Cardiovascular Flow in Cylindrical Domain

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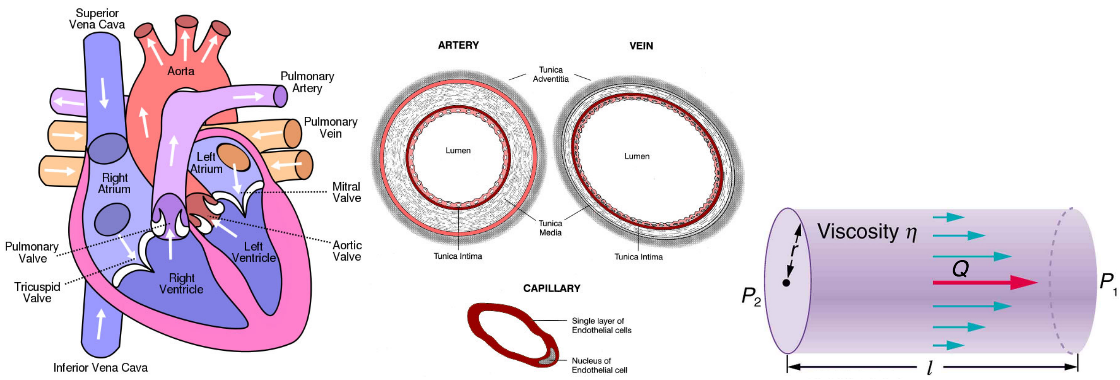
This project will briefly introduce the study of cardiovascular fluids, and discuss current analytical and numerical solutions. Assuming Newtonian flow will inherently limit this discussion to larger vessels, but will allow for a simpler problem construction using the Navier-Stokes equations.

1 Introduction

Cardiovascular disease (CVD) claims over 17 million lives every year, thus very quickly becoming one of the leading causes of death around the world. The difficulty with such an intricate collection of diseases is that detection is typically invasive and dangerous. Hence, having analytical or numerical blood flow models accurately describing the circulatory system would contribute to the creation of alternative methods of diagnosis, and treatment of CVDs [3].

It was not until the 19th century that J.P. Poiseuille produced the first simplified mathematical model of fluid flow in a cylindrical pipe (fluid parcel). Thereafter, contributors such as T. Young (elasticity of arterial tissues and blood pressure propagation), O. Frank (electric network analogy), and J. Womersley (periodic pressure gradients) have been associated with analytical hemodynamical problems [1]. A modified version of the incompressible Navier-Stokes equations in axisymmetric coordinates as demonstrated by Nithiarasu will be considered to expose the challenges of analytical solutions to blood flow problems [4]. One particular numerical model using CT scans and axial geometry shown by Jane et. al. in [3] will also be considered to express the value of numerical solutions to cardiovascular fluid PDEs.



(a) Heart Diagram (b) Artery Wall (c) Prospective Domain

2 Problem Formulation

Note that this is an analytical construction of a crude blood flow problem [4]. Referencing figure 2, the incompressible Navier-Stokes equations in axisymmetric coordinates, can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{\partial^2 u_r}{\partial z^2}\right] \tag{2}$$

$$\alpha^2\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_z)\right) + \frac{\partial^2 u_z}{\partial z^2}\right] \tag{3}$$

where the circumferential velocity is assumed to be zero (i.e. $u_\phi = 0$).

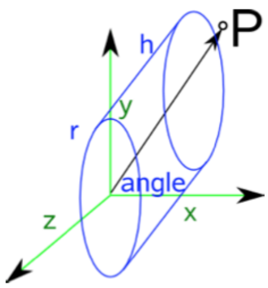


Figure 2: Cylindrical coordinates

Supposing that the fluid in question is Newtonian, then any derivative $\frac{\partial u}{\partial z}$ or u_r will also be zero.

As such, the Navier-Stokes equations are reduced as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\cancelto{0}{u_r}) + \cancelto{0}{\frac{\partial u_z}{\partial z}} = 0 \tag{4}$$

$$\cancelto{0}{\frac{\partial u_r}{\partial t}} + \frac{\partial u_r}{\partial r}\cancelto{0}{u_r} + u_z\cancelto{0}{\frac{\partial u_r}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cancelto{0}{u_r})\right) + \cancelto{0}{\frac{\partial^2 u_r}{\partial z^2}}\right] \tag{5}$$

$$\alpha^2\frac{\partial u_z}{\partial t} + \frac{\partial u_z}{\partial r}\cancelto{0}{u_r} + u_z\cancelto{0}{\frac{\partial u_z}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_z)\right) + \cancelto{0}{\frac{\partial^2 u_z}{\partial z^2}}\right] \tag{6}$$

Thus, equations 4,5, and 6 are reduced to

$$\alpha^2\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{7}$$

assuming that the pressure gradient in the x-direction remains zero, and where $\alpha = R\sqrt{\frac{\omega}{\nu}}$ is the Womersley number). If focusing on vessels such as veins, or arterioles and capillaries with a very low Womersley number ($\alpha << 1$), then the first term in (7) goes to zero.

$$\alpha^2 \cancelto{0}{\frac{\partial u_z}{\partial t}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{8}$$

Given $\nu = \mu/\rho$ as the dynamic viscocity, then equation (8) becomes (assuming blood density $\rho = 1$–equivalent to water)

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{9}$$

$$\frac{r}{\mu}\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{10}$$

In order to solve for u_z and eventually the flow rate Q we shall integrate equation (10) twice with respect to r. Note that pressure p isn't a function of r. Also, integrating twice we will generate a linear term as well as a constant term which will be added on at the end.

$$\frac{r}{\mu}\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{11}$$

$$\int \frac{r}{\mu}\frac{\partial p}{\partial z}\,dr = \int \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\,dr \tag{12}$$

$$\frac{r^2}{2\mu}\frac{\partial p}{\partial z} = r\frac{\partial u_z}{\partial r} \tag{13}$$

$$\frac{r}{2\mu}\frac{\partial p}{\partial z} = \frac{\partial u_z}{\partial r} \tag{14}$$

$$\int \frac{r}{2\mu}\frac{\partial p}{\partial z}\,dr = \int \frac{\partial u_z}{\partial r}\,dr \tag{15}$$

$$\frac{r^2}{4\mu}\frac{\partial p}{\partial z} = u_z \tag{16}$$

Introducing the linear and constant terms generated from the constants of integration, we have

$$u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + ar + b \tag{17}$$

Note that we add on the linear and constant terms at the end to avoid a blow-up term. Now, to solve for constants a and b, the chosen boundary conditions must be applied before calculating the flow rate Q through a slice of the cylinder.

With the assumptions that $\partial u_z/\partial r = 0$ at $r = 0$ and $r = R$, $u_z = 0$ we find that $a = 0$ and $b = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}$.

To find a,

$$u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + ar + b \tag{18}$$

$$0 = \partial u_z/\partial r = \underbrace{\frac{r}{2\mu}\frac{\partial p}{\partial z}} + a \tag{19}$$

$$\text{at } r = 0 \rightarrow a = 0 \tag{20}$$

To find b,

$$0 = u_z = \frac{R^2}{4\mu}\frac{\partial p}{\partial z} + 0 \cdot r + b \tag{21}$$

$$b = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z} \tag{22}$$

After substituting a and b back into u_z we obtain

$$u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + ar + b \tag{23}$$

$$u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} - \frac{R^2}{4\mu}\frac{\partial p}{\partial z} \tag{24}$$

$$u_z = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}\left[1 - \frac{r^2}{R^2}\right] \tag{25}$$

To find the flow rate Q through a cross-section of the cylinder (circle of radius R) in Figure 2, we have

$$u_z = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}\left[1 - \frac{r^2}{R^2}\right] \tag{26}$$

$$\text{why this is the flux?} \quad \rightarrow Q = \int_0^{2\pi}\int_0^R \frac{r^2}{4\mu}\frac{\partial p}{\partial z} - \frac{R^2}{4\mu}\frac{\partial p}{\partial z}\, r dr d\theta \tag{27}$$

$$= \int_0^{2\pi}\int_0^R \frac{r^3}{4\mu}\frac{\partial p}{\partial z} - \frac{R^2 r}{4\mu}\frac{\partial p}{\partial z}\, dr d\theta \tag{28}$$

$$= \int_0^{2\pi} \frac{r^4}{16\mu}\frac{\partial p}{\partial z} - \frac{R^2 r^2}{8\mu}\frac{\partial p}{\partial z}\Big|_0^R d\theta \tag{29}$$

$$= \int_0^{2\pi} \frac{R^4}{16\mu}\frac{\partial p}{\partial z} - \frac{R^4}{8\mu}\frac{\partial p}{\partial z}\, d\theta \tag{30}$$

$$= \frac{\theta R^4}{16\mu}\frac{\partial p}{\partial z} - \frac{\theta R^4}{8\mu}\frac{\partial p}{\partial z}\Big|_0^{2\pi} \tag{31}$$

$$= \frac{2\pi R^4}{16\mu}\frac{\partial p}{\partial z} - \frac{2\pi R^4}{8\mu}\frac{\partial p}{\partial z} \tag{32}$$

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z} \tag{33}$$

References

[1] Alfio Quarteroni, *Modeling the Cardiovascular System–A Mathematical Adventure: Part I.* 2001: SIAM News 34.5 (2001): 1-3.

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[3] Jhalique Jane R. Fojas, Rizalinda L. De Leon, *Carotid Artery Modeling Using the Navier-Stokes Equations for an Incompressible, Newtonian and Axisymmetric Flow.* 2013: Asia-Pacific Chemical, Biological & Environmental Engineering Society.

[4] Perumal Nithiarasu *Biofluid Dynamics.* 2008: Swansea University, United Kingdom.

[5] Suares Clovis Oukouomi Noutchie, *Flow of a Newtonian Fluid: The Case of Blood in Large Arteries.* 2005 (Unpublished (?) masters dissertation): University of South Africa, South Africa.