf{V}$ with respect to the norm of $\mathbf{L}^2(\Omega)$. Then for every $\mathbf{f} \in \mathbf{V}^\star$, there exists exactly one $\mathbf{v} \in \mathbf{V}$ satisfying

$$(\nabla\mathbf{v}, \nabla\boldsymbol{\varphi}) = (\mathbf{f}, \boldsymbol{\varphi}) \quad \forall\boldsymbol{\varphi} \in \mathbf{V}. \tag{3.3}$$

Moreover, for each $\mathbf{v} \in \mathbf{V}$, there exist at most one $\mathbf{f} \in \mathbf{V}^\star$, such that (3.3) holds. In this way, a one-to-one correspondence can be defined between elements $\mathbf{f}$ of $\mathbf{V}^\star$ and functions $\mathbf{v}$ belonging to a suitable subspace of $\mathbf{V}$ that we denote by $D(\widetilde{\Delta})$. The Stokes operator

$$\widetilde{\Delta} : D(\widetilde{\Delta}) \to \mathbf{V}^\star \tag{3.4}$$

is defined setting

$$-\widetilde{\Delta}\mathbf{v} = \mathbf{f} \tag{3.5}$$

so that (3.3) is satisfied.
The inverse operator $\widetilde{\Delta}^{-1}$ is completely continuous and self-adjoint. Therefore it possesses a sequence of eigenfunctions $\{\mathbf{a}_k\}_{k=1}^{\infty}$, which are complete and orthogonal in both $\mathbf{V}$ and $\mathbf{V}^\star$.

In what follows, $c_i(i = 1, 2, ..)$ will denote generic constants, not necessarily the same at different places. The inequalities

$$\sup_{\Omega}|\mathbf{v}| \leq c_1\|\nabla\mathbf{v}\|^{\frac{1}{2}}\|\widetilde{\Delta}\mathbf{v}\|^{\frac{1}{2}} \tag{3.6}$$

$$\|\nabla\mathbf{v}\| \leq c_2\|\widetilde{\Delta}\mathbf{v}\| \tag{3.7}$$

are satisfied for every $\mathbf{v} \in D(\widetilde{\Delta})$, provided that Ω is a bounded domain (see [14] page 178).

3.2 Problem formulation

PROBLEM 3.1

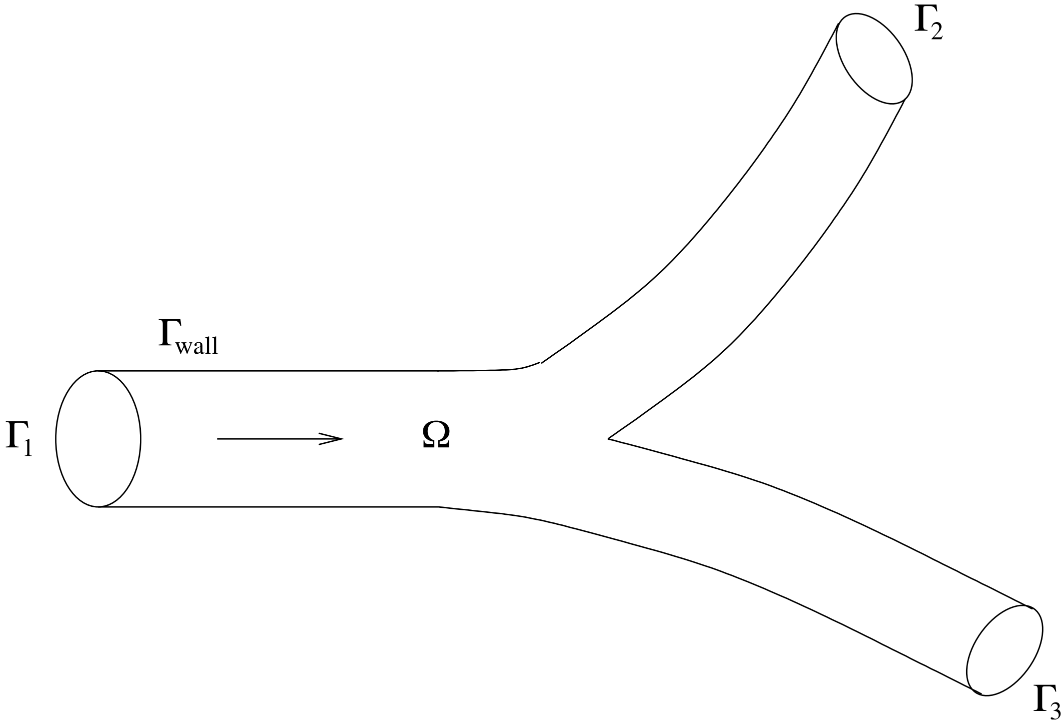

Figure 3.1: Artery portion.

Let $\Omega \subset \mathbb{R}^3$ be the artery portion where we aim at providing a detailed flow analysis. For each $\mathbf{x} \in \Omega$, and any time $t > 0$, we denote by $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ the blood velocity and pressure respectively. The blood density ρ and the blood viscosity μ are all assumed to be constant. Under these previous assumptions, blood flow can be described by the Navier-Stokes equations

$$\begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \cdot \mathbf{T} = 0 \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad \mathbf{x} \in \Omega,\ t > 0 \tag{3.8}$$

where $\mathbf{T}$ is the Cauchy stress tensor, which in the case of a Newtonian fluid reads

$$\mathbf{T} = p\mathbf{I} - \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

where $\mathbf{I}$ denotes the identity tensor. The equations are expressions of balance of linear momentum and incompressibility. We are neglecting the presence of any external forces. Then (3.9) is obtained by substitution of the divergence-free constraint into the expression for the stress in $(3.8)_1$.

$$\begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = 0 \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad \mathbf{x} \in \Omega,\ t > 0 \tag{3.9}$$

In fact

$$\begin{aligned}\nabla\cdot\mathbf{T} &= \nabla\cdot(p\mathbf{I}-\mu(\nabla\mathbf{u}+(\nabla\mathbf{u})^T))\\ &= \nabla\cdot(p\mathbf{I})-\mu\nabla\cdot(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)\\ &= \nabla p-\mu\nabla\cdot(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)\\ &= \nabla p-\mu(\Delta\mathbf{u}+\nabla(\nabla\cdot\mathbf{u}))\\ &= \nabla p-\mu\Delta\mathbf{u}\ \text{ provided that }\ \nabla\cdot\mathbf{u}=0\end{aligned}$$

For the sake of simplicity, we normalize ρ to 1.

3.2.1 Initial and Boundary conditions

The initial condition requires the specification of the flow velocity at the initial time, e.g., $t_0 = 0$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})\quad \mathbf{x}\in\Omega, \tag{3.10}$$

where the given initial velocity field $\mathbf{u}_0$ is divergence free. The system (3.9) has to be provided with boundary conditions. In this respect, we split Γ into different parts (Figure 3.1). In the present work, we are assuming that the wall is rigid so that no-slip boundary condition

$$\mathbf{u}(\mathbf{x},t)=0,\ \ \mathbf{x}\in\Gamma_{\text{wall}},\ t>0, \tag{3.11}$$

holds. The other parts of Γ are the artificial boundaries which bound the computational domain. For the sake of clarity, we distinguish the upstream section on the side of the heart denoted by Γ_1 and the downstream sections on the side of the peripheral vessels denoted by Γ_2 and Γ_3. The boundary conditions associated with the variational formulation of the mean pressure drop problem investigated in [7] reads as

$$p\mathbf{n}-\mu(\nabla\mathbf{u})\mathbf{n}=P_i\mathbf{n}\quad \text{for } t>0,\ \mathbf{x}\in\Gamma_i\ (i=1,2,3), \tag{3.12}$$

where $P_i = P_i(t) = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} p\,ds$ is the prescribed mean pressure on each artificial section Γ_i $(i = 1,2,3)$ with $|\Gamma_i| = \int_{\Gamma_i} ds$, and $\mathbf{n}$ represents the outward normal unit vector on every part of the vessel boundary. As it has been introduced in [14], the following boundary conditions are provided:

$$p\mathbf{n}-\mu(\nabla\mathbf{u})\mathbf{n}-K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n}=P_i\mathbf{n}\quad \text{for } t>0,\ \mathbf{x}\in\Gamma_i\ (i=1,2,3), \tag{3.13}$$

where K_i is a suitable nonnegative constant, $P_i = P_i(t)$ is the prescribed mean pressure on each artificial section Γ_i and $\mathbf{n}$ represents the outward normal unit vector on every part of the vessel boundary. These conditions can be considered as a generalization of the mean pressure drop problem investigated in [7] in the sense that when $K_i = 0$ $(i = 1,2,3)$, we recover the usual Neumann or natural conditions associated with (3.9). The physical justification for the case in which $K_i \neq 0$ is provided in [14], (see e.g., [14] Fig 4.1).

We define the following bilinear and trilinear forms:

$a : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ such that

$$a(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) = \mu(\nabla \mathbf{v}^{(1)}, \nabla \mathbf{v}^{(2)}) + \sum_{i=1}^{3} K_i \int_{\Gamma_i} (\mathbf{v}^{(1)} \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{v}^{(2)}\, ds, \tag{3.14}$$

$b : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ such that

$$b(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}) = ((\mathbf{v}^{(1)} \cdot \nabla)\mathbf{v}^{(2)}, \mathbf{v}^{(3)}), \tag{3.15}$$

we set

$$c(t, \mathbf{v}) = -\sum_{i=1}^{3} P_i(t) \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{v}\, ds \quad \text{for each} \quad \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{3.16}$$

Furthermore, if we define

$$\widetilde{P} = \sup_{t \geq 0} \sum_{i=1}^{3} |P_i|, \qquad K = \max_{1 \leq i \leq 3} K_i, \tag{3.17}$$

and consider the sequence $\{\mathbf{a}_k\}_{k=1}^{\infty}$ of eigenfunctions of $\widetilde{\Delta}^{-1}$, the following estimates hold for every $\mathbf{v} \in \text{span}\{\mathbf{a}_k, k = 1, 2, ...\}$

$$\left| \sum_{i=1}^{3} P_i \int_{\Gamma_i} \widetilde{\Delta}\mathbf{v} \cdot \mathbf{n} ds \right| \leq c_3 \widetilde{P} \|\widetilde{\Delta}\mathbf{v}\| \tag{3.18}$$

where c_3 depends on the trace inequality $(A.13)$ and

$$\left| \sum_{i=1}^{3} K_i \int_{\Gamma_i} (\mathbf{v} \cdot \mathbf{n})\mathbf{n} \cdot \widetilde{\Delta}\mathbf{v} ds \right| \leq c_4 K \|\nabla \mathbf{v}\| \|\widetilde{\Delta}\mathbf{v}\|. \tag{3.19}$$

where c_4 depends on the trace inequality $(A.13)$ and the Poincaré's inequality (3.1). The former is a consequence of the trace inequality for solenoidal functions $(A.13)$ and the proof of (3.19) is given in [14].

3.3 The stationary problem

3.3.1 Summary of the boundary value problem

Our purpose is to establish the existence of a weak solution of the following problem:

$$\begin{cases} (\mathbf{u} \cdot \nabla)\mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = 0 \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad \mathbf{x} \in \Omega. \tag{3.20}$$

On the artery wall Γ_{wall}, the no-slip boundary condition

$$\mathbf{u}(\mathbf{x}) = 0, \ \ \mathbf{x} \in \Gamma_{\text{wall}} \tag{3.21}$$

holds. On each artificial boundary Γ_i $(i = 1, 2, 3)$, the boundary condition

$$p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} - K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n} = P_i\mathbf{n} \quad \text{for } \mathbf{x}{\in}\Gamma_i \tag{3.22}$$

is prescribed, where P_i, K_i are suitable nonnegative constants and $\mathbf{n}$ represents the outward normal unit vector on every part of the vessel boundary.

3.3.2 Weak formulation

Assume that $\mathbf{u}$ is a solution of the boundary value problem described above and $\boldsymbol{\varphi}$ is a smooth solenoidal vector-valued function defined on Ω. $\mathbf{u}$ satisfies the following identities:

$$\begin{cases} ((\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p, \boldsymbol{\varphi}) = 0, \quad \mathbf{x}{\in}\,\Omega \\ (p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} - K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{\varphi}) = (P_i\mathbf{n}, \boldsymbol{\varphi}), \ \ \mathbf{x}{\in}\Gamma_i \ \ i = 1, 2, 3. \end{cases} \tag{3.23}$$

This system reads as:

$$\begin{cases} ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - \mu(\Delta\mathbf{u}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi}) = 0, \quad \mathbf{x}{\in}\,\Omega \\ (p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} - K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{\varphi}) = (P_i\mathbf{n}, \boldsymbol{\varphi}), \ \ \mathbf{x}{\in}\Gamma_i \ \ i = 1, 2, 3. \end{cases} \tag{3.24}$$

We notice that $\boldsymbol{\varphi}|_{\Gamma_{\text{wall}}} = 0$ since $\boldsymbol{\varphi}{\in}\mathbf{V}$.
Now

$$\begin{aligned}
(\nabla p, \boldsymbol{\varphi}) &= \int_\Omega \sum_{i=1}^3 \varphi_i \frac{\partial p}{\partial x_i} d\mathbf{x} \\
&= -\int_\Omega p \sum_{i=1}^3 \frac{\partial \varphi_i}{\partial x_i} d\mathbf{x} + \int_\Gamma p \sum_{i=1}^3 \varphi_i n_i ds \\
&= -\int_\Omega p(\nabla\cdot\boldsymbol{\varphi}) d\mathbf{x} + \int_\Gamma p(\boldsymbol{\varphi}\cdot\mathbf{n}) ds \\
&= \int_\Gamma (p\mathbf{n})\cdot\boldsymbol{\varphi} ds \ \ \text{since } \nabla{\cdot}\boldsymbol{\varphi} = 0 \\
&= \sum_{i=1}^3 \int_{\Gamma_i} (P_i\mathbf{n} + \mu(\nabla\mathbf{u})\mathbf{n} + K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n})\cdot\boldsymbol{\varphi} ds \ \ \text{from (3.22)} \\
&= \sum_{i=1}^3 \int_{\Gamma_i} (P_i\mathbf{n})\cdot\boldsymbol{\varphi} ds + \mu \sum_{i=1}^3 \int_{\Gamma_i} (\mathbf{n}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\varphi} ds \\
&\quad + \sum_{i=1}^3 \int_{\Gamma_i} K_i((\mathbf{u}\cdot\mathbf{n})\mathbf{n})\cdot\boldsymbol{\varphi} ds,
\end{aligned}$$

and

$$
\begin{aligned}
(\Delta \mathbf{u}, \boldsymbol{\varphi}) &= \int_\Omega \sum_{i=1}^{3} \varphi_i \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2} d\mathbf{x} \\
&= -\int_\Omega \sum_{i,j=1}^{3} \frac{\partial \varphi_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} d\mathbf{x} + \int_\Gamma \sum_{i=1}^{3} \varphi_i \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} n_j ds \\
&= -\int_\Omega (\nabla \boldsymbol{\varphi}) \cdot (\nabla \mathbf{u}) d\mathbf{x} + \int_\Gamma \sum_{i=1}^{3} \varphi_i (\mathbf{n} \cdot \nabla u_i) ds \\
&= -\int_\Omega (\nabla \boldsymbol{\varphi}) \cdot (\nabla \mathbf{u}) d\mathbf{x} + \int_\Gamma (\mathbf{n} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{\varphi}\, ds \\
&= -(\nabla \boldsymbol{\varphi}, \nabla \mathbf{u}) + \sum_{i=1}^{3} \int_{\Gamma_i} (\mathbf{n} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{\varphi} ds.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) - (\mu \Delta \mathbf{u}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi}) \\
&= ((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) - \mu\Big(-(\nabla \boldsymbol{\varphi}, \nabla \mathbf{u}) + \sum_{i=1}^{3} \int_{\Gamma_i} (\mathbf{n} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{\varphi} ds\Big) + \sum_{i=1}^{3} \int_{\Gamma_i} (P_i \mathbf{n}) \cdot \boldsymbol{\varphi} ds \\
&\quad + \mu \sum_{i=1}^{3} \int_{\Gamma_i} (\mathbf{n} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{\varphi} ds + \sum_{i=1}^{3} \int_{\Gamma_i} K_i (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \cdot \boldsymbol{\varphi} ds \\
&= ((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) + \mu(\nabla \boldsymbol{\varphi}, \nabla \mathbf{u}) + \sum_{i=1}^{3} \int_{\Gamma_i} (P_i \mathbf{n}) \cdot \boldsymbol{\varphi} ds + \sum_{i=1}^{3} \int_{\Gamma_i} K_i (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \cdot \boldsymbol{\varphi} ds \\
&= ((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) + \mu(\nabla \boldsymbol{\varphi}, \nabla \mathbf{u}) + \sum_{i=1}^{3} \int_{\Gamma_i} K_i (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \cdot \boldsymbol{\varphi} ds + \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \boldsymbol{\varphi} ds.
\end{aligned}
$$

It follows that

$$
((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \cdot \boldsymbol{\varphi} ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \boldsymbol{\varphi} ds, \quad (3.25)
$$

since

$$
((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) - \mu(\Delta \mathbf{u}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi}) = 0
$$

from $(3.24)_1$.
We notice that (3.25) does not depend on the pressure p. Therefore the weak formulation of the problem reads as follow:

Given a divergence free velocity field $\mathbf{u}_0 \in \mathbf{V}$, nonnegative constants P_i, K_i, $(i = 1, 2, 3)$, find $\mathbf{u} \in \mathbf{V}$ such that

$$
((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i (\mathbf{u} \cdot \mathbf{n}) \mathbf{n} \cdot \boldsymbol{\varphi} ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \boldsymbol{\varphi} ds \quad (3.26)
$$

holds for all $\boldsymbol{\varphi} \in \mathbf{V}$, where

$$
\mathbf{V} = \{\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega) : \boldsymbol{\varphi}|_{\Gamma_{\text{wall}}} = 0, \nabla \cdot \boldsymbol{\varphi} = 0\}
$$

is the space of test functions.

3.3.3 Galerkin approximations

The inverse operator $\widetilde{\Delta}^{-1}$ of the Stokes operator $\widetilde{\Delta}$ is self-adjoint and possesses a sequence of eigenfunctions $\{\mathbf{a}_k\}$ which are orthogonal in $\mathbf{V}$. Fix a positive integer n. Galerkin approximations

$$\mathbf{u}_n = \sum_{k=1}^{n} d_n^k \mathbf{a}_k \tag{3.27}$$

are defined as solutions of the finite system of equations (k=1,...,n):

$$((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{a}_k) + \mu(\nabla \mathbf{u}_n, \nabla \mathbf{a}_k) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{a}_k ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{a}_k ds. \tag{3.28}$$

This is a system of linear equations for constant unknowns $d_n^k (k = 1, ..., n)$. The identity (3.29) for $\mathbf{u}_n$ is obtained by multiplying (3.28) through by d_n^k and summing over $k = 1...., n$:

$$((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n) + \mu(\nabla \mathbf{u}_n, \nabla \mathbf{u}_n) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{u}_n ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{u}_n ds. \tag{3.29}$$

This implies that

$$\begin{aligned}
\mu\|\nabla \mathbf{u}_n\|^2 &= \left| -((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n) - \sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{u}_n ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{u}_n ds \right| \\
&\leq |((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n)| + \left| \sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{u}_n ds \right| \\
&\quad + \left| \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{u}_n ds \right|.
\end{aligned}$$

Thus

$$\mu\|\nabla \mathbf{u}_n\|^2 \leq |((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n)| + \left| \sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{u}_n ds \right| + \left| \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{u}_n ds \right|. \tag{3.30}$$

Together with (3.1), we make use of (3.18) to obtain that

$$\begin{aligned}
\left| \sum_{i=1}^{3} P_i(t) \int_{\Gamma_i} \mathbf{n} \cdot (\mathbf{u}_n) ds \right| &\leq c_3 \widetilde{P} \|\mathbf{u}_n\| \\
&\leq c_3 C_\Omega \widetilde{P} \|\nabla \mathbf{u}_n\| \\
&\leq c_5 \widetilde{P} \|\nabla \mathbf{u}_n\|.
\end{aligned}$$

Likewise, from (3.19), we have that

$$\begin{aligned}
\left| \sum_{i=1}^{3} K_i \int_{\Gamma_i} (\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{u}_n ds \right| &\leq c_4 K \|\nabla \mathbf{u}_n\| \|\mathbf{u}_n\| \\
&\leq c_4 C_\Omega K \|\nabla \mathbf{u}_n\| \|\nabla \mathbf{u}_n\| \\
&\leq c_6 K \|\nabla \mathbf{u}_n\|^2.
\end{aligned}$$

Furthermore, making use of Hölder's inequality ($A.8$), one obtains that

$$\begin{aligned}((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n,\mathbf{u}_n) &\leq \|\mathbf{u}_n\|_{L^6}\,\|\nabla\mathbf{u}_n\|\,\|\mathbf{u}_n\|_{L^3}\\ &\leq c_7\,\|\nabla\mathbf{u}_n\|^3,\end{aligned}$$

where c_7 depends on Rellich-Kondrachov compactness inequality ($A.3$), Sobolev's inequality ($A.15$) and Poincaré's inequality (3.1) (see [7] page 348). We make use of these inequalities in (3.30) and we obtain that

$$\mu\|\nabla\mathbf{u}_n\|^2\leq c_7\,\|\nabla\mathbf{u}_n\|^3+c_5\widetilde{P}\|\nabla\mathbf{u}_n\|+c_6K\|\nabla\mathbf{u}_n\|^2.$$

Set $\xi:=\mu-c_6K$. It follows that

$$\xi\|\nabla\mathbf{u}_n\|\leq c_7\,\|\nabla\mathbf{u}_n\|^2+c_5\widetilde{P}, \tag{3.31}$$

where $\widetilde{P}=\sum_{i=1}^3|P_i|,\quad K=\max_{1\leq i\leq 3}K_i$.

Theorem 3.1 **(Construction of Approximate Solutions.)**
Assume

$$K\leq\frac{\mu}{c_6},\quad\textit{and}\quad\widetilde{P}\leq\frac{\xi^2}{4c_5c_7}.$$

For each integer $n=1,2,...,$ *there exists a function* $\boldsymbol{u}_n$ *of the form* (3.27) *satisfying* (3.28) *and such that*

$$\|\nabla\boldsymbol{u}_n\|\leq\frac{\xi}{2c_7}\left[1-\sqrt{(1-\frac{4c_5c_7\widetilde{P}}{\xi^2})}\right]. \tag{3.32}$$

Proof.
Owing to Poincaré's inequality (3.1), $\|\nabla\boldsymbol{\varphi}\|$ is a norm equivalent to $\|\boldsymbol{\varphi}\|_1$ for all $\boldsymbol{\varphi}\in\mathbf{V}$, therefore (3.32) defines a closed ball in span$\{\mathbf{a}_1,...,\mathbf{a}_n\}$. To prove the solvability of the finite-dimensional problem (3.28), we follow Fujita [3] in using Brouwer's fixed point theorem (see Appendix, theorem A.1), applying it to the continuous mapping $\mathbf{v}\rightarrow\mathbf{w}$ defined by the linear problem (k=1,...,n):

$$((\mathbf{v}\cdot\nabla)\mathbf{w},\mathbf{a}_k)+\mu(\nabla\mathbf{w},\nabla\mathbf{a}_k)=-\sum_{i=1}^3\int_{\Gamma_i}K_i(\mathbf{w}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{a}_k ds-\sum_{i=1}^3P_i\int_{\Gamma_i}\mathbf{n}\cdot\mathbf{a}_k ds. \tag{3.33}$$

(3.33) is a system of n linear equations. These linear equations are uniquely solvable if $\mathbf{v}$ lies in the ball defined by (3.32), because then $\mathbf{w}=0$ is the only solution of the corresponding homogeneous equation ($K_i=P_i=0,\ 1\leq i\leq 3$). In fact, if $\mathbf{v}$ satisfies (3.32) and $\mathbf{w}$ satisfies (3.33) with $K_i=P_i=0$, we have that

$$\begin{aligned}\xi\|\nabla\mathbf{w}\|^2 &\leq c_7\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|^2\\ &\leq c_7\frac{\xi}{2c_7}\left[1-\sqrt{(1-\frac{4c_5c_7\widetilde{P}}{\xi^2})}\right]\|\nabla\mathbf{w}\|^2\\ &\leq\frac{\xi}{2}\|\nabla\mathbf{w}\|^2.\end{aligned}$$

Together with Poincaré's inequality (3.1), this imply that $\mathbf{w} = 0$. To see that the mapping $\mathbf{v} \rightarrow \mathbf{w}$ takes the ball defined by (3.32) into itself, suppose that $\mathbf{v}$ satisfies (3.32). Then, similarly to (3.31), we obtain that

$$\xi \|\nabla \mathbf{w}\| \leq c_7 \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\| + c_5 \widetilde{P},$$

and therefore

$$\begin{aligned}
\|\nabla \mathbf{w}\| &\leq \frac{c_5 \widetilde{P}}{\xi - c_7 \|\nabla \mathbf{v}\|} \\
&\leq \frac{c_5 \widetilde{P}}{\xi - \frac{\xi}{2}\left[1 - \sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}\right]} \\
&\leq \frac{c_5 \widetilde{P}}{\frac{\xi}{2} + \frac{\xi}{2}\sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}} \\
&= \frac{c_5 \widetilde{P}}{\frac{\xi}{2} + \frac{\xi}{2}\sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}} \frac{\frac{\xi}{2} - \frac{\xi}{2}\sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}}{\frac{\xi}{2} - \frac{\xi}{2}\sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}} \\
&= \frac{\xi}{2c_7}\left[1 - \sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}\right].
\end{aligned}$$

Thus, (3.33) defines a continuous mapping $\mathbf{v} \rightarrow \mathbf{w}$ of the closed ball

$$\left\{ \boldsymbol{\varphi} \in \operatorname{span}\{\mathbf{a}_1, ..., \mathbf{a}_n\} : \|\nabla \boldsymbol{\varphi}\| \leq \frac{\xi}{2c_7}\left[1 - \sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}\right] \right\}.$$

into itself. The map has at least one fixed point, and any such fixed point is a solution of (3.28). $\mathbf{u}_n$ is chosen to be any one of these fixed points. Hence Brouwer's fixed point has been applied and has given the existence of Galerkin approximations

$$\mathbf{u}_n = \sum_{k=1}^{n} d_n^k \mathbf{a}_k$$

satisfying

$$\|\nabla \mathbf{u}_n\| \leq \frac{\xi}{2c_7}\left[1 - \sqrt{(1 - \frac{4c_5 c_7 \widetilde{P}}{\xi^2})}\right].$$

3.3.4 Existence of a weak solution

Lemma 3.1 (Weak compactness.)
Let X be a reflexive Banach space and suppose that the sequence $\{\boldsymbol{u}_k\}_{k=1}^{\infty} \subset X$ is bounded.

Then there exists a subsequence $\{\boldsymbol{u}_{kj}\}_{j=1}^{\infty} \subset \{\boldsymbol{u}_k\}_{k=0}^{\infty}$ *and* $\boldsymbol{u} \in X$ *such that*

$$\mathbf{u}_{kj} \rightharpoonup \mathbf{u}.$$

In other words, bounded sequences in a reflexive Banach space are weakly precompact. In particular, a bounded sequence in a Hilbert space contains a weakly convergent subsequence.

Proof. see [1]. $\Diamond$

Together with Poincaré's inequality (3.1), the fact that the sequence $\{\|\nabla \mathbf{u}_n\|\}_{n=1}^{\infty}$ is bounded imply that the sequence $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is bounded in $\mathbf{V}$. Therefore we make use of lemma 3.1 to find that there exists a subsequence $\{\mathbf{u}_{n_q}\}_{q=1}^{\infty} \subset \{\mathbf{u}_n\}_{n=1}^{\infty}$, such that

$$\mathbf{u}_{n_q} \rightharpoonup \mathbf{u} \ \text{ weakly in } \mathbf{V}. \tag{3.34}$$

Then we show that the weak limit $\mathbf{u}$ is in fact a weak solution. In this respect, we are going to show that

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \cdot \boldsymbol{\varphi} ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \boldsymbol{\varphi} ds$$

for each $\boldsymbol{\varphi} \in \mathbf{V}$.

Fix an integer k, $(k = 1, 2, ...)$. From (3.28), we have the identity

$$((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{a}_k) + \mu(\nabla \mathbf{u}_n, \nabla \mathbf{a}_k) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{a}_k ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{a}_k ds. \tag{3.35}$$

We recall (3.34), to find upon passing to weak limits that

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{a}_k) + \mu(\nabla \mathbf{u}, \nabla \mathbf{a}_k) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{a}_k ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{a}_k ds. \tag{3.36}$$

Since $\{\mathbf{a}_k\}_{k=1}^{\infty}$ is a basis of $\mathbf{V}$, it follows that

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) = -\sum_{i=1}^{3} \int_{\Gamma_i} K_i(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \cdot \boldsymbol{\varphi} ds - \sum_{i=1}^{3} P_i \int_{\Gamma_i} \mathbf{n} \cdot \boldsymbol{\varphi} ds$$

for each $\boldsymbol{\varphi} \in \mathbf{V}$.

Remark 3.1 (**Weak limit.**)
In taking this limit, there is no difficulty with the nonlinear term; In fact, we have that

$$\begin{aligned} (\mathbf{u}_{n_q} \cdot \nabla)\mathbf{u}_{n_q} - (\mathbf{u} \cdot \nabla)\mathbf{u} &= (\mathbf{u}_{n_q} \cdot \nabla)\mathbf{u}_{n_q} + (\mathbf{u}_{n_q} \cdot \nabla)\mathbf{u} - (\mathbf{u}_{n_q} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} \\ &= (\mathbf{u}_{n_q} \cdot \nabla)(\mathbf{u}_{n_q} - \mathbf{u}) + ((\mathbf{u}_{n_q} - \mathbf{u}) \cdot \nabla)\mathbf{u}. \end{aligned}$$

Also $\mathbf{a}_k \in \mathbf{H}^2(\Omega)$, as an eigenfunction of the inverse of the stokes operator $\widetilde{\Delta}$ and $\mathbf{H}^2(\Omega) \subset \mathbf{C}(\overline{\Omega})$ $(A.16)$. This implies that $\mathbf{a}_k \in \mathbf{C}(\overline{\Omega})$. It follows that

$$\begin{aligned} & \big((\mathbf{u}_{n_q}\cdot\nabla)\mathbf{u}_{n_q} - (\mathbf{u}\cdot\nabla)\mathbf{u}\ \ ,\ \mathbf{a}_k\big) \\ = \ & \big((\mathbf{u}_{n_q}\cdot\nabla)(\mathbf{u}_{n_q} - \mathbf{u})\ ,\ \mathbf{a}_k\big) + \big(((\mathbf{u}_{n_q} - \mathbf{u})\cdot\nabla)\mathbf{u}\ ,\ \mathbf{a}_k\big) \ \rightarrow\ 0 \ \ \text{(see [6], page 651).} \end{aligned}$$

The uniqueness and stability of the stationary problem are considered in [7].

3.4 The nonstationary problem

3.4.1 Summary of the initial boundary value problem

Our purpose is to study the well posed-ness of the following initial boundary value problem:
We consider the system of partial differential equations (3.9):

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = 0 \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \quad \mathbf{x} \in \Omega,\ t > 0.$$

The initial condition is described by (3.10) :

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})\ \mathbf{x} \in \Omega,$$

where the given initial velocity field $\mathbf{u}_0$ is divergence free. On the artery wall Γ_{wall}, no-slip boundary condition (3.11):

$$\mathbf{u}(\mathbf{x}, t) = 0,\ \ \mathbf{x} \in \Gamma_{\text{wall}},\ t > 0$$

holds. On each artificial boundary Γ_i $(i = 1, 2, 3)$, the boundary condition (3.13):

$$p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} - K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n} = P_i\mathbf{n} \quad \text{for } t > 0,\ \mathbf{x}\in\Gamma_i$$

is prescribed, where K_i is a suitable nonnegative constant, $P_i \in L^\infty(0, T)$ is assumed to be a given function and $\mathbf{n}$ represents the outward normal unit vector on every part of the vessel boundary.

Both the mathematical analysis and the numerical treatment of the Navier-Stokes problem are based on its weak formulation.

3.4.2 Weak formulation

Assume that $\mathbf{u}$ is a solution of problem 3.1 and $\boldsymbol{\varphi}$ is a smooth solenoidal vector-valued function defined on Ω. $\mathbf{u}$ satisfies the following identities:

$$\begin{cases} (\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p, \boldsymbol{\varphi}) = 0 \ \ \mathbf{x} \in \Omega,\ t > 0, \\ (\mathbf{u}(0), \boldsymbol{\varphi}) = (\mathbf{u}_0, \boldsymbol{\varphi}) \ \ \mathbf{x} \in \Omega, \\ (p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} - K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{\varphi}) = (P_i\mathbf{n}, \boldsymbol{\varphi}) \ \ \mathbf{x}\in\Gamma_i \ \ i = 1, 2, 3. \end{cases} \tag{3.37}$$

This system reads as:

$$\begin{cases} (\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - \mu(\Delta\mathbf{u}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi}) = 0 \quad \mathbf{x}\in\Omega,\ t>0, \\ (\mathbf{u}(0), \boldsymbol{\varphi}) = (\mathbf{u}_0, \boldsymbol{\varphi}) \quad \mathbf{x}\in\Omega, \\ (p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} - K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{\varphi}) = (P_i\mathbf{n}, \boldsymbol{\varphi}) \quad \mathbf{x}\in\Gamma_i \quad i=1,2,3. \end{cases} \tag{3.38}$$

We have

$$\begin{aligned} (\nabla p, \boldsymbol{\varphi}) &= \sum_{i=1}^{3}\int_{\Gamma_i}(P_i\mathbf{n})\cdot\boldsymbol{\varphi}ds + \mu\sum_{i=1}^{3}\int_{\Gamma_i}(\mathbf{n}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\varphi}ds \\ &\quad +\sum_{i=1}^{3}\int_{\Gamma_i}K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\varphi}ds, \end{aligned}$$

and

$$(\Delta\mathbf{u}, \boldsymbol{\varphi}) = -(\nabla\boldsymbol{\varphi}, \nabla\mathbf{u}) + \sum_{i=1}^{3}\int_{\Gamma_i}(\mathbf{n}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\varphi}ds.$$

Thus

$$(\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - \mu(\Delta\mathbf{u}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi})$$

$$= (\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) + \mu(\nabla\boldsymbol{\varphi}, \nabla\mathbf{u}) + \sum_{i=1}^{3}\int_{\Gamma_i}K_i(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\varphi}ds + \sum_{i=1}^{3}P_i\int_{\Gamma_i}\mathbf{n}\cdot\boldsymbol{\varphi}ds$$

$$= (\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\varphi}) + a(\mathbf{u}, \boldsymbol{\varphi}) - c(t, \boldsymbol{\varphi}).$$

It follows that

$$(\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\varphi}) + a(\mathbf{u}, \boldsymbol{\varphi}) = c(t, \boldsymbol{\varphi}), \tag{3.39}$$

where the a, b and c are defined by (3.14), (3.15) and (3.16) respectively, since

$$(\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - \mu(\Delta\mathbf{u}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi}) = 0$$

from $(3.38)_1$.
We notice that (3.39) does not depend on the pressure p. Therefore the weak formulation of the problem reads as follow:

Given a divergence free velocity field $\mathbf{u}_0 \in \mathbf{V}$, $P_i \in L^\infty(0,T)$ and a nonnegative constant K_i for $(i=1,2,3)$, find $\mathbf{u}\in L^2(0,T;\mathbf{V})$ such that for all $t\geq 0$

$$\begin{cases} (\frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\varphi}) + a(\mathbf{u}, \boldsymbol{\varphi}) = c(t, \boldsymbol{\varphi}) \\ (\mathbf{u}(0), \boldsymbol{\varphi}) = (\mathbf{u}_0, \boldsymbol{\varphi}) \end{cases} \quad \text{for all} \quad \boldsymbol{\varphi}\in\mathbf{V}, \tag{3.40}$$

where

$$\mathbf{V} = \{\boldsymbol{\varphi}\in\mathbf{H}^1(\Omega) : \boldsymbol{\varphi}|_{\Gamma_{\mathrm{wall}}} = 0, \nabla\cdot\boldsymbol{\varphi} = 0\}$$

is the space of test functions.

3.4.3 Galerkin approximations

We are going to follow the same approach as in [1]. We will construct the weak solution of the initial boundary value problem by first solving a finite dimensional approximation. We recall that the inverse operator $\widetilde{\Delta}^{-1}$ of the Stokes operator $\widetilde{\Delta}$ is self-adjoint and possesses a sequence of eigenfunctions $\{\mathbf{a}_k\}$ which are orthogonal in $\mathbf{V}$ and orthonormal in $\mathbf{L}^2(\Omega)$. Fix a positive integer n and write

$$\mathbf{u}_n(t) := \sum_{k=1}^{n} d_n^k(t)\mathbf{a}_k, \tag{3.41}$$

where we intend to select the coefficients $d_n^k(t)$ $(0 \leq t \leq T, k = 1, ..., n)$ to satisfy

$$d_n^k(0) = (\mathbf{u}_0, \mathbf{a}_k), \tag{3.42}$$

and

$$(\frac{\partial \mathbf{u}_n}{\partial t}, \mathbf{a}_k) + a(\mathbf{u}_n, \mathbf{a}_k) + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{a}_k) = c(t, \mathbf{a}_k). \tag{3.43}$$

Theorem 3.2 (**Construction of Approximate Solutions.**)
For each integer $n = 1, 2, ...,$ *there exists a unique function* $\boldsymbol{u}_n$ *of the form* (3.41) *satisfying* (3.42)-(3.43).

Proof.
Assume that $\mathbf{u}_n$ is given by (3.41), we observe using the fact that $\{\mathbf{a}_k\}_{k=1}^{\infty}$ is an orthonormal basis of $\mathbf{L}^2(\Omega)$ that

$$(\frac{\partial \mathbf{u}_n}{\partial t}, \mathbf{a}_k) = (d_n^k)'(t), \tag{3.44}$$

where $(d_n^k)'(t)$ is the derivative of $d_n^k(t)$ with respect of t.
Furthermore, from (3.14) and the fact that a is a bilinear form, we have

$$\begin{aligned} a(\mathbf{u}_n, \mathbf{a}_k) &= a(\sum_{i=1}^{n} d_n^i(t)\mathbf{a}_i, \mathbf{a}_k) \\ &= \sum_{i=1}^{n} d_n^i(t) a(\mathbf{a}_i, \mathbf{a}_k) \\ &= \sum_{i=1}^{n} A_i^k d_n^i(t), \end{aligned}$$

where $A_i^k = a(\mathbf{a}_i, \mathbf{a}_k)$.
From (3.15) and recalling that b is a trilinear form, we have that

$$\begin{aligned} b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{a}_k) &= b(\sum_{i=1}^{n} d_n^i(t)\mathbf{a}_i, \sum_{j=1}^{n} d_n^j(t)\mathbf{a}_j, \mathbf{a}_k) \\ &= \sum_{i,j=1}^{n} d_n^i(t) d_n^j(t) b(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k) \\ &= \sum_{i,j=1}^{n} B_{i,j}^k d_n^i(t) d_n^j(t), \end{aligned}$$

where $B_{i,j}^k = b(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k)$.

Moreover, we set

$$f^k(t) := c(t, \mathbf{a}_k).$$

It follows that (3.43) becomes the system of ODE's

$$d_n^{k\prime}(t) + \sum_{i=1}^{n} A_i^k d_n^i(t) + \sum_{i,j=1}^{n} B_{i,j}^k d_n^i(t) d_n^j(t) = f^k(t) \quad (k = 1, ..., n) \tag{3.45}$$

subject to the initial condition (3.42). According to standard existence theory for ordinary differential equations, there exists a unique function $\mathbf{d}_n(t) = (d_n^1(t), ..., d_n^n(t))$ satisfying (3.42) and (3.45) for a.e. $0 \leq t \leq T$. And then $\mathbf{u}_n$ defined by (3.41) solves (3.43) for a.e. $0 \leq t \leq T$. ♦

3.4.4 Energy estimate

Our plan is hereafter to let $n \to \infty$. Before that, we will need some estimates, uniform in n.

Theorem 3.3 (**Energy estimate.**)
For each approximate solution $\boldsymbol{u}_n$, the following energy estimates hold:

$$\frac{d}{dt}\|\nabla \boldsymbol{u}_n\|^2 + (\mu - 2Kc_2c_4)\|\widetilde{\Delta} \boldsymbol{u}_n\|^2 \leq \frac{16c_1^4}{\mu^3}\|\nabla \boldsymbol{u}_n\|^6 + \frac{2c_3^2\widetilde{P}^2}{\mu}. \tag{3.46}$$

Proof.
We denote by λ_k the eigenvalue associated to the eigenfunction $\mathbf{a}_k$. Multiplying (3.39) through by $-\lambda_k d_n^k$ and summing over $k = 1, ..., n$, one obtains that

$$(\frac{\partial \mathbf{u}_n}{\partial t}, -\widetilde{\Delta}\mathbf{u}_n) + b(\mathbf{u}_n, \mathbf{u}_n, -\widetilde{\Delta}\mathbf{u}_n) + a(\mathbf{u}_n, -\widetilde{\Delta}\mathbf{u}_n) = c(t, -\widetilde{\Delta}\mathbf{u}_n).$$

Now

$$\begin{aligned}(\frac{\partial \mathbf{u}_n}{\partial t}, -\widetilde{\Delta}\mathbf{u}_n) &= \left(\nabla(\frac{\partial \mathbf{u}_n}{\partial t}), \nabla \mathbf{u}_n\right) \\ &= \left(\frac{\partial}{\partial t}(\nabla \mathbf{u}_n), \nabla \mathbf{u}_n\right) \\ &= \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_n\|^2,\end{aligned}$$

also

$$\begin{aligned}a(\mathbf{u}_n, -\widetilde{\Delta}\mathbf{u}_n) &= \mu(\nabla \mathbf{u}_n, \nabla(-\widetilde{\Delta}\mathbf{u}_n)) + \sum_{i=1}^{3} K_i \int_{\Gamma_i} (\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot (-\widetilde{\Delta}\mathbf{u}_n) ds \\ &= \mu\|\widetilde{\Delta}\mathbf{u}_n\|^2 - \sum_{i=1}^{3} K_i \int_{\Gamma_i} (\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot (\widetilde{\Delta}\mathbf{u}_n) ds,\end{aligned}$$

and

$$b(\mathbf{u}_n, \mathbf{u}_n, -\widetilde{\Delta}\mathbf{u}_n) = -((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \widetilde{\Delta}\mathbf{u}_n),$$

furthermore

$$\begin{aligned}c(t, -\widetilde{\Delta}\mathbf{u}_n) &= -\sum_{i=1}^{3} P_i(t) \int_{\Gamma_i} \mathbf{n} \cdot (-\widetilde{\Delta}\mathbf{u}_n) ds \\ &= \sum_{i=1}^{3} P_i(t) \int_{\Gamma_i} \mathbf{n} \cdot \widetilde{\Delta}\mathbf{u}_n ds.\end{aligned}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_n\|^2 + \mu\|\widetilde{\Delta}\mathbf{u}_n\|^2 = ((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \widetilde{\Delta}\mathbf{u}_n) + \sum_{i=1}^{3} P_i(t) \int_{\Gamma_i} \mathbf{n} \cdot (\widetilde{\Delta}\mathbf{u}_n) ds \tag{3.47}$$

$$+\sum_{i=1}^{3} K_i \int_{\Gamma_i} (\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \widetilde{\Delta}\mathbf{u}_n ds.$$

We now make use of (3.18) to obtain that

$$\left|\sum_{i=1}^{3} P_i(t) \int_{\Gamma_i} \mathbf{n} \cdot (\widetilde{\Delta}\mathbf{u}_n) ds\right| \le c_3 \widetilde{P}\|\widetilde{\Delta}\mathbf{u}_n\|.$$

Likewise, from (3.19), we have

$$\left|\sum_{i=1}^{3} K_i \int_{\Gamma_i} (\mathbf{u}_n \cdot \mathbf{n})\mathbf{n} \cdot \widetilde{\Delta}\mathbf{u}_n ds\right| \le c_4 K\|\nabla \mathbf{u}_n\|\|\widetilde{\Delta}\mathbf{u}_n\|.$$

Furthermore,

$$((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n,\widetilde{\Delta}\mathbf{u}_n)\leq\sup_{\Omega}|\mathbf{u}_n|\|\nabla\mathbf{u}_n\|\|\widetilde{\Delta}\mathbf{u}_n\|.$$

We make use of these inequalities in (3.47) to find that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_n\|^2+\mu\|\widetilde{\Delta}\mathbf{u}_n\|^2\leq\sup_{\Omega}|\mathbf{u}_n|\|\nabla\mathbf{u}_n\|\|\widetilde{\Delta}\mathbf{u}_n\|+c_3\widetilde{P}\|\widetilde{\Delta}\mathbf{u}_n\|+c_4K\|\nabla\mathbf{u}_n\|\|\widetilde{\Delta}\mathbf{u}_n\|. \quad (3.48)$$

Now from (3.6)

$$\sup_{\Omega}|\mathbf{u}_n|\leq c_1\|\nabla\mathbf{u}_n\|^{\frac{1}{2}}\|\widetilde{\Delta}\mathbf{u}_n\|^{\frac{1}{2}},$$

which implies that

$$\sup_{\Omega}|\mathbf{u}_n|\|\nabla\mathbf{u}_n\|\|\widetilde{\Delta}\mathbf{u}_n\|\leq c_1\|\nabla\mathbf{u}_n\|^{\frac{3}{2}}\|\widetilde{\Delta}\mathbf{u}_n\|^{\frac{3}{2}},$$

and so, inequality (3.48) now reads as:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_n\|^2+\mu\|\widetilde{\Delta}\mathbf{u}_n\|^2\leq c_1\|\nabla\mathbf{u}_n\|^{\frac{3}{2}}\|\widetilde{\Delta}\mathbf{u}_n\|^{\frac{3}{2}}+c_3\widetilde{P}\|\widetilde{\Delta}\mathbf{u}_n\|+c_4K\|\nabla\mathbf{u}_n\|\|\widetilde{\Delta}\mathbf{u}_n\|. \quad (3.49)$$

We next make use of Cauchy's inequality with epsilon $(A.4)$ to estimate the right-hand side of (3.49). We have

$$\begin{aligned}
c_1\|\nabla\mathbf{u}_n\|^{\frac{3}{2}}\|\widetilde{\Delta}\mathbf{u}_n\|^{\frac{3}{2}} &= (c_1\|\nabla\mathbf{u}_n\|^{\frac{3}{2}}\|\widetilde{\Delta}\mathbf{u}_n\|^{\frac{1}{2}})\|\widetilde{\Delta}\mathbf{u}_n\| \\
&\leq \frac{1}{4\epsilon_1}c_1^2\|\nabla\mathbf{u}_n\|^3\|\widetilde{\Delta}\mathbf{u}_n\|+\epsilon_1\|\widetilde{\Delta}\mathbf{u}_n\|^2 \\
&\leq (\frac{1}{4\epsilon_1}c_1^2\|\nabla\mathbf{u}_n\|^3)(\|\widetilde{\Delta}\mathbf{u}_n\|)+\epsilon_1\|\widetilde{\Delta}\mathbf{u}_n\|^2 \\
&\leq \frac{1}{4\epsilon_2}(\frac{1}{16{\epsilon_1}^2}c_1^4\|\nabla\mathbf{u}_n\|^6)+\epsilon_2\|\widetilde{\Delta}\mathbf{u}_n\|^2+\epsilon_1\|\widetilde{\Delta}\mathbf{u}_n\|^2 \\
&\leq \frac{1}{64\epsilon_2{\epsilon_1}^2}c_1^4\|\nabla\mathbf{u}_n\|^6+\epsilon_2\|\widetilde{\Delta}\mathbf{u}_n\|^2+\epsilon_1\|\widetilde{\Delta}\mathbf{u}_n\|^2 \qquad (\text{set } \epsilon_1=\epsilon_2=\frac{\mu}{8}) \\
&\leq \frac{8c_1^4}{\mu^3}\|\nabla\mathbf{u}_n\|^6+\frac{\mu}{4}\|\widetilde{\Delta}\mathbf{u}_n\|^2,
\end{aligned}$$

and

$$c_3\widetilde{P}\|\widetilde{\Delta}\mathbf{u}_n\| \leq \frac{c_3^2\widetilde{P}^2}{\mu}+\frac{\mu}{4}\|\widetilde{\Delta}\mathbf{u}_n\|^2 \qquad (\text{set} \quad \epsilon=\frac{1}{\mu}).$$

Also inequality (3.7) implies that

$$\|\nabla\mathbf{u}_n\|\leq c_2\|\widetilde{\Delta}\mathbf{u}_n\|,$$

so that

$$c_4K\|\nabla\mathbf{u}_n\|\|\widetilde{\Delta}\mathbf{u}_n\|\leq Kc_2c_4\|\widetilde{\Delta}\mathbf{u}_n\|^2.$$

Making use of this inequalities on (3.49), it follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_n\|^2+\mu\|\widetilde{\Delta}\mathbf{u}_n\|^2\leq\frac{8c_1^4}{\mu^3}\|\nabla\mathbf{u}_n\|^6+\frac{\mu}{4}\|\widetilde{\Delta}\mathbf{u}_n\|^2+\frac{c_3^2\widetilde{P}^2}{\mu}+\frac{\mu}{4}\|\widetilde{\Delta}\mathbf{u}_n\|^2+Kc_2c_4\|\widetilde{\Delta}\mathbf{u}_n\|^2.$$

That is

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_n\|^2+\frac{\mu}{2}\|\widetilde{\Delta}\mathbf{u}_n\|^2\leq\frac{8c_1^4}{\mu^3}\|\nabla\mathbf{u}_n\|^6+\frac{c_3^2\widetilde{P}^2}{\mu}+Kc_2c_4\|\widetilde{\Delta}\mathbf{u}_n\|^2,$$

which leads us to the required energy estimates:

$$\frac{d}{dt}\|\nabla\mathbf{u}_n\|^2+(\mu-2Kc_2c_4)\|\widetilde{\Delta}\mathbf{u}_n\|^2\leq\frac{16c_1^4}{\mu^3}\|\nabla\mathbf{u}_n\|^6+\frac{2c_3^2\widetilde{P}^2}{\mu}.$$

♦

3.4.5 Local existence of a weak solution

In this subsection, we use Galerkin method to build up a local weak solution of the initial/boundary-value problem. We have already constructed the Galerkin approximantions sequence $\{\mathbf{u}_n\}_{n=1}^{\infty}$ in subsection 3.4.3. Our goal now is to extract from this sequence a subsequence that converges to the weak solution. In this respect, we are going to show that this sequence is bounded and thereafter, we will make use of a compactness result.

Lemma 3.2
Let Y be a nonnegative function satisfying the inequality

$$Y'\leq\eta Y^3+\zeta$$

and $Y(0)=M_0$ be a strictly positive real number, then there exists a time interval $(0,T^\star)$ where $Y(t)$ is bounded by a positive constant M and M depends only on M_0, η and ζ ([5]).

Proof.
We have

$$\begin{aligned} Y' &\leq \eta Y^3+\zeta \\ &\leq \eta\left(Y+(\frac{\zeta}{\eta})^{\frac{1}{3}}\right)^3 \\ &= \eta(Y+\tau)^3 \quad \text{where } \tau=(\frac{\zeta}{\eta})^{\frac{1}{3}}. \end{aligned}$$

We set $Z=Y+\tau$. It follows that

$$Z'\leq\eta Z^3,$$

and so
$$\frac{Z'}{Z^3}\leq\eta.$$
We integrate over $(0,t)$, with $0\leq t\leq T$
$$\int_0^t\frac{Z'}{Z^3}d\iota\leq\int_0^t\eta d\iota,$$
and obtain that
$$\frac{Z^{-2}(t)}{-2}-\frac{Z^{-2}(0)}{-2}\leq\eta t.$$
This gives that
$$Z(t)^{-2}-Z(0)^{-2}\geq-2\eta t,$$
and
$$Z(t)^{-2}-Z(0)^{-2}\geq-2\eta t\geq-2\eta T.$$
It follows that
$$Z(t)^{-2}\geq Z(0)^{-2}-2\eta T.$$
We choose $T^\star$ in $(0,T)$ such that $Z(0)^{-2}-2\eta T^\star>0$, that is
$$T^\star<\frac{(Z(0))^{-2}}{2\eta}=\frac{(M_0+(\frac{\zeta}{\eta})^{\frac{1}{3}})^{-2}}{2\eta}.$$
We have that
$$Z(t)^{-2}-Z(0)^{-2}\geq-2\eta t\geq-2\eta T^\star\quad t\in(0,T^\star),$$
and so
$$Z(t)^{-2}\geq Z(0)^{-2}-2\eta T^\star\quad t\in(0,T^\star).$$
Therefore
$$Z(t)^2\leq(Z(0)^{-2}-2\eta T^\star)^{-1},$$
so that
$$Z(t)\leq(Z(0)^{-2}-2\eta T^\star)^{-\frac{1}{2}}\quad t\in(0,T^\star).$$
Considering the fact that $Z=Y+\tau$ and $\tau=(\frac{\zeta}{\eta})^{\frac{1}{3}}$, we finally obtain that
$$Y(t)\leq\left((M_0+(\frac{\zeta}{\eta})^{\frac{1}{3}})^{-2}-2\eta T^\star\right)^{-\frac{1}{2}}-(\frac{\zeta}{\eta})^{\frac{1}{3}}\quad t\in(0,T^\star).\tag{3.50}$$
We must set an extra condition on $T^\star$ to have $\left((M_0+(\frac{\zeta}{\eta})^{\frac{1}{3}})^{-2}-2\eta T^\star\right)^{-\frac{1}{2}}-(\frac{\zeta}{\eta})^{\frac{1}{3}}>0$, we set
$$T^\star>\frac{(M_0+(\frac{\zeta}{\eta})^{\frac{1}{3}})^{-2}-(\frac{\zeta}{\eta})^{\frac{-2}{3}}}{2\eta}.$$

Hence we choose $T^\star$ such that

$$\frac{(M_0+(\frac{\zeta}{\eta})^{\frac{1}{3}})^{-2}-(\frac{\zeta}{\eta})^{\frac{-2}{3}}}{2\eta}<T^\star<\frac{(M_0+(\frac{\zeta}{\eta})^{\frac{1}{3}})^{-2}}{2\eta},$$

it follows that $Y(t)$ is bounded by a positive constant

$$M=\left((M_0+(\frac{\zeta}{\eta})^{\frac{1}{3}})^{-2}-2\eta T^\star\right)^{-\frac{1}{2}}-(\frac{\zeta}{\eta})^{\frac{1}{3}}$$

for all t such that $0\le t\le T^\star$ and M depends only on M_0, η and ζ. $\Diamond$

Theorem 3.4 (**Local existence.**)
Let K be such that

$$\kappa=\mu-2Kc_2c_4>0, \tag{3.51}$$

there is a time interval $(0,T^\star)$ on which a weak solution of problem 3.1 exists.

Proof.
Considering (3.51), the energy estimates (3.46) now reads as:

$$\frac{d}{dt}\|\nabla\mathbf{u}_n\|^2+\kappa\|\widetilde{\Delta}\mathbf{u}_n\|^2\le\frac{16c_1^4}{\mu^3}\|\nabla\mathbf{u}_n\|^6+\frac{2c_3^2\widetilde{P}^2}{\mu}. \tag{3.52}$$

Since $\kappa>0$, (3.52) implies that

$$\frac{d}{dt}\|\nabla\mathbf{u}_n\|^2\le\frac{16c_1^4}{\mu^3}\|\nabla\mathbf{u}_n\|^6+\frac{2c_3^2\widetilde{P}^2}{\mu}. \tag{3.53}$$

Defining

$$Y_n(t)=\|\nabla\mathbf{u}_n\|^2,$$

we see that inequality (3.53) has the following form:

$$Y_n{}'\le\eta Y_n{}^3+\zeta, \tag{3.54}$$

where $\eta>0$ and $\zeta>0$ are some constants depending on the data.

Remark 3.2
For each integer n, the function $Y_n(t)=\|\nabla\mathbf{u}_n\|^2$ verifies the requirements of lemma 3.2 with $Y_n(0)=\|\nabla\mathbf{u}_0\|^2$ because the initial condition is the same for all functions $\mathbf{u}_n$, $n=0,1,2,...$ Consequently, the sequence $\{\|\nabla\mathbf{u}_n\|^2\}_{n=1}^\infty$ is bounded by a real number M which depends only on the initial data. We make use of Poincaré's inequality (3.1) to find that for each integer n,

$$\begin{aligned}\|\mathbf{u}_n\|_1^2 &= \|\mathbf{u}_n\|^2+\|\nabla\mathbf{u}_n\|^2\\ &\le C_\Omega{}^2\|\nabla\mathbf{u}_n\|^2+\|\nabla\mathbf{u}_n\|^2\\ &\le (1+C_\Omega^2)\|\nabla\mathbf{u}_n\|^2\\ &\le (1+C_\Omega^2)M,\end{aligned}$$

and

$$\begin{aligned} \|\mathbf{u}_n\|^2_{L^2(0,T^\star;\mathbf{V})} &= \int_0^{T^\star} \|\mathbf{u}_n\|_1^2 dt \\ &\leq (1+C_\Omega^2)MT^\star, \text{ since } \|\mathbf{u}_n\|_1^2 \leq (1+C_\Omega^2)M. \end{aligned}$$

Therefore the sequence $\{\|\mathbf{u}_n\|_{L^2(0,T^\star;\mathbf{V})}\}_{n=1}^\infty$ is bounded .

Remark 3.3
The space $L^2(0,T^\star;\mathbf{V})$ being a Hilbert space, lemma 3.1 can be applied to the sequence $\{\|\mathbf{u}_n\|_{L^2(0,T^\star;\mathbf{V})}\}_{n=1}^\infty$. In fact, the sequence $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded in $L^2(0,T^\star;\mathbf{V})$. Consequently, there exists a subsequence $\{\mathbf{u}_{n_q}\}_{q=1}^\infty \subset \{\mathbf{u}_n\}_{n=1}^\infty$, such that

$$\mathbf{u}_{n_q} \rightharpoonup \mathbf{u} \text{ weakly in } \mathbf{L}^2(0,T^\star;\mathbf{V}). \tag{3.55}$$

We next show that the weak limit $\mathbf{u}$ is in fact a weak solution. In this respect, we are going to show at first that

$$(\frac{\partial \mathbf{u}}{\partial t},\boldsymbol{\varphi}) + b(\mathbf{u},\mathbf{u},\boldsymbol{\varphi}) + a(\mathbf{u},\boldsymbol{\varphi}) = c(t,\boldsymbol{\varphi}) \quad \text{for all} \quad \boldsymbol{\varphi} \in \mathbf{V},$$

and thereafter

$$(\mathbf{u}(0),\boldsymbol{\varphi}) = (\mathbf{u}_0,\boldsymbol{\varphi}) \quad \text{for all} \quad \boldsymbol{\varphi} \in \mathbf{V}.$$

1. Fix an integer N and choose a function $\mathbf{w} \in \mathbf{C}(0,T^\star;\ \mathbf{V})$ having the form

   $$\mathbf{w}(t) = \sum_{k=1}^N d^k(t)\mathbf{a}_k, \tag{3.56}$$

   where $\{d^k\}_{k=1}^N$ are given functions and $\{\mathbf{a}_k\}$ is the basis of $\mathbf{V}$. We choose $n \geq N$, multiply (3.43) by $d^k(t)$, sum $k=1,...,N$, and then integrate with respect to t to find that

   $$\int_0^{T^\star} [(\frac{\partial \mathbf{u}_{n_q}}{\partial t},\mathbf{w}) + a(\mathbf{u}_{n_q},\mathbf{w}) + b(\mathbf{u}_{n_q},\mathbf{u}_{n_q},\mathbf{w})]\ dt = \int_0^{T^\star} c(t,\mathbf{w})dt. \tag{3.57}$$

   We recall (3.55), to find upon passing to weak limits (see remark 3.1 and [6]) that

   $$\int_0^{T^\star} [(\frac{\partial \mathbf{u}}{\partial t},\mathbf{w}) + a(\mathbf{u},\mathbf{w}) + b(\mathbf{u},\mathbf{u},\mathbf{w})]\ dt = \int_0^{T^\star} c(t,\mathbf{w})dt. \tag{3.58}$$

   Equality (3.58) then holds for all functions $\mathbf{w} \in \mathbf{L}^2(0,T;\mathbf{V})$ as functions of the form (3.56) are dense in this space [6]. Hence in particular

   $$(\frac{\partial \mathbf{u}}{\partial t},\boldsymbol{\varphi}) + b(\mathbf{u},\mathbf{u},\boldsymbol{\varphi}) + a(\mathbf{u},\boldsymbol{\varphi}) = c(t,\boldsymbol{\varphi})$$

   for each $\boldsymbol{\varphi} \in \mathbf{V}$ and a.e. $t \geq 0$.

2. In order to prove that $(\mathbf{u}(0), \boldsymbol{\varphi}) = (\mathbf{u}_0, \boldsymbol{\varphi})$ for every $\boldsymbol{\varphi} \in \mathbf{V}$, we first note from (3.58) that

$$\int_0^{T^\star} [-(\mathbf{u}, \frac{\partial \mathbf{w}}{\partial t}) + a(\mathbf{u}, \mathbf{w}) + b(\mathbf{u}, \mathbf{u}, \mathbf{w})]\ dt = \int_0^{T^\star} c(t, \mathbf{w}) dt + (\mathbf{u}(0), \mathbf{w}(0)). \quad (3.59)$$

for each $\mathbf{w} \in \mathbf{H}^1(0, T^\star; \mathbf{V})$ with $\mathbf{w}(T^\star) = 0$. Similarly, from (3.57) we deduce

$$\int_0^{T^\star} [-(\mathbf{u}_{n_q}, \frac{\partial \mathbf{w}}{\partial t}) + a(\mathbf{u}_{n_q}, \mathbf{w}) + b(\mathbf{u}_{n_q}, \mathbf{u}_{n_q}, \mathbf{w})]\ dt = \int_0^{T^\star} c(t, \mathbf{w}) dt + (\mathbf{u}_{n_q}(0), \mathbf{w}(0)). \quad (3.60)$$

We once again employ (3.55) to find

$$\int_0^{T^\star} [-(\mathbf{u}, \frac{\partial \mathbf{w}}{\partial t}) + a(\mathbf{u}, \mathbf{w}) + b(\mathbf{u}, \mathbf{u}, \mathbf{w})]\ dt = \int_0^{T^\star} c(t, \mathbf{w}) dt + (\mathbf{u}_0, \mathbf{w}(0)). \quad (3.61)$$

since $\mathbf{u}_{n_q}(0) \to \mathbf{u}_0$. As $\mathbf{w}(0)$ is arbitrary, comparing (3.59) and (3.61), we conclude that $(\mathbf{u}(0), \boldsymbol{\varphi}) = (\mathbf{u}_0, \boldsymbol{\varphi})$ for each $\boldsymbol{\varphi} \in \mathbf{V}$.
Therefore $\mathbf{u}$ is a local weak solution of problem 3.1. ♦

3.4.6 Global existence of a weak solution for small data

In this section, we make use of Galerkin's method to establish the global existence of a weak solution. We are going to show that under certain circumstances, the weak solution $\mathbf{u}$ is defined at any time t. In this regard, we are going to consider the energy estimate and the following lemma.

Lemma 3.3
Let Y be a nonnegative, absolutely continuous function satisfying inequality

$$Y' + \lambda Y \leq \eta Y^3 + \zeta, \quad (3.62)$$

let a real number M be such that $0 < M < \left(\frac{\lambda}{2\eta}\right)^{\frac{1}{2}}$,
If $Y(0) \leq M$, and $\zeta \leq \frac{\lambda M}{2}$, then $Y(t)$ is bounded by M for all $t > 0$ [5] .

Proof.
This result is proven by contradiction. Suppose that there exists a t such that $Y(t) > M$, and define

$$t^\star = \inf\{t \in \mathbb{R}_+, Y(t) > M\},$$

we have $Y(t^\star) = M$ and $Y'(t^\star) > 0$.

1. We first show that $Y(t^\star) = M$.
   Set $I = \{t \in \mathbb{R}_+, Y(t) > M\}$ and choose any $\epsilon > 0$, we have

$$t^\star - \epsilon < t^\star \text{ and } t^\star = \inf(I),$$

and so $t^\star - \epsilon$ doesn't belong to the set I. It follows that $Y(t^\star - \epsilon) \leq M$, and this is true for each $\epsilon > 0$, hence

$$Y(t^\star) \leq M.$$

At the other hand, for each natural number n, we have $t^\star + \frac{1}{n} > t^\star$. We make use of the fact that $t^\star = \inf(I)$ to see that there exists $\phi_n \in I$ such that

$$t^\star \leq \phi_n \leq t^\star + \frac{1}{n},$$

ϕ_n is not necessarily unique. We choose one value that ϕ_n may take and we denote it by t_n. This defines a real sequence $\{t_n\}_{n=1}^{\infty}$.
For each natural number n, since $t_n \in I$, we have that $Y(t_n) > M$. The fact that $t^\star \leq t_n \leq t^\star + \frac{1}{n}$ implies that the sequence $\{t_n\}_{n=1}^{\infty}$ converges to $t^\star$. We make use of the continuity of Y to see that the sequence

$$\{Y(t_n)\}_{n=1}^{\infty} \quad \text{converges to} \quad Y(t^\star).$$

Since $t_n \in I$ for each n, we have $Y(t_n) > M$, and then $Y(t_n) \rightarrow Y(t^\star)$ implies that

$$Y(t^\star) \geq M.$$

It follows that $Y(t^\star) \leq M$ and $Y(t^\star) \geq M$, therefore

$$Y(t^\star) = M.$$

2. Next we show that $Y'(t^\star) > 0$.
Suppose that $Y'(t^\star) \leq 0$, then there exists a nonnegative natural number n such that Y decreases on the interval $(t^\star, t^\star + \frac{1}{n})$.
Also we have that
$$t^\star \leq t_n \leq t^\star + \frac{1}{n},$$
this implies that
$$Y(t^\star) \geq Y(t_n).$$
But from part (1.), we have $Y(t^\star) = M$ and $Y(t_n) > M$ because $t_n \in I$, it follows that
$$Y(t^\star) < Y(t_n).$$
Hence $Y(t^\star) \geq Y(t_n)$ and $Y(t^\star) < Y(t_n)$, which is impossible. Therefore
$$Y'(t^\star) > 0.$$

Finally we show that $Y'(t^\star) \leq 0$ and this contradicts the result obtained in (2.). From (3.62), we have

$$\begin{array}{rcl} Y'(t^\star) & \leq & -\lambda Y(t^\star) + \eta Y^3(t^\star) + \zeta \\ & \leq & -\lambda M + \eta M^3 + \zeta \ \text{ (because } Y(t^\star) = M) \\ & \leq & -\lambda M + \eta M(M^2) + \zeta \\ & \leq & -\lambda M + \eta M(\dfrac{\lambda}{2\eta}) + \zeta \\ & \leq & \zeta - \dfrac{\lambda M}{2} \\ & \leq & 0 \ \text{ because of the hypothesis } (\zeta \leq \dfrac{\lambda M}{2}). \end{array}$$

Therefore $Y(t) \leq M$ for all $t \in \mathbb{R}_+$.

$\Diamond$

Theorem 3.5 (**Global existence.**)
Assume that the initial and boundary data are sufficiently small, precisely

$$\|\nabla \boldsymbol{u}_0\| \leq \left(\frac{\kappa\mu^3}{64c_2c_1^4}\right)^{\frac{1}{4}} \quad \textit{and} \quad \widetilde{P} < \frac{\mu}{4c_1c_3}\left(\frac{\kappa^3\mu}{4c_2^3}\right)^{\frac{1}{4}}, \tag{3.63}$$

then for all $T \geq 0$, there exists a weak solution $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{V})$ of problem 3.1 and it satisfies the inequality

$$\|\nabla \boldsymbol{u}\| \leq \left(\frac{\kappa\mu^3}{64c_2c_1^4}\right)^{\frac{1}{4}}. \tag{3.64}$$

Proof
We recall the energy estimate (3.46)

$$\frac{d}{dt}\|\nabla \mathbf{u}_n\|^2 + (\mu - 2Kc_2c_4)\|\widetilde{\Delta}\mathbf{u}_n\|^2 \leq \frac{16c_1^4}{\mu^3}\|\nabla \mathbf{u}_n\|^6 + \frac{2c_3^2\widetilde{P}^2}{\mu},$$

the inequality (3.7)

$$\|\nabla \mathbf{u}_n\| \leq c_2\|\widetilde{\Delta}\mathbf{u}_n\|,$$

and the assumption (3.51) which states that K is such that

$$\kappa = \mu - 2Kc_2c_4 > 0.$$

We make use of these estimates and we obtain that

$$\frac{d}{dt}\|\nabla \mathbf{u}_n\|^2 + \frac{\kappa}{c_2}\|\nabla \mathbf{u}_n\|^2 \leq \frac{16c_1^4}{\mu^3}\|\nabla \mathbf{u}_n\|^6 + \frac{2c_3^2\widetilde{P}^2}{\mu}. \tag{3.65}$$

Set

$$Y_n = \|\nabla \mathbf{u}_n\|^2,$$

the energy estimate (3.65) takes the form

$$Y_n' + \lambda Y_n \leq \eta Y_n^3 + \zeta,$$

where

$$\lambda = \frac{\kappa}{c_2}, \quad \eta = \frac{16c_1^4}{\mu^3} \text{ and } \zeta = \frac{2c_3^2\widetilde{P}^2}{\mu}.$$

Also we have

$$Y_n(0) = \|\nabla \mathbf{u}_0\|^2 \quad \text{for each integer } n.$$

Set

$$M_0 = \left(\frac{\lambda}{2\eta}\right)^{\frac{1}{2}} = \left(\frac{\kappa\mu^3}{32c_2c_1^4}\right)^{\frac{1}{2}}.$$

We choose

$$M = \left(\frac{\kappa\mu^3}{64c_2c_1^4}\right)^{\frac{1}{2}} = \frac{M_0}{\sqrt{2}},$$

it follows that

$$0 < M < M_0.$$

According to the hypothesis, we have

$$Y_n(0) = \|\nabla \mathbf{u}_0\|^2 \leq \left(\frac{\kappa\mu^3}{64c_2c_1^4}\right)^{\frac{1}{2}} = M,$$

and from the fact that $\widetilde{P} < \frac{\mu}{4c_1c_3}\left(\frac{\kappa^3\mu}{4c_2^3}\right)^{\frac{1}{4}}$, we obtain that

$$\zeta = \frac{2c_3^2\widetilde{P}^2}{\mu} \leq \frac{\kappa}{2c_2}\left(\frac{\kappa\mu^3}{64c_2c_1^4}\right)^{\frac{1}{2}} = \frac{\lambda M}{2}.$$

Remark 3.4
We make use of lemma 3.3 to find that $Y_n = \|\nabla \mathbf{u}_n\|^2$ is bounded by M for all $t > 0$ and the bound doesn't depend on n. Poincaré's inequality (3.1) implies that

$$\begin{aligned}
\|\mathbf{u}_n\|_1^2 &= \|\mathbf{u}_n\|^2 + \|\nabla \mathbf{u}_n\|^2 \\
&\leq {C_\Omega}^2\|\nabla \mathbf{u}_n\|^2 + \|\nabla \mathbf{u}_n\|^2 \\
&\leq (1 + C_\Omega^2)\|\nabla \mathbf{u}_n\|^2 \\
&\leq (1 + C_\Omega^2)M.
\end{aligned}$$

Let $T > 0$ be any positive real number,

$$\begin{aligned}\|\mathbf{u}_n\|^2_{L^2(0,T;\mathbf{V})} &= \int_0^T \|\mathbf{u}_n\|_1^2 dt \\ &\leq (1+C_\Omega^2)MT, \text{ since } \|\mathbf{u}_n\|_1^2 \leq (1+C_\Omega^2)M.\end{aligned}$$

Thus for each nonnegative real number T, the sequence $\{\|\mathbf{u}_n\|_{L^2(0,T;\mathbf{V})}\}_{n=1}^{\infty}$ is bounded. We make use of lemma 3.1 to extract a subsequence $\{\mathbf{u}_{nq}\}_{q=1}^{\infty}$ that converges to an element $\mathbf{u}$ of $L^2(0,T;\mathbf{V})$ and we use the same steps as we did for the local existence to show that $\mathbf{u}$ is a weak solution of problem 3.1.
Furthermore, the fact that the sequence $\{\nabla \mathbf{u}_n\}_{n=1}^{\infty}$ is bounded by $\sqrt{M}$ implies that the subsequence $\left\{\nabla \mathbf{u}_{nq}\right\}_{q=1}^{\infty}$ is also bounded by $\sqrt{M}$.
Therefore, since $\nabla \mathbf{u}$ is the limit of this subsequence, it follows that

$$\|\nabla \mathbf{u}\| \leq \left(\frac{\kappa\mu^3}{64 c_2 c_1^4}\right)^{\frac{1}{4}}.$$

♦

3.5 Uniqueness of the solution

In this section, we are going to prove that there exists a time interval where the solution of problem 3.1 is unique.

Theorem 3.6 (**Uniqueness.**)
There exists a time interval T_1 where the solution of problem 3.1 is unique.

Proof.
Assume that there exist two solutions $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ associated with the same data. Set $\mathbf{w} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$. Consider (3.39) with $\boldsymbol{\varphi} = \mathbf{w}$. We have

$$(\frac{\partial \mathbf{u}^{(1)}}{\partial t}, \mathbf{w}) + b(\mathbf{u}^{(1)}, \mathbf{u}^{(1)}, \mathbf{w}) + a(\mathbf{u}^{(1)}, \mathbf{w}) = c(t, \mathbf{w}),$$

and

$$(\frac{\partial \mathbf{u}^{(2)}}{\partial t}, \mathbf{w}) + b(\mathbf{u}^{(2)}, \mathbf{u}^{(2)}, \mathbf{w}) + a(\mathbf{u}^{(2)}, \mathbf{w}) = c(t, \mathbf{w}).$$

By substraction, we obtain that

$$(\frac{\partial \mathbf{w}}{\partial t}, \mathbf{w}) + a(\mathbf{w}, \mathbf{w}) = b(\mathbf{u}^{(2)}, \mathbf{u}^{(2)}, \mathbf{w}) - b(\mathbf{u}^{(1)}, \mathbf{u}^{(1)}, \mathbf{w}). \tag{3.66}$$

We recall that a is a bilinear form, b is a trilinear form and c is a linear form described by (3.14), (3.15) and (3.16) respectively. We evaluate every term of (3.66):

$$(\frac{\partial \mathbf{w}}{\partial t}, \mathbf{w}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2,$$

also

$$a(\mathbf{w},\mathbf{w}) = \mu\|\nabla\mathbf{w}\|^2 + \sum_{i=1}^{3} K_i \int_{\Gamma i} |\mathbf{w}\cdot\mathbf{n}|^2 ds,$$

and

$$\begin{aligned}
& b(\mathbf{u}^{(2)},\mathbf{u}^{(2)},\mathbf{w}) - b(\mathbf{u}^{(1)},\mathbf{u}^{(1)},\mathbf{w}) \\
= \; & b(\mathbf{u}^{(2)},\mathbf{u}^{(2)},\mathbf{w}) - b(\mathbf{u}^{(2)}+\mathbf{w},\mathbf{u}^{(2)}+\mathbf{w},\mathbf{w}) \\
= \; & b(\mathbf{u}^{(2)},\mathbf{u}^{(2)},\mathbf{w}) - b(\mathbf{u}^{(2)},\mathbf{u}^{(2)}+\mathbf{w},\mathbf{w}) - b(\mathbf{w},\mathbf{u}^{(2)}+\mathbf{w},\mathbf{w}) \\
= \; & b(\mathbf{u}^{(2)},\mathbf{u}^{(2)},\mathbf{w}) - b(\mathbf{u}^{(2)},\mathbf{u}^{(2)},\mathbf{w}) - b(\mathbf{u}^{(2)},\mathbf{w},\mathbf{w}) - b(\mathbf{w},\mathbf{u}^{(2)},\mathbf{w}) - b(\mathbf{w},\mathbf{w},\mathbf{w}) \\
= \; & -b(\mathbf{w},\mathbf{w},\mathbf{w}) - b(\mathbf{u}^{(2)},\mathbf{w},\mathbf{w}) - b(\mathbf{w},\mathbf{u}^{(2)},\mathbf{w}).
\end{aligned}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \mu\|\nabla\mathbf{w}\|^2 + \sum_{i=1}^{3} K_i \int_{\Gamma i} |\mathbf{w}\cdot\mathbf{n}|^2 ds = -b(\mathbf{w},\mathbf{w},\mathbf{w}) - b(\mathbf{u}^{(2)},\mathbf{w},\mathbf{w}) - b(\mathbf{w},\mathbf{u}^{(2)},\mathbf{w}). \tag{3.67}$$

We now make use of some classical estimates to evaluate the righthand side of (3.67) as follow:

$$\begin{aligned}
b(\mathbf{w},\mathbf{w},\mathbf{w}) &= ((\mathbf{w}\cdot\nabla)\mathbf{w},\mathbf{w}) \\
&= \int_\Omega \sum_{i=1}^{3}\left(\sum_{j=1}^{3} w_j \frac{\partial w_i}{\partial x_j}\right) w_i d\mathbf{x} \\
&= -\int_\Omega \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \frac{\partial (w_i w_j)}{\partial x_j}\right) w_i d\mathbf{x} + \int_\Gamma (\mathbf{w}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{n}) ds \\
&= -((\mathbf{w}\cdot\nabla)\mathbf{w},\mathbf{w}) - \int_\Omega (\mathbf{w}\cdot\mathbf{w})(\nabla\cdot\mathbf{w}) d\mathbf{x} + \int_\Gamma (\mathbf{w}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{n}) ds \\
&= -b(\mathbf{w},\mathbf{w},\mathbf{w}) + \int_\Gamma (\mathbf{w}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{n}) ds \quad \text{because } \nabla\cdot\mathbf{w} = 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
b(\mathbf{w},\mathbf{w},\mathbf{w}) &= \frac{1}{2}\int_\Gamma (\mathbf{w}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{n}) ds \\
&\le \frac{1}{2}\left(\int_\Gamma (\mathbf{w}\cdot\mathbf{w})^2 ds\right)^{\frac{1}{2}} \left(\int_\Gamma (\mathbf{w}\cdot\mathbf{n})^2 ds\right)^{\frac{1}{2}} \qquad (A.6) \\
&\le c_5\|\mathbf{w}\|_{L^4}^2 \|\mathbf{w}\| \qquad (A.13) \\
&\le c_6\|\mathbf{w}\|\|\nabla\mathbf{w}\|^2 \qquad (A.3), (3.1).
\end{aligned}$$

Also,

$$\begin{aligned}
|b(\mathbf{u}^{(2)},\mathbf{w},\mathbf{w})| &\le \sup_\Omega |\mathbf{u}^{(2)}| \|\nabla\mathbf{w}\|\|\mathbf{w}\| \\
&\le \frac{\mu}{2}\|\nabla\mathbf{w}\|^2 + \frac{1}{2\mu}(\sup_\Omega |\mathbf{u}^{(2)}|)^2 \|\mathbf{w}\|^2 \quad (A.4),
\end{aligned}$$

and

$$|b(\mathbf{w}, \mathbf{u}^{(2)}, \mathbf{w})| \leq \sup_\Omega |\nabla \mathbf{u}^{(2)}| \|\mathbf{w}\|^2.$$

It follows that

$$|b(\mathbf{u}^{(2)}, \mathbf{w}, \mathbf{w}) + b(\mathbf{w}, \mathbf{u}^{(2)}, \mathbf{w})| \leq \frac{\mu}{2}\|\nabla \mathbf{w}\|^2 + \frac{1}{2\mu}(\sup_\Omega |\mathbf{u}^{(2)}|)^2\|\mathbf{w}\|^2 + \sup_\Omega |\nabla \mathbf{u}^{(2)}| \|\mathbf{w}\|^2.$$

We now make use of these estimates in (3.67) to find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \mu\|\nabla \mathbf{w}\|^2 + \sum_{i=1}^{3} K_i \int_{\Gamma i} |\mathbf{w}\cdot\mathbf{n}|^2 ds \quad &\leq \quad \frac{\mu}{2}\|\nabla \mathbf{w}\|^2 + \frac{1}{2\mu}(\sup_\Omega |\mathbf{u}^{(2)}|)^2\|\mathbf{w}\|^2 \\ &\quad + \sup_\Omega |\nabla \mathbf{u}^{(2)}| \|\mathbf{w}\|^2 + c_6\|\mathbf{w}\|\|\nabla \mathbf{w}\|^2. \end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \mu\|\nabla \mathbf{w}\|^2 \leq \frac{\mu}{2}\|\nabla \mathbf{w}\|^2 + \frac{1}{2\mu}(\sup_\Omega |\mathbf{u}^{(2)}|)^2\|\mathbf{w}\|^2 + \sup_\Omega |\nabla \mathbf{u}^{(2)}| \|\mathbf{w}\|^2 + c_6\|\mathbf{w}\|\|\nabla \mathbf{w}\|^2,$$

which gives that

$$\frac{d}{dt}\|\mathbf{w}\|^2 + (\mu - 2c_6\|\mathbf{w}\|)\|\nabla \mathbf{w}\|^2 \leq \left(\frac{1}{\mu}(\sup_\Omega |\mathbf{u}^{(2)}|)^2 + 2(\sup_\Omega |\nabla \mathbf{u}^{(2)}|)\right)\|\mathbf{w}\|^2.$$

Since $\mathbf{w}(0, \mathbf{x}) = 0$, there exists $T_1 > 0$ such that $\|\mathbf{w}\| \leq \frac{\mu}{2c_6}$ with $0 < T_1 < T^\star$. On the interval $(0, T_1)$, we have

$$\frac{d}{dt}\|\mathbf{w}\|^2 \leq \vartheta(t)\|\mathbf{w}\|^2, \tag{3.68}$$

where

$$\vartheta(t) = \left(\frac{1}{\mu}(\sup_\Omega |\mathbf{u}^{(2)}|)^2 + 2(\sup_\Omega |\nabla \mathbf{u}^{(2)}|)\right)$$

is a real function depending on t.
The uniqueness theorem follows from Gronwall's inequality $(A.11)$ as $\mathbf{w}(0, \mathbf{x}) = 0$. ♦

3.6 Stability of the solution

In this section, we are going to prove that there exists a time interval of continuous dependence on the data. We start with the boundary conditions and afterward we move on to the initial condition.

3.6.1 Boundary conditions stability.

We recall the boundary conditions: On the artery wall Γ_{wall}, no-slip boundary condition

$$\mathbf{u}(\mathbf{x}, t) = 0, \ \ \mathbf{x} \in \Gamma_{\text{wall}}, \ t > 0$$

holds. On each artificial boundary Γ_i $(i = 1, 2, 3)$, the boundary condition

$$p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} - K_i(\mathbf{u} \cdot \mathbf{n})\mathbf{n} = P_i\mathbf{n} \quad \text{for } t > 0, \ \mathbf{x}{\in}\Gamma_i$$

is prescribed, where K_i is a suitable nonnegative constant, $P_i \in L^\infty(0, T)$ is assumed to be a given function and $\mathbf{n}$ represents the outward normal unit vector on every part of the vessel boundary.
The purpose of this subsection is to show that there exists a time interval where a small change in the given functions P_i $(i = 1, 2, 3)$ produces a correspondingly small change in the solution.

Theorem 3.7
There exists a time interval where the solution of problem 3.1 depends continuously on the prescribed data as the real functions P_i, $(i = 1, 2, 3)$ are varied.

Proof.
Denote by $\mathbf{u}^{(1)}$ the solution associated with the data $P_i^{(1)}$ (i=1,2,3) and, correspondingly, by $\mathbf{u}^{(2)}$ the solution associated with $P_i^{(2)}$ (i=1,2,3).
Set

$$\mathbf{w} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \ \ \overline{P} = \max_{i=1,2,3} \left| P_i^{(1)} - P_i^{(2)} \right|.$$

We make use of (3.39) for $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ with $\boldsymbol{\varphi} = \mathbf{w}$. We obtain that

$$(\frac{\partial \mathbf{u}^{(1)}}{\partial t}, \mathbf{w}) + b(\mathbf{u}^{(1)}, \mathbf{u}^{(1)}, \mathbf{w}) + a(\mathbf{u}^{(1)}, \mathbf{w}) = c_1(t, \mathbf{w}),$$

and

$$(\frac{\partial \mathbf{u}^{(2)}}{\partial t}, \mathbf{w}) + b(\mathbf{u}^{(2)}, \mathbf{u}^{(2)}, \mathbf{w}) + a(\mathbf{u}^{(2)}, \mathbf{w}) = c_2(t, \mathbf{w}),$$

where

$$c_1(t, \mathbf{v}) = -\sum_{i=1}^{3} P_i^{(1)}(t) \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{v} \, ds \ \text{ for each } \ \mathbf{v} \in \mathbf{H}^1(\Omega),$$

and

$$c_2(t, \mathbf{v}) = -\sum_{i=1}^{3} P_i^{(2)}(t) \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{v} \, ds \ \text{ for each } \ \mathbf{v} \in \mathbf{H}^1(\Omega).$$

By substraction, we obtain that

$$(\frac{\partial \mathbf{w}}{\partial t}, \mathbf{w}) + a(\mathbf{w}, \mathbf{w}) = b(\mathbf{u}^{(2)}, \mathbf{u}^{(2)}, \mathbf{w}) - b(\mathbf{u}^{(1)}, \mathbf{u}^{(1)}, \mathbf{w}) + c_1(t, \mathbf{w}) - c_2(t, \mathbf{w}). \tag{3.69}$$

This gives us

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \mu\|\nabla\mathbf{w}\|^2 + \sum_{i=1}^{3} K_i \int_{\Gamma i} |\mathbf{w}\cdot\mathbf{n}|^2 ds = -b(\mathbf{w},\mathbf{w},\mathbf{w}) - b(\mathbf{u}^{(2)},\mathbf{w},\mathbf{w}) - b(\mathbf{w},\mathbf{u}^{(2)},\mathbf{w}) + \sum_{i=1}^{3}(P_i^{(2)} - P_i^{(1)})\int_{\Gamma i} \mathbf{w}\cdot\mathbf{n}\, ds.$$

Also

$$\begin{aligned}
\sum_{i=1}^{3}(P_i^{(2)} - P_i^{(1)})\int_{\Gamma i} \mathbf{w}\cdot\mathbf{n}\, ds &\leq \sum_{i=1}^{3}\overline{P}\int_{\Gamma i}|\mathbf{w}\cdot\mathbf{n}|\, ds \\
&\leq \sum_{i=1}^{3} c_i\overline{P}\left(\int_{\Gamma i}|\mathbf{w}\cdot\mathbf{n}|^2\, ds\right)^{\frac{1}{2}} \quad \text{where } c_i = \left(\int_{\Gamma i} ds\right)^{\frac{1}{2}} \\
&\leq \sum_{i=1}^{3}\left(\frac{c_i^2\overline{P}^2}{4\epsilon_i} + \int_{\Gamma i}\epsilon_i|\mathbf{w}\cdot\mathbf{n}|^2\, ds\right) \quad \text{for each } \epsilon_i \in \mathbb{R} \qquad (A.4) \\
&\leq c_7\overline{P}^2 + \sum_{i=1}^{3} K_i\int_{\Gamma i}|\mathbf{w}\cdot\mathbf{n}|^2\, ds, \quad \text{where } \epsilon_i = K_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \mu\|\nabla\mathbf{w}\|^2 \leq{}& \frac{\mu}{2}\|\nabla\mathbf{w}\|^2 + \frac{1}{2\mu}(\sup_{\Omega}|\mathbf{u}^{(2)}|)^2\|\mathbf{w}\|^2 + \sup_{\Omega}|\nabla\mathbf{u}^{(2)}|\|\mathbf{w}\|^2 \\
&+ c_6\|\mathbf{w}\|\|\nabla\mathbf{w}\|^2 + c_7\overline{P}^2.
\end{aligned}$$

Thus

$$\frac{d}{dt}\|\mathbf{w}\|^2 + (\mu - 2c_6\|\mathbf{w}\|)\|\nabla\mathbf{w}\|^2 \leq \left(\frac{1}{\mu}(\sup_{\Omega}|\mathbf{u}^{(2)}|)^2 + 2(\sup_{\Omega}|\nabla\mathbf{u}^{(2)}|)\right)\|\mathbf{w}\|^2 + 2c_7\overline{P}^2. \quad (3.70)$$

Since $\mathbf{w}(0,\mathbf{x}) = 0$, there exists $T_1 > 0$ such that $\|\mathbf{w}\| \leq \frac{\mu}{2c_6}$ with $0 < T_1 < T^\star$. On the interval $(0, T_1)$, we have that

$$\frac{d}{dt}\|\mathbf{w}\|^2 \leq \left(\frac{1}{\mu}(\sup_{\Omega}|\mathbf{u}^{(2)}|)^2 + 2(\sup_{\Omega}|\nabla\mathbf{u}^{(2)}|)\right)\|\mathbf{w}\|^2 + 2c_7\overline{P}^2. \quad (3.71)$$

It follows that

$$\frac{d}{dt}\|\mathbf{w}\|^2 \leq c_8\|\mathbf{w}\|^2 + 2c_7\overline{P}^2,$$

where

$$c_8 = \sup_{(0,T_1)}\left[\left(\frac{1}{\mu}(\sup_{\Omega}|\mathbf{u}^{(2)}|)^2 + 2(\sup_{\Omega}|\nabla\mathbf{u}^{(2)}|)\right)\right].$$

According to Gronwall's inequality $(A.10)$, we deduce that in the interval $(0, T_1)$,

$$\|\mathbf{w}\| \leq c \max_{0<t\leq T_1}\overline{P}, \quad \text{where } \overline{P} = \max_{i=1,2,3}|P_i^{(1)} - P_i^{(2)}|. \quad (3.72)$$

Hence a small change in the given data produces a correspondingly small change in the solution.
Therefore the solution of problem 3.1 is stable in the interval $(0, T_1)$ as the real functions P_i, $(i = 1, 2, 3)$ are varied. ♦

3.6.2 Initial condition stability.

We recall the initial condition (3.10)

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}),\ \ \mathbf{x} \in \Omega.$$

where the given initial velocity field $\mathbf{u}_0$ is divergence free. In this subsection, we aim at showing that there exists a time interval where a small change in the initial data produces a correspondingly small change in the solution.

Theorem 3.8
There exists a time interval where the solution of problem 3.1 depends continuously on the prescribed data as the initial condition $\boldsymbol{u}_0(\boldsymbol{x})$ is varied.

Proof.
Denote by $\mathbf{u}^{(1)}$ the solution associated with the initial data $\mathbf{u}_0^{(1)}$ and, correspondingly, by $\mathbf{u}^{(2)}$ the solution associated with the initial data $\mathbf{u}_0^{(2)}$.
Set $\mathbf{w} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$.
Consider (3.39) with $\boldsymbol{\varphi} = \mathbf{w}$. We have that

$$(\frac{\partial \mathbf{u}^{(1)}}{\partial t}, \mathbf{w}) + b(\mathbf{u}^{(1)}, \mathbf{u}^{(1)}, \mathbf{w}) + a(\mathbf{u}^{(1)}, \mathbf{w}) = c(t, \mathbf{w}),$$

and

$$(\frac{\partial \mathbf{u}^{(2)}}{\partial t}, \mathbf{w}) + b(\mathbf{u}^{(2)}, \mathbf{u}^{(2)}, \mathbf{w}) + a(\mathbf{u}^{(2)}, \mathbf{w}) = c(t, \mathbf{w}).$$

By substraction, we obtain that

$$(\frac{\partial \mathbf{w}}{\partial t}, \mathbf{w}) + a(\mathbf{w}, \mathbf{w}) = b(\mathbf{u}^{(2)}, \mathbf{u}^{(2)}, \mathbf{w}) - b(\mathbf{u}^{(1)}, \mathbf{u}^{(1)}, \mathbf{w}), \tag{3.73}$$

and this gives us

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \mu\|\nabla \mathbf{w}\|^2 + \sum_{i=1}^{3} K_i \int_{\Gamma i} |\mathbf{w} \cdot \mathbf{n}|^2 ds = -b(\mathbf{w}, \mathbf{w}, \mathbf{w}) - b(\mathbf{u}^{(2)}, \mathbf{w}, \mathbf{w}) - b(\mathbf{w}, \mathbf{u}^{(2)}, \mathbf{w}). \tag{3.74}$$

We estimate the righthand side of this equality like we did in the case of uniqueness and we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \mu\|\nabla \mathbf{w}\|^2 + \sum_{i=1}^{3} K_i \int_{\Gamma i} |\mathbf{w} \cdot \mathbf{n}|^2 ds \ \ &\leq \ \ \frac{\mu}{2}\|\nabla \mathbf{w}\|^2 + \frac{1}{2\mu}(\sup_{\Omega} |\mathbf{u}^{(2)}|)^2 \|\mathbf{w}\|^2 \\ &+ \sup_{\Omega} |\nabla \mathbf{u}^{(2)}| \|\mathbf{w}\|^2 + c_6 \|\mathbf{w}\| \|\nabla \mathbf{w}\|^2. \end{aligned}$$

Thus,

$$\frac{d}{dt}\|\mathbf{w}\|^2 + (\mu - 2c_6\|\mathbf{w}\|)\|\nabla\mathbf{w}\|^2 \leq \left(\frac{1}{\mu}(\sup_{\Omega}|\mathbf{u}^{(2)}|)^2 + 2(\sup_{\Omega}|\nabla\mathbf{u}^{(2)}|)\right)\|\mathbf{w}\|^2. \tag{3.75}$$

Therefore, on any interval during which $\|\mathbf{w}\| \leq \frac{\mu}{4c_6}$, Poincaré's inequality (3.1)

$$\|\mathbf{w}\| \leq C_\Omega\|\nabla\mathbf{w}\|$$

gives that

$$\frac{d}{dt}\|\mathbf{w}\|^2 + \left(\frac{\mu}{2C_\Omega^2} - \frac{1}{\mu}(\sup_{\Omega}|\mathbf{u}^{(2)}|)^2 - 2(\sup_{\Omega}|\nabla\mathbf{u}^{(2)}|)\right)\|\mathbf{w}\|^2 \leq 0. \tag{3.76}$$

It follows that

$$\frac{d}{dt}\|\mathbf{w}\|^2 \leq -c_7\|\mathbf{w}\|^2,$$

where

$$c_7 = \sup_J\left(\frac{\mu}{2C_\Omega^2} - \frac{1}{\mu}(\sup_{\Omega}|\mathbf{u}^{(2)}|)^2 - 2(\sup_{\Omega}|\nabla\mathbf{u}^{(2)}|)\right),$$

and J is an interval of time during which $\|\mathbf{w}\| \leq \frac{\mu}{4c_6}$.
According to Gronwall's inequality (A.10), we deduce that in the interval J,

$$\|\mathbf{w}\| \leq c\max_J \mathbf{w}_0, \quad \text{where} \quad \mathbf{w}_0 = \mathbf{u}_0^{(1)} - \mathbf{u}_0^{(2)}. \tag{3.77}$$

Hence a small change in the given data produces a correspondingly small change in the solution.
Therefore, the solution of problem 3.1 is stable in the interval J as the initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \;\; \mathbf{x} \in \Omega$$

is varied. ♦

Chapter 4

Conclusion

This work has reviewed the basic physiology of the cardiovascular system, and described the anatomy of blood and some cardiovascular equations. The main emphasis was placed on the blood circulation through the large arteries. The problem of modelling blood flow as well as some simplifying assumptions were discussed. We established the weak formulation of the considered boundary value problem for both stationary case and nonstationary case. Then we showed that the weak formulation did not depend on the blood pressure. Rather than giving serious thought to the artificial sections boundary conditions, in seeking a variational formulation, the test space was left free on these portions of the boundary. For this purpose, we introduced

$$\mathbf{V} \equiv \{\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega) : \boldsymbol{\varphi}|_{\Gamma_{\text{wall}}} = 0, \nabla \cdot \boldsymbol{\varphi} = 0\}$$

as the test space. The Stoke's operator $\widetilde{\Delta}$ was defined, we made use of the fact that its inverse $\widetilde{\Delta}^{-1}$ is self adjoint and possesses a sequence of eigenfunctions which is orthogonal in the space of test functions $\mathbf{V}$ and orthonormal in $\mathbf{L}^2(\Omega)$, to derive some helpful estimates. We employed galerkin's method by first solving the problem in some finite dimensional spaces, then we were able to construct a sequence of finite dimensional solutions, and showed that this sequence weakly converged by considering a compactness argument. In this regard, we made use of Brouwer's fixed point theorem in the case of the stationary problem to prove the existence of Galerkin approximations. When considering nonstationary problem, we made use of an existence theorem from the theory of ordinary differential equations. An energy estimate was derived as well, this helped us to prove the local existence of a weak solution. Later on, we proved that the weak limit of the Galerkin approximantions was indeed a weak solution of the initial/boundary value problem.

We moved on to prove the uniqueness of a local solution and its continuous dependence upon the data. Gronwall's inequality was useful in the proof of the uniqueness and the stability of the results.

Blood is a very complex fluid in its nature. It is not homogeneous and consists of plasma and corpuscules. Its flow properties are uniquely adapted to the architecture of the blood vessels. In the large arteries, whole blood can be approximated as a Newtonian fluid. In the future, we are going to consider the shear thinning viscoelastic nature of blood for more accuracy and this will lead us to the reformulation of the initial boundary value problem described in this dissertation.

Appendix A

appendix

In this section, we gather in one place the results that are continually employed throughout the dissertation.

A.1 Linear Functional analysis

A.1.1 Banach spaces.

Let X denote a real linear space.

Definition. A mapping $\| \ \| : X \to [0, \infty)$ is called a norm if
(i) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in X$.
(ii) $\|\lambda u\| = |\lambda| \|u\|$ for all $u \in X$, $\lambda \in R$.
(iii) $\|u\| = 0$ if and only if $u = 0$.

Inequality (i) is the triangle inequality.
Hereafter we assume that X is a normed linear space.

Definition. We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ converges to $u \in X$, written

$$u_k \to u,$$

if

$$\lim_{k\to\infty} \|u_k - u\| = 0.$$

Definitions.

A sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is called a Cauchy's sequence provided for each $\epsilon > 0$ there exists $N > 0$ such that

$$\|u_k - u_l\| < \epsilon \quad \text{for all } k, l \geq N.$$

X is complete if each cauchy sequence in X converges; that is, whenever $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence, there exists $u \in X$ such that $\{u_k\}_{k=1}^{\infty}$ converges to u.
A Banach space is a complete, normed linear space.

Definition. We say X is separable if X contains a countable dense subset.

Theorem A.1 (**Brouwer's fixed point theorem.**)
Let X be a finite dimensional linear space endowed with a norm. Assume

$$u\,:\,B(0,1)\,\rightarrow\,B(0,1) \tag{A.1}$$

is continuous, where $B(0,1)$ denotes the closed unit ball in X. Then u has a fixed point; that is, there exists a point $x \in B(0,1)$ with

$$u(x)\,=\,x.$$

Definition. Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, written

$$X \subset\subset Y,$$

provided

1. $\|x\|_Y \leq C\,\|x\|_X\;\;(x \in X)$ for some constant C, and
2. each bounded sequence in X is precompact in Y.

Theorem A.2 (**Rellich-Kondrachov compactness theorem**).
Assume U is a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Suppose $1 \leq p < n$. Then

$$W_p^1(U) \subset\subset L^q(U) \tag{A.2}$$

for each $1 \leq q < p^\star$, where $\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{n}$. As a result of that, we have

$$\|u\|_{L^q(U)} \leq C\|u\|_{W_p^1(U)}. \tag{A.3}$$

A.1.2 Hilbert spaces.

Let H be a real linear space.

Definition. A mapping $(\;,\;)\,:H \times H \rightarrow R$ is called an inner product if
(i) $(u,v) = (v,u)$ for all $u,v\,\in\,H$,
(ii) the mapping $u\,\mapsto\,(u,v)$ is linear for each $v\,\in\,H$,
(iii) $(u,u) \geq 0$ for all $u\,\in\,H$,
(iv) $(u,u) = 0$ if and only if $u = 0$.

Notation. If $(\;,\;)$ is an inner product, the associated *norm* is $\|u\| := (u,u)^{\frac{1}{2}} \qquad (u\,\in\,H)$.

Definition. A Hilbert space H is a Banach space endowed with an inner product which generates the norm.

Definitions. (i) Two elements $u, v \in H$ are orthogonal if $(u, v) = 0$.
(ii) A comptable basis $\{w_k\}_{k=1}^{\infty} \subset H$ is called orthonormal if
$(w_k, w_l) = 0 \quad (k, l = 1, ...; k \neq l)$ and $\|w_k\| = 1 \quad (k = 1, ...)$.

If $u \in H$ and $\{w_k\}_{k=1}^{\infty} \subset H$ is an orthonormal basis, we can write

$$u = \sum_{k=1}^{\infty} (u, w_k) w_k,$$

the series converging in H. In addition

$$\|u\|^2 = \sum_{k=1}^{\infty} (u, w_k)^2.$$

A.1.3 Bounded linear operators.

Linear operators on Banach spaces. Definitions. A mapping $A : X \to Y$ is a linear operator provided

$$A[\lambda u + \gamma v] = \lambda Au + \gamma Av$$

for all $u, v \in X, \ \lambda, \gamma \in R$.

The range of A is $R(A) := \{v \in Y \mid v = Au \text{ for some } u \in X\}$ and
the null space of A is $N(A) := \{u \in X \mid Au = 0\}$.

Definition. A linear operator $A : X \to Y$ is bounded if

$$\|A\| := \sup\{\|Au\|_Y \mid \|u\|_X \leq 1\} < \infty.$$

A bounded linear operator $A : X \to Y$ is continuous.

Definitions. A bounded linear operator $u^\star : X \to R$ is called a bounded linear functional on X.
We write $X^\star$ to denote the collection of all bounded linear functionals on X; $X^\star$ is the dual space of X.

Definitions. If $u \in X$, $u^\star \in X^\star$ we write

$$< u^\star, u >$$

to denote the real number $u^\star(u)$. The symbol $<,>$ denotes the pairing of $X^\star$ and X.
We define

$$\|u^\star\| := \sup\{< u^\star, u > \mid \|u\| \leq 1\}.$$

A Banach space is reflexive if $(X^\star)^\star = X$. More precisely, there exists $u \in X$ such that

$$< u^{\star\star}, u^\star > = < u^\star, u > \quad \text{for all } u^\star \in X^\star.$$

Linear operators on Hilbert spaces.

Now we let H be a real Hilbert space, with inner product (,).

Theorem A.3 (**Riesz Representation Theorem.**)
$H^\star$ can be canonically identified with H; more precisely, for each $u^\star \in H^\star$ there exists a unique element $u \in H$ such that

$$< u^\star, v >= (u, v) \quad \textit{for all } v \in H.$$

The mapping $u^\star \mapsto u$ is a linear isomorphism of $H^\star$ onto H.

Definitions. If $A : H \to H$ is a bounded, linear operator, its adjoint $A^\star : H \to H$ satisfies

$$(Au, v) = (u, A^\star v) \quad \text{for all}\, u, v \in H.$$

A is symmetric if $A^\star = A$.

A.1.4 Weak convergence.

Let X denotes a real Banach space.

Definition. We say a sequence $\{u_k\}_{k=1}^\infty$ converges weakly to $u \in X$, written

$$u_k \rightharpoonup u,$$

if

$$< u^\star, u_k > \to < u^\star, u >$$

for each bounded linear functional $u^\star \in X^\star$.

$u_k \to u$ implies that $u_k \rightharpoonup u$.
Any weakly convergent sequence is bounded. In addition, if $u_k \rightharpoonup u$, then

$$\|u\| \leq \lim_{k\to\infty} \inf \|u_k\|.$$

A.2 Fundamental inequalities

A.2.1 Cauchy's inequality with epsilon.

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \quad (a, b \in R, \ \epsilon > 0). \tag{A.4}$$

Proof.

$$0 \leq (\epsilon a - \frac{b}{2})^2 \leq \epsilon^2 a^2 - \epsilon ab + \frac{b^2}{4}.$$

Then

$$\epsilon ab \leq \epsilon^2 a^2 + \frac{b^2}{4}.$$

Hence

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

$\Diamond$

A.2.2 Young's inequality.

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b > 0). \tag{A.5}$$

Proof. The mapping $x \mapsto e^x$ is convex, and consequently

$$ab = exp(log\, a + log\, b) = exp(\frac{1}{p} log\, a^p + \frac{1}{q} log\, b^q) \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

$\Diamond$

A.2.3 Cauchy-Schwarz inequality.

Assume X is a real Banach space, denote by $(\ ,\ )_X$ and $\|\ \|_X$ the inner product and the norm defined on X respectively. Then

$$|(x, y)_X| \leq \|x\|_X \|y\|_X \quad (x, y \in X). \tag{A.6}$$

Proof. Let $\epsilon > 0$ and note

$$0 \leq \|x \pm \epsilon y\|_X^2 = \|x\|_X^2 (\pm) 2\epsilon (x, y)_X + \epsilon^2 \|y\|_X^2.$$

Consequently

$$\pm (x, y)_X \leq \frac{1}{2\epsilon} \|x\|_X^2 + \frac{\epsilon}{2} \|y\|_X^2.$$

Minimize the right hand side by setting $\epsilon = \frac{\|x\|_X}{\|y\|_X}$, provided $y \neq 0$. $\Diamond$

A.2.4 Hölder's inequality.

Assume $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(U)$, $v \in L^q(U)$, we have

$$\int_U |uv| dx \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}. \tag{A.7}$$

Proof.
By homogeneity, we may assume $\|u\|_{L^p(U)} = \|v\|_{L^q(U)} = 1$. Then Young's inequality implies that for $1 < p, q < \infty$

$$\int_U |uv| dx \leq \frac{1}{p} \int_U |u|^p dx + \frac{1}{q} \int_U |v|^q dx = 1 = \|u\|_{L^p(U)} \|v\|_{L^q(U)}.$$

General Hölder's inequality.
Let $1 \leq p_1, ..., p_n \leq \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$, and assume $u_k \in L^{p_k}(U)$ for $k = 1, ..., n$. Then

$$\int_U |u_1 \cdots u_n| \, dx \leq \prod_{k=1}^{n} \|u_k\|_{L^{p_k}(U)}. \tag{A.8}$$

Proof. Induction, using Hölder's inequality. ◊

A.2.5 Gronwall's inequality (differential form).

Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$ which satisfies for a.e t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \tag{A.9}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$\eta(t) \leq \exp\left(\int_0^T \phi(s) \, \mathrm{ds}\right) \left[\eta(0) + \int_0^T \psi(s) \, \mathrm{ds}\right] \tag{A.10}$$

for all $0 \leq t \leq T$.

In particular, if

$$\eta' \leq \phi\eta \text{ on } [0, T] \text{ and } \eta(0) = 0, \text{ then } \qquad \eta \equiv 0 \text{ on } [0, T]. \tag{A.11}$$

Proof. From $(A.9)$ we see

$$\begin{aligned} \frac{d}{\mathrm{ds}} \left(\eta(s) \exp(-\int_0^s \phi(r) dr)\right) &= \exp\left(-\int_0^s \phi(r) dr)(\eta'(s) - \phi(s)\eta(s)\right) \\ &\leq \exp\left(-\int_0^s \phi(r) dr\right) \psi(s) \end{aligned}$$

for a.e. $0 \leq s \leq T$. Consequently for each $0 \leq s \leq T$, we have

$$\begin{aligned} \eta(t)\exp\left(-\int_0^t \phi(r)dr\right) &= \eta(0) + \int_0^t \exp\left(-\int_0^s \phi(r)dr\right)\psi(s)ds \\ &= \eta(0) + \int_0^t \psi(s)\,\mathrm{ds} \end{aligned}$$

This implies inequality $(A.11)$. ◊

A.2.6 Trace theorem.

Assume U is a bounded domain of $\mathbb{R}^n$ and its boundary $\partial(U)$ is C^1.
Then there exists a bounded linear operator

$$T\,:\,W^{1,p}(U)\,\rightarrow\,L^p(\partial U)$$

such that

$$(i)\;\; T\,u\,=\,u|_{\partial(U)}\;\text{ if }\;u\,\in\,W^{1,p}(U)\bigcap C(\overline{U})$$

and

$$(ii)\quad \|T\,u\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}, \tag{A.12}$$

for each $u\,\in\,W^{1,p}(U)$, with the constant C depending only on P and U.

Definition. $T\,u$ is called the trace of u on ∂U.

Theorem A.4 (**Trace inequality for solenoidal functions.**)
Let $\boldsymbol{u}$ be a solenoidal function defined on U. The following inequality holds:

$$\int_{\partial U} (\boldsymbol{u}\cdot\boldsymbol{n})^2 ds \leq c\|\boldsymbol{u}\|^2. \;\; ([7],\ \textit{page 350}) \tag{A.13}$$

A.2.7 Poincaré's inequalities.

(i) Let U be a bounded, connected, open subset of R^n, with a C^1 boundary ∂U. Assume that $1{\leq}p{\leq}\infty$. then there exists a constant C, depending only on n, p and U, such that

$$\|u - (u)_U\|_{L^p(U)} \leq C\,\|Du\|_{L^p(U)}$$

where $(u)_U \;=\; \int_U ds$.
(ii) Assume U is a bounded, open subset of R^n. Suppose $u\,\in\,W_0^{1,p}(U)$ for some $1{\leq}p{\leq}\infty$.
Then we have the estimate

$$\|u\| \leq C_U\|\nabla u\|. \tag{A.14}$$

A.2.8 General Sobolev inequalities

Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W_p^k(U)$.

1. If
$$k < \frac{n}{p},$$
then $u \in L^q(U)$, where
$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$
We have in addition the estimate
$$\|u\|_{L^q(U)} \leq C\,\|u\|_{W_p^k(U)},$$
the constant C depending only on k, p, n and U.
In particular, if $p = 2$, $k = 1$ and $n = 3$,
$$\|u\|_{L^6(U)} \leq C\,\|u\|_{W_2^1(U)}, \tag{A.15}$$

2. We identify u with its continuous version.
If
$$k > \frac{n}{p},$$
then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\overline{U})$, where
$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, \text{ if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1, \text{ if } \frac{n}{p} \text{ is an integer.} \end{cases}$$
We have in addition the estimate
$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{U})} \leq C\,\|u\|_{W_p^k(U)},$$
the constant C depending only on k, p, n, γ and U.
In particular, if $n = 3$, $p = 2$ and $k = 2$, we have
$$H^2(U) \subset C(\overline{U}). \tag{A.16}$$

A.3 Measure theory

Lebesgue measure provides a way of describing the size or volume of certain subsets of R^n.

A.3.1 Measurable functions and integration.

Definition. Let $f : R^n \to R$. We say f is a measurable function if

$$f^{-1}(U) \in \mathbf{M}$$

for each subset $U \subset R$, where $\mathbf{M}$ is the set of Lebesgue measurable sets.

Note in particular that if f is continuous, then f is measurable. The sum and product of two measurable functions are measurable.

Theorem A.5 (**Egoroff's Theorem**.)
Let $\{f_k\}_{k=1}^{\infty}$, f be measurable functions and

$$f_k \to f \quad a.e. \;\; on \; A,$$

where $A \subset R^n$ is measurable, $|A| < \infty$. Then for each $\epsilon > 0$ there exists a measurable subset $E \subset A$ such that

$$(i) \quad |A - E| \leq \epsilon$$

and

$$(ii) \quad f_k \to f \quad uniformly \; on \, E.$$

Now if f is nonnegative, measurable function, it is possible, by an approximation of f with simple functions, to define the Lebesgue integral

$$\int_{R^n} f \, dx.$$

This agree with the usual integral if f is continuous or Riemann integrable. If f is measurable, but is not necessary nonnegative, we define

$$\int_{R^n} f \, dx \;=\; \int_{R^n} f^+ \, dx - \int_{R^n} f^- \, dx,$$

provided at list one of the terms on the right hand side is finite. In this case, we say f is integrable.

Definition.

A measurable function f is summable if

$$\int_{R^n} |f| \, dx \;<\; \infty.$$

A.3.2 Banach space-valued functions.

We extend the notions of measurability, integrability to mappings

$$\mathbf{f}\,:\;[0,T]\;\to\;X$$

where $T>0$ and X is a real Banach space, with norm $\|\;\|$.

Definition. A function $\mathbf{f}\,:\;[0,T]\;\to\;X$ is weakly measurable if for each $u^\star\in X^\star$, the mapping

$$t\;\mapsto < u^\star\,,\,\mathbf{f}(t) >$$

is Lebesgue measurable.

Theorem A.6 (**Bochner.**)
A strongly measurable function $\boldsymbol{f}\,:\;[0,T]\;\to\;X$ is summable if and only if $t\to\|\boldsymbol{f}(t)\|$ is summable. In this case

$$\|\int_0^T \boldsymbol{f}(t)\,dt\|\leq\int_0^T\|\boldsymbol{f}(t)\|dt,$$

and

$$< u^\star,\int_0^T \boldsymbol{f}(t)\,dt >=\int_0^T\; < u^\star,\boldsymbol{f}(t)>\;dt$$

for each $u^\star\in X^\star$.

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