Cardiovascular Flow in Cylindrical Domain

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Abstract

This project will briefly introduce the study of cardiovascular fluids, and discuss current analytical and numerical solutions. The Hagen-Poiseulle profile will be discussed as an example of the difficulties in constructing analytical solutions to hemodynamical problems. Also, the Arbitrary Lagrangian-Eulerian numerical technique will be presented to demonstrate the value of numerical solutions to PDEs.

1 Analytical Problem Formulation

Variables

$$\begin{aligned} u = u(y,z) &\equiv \text{ Blood velocity} \\ p = p(y,z) &\equiv \text{ Blood pressure} \\ \rho &\equiv \text{ Blood density} \\ \mu &\equiv \text{ Blood viscocity} \\ a &\equiv \text{ radius} \end{aligned}$$

Assumptions

- Incompressible flow: No divergence
- Laminar flow: No turbulence
- Newtonian flow: Viscous stresses proportional to the rates of change of the fluid's velocity vector
- No-slip condition: Fluid has zero velocity relative to a solid boundary (viscous fluid)
- Horizontal tube: No gravity affecting the system
- Symmetric flow: u becomes a function of radius exclusively
- Steady-state flow: Properties of the fluid do not change with time

Here are the incompressible Navier-Stokes equations for blood flow in 3 dimensions

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\Delta u \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\Delta v \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\Delta w \tag{3}$$

over the circular cylindrical domain as expressed in the figure below from Y.C. Fung [6].

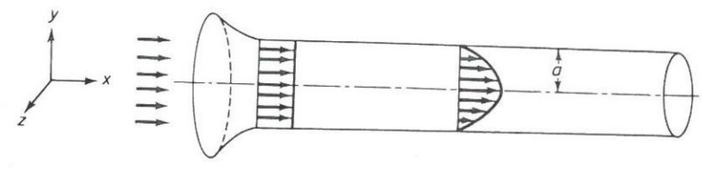

FIGURE 3.2:2. Laminar flow in a circular cylindrical tube.

The parabolic velocity profile of the Hagen-Poiseulle flow $u(r)$ as well as the flow rate Q through a cross-section of the cylindrical domain above are presented below:

$$u(r) = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[a^2 - r^2\right] \tag{4}$$

$$Q = -\frac{\pi a^4}{8\mu}\frac{\partial p}{\partial x} \tag{5}$$

given the boundary conditions $u = 0$ at $r = a$, and $\partial u/\partial r = 0$ at $r = 0$.

2 Numerical Method

The Arbitrary Lagrangian-Eulerian method is commonly used for computational blood flow with respect to tube and bifurcation models. This particular method is a combination of the Lagrangian and Eulerian descriptions of motion, where their respective advantages are combined while minimizing drawbacks [4].

Lagrangian: Each individual node of the computational mesh follows the associated material particle motion, which allows for an easy tracking of free surfaces and interfaces between materials. The method is unable to accurately follow large distortions of the computational domain [3].

Eulerian: The computational mesh is fixed and the continuum moves with respect to the grid. The method is capable of handling large distortions easily, but generally at the expense of precise interface definition and resolution of flow details [3].

Arbitrary Lagrangian-Eulerian Method: Nodes of the computational mesh may be moved with the continuum in a normal Lagrangian fashion, or be held in a fixed Eulerian manner. The ability to accommodate significant distortions of the computational mesh, while preserving the clear delineation of interfaces typical of a purely lagrangian approach allows for the modeling of an elastic boundary that does not depend solely on the fluid motion [3].

Using FEMLAB (numerical solutions), and ALE (computational mesh construction–boundary/domain) to model the domain given 893 CT scan images of an arbitrary patient's carotid artery, CAD models were generated for both the vessel itself (Figure 2), as well as with altered boundary conditions introducing simulations for vasoconstriction and vasodilation (Figure 3).

Assumptions

- Incompressible flow: No divergence
- Newtonian flow: Viscous stresses proportional to the rates of change of the fluid's velocity vector
- Axisymmetric flow: Cylindrical symmetry
- Blood properties:
  - $\mu = 0.005$ N·s/m^2
  - $\rho = 1060$ kg/m^3
- Arterial wall properties:

Mesh Displacement Field Variable	Values
Density	9.6 x10^2 kg/m^3
Coefficient of hyper-elasticity	6.2 x10^6 N/m^2
Bulk modulus	1.2 x10^8 N/m^2
Poisson Ratio	0.45
Elastic modulus	1.017

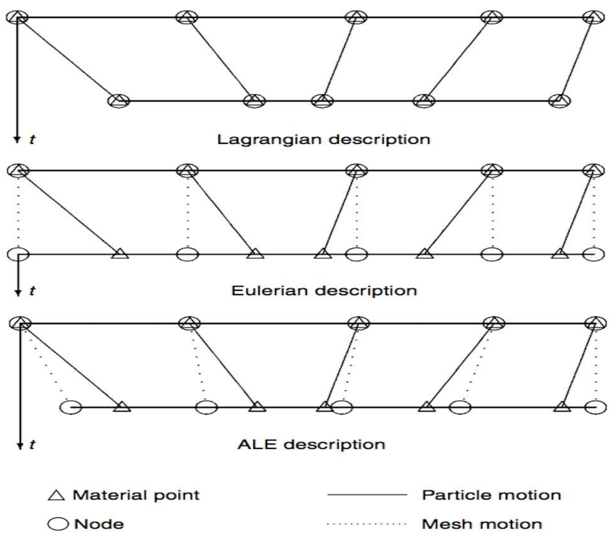

Figure 1: 1D example of Lagrangian, Eulerian and ALE mesh and particle motion [3].

The 2D incompressible Navier-Stokes equations below were used for cardiovascular flow [3],[4]. Note that time is considered in this example.

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\Delta u$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\Delta v$$

Also, Figure 1 is a visual representation of the comparison between the Lagrangian, Eulerian, and ALE descriptions of motion for particles within the domain, as well as the domain boundary itself.

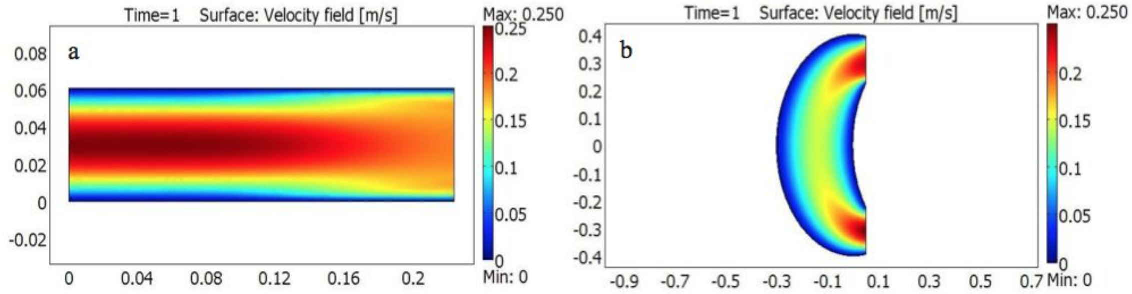

Figure 2: Patient X carotid artery blood velocity profile for laminar flow in (a) tubular domain and (b) curved domain [4].

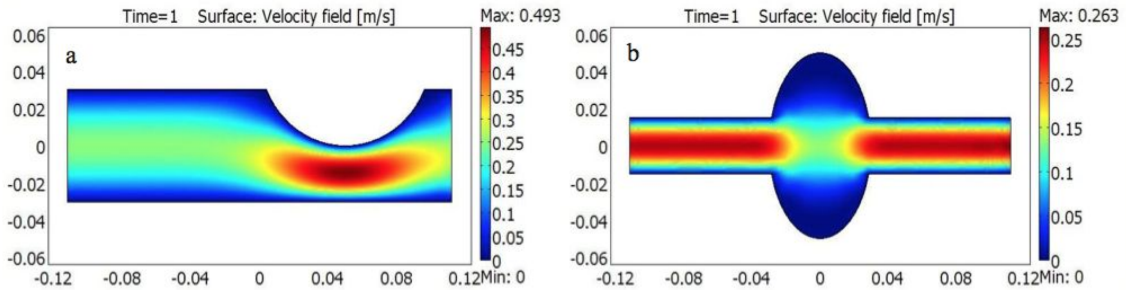

Figure 3: Blood velocity field with irregular flow pattern resulting from (a) vasoconstriction and (b) vasodilation [4].

It is important to note that the complexity of these boundary value problems typically pose difficulties in determining the convergence of solutions [4].

References

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