Definition. (Runge-Kutta Method)
A method for numerically integrating ordinary differential equations by using a trial step at the midpoint of an interval to cancel out the lower-order error terms.

Second-Order Formula

$$\begin{aligned} k_1 &= h \cdot f(x_n, y_n) \\ k_2 &= h \cdot f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \\ y_{n+1} &= y_n + k_2 + \mathcal{O}\left(h^3\right) \end{aligned}$$

Fourth-Order Formula

$$\begin{aligned} k_1 &= h \cdot f(x_n, y_n) \\ k_2 &= h \cdot f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \\ k_3 &= h \cdot f\left(x_n + h, y_n + \frac{1}{2}k_2\right) \\ k_4 &= h \cdot f\left(x_n + h, y_n + \frac{1}{2}k_3\right) \\ y_{n+1} &= y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + \mathcal{O}\left(h^5\right) \end{aligned}$$

This method is reasonably simple and robust and is a good general candidate for numerical solution of differential equations when combined with an intelligent adaptive step-size routine.

Let φ be the map from the material domain to the spatial domain.
let Φ be the map from the referential domain to the spatial domain.
let Ψ be the map from the referential domain to the material domain.
let Ω_M denote the material domain and M_i is a unit material.
let Ω_R denote the referential domain.
let Ω_S denote the spatial domain and S_i is a unit of space.

MAP OF DOMAINS

Thus,

$$\begin{aligned} \varphi : \Omega_M \times [t_0, t_{\text{final}}) &\longrightarrow \Omega_S \times [t_0, t_{\text{final}}) \\ (\overrightarrow{M}, t) &\longmapsto \varphi(\overrightarrow{M}, t) = (\vec{s}, t) \\ &\longmapsto \Phi(\Psi^{-1}(\overrightarrow{M}, t)) = (\vec{s}, t) \end{aligned}$$

which links $\overrightarrow{M}$ and $\vec{S}$ in time by the law of motion, namely

$$\vec{S} = \vec{S}(\overrightarrow{M}, t), t = t \tag{1}$$

1. The spatial coordinates $\overrightarrow{S}$ depend on both the material particle $\overrightarrow{M}$ and time t.

2. Physical time is measured by the same variable v in both material and spatial domains.

Since $\varphi = \Phi \circ \Psi^{-1}$, let us define the mappings Φ (map from reference to spatial) and Φ^{-1} (map from material to referential).

Thus,

1.

$$\begin{aligned} \Phi^{-1} : \Omega_M \times [t_0, t_{\text{final}} &\longrightarrow \Omega_R \times [t_0, t_{\text{final}}) \\ (\overrightarrow{M}, t) &\longmapsto \Phi(\overrightarrow{M}, t) = (\overrightarrow{R}, t) \end{aligned}$$

2.

$$\begin{aligned} \Phi^{-1} : \Omega_R \times [t_0, t_{\text{final}} &\longrightarrow \Omega_S \times [t_0, t_{\text{final}}) \\ (\overrightarrow{R}, t) &\longmapsto \Phi(\overrightarrow{R}, t) = (\overrightarrow{S}, t) \end{aligned}$$

Φ can be understood as the motion of the grid points in the spatial domain.
Ψ^{-1} can be understood as the motion of the material form the perspective of the material.