Siu's Lectures Notes in

Complex Analytic Geometry

(A private collection)

Contents

CHAPTER 1. CONNECTIONS, THE KÄHLER CONDITION, AND CURVATURE

(1.1) *Connections in Riemannian manifolds.*

First let us start with the case of a smooth manifold M of real dimension m. Let us look at a tangent vector X of M at the point P. It means the following. We take a curve $\gamma : (-1, 1) \to M$ with $\gamma(0) = P$. In local coordinates γ is given by $x^i = \gamma^i(t)$. The tangent vector X is given by a column vector ξ whose components ξ^i are $\frac{d\gamma^i}{dt}(0)$. If we use another local coordinate system y^i, then the tangent vector X is given by a column vector η with components η^i. By the chain rule the two column vectors ξ and η are related by $\eta^i = \xi^j \frac{\partial y^i}{\partial x^j}(P)$. Here the summation convention of summing over repeated indices is used. So we can define a tangent vector X of M at P as given by a column vector with respect to a local coordinate system and these column vectors for different coordinate systems transform according the rule $\eta^i = \xi^j \frac{\partial y^i}{\partial x^j}(P)$. Formally the transformation rule can be described by $\eta^i \frac{\partial}{\partial y^i} = \xi^i \frac{\partial}{\partial x^i}$. Another way of looking at the formal expression $\xi^i \frac{\partial}{\partial x^i}$ is that one knows a tangent vector if and only if one knows how to take partial derivative of any function in the direction of the tangent vector, because the components of the tangent vector are precisely the partial derivatives of the coordinate functions along the tangent vector. The formal expression $\xi^i \frac{\partial}{\partial x^i}$ is simply the partial differential operator in the direction of the tangent vector.

The set of tangent vectors form a bundle over M in the following sense. Suppose M is covered by coordinate charts U_α with coordinate x^i_α. Through the coordinate chart $\{U_\alpha, x^i_\alpha\}$ we identify all tangent vectors at points of U_α by column vectors ξ. So the totality of all tangent vectors at points of U_α is given by $U_\alpha \times \mathbf{R}^m$. So the totality of all tangent vectors at points of M is obtained by taking the disjoint union of $U_\alpha \times \mathbf{R}^m$ for all α and then identify $(P, \xi_\alpha) \in U_\alpha \times \mathbf{R}^m$ with $(P, \xi_\beta) \in U_\beta \times \mathbf{R}^m$ by $\xi^i_\alpha = \xi^j_\beta \frac{\partial x^i_\alpha}{\partial x^j_\beta}$, where ξ^i_α is the i^{th} component of ξ_α. So we have a bundle with transition functions $\left(\frac{\partial x^i_\alpha}{\partial x^j_\beta}\right)$. This is the tangent bundle and we denote it by T_M.

Suppose now we have a tangent vector field $\xi^i(x) \frac{\partial}{\partial x^i}$ on some open subset U of M. That is, we have a section of the tangent bundle T_M over U. We would like to be able to differentiate the tangent vector field $\xi^i(x) \frac{\partial}{\partial x^i}$ along some given direction and get a tangent vector field. This is a question of how to differentiate the section of a vector bundle and get a section as a result.

We would like to tackle this question in the general setting. Suppose we have a vector bundle V of rank r with transition functions $g_{\alpha\beta}$ with respect to a covering $\{U_\alpha\}$. A section s over an open subset U is given by $\{s_\alpha\}$ so that s_α is an r-tuple of functions over $U \cap U_\alpha$ and $s_\alpha = g_{\alpha\beta} s_\beta$. Here no summation is used and s_α, s_β are column vectors and $g_{\alpha\beta}$ is an $r \times r$ matrix. For our differentiation we do not specify the direction and consider the total derivative. From $s_\alpha = g_{\alpha\beta} s_\beta$ we have $ds_\alpha = g_{\alpha\beta} ds_\beta + dg_{\alpha\beta} s_\beta$. If the term $dg_{\alpha\beta} s_\beta$ is not present, we get a section of V from ds_α after we specify a direction. However, in general we have the term $dg_{\alpha\beta} s_\beta$ and it is not possible to use ds_α for differentiation and get a section. We have to use other ways to differentiate sections of a bundle.

We want the procedure of differentiation to obey the Leibniz formula for differentiating products. So to define differentiation it suffices to define differentiation for a local basis e_α $(1 \le \alpha \le r)$ of V. We denote the (total) differential operator by D. Then De_α is a V-valued 1-form and we can express it in terms of our local basis e_α $(1 \le \alpha \le r)$ and get $De_\alpha = \omega_\alpha^\beta e_\beta$, where ω_α^β is a 1-form. To be able to differentiate sections is equivalent to having an $r \times r$ matrix valued 1-form (ω_α^β). This matrix valued 1-form (ω_α^β) depends on the choice of e_α. Suppose we have another local basis $\tilde{e}_\alpha$ which is related to e_α by $\tilde{e}_\alpha = g_\alpha^\beta e_\beta$. Then

$$D\tilde{e}_\alpha = D(g_\alpha^\beta e_\beta) = g_\alpha^\beta De_\beta + dg_\alpha^\beta e_\beta = g_\alpha^\beta \omega_\beta^\gamma e_\gamma + dg_\alpha^\beta e_\beta.$$

In matrix notations we have $D\tilde{e} = g\,\omega\, e + dg\, e$, where $\tilde{e}$, e are column vectors with components $\tilde{e}_\alpha$, e_α and $g = (g_\alpha^\beta)$ and $\omega = (\omega_\alpha^\beta)$. Thus $D\tilde{e} = (g\,\omega\, g^{-1} + dg\, g^{-1})\tilde{e}$ and $\tilde{\omega} = g\,\omega\, g^{-1} + dg\, g^{-1}$. So to be able to differentiate sections is equivalent to having an $r \times r$ matrix valued 1-form ω for any local basis e so that the transformation rule $\tilde{\omega} = g\,\omega\, g^{-1} + dg\, g^{-1}$ is satisfied. In that case we say that we have a *connection*.

When we have a connection for V, we have an induced connection on the dual bundle V^* of V. It is given as follows. Suppose s^* is a local section of V^* and s is a local section of V. Then Ds^* is defined so that $d\,\langle s^*, s\rangle = \langle Ds^*, s\rangle + \langle s^*, Ds\rangle$ *is* satisfied, where $\langle s^*, s\rangle$ means the evaluation of s^* at s. If e_*^α is the dual basis of e_α, then the connection $\omega_\alpha^{*\beta}$ is simply equal to $-\omega_\alpha^\beta$.

Suppose we have an inner product $\langle\cdot,\cdot\rangle$ along each fiber of V. We say that a connection ω is compatible with the metric if for any section s over a curve

γ in M with $Ds = 0$ along γ the length s is constant along γ. This condition is equivalent to $d\langle s,t\rangle = \langle Ds,t\rangle + \langle s,Dt\rangle$. Clearly this equation implies that if $Ds = 0$ along γ, then $\langle s,s\rangle$ is constant along γ. Conversely, for a point P and a tangent vector X at P we can find a curve γ passing through P with tangent vector X at P. We choose a frame field e_α $(1 \le \alpha \le r)$ along γ so that the frame field is orthonormal at P and each e_α has zero covariant derivative along γ. Then the frame field e_α $(1 \le \alpha \le r)$ is orthonormal at each point of γ. Write $s = s^\alpha e_\alpha$ and $t = t^\alpha e_\alpha$. Then evaluated at X

$$d\langle s,t\rangle = d\sum_\alpha s^\alpha t^\alpha = \sum_\alpha ds^\alpha t^\alpha + \sum_\alpha s^\alpha dt^\alpha = \langle Ds,t\rangle + \langle s,Dt\rangle .$$

By applying $d\langle s,t\rangle = \langle Ds,t\rangle + \langle s,Dt\rangle$ to s,t belonging to a local orthonormal basis, we conclude that compatibility with the metric is equivalent to the connection ω for an orthonormal basis is skew-symmetric.

Now we come back to the tangent bundle T_M of M. An inner product along the fibers of T_M is simply a Riemannian metric. We introduce a connection so that we can differentiate sections of T_M and get sections again. Having a connection for T_M is equivalent to having a connection for the dual bundle T_M^* of T_M. A section of T_M^* is simply a 1-form. For a 1-form we have always the concept of the exterior differentiation. We do not need any connection for exterior differentiation. The result of an exterior differentiation of a 1-form s is a 2-form ds. It is different from Ds coming from a connection, because Ds is a section of $T_M^* \otimes T_M^*$ whereas ds is a section of $\wedge^2 T_M^*$. We would like to consider connections that relate Ds to ds in a natural way. A connection ω is said to be *torsion-free* if the skew-symmetrization of Ds is ds. Suppose e_α $(1 \le \alpha \le m)$ is a local frame for T_M and ω^α $(1 \le i \le m)$ is its dual frame. Let ω_α^β with $De_\alpha = \omega_\alpha^\beta e_\beta$ be the matrix valued 1-form of the connection. Then from definition of the torsion-free condition we see that the connection is torsion-free if and only if $d\omega^\alpha + \omega_\beta^\alpha \wedge \omega^\beta = 0$.

Fundamental Theorem of Riemannian Geometry. There exists a unique torsion-free connection that is compatible with a Riemannian metric.

Proof. Let the Riemannian metric be given by $g_{ij}dx^i dx^j$. We use dx^i as local basis for T_M^* and $\frac{\partial}{\partial x^i}$ as local basis for T_M. Write the connection ω for T_M as $\omega_i^j = \Gamma_{ik}^j dx^k$. The coefficients Γ_{ik}^j are known as the Christoffel symbols. So $D(dx^i) = -\omega_i^j \otimes dx^j = -\Gamma_{ik}^j dx^k \otimes dx^i$. The torsion-free condition is simply the symmetry of Γ_{ik}^j in i and k. Compatibility with the Riemannian metric means

$$
\begin{aligned}
dg_{ij} = d\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle &= \left\langle D\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, D\frac{\partial}{\partial x^j} \right\rangle \\
&= \omega_i^k \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right\rangle + \omega_j^k \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle \\
&= \omega_i^k g_{kj} + \omega_j^k g_{ik}.
\end{aligned}
$$

So

$$
\frac{\partial}{\partial x^\ell} g_{ij} = \Gamma_{i\ell}^k g_{kj} + \Gamma_{j\ell}^k g_{ik}.
$$

To simplify notations we let $\Gamma_{j,i\ell} = \Gamma_{i\ell}^k g_{kj}$ and let $\partial_\ell = \frac{\partial}{\partial x^\ell}$. The theorem is reduced to proving the existence and uniqueness of $\Gamma_{k,i\ell}$ symmetric in i and ℓ which satisfy $\partial_\ell g_{ij} = \Gamma_{j,i\ell} + \Gamma_{i,j\ell}$. This is a linear algebra problem. We have $\frac{1}{2}n^2(n+1)$ unknowns and as many equations. It is in general difficult to handle such a large system of linear equations. Unfortunately this large system can be decoupled into smaller sets of three equations in three unknowns. For a fixed triple i, ℓ, k, since $\Gamma_{k,i\ell}$ is symmetric in i and ℓ, by permuting i, ℓ, k we have only three unknowns $\Gamma_{k,i\ell}$, $\Gamma_{i,k\ell}$, and $\Gamma_{i\ell,k}$. Such permutations would generate from $\partial_\ell g_{ij} = \Gamma_{j,i\ell} + \Gamma_{i,j\ell}$ three equations. We can now solve uniquely our three unknowns from these three equations. The three equations are

$$
\begin{aligned}
\partial_\ell g_{ij} &= \Gamma_{j,i\ell} + \Gamma_{i,j\ell} \\
\partial_i g_{j\ell} &= \Gamma_{\ell,ji} + \Gamma_{j,\ell i} \\
\partial_j g_{\ell i} &= \Gamma_{i,\ell j} + \Gamma_{\ell,ij.}
\end{aligned}
$$

The usual way is to add up the three equations so that we know the sum of the three unknowns and then subtract from it the sum of two unknowns obtained from any of the three equations. This is equivalent to subtracting one equation from the sum of the other two. So we subtract the third equation from the sum of the first two equations and we get

$$
2\Gamma_{j,\ell i} = \partial_\ell g_{ij} + \partial_i g_{j\ell} - \partial_j g_{\ell i}.
$$

The unique connection is now given by

$$
\Gamma_{jk}^i = \frac{1}{2} g^{i\ell}(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}).
$$

This unique connection is called the Levi-Civita connection or the Riemannian connection.

We would like to introduce the invariant formulation for the torsion-free condition. In the literature the standard notation for the covariant differential operator is ∇ instead of D. We will interchangeably use both ∇ and D to denote the covariant differential operator. The torsion-free condition given by the symmetry of the Christoffel symbols in the two covariant indices when expressed in terms of local coordinates. The Christoffel symbols are given by $\nabla(\frac{\partial}{\partial x^i}) = \omega_i^j \frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j dx^k$. In other words, $\nabla_k(\frac{\partial}{\partial x^i}) = \Gamma_{ik}^j \frac{\partial}{\partial x^j}$. The torsion-free condition is that $\nabla_k(\frac{\partial}{\partial x^i})$ is symmetric in i and k, *i.e.* $\nabla_X Y$ is symmetric in X and Y when X and Y equal $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial x^i}$. For general vector fields X and Y we do not have $\nabla_X Y - \nabla_Y X$ from the torsion-free condition, because for smooth functions φ and ψ

$$\nabla_{\varphi X}(\psi Y) - \nabla_{\psi Y}(\varphi X) = \varphi\psi(\nabla_X Y - \nabla_Y X) + \varphi(X\psi)Y - \psi(Y\varphi)X$$

and we cannot expect to get the general case by multiplying the special case $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ by smooth functions and then summing up. The trouble is the discrepancy terms $\varphi(X\psi)Y - \psi(Y\varphi)X$. The Lie bracket $[X, Y]$ yields the same discrepancy terms when X and Y are respectively multiplied by smooth functions φ and ψ, *viz.*

$$[\varphi X, \psi Y] = \varphi\psi[X, Y] + \varphi(X\psi)Y - \psi(Y\varphi)X.$$

So we conclude that the torsion-free condition is equivalent to the vanishing of $\nabla_X Y - \nabla_Y X - [X, Y]$ for any pair of tangent vectors X and Y. Moreover, we have seen that

$$\nabla_{\varphi X}(\psi Y) - \nabla_{\psi Y}(\varphi X) - [\varphi X, \psi Y] = \varphi\psi(\nabla_X Y - \nabla_Y X - [X, Y]).$$

When the torsion-free condition is not satisfied, we define $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. Then $T(\varphi X, \psi Y) = \varphi\psi T(X, Y)$ and T is a tensor and is called the *torsion tensor*. In local coordinates write $T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = T_{ij}^k \frac{\partial}{\partial x^k}$ with $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$.

We give here a geometric interpretation of torsion. Take a point P of M and two tangent vectors X and Y at P. We take a curve γ_X going through P whose tangent at P is X. We transport Y for a distance s along γ_X and get a tangent vector Y' of M at $\gamma_X(s)$. Take a curve γ_Y through $\gamma_X(s)$ whose tangent at $\gamma_X(s)$ is Y'. Let Q be the point $\gamma_Y(t)$. Now reverse the roles of X and Y. We take a curve σ_Y going through P whose tangent at P is Y. We transport X for a distance t along σ_Y and get a tangent vector X' of M at $\sigma_Y(t)$. Take a curve σ_X through $\sigma_Y(t)$ whose tangent at $\sigma_Y(t)$ is X'. Let R be the point $\sigma_X(s)$. The limit of the vector $\frac{1}{st}\vec{RQ}$ (in any coordinate system) as s and t approach zero defines a tangent vector of M which is independent of the choice of the curves by order considerations. We claim that this tangent vector is $T(X,Y)$. Let us now verify. We choose a local coordinate system x^i $(1 \leq i \leq m)$ at P so that P corresponds to the origin. Let X^i and Y^i be respectively the components of X and Y in terms of this local coordinate system. We choose as our curve γ_X the curve defined by the equations $\gamma_X^i(s) = X^i s$. The equation for parallel transport of Y along γ_X is $\frac{d}{ds}Y^i + \Gamma^i_{jk}Y^jX^i = 0$. Hence $Y'^i \approx Y^i - \Gamma^i_{jk}Y^jX^ks$. Here higher orders are ignored in the computation. We can take as γ_Y the curve defined by the equations $\gamma_Y^i(t) = X^is + Y'^it$. Hence $x^i(Q) \approx X^is + Y^it - \Gamma^i_{jk}Y^jX^kst$. Likewise we get the expression for R by changing the roles of X, s and Y, t. Thus $\frac{1}{st}(x^i(R) - x^i(Q)) \approx (\Gamma^i_{jk} - \Gamma^i_{jk})X^jY^k = T(X,Y)^i$.

(1.2) *Connections for Holomorphic Vector Bundles*

Before we discuss the case of a holomorphic vector bundle over a complex manifold, let us review the Cauchy-Riemann equations and fix the notations. Suppose we have a complex valued function $f(z)$ of a complex variable $z = x + iy$. The complex derivative $f'(z)$ of f is the limit of the difference quotient $\frac{f(z+\Delta z)-f(z)}{\Delta z}$ as Δz approaches 0 in $\mathbf{C}$. In particular, we get the same limit when Δz approaches 0 through Δx with $\Delta y = 0$ and also the same limit when Δz approaches 0 through $i\Delta y$ with $\Delta x = 0$. So we have $f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$. The equation $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$ is the Cauchy-Riemann equations. When we write $f = u + iv$ and separate the real and imaginary parts, we get back the usual Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$. We can rewrite $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$ in the form $\frac{\partial}{\partial \bar z}f = 0$, where $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. The reason for the factor $\frac{1}{2}$ is that we want the value of the 1-form $d\bar z$ evaluated at $\frac{\partial}{\partial \bar z}$ to be 1. Note that $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are tangent vectors and $\frac{\partial}{\partial \bar z}$ is a $\mathbf{C}$-linear combination of tangent vectors and therefore is an element of the complexification of the tangent space of $\mathbf{C}$. A holomorphic function can be defined either as a function expressible as a convergent power series of the complex variable or as a function satisfying the Cauchy-Riemann equations. In the higher dimensional case the situation is the same. A holomorphic function $f(z^1, \cdots, z^n)$

of complex variables $z^1, \cdots, z^n$ can be defined either as a function expressible as a convergent power series of the complex variables $z^1, \cdots, z^n$ or as a function satisfying the Cauchy-Riemann equations $\frac{\partial f}{\partial \overline{z^\nu}} = 0$ for $1 \leq \nu \leq n$, where $\frac{\partial}{\partial \overline{z^\nu}} = \frac{1}{2}\left(\frac{\partial}{\partial x^\nu} + \sqrt{-1}\frac{\partial}{\partial y^\nu}\right)$. When we allow tangent vectors to have complex coefficients, $\frac{\partial}{\partial z^\nu}, \frac{\partial}{\partial \overline{z^\nu}}$ $(1 \leq \nu \leq n)$ form a basis of the complexified tangent space $T_{\mathbf{C}^n} \otimes \mathbf{C}$ of $\mathbf{C}^n$ over $\mathbf{C}$ at every point of $\mathbf{C}^n$. Let $T^{1,0}$ be the $\mathbf{C}$-vector subspace of $T_{\mathbf{C}^n} \otimes \mathbf{C}$ spanned by $\frac{\partial}{\partial z^\nu}$ $(1 \leq \nu \leq n)$ and let $T^{0,1}$ be the $\mathbf{C}$-vector subspace of $T_{\mathbf{C}^n} \otimes \mathbf{C}$ spanned by $\frac{\partial}{\partial \overline{z^\nu}}$ $(1 \leq \nu \leq n)$. Suppose we have another set of local holomorphic coordinates $w^1, \cdots, w^n$. Because of the Cauchy-Riemann equations $\frac{\partial w^\mu}{\partial \overline{z^\nu}} = 0$ and $\frac{\partial z^\mu}{\partial \overline{w^\nu}} = 0$, the subspaces $T^{1,0}$ and $T^{0,1}$ are independent of the choice of local holomorphic coordinates and one can do similar constructions on complex manifolds. Elements of $T^{1,0}$ are called the (1,0)-directions and elements of $T^{0,1}$ are called the (0,1)-directions. A k-form on $\mathbf{C}^n$ with complex coefficients is an element of $\wedge^k(T_{\mathbf{C}^n} \otimes \mathbf{C})^*$, where $(T_{\mathbf{C}^n} \otimes \mathbf{C})^*$ means the dual space of $T_{\mathbf{C}^n} \otimes \mathbf{C}$ and $\wedge^k$ means taking the k^{th} exterior product. We have the direct sum decomposition

$$\bigwedge^k (T_{\mathbf{C}^n} \otimes \mathbf{C})^* = \oplus_{\nu=0}^k \left(\bigwedge^\nu (T^{1,0})^*\right) \otimes \left(\bigwedge^{k-\nu} (T^{0,1})^*\right).$$

An element of $(\wedge^p(T^{1,0})^*) \otimes (\wedge^q(T^{0,1})^*)$ is called a (p,q)-form. The notion of (p,q)-forms is independent of the choice of local holomorphic coordinates and can be carried
over to complex manifolds. A k-form is uniquely decomposable into a sum of $(\nu, k-\nu)$-forms with $0 \leq \nu \leq k$.

Let M be a complex manifold of complex dimension n and V be a holomorphic vector bundle of $\mathbf{C}$-rank r over M. Let H be a Hermitian metric along the fibers of V. With respect a local trivialization of V the Hermitian metric H is a positive Hermitian matrix $(H_{\alpha\bar{\beta}})_{1\leq\alpha,\beta\leq r}$. We are going to use the first index α as the row index and the second index $\bar{\beta}$ as the column index for the matrix $(H_{\alpha\bar{\beta}})_{1\leq\alpha,\beta\leq r}$.

We introduce the concept of a *complex metric connection* for the Hermitian vector bundle V with metric H. Suppose e_α, $1 \leq \alpha \leq r$, is a smooth local basis of V. Suppose we have a connection and use D to denote the operator of differentiating sections. The result De_α of applying D to e_α is a local E-valued 1-form on M. We express De_α in terms of the basis e_β, $1 \leq \beta \leq r$ and get $De_\alpha = \omega_\alpha^\beta e_\beta$, where ω_α^β is a local 1-form on M and the summation convention of summing over repeated indices is used. We use ω to denote the matrix $(\omega_\alpha^\beta)_{1\le\alpha,\beta\le r}$ and regard the subscript α as the row index and the superscript β as the column index of the matrix $(\omega_\alpha^\beta)_{1\le\alpha,\beta\le r}$.

Since the bundle V is holomorphic, it is possible to define partial differentiation in the (0,1) direction in a natural way, *viz.* the (0,1) derivative of a local holomorphic section of V is defined to be zero and the (0,1) derivative of any smooth section is defined by expressing it in terms of a local holomorphic basis and using the Leibniz rule of differentiating products. A connection is said to be *complex* if its partial differentiation in the (0,1) direction is the natural one just described. When the local basis e_α $(1 \le \alpha \le r)$ is holomorphic, a connection $(\omega_\alpha^\beta)_{1\le\alpha,\beta\le r}$ is complex if and only if the local 1-forms ω_α^β are all of type (1,0).

As with the real case we have the concept of compatibility with the metric. A connection is compatible with the metric if and only if

$$d\langle u, v\rangle = \langle Du, v\rangle + \langle u, Dv\rangle$$

for any local smooth section u and v of V, where $\langle\cdot,\cdot\rangle$ denotes the pointwise inner product defined by the metric H and the equation means that both sides give the same value when evaluated at any tangent vector of M. For $u = e_\alpha$ and $v = e_\beta$ the above equation reads

$$dH_{\alpha\bar\beta} = \omega_\alpha^\gamma H_{\gamma\bar\beta} + H_{\alpha\bar\gamma}\overline{\omega_\beta^\gamma}$$

as one can easily verify by evaluating both sides at a tangent vector of M. In matrix notations this means that

$$dH = \omega H + H\bar\omega^t,$$

where $\bar\omega$ is the complex conjugate of ω and the superscript t of $\bar\omega$ means the transpose of $\bar\omega$. By breaking down the equation into the (1,0) and (0,1) components, we get two equations $\partial H = \omega H$ and $\bar\partial H = H\bar\omega^t$, because ω, being a complex connection, is a matrix of (1,0) forms. The two equations $\partial H = \omega H$ and $\bar\partial H = H\bar\omega^t$ are equivalent, because H is a Hermitian matrix, as one can easily see by evaluating at a tangent vector of M and taking complex conjugates and transposes of the matrices.

Given any Hermitian metric H along the fibers of a holomorphic vector bundle V there exists one and only one complex metric connection ω. We verify the statement by taking a local holomorphic basis e_α $(1 \leq \alpha \leq r)$. The two conditions are: (i) A is a matrix of (1,0)-forms; and (ii) $\partial H = \omega$. Thus $\omega = (\partial H)H^{-1}$ is the unique complex metric connection.

There is another interpretation of this connection. A Hermitian metric defines an isomorphism Ψ from V to the complex conjugate of its dual $\overline{V^*}$. The transition functions of $\overline{V^*}$ are the complex conjugates of those of V^*. Since the transition functions of $\overline{V^*}$ are anti-holomorphic, there is a natural connection of for the (1,0) directions. Take a local smooth section s of V. We can define $D^{1,0}s$ as $\Psi^{-1}(D^{0,1}\Psi s)$. We claim that this is the complex metric connection. Take a local holomorphic basis e_α $(1 \leq \alpha \leq r)$ of V and write $s = s^\alpha e_\alpha$. Then $\Psi s = (H_{\alpha\bar\beta}s^\alpha)\overline{e_*^\beta}$, where $\overline{e_*^\beta}$ $(1 \leq \beta \leq r)$ is the local basis of $\overline{V^*}$ corresponding to the basis e_α $(1 \leq \alpha \leq r)$. It follows from $D^{1,0}\Psi s = (\partial(H_{\alpha\bar\beta}s^\alpha))\overline{e_*^\beta}$ that

$$\Psi^{-1}(D^{1,0}\Psi s) = H^{\bar\beta\gamma}(\partial(H_{\alpha\bar\beta}s^\alpha))e_\gamma = (\partial s^\gamma + H^{\bar\beta\gamma}(\partial H_{\alpha\bar\beta})s^\alpha)e_\gamma$$

and the connection is the complex metric connection.

(1.3) *The Kähler Condition.*

Suppose M is a complex manifold of complex dimension n. We can regard M as a real manifold of real dimension $m = 2n$. For local holomorphic coordinates $z^1, \cdots, z^n$ we write $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha = x^\alpha + \sqrt{-1}x^{n+\alpha}$. Then $\frac{\partial}{\partial x^\alpha} = 2\mathrm{Re}\,\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial y^\alpha} = 2\mathrm{Re}\,\sqrt{-1}\frac{\partial}{\partial z^\alpha}$. The complexification $T_M \otimes \mathbf{C}$ of T_M splits into a direct sum $T_M^{1,0} \oplus T_M^{0,1}$ with $T_M^{1,0}$ spanned by $\frac{\partial}{\partial z^\alpha}$ and $T_M^{0,1}$ spanned by $\frac{\partial}{\partial \overline{z^\alpha}}$. We have a natural map from T_M to $T_M^{1,0}$ which is the composite of the inclusion map $T_M \to T_M \otimes \mathbf{C}$ and the projection map $T_M \otimes \mathbf{C} = T_M^{1,0} \oplus T_M^{0,1} \to T_M^{1,0}$ onto the first summand. This natural map is an isomorphism and its inverse is the map $2\,\mathrm{Re}\,(\cdot)$ from $T_M^{1,0}$ to T_M. Through the $\mathbf{R}$-isomorphism $2\,\mathrm{Re}\,(\cdot)$ the operator of multiplication by $\sqrt{-1}$ in $T_M^{1,0}$ corresponds to some operator J in T_M. From $\frac{\partial}{\partial x^\alpha} = 2\,\mathrm{Re}\,\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial y^\alpha} = 2\,\mathrm{Re}\,\sqrt{-1}\frac{\partial}{\partial z^\alpha}$ it follows that the operator $J : T_M \to T_M$ is given by $J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha}$ and $J(\frac{\partial}{\partial y^\alpha}) = -\frac{\partial}{\partial x^\alpha}$. This means that if T_M is made into a $\mathbf{C}$-vector space by defining multiplication by $\sqrt{-1}$ as J, then $2\,\mathrm{Re}\,(\cdot)$ is a $\mathbf{C}$-isomorphism between the $\mathbf{C}$-vector spaces $T_M^{1,0}$ and T_M. We identify $T_M^{1,0}$ and T_M through this isomorphism. Note that

$$J(\frac{\partial}{\partial z^\alpha}) = J\left(\frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right)\right) = \frac{1}{2}(\frac{\partial}{\partial y^\alpha} + \sqrt{-1}\frac{\partial}{\partial x^\alpha}) = \sqrt{-1}\frac{\partial}{\partial z^\alpha}$$

and by analogous direct derivation or from the fact that J is real one has $J(\frac{\partial}{\partial \overline{z^\alpha}}) = -\sqrt{-1}\frac{\partial}{\partial \overline{z^\alpha}}$. This simply means that $T_M^{1,0}$ and $T_M^{0,1}$ are respectively the eigenspaces of J for the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$.

Since T_M with the operator J becomes a $\mathbf{C}$-vector space, when we have a metric on T_M to measure the length of the vectors in T_M we can ask whether this length function comes from a Hermitian inner product. One necessary condition is that the length of a vector should be invariant under J. It turns out that this condition is also sufficient. So we are going to use this condition as definition of a Hermitian metric and then verify that it agrees with the usual notion of a Hermitian metric. We say that a Riemannian metric g on T_M is *Hermitian* if $g(JX, JY) = g(X, Y)$ for any X, Y in T_M. Now we are going to reconcile this definition with the usual definition of a Hermitian metric represented by a Hermitian matrix. A Riemannian metric is a real-valued $\mathbf{R}$-bilinear function on $T_M \times T_M$. We can extend the it by $\mathbf{C}$-bilinearity to a $\mathbf{C}$-bilinear function on $(T_M \otimes \mathbf{C}) \times (T_M \otimes \mathbf{C})$. Then the Riemannian metric g is broken into four parts $g_{\alpha\beta}$, $g_{\alpha\bar\beta}$, $g_{\bar\alpha\beta}$, $g_{\beta\alpha\bar\beta}$. We investigate the condition $g(JX, JY) = g(X, Y)$ for each part. For the part $g_{\alpha\beta}$, since $J|T_M^{1,0}$ is the same as multiplication by $\sqrt{-1}$, we have $g(\sqrt{-1}X^{1,0}, \sqrt{-1}X^{1,0}) = g(X^{1,0}, X^{1,0})$ and by $\mathbf{C}$-bilinearity we have $g_{\alpha\beta} = 0$. From an analogous argument or from the fact that g is real, we have $g_{\bar\alpha\bar\beta} = 0$.

Since g is symmetric, we have $g_{\alpha\bar\beta} = g_{\bar\beta_\alpha}$. Moreover, since g is real, we must have $\overline{g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z^\beta}}\right)} = g\left(\frac{\partial}{\partial \overline{z^\alpha}}, \frac{\partial}{\partial z^\beta}\right)$, *i.e.* $\overline{g_{\alpha\bar\beta}} = g_{\bar\alpha\beta}$. Hence $\overline{g_{\alpha\bar\beta}} = g_{\beta\bar\alpha}$. and $g_{\alpha\bar\beta}$ is a Hermitian matrix. So the condition $g(JX, JY) = g(X, Y)$ simply means that $g_{\alpha\beta} = g_{\bar\alpha\bar\beta} = 0$ and $g_{\alpha\bar\beta}$ is Hermitian and $g_{\bar\beta\alpha}$ is the transpose of $g_{\alpha\bar\beta}$.

The correspondence $2\,\mathrm{Re}\,(\cdot)$ between $T_M^{1,0}$ and T_M transports the metric g of T_M to a metric on $T_M^{1,0}$ and we want to determine this metric on T_M. For this metric on $T_M^{1,0}$ the inner produce of $\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial z^\beta}$ is

$$g\left(\frac{\partial}{\partial z^\alpha} + \overline{\frac{\partial}{\partial z^\alpha}}, \frac{\partial}{\partial z^\beta} + \overline{\frac{\partial}{\partial z^\beta}}\right) = g_{\alpha\bar\beta} + g_{\bar\alpha\beta} = g_{\alpha\bar\beta} + g_{\beta\bar\alpha} = 2\,\mathrm{Re}\, g_{\alpha\bar\beta}.$$

So the Riemannian metric g on T_M corresponds to the metric $2\,\mathrm{Re}\,(g_{\alpha\bar\beta}dz^\alpha \overline{dz^\beta})$ on $T_M^{1,0}$. If we only look at the length of a vector, the metric on $T_M^{1,0}$ is simply the Hermitian metric $2g_{\alpha\bar\beta}dz^\alpha\overline{dz^\beta}$. Let us now explain the difference between $2\mathrm{Re}\ g_{\alpha\bar\beta}$ and $2g_{\alpha\bar\beta}$. A metric is used to measure the length of a vector. In that sense 2Re $g_{\alpha\bar{\beta}}$ and $2g_{\alpha\bar{\beta}}$ agree. The difference comes in when we want to express the square of the length of the vector as its inner product with itself. In the case of an **R**-vector space the inner product should be an **R**-bilinear functional. In the case of a **C**-vector space the inner product should be a functional which is **C**-linear in the first argument and **C**-conjugate linear in the second argument. When the lengths of the vectors are determined, in either case there is only one way of writing out such an inner product and it is done by polarization. When we regard $T_M^{1,0}$ as an **R**-vector space, we have the inner product 2Re $g_{\alpha\bar{\beta}}$ for the metric, whereas when we regard $T_M^{1,0}$ as a **C**-vector space, we have the Hermitian inner product $2g_{\alpha\bar{\beta}}$ for the metric.

If one wants to keep working only with real vectors all the time, one can verify directly in the following way that the condition $g(JX, JY) = g(X, Y)$ implies that the length function defined by g comes from a Hermitian inner product on T_M when T_M is made into a **C**-vector space by the operator J. The map $2\,\mathrm{Re}\,(\cdot)$ is a **C**-isomorphism between $T_M^{1,0}$ and T_M. The inverse of this map is $X \to \frac{1}{2}(X - iJX)$, because $J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha}$. Hence we expect the Hermitian inner product to be $2\ g\left(\frac{1}{2}(X - iJX), \frac{1}{2}(Y + iJY)\right)$ when g is extended by **C**-bilinearity. In other words, the Hermitian inner product is

$$\frac{1}{2}g(X,Y) + \frac{i}{2}g(X,JY) - \frac{i}{2}g(JX,Y) + \frac{1}{2}g(JX,JY)$$

which is the same as $g(X,Y) + i\ g(X,JY)$. This procedure is the same as getting the Hermitian inner product from the square of the length by polarization. We would like to note that $g(X,Y)$ is the real part of the Hermitian inner product and $g(X,JY)$ is its imaginary part. In terms of abstract algebra the above argument can be formulated in the following way. An **R**-bilinear inner product on a **C**-vector space is the real part of a Hermitian inner product if and only if it is invariant under the operation of multiplication by $\sqrt{-1}$.

Let us now consider the Levi-Civita connection expressed in terms of the complex coordinates indices α, $\bar{\alpha}$. We let i denote both α and $\bar{\alpha}$. We have

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\ell}(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}).$$

Suppose $i = \alpha$. Then ℓ must be of type $\bar{\lambda}$ to make a nonzero contribution. So

$$\Gamma^\alpha_{\beta\bar{\gamma}} = \frac{1}{2}g^{\alpha\bar{\lambda}}(\partial_\beta g_{\bar{\gamma}\bar{\lambda}} + \partial_{\bar{\gamma}} g_{\beta\bar{\lambda}} - \partial_{\bar{\lambda}} g_{\beta\bar{\gamma}}) = \frac{1}{2}g^{\alpha\bar{\lambda}}(\partial_{\bar{\gamma}} g_{\beta\bar{\lambda}} - \partial_{\bar{\lambda}} g_{\beta\bar{\gamma}})$$

and

$$\Gamma^{\alpha}_{\bar\beta\bar\gamma} = \frac{1}{2} g^{\alpha\bar\lambda}(\partial_{\bar\beta} g_{\bar\gamma\bar\lambda} + \partial_{\bar\gamma} g_{\bar\beta\bar\lambda} - \partial_{\bar\lambda} g_{\bar\beta\bar\gamma}) = 0.$$

These Christoffel symbols are for the connection of $T_M \otimes \mathbf{C}$ which is induced by the Levi-Civita connection of T_M. In general this connection of $T_M \otimes \mathbf{C}$ does not define a connection of $T_M^{1,0}$, because $\Gamma^{\bar\alpha}_{\beta\bar\gamma} = \overline{\Gamma^{\alpha}_{\gamma\bar\beta}}$ may not be zero. So when we differentiate a section of $T_M^{1,0}$ in the (0,1) direction, we may get a section of $T_M \otimes \mathbf{C}$ which is not entirely in $T_M^{1,0}$. We get a connection of $T_M^{1,0}$ if and only if $\Gamma^{\alpha}_{\beta\bar\gamma}$ vanishes.

We want to get a connection of $T_M^{1,0}$ from T_M through the $\mathbf{R}$-isomorphism $2\,\mathrm{Re}\,(\cdot)$. In the correspondence $2\,\mathrm{Re}\,(\cdot)$ between $T_M^{1,0}$ and T_M, when we take a section s of $T_M^{1,0}$ we should consider $D(s+\bar s)$ in T_M which must be of the form $t+\bar t$. Then Ds in $T_M^{1,0}$ is equal to t. So when we take $D\frac{\partial}{\partial z^\alpha}$, we should be looking at

$$\begin{aligned} D\left(\frac{\partial}{\partial z^\alpha} + \overline{\frac{\partial}{\partial z^\alpha}}\right) &= (\Gamma^i_{\alpha j} + \Gamma^i_{\bar\alpha j})\frac{\partial}{\partial z^i} \otimes dz^j \\ = (\Gamma^\beta_{\alpha\gamma} + \Gamma^\beta_{\bar\alpha\gamma})\frac{\partial}{\partial z^\beta} \otimes dz^\gamma &+ (\Gamma^\beta_{\alpha\bar\gamma} + \Gamma^\beta_{\bar\alpha\bar\gamma})\frac{\partial}{\partial z^\beta} \otimes dz^{\bar\gamma} \\ +(\Gamma^{\bar\beta}_{\alpha\gamma} + \Gamma^{\bar\beta}_{\bar\alpha\gamma})\frac{\partial}{\partial z^{\bar\beta}} \otimes dz^\gamma &+ (\Gamma^{\bar\beta}_{\alpha\bar\gamma} + \Gamma^{\bar\beta}_{\bar\alpha\bar\gamma})\frac{\partial}{\partial z^{\bar\beta}} \otimes dz^{\bar\gamma} \\ = (\Gamma^\beta_{\alpha\gamma} + \Gamma^\beta_{\bar\alpha\gamma})\frac{\partial}{\partial z^\beta} \otimes dz^\gamma &+ \Gamma^\beta_{\alpha\bar\gamma}\frac{\partial}{\partial z^\beta} \otimes dz^{\bar\gamma} \\ +\Gamma^{\bar\beta}_{\bar\alpha\gamma}\frac{\partial}{\partial z^{\bar\beta}} \otimes dz^\gamma &+ (\Gamma^{\bar\beta}_{\alpha\bar\gamma} + \Gamma^{\bar\beta}_{\bar\alpha\bar\gamma})\frac{\partial}{\partial z^{\bar\beta}} \otimes dz^{\bar\gamma}. \end{aligned}$$

Here i, j go through the range $1, \cdots, n, \bar 1, \cdots, \bar n$. Thus in $T_M^{1,0}$ we have

$$D\left(\frac{\partial}{\partial z^\alpha}\right) = (\Gamma^\beta_{\alpha\gamma} + \Gamma^\beta_{\bar\alpha\gamma})\frac{\partial}{\partial z^\beta} \otimes dz^\gamma + \Gamma^\beta_{\alpha\bar\gamma}\frac{\partial}{\partial z^\beta} \otimes dz^{\bar\gamma}.$$

So the connection is complex if and only if $\Gamma^\beta_{\alpha\bar\gamma} = 0$. The induced connection of $T_M^{1,0}$ is automatically compatible with the metric because the Levi-Civita connection of T_M is compatible with the metric and the length functions of the metrics of $T_M^{1,0}$ and T_M correspond through $2\,\mathrm{Re}\,(\cdot)$. Moreover, when the connection is complex, the connection is given by $\omega^\beta_\alpha = \Gamma^\beta_{\alpha\gamma} dz^\gamma$. The formula for the Christoffel symbol $\Gamma^\beta_{\alpha\gamma}$ of the Levi-Civita connection gives

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\bar\lambda}(\partial_\beta g_{\gamma\bar\lambda} + \partial_\gamma g_{\beta\gamma} - \partial_{\bar\lambda} g_{\beta\gamma}) = \frac{1}{2} g^{\alpha\bar\lambda}(\partial_\gamma g_{\beta\gamma} - \partial_{\bar\lambda} g_{\beta\gamma}) = g^{\alpha\bar\lambda}\partial_\beta g_{\gamma\bar\lambda}.$$

We note that this connection is the same as the complex metric connection for the holomorphic vector bundle $T_M^{1,0}$ with the Hermitian metric $g_{\alpha\bar{\beta}}$ (or $2g_{\alpha\bar{\beta}}$). The Hermitian metric is called Kähler if the Levi-Civita connection is complex. An equivalent condition is that $\partial_{\bar{\gamma}} g_{\beta\bar{\lambda}} = \partial_{\bar{\lambda}} g_{\beta\bar{\gamma}}$. This is equivalent to the condition that the 2-form $g_{\alpha\bar{\beta}} dz^\alpha \wedge \overline{dz^\beta}$ is closed. The 2-form $2\sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge \overline{dz^\beta}$ is equal to

$$
\begin{aligned}
&\sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \otimes \overline{dz^\beta} - \sqrt{-1} g_{\alpha\bar{\beta}} \overline{dz^\beta} \otimes dz^\alpha \\
= &-\left(g_{\alpha\bar{\beta}} dz^\alpha \otimes d\overline{\sqrt{-1} z^\beta} + g_{\alpha\bar{\beta}} \overline{dz^\beta} \otimes \sqrt{-1} dz^\alpha \right).
\end{aligned}
$$

Thus

$$
2\sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge \overline{dz^\beta}(X, Y) = -g(X, JY),
$$

because g is $2\mathrm{Re}(g_{\alpha\bar{\beta}} dz^\alpha \otimes \overline{dz^\beta}$ and $dz^\beta(JY) = \sqrt{-1} dz^\beta(Y)$. The function $g(X, JY)$ is skew-symmetric in X and Y because $g(JX, JY) = g(X, Y)$. So $g(X, JY)$ defines a 2-form on M. Consider the Hermitian form $g_{\alpha\bar{\beta}} dz^\alpha \otimes \overline{dz^\beta}$. Twice its imaginary part is $-\sqrt{-1}(g_{\alpha\bar{\beta}} dz^\alpha \otimes \overline{dz^\beta} - g_{\bar{\alpha}\beta} \overline{dz^\alpha} \otimes dz^\beta)$ which is $-2\sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge \overline{dz^\beta}$. Thus the 2-form $g(X, JY)$ is the imaginary part of $g_{\alpha\bar{\beta}} dz^\alpha \otimes \overline{dz^\beta}$ and the Riemannian metric is twice its real part. We have also seen this before. The 2-form is called the fundamental form of the Hermitian metric. So a Hermitian metric is Kähler if and only if its fundamental form is closed.

We now look at another characterization of the Kähler condition for a Hermitian metric. This characterization is that for any point P of M there exists a local holomorphic coordinate system z^α centered at P so that the first derivative of $g_{\alpha\bar{\beta}}$ vanishes at P. Clearly if such a coordinate system exists, then the fundamental form of the Hermitian metric is zero. Conversely suppose the metric is Kähler. Without loss of generality we can assume that $g_{\alpha\bar{\beta}}$ equals the Kronecker delta at the point P. We seek a new coordinate system w^α which is related to z^α by $z^\alpha = w^\alpha + c^\alpha_{\beta\gamma} w^\beta w^\gamma$ with $c^\alpha_{\beta\gamma}$ symmetric in β and γ. Let $h_{\alpha\bar{\beta}} dw^\alpha \overline{dw^\beta} = g_{\alpha\bar{\beta}} dz^\alpha \overline{dz^\beta}$. Since $c^\alpha_{\beta\gamma}$ symmetric in β and γ, it follows that

$$
h_{\lambda\bar{\mu}} = g_{\alpha\bar{\beta}} (\delta^\alpha_\lambda + 2c^\alpha_{\lambda\sigma} w^\sigma) \overline{(\delta^\beta_\mu + 2c^\beta_{\mu\tau} w^\tau)}
$$

and $dh_{\lambda\bar{\mu}}$ at P equals

$$
dg_{\lambda\bar{\mu}} + 2c^\mu_{\lambda\sigma} dw^\sigma + 2\overline{c^\lambda_{\mu\tau}} \overline{dw^\tau}.
$$

Thus we should choose $c^{\mu}_{\lambda\sigma} = -\frac{1}{2}\partial_\sigma g_{\lambda\bar{\mu}}$. This can be done with $c^{\mu}_{\lambda\sigma}$ symmetric in λ and σ if and only if $\partial_\sigma g_{\lambda\bar{\mu}}$ is symmetric in λ and μ.

We would like to remark why in the real case one can always find a local coordinate so that the first derivative of the coefficients of the metric tensor g_{ij} vanishes at a point. In the real case we want $dg_{ij} + 2c^{j}_{ik}dw^k + 2c^{i}_{jk}dw^k$ to vanish at P. In other words $c^{j}_{ik} + c^{i}_{jk} = -\frac{1}{2}\partial_k g_{ij}$ with c^{i}_{jk} symmetric in j and k. Recall the equations $\partial_k g_{ij} = \Gamma_{j,ik} + \Gamma_{i,jk}$ which were used to solve for the Christoffel symbols in terms of the Riemannian metric. All we have to do is to set $c^{i}_{jk} = -\frac{1}{2}\Gamma_{i,jk}$. This solution is the same as getting a coordinate system by integrating out along geodesics emanating from P.

Another characterization of the Kähler condition for a Hermitian metric is that all the Christoffel symbols Γ^{i}_{jk} for the Levi-Civita connection vanish except the types $\Gamma^{\alpha}_{\beta\gamma}$ and its complex conjugate $\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ We have seen that $\Gamma^{\alpha}_{\bar{\beta}\bar{\gamma}}$ always vanishes. We have the vanishing of $\Gamma^{\alpha}_{\beta\bar{\gamma}}$ if and only if the Hermitian metric is Kähler. We get our conclusion because Γ^{i}_{jk} is symmetric in j and k and it is real in the sense that $\overline{\Gamma^{i}_{jk}} = \Gamma^{\bar{i}}_{\bar{j}\bar{k}}$. Geometrically this characterization of a Kähler metric says that a Hermitian metric is Kähler if and only if the covariant derivative of a vector field of type (1,0) is still of type (1,0). In other words, types are preserved under parallel transport. This is equivalent to our earlier observation that $\Gamma^{\alpha}_{\beta\bar{\gamma}}$ vanishes if and only if the connection of $T_M \otimes \mathbf{C}$ induced from the Levi-Civita connection of T_M defines a connection of $T^{1,0}_M$ through the inclusion map $T^{1,0}_M \to T_M \otimes \mathbf{C}$.

(1.4) *Curvature.*

Unlike the case of partial differentiation for functions, in general for sections of a vector bundle partial differentiations for different directions do not commute. The failure of the commutativity is measured by the curvature of the connection. The commutativity of partial differentiation of a function in two different directions is equivalent to the vanishing of the composite of two exterior differentiation applied to functions. We do this for a real manifold M of real dimension m and a smooth vector bundle V of rank r over M which can be a complex vector bundle or a real one. Take a local smooth basis e_α $(1 \leq \alpha \leq r)$ of V. We apply D twice to $e_\wedge$ and get a local section $D\,D\,e_\alpha$ of $V \otimes T^*_M \otimes T^*_M$, where T^*_M is the dual of the tangent bundle T_M of M. By skew-symmetrizing $D\,D\,e_\alpha$ in the two arguments for the tangent vectors of M, we get a local V-valued 2-form $D_\wedge D\,e_\alpha$. We can express $D_\wedge D\,e_\alpha$ in terms of the connection ω and its exterior derivative $d\omega$ as follows. We use the column vector e with components e_α.

$$\begin{aligned} D_\wedge D\, e &= D_\wedge(\omega\ e) = d\omega\ e - \omega\wedge De \\ &= d\omega\ e - \omega\wedge\omega\ e = (d\omega - \omega\wedge\omega)e. \end{aligned}$$

We define the *curvature* to be $d\omega - \omega\wedge\omega$ and denote it by Ω. It is an $\mathrm{End}(V)$-valued 2-form on M, because if f is a local smooth matrix-valued function, then

$$\begin{aligned} D_\wedge D(fe) &= D_\wedge(df\ e + f\ De) \\ &= ddf\ e - df\wedge De + df\wedge De + fD\wedge De = f\ D\wedge De. \end{aligned}$$

Let us look at the curvature tensor Ω in local coordinates. With respect to a local frame e_α $(1 \le \alpha \le r)$ we have $\Omega^\beta_\alpha = \Omega^\beta_{\alpha ij} dx^i\wedge dx^j$. Using the Christoffel symbol $\Gamma^\beta_{\alpha i}$ of the connection ω^β_α, we have

$$\Omega^\beta_{\alpha ij} dx^i\wedge dx^j = d(\Gamma^\beta_{\alpha i} dx^i) - (\Gamma^\gamma_{\alpha i} dx^i)\wedge(\Gamma^\beta_{\gamma j} dx^j)$$

and

$$2\Omega^\beta_{\alpha ij} = \partial_i \Gamma^\beta_{\alpha j} - \partial_j \Gamma^\beta_{\alpha i} + \Gamma^\gamma_{\alpha i}\Gamma^\beta_{\gamma j} - \Gamma^\gamma_{\alpha j}\Gamma^\beta_{\gamma i}.$$

The curvature tensor Ω satisfies a Bianchi identity. The Bianchi identity is

$$\begin{aligned} D_\wedge\Omega &= d\Omega + \Omega\wedge\omega - \omega\wedge\Omega \\ &= d(d\omega - \omega\wedge\omega) + (d\omega - \omega\wedge\omega)\wedge\omega - \omega\wedge(d\omega - \omega\wedge\omega) \\ &= -d\omega\wedge\omega + \omega\wedge d\omega + d\omega\wedge\omega - \omega\wedge\omega\wedge\omega - \omega\wedge d\omega + \omega\wedge\omega\wedge\omega = 0. \end{aligned}$$

In local coordinates this says that $\nabla_i \Omega^\beta_{\alpha\ jk} + \nabla_j \Omega^\beta_{\alpha ki} + \nabla_k \Omega^\beta_{\alpha ij} = 0$.

We would like to remark that two partial covariant differentiations in general do not commute and the non-commutativity gives rise to the curvature. However, the skew-symmetrization of the results of three partial covariant differentiations becomes zero, which is the Bianchi identity.

We can also define the curvature in invariant formulation. The curvature is defined to measure the failure of the commutativity of partial covariant differentiation. So for tangent vector fields X, Y and a smooth local section s of V, we should consider $\nabla_X\nabla_Y s - \nabla_Y\nabla_X s$. However, when X and Y are arbitrary tangent vector fields instead of tangent vector fields defined by coordinate functions, we do not expect to have commutativity of differentiations along X and Y even in the case of functions, because for a smooth function f in general $X(Yf) - Y(Xf)$ is nonzero and is equal to the derivative $[X,Y]f$ of f along the direction of the Lie bracket $[X,Y]$ of X and Y. So we have to make accommodation for that and consider $R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$. For smooth functions φ, ψ, θ we have $R(\varphi X, \psi Y)(\theta s) = \varphi\psi\theta\ R(X,Y)s$, because of the following computations.

$$\begin{gathered}
\nabla_{\psi Y}(\theta s) = \psi(Y\theta)s + \psi\theta\nabla_Y s. \\
\nabla_{\varphi X}(\nabla_{\psi Y}(\theta s)) = \varphi\nabla_X(\psi(Y\theta)s + \psi\theta\nabla_Y s) \\
= \varphi(X\psi)(Y\theta)s + \varphi\psi(XY\theta)s + \varphi\psi(Y\theta)\nabla_X s \\
+\varphi(X\psi)\theta\nabla_Y s + \varphi\psi(X\theta)\nabla_Y s + \varphi\psi\theta\nabla_X\nabla_Y s. \\
[\varphi X, \psi Y] = \varphi(X\psi)Y - \psi(Y\varphi)X + \varphi\psi[X,Y]. \\
\nabla_{[\varphi X, \psi Y]}(\theta s) = \varphi(X\psi)(Y\theta)s - \psi(Y\varphi)(X\theta)s + \varphi\psi([X,Y]\theta)s \\
+\varphi(X\psi)\theta\nabla_Y s - \psi(Y\varphi)\theta\nabla_X s + \varphi\psi\theta\nabla_{[X,Y]}s. \\
\nabla_{\varphi X}(\nabla_{\psi Y}(\theta s)) - \nabla_{\psi Y}(\nabla_{\varphi X}(\theta s)) - \nabla_{[\varphi X, \psi Y]}(\theta s) \\
= \varphi\psi\theta(\nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s).
\end{gathered}$$

So the value of $R(X,Y)s$ at a point P depends only on the values of X, Y, s at the point P. For $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ and $s = e_\alpha$ it follows from definitions that $R(X,Y)s = 2\,\Omega^\beta_{\alpha ij} e_\beta$. So the curvature can be defined in an invariant way by $R(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$.

Let us consider the case when M is a complex manifold of complex dimension n and V is a holomorphic vector bundle. Suppose we have a Hermitian metric H along the fibers of V and the connection ω is a complex metric connection. When we choose a local holomorphic basis e_α $(1 \leq \alpha \leq r)$, we have

$$\begin{gathered}
\Omega = d\omega - \omega\wedge\omega \\
= d((\partial H)H^{-1}) - (\partial H)H^{-1}\wedge(\partial H)H^{-1} \\
= (\partial + \bar\partial)((\partial H)H^{-1}) - (\partial H)H^{-1}\wedge(\partial H)H^{-1} \\
= (\partial H)H^{-1}\wedge(\partial H)H^{-1} + \bar\partial((\partial H)H^{-1}) - (\partial H)H^{-1}\wedge(\partial H)H^{-1} \\
= \bar\partial((\partial H)H^{-1}) = \bar\partial\omega.
\end{gathered}$$

This shows that Ω is actually an End(V)-valued (1,1)-form on M and it simply equals $\bar\partial\omega$ when expressed in terms of a local holomorphic basis of V.

The Bianchi identity for Ω takes on a simpler form. In terms of local coordinates the Bianchi identity is $\nabla_k F^\beta_{\alpha i\bar{j}} + \nabla_i F^\beta_{\alpha \bar{j} k} + \nabla_{\bar{j}} F^\beta_{\alpha k i} = 0$. Since the curvature tensor is of type (1,1), we simply have $\nabla_k F^\beta_{\alpha i\bar{j}} = \nabla_i F^\beta_{\alpha k\bar{j}}$. Analogously we have also $\nabla_{\bar{k}} F^\beta_{\alpha i\bar{j}} = \nabla_{\bar{j}} F^\beta_{\alpha i\bar{k}}$.

We denote by Tr Ω the trace of Ω so that Tr Ω is a (1,1) form on M. The (1,1)-form Tr Ω is the curvature form of the determinant bundle $\det V$ of V with the metric $\det H$ induced from the metric H of V, because with respect to a local holomorphic basis

$$\partial \log \det H = H^{\alpha\bar{\beta}} \partial \ H_{\alpha\bar{\beta}} = \mathrm{Tr}((\partial H) H^{-1})$$

and

$$\bar{\partial}((\partial(\det H))(\det H)^{-1}) = \mathrm{Tr}(\bar{\partial}((\partial H) H^{-1}) = \mathrm{Tr}\,\Omega.$$

Now we come back to the real case. Consider the real manifold M of real dimension m and the tangent vector bundle T_M with a Riemannian metric. Then the curvature tensor is given by

$$\Omega^\ell_{kij} = \partial_i \Gamma^\ell_{kj} - \partial_j \Gamma^\ell_{ki} + \Gamma^p_{ki}\Gamma^\ell_{pj} - \Gamma^p_{kj}\Gamma^\ell_{pi}.$$

In this case we have another Bianchi identity. The earlier one is usually referred to as the second Bianchi identity and the one we are going to discuss is called the first Bianchi identity. At one point choose a coordinate system so that the first derivatives of the coefficients of the Riemannian metric tensor all vanish at that point. Then at that point we simply have

$$\Omega^\ell_{kij} = \partial_i \Gamma^\ell_{kj} - \partial_j \Gamma^\ell_{ki},$$

from which it follows that

$$\Omega^\ell_{kij} + \Omega^\ell_{i\ jk} + \Omega^\ell_{jki} = 0,$$

because of the symmetry of Γ^ℓ_{kj} in k and j. This is the first Bianchi identity.

The first Bianchi identity is a consequence of the torsion-free condition. We can see this more clearly in the following way. Suppose e_i $(1 \leq i \leq m)$ is a local frame for T_M and ω^i $(1 \leq i \leq m)$ is its dual frame. Let ω^j_i with $De_i = \omega^j_i e_j$ be the matrix valued 1-form of the connection. The torsion-free condition is $d\omega^i + \omega^i_j \wedge \omega^j = 0$. Taking exterior derivative of both sides and using the original equation to get rid of $d\omega^i$, we get

$$\begin{aligned} 0 &= d\omega^i_j \wedge \omega^j - \omega^i_j \wedge d\omega^j = d\omega^i_j \wedge \omega^j - \omega^i_j \wedge d\omega^j \\ &= d\omega^i_j \wedge \omega^j + \omega^i_j \wedge \omega^j_k \wedge \omega^k = \Omega^i_j \wedge \omega^j . \end{aligned}$$

When we write $\Omega^i_j = \Omega^i_{j\,k\ell} \omega^k \wedge \omega^\ell$, we have $\Omega^i_{j\,k\ell} + \Omega^i_{k\,\ell j} + \Omega^i_{\ell\,jk} = 0$ which is the first Bianchi identity.

There is another symmetry we want. Let $R_{k\ell ij} = 2\ g_{\ell p} \Omega^\ell_{kij}$. From

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} (\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk})$$

we have

$$\begin{aligned} R_{k\ell ij} &= \partial_i \Gamma_{\ell,kj} - \partial_j \Gamma_{\ell,ki} \\ &= \frac{1}{2} \partial_i (\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}) - \frac{1}{2} \partial_j (\partial_i g_{k\ell} + \partial_k g_{i\ell} - \partial_\ell g_{ik}) \\ &= \frac{1}{2} \left(\partial_i \partial_k g_{j\ell} + \partial_j \partial_\ell g_{ik} - \partial_i \partial_\ell g_{jk} - \partial_j \partial_k g_{i\ell} \right) . \end{aligned}$$

Thus $R_{k\ell ij} = R_{ijk\ell}$.

We consider now the Kähler case. From $2\Omega^\ell_{kij} = \partial_i \Gamma^\ell_{kj} - \partial_j \Gamma^\ell_{ki}$ and the fact that the only nonvanishing components of the Christoffel symbols are of type $\Gamma^\alpha_{\beta\gamma}$ and $\Gamma^{\bar\alpha}_{\bar\beta\bar\gamma}$ we conclude that when k and ℓ are of different type, Ω^ℓ_{kij} must vanish. So $R_{k\ell ij}$ vanishes unless k and ℓ are of different type. We can also see this from the fact that the curvature of the complex metric connection is of type (1,1). By the above symmetry we conclude that the only nonzero components of the curvature tensor are of the type $R_{\alpha\bar\beta\gamma\bar\delta}, R_{\bar\alpha\beta\gamma\bar\delta}, R_{\alpha\bar\beta\bar\gamma\delta}, R_{\bar\alpha\beta\bar\gamma\delta}$.

(1.5) *Chern Forms and Chern Classes*

Now we go back to the case of a real smooth manifold M of real dimension m and a smooth $\mathbf{C}$-vector bundle V over M. We take a smooth connection ω of V and get a curvature form Ω. The curvature form Ω is an $\mathrm{End}(V)$-valued 2-form on M. With respect to a local trivialization of the bundle V the curvature form is a matrix valued 2-form. From a matrix we can get numerical invariants like the trace and the determinant and other symmetric functions of the eigenvalues of the matrix. By using the same method, from

Ω we can to construct from $2k$-forms on M. Let us look at the construction of numerical invariants of an $r \times r$ matrix $A = (A_{ij})$. Let the eigenvalues of A be $\lambda_1, \cdots, \lambda_r$. Then $\sum_{\nu=1}^{r} \lambda_\nu^k$ is simply the trace of A^k as one can easily see by reducing A to its Jordan normal form. So the k^{th} elementary symmetric function σ_k of the eigenvalues of A is simply a universal homogeneous polynomial P_k of weight k in $\operatorname{Tr} A^\nu$ $(1 \leq \nu \leq k)$ when $\operatorname{Tr} A^\nu$ is given the weight ν. We now define the k^{th} Chern form γ_k to be $P_k(\operatorname{Tr}\Omega, \cdots, \operatorname{Tr}\Omega^k)$. Thus the Chern form γ_k is a global form of degree $2k$ on M and is a homogeneous polynomial of degree k in the coefficients $\Omega_{\alpha ij}^{\beta}$ of Ω. Recall that we have the Bianchi identity $D_\wedge \Omega = 0$. So we have $D_\wedge P_k(\operatorname{Tr}\Omega, \cdots, \operatorname{Tr}\Omega^k) = 0$ and we conclude that γ_k is always a closed form.

The most important aspect of the theory of Chern classes is that the Chern form γ_k *up to a global exact form* is independent of the choice of the connection. Let us rename our connection ω and give it the new name ω_0. Suppose we have another connection ω_1. Our original curvature is now called Ω_0 and our new curvature is Ω_1. We denote by $\gamma_k^{(\nu)}$ the k^{th} Chern form from Ω_ν. Consider the new manifold $\tilde{M} = M \times \mathbf{R}$ and the vector bundle $\tilde{V}$ over $\tilde{M}$ obtained by pulling back V via the natural projection map $\pi : \tilde{M} \to M$ onto the first factor. Let t be the variable of $\mathbf{R}$. We can define a connection $\tilde{\omega} = (1-t)\omega_0 + t\omega_1$. More precisely we should have used $\pi^*\omega_\nu$ instead of ω_ν in the above definition of $\tilde{\omega}$, but for the sake of simpler notations we allow this slight abuse of language. We have the curvature $\tilde{\Omega}$ of $\tilde{\omega}$ on $\tilde{M}$ and Chern forms $\tilde{\gamma}_k$ on $\tilde{M}$. We can write $\tilde{\gamma}_k = \alpha_k + dt \wedge \beta_k$, where α_k and β_k are respectively a $2k$-form and a $(2k-1)$-form not containing dt (*i.e.*, expressible in terms of the differentials of the local coordinates of M with coefficients depending on t and the local coordinates of M). Since $\tilde{\gamma}_k$ is closed, we have

$$d_M \alpha_k + dt \wedge \frac{\partial}{\partial t} \alpha_k - dt \wedge d_M \beta_k = 0,$$

where d_M means (partial) exterior differentiation with the coordinate t kept constant and $\frac{\partial}{\partial t}\alpha_k$ means differentiating the coefficients of α_k respect to the variable t. Hence $\frac{\partial}{\partial t}\alpha_k = d_M \beta_k$ and

$$\alpha_k|_{t=1} - \alpha_k|_{t=0} = d_M \left(\int_{t=0}^{1} \beta_k dt \right).$$

Since

$$\alpha_k|_{t=\nu} = \tilde{\gamma}_k|_{t=\nu} = \gamma_k^{(\nu)}$$

for $\nu = 0,\ 1$, it follows that $\gamma_k^{(1)} - \gamma_k^{(0)}$, being equal to $d_M \left(\int_{t=0}^1 \beta_k dt \right)$, is exact on M.

We define as the deRham cohomology group of degree k over M the abelian group of all smooth closed k-forms on M quotiented by the abelian subgroup of all smooth exact k-forms on M. An element of the deRham cohomology group of degree k is called a deRham cohomology class of degree k. We will later discuss deRham cohomology in the context of sheaf cohomology. The k^{th} Chern form γ_k defines a deRham cohomology class of degree $2k$ and this class is independent of the choice of the connection and is called the k^{th} *Chern class* of the $\mathbf{C}$-vector bundle V over M. Chern classes are invariants of smooth vector bundles.

When M is a complex manifold and V is a holomorphic vector bundle, we can choose as our connection the complex metric connection of a Hermitian metric along the fibers of V. Then the curvature form is of type (1,1) and the k^{th} Chern form is of type (k,k). So we conclude that the k^{th} Chern class of a holomorphic vector bundle over a complex manifold can be represented by a form of type (k,k).

(1.6) *Relation of Curvature Tensor with Gaussian Curvature*

We want to link our definition to the classical definition of Gaussian curvature. Suppose we have a surface S in $\mathbf{R}^3$ given by $\vec{r} = \vec{r}(u,v)$. We give S the metric induced from the Euclidean metric of $\mathbf{R}^3$. Then the metric in terms of the coordinates u and v is given by $d\vec{r} \cdot d\vec{r} = E\ du^2 + 2F$ dudv $+G\ dv^2$, where $E = \vec{r}_u \cdot \vec{r}_u$, $F = \vec{r}_u \cdot \vec{r}_v$, $G = \vec{r}_v \cdot \vec{r}_v$ and the subscripts u and v mean partial differentiation with respect to u and v.

Let $\vec{n}$ be the unit normal to the surface S. Choose a point P in the surface S and let C be a curve cut out from S by a plane Π through P containing $\vec{n}(P)$. Let $u = u(t)$, $v = v(t)$ be the equations defining C with the parameter t equal to the arc-length of C. We now compute the curvature of the curve C. Let $\vec{r}'$ and $\vec{r}''$ denote respectively the first and second order derivatives of $\vec{r}$ with respect to t. Since $\vec{r}'$ is a unit vector, the curvature of C is given by the length of $\vec{r}''$. Since C is a plane curve on the same plane as $\vec{n}(P)$, the vector $\vec{r}''(P)$ is parallel to $\vec{n}(P)$. Hence the curvature κ of C at P is given by $\vec{r}'' \cdot \vec{n}$ which by the chain rule is equal to $(\vec{r}_{uu} \cdot \vec{n})u'^2 + 2(\vec{r}_{uv} \cdot \vec{n})u'v' + (\vec{r}_{vv} \cdot \vec{n})v'^2$, where the primes for u and v mean differentiation with respect to t. Let $D = \vec{n} \cdot \vec{r}_{uu}$, $D' = \vec{n} \cdot \vec{r}_{uv}$, and $D'' = \vec{n} \cdot \vec{r}_{vv}$. Since t is the arc-length of C, we can write

$$\kappa = \frac{D\ du^2 + 2D'dudv + D''dv^2}{E\ du^2 + 2F\ dudv + F\ dv^2}.$$

The expression makes κ the quotient of two quadratic forms whose variables are the components of the tangent vector at P which belongs to Π. As we change this tangent vector or equivalently as we change Π, the value of κ changes and we have two extremal values. The product of these two extremal values is known as the Gaussian curvature. We let $\xi = \frac{du}{dt}$ and $\eta = \frac{dv}{dt}$. We have

$$\kappa(\xi,\eta)\left(E\,\xi^2 + 2F\,\xi\eta + G\,\eta^2\right) = D\,\xi^2 + 2D'\xi\eta + D''\eta^2.$$

By differentiating with respect to ξ and η, we conclude that the extremal values of κ satisfy the two equations

$$\begin{aligned}(D-\kappa E)\xi + (D'-\kappa F)\eta &= 0\\ (D'-\kappa F)\xi + (D''-\kappa G)\eta &= 0.\end{aligned}$$

The vanishing of the determinant of the coefficients of ξ and η yields a quadratic equation in κ. The product of the two roots of κ in this equation can be read off from the coefficients of the equation. So we conclude the Gaussian curvature is given by

$$\frac{DD'' - D'^2}{EG - F^2}.$$

Another way to see it is that the Gaussian curvature is the product of the eigenvalues of the matrix $\begin{pmatrix} D & D' \\ D' & D'' \end{pmatrix}$ with respect to the matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and so is equal to the quotient of the determinants of the two matrices.

Now we verify that the Gaussian curvature agrees with the notion of curvature defined as a measure of the failure of the commutativity of partial covariant differentiation. We define covariant differentiation of a vector field on S as the orthogonal projection of the usual differentiation onto the tangent planes of S. First let us check that this is a torsion-free connection compatible with the metric. Suppose we have a vector field $\vec{a}$ along a curve γ of S parametrized by t and assume that the covariant derivative of $\vec{a}$ along the curve γ vanishes. Then $\frac{d}{dt}\vec{a}$ is proportional to $\vec{n}$ and $\frac{d}{dt}(\vec{a}\cdot\vec{a}) = 2(\frac{d}{dt}\vec{a})\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{a}$ is constant. Next we have to verify that the Christoffel symbols with respect to a local coordinate system are symmetric in the two covariant indices. The vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are respectively the vectors $\vec{r}_u$ and $\vec{r}_v$. The total covariant derivative of $\frac{\partial}{\partial u}$ is the projection of $d\vec{r}_u$ onto the tangent plane of S, *i.e.* $d\vec{r}_u - (d\vec{r}_u \cdot \vec{n})\vec{n}$. Relabel the two variables u, v respectively as x^1, x^2. Then $\vec{r}_{x^j x^k} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n} = \Gamma^i_{jk}\vec{r}_{x^i}$. Clearly the Christoffel symbols Γ^i_{jk} are symmetric in k and j.

Let us now compute the curvature from the connection. We have seen that $\nabla_j(\frac{\partial}{\partial x^k})$ corresponds to $\vec{r}_{x^j x^k} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n}$. Then $\nabla_\ell \nabla_j(\frac{\partial}{\partial x^k})$ corresponds to the orthogonal projection of $\frac{\partial}{\partial x^\ell}(\vec{r}_{x^j x^k} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n})$ onto the tangent plane of S. Now

$$\frac{\partial}{\partial x^\ell}(\vec{r}_{x^j x^k} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n}) = \vec{r}_{x^j x^k x^\ell} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n}_{x^\ell} \text{ mod } \vec{n}.$$

So $\nabla_\ell \nabla_j(\frac{\partial}{\partial x^k}) - \nabla_j \nabla_\ell(\frac{\partial}{\partial x^k})$ corresponds to the orthogonal projection of $(\vec{r}_{x^\ell x^k} \cdot \vec{n})\vec{n}_{x^j} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n}_{x^\ell}$ onto the tangent plane of S. Since $\vec{n}$ is a unit vector, it is always perpendicular to its derivatives. So $(\vec{r}_{x^\ell x^k} \cdot \vec{n})\vec{n}_{x^j} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n}_{x^\ell}$ equals its own orthogonal projection onto the tangent plane of S. We assume without loss of generality that at P the two vectors $\vec{r}_u$ and $\vec{r}_v$ are orthonormal. Then $R_{kij\ell}$ is the inner product of $(\vec{r}_{x^\ell x^k} \cdot \vec{n})\vec{n}_{x^j} - (\vec{r}_{x^j x^k} \cdot \vec{n})\vec{n}_{x^\ell}$ with $\vec{r}_{x^i}$. So

$$R_{kij\ell} = (\vec{r}_{x^\ell x^k} \cdot \vec{n})(\vec{n}_{x^j} \cdot \vec{r}_{x^i}) - (\vec{r}_{x^j x^k} \cdot \vec{n})(\vec{n}_{x^\ell} \cdot \vec{r}_{x^i})$$

which is equal to

$$-(\vec{r}_{x^\ell x^k} \cdot \vec{n})(\vec{r}_{x^i x^j} \cdot \vec{n}) + (\vec{r}_{x^j x^k} \cdot \vec{n})(\vec{r}_{x^\ell x^i} \cdot \vec{n})$$

from differentiating $\vec{r}_{x^i} \cdot \vec{n} = 0$ with respect to x^j. In particular R_{1212} with respect to an orthonormal frame is

$$-(\vec{r}_{uv} \cdot \vec{n})^2 + (\vec{r}_{uu} \cdot \vec{n})(\vec{r}_{vv} \cdot \vec{n}) = DD'' - D'^2 = \frac{DD'' - D'^2}{EG - F^2}$$

which is the Gaussian curvature. Incidentally this verification proves the so-called the Celebrated Theorem of Gauss that the Gaussian curvature of a surface in $\mathbf{R}^3$ depends only on the first fundamental form.

(1.7) *The Second Fundamental Form in the Riemannian Case.*

Let us first look at the real case. Suppose M is a real Riemannian manifold of real dimension m with Riemannian metric g_{ij}. Let S be a submanifold of M with real dimension s. We choose local coordinates $x^1, \cdots, x^m$ of M so that S is given by the vanishing of $x^{s+1}, \cdots, x^m$. We give S the metric induced from that of M. So its Riemannian metric tensor is also g_{ij} with $1 \leq i, j \leq s$. Let us take a local orthonormal frame e_i $(1 \leq i \leq m)$ of tangent vectors of M so that at points of S the vectors e_i $(1 \leq i \leq s)$ are tangential to S. Let f^i $(1 \leq i \leq m)$ be the dual frame of e_i $(1 \leq i \leq m)$ so that f^i is a 1-form on M. Let ω_i^j $(1 \leq i, j \leq m)$ be the Levi-Civita connection of M expressed as a matrix-valued 1-form in terms of the frame e_i $(1 \leq i \leq m)$. We want to determine the Levi-Civita connection of S. For a vector field X on S we define covariant derivative $\nabla^S X$ in S as the orthogonal projection of the covariant derivative $\nabla^M X$ onto the tangent space of S. Clearly this connection is compatible with the metric, because if $\nabla^S X$ vanishes along a curve γ in S, then $\nabla^M X$ along γ is perpendicular to S and the derivative of the square of the length of X along γ is equal to twice the inner product of X with $\nabla^M X$ along γ and must vanish. The matrix valued 1-form representing the connection of S is simply $(\omega_i^j)_{1 \leq i,j \leq s}$. To verify that the connection of S is torsion-free, we have to check that the skew-symmetrization of the covariant derivative of $f^i|S$ agrees with the exterior derivative of $f^i|S$. In other words, $D^S_\wedge f^i - df^i = \sum_{j=1}^s \omega_j^i \wedge f^j$ vanishes on S, where D^S means covariant differentiation with respect to the connection on S. We know that $D^M_\wedge f^i - df^i = \sum_{j=1}^m \omega_j^i \wedge f^j$ vanishes on M, where D^M means covariant differentiation with respect to the connection on M. The vanishing of $\sum_{j=s+1}^m \omega_j^i \wedge f^j$ on S follows from the fact that f^j $(s+1 \leq j \leq m)$ vanishes identically on S. Thus we know that the connection $\left(\omega_i^j\right)_{1 \leq i,j \leq s}$ on S is the Levi-Civita connection.

We now compare the curvature tensor Ω^S on S and the curvature tensor Ω^M on M. We have

$$\Omega_i^{M\ j} = d\omega_i^j - \sum_{k=1}^m \omega_i^k \wedge \omega_k^j$$

and

$$\Omega_i^{S\ j} = d\omega_i^j - \sum_{k=1}^s \omega_i^k \wedge \omega_k^j.$$

Thus

$$\Omega_i^{S\ j} = \Omega_i^{M\ j} + \sum_{k=s+1}^m \omega_i^k \wedge \omega_k^j.$$

The difference of the two curvature tensors is the term $\sum_{k=s+1}^m \omega_i^k \wedge \omega_k^j$. We are going to interpret this term in terms of what is called the second fundamental form of S.

Let T_S denote the tangent bundle of S. It is a subbundle of the tangent bundle T_M of M. Let $N_{S,M}$ be the normal bundle of S in M which is defined as the quotient bundle T_M/T_S over S. We identify $N_{S,M}$ with the orthogonal complement of T_S in $T_M|S$. Suppose we have two vector fields X, Y on S. The difference $\nabla_Y^M X - \nabla_Y^M X$ is an element of the normal bundle $N_{S,M}$ of S in M. For any smooth functions φ, ψ on S we have

$$\nabla_{\psi Y}^M(\varphi X) - \nabla_{\psi Y}^S(\varphi X) = \varphi\psi(\nabla_Y^M X - \nabla_Y^S X).$$

This means that $\nabla_Y^M X - \nabla_Y^S X$ at a point P depends only on the values of X and Y at P. Let us denote $\nabla_Y^M X - \nabla_Y^S X$ by $\Pi(X, Y)$. Then Π is a section of $N_{S,M} \otimes T_S^* \otimes T_S^*$ over S and is called the *second fundamental form* of S. In terms of the local frame e_i $(1 \leq i \leq m)$ we have

$$\Pi(e_i, e_j) = \sum_{k=s+1}^{m} \omega_i^k(e_j)e_k.$$

The second fundamental form $\Pi(X, Y)$ is symmetric in X and Y. To check this at a point P we can choose local coordinates $x^1, \cdots, x^m$ so that S is defined by the vanishing of $x^{s+1}, \cdots, x^m$ and $\frac{\partial}{\partial x^i} = e_i$ $(1 \leq i \leq m)$ at the point P. Then $\omega_i^k(e_j)$ equals the Christoffel symbol Γ_{ij}^k which is symmetric in i and j because of the torsion-free condition of the Levi-Civita connection.

We can regard Π as an $\mathrm{Hom}(T_S, N_{S,M})$-valued 1-form on S. Let Π^* denote the $\mathrm{Hom}(N_{S,M}, T_S)$-valued 1-form on S which is the adjoint of Π with respect to the metrics of T_S and $N_{S,M}$. Then

$$\Omega_i^{S\ j} = \Omega_i^{M\ j} + \sum_{k=s+1}^{m} \omega_i^k \wedge \omega_k^j$$

can be rewritten as $\Omega^S = \Omega^M + \Pi \wedge \Pi^*$ as $\mathrm{End}(T_S)$-valued 2-forms. Even though $\Pi \wedge \Pi^*$ is the product of a matrix and its adjoint we cannot conclude that it has a fixed sign, because the product is an exterior product. So the curvature of a submanifold can be greater than or less than that of the ambient manifold. However, as we will see later it is possible to get a fixed sign for $\Pi \wedge \Pi^*$ is some cases and certain curvatures of a complex submanifold are always no greater than those of the ambient complex submanifold.

When the second fundamental form of S vanishes at a point, the curvature of S agrees with that of M. Suppose S is constructed in the following way from a linear subspace V of the tangent space of M at a point P. For every $v \in V$ we construct the geodesic γ_v through P in the direction of v. The totality of γ_v as v varies in V forms a submanifold in a neighborhood of P and this manifold is S. The second fundamental form Π of S is identically zero at P, because $\Pi(v,v)$ is clearly zero at P and $\Pi(X,Y)$ is linear in X and Y. In the special case when V is spanned by two tangent vectors $\xi = \xi^i \frac{\partial}{\partial x^i}$ and $\eta = \eta^i \frac{\partial}{\partial x^i}$. Our manifold S is a surface and we denote it by $S_{\xi,\eta}$. Then the Gaussian curvature of $S_{\xi,\eta}$ is

$$\frac{R(\xi,\eta,\xi,\eta)}{|\xi \wedge \eta|^2},$$

where $R(\xi,\eta,\xi,\eta) = R_{ijk\ell}\xi^i\eta^j\xi^k\eta^\ell$ and

$$|\xi \wedge \eta|^2 = \frac{1}{2} g^{ik} g^{j\ell} (\xi^i\eta^j - \eta^i\xi^j)(\xi^k\eta^\ell - \eta^k\xi^\ell)$$

is the square of the norm of the 2-vector $\xi \wedge \eta$ in the metric induced from the Riemannian metric g_{ij}. One verifies this by first considering the case when ξ and η are orthonormal and using the above relation between the Gaussian curvature of a surface and the curvature tensor defined as a measure of failure of the commutativity of partial covariant differentiation. The Gaussian curvature

$$\frac{R(\xi,\eta,\xi,\eta)}{|\xi \wedge \eta|^2}$$

of $S_{\xi,\eta}$ is known as the *Riemannian sectional curvature* of the plane spanned by ξ and η.

(1.8) *The Second Fundamental Form in the Complex Manifold Case.*

We now look at the case of holomorphic vector bundles over a complex manifold and discuss the concept of second fundamental form. Now M is a complex manifold of complex dimension n and E is a holomorphic vector bundle of rank r over M. Assume that E has a Hermitian metric. Let E' be a holomorphic vector subbundle of E of rank s.

We can choose a local unitary basis e_α $(1 \leq \alpha \leq r)$ of E such that e_α $(1 \leq \alpha \leq s)$ belongs to E'. Let ω' be the connection of E' induced by the complex metric connection ω of E. In other words, ${\omega'}^\beta_\alpha = \omega^\beta_\alpha$ for $1 \leq \alpha \leq s$. Another way of describing the connection ω' is that the E'-covariant derivative $D's$ of a local section s of E' is obtained by taking its

E-covariant derivative Ds and then projecting Ds onto E' by the orthogonal projection. This connection agrees with the complex metric connection of E' with respect to the metric induced from that of E. The reason is as follows. Firstly it is easy to see that the E'-covariant derivative of a local holomorphic section of E' along any (0,1) direction is zero from the above description of the E'-covariant differentiation. Secondly if s is a section of E' above a local curve of M with zero E'-covariant derivative along the curve, then its E-covariant derivative Ds is a linear combination of e_α $(s < \alpha \leq r)$ and must be perpendicular to s. As a consequence

$$d\langle s, s\rangle = \langle Ds, s\rangle + \langle s, Ds\rangle$$

must vanish along the curve and the length of s is constant. Let Q be the orthogonal complement of E' in E. We give Q the complex structure of the quotient bundle E/E'. The difference $Ds - D's$ of the E-covariant derivative of s and the E'-covariant derivative of s is a 1-form with values in $\mathrm{Hom}(E', Q)$, This $\mathrm{Hom}(E', Q)$-valued 1-form is called the *second fundamental form* of E' in E and we denote it by B. Let $s = \sum_{\alpha=1}^r s^\alpha e_\alpha$ be the representation of s in terms of some local *holomorphic* basis e_α. Then

$$Ds = \sum_{\alpha=1}^r ds^\alpha e_\alpha + \sum_{\alpha=1}^r s^\alpha De_\alpha$$

and

$$D's = \sum_{\alpha=1}^r ds^\alpha e_\alpha + \sum_{\alpha=1}^r s^\alpha D'e_\alpha.$$

Hence

$$Ds - D's = \sum_{\alpha=1}^r s^\alpha (De_\alpha - D'e_\alpha)$$

is a Q-valued (1,0)-form. Thus the second fundamental form B must be a $\mathrm{Hom}(E', Q)$-valued (1,0)-form and we can write $Ds = D's + Bs$ for a section s of E'.

Let us now consider the case of the quotient bundle. Take a local holomorphic basis e''_α $(s+1 \leq \alpha \leq r)$ of Q. We use the same notation e''_α to denote local sections of E orthogonal to E'. The holomorphicity of basis e''_α means that for some sections e'_α $(s+1 \leq \alpha \leq r)$ of E', the sections $e'_\alpha + e''_\alpha$ $(s+1 \leq \alpha \leq r)$ are holomorphic sections of E. We take also a local holomorphic basis e_γ $(1 \leq \gamma \leq s)$ of E'. We have

$$(De''_\alpha)^{(0,1)} = \bar\partial e''_\alpha = -\bar\partial e'_\alpha.$$

When we write $e'_\alpha = \sum_{\gamma=1}^{s} a_\alpha^\gamma e_\gamma$ in terms of the local holomorphic basis e_γ $(1 \leq \gamma \leq s)$ of E', we see that

$$-\bar\partial e'_\alpha = -\sum_{\gamma=1}^{s} (\bar\partial a_\alpha^\gamma) e_\gamma.$$

Hence $-\bar\partial e'_a$ is an E'-valued $(0,1)$-form. Since from the definition of D one has

$$\left\langle (De''_\alpha)^{(1,0)}, e_\gamma \right\rangle = \partial \left\langle e''_\alpha, e_\gamma \right\rangle - \left\langle e''_\alpha, \bar\partial e_\gamma \right\rangle = 0,$$

it follows that $(De''_\alpha)^{(1,0)}$ is a Q-valued $(1,0)$-form. So we can define a connection D'' on Q by $D'' e''_\alpha = De''_\alpha + \bar\partial e'_\alpha$. We claim that this connection is the complex metric connection of Q. It is a complex connection, because we have observed earlier that $(De''_\alpha + \bar\partial e'_\alpha)^{(0,1)} = 0$ and the (1,0)-form $(De''_\alpha)^{(1,0)}$ has values in Q. It is also a metric connection, because

$$\left\langle De''_\alpha + \bar\partial e'_\alpha, e''_\beta \right\rangle + \left\langle e''_\alpha, De''_\beta + \bar\partial e'_\beta \right\rangle = \left\langle De''_\alpha, e''_\beta \right\rangle + \left\langle e''_\alpha, De''_\beta \right\rangle = d \left\langle e''_\alpha, e''_\beta \right\rangle.$$

We write $Ds'' - D''s'' = Cs''$ for sections s'' of Q. The operator C is a $\mathrm{Hom}(Q, E')$-valued $(0,1)$-form given by

$$Ce''_\alpha = -\bar\partial e'_\alpha = -\sum_{\gamma=1}^{s} (\bar\partial a_\alpha^\gamma) e_\gamma.$$

We call C the *second fundamental form* of the quotient bundle Q of E.

Another more invariant way of representing the second fundamental form C of the quotient bundle Q of E is the following. We have $Ce''_\alpha = \bar\partial e''_\alpha$. The local basis e''_α $(s+1 \leq \alpha \leq r)$ of the orthogonal complement of E' in E simply describes the monomorphism from Q to E which lifts Q to the orthogonal complement of E' in E. Let us call this monomorphism φ. Then C is simply $\bar\partial\varphi$. The entity $\bar\partial\varphi$ is *a priori* only a $\mathrm{Hom}(Q, E)$-valued (0,1)-form which is $\bar\partial$ exact. Our previous discussion shows that it is actually a $\mathrm{Hom}(Q, E')$-valued (0,1)-form. However, as a $\mathrm{Hom}(Q, E')$-valued (0,1)-form, in general it is only $\bar\partial$ closed and is not $\bar\partial$ exact. Suppose we have another Hermitian metric for E. Then we would have a different monomorphism φ' from Q to E' and a different second fundamental form C'. The difference of φ and φ' is a homomorphism from Q to E', because φ and φ' are different liftings of Q to E. So the difference $C - C'$ of the two second fundamental forms of Q equals $\bar\partial(\varphi - \varphi')$ which is a $\bar\partial$ exact $\mathrm{Hom}(Q, E')$-valued (0,1)-form.

We want to relate C to the second fundamental form B of the subbundle E' of E. For sections s' and s'' of E' and Q respectively

$$\begin{aligned} 0 = d\left\langle s', s''\right\rangle &= \left\langle Ds', s''\right\rangle + \left\langle s', Ds''\right\rangle \\ &= \left\langle D's + Bs', s''\right\rangle + \left\langle s', D''s'' + Cs''\right\rangle \\ &= \left\langle Bs', s''\right\rangle + \left\langle s', Cs''\right\rangle . \end{aligned}$$

Hence C is simply the negative of the of adjoint B^* of B with respect to the Hermitian metrics of E' and Q. So $D = D'' - B^*$.

We now compute the curvatures Ω^E, $\Omega^{E'}$, and Ω^Q. We choose a local orthonormal frame e_α $(1 \leq \alpha \leq r)$ so that e_α $(1 \leq \alpha \leq s)$ belongs to E'. Write $De_\alpha = \sum_{\beta=1}^r \omega_\alpha^\beta e_\beta$. Since the connection is compatible with the metric by differentiating $\langle e_\alpha, e_\beta\rangle = 0$ or 1, we conclude that $\omega_\alpha^\beta = -\overline{\omega_\beta^\alpha}$. The second fundamental form B of E' is given by $Be_\alpha = \sum_{\beta=s+1}^r \omega_\alpha^\beta e_\beta$ for $1 \leq \alpha \leq s$. From

$$\Omega_\alpha^{E\,\beta} = d\omega_\alpha^\beta - \sum_{\gamma=1}^r \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$$

and

$$\Omega_\alpha^{E'\beta} = d\omega_\alpha^\beta - \sum_{\gamma=1}^s \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$$

we have

$$\Omega_\alpha^{E\,\beta} = \Omega_\alpha^{E'\beta} + \sum_{\gamma=s+1}^r \omega_\alpha^\gamma \wedge \overline{\omega_\beta^\gamma}.$$

Thus $\Omega_\alpha^{E\,\alpha}(X, \bar{X}) \geq \Omega_\alpha^{E'\alpha}(X, \bar{X})$ (no summation over α) for any vector X of type (1,0) and equality holds if and only if $B(X)e_\alpha = 0$. In invariant formulation we have

$$\left\langle \Omega^E(X, \bar{X})\xi, \xi\right\rangle \geq \left\langle \Omega^{E'}(X, \bar{X})\xi, \xi\right\rangle$$

for any ξ in E' and any vector X of M of type (1,0).

Let us now look at the case of the quotient bundle. We can identify the quotient bundle Q as the subbundle of E which is the orthogonal complement of E' in E. We needed the complex structure of Q only to define the connection of Q as a complex metric connection. Once we get the connection of Q we can ignore the complex structure of Q. The calculation of the curvature tensor depends only on the connection. So when it comes to comparing the curvature tensors, the computation of the quotient bundle case is the same as the subbundle case. There is however one difference. The second fundamental form of the subbundle is an endomorphism valued (1,0)-form whereas the second fundamental form of the quotient bundle is an endomorphism valued (0,1)-form. So when it comes to evaluating the exterior product of the second fundamental form and its complex conjugate transpose at $(X, \bar{X})$ for some (1,0)-vector X, there is a sign difference between the quotient bundle case and the subbundle case. So we have

$$\left\langle \Omega^E(X, \bar{X})\xi, \xi \right\rangle \leq \left\langle \Omega^Q(X, \bar{X})\xi', \xi' \right\rangle$$

for any ξ in the orthogonal complement of E' in E and for any vector X of M of type (1,0), where ξ' is the image of ξ in Q.

(1.9) *Holomorphic Sectional and Bisectional Curvatures*

Let M be a complex manifold of complex dimension n with a Kähler metric $g_{\alpha\bar{\beta}}$. Take a real tangent vector X. By the *holomorphic sectional curvature* in the direction of X we mean the Riemannian sectional curvature for the plane spanned by X and JX. We want to express this holomorphic sectional curvature in terms of local coordinates. Let $X = 2\,\mathrm{Re}\left(\xi^\alpha \frac{\partial}{\partial z^\alpha}\right)$. Then $JX = 2\,\mathrm{Re}\left(\sqrt{-1}\xi^\alpha \frac{\partial}{\partial z^\alpha}\right)$. We have

$$\begin{aligned} R(X, JX, X, JX) &= R(\xi + \bar{\xi}, \sqrt{-1}\xi - \sqrt{-1}\bar{\xi}, \xi + \bar{\xi}, \sqrt{-1}\xi - \sqrt{-1}\bar{\xi}) \\ &= -4\ R(\xi, \bar{\xi}, \xi, \bar{\xi}) = -4\ R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha \overline{\xi^\beta} \xi^\gamma \overline{\xi^\delta}. \end{aligned}$$

Now X is perpendicular to JX because $g(X, JX) = 0$. The length of $X \wedge JX$ is simply the square of the length of X, because X and JX have the same length. The square of the length of X equals 2 $g_{\alpha\bar{\beta}}$. Hence the holomorphic sectional curvature in the direction of $2\,\mathrm{Re}\left(\xi^\alpha \frac{\partial}{\partial z^\alpha}\right)$ is

$$-\frac{R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha \overline{\xi^\beta} \xi^\gamma \overline{\xi^\delta}}{(g_{\alpha\bar{\beta}}\xi^\alpha \overline{\xi^\beta})^2}.$$

Since in the case of a Kähler manifold the curvature Ω of the complex metric connection of the tangent bundle agrees with the Riemannian curvature, we have

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha \overline{\xi^\beta} \xi^\gamma \overline{\xi^\delta} = \left\langle \Omega(\xi, \bar{\xi})\xi, \xi \right\rangle.$$

Thus the holomorphic sectional curvature of a complex submanifold is no more than the corresponding holomorphic sectional curvature of the ambient Kähler manifold. Note that this statement is not true for Riemannian sectional curvatures and Riemannian manifolds, because the Riemannian sectional curvature of the unit sphere in the real Euclidean space is clearly greater than the corresponding Riemannian sectional curvature of the Euclidean space.

The decrease in holomorphic sectional curvature for complex submanifolds holds also for a more general kind of curvature, because it comes from the inequality involving $\left\langle \Omega(\xi,\bar{\xi})\eta,\eta \right\rangle$. So we want to see what curvature $\left\langle \Omega(\xi,\bar{\xi})\xi,\xi \right\rangle$ corresponds to. Let $Y = 2\,\mathrm{Re}\left(\eta^\alpha \frac{\partial}{\partial z^\alpha}\right)$. We consider

$$\begin{aligned} R(\xi,\bar{\xi},\eta,\bar{\eta}) &= \frac{1}{16} R(X+iJX, X-iJX, Y+iJY, Y-iJY) \\ &= -\frac{1}{4} R(X,JX,Y,JY) \\ &= -\frac{1}{4}\left(R(X,Y,JX,JY) + R(X,JY,Y,JX)\right), \end{aligned}$$

where for the last equality the first Bianchi identity is used. One can easily check that a $\mathbf{R}$-bilinear form $h(X,Y)$ on T_M satisfies $h(JX,JY) = h(X,Y)$ if and only if $h_{\alpha\beta} = h_{\bar{\alpha}\bar{\beta}} = 0$ when expressed in terms of complex basis of $T_M^{1,0}$. Hence $R(\cdot,\cdot,JX,JY) = R(\cdot,\cdot,X,Y)$. Hence

$$R(\xi,\bar{\xi},\eta,\bar{\eta}) == -\frac{1}{4}\left(R(X,Y,X,Y) + R(X,JY,X,JY)\right).$$

We call $R(\xi,\bar{\xi},\eta,\bar{\eta})$ the *holomorphic bisectional curvature* in the direction of ξ and η (or in the direction of 2 Re ξ and 2 Re η). After suitable normalization it is equal to the sum of two Riemannian sectional curvatures, one for the plane spanned by X and Y and the other for the plane spanned by X and JY. This is the reason for the name holomorphic bisectional curvature. The holomorphic bisectional curvature of a complex submanifold is no more than the corresponding holomorphic bisectional curvature of the ambient Kähler manifold.

CHAPTER 2. SHEAF COHOMOLOGY

(2.1) *The Concept of a Sheaf*

The tools of sheaves and sheaf cohomology were introduced to patch together local objects (holomorphic functions, holomorphic sections of holomorphic vector bundles) to form global objects. Let us consider first a simple example. Suppose we have a Riemann surface M and a point P of M. We would like to produce a global meromorphic function f on M with a given principal part p at P. Locally we can always do it. We cover M by open coordinate charts $\{U_\alpha\}$ so that P belongs to only one chart U_{α_0}. We can choose meromorphic functions f_α on U_α so that the principal part of f_α at P is p if P belongs to U_α. Of course in general these local meromrophic functions f_α cannot be patched together to give a global meromorphic function with principal part p at P. The discrepancies are given by $f_{\alpha\beta} = f_\beta - f_\alpha$ on $U_\alpha \cap U_\beta$ which is a holomorphic function on $U_\alpha \cap U_\beta$. Observe that $f_{\alpha\beta}$ is skew-symmetric in α and β and $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$. If we can modify f_α on U_α by a *holomorphic* function g_α on U_α to make the discrepancies vanish, then we can piece the local meromorphic functions f_α together to form a global meromorphic function. In other words, we want to find a holomorphic function g_α on U_α so that $f_{\alpha\beta} = g_\beta - g_\alpha$. Then the meromorphic function which is equal to $f_\alpha - g_\alpha$ is a global meromorphic function whose principal part at P is p.

Now when we choose the local meromorphic functions, it does not matter how big the open set is on which a local meromorphic function is defined as long as we can make the choice for some open neighborhood of each point. In order to minimize the role played by the size of the open neighborhood of a point on which the local object is defined, we introduce the concept of a germ. Fix a point P of a topological space M. Suppose we have two functions f and g defined respectively on open neighborhoods U_f and U_g of P. We introduce the following equivalence relation. The two functions f and g are equivalent if there exists some open neighborhood W of P in $U_f \cap U_g$ so that $f|W = g|W$. By the *germ* of a function at P we mean an equivalence class of functions defined on open neighborhoods of P in the equivalence relation given above. Our goal is to try to piece together germs of a certain class of functions (*e.g.* holomorphic functions) to form global functions in the same class. Let us conduct our discussion by looking at an example. Take a point P of $\mathbf{C}$. Consider the set of germs at P of all holomorphic functions of

$\mathbf{C}$ defined locally near P. This set is simply the set of all convergent power series centered at P. We denote this set by $\mathcal{O}_P$ or $\mathcal{O}_{\mathbf{C},P}$. We denote the union of $\mathcal{O}_P$ for all $P \in \mathbf{C}$ by $\mathcal{O}$ or $\mathcal{O}_{\mathbf{C}}$. There is a natural projection $\pi : \mathcal{O} \to \mathbf{C}$ so that π maps the element of $\mathcal{O}_P$ to P. Our goal of the whole process is to find global objects. We want to piece the germs of holomorphic functions together to form global holomorphic functions. Suppose G is an open subset of $\mathbf{C}$ and we want to find a global holomorphic function f on G. We have to find for every P in G an element of $\mathcal{O}_P$ and do it in such a way that they form a global holomorphic function. In other words, we would like to find a section of $\pi : \mathcal{O} \to \mathbf{C}$ over G. We have to introduce a criterion to determine whether this section defines a holomorphic function on G. A section of $\pi : \mathcal{O} \to \mathbf{C}$ over G simply means a choice of a convergent power series for every point of G and these convergent power series may be completely independent of each other. It turns out that the best way to introduce a criterion is to impose some topology on $\mathcal{O}$ so that the sections we want are precisely the continuous sections. Why is the imposition of a topology on $\mathcal{O}$ is the best way to do it? To demand that a section s of $\pi : \mathcal{O} \to \mathbf{C}$ over G is continuous with respect to a topology means that when P and Q are two points of $\mathbf{C}$ that are sufficiently close together, we demand that $\pi(P)$ and $\pi(Q)$ are close together in a sense we are free to impose via the topology. We can achieve our goal by suitably interpreting the closeness of $\pi(P)$ and $\pi(Q)$ via the imposed topology and the concept of continuity is one of the simplest in mathematics. So doing it through topology is the simplest way and gives us the most flexibility to develop a general theory. In our case the closeness between $\pi(P)$ and $\pi(Q)$ is that the expansion at Q of the P-centered convergent power series $\pi(P)$ is $\pi(Q)$. Closeness is determined by open sets of the topology. So we define our topology of $\mathcal{O}$ as follows. We do it by specifying an open neighborhood basis of every point of $\mathcal{O}$. Take an element f of $\mathcal{O}$. Then f is represented by a convergent power series centered at some point P of $\mathbf{C}$ with a domain of convergence U. For $Q \in U$ let g_Q be the expansion at Q of the power series f. Let $\mathcal{W}$ be the set of all open neighborhoods W of P in U. For every $W \in \mathcal{W}$ let $\tilde{W} = \{g_Q | Q \in W\}$. Then an open neighborhood basis of f in our topology is $\{\tilde{W} | W \in \mathcal{W}\}$. It is straightforward to check that this gives a well-defined topology and it is the largest topology on $\mathcal{O}$ so that every holomorphic function on an open subset G of $\mathbf{C}$ defines a continuous section of $\pi : \mathcal{O} \to \mathbf{C}$ over G. This way of defining a topology can be done in very general situations, because the use of convergent power series is purely of an illustrative nature and is unnecessary. One can say that the element f of $\mathcal{O}$

is represented by a holomorphic function f on U and g_Q is the element of $\mathcal{O}_Q$ induced by the holomomrphic function f.

This topology of $\mathcal{O}$ makes $\pi : \mathcal{O} \to \mathbf{C}$ a local hoemeorphism, because clearly for every element f of $\mathcal{O}$ the restriction of π to $\tilde{W}$ maps $\tilde{W}$ homeomorphically onto W. We would like to point out that even though $\pi : \mathcal{O} \to \mathbf{C}$ is a local hoemeorphism, yet $\pi : \mathcal{O} \to \mathbf{C}$ is not a topological covering map. The reason is that for a topological covering map $\sigma : X \to Y$ one must have the property that for every $y \in Y$ there is a connected open neighborhood W of y in Y so that the map σ maps every connected component of $\sigma^{-1}(W)$ homeomorphically onto W. In particular, for every point $x \in \sigma^{-1}(y)$ the map σ maps some open neighborhood W_x of x in X homemorphically onto W and the image W of the homemorphism $\sigma|W_x$ is *the same* for all $x \in \sigma^{-1}(y)$. In the case of $\pi : \mathcal{O} \to \mathbf{C}$ the image W of the homemorphism $\pi|\tilde{W}$ must be contained in the domain of the holomorphic function f defining the germ f at P and we cannot have the same W for all elements of $\mathcal{O}_P$. So $\pi : \mathcal{O} \to \mathbf{C}$ is not a topological covering map. For every element f of $\mathcal{O}$ the connected component of $\mathcal{O}$ containing f corresponds precisely to the maximum analytic continuation of the germ f. The domain of definition of this maximum analytic in general of course is not the same as $\mathbf{C}$.

To piece local functions to form global functions we have to look at the discrepancies obtained by taking the differences of local functions. So we have to consider the algebraic process of addition. Every fiber $\mathcal{O}_P$ of $\pi : \mathcal{O} \to \mathbf{C}$ is an algebra over $\mathbf{C}$, *i.e.* the operations of addition and multiplication in it make it a ring and at the same time it is a vector space over $\mathbf{C}$ compatible with its ring structure. In particular, $\mathcal{O}_P$ is an additive abelian group. The algebraic structure of the fibers of $\pi : \mathcal{O} \to \mathbf{C}$ is compatible with the topology of $\mathcal{O}$ in the sense that the map from the fiber product $\mathcal{O} \times_\pi \mathcal{O} = \{(f, g) \in \mathcal{O} \times_\pi \mathcal{O} | \pi(f) = \pi(g)\}$ to $\mathcal{O}$ defined addition or multiplication is continuous. Moreover, the map $\mathbf{C} \times \mathcal{O} \to \mathcal{O}$ defined by scalar multiplication is continuous. Now we state the abstract definition of a sheaf.

Definition. Let M be a topological space. A sheaf of abelian groups over M is a topological space $\mathcal{S}$ with a local homeomorphism $\pi : \mathcal{S} \to M$ so that every fiber of $\pi : \mathcal{S} \to M$ is an abelian group and the map from the fiber product $\mathcal{S} \times_\pi \mathcal{S}$ to $\mathcal{S}$ defined by the addition operation is continuous.

Similarly one can define a sheaf of rings, a sheaf of vector spaces, a sheaf of algebras, etc. so that every fiber carry the algebraic structure of a ring, a vector space, an algebra, etc. and the algebraic operations are continuous. The set $\mathcal{O}$ with the projection $\pi : \mathcal{O} \to \mathbf{C}$ and its topology and the algebraic operations on the fibers of $\pi : \mathcal{O} \to \mathbf{C}$ is a sheaf of $\mathbf{C}$-algebras over $\mathbf{C}$.

The fiber over P of a sheaf $\pi : \mathcal{S} \to M$ over M is usually referred to as the stalk of $\mathcal{S}$ at P and is denoted by $\mathcal{S}_P$. For an open subset W of M the restriction of $\mathcal{S}$ to W denoted by $\mathcal{S}|W$ is the sheaf $\pi^{-1}(W)$ over M with the projection induced by π and the topology and algebraic structure induced from $\mathcal{S}$.

In the discussion of the construction of the sheaf $\mathcal{O}_{\mathbf{C}}$ of germs of holomorphic functions on $\mathbf{C}$, for every open subset G of $\mathbf{C}$ one considers a class of functions on G and this class of functions is the set of all holomorphic functions on G. Then one introduces the concept of a germ of a function in that class. For this step one has to know how to restrict a function on G to a subset of G. Having a class of functions on G and knowing how to restrict a function on G to a subset of G are the only two ingredients needed to construct the sheaf $\mathcal{O}_{\mathbf{C}}$. We formulate these two ingredients in an abstract definition of a presheaf.

Definition. Let M be a topological space. A presheaf $\mathcal{P}$ of abelian groups over M is an assignment to every open subset U of M an abelian group $\mathcal{P}_U$ and an assignment to every inclusion map $V \to U$ a homomorphism of abelian groups $\mathcal{P}_V \to \mathcal{P}_U$ so that the composite of $\mathcal{P}_W \to \mathcal{P}_V$ and $\mathcal{P}_V \to \mathcal{P}_U$ is $\mathcal{P}_W \to \mathcal{P}_U$ when one has inclusion maps $W \to V \to U$. In other words, $\mathcal{P}$ is a functor from the category of open subsets of M and inclusions to the category of abelian groups and homomorphisms.

In most of our applications the topological space M is a complex manifold. In order to avoid spending time to handle exceptional pathological cases we assume that the base topological space M is always a locally compact metrizable topological space.

An example of a presheaf is $M = \mathbf{C}$ and $\mathcal{P}_U = \mathit{all}$ holomorphic functions on U. In general from a presheaf $\mathcal{P}$ one can construct an associated sheaf $\mathcal{S}$ in the following way. The fiber $\mathcal{S}_P$ over the point P of M is the direct limit of $\mathcal{P}_U$ for U in the directed set of all open neighborhoods of P in M. An open neighborhood basis of the point s of $\mathcal{S}_P$ coming from an element s of $\mathcal{P}_U$ consists of all sets of the form $\{$image of s
$hboxin \mathcal{S}_Q | Q \in W\}$, where W is an open neighborhood of P in U. Conversely when we are given a sheaf $\mathcal{S}$ we can construct a presheaf $\mathcal{P}$ whose associated

sheaf is $\mathcal{S}$ by setting $\mathcal{P}_U$ equal to the set of all continuous sections of $\mathcal{S}$ over U.

In general the topology of a sheaf may not be Hausdorff. Consider the sheaf $\mathcal{E}$ of germs of C^∞ functions on $\mathbf{R}$. It comes from the presheaf which assigns to every opens subset G of $\mathbf{R}$ the set of all C^∞ functions on G. Let f be the function defined by $f(x) = \exp(-\frac{1}{x^2})$ for $x > 0$ and $f(x) = 0$ otherwise. Let f be the germ of f at $x = 0$ and let $\mathbf{0}$ be the germ at $x = 0$ of the function that is identically zero. Then the elements f and $\mathbf{0}$ of the stalk $\mathcal{E}_0$ cannot be separated by open subsets, because an open neighborhood of f contains the set

$$\tilde{G} = \{\textit{the} \text{ germ of } f \text{ at } x | x \in G\}$$

for some open neighborhood G of 0 in $\mathbf{R}$ and an open neighborhood of $\mathbf{0}$ contains the set

$$\tilde{H} = \{ \text{ the germ of the zero function at } x | x \in H\}$$

for some open neighborhood H of 0 in $\mathbf{R}$. The germs of f and the zero function coincide at any point x in $G \cap H$ with $x < 0$ and this common germ belong to both $\tilde{G}$ and $\tilde{H}$. However, the topology of the sheaf $\mathcal{O}_{\mathbf{C}}$ is Hausdorff because a holomorphic function on a domain is identically zero everywhere if it is identically zero on a nonempty open set.

Suppose we have two sheaves of abelian groups $\mathcal{S}$ and $\mathcal{T}$ over a topological space M. A *sheaf-homomorphism* $\varphi : \mathcal{S} \to \mathcal{T}$ is a continuous map so that the map $\varphi_P : \mathcal{S}_P \to \mathcal{T}_P$ between the stalks at every point P of M induced by φ is a group homomorphism. Since both $\mathcal{S}$ and $\mathcal{T}$ are locally homeomorphic to M under their respective projections, it follows a sheaf-homomorphism between $\mathcal{S}$ and $\mathcal{T}$ is a local homeomorphism. It is clear that the kernel, the image, and the cokernel of a sheaf-homomorphism with the induced topology and algebraic structure are sheaves over the same base space.

(2.2) *Sheaf Cohomology.*

When we discuss the construction of a meromorphic function on a Riemann surface with a given principal part at a given point P we have to find holomorphic functions g_α on U_α such that $f_{\alpha\beta} = g_\beta - g_\alpha$ on $U_\alpha \cap U_\beta$. One can formulate this in a general setting by introducing the concept of sheaf cohomology. Suppose M is a topology space and $\mathcal{S}$ is a sheaf of abelian groups over M. Let $\mathcal{U} = \{U_\alpha\}$ be a covering of M by open subsets. For every nonnegative integer q we define $C^q(\mathcal{U}, \mathcal{S})$ as follows. An element of $C^q(\mathcal{U}, \mathcal{S})$ is $f = \{f_{\alpha_0 \cdots \alpha_q}\}$, where $f_{\alpha_0 \cdots \alpha_q}$ is a continuous section of $\mathcal{S}$ over $U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$ and $f_{\alpha_0 \cdots \alpha_q}$ is skew-symmetric in $\alpha_0, \cdots, \alpha_q$. We call $C^q(\mathcal{U}, \mathcal{S})$ the group of alternating q-cochains for the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$. An element of $C^q(\mathcal{U}, \mathcal{S})$ is an alternating q-cochain for the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$ or simply a q-cochain when there is no confusion. For notational convenience we define $C^q(\mathcal{U}, \mathcal{S})$ to be the zero group when $q = -1$.

We define a map $\delta : C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$ as follows. The image of $f = \{f_{\alpha_0 \cdots \alpha_q}\}$ under δ is $g = \{g_{\alpha_0 \cdots \alpha_{q+1}}\}$, where

$$g_{\alpha_0 \cdots \alpha_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu f_{\alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_{q+1}}$$

on $U_{\alpha_0} \cap \cdots \cap U_{\alpha_{q+1}}$ and $\hat{\alpha}_\nu$ means that the index α_ν is omitted. We call the map $\delta : C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$ the coboundary map. Denote by $Z^q(\mathcal{U}, \mathcal{S})$ the kernel of $\delta : C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$ and denote by $B^q(\mathcal{U}, \mathcal{S})$ the image of $\delta : C^{q-1}(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{U}, \mathcal{S})$. The group $Z^q(\mathcal{U}, \mathcal{S})$ (respectively $B^q(\mathcal{U}, \mathcal{S})$) is respectively called the group of alternating q-cocycles (respectively q-coboundaries) for the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$.

The composite of $\delta : C^{q-1}(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{U}, \mathcal{S})$ and $\delta : C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$ is zero, because if the image of the element $h = \{h_{\alpha_0 \cdots \alpha_{q-1}}\}$ of $C^{q-1}(\mathcal{U}, \mathcal{S})$ under $\delta : C^{q-1}(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{U}, \mathcal{S})$ is $f = \{f_{\alpha_0 \cdots \alpha_q}\}$, then

$$g_{\alpha_0 \cdots \alpha_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu f_{\alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_{q+1}}$$

$$= \sum_{\nu=0}^{q+1} (-1)^\nu \left(\sum_{\mu<\nu} (-1)^\mu h_{\alpha_0 \cdots \hat{\alpha_\mu} \cdots \hat{\alpha_\nu} \cdots \alpha_{q+1}} + \sum_{\mu>\nu} (-1)^{\mu-1} h_{\alpha_0 \cdots \hat{\alpha_\mu} \cdots \hat{\alpha_\nu} \cdots \alpha_{q+1}} \right)$$

$$= \sum_{0 \le \mu < \nu \le q+1} (-1)^{\mu+\nu} h_{\alpha_0 \cdots \hat{\alpha_\mu} \cdots \hat{\alpha_\nu} \cdots \alpha_{q+1}}$$

$$+ \sum_{0 \le \nu < \mu \le q+1} (-1)^{\mu+\nu-1} h_{\alpha_0 \cdots \hat{\alpha_\mu} \cdots \hat{\alpha_\nu} \cdots \alpha_{q+1}} = 0.$$

Hence $B^q(\mathcal{U}, \mathcal{S})$ is contained in $Z^q(\mathcal{U}, \mathcal{S})$. Denote by $H^q(\mathcal{U}, \mathcal{S})$ the quotient $Z^q(\mathcal{U}, \mathcal{S})/B^q(\mathcal{U}, \mathcal{S})$. The group $H^q(\mathcal{U}, \mathcal{S})$ is called the cohomology group of dimension q for the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$. It is clear from the definition that $H^0(\mathcal{U}, \mathcal{S})$ is simply the set of all global continuous sections of the sheaf $\mathcal{S}$ over M and is usually denoted also by $\Gamma(M, \mathcal{S})$.

In our the construction of a meromorphic function on a Riemann surface with a given principal part at a given point P the collection $\{f_{\alpha\beta}\}$ is a 1-cocyle with coefficients in the sheaf of germs of holomorphic functions on M and the existence of $\{g_\alpha\}$ is equivalent to $\{f_{\alpha\beta}\}$ being a 1-coboundary. In this formulation the obstruction to the solution of the problem is the cohomology group of dimension 1. When we try to piece together local meromorphic functions to form a global meromorphic function, we want to get rid of the discrepancies of the local meromorphic functions and it would serve the same purpose if we can get rid of the discrepancies by going to a refinement of the covering. The problem is solved if the cohomology group of dimension 1 vanishes when one goes to a refinement of the covering. This suggests that one should take all coverings of the space and consider the direct limit of the cohomology groups for all the coverings.

In the general case when we have a refinement $\mathcal{V}$ of the covering $\mathcal{U}$ we have natural homomorphisms $C^q(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{V}, \mathcal{S})$, $Z^q(\mathcal{U}, \mathcal{S}) \to Z^q(\mathcal{V}, \mathcal{S})$, $B^q(\mathcal{U}, \mathcal{S}) \to B^q(\mathcal{V}, \mathcal{S})$, and $H^q(\mathcal{U}, \mathcal{S}) \to H^q(\mathcal{V}, \mathcal{S})$. We define $H^q(M, \mathcal{S})$ as the direct limit of $H^q(\mathcal{U}, \mathcal{S})$ as $\mathcal{U}$ runs through the directed set of all open coverings of M. The group $H^q(M, \mathcal{S})$ is called the cohomology group of dimension q of M with coefficients in the sheaf $\mathcal{S}$. We denote respectively by $C^q(M, \mathcal{S})$, $Z^q(M, \mathcal{S})$, and $B^q(M, \mathcal{S})$ the direct limits of $C^q(\mathcal{U}, \mathcal{S})$, $B^q(\mathcal{U}, \mathcal{S})$, and $Z^q(\mathcal{U}, \mathcal{S})$. Then we have an induced coboundary map $\delta : C^q(M, \mathcal{S}) \to C^{q+1}(M, \mathcal{S})$ whose kernel is $Z^q(M, \mathcal{S})$ and whose image is $B^{q+1}(M, \mathcal{S})$. Moreover, $H^q(M, \mathcal{S})$ is the quotient of $Z^q(M, \mathcal{S})$ by $B^q(M, \mathcal{S})$.

Among all cohomology groups of positive dimension the most important cohomology group is the one of dimension 1. Its vanishing enables one to piece together local continuous sections of a sheaf to form a global continuous section. Why are the cohomology groups of higher dimensions introduced? They are introduced to help us compute the cohomology group of dimension 1, because when we have a short exact sequence of three sheaves we have a long exact sequence of cohomology groups. We are going to discuss this long exact sequence of cohomology groups.

Suppose $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ is an exact sequence of sheaves and sheaf-homomorphisms. For every q we have the short exact sequence

$$0 \to C^q(M, \mathcal{S}') \to C^q(M, \mathcal{S}) \to C^q(M, \mathcal{S}'') \to 0.$$

This is clear except the surjectivity of $C^q(M, \mathcal{S}) \to C^q(M, \mathcal{S}'')$. Suppose we have an element $f'' = \{f_{\alpha_0 \cdots \alpha_q}$ of $C^q(\mathcal{U},\ \mathcal{S}'')$. We have to show that its restriction to some refinement $\mathcal{W}$ of $\mathcal{U}$ is the image of some element of $C^q(\mathcal{W},$ $\mathcal{S})$. First let us make a trivial observation. Suppose we have a continuous section g'' of $\mathcal{S}''$ over an open subset G of M. We give M a metric $d(\cdot, \cdot)$. For every point P in G there exists a maximum positive number $\eta = \eta(g'', P)$ such that the ball $B_\eta(P)$ of radius η centered at P is contained in G and for every $r < \eta$ the restriction $g''|B_r(P)$ of g'' to $B_r(P)$ is the image of some continuous section of $\mathcal{S}$ over $B_r(P)$. The function $\eta(g'', P)$ is clearly a lower semi-continuous function of P so that if $\eta(g'', P_0) > \epsilon$ then $\eta(g'', P) > \epsilon$ for all P in some open neighborhood of P_0. We can assume without loss of generality that the covering $\mathcal{U}$ is locally finite. We choose U'_α relatively compact in U_α so that $\mathcal{U}' = \{U'_\alpha\}$ still covers M. For every point P in M we let $\epsilon(P)$ be the minimum of $\eta(f_{\alpha_0 \cdots \alpha_q}, P)$ for all U'_{α_0} containing P. Clearly every P admits an open neighborhood on which the function $\epsilon(\cdot)$ has a positive lower bound. Now for every point P in M let W_P be an open metric ball centered at P of radius $r(P)$ contained in U'_α for some $\alpha = \alpha(P)$ so that the function $\epsilon(\cdot) is > 2\ r(P)$ on W_P. Let $\mathcal{W} = \{W_P\}_{P \in M}$. Then $\mathcal{W}$ is a refinement of $\mathcal{U}'$. Suppose $W_{P_0}, \cdots, W_{P_q}$ have a common point P. We can choose a number r such that $r < \epsilon(P)$ and $r > 2\ r(P_\nu)$ for $0 \le \nu \le q$. Let B be the metric ball centered at P whose radius r. Then B contains W_{P_ν} for $0 \le \nu \le q$ and B is contained in $U_{\alpha(P_0)} \cap \cdots \cap U_{\alpha(P_q)}$. Moreover, $f_{\alpha(P_0) \cdots \alpha(P_q)}|B$ is the image of some continuous section g of $\mathcal{S}$ over B. Let $f'_{P_0 \cdots P_q}$ be the restriction of g to $W_{P_0} \cap \cdots \cap W_{P_q}$. We skew-symmetrize $f'_{P_0 \cdots P_q}$ with respect to $P_0, \cdots, P_q$. Then the restriction of f'' to $\mathcal{W}$ is the image of the element $\{f'_{P_0 \cdots P_q}\}$ of $C^q(\mathcal{W},\ \mathcal{S})$.

From the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{q-1}(M, \mathcal{S}') & \to & C^{q-1}(M, \mathcal{S}) & \to & C^{q-1}(M, \mathcal{S}'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{q}(M, \mathcal{S}') & \to & C^{q}(M, \mathcal{S}) & \to & C^{q}(M, \mathcal{S}'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{q+1}(M, \mathcal{S}') & \to & C^{q+1}(M, \mathcal{S}) & \to & C^{q+1}(M, \mathcal{S}'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

we get a long exact sequence

$$
\begin{aligned}
&0 \to \Gamma(M, \mathcal{S}') \to \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}'') \to H^1(M, \mathcal{S}') \to \cdots \to \\
&H^q(M, \mathcal{S}') \to H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}'') \xrightarrow{\varphi} H^{q+1}(M, \mathcal{S}') \to \cdots .
\end{aligned}
$$

The only map in the long exact sequence that needs some explanation is the so-called connecting homomorphism φ. It is defined as follows. Take an element f'' of $Z^q(M, \mathcal{S}'')$. We can find an element f of $C^q(M, \mathcal{S})$ whose image under $\mathcal{S} \to \mathcal{S}''$ is f''. Let $g \in C^{q+1}(M, \mathcal{S})$ be the image of f under the coboundary map $C^q(M, \mathcal{S}) \to C^{q+1}(M, \mathcal{S})$. From the above commutative diagram it follows that g is the image of some element h' of $Z^{q+1}(M, \mathcal{S}')$. The element of $H^{q+1}(M, \mathcal{S}')$ defined by h' is the image under φ of the element of $H^q(M, \mathcal{S}'')$ defined by f''. The exactness of the sequence is a consequence of straightforward diagram chasing.

(2.3) *Two Ways of Computing Cohomology Groups*

It is impractical to calculate the cohomology groups from definition, because we have to compute the cohomology groups for each covering and then take their direct limit. In practice two simpler ways of computation are used. The first one uses a special covering called Leray covering so that the cohomology groups can be computed with respect to only one such covering and it is not necessary to take any direct limit. The second way uses a resolution of the sheaf by so-called soft sheaves or more generally acyclic sheaves and the computation of the cohomology is reduced to computation involving the global continuous sections of the soft sheaves. We formulate these two simpler ways of computing cohomology groups in the following two theorems.

Theorem (Leray covering). Suppose $\mathcal{S}$ is a sheaf of abelian groups over a topological space M and $\mathcal{U} = \{U_\alpha\}$ is a locally finite open covering of M so that $H^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p},\, \mathcal{S})$ vanishes for any $q > 0$ and for all $\alpha_0, \cdots, \alpha_p$. Then the natural map $H^q(\mathcal{U},\, \mathcal{S}) \to H^q(M,\, \mathcal{S})$ is an isomorphism for all q.

Theorem (*Acyclic sheaf resolution*). Suppose M is a topological space and $0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$ is an exact sequence of sheaves over M. Suppose $H^q(M, \mathcal{S}^p) = 0$ for $q > 0$ and $p \geq 0$. Then $H^q(M, \mathcal{S})$ is isomorphic to the kernel of $\Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{S}^{q+1})$ modulo the image of $\Gamma(M, \mathcal{S}^{q-1}) \to \Gamma(M, \mathcal{S}^q)$ for all q.

Before we prove the theorems let us explain the term acyclic. A sheaf $\mathcal{S}$ over a topological space M is *acyclic* if $H^q(M, \mathcal{S})$ vanishes for all $q \geq 1$.

Both theorems involve some condition of acyclicity. The assumption of the theorem on Leray coverings says that $\mathcal{S}|U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is acyclic for all $\alpha_0, \cdots, \alpha_p$. A covering $\mathcal{U} = \{U_\alpha\}$ satisfying this assumption is called a *Leray covering* for the sheaf $\mathcal{S}$. The assumption of the theorem on acyclic sheaf resolution says that each $\mathcal{S}^p$ is acyclic.

We first prove the theorem on acyclic sheaf resolution. The statement is trivially true when $q = 0$. Assume $q \geq 1$. Let $\mathcal{K}^p$ be the kernel of the sheaf-homomorphism $\mathcal{S}^p \to \mathcal{S}^{p+1}$. From the short exact sequence of sheaves $0 \to \mathcal{K}^p \to \mathcal{S}^p \to \mathcal{K}^{p+1} \to 0$ we have the cohomology exact sequence

$$H^\nu(M, \mathcal{S}^p) \to H^\nu(M, \mathcal{K}^{p+1}) \to H^{\nu+1}(M, \mathcal{K}^p) \to H^{\nu+1}(M, \mathcal{S}^p).$$

Since each $\mathcal{S}^p$ is acyclic, it follows that $H^\nu(M, \mathcal{K}^{p+1}) \to H^{\nu+1}(M, \mathcal{K}^p)$ is isomorphic for $\nu \geq 1$. From $\mathcal{S} = \mathcal{K}^0$ we conclude that $H^q(M, \mathcal{S})$ is isomorphic to $H^1(M, \mathcal{K}^{q-1})$. From the short exact sequence $0 \to \mathcal{K}^{q-1} \to \mathcal{S}^q \to \mathcal{K}^q \to 0$ we have the exact sequence

$$\Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{K}^q) \to H^1(M, \mathcal{K}^{q-1}) \to H^1(M, \mathcal{S}^q) = 0.$$

So $H^1(M, \mathcal{K}^{q-1})$ is isomorphic to the cokernel of $\Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{K}^q)$. From the definition of $\mathcal{K}^q$ we know that $\Gamma(M, \mathcal{K}^q)$ is isomorphic to the kernel of $\Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{S}^{q+1})$. The image of $\Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{K}^q)$ is clearly isomorphic to the image of $\Gamma(M, \mathcal{S}^{q-1}) \to \Gamma(M, \mathcal{S}^q)$. This concludes the proof of the theorem.

Before we prove the theorem on Leray covering, we would like to discuss the construction of acyclic sheaf resolution. How does one get an acyclic sheaf? Recall that the cohomology groups of positive dimensions were introduced to handle the passage from local sections to global sections. When the sheaf has the property that every local section automatically extends to a global section, we expect all the positive-dimensional cohomology groups to vanish. Which sheaves would have such a property? Sheaves are constructed from presheaves. For example, in the case of sheaves of germs of functions, if we impose no condition on the class of functions used, then clearly for the sheaves of germs of *all* functions every local section extends to a global one. Let us make this more precise. A sheaf $\mathcal{S}$ over a topological space M is *flabby* if for every open subset G of M the restriction map $\Gamma(M, \mathcal{S}) \to \Gamma(G, \mathcal{S})$ is surjective, *i.e.* every continuous section of $\mathcal{S}$ over any open subset of M can be extended to a global continuous section of $\mathcal{S}$ over M. For any given sheaf

$\mathcal{S}$ over a topological space M we denote by $\mathcal{C}(\mathcal{S})$ the sheaf obtained from the presheaf which assigns to every open subset U of M the abelian group of *all* sections of $\mathcal{S}$ over U whether continuous or not. In other words, $\mathcal{C}(\mathcal{S})$ is the sheaves of germs of all local sections of $\mathcal{S}$ whether continuous or not. Clearly $\mathcal{C}(\mathcal{S})$ is a flabby sheaf.

We would like to show that every flabby sheaf is acyclic. First we observe that if in the short sequence of sheaves $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ over M the sheaf $\mathcal{S}'$ is flabby, then the map $\Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}'')$ is surjective. Without loss of generality we can assume that M is connected. Take a continuous section s of $\mathcal{S}''$ over M. Let G be an open subset of M with the property that $s|G$ is the image of some $t \in \Gamma(G, \mathcal{S})$ and assume that G is not contained properly in any open subset of M with this property. Such a maximal open subset G clearly exists. We have to show that G agrees with M. Assume the contrary. Since M is connected, G has a boundary point P not belonging to G. Since the sheaf-homomorphism $\mathcal{S} \to \mathcal{S}''$ is surjective, there exist an open neighborhood U of P in M and an element u of $\Gamma(U, \mathcal{S})$ such that $s|U$ is the image of u. The continuous section $(t - u)|U \cap G$ of $\mathcal{S}$ over $U \cap G$ is a continuous section of $\mathcal{S}'$ over $U \cap G$ and so can be extended to a continuous section v of $\mathcal{S}'$ over M. Then the continuous section of $\mathcal{S}$ over $G \cup U$ which is equal to s on G and equal to $u + v$ on U is mapped to s under $\mathcal{S} \to \mathcal{S}''$, contradicting the maximality of G. Hence $G = M$ and $\Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}'')$ is surjective. From this observation we conclude that in the short exact sequence of sheaves $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ over M if $\mathcal{S}'$ and $\mathcal{S}$ are flabby then $\mathcal{S}''$ is also flabby. When we have a long exact sequence of flabby sheaves $0 \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \mathcal{S}^3 \to \cdots$ over M, the induced sequence

$$0 \to \Gamma(M, \mathcal{S}^0) \to \Gamma(M, \mathcal{S}^1) \to \Gamma(M, \mathcal{S}^2) \to \Gamma(M, \mathcal{S}^3) \to \cdots$$

is exact, because if we denote the kernel of $\mathcal{S}^q \to \mathcal{S}^{q+1}$ by $\mathcal{K}^q$, then $\mathcal{K}^0 = \mathcal{S}^0$ and from the exact sequence $0 \to \mathcal{K}^q \to \mathcal{S}^q \to \mathcal{K}^{q+1} \to 0$ and by induction on q every $\mathcal{K}^q$ is flabby and

$$0 \to \Gamma(M, \mathcal{K}^q) \to \Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{K}^{q+1}) \to 0$$

is exact.

Suppose G is an open subset of M and $\mathcal{S}$ is a sheaf over M. We denote by $(\mathcal{S}|G)$ the trivial extension to M of the restriction of $\mathcal{S}$ to G. Let us make more precise what we mean. Consider the presheaf $\mathcal{P}_G$ which assigns to every open subset U of M the abelian group $\Gamma(U \cap G, \mathcal{S})$. The sheaf constructed from this presheaf is our sheaf $(\mathcal{S}|G)$. At points of G the stalks of $(\mathcal{S}|G)$ and $\mathcal{S}$ are the same, but at interior points of $M - G$ the stalk of $(\mathcal{S}|G)$ is the zero group. If H is an open subset of G, then the homomorphism $\mathcal{P}_G(U) \to \mathcal{P}_H(U)$ which is the identity map for $U \subset H$ and is the zero map otherwise induces a sheaf-homomorphism $(\mathcal{S}|G) \to (\mathcal{S}|H)$. This sheaf-homomorphism is the same as the one defined by restriction.

Now assume that $\mathcal{S}$ is a flabby sheaf over a topological space M. Take a countable locally finite open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M with I equal to a subset of the natural numbers. For any nonnegative integer q let $\mathcal{S}^q$ be the direct sum of $(\mathcal{S}|U_{\alpha_0} \cap \cdots \cap U_{\alpha_q})$ for all $\alpha_0 < \cdots < \alpha_q$ and $U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$ nonempty. A global section of $\mathcal{S}^q$ over M is naturally isomorphic to $C^q(\mathcal{U}, \mathcal{S})$. There is a natural sheaf-homomorphism $\mathcal{S}^q \to \mathcal{S}^{q+1}$ so that the induced map $\Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{S}^{q+1})$ corresponds precisely to the coboundary map $C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$. We claim that the long exact sequence of sheaves $0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$ is exact. For notation simplicity we denote $\mathcal{S}$ by $\mathcal{S}^{-1}$. Take a point P of M. Let J be the subset of all elements α of I so that U_α contains P. Let N be the number of elements in J. For every integer $q \geq -1$ let N_q be the binomial coefficient $\binom{N}{q+1}$. The stalk of $\mathcal{S}^q$ at P is the direct sum of N_q copies of the stalk of $\mathcal{S}$ at P. Every element f^q of $\mathcal{S}^q_P$ is of the form $\{f^q_{\alpha_0 \cdots \alpha_q}\}_{\alpha_0 < \cdots < \alpha_q \text{ in } J}$ with $f^q_{\alpha_0 \cdots \alpha_q} \in \mathcal{S}_P$. Define $f^q_{\alpha_0 \cdots \alpha_q}$ for all $\alpha_0, \cdots, \alpha_q \in J$ by making $f^q_{\alpha_0 \cdots \alpha_q}$ skew-symmetric in $\alpha_0, \cdots, \alpha_q$. If f^{q+1} is the image of f^q under $\mathcal{S}^q_P \to \mathcal{S}^{q+1}_P$, then

$$f^{q+1}_{\alpha_0 \cdots \alpha_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu f^q_{\alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_{q+1}}.$$

Just as in the proof that the composite of two coboundary maps is zero, one easily verifies that the composite of $\mathcal{S}^{q-1}_P \to \mathcal{S}^q_P$ and $\mathcal{S}^q_P \to \mathcal{S}^{q+1}_P$ is zero. Suppose f^q is mapped to zero under $\mathcal{S}^q_P \to \mathcal{S}^{q+1}_P$. We want to show that it is the image of some f^{q-1} under $\mathcal{S}^{q-1}_P \to \mathcal{S}^q_P$. Fix some $\beta \in J$. Define f^{q-1} by $f^{q-1}_{\alpha_0 \cdots \alpha_{q-1}} = f^q_{\beta \alpha_0 \cdots \alpha_{q-1}}$. Let the image of f^{q-1} under $\mathcal{S}^{q-1}_P \to \mathcal{S}^q_P$ be g^q. Then

$$g^q_{\alpha_0 \cdots \alpha_q} = \sum_{\nu=0}^{q} (-1)^\nu f^{q-1}_{\alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_q} = \sum_{\nu=0}^{q} (-1)^\nu f^q_{\beta \alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_q}.$$

Since the image of f^q under $\mathcal{S}^q_P \to \mathcal{S}^{q+1}_P$ is zero, we have

$$f^q_{\alpha_0 \cdots \alpha_q} + \sum_{\nu=0}^{q} (-1)^{\nu+1} f^q_{\beta \alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_q} = 0.$$

Hence $g^q_{\alpha_0\cdots\alpha_q} = f^q_{\alpha_0\cdots\alpha_q}$. This argument is the same as proving the vanishing of positive-dimensional cohomology groups of a simplex by contracting it to one of its vertices. So the sequence $0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$ is exact. Since all terms in the sequence are clearly flabby, the sequence

$$0 \to \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}^0) \to \Gamma(M, \mathcal{S}^1) \to \Gamma(M, \mathcal{S}^2) \to \cdots$$

is exact and consequently the sequence

$$0 \to \Gamma(M, \mathcal{S}) \to C^0(\mathcal{U}, \mathcal{S}) \to C^1(\mathcal{U}, \mathcal{S}) \to C^2(\mathcal{U}, \mathcal{S}) \to C^3(\mathcal{U}, \mathcal{S}) \to \cdots.$$

So $H^q(\mathcal{U}, \mathcal{S})$ vanishes for $q \geq 1$ and $\mathcal{S}$ is acyclic over M. We would like to remark that actually we have the stronger statement that for any locally finite open covering $\mathcal{U}$ of M we have the vanishing of $H^q(\mathcal{U}, \mathcal{S})$ vanishes for $q \geq 1$ when $\mathcal{S}$ is a flabby sheaf over M.

Now that we know that every flabby sheaf is acyclic, we have the following acyclic sheaf resolution for any sheaf of abelian groups $\mathcal{S}$ over a topological space M. Recall that $\mathcal{C}(\mathcal{S})$ denotes the sheaf of germs of all (possibly discontinuous) sections of $\mathcal{S}$. Inductively define $\mathcal{S}^0 = \mathcal{C}(\mathcal{S})$ and $\mathcal{S}^{q+1} = \mathcal{C}(\mathcal{S}^q/\mathcal{S}^{q-1})$, where $\mathcal{S}^{-1}$ means $\mathcal{S}$. Then each $\mathcal{S}^q$ is flabby and the sequence $0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$ is exact. This acyclic sheaf resolution of $\mathcal{S}$ is only of theoretical interest, because in actual practice it is not easier to compute $H^q(M, \mathcal{S})$ from the sequence

$$0 \to \Gamma(M, \mathcal{S}^0) \to \Gamma(M, \mathcal{S}^1) \to \Gamma(M, \mathcal{S}^2) \to \cdots$$

than directly from the definition of $H^q(M, \mathcal{S})$. There is another way to get acyclic sheaves which is very useful in actual applications. A sheaf $\mathcal{S}$ over a topological space M is said to be a *soft* sheaf if for every locally finite open covering $\mathcal{U} = \{U_\alpha\}$ of M there exists a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\mathcal{U}$ in the following sense. Each ρ_α is a continuous section of $Hom(\mathcal{S}, \mathcal{S})$ over M with support in U_α so that $\sum_\alpha \rho_\alpha$ is the identity section of $Hom(\mathcal{S}, \mathcal{S})$ over M. Here $Hom(\mathcal{S}, \mathcal{S})$ is the sheaf over M given by the presheaf which assigns to every open subset W of M the abelian group of all sheaf-homomorphisms from $\mathcal{S}|W$ to $\mathcal{S}|W$. As an example, we let M be a smooth real manifold and consider the sheaf $\mathcal{E}$ of germs of all C^∞ functions on M. Let $\{\rho_\alpha\}$ be the usual partition of unity subordinate to a locally finite open covering $\mathcal{U} = \{U_\alpha\}$. We denote also by ρ_α the sheaf-homomorphism from $\mathcal{E}$ to $\mathcal{E}$ defined by multiplication by the function ρ_α. Then the collection of the sheaf-homomorphisms $\{\rho_\alpha\}$ is a partition of unity in the sense used in the definition of a soft sheaf. So $\mathcal{E}$ is a soft sheaf.

We show now that every soft sheaf $\mathcal{S}$ over a topological space M is acyclic. Take a locally finite open covering $\mathcal{U} = \{U_\alpha\}$ and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\mathcal{U}$ in the sense used in the definition of a soft sheaf. Take $q \geq 1$. Let $f = \{f_{\alpha_0 \cdots \alpha_q}\}$ be an element of $Z^q(\mathcal{U}, \mathcal{S})$. We want to show that f is an element of $B^q(\mathcal{U}, \mathcal{S})$. Define an element $g = \{g_{\alpha_0 \cdots \alpha_{q-1}}\}$ of $C^{q-1}(\mathcal{U}, \mathcal{S})$ by $g_{\alpha_0 \cdots \alpha_{q-1}} = \sum_\beta \rho_\beta(f_{\beta\alpha_0 \cdots \alpha_{q-1}})$. Since ρ_β is a continuous section of $Hom(\mathcal{S}, \mathcal{S})$ over M with support in U_α, we can regard $\rho_\beta(f_{\beta\alpha_0 \cdots \alpha_{q-1}})$ naturally as a continuous section of $\mathcal{S}$ over $U_{\alpha_0} \cap \cdots \cap U_{\alpha_{q-1}}$. Let $h = \{h_{\alpha_0 \cdots \alpha_q}\}$ be the image of g under the coboundary operator. Then

$$h_{\alpha_0 \cdots \alpha_q} = \sum_\beta \sum_{\nu=0}^{q} (-1)^\nu \rho_\beta(f_{\beta\alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_q}).$$

Since f is a cocycle, we have

$$f_{\alpha_0 \cdots \alpha_q} + \sum_{\nu=0}^{q} (-1)^{\nu+1} f_{\beta\alpha_0 \cdots \hat{\alpha_\nu} \cdots \alpha_q} = 0.$$

Hence

$$h_{\alpha_0 \cdots \alpha_q} = \sum_\beta \rho_\beta(f_{\alpha_0 \cdots \alpha_q}) = f_{\alpha_0 \cdots \alpha_q}.$$

So f is an element of $B^q(\mathcal{U}, \mathcal{S})$ and $\mathcal{S}$ is acyclic.

Now we prove the theorem on Leray covering. Let $0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$ be a flabby sheaf resolution of $\mathcal{S}$. Consider the following commutative diagram.

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Gamma(\mathcal{U},\mathcal{S}^0) & \to & \Gamma(\mathcal{U},\mathcal{S}^1) & \to & \Gamma(\mathcal{U},\mathcal{S}^2) & \to & \Gamma(\mathcal{U},\mathcal{S}^3) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^0(\mathcal{U},\mathcal{S}) & \to & C^0(\mathcal{U},\mathcal{S}^0) & \to & C^0(\mathcal{U},\mathcal{S}^1) & \to & C^0(\mathcal{U},\mathcal{S}^2) & \to & C^0(\mathcal{U},\mathcal{S}^3) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^1(\mathcal{U},\mathcal{S}) & \to & C^1(\mathcal{U},\mathcal{S}^0) & \to & C^1(\mathcal{U},\mathcal{S}^1) & \to & C^1(\mathcal{U},\mathcal{S}^2) & \to & C^1(\mathcal{U},\mathcal{S}^3) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^2(\mathcal{U},\mathcal{S}) & \to & C^2(\mathcal{U},\mathcal{S}^0) & \to & C^2(\mathcal{U},\mathcal{S}^1) & \to & C^2(\mathcal{U},\mathcal{S}^2) & \to & C^2(\mathcal{U},\mathcal{S}^3) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^3(\mathcal{U},\mathcal{S}) & \to & C^3(\mathcal{U},\mathcal{S}^0) & \to & C^3(\mathcal{U},\mathcal{S}^1) & \to & C^3(\mathcal{U},\mathcal{S}^2) & \to & C^3(\mathcal{U},\mathcal{S}^3) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

In the diagram the vertical maps are coboundary maps and the horizontal maps are induced by the sheaf-homomorphisms in $0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$. All the rows except the row

$$0 \to \Gamma(M,\mathcal{S}^0) \to \Gamma(M,\mathcal{S}^1) \to \Gamma(M,\mathcal{S}^2) \to \Gamma(M,\mathcal{S}^3) \to \cdots$$

are exact, because the row

$$0 \to C^q(\mathcal{U},\mathcal{S}) \to C^q(\mathcal{U},\mathcal{S}^0) \to C^q(\mathcal{U},\mathcal{S}^1) \to C^q(\mathcal{U},\mathcal{S}^2) \to C^q(\mathcal{U},\mathcal{S}^3) \to \cdots$$

is the direct product of

$$\begin{gathered}
0 \to \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q},\mathcal{S}) \to \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q},\mathcal{S}^0) \to \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q},\mathcal{S}^1) \\
\to \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q},\mathcal{S}^2) \to \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q},\mathcal{S}^3) \to \cdots
\end{gathered}$$

which is exact due to the acyclicity of $\mathcal{S}|U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$ and the theorem on acyclic sheaf resolution. All columns in the commutative diagram except the column with terms $C^q(\mathcal{U},\mathcal{S})$ are exact, because each $\mathcal{S}^q$ is acyclic and $H^\nu(\mathcal{U},\mathcal{S}^q) = 0$ for $\nu \geq 1$. We claim that this implies that the cohomology of the exceptional row is isomorphic to the cohomology of the exceptional column. We use the following names for maps in the diagram

$$\begin{aligned}
\varphi^p &: C^p(\mathcal{U},\mathcal{S}) \to C^p(\mathcal{U},\mathcal{S}^0) \\
\varphi^{p,q} &: C^p(\mathcal{U},\mathcal{S}^q) \to C^p(\mathcal{U},\mathcal{S}^{q+1}) \\
\iota^q &: \Gamma(M,\mathcal{S}^q) \to C^0(\mathcal{U},\mathcal{S}^q) \\
\delta^{p,q} &: C^p(\mathcal{U},\mathcal{S}^q) \to C^{p+1}(\mathcal{U},\mathcal{S}^q).
\end{aligned}$$

Take an element f^q of $\Gamma(M, \mathcal{S}^q)$ which is mapped to zero under $\Gamma(M, \mathcal{S}^q) \to \Gamma(M, \mathcal{S}^{q+1})$. Because of the exactness of the rows and columns other than the one exceptional row and one exceptional column in the commutative diagram, by induction on μ we can find $f^{\mu,q-\mu-1}$ in $C^\mu(\mathcal{U}, \mathcal{S}^{q-\mu-1})(0 \leq \mu \leq q-1)$ such that

$$\begin{aligned} \varphi^{0,q-1} f^{0,q-1} &= \iota^q f^q \\ \delta^{\mu,q-\mu-1} f^{\mu,q-\mu-1} &= \varphi^{\mu+1,q-\mu-2} f^{\mu+1,q-\mu-2} (0 \leq \mu \leq q-2) \\ \varphi^q g^q &= \delta^{q-1,0} f^{q-1,0}. \end{aligned}$$

Write $f^{\mu,q-\mu} = \delta^{\mu,q-\mu-1} f^{\mu,q-\mu-1}$ for $0 \leq \mu \leq q-1$. In the commutative diagram we have the following zigzag sequence.

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & f^q \\
 & & & & & & & & & & & & \downarrow \\
 & & & & & & & & & & f^{(0,q-1)} & \to & f^{(0,q)} \\
 & & & & & & & & & & \downarrow & & \\
 & & & & & & & & f^{(1,q-2)} & \to & f^{(1,q-1)} & & \\
 & & & & & & & & \downarrow & & & & \\
 & & & & & & f^{(2,q-3)} & \to & f^{(2,q-2)} & & & & \\
 & & & & & & \downarrow & & & & & & \\
 & & & & & \cdot & & & & & & & \\
 & & & & \cdot & & & & & & & & \\
 & & & \cdot & & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & f^{(q-1,0)} & \to & f^{(q-1,1)} & & & & & & & & \\
 & & \downarrow & & & & & & & & & & \\
g^q & \to & f^{(q,0)} & & & & & & & & & &
\end{array}$$

It is straightforward to check that the correspondence $f^q \to g^q$ represented by such a zigzag sequence in the commutative diagram yields an isomorphism of $H^q(\Gamma(M, \mathcal{S}^\cdot))$ and $H^q(C^\cdot(\mathcal{U}, \mathcal{S}))$, where for a complex

$$0 \to E^0 \to E^1 \to E^2 \to E^3 \to E^4 \to \cdots$$

$H^q(E^\cdot)$ means the kernel of $E^q \to E^{q+1}$ quotiented by the image $E^{q-1} \to E^q$. This together with theorem on acyclic sheaf resolution proves the theorem on Leray covering.

(2.4) *deRham and Dolbeault Cohomologies.*

We now apply the two ways introduced above to calculate cohomology groups to the examples of deRham and Dolbeault cohomology groups. In the case of deRham cohomology the base manifold is a real smooth manifold and the sheaf $\mathcal{S}$ is the sheaf of germs of constant real-valued functions or the sheaf of germs of constant complex-valued functions. We denote these two sheaves by $\underline{\mathbf{R}}$ and $\underline{\mathbf{C}}$ or simply by $\mathbf{R}$ and $\mathbf{C}$. In the case of deRham cohomology the base manifold is a complex manifold M and the sheaf $\mathcal{S}$ is the sheaf $\mathcal{O}_M$ of germs of holomorphic functions on M.

Let us deal first with deRham cohomology. Let M be a real smooth manifold of real dimension m. For $0 \leq \nu \leq m$ let $\mathcal{E}^\nu$ be the sheaf of germs of real smooth ν-forms on M. We have the following sequence of sheaves

$$0 \to \mathbf{R} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to \mathcal{E}^3 \to \cdots \to \mathcal{E}^{m-1} \to \mathcal{E}^m \to 0$$

where the sheaf-homomorphisms $\mathcal{E}^\nu \to \mathcal{E}^{\nu+1}$ are given by exterior differentiation and $\mathbf{R} \to \mathcal{E}^0$ is the inclusion map. We are going to show that this sequence is exact. The exactness of this sequence is known as the Poincaré lemma. To show exactness, one has to prove that for $q \geq 1$ given any smooth closed q-form α on some open neighborhood U of the origin in $\mathbf{R}^m$ one can find a smooth $(q-1)$-form β on an open neighborhood U' of 0 in U such that $d\beta = \alpha$ on U'. Without loss of generality we assume that U is the open unit ball in $\mathbf{R}^m$. Let $x = (x^1, \cdots, x^m)$ be the coordinates of $\mathbf{R}^m$. Let $\Psi : \mathbf{R} \times U \to U$ be defined by $\Psi(t, x) = t\ x$. Let $\gamma = \Psi^* \alpha$ and write $\gamma = \sigma + dt \wedge \tau$, where σ and τ are respectively a q-form and a $(q-1)$-form not containing dt. The coefficients of σ and τ when they are expressed in terms of $dx^1, \cdots, dx^m$ are functions of x and t. Since γ is a closed form on $\mathbf{R} \times U$, it follows that $\frac{\partial}{\partial t}\sigma - d_U \tau = 0$, where $\frac{\partial}{\partial t}\sigma$ means differentiating the coefficients of σ with respect to t and $d_U \tau$ means the exterior derivative of τ with respect to the variables x when the variable t is held fixed. Integrating the equation $\frac{\partial}{\partial t}\sigma - d_U \tau = 0$ with respect to t from $t = 0$ to $t = 1$ yields $\sigma|_{t=1} - \sigma|_{t=0} = d \int_{t=0}^1 \tau\ dt$. Since $\sigma|_{t=0}$ is zero and $\sigma|_{t=1}$ is simply α, if follows that $\alpha = d\beta$ when $\beta = \int_{t=0}^1 \tau\ dt$. Here the only property of U that is used is that U is smoothly contractible to a point in U.

Clearly each $\mathcal{E}^\nu$ admits a partition of unity. So $H^q(M, \mathbf{R})$ equals the kernel of $\Gamma(M, \mathcal{E}^q) \to \Gamma(M, \mathcal{E}^{q+1})$ quotiented by the image of $\Gamma(M, \mathcal{E}^{q-1}) \to \Gamma(M, \mathcal{E}^q)$. In other words, $H^q(M, \mathbf{R})$ is isomorphic to the group of closed q-forms quotiented by the group of exact q-forms. The group of closed q-forms by quotiented by the group of exact q-forms is known as the deRham cohomology group of dimension q. On the other hand, if $\mathcal{U} = \{U_\alpha\}$ is an open cover of M so that each $U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$ is smoothly contractible to a point. Then $H^p(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}, \mathbf{R})$ vanishes for $p \geq 1$ and $\mathcal{U}$ is a Leray covering for the sheaf $\mathbf{R}$. So $H^q(M, \mathbf{R})$ is isomorphic to $H^q(\mathcal{U}, \mathbf{R})$ which is the usual Cech cohomology with real coefficients. The discussion also applies to the sheaf $\mathbf{C}$ instead of the sheaf $\mathbf{R}$.

Now we discuss the Dolbeault cohomology. The locally constant functions f on a real manifold are characterized by the vanishing of df. The analog of d in the case of a complex manifold is the operator $\bar{\partial}$. So the analog of the locally constant sheaf $\mathbf{R}$ or $\mathbf{C}$ is the sheaf $\mathcal{O}_M$ of germs of holomorphic functions on a complex manifold M. Let $\mathcal{E}^{0,\nu}$ be the sheaf of germs of smooth $(0, \nu)$-forms on M. We have the sequence of sheaves

$$0 \to \mathcal{O}_M \to \mathcal{E}^{0,0} \to \mathcal{E}^{0,1} \to \mathcal{E}^{0,2} \to \mathcal{E}^{0,3} \to \cdots \to \mathcal{E}^{0,m-1} \to \mathcal{E}^{0,m} \to 0$$

where the sheaf-homomorphisms $\mathcal{E}^{0,\nu} \to \mathcal{E}^{0,\nu+1}$ are given by exterior differentiation $\bar{\partial}$ in the (0,1)-direction and $\mathcal{O}_M \to \mathcal{E}^{0,0}$ is the inclusion map. This sequence is exact. Its exactness is known as the Dolbeault-Grothendieck lemma.

The Dolbeault-Grothendieck lemma cannot be proved by directly imitating the proof of the Poincaré lemma, because a direct imitation gives us $\frac{\partial}{\partial \bar{t}}\sigma - \bar{\partial}_U \tau = 0$ and we cannot apply the fundamental theorem of calculus to recover $\sigma|_{t=1} = \alpha$. The key point of the proof of the Poincaré lemma is the use of the fundamental theorem of calculus to recover $\sigma|_{t=1} = \alpha$. The fundamental theorem of calculus is equivalent to the statement that the derivative of the characteristic function of the interval (a, b) in $\mathbf{R}$ in the sense of distributions is equal to the Dirac delta at a minus the Dirac delta at b. It expresses the value of a function in terms of its derivative and its value at some initial point. The existence of a primitive given by the fundamental theorem of calculus gives us right away the Poincaré lemma for real dimension one.

The analog of the fundamental theorem of calculus in complex analysis is that $\frac{\partial}{\partial \bar{z}}(-\frac{1}{2\pi}\frac{1}{z})$ in the sense of distributions is the Dirac delta at the origin of $\mathbf{C}$, where z is the coordinate of $\mathbf{C}$. This can be derived from the Newton equation $\Delta \frac{1}{4\pi} \log |z| =$ Dirac delta at 0, because $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. The integrated form gives the generalized Cauchy formula for smooth functions

$$f(0) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f(z)}{z} dz + \frac{1}{2\pi\sqrt{-1}} \int_D \frac{\frac{\partial}{\partial \bar{z}} f(z)}{z} dz \wedge d\bar{z},$$

where D is a neighborhood of 0 enclosed by a simple closed curve C oriented in the positive sense. A coordinate translation gives

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f(\zeta)}{\zeta} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_D \frac{\frac{\partial}{\partial\bar{\zeta}} f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

when D contains z and the boundary C of D is a simple closed curve. It can be used to give a proof of the Dolbeault-Grothendieck lemma in the case of complex dimension one. The Dolbeault-Grothendieck lemma in the case of complex dimension one says that given a smooth function $g(z)$ on an open neighborhood U of 0 in $\mathbf{C}$ there exists a smooth function $f(z)$ on some open neighborhood U' of 0 in U such that $\frac{\partial}{\partial\bar{z}} f(z) = g(z)$ on U'. Without loss of generality one can assume that $g(z)$ has compact support. From the generalized Cauchy formula, because the integral over C is a holomorphic function on D, we know that if $\frac{\partial}{\partial\bar{z}} f(z) = g(z)$ can be solved, then

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

would be a solution near the origin. One sees easily $\frac{\partial}{\partial\bar{z}} f(z) = g(z)$ when $f(z)$ is defined by the above formula, because f is simply the convolution of g with $-\frac{1}{2\pi}\frac{1}{z}$ and we know that $\frac{\partial}{\partial\bar{z}}(-\frac{1}{2\pi}\frac{1}{z})$ in the sense of distributions is the Dirac delta at the origin of $\mathbf{C}$. One can also verify directly in the following way.

$$\begin{aligned}
\frac{\partial}{\partial\bar{z}} \left(\frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} \right) &= \frac{\partial}{\partial\bar{z}} \left(\frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} \frac{g(\zeta + z)}{\zeta} d\zeta \wedge d\bar{\zeta} \right) \\
= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} \frac{\frac{\partial}{\partial\bar{z}} g(\zeta + z)}{\zeta} d\zeta \wedge d\bar{\zeta} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} \frac{\frac{\partial}{\partial\bar{\zeta}} g(\zeta + z)}{\zeta} d\zeta \wedge d\bar{\zeta} \\
= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} \frac{\frac{\partial}{\partial\bar{\zeta}} g(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} &= g(z).
\end{aligned}$$

We observe that if g depends smoothly or holomorphically on other parameters, the function f obtained from the above formula also depends smoothly or holomomorphically on other parameters.

Now we prove the Dolbeault-Grothendieck lemma in the general case. Suppose $q \geq 1$ and α is a smooth $(0,q)$-form on some open neighborhood U of 0 in $\mathbf{C}^n$ so that $\bar{\partial}\alpha = 0$. We are going to use induction on n to find a smooth $(0,q)$-form β on some open neighborhood U' of 0 in U such that $\bar\partial\beta = \alpha$. We want β to depend smoothly or holomorphically on other parameters when α depends smoothly or holomorphically on those parameters. Write $\alpha = \sigma + \overline{dz^n} \wedge \tau$, where σ and τ are respectively a $(0,q)$-form and a $(0,q-1)$-form not containing $\overline{dz^n}$, but the coefficients of σ and τ depend on $z^1, \cdots, z^n$. Let γ be a $(0,q-1)$-form not containing $\overline{dz^n}$ such that $\frac{\partial}{\partial \overline{z^n}}\gamma = \tau$, where $\frac{\partial}{\partial \overline{z^n}}\gamma$ means applying $\frac{\partial}{\partial \overline{z^n}}$ to every coefficient of γ. The form γ can be constructed coefficientwise from the formula for solving the equation $\frac{\partial}{\partial \bar z} f(z) = g(z)$ on some open neighborhood of the origin in $\mathbf{C}$. Then the $(0,q)$-form $\alpha - \bar\partial\gamma$ does not contain $\overline{dz^n}$. Since $\bar\partial(\alpha - \bar\partial\gamma)$ vanishes, it follows that the coefficients of $\alpha - \bar\partial\gamma$ are holomorphic in z^n. We now regard z^n as a holomorphic parameter and consider $\alpha - \bar\partial\gamma$ as a $(0,q)$-form on some open neighborhood of the origin of $\mathbf{C}^{n-1}$. By induction we can find a $(0,q-1)$-form θ on some open neighborhood of the origin in $\mathbf{C}^{n-1}$ depending holomorphically on the parameter z^n such that $\bar\partial\theta = \alpha - \bar\partial\gamma$. We can now regard θ as a $(0,q-1)$-form on some open neighborhood of the origin in $\mathbf{C}^n$ and set $\beta = \theta + \gamma$. Then $\bar\partial\beta = \alpha$ on some open neighborhood of the origin in $\mathbf{C}^n$. This concludes the proof of the Dolbeault-Grothendieck lemma. This proof by induction on the dimension can also be used to prove the Poincaré lemma.

Now that we have the Dolbeault-Grothendieck lemma we conclude that $H^q(M, \mathcal{O}_M)$ is isomorphic to the Dolbeault cohomology group $H^q(\Gamma(M, \mathcal{E}^{0,\cdot}))$ of global smooth $\bar\partial$-closed $(0,q)$-forms modulo global smooth $\bar\partial$-exact $(0,q)$-forms. This can be done in a more general setting. Suppose V is a holomorphic vector bundle over M. Let $\mathcal{O}_M(V)$ be the sheaf of germs of local holomorphic sections of V over M. Let $\mathcal{E}^{0,\nu}(V)$ be the sheaf of germs of local smooth V-valued $(0,\nu)$-forms on M. The Dolbeault-Grothendieck lemma gives the exactness of

$$0 \to \mathcal{O}_M(V) \to \mathcal{E}^{0,0}(V) \to \mathcal{E}^{0,1}(V) \to \mathcal{E}^{0,2}(V) \to \cdots \to \mathcal{E}^{0,m-1}(V) \to \mathcal{E}^{0,m}(V) \to 0$$

where $\mathcal{O}_M(V) \to \mathcal{E}^{0,0}(V)$ is the inclusion map and the sheaf-homomorphisms $\mathcal{E}^{0,\nu}(V) \to \mathcal{E}^{0,\nu+1}(V)$ are given by exterior differentiation $\bar\partial$ in the (0,1)-direction which is well-defined because the transition functions of V are holomorphic. Thus $H^q(M, \mathcal{O}_M(V))$ is isomorphic to the Dolbeault cohomology group $H^q(\Gamma(M, \mathcal{E}^{0,\cdot}(V)))$ of global smooth $\bar\partial$-closed V-valued $(0,q)$-forms modulo global smooth $\bar\partial$-exact V-valued $(0,q)$-forms. In the case when V is the sheaf of germs Ω^p of holomorphic $(p,0)$-forms on M, the cohomology group $H^q(M, \Omega^p)$ is isomorphic to the Dolbeault cohomology group of global smooth $\bar\partial$-closed (p, q)-forms modulo global smooth $\bar\partial$-exact (p, q)-forms.

(2.5) *Spectral Sequence.* We start with a differential module which means a module K with a differential d such that $dd = 0$. Introduce a filtration of K, *i.e.* nested submodules $K_{p+1} \subset K_p \subset K$ so that $K = \cup_p K_p$ and assume that we have a differential filtered module in the sense that d maps K_p to K_p for every p. The motivation for this is a double complex $K = \sum K^{pq}$ with the differentials $d' : K^{pq} \to K^{p+1,q}$ and $d'' : K^{pq} \to K^{p,q+1}$ satisfying $d'd'' + d''d' = 0$. Consider the filtration $K_p = \sum_{i \geq p} K^{ij}$ which is usually called the first filtration. Then we have $K_{p+1} \subset K_p$ and $d : K_p \to K_p$. Consider the graded module $G(K) = \sum K_p/K_{p+1}$. In the example from a double complex $G(K)$ is isomorphic to K itself.

The setting of the theory of spectral sequence is a filtered graded module. Since K is endowed with a differential d, it is actually a differential module. Besides the example from a double complex above another kind of examples is to consider a simplicial complex and a sequence of nested subcomplexes and then one can consider a chain in the simplicial complex with the filtration determined by the subcomplex where the support of the chain lies. When one computes the cohomology group of a complex, we take the quotient of the cocyles modulo the coboundaries. When there is filtration, we have a sequence of nested subcomplexes compatible with the differential, we can consider cocyles modulo a subcomplex. In other words, we can introduce Z_r^p consisting of all $x \in K_p$ such that $dx \in K_{p+r}$. This means that Z_r^p consists of those elements which travel a distance at least r in the filtration under the differential d. Among those elements of Z_r^p there are some from K_{p+1}. Note that when the element comes from K_{p+1}, it needs to travel only a distance of $r-1$ in the filtration under the differential d to end up in K_{p+r}. Also there are some which are coboundaries. We are interested in those boundaries that have travelled a distance of $r-1$ in the filtration. We take quotient of Z_r^p with respect to these two kinds and we define

$$E_r^p = Z_r^p/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}).$$

The reason why we are only interested in those coboundaries that come from an element which travels a distance $r-1$ in the filtration under the differential d is that this setup enables us to continue in the next step. Let $E_r = \sum_p E_r^p$.

The first important observation is that d sends E_r^p to E_r^{p+r}. The reason is that d maps Z_r^p to K_{p+r} by definition of Z_r^p and d maps $dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}$ to

dZ^{p+1}_{r-1} which is in the denominator of $E^{p+r}_r = Z^{p+r}_r/(dZ^{p+1}_{r-1} + Z^{p+r+1}_{r-1})$. Thus d induces a differential $d_r : E^p_r \to E^{p+r}_r$. We can consider the cohomology group $H(E_r)$ of the differential complex $(E_r = \sum_p E^p_r,\, d_r)$.

The main idea of a spectral sequence is to use the graded module

$$E_\infty := \oplus_p \left(\mathrm{Im}\,(H(K_p) \to H(K))/\mathrm{Im}\,(H(K_{p+1}) \to H(K))\right)$$

to get information on $H(K)$. For example, if the cohomology groups are vector spaces, the dimension of $H(K)$ equals the dimension of the graded module E_∞. For E_∞ we consider the elements in the kernel of d which are in K_p modulo the image of d and also modulo the elements in the kernel of d which are in K_{p+1}. To compute E_∞ we consider the quotient complex K_p/K_{p+r} instead of the complex K_p and we get the graded module

$$E_r := \oplus_p \left(\mathrm{Im}\,(H(K_p/K_{p+r}) \to H(K/K_{p+r}))/\mathrm{Im}\,(H(K_{p+1}/K_{p+r}) \to H(K/K_{p+r}))\right).$$

Among the elements of E_r there are some who come from $H(K_p/K_{p+r+1})$, *i.e.,* from the quotient complex with the next depth. These liftable classes are precisely those which are mapped to zero by a differential. This differential is d_r.

One of the main theorem of the theory of spectral sequence is that $H(E_r)$ is naturally isomorphic to E_{r+1}. In other words, the cohomology of the quotient complex of depth r is precisely equal to the quotient complex of depth $r+1$. The quotient complex of infinite depth is equal to the graded module of the cohomology of the original complex. We now do the straightforward verification.

First we claim that

$$Z^p(E_r) = (Z^p_{r+1} + Z^{p+1}_{r-1})/(dZ^{p-r+1}_{r-1} + Z^{p+1}_{r-1}).$$

Take an element x^* of $Z^p(E_r)$ represented by $x \in Z^p_r$. Then we must have $dx \in dZ^{p+1}_{r-1} + Z^{p+r+1}_{r-1}$. In other words, $dx = dy + z$ with $y \in Z^{p+1}_{r-1} \subset K_{p+1}$ and $z \in Z^{p+r+1}_{r-1}$. Then $d(x-y) = z \in Z^{p+r+1}_{r-1} \subset K_{p+r+1}$. So we have $x - y \in Z^p_{r+1}$. By the definition $E^p_r = Z^p_r/(dZ^{p-r+1}_{r-1} + Z^{p+1}_{r-1})$ of E^p_r we know that $x - y$ also represents that element x^* of $Z^p(E_r)$. Thus every element of $Z^p(E_r)$ can be represented by an element of Z^p_{r+1}.

Conversely, if an element x^* of E^p_r is represented by an element x of Z^p_{r+1}. Thus $dx \in K_{p+r+1}$ and $d(dx) = 0 \in K_{p+r+1+r-1}$ and $dx \in Z^{p+r+1}_{r-1}$. From the definition $E_r^{p+r} = Z_r^{p+r}/(dZ_{r-1}^{p+1} + Z_{r-1}^{p+r+1})$ of E_r^{p+r} and from $dx \in Z_{r-1}^{p+r+1}$ it follows that d_r maps x^* to zero. Thus an element x^* of E_r^p belongs to $Z^p(E_r)$ if only if it is representable by an element of Z_{r+1}^p. The image of Z_{r+1}^p in $E_r^p = Z_r^p/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1})$ is precisely $(Z_{r+1}^p + Z_{r-1}^{p+1})/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1})$, because dZ_{r-1}^{p-r+1} is clearly contained in Z_{r+1}^p.

Next we claim that $B^p(E_r) = (dZ_r^{p-r} + Z_{r-1}^{p+1})/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1})$. The reason is that $B^p(E_r)$ is represented by elements of dZ_r^{p-r}. The image of dZ_r^{p-r} in $E_r^p = Z_r^p/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1})$ is precisely $(dZ_r^{p-r} + Z_{r-1}^{p+1})/(dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1})$, because dZ_{r-1}^{p-r+1} is clearly contained in dZ_r^{p-r}.

Now we can compute $H^p(E_r) = Z^p(E_r)/B^p(E_r)$ as

$$(Z_{r+1}^p + Z_{r-1}^{p+1})/(dZ_r^{p-r} + Z_{r-1}^{p+1})$$

which is isomorphic to $Z_{r+1}^p/Z_{r+1}^p \cap (dZ_r^{p-r} + Z_{r-1}^{p+1})$ and is equal to

$$E_{r+1}^p = Z_{r+1}^p/(dZ_r^{p-r} + Z_r^{p+1}),$$

because $dZ_r^{p-r} \subset Z_{r+1}^p$ and $Z_{r+1}^p \cap Z_{r-1}^{p+1} = Z_r^{p+1}$.

Definition. The *spectral sequence* of the differential filtered module K is the sequence of complexes (E_r, d_r).

A spectral sequence $\{E_r\}$ is said to be *degenerate* in E_r of $d_r = 0$, in which case $E_s = E_r$ for $s \geq r$ including $s = \infty$. When the cohomology groups are vector spaces, since E_{r+1} is the quotient of a subspace of E_r, we always have $\dim E_{r+1} \leq \dim E_r$ with equality if and only if the spectral sequence degenerates in E_r. This means that all classes in the quotient complex of depth r can be lifted to all subsequent depths.

Formally we set $r = \infty$ and define $K_\infty = 0$ and $K_{-\infty} = K$ and $Z_\infty^p = K_p \cap Ker\ d$ and $B_\infty^p = K_p \cap dK_{-\infty} = K_p \cap dK$. We define $E_\infty^p = Z_\infty^p/(B_\infty^p + Z_\infty^{p+1})$ and $E_\infty = \sum_p E_\infty^p$. In general we can also define $B_r^p = K_p \cap dK_{p-r}$ and write $E_r^p = Z_r^p/(B_{r-1}^p + Z_{r-1}^{p+1})$.

Another important result in the theory of spectral sequence is the following. If $H(K)_p$ is the image of $H(K_p)$ in $H(K)$, then $E_\infty = \sum_p H(K)_p/H(K)_{p+1}$. To verify this, we observe that an element z of $Z(K_p)$ is mapped to $H(K)_{p+1}$ means that $z = dx + y$ with $x \in K$ and $y \in Z(K_{p+1})$. When y is in K_{p+1}, it automatically is in $Z(K_p)$. Also automatically dx is in K_p because both y and z are in K_p. Thus

$$H(K)_p/H(K)_{p+1} = Z(K_p)/(K_p \cap dK + K_{p+1}).$$

In other words, we have $H(K)_p/H(K)_{p+1} = E^p_\infty$.

We now introduce the concept of regularity so that we can approximate E_∞ by E_r. Let K be a filtered complex, *i.e.* a graded filtered differential module $K = \sum_n K^n$ with grading compatible with the filtering so that the differential d is homogeneous of degree 1 with respect to the grading of K. The compatibility of the grading of K with its filtering means that $K_p = \sum_n K_p \cap K^n$. Note that we use the subscript to denote the filtration and the superscript to denote the degree. Let $Z^{pq}_r = Z^p_r \cap K^{p+q}$, $B^{pq}_r = B^p_r \cap K^{p+q}$, and $E^{pq}_r = Z^{pq}_r/(B^{pq}_{r-1} + Z^{p+1,q-1}_{r-1})$. We call p the *filtering degree*, $p+1$ the *total degree*, and q the *complementary degree*. We say that the filtration of K is *regular* if $K_p \cap K^q = 0$ for $p > n(q)$, where $n(\cdot)$ is a function from the set of integers to itself. An example is the case of a double complex $K^{pq}(p \geq 0, q \geq 0)$ with the first filtration $'K_p = \sum_{i \geq p} K^{ij}$ and the grading of K by the total degree $K^n = \sum_{p+q=n} K^{pq}$. In this example we have $n(q) = q$, *i.e.* the submodule K_p contains only elements of degree at least p. When we have a regular filtered complex, we have $Z^{pq}_r = Z^{pq}_\infty$ for $r > n(p+q+1) - p$, because if $z \in Z^{pq}_r$ then $dz \in K_{p+r} \cap K^{p+q+1} = 0$ due to $p + r > n(p+q+1)$. Since d_r on $E^{pq}_r = Z^{pq}_r/(B^{pq}_{r-1} + Z^{p+1,q-1}_{r-1})$ is induced by d, it follows that $d_r = 0$ on E^{pq}_r for $r > n(p+q+1) - p$. From $H(E_r) = E_{r+1}$ it follows that there is a map from E^{pq}_r to E^{pq}_{r+1} and inductively a map from $\theta^r_s : E^{pq}_r \to E^{pq}_s$ for $s > r > n(p+q+1) - p$. We claim that the inductively limit of this system is E^{pq}_∞.

To prove this claim, we observe that $Z^{pq}_r = Z^{pq}_\infty$ and $Z^{p+1,q-1}_r = Z^{p+1,q-1}_\infty$ for r sufficiently large. Moreover, since $B^{pq}_r = K_p \cap K^{p+q} \cap d(K_{p-r})$ is a subset of $B^{pq}_\infty = K_p \cap K^{p+q} \cap dK$ always, it follows that we have a epimorphism $\theta^r_\infty : E^{pq}_r \to E^{pq}_\infty$ induced from the identity map $Z^{pq}_r \to Z^{pq}_\infty$. Since $B^{pq}_\infty = K_p \cap K^{p+q} \cap dK$ is clearly the union of $B^{pq}_r = K_p \cap K^{p+q} \cap d(K_{p-r})$ for all r from $K = \cup_r K_{p-r}$, it follows that E^{pq}_∞ is the inductive limit of the system $\theta^r_s : E^{pq}_r \to E^{pq}_s$.

Suppose we have a homomorphism $f : K \to L$ of two filtered complexes. Then we have the induced homomorphism $E_r(K) \to E_r(L)(1 \leq r \leq \infty)$. Suppose both K and L are regular. Then we have the following. If for some r the homomorphism $E_r(K) \to E_r(L)$ is bijective, then $H(K) \to H(L)$ is bijective. The reason is as follows. The bijectivity of $E_r(K) \to E_r(L)$ implies the bijectivity of $E_s(K) \to E_s(L)$ for $s > r$ due to $E_{r+1} = H(E_r)$. Since E_∞ is the inductive limit of E_r, it follows that $E_\infty(K) \to E_\infty(L)$ is bijective. In other words, $GH(K) \to GH(L)$ is bijective. To show that $H(K) \to H(L)$ we need only show it for a fixed degree. Since K (respectively L) is regular, for a fixed degree n we have $K_p \cap K^n = 0$ (respectively $L_p \cap L^n = 0$) for p sufficiently large and it follows that for that degree there is a largest p such that both $H(K)_p = \operatorname{Im}(H(K_p) \to H(K))$ and $H(L)_p = \operatorname{Im}(H(L_p) \to H(L))$ is nonzero and we have $H(K)_p/H(K)_{p+1} = H(K)_p$ and $H(L)_p/H(L)_{p+1} = H(L)_p$ for that degree. The bijectivity between $GH(K) = \sum_j H(K)_j/H(K)_{j+1}$ and $GH(L) = \sum_j H(L)_j/H(L)_{j+1}$ would imply the bijectivity of $H(K)_p$ and $H(L)_p$ and by descending induction imply the bijectivity of $H(K)_j$ and $H(L)_j$ for all j. Since $H(K) = \cup_j H(K_j)$ and $H(K) = \cup_j H(K_j)$, we have the bijectivity of $H(K)$ and $H(L)$.

Let us consider the terms E_0, E_1, and E_2 of the spectral sequence for a filtered differential module K. We have $Z_0^p = K_p$ because Z_0^p means those elements of K_p which travels at least 0 distance in filtration under the differential d and by definition of a filtered differential module the differential d maps K_p to K_p. Similarly we have $Z_{-1}^{p+1} = K_{p+1}$. From $dZ_{-1}^{p+1} \subset K_{p+1}$ it follows that $E_0^p = K_p/K_{p+1}$. The differential d_0 is induced from d and is simply the self map of K_p/K_{p+1} induced by d. As a consequence we have $E_1^p = H(K_p/K_{p+1})$. We would like to calculate $d_1 : E_1^p \to E_1^p$. Consider the short exact sequence

$$0 \to K_{p+1}/K_{p+2} \to K_p/K_{p+2} \to K_p/K_{p+1} \to 0.$$

From its long exact sequence we have the map

$$\delta : E_1^p = H(K_p/K_{p+1}) \to H(K_{p+1}/K_{p+2}) = E_1^{p+1}.$$

We claim that this map is precisely the differential d_1 on E_1^p. To verify this, we take an element $\alpha \in H(K_p/K_{p+1})$ represented by an element $z \in Z(K_p/K_{p+1}) = Z_1^p$. By definition $d_1\alpha$ is represented by the element $dz \in Z(K_{p+1}/K_{p+2})$. This precisely agrees with the definition for the map δ. Hence E_2 is the cohomology form the complex $E_1 = \sum_p H(K_p/K_{p+1})$.

We now apply the preceding theory to the case of a double complex $K = \sum\ K^{pq}$ with the differentials $d' : K^{pq} \to K^{p+1,q}$ and $d'' : K^{pq} \to K^{p,q+1}$ satisfying $d'd'' + d''d' = 0$. Consider the first filtration $'K_p = \sum_{i \geq p} K^{ij}$. We form the submodule $K^n = \sum_{p+q=n} K^{pq}$ of elements of total degree n and use the total differential $d = d' + d''$. We would like to compute the E_1 and E_2 terms of the spectral sequence in this case. Since $'K_p/'K_{p+1}$ is the same as $K^{p,*} = \sum_j K^{pj}$, it follows that $'E_1^p ='' H(K^{p,*})$. The calculation of

$''H(K^{p,*})$ uses the differential d''. To calculate d_1 we consider the connecting homomorphism in the long exact sequence for the short exact sequence

$$0 \to' K_{p+1}/'K_{p+2} \to' K_p/'K_{p+2} \to' K_p/'K_{p+1} \to 0$$

which is the same

$$0 \to K^{p+1,*} \to K^{p+1,*} + K^{p,*} \to K^{p,*} \to 0$$

with the differential d'' for the first and third term and the differential $d = d' + d''$ for the second term. The connecting homomorphism $\delta :'' H(K^{p,*}) \to'' H(K^{p+1,*})$ is induced by d'. Thus $'E_2^p =' H^p(''H(K))$.

As an example we apply the theory of spectral sequence to the following statement. Suppose $'H^p(K^{*,q}) = 0$ and $''H^p(K^{q,*}) = 0$ for $p \geq 1$ and $q \geq 0$. Let $'L$ be the subcomplex of K consisting of all $x = \sum_p x^{p0}$ with $d''x^{p0} = 0$. Let $''L$ be the subcomplex of K consisting of all $x = \sum_p x^{0p}$ with $d'x^{0p} = 0$. Then $H^p('L)$ is naturally isomorphic to $H^p(''L)$. The verification is as follows. For any $p \geq 0$ we claim that $''H^p(K^{q,*}) =' H^p('L^{q,*})$ for $q \geq 0$. When $p \geq 1$, by assumption $'H^p(K^{q,*}) = 0$ and also from $'L^{q,p} = 0$ for $p \geq 1$ we have clearly $'H^p('L^{q,*}) = 0$. On the other hand, from the definition of L we have $'H^0(K^{q,*}) = L^{q,0} =' H^0(L^{q,*})$. Hence the inclusion $'L \to K$ induces an isomorphism between $E_1('L)$ and $E_1(K)$ and thus induces an isomorphism between $H('L)$ and $H(K)$. Likewise the inclusion $''L \to K$ induces an isomorphism between $E_1(''L)$ and $E_1(K)$ and thus induces an isomorphism between $H(''L)$ and $H(K)$. From these two isomorphisms we have a natural isomorphism between $H('L)$ and $H(''L)$.

We can reformulate the above statement as follows. If $K^{00} = 0$ and $'H^p(K^{*,q}) = 0$ and $''H^p(K^{q,*}) = 0$ for $p \geq 0$ and $q \geq 1$, then $'H^p(K^{*,0})$ is naturally isomorphic to $''H^p(K^{0,*})$. This we can see by apply the earlier statement to the complex $\sum_{p\geq 1, q\geq 1} K^{pq}$. Then for this new complex we have $'L = \sum_{p\geq 1} K^{p0}$ and $''L = \sum_{p\geq 1} K^{0p}$.

CHAPTER 3. HODGE DECOMPOSITION

§1. *Harmonic Forms.*

Suppose M is a compact complex manifold of complex dimension m and V is a holomorphic vector bundle of rank r over M. From the Dolbeault isomorphism we know that the cohomology group $H^p(M, \mathcal{O}(V))$ is isomorphic to the set of all $\bar{\partial}$-closed smooth V-valued $(0,p)$-forms on M modulo the set of all $\bar{\partial}$-closed smooth V-valued $(0,p)$-forms on M. Though this gives us a more practical way of computing $H^p(M, \mathcal{O}(V))$ by using differential forms, it would be more convenient for computational purpose if a cohomology class is represented by a *unique* differential form rather than an equivalence class of differential forms.

When we have a quotient of a finite dimensional vector space by a vector subspace, instead of looking at an element in the quotient space one way is to consider the orthogonal complement of the vector subspace provided we have an inner product in the ambient space. So every element in the quotient space is represented by a unique element in the orthogonal complement of the subspace instead of an equivalence class in the original ambient space. This is the same as using the element with minimum length in its equivalence class to represent the equivalence class.

In our case the vector space of all $\bar{\partial}$-closed smooth V-valued $(0,p)$-forms on M in general is an infinite dimensional vector space. Suppose we give M a Hermitian metric and also give V a Hermitian metric along its fibers. So there is a natural inner product on the vector space of all $\bar{\partial}$-closed smooth V-valued $(0,p)$-forms on M. If we use this method of orthogonal complement or equivalently the method of finding an element with minimum length in its equivalence class, we run into the trouble of closedness of the subspace or equivalently the convergence of a sequence of elements in an equivalence class minimizing the length function. We know that the vector space of all $\bar{\partial}$-closed smooth V-valued $(0,p)$-forms on M is not complete in the natural inner product. In order to talk about closedness of subspaces or the convergence of a minimizing sequence of elements, we should first complete the vector space of all $\bar{\partial}$-closed smooth V-valued $(0,p)$-forms on M to a Hilbert space. So we should consider the set of all $\bar{\partial}$-closed L^2V-valued $(0,p)$-forms on M. The first question that arises in considering the set of all $\bar{\partial}$-closed L^2V-valued $(0,p)$-forms on M is the definition of the operator $\bar{\partial}$ on L^2V-valued $(0,p)$-forms on M. The second question is the closedness of the image of $\bar{\partial}$ after it is defined.

Let $\mathcal{L}^2(p)$ denote the set of all L^2V-valued $(0,p)$-forms on M and let $\mathcal{C}^\infty(p)$ be the set of all V-valued $(0,p)$-forms on M. The operator $\bar\partial$ from $\mathcal{L}^2(p)$ to $\mathcal{L}^2(p+1)$ is defined on the dense subset $\mathcal{C}^\infty(p)$ of $\mathcal{L}^2(p)$. In order to make the image of $\bar\partial$ closed, we should define it on as large set as possible. So we consider the closure of the graph of $\bar\partial$ in the product space of $\mathcal{L}^2(p)$ and $\mathcal{L}^2(p+1)$. Given an element f of $\mathcal{L}^2(p)$ and an element g of $\mathcal{L}^2(p+1)$, the pair (f,g) is in the graph if and only there exists a sequence $f_\nu \in \mathcal{C}^\infty(p)$ converging in the L^2 norm of $\mathcal{L}^2(p)$ to f so that $\bar\partial f_\nu$ converges in the L^2 norm of $\mathcal{L}^2(p+1)$ to g. We have to check that this closure is the graph of a map. The trouble is that for the closure of the graph we may have two distinct elements of $\mathcal{L}^2(p+1)$ given as the image of the same element of $\mathcal{L}^2(p+1)$. By taking the difference of these two distinct elements of $\mathcal{L}^2(p+1)$, we can assume that there exists a sequence $f_\nu \in \mathcal{C}^\infty(p)$ converging in the L^2 norm of $\mathcal{L}^2(p)$ to 0 so that $\bar\partial f_\nu$ converges in the L^2 norm of $\mathcal{L}^2(p+1)$ to some nonzero element g of $\mathcal{L}^2(p+1)$. Take an arbitrary element φ of $\mathcal{C}^\infty(p+1)$. We have the process of integration by parts. So we get $(\bar\partial f_\nu, \varphi)_{\mathcal{L}^2(p+1)} = (f_\nu, \bar\partial^*\varphi)_{\mathcal{L}^2(p)}$, where $(\cdot,\cdot)_{\mathcal{L}^2(q)}$ means the inner product in $\mathcal{L}^2(q)$ and $\bar\partial^*$ is a first order linear partial differential operator. Letting $\nu \to \infty$, we conclude that $(g,\varphi)_{\mathcal{L}^2(p+1)} = (0, \bar\partial^*\varphi)_{\mathcal{L}^2(p)} = 0$ for any φ, contradicting that g is nonzero. Hence we know that the closure of the graph of $\bar\partial$ is also a graph. This argument works in general for differential operators because we have integration by parts.

There is another way to extend the definition of $\bar\partial$ from $\mathcal{C}^\infty(p)$. For any $f \in \mathcal{L}^2(p)$ the expression $\bar\partial f$ makes sense when differentiation is done in the sense of distributions. So in general $\bar\partial f$ would be a current. If this current can be represented by an element g of $\mathcal{L}^2(p+1)$, then we say that f belongs to the domain of $\bar\partial$ and define $\bar\partial f$ as g. This new definition of the extension of $\bar\partial$ is called the weak extension. The earlier extension done by using the closure of the graph of $\bar\partial$ is known as the strong extension. At first it seems that the weak extension may have a bigger domain than the strong extension. It turns out that the two extensions are the same. This is known as the Friedrichs lemma (K. O. Friedrichs, The identity of weak and strong extensions of differential operators. Trans. Amer. Math. Soc. 55 (1944), 132-151). First one observes that by using a partition of unity we can assume that all forms involved are supported in a single coordinate chart. Then in that coordinate chart one uses smoothing by convolution with a cut-off function. Suppose $L = a(x)\frac{\partial}{\partial x} + b(x)$ is a first-order differential operator with smooth coefficients. Let $\chi(x)$ be a nonnegative function supported on the unit open ball of $\mathbf{R}^n$. Let $\chi_\epsilon(x) = \frac{1}{\epsilon^n}\chi(\frac{x}{\epsilon^n})$. Suppose u is an L^2 function. Then $\chi_\epsilon * u \to u$ in L^2 norm as $\epsilon \to 0$. We assume that Lu taken in the sense of distributions is L^2. Then $\chi_\epsilon * Lu \to Lu$ in L^2 norm as $\epsilon \to 0$. What we want is that $L(\chi_\epsilon * u) \to Lu$ in L^2 norm as $\epsilon \to 0$. It suffices to show that $\chi_\epsilon * Lu - L(\chi_\epsilon * u)$ approaches 0 in L^2 norm as $\epsilon \to 0$. This is clearly true when u belongs to the dense subset of smooth functions. So it suffices to show that $\chi_\epsilon * Lu - L(\chi_\epsilon * u)$ is bounded in L^2 norm when u belongs to a set bounded in L^2 norm. The zero-order part $b(x)$ of L clearly has bounded contribution. So we can assume without loss of generality that $b(x) = 0$. Then

$$\begin{aligned}\chi_\epsilon * Lu - L(\chi_\epsilon * u) &= \chi_\epsilon * \left(a\frac{\partial u}{\partial x}\right) - a\frac{\partial}{\partial x}(\chi_\epsilon * u) \\ &= \chi_\epsilon * \frac{\partial}{\partial x}(au) - \chi_\epsilon * \left(u\frac{\partial a}{\partial x}\right) - a\left(\frac{\partial}{\partial x}\chi_\epsilon * u\right) \\ &= \left(\frac{\partial}{\partial x}\chi_\epsilon\right) * (au) - \chi_\epsilon * \left(u\frac{\partial a}{\partial x}\right) - a\left(\frac{\partial}{\partial x}\chi_\epsilon * u\right).\end{aligned}$$

Clearly the second term on the right-hand side is bounded. So we can drop it. We have

$$\begin{aligned}&\left(\left(\frac{\partial}{\partial x}\chi_\epsilon\right) * (au) - a\left(\frac{\partial}{\partial x}\chi_\epsilon * u\right)\right)(x) \\ &= \int \left(\frac{\partial}{\partial y}\chi_\epsilon\right)(y)a(x-y)u(x-y) - a(x)\left(\frac{\partial}{\partial y}\chi_\epsilon\right)(y)u(x-y)dy \\ &= \int_{|y|<\epsilon} \left(\frac{\partial}{\partial y}\chi_\epsilon\right)(y)\left(a(x-y) - a(x)\right)u(x-y)dy.\end{aligned}$$

In the last integral $|a(x-y) - a(x)| \leq C\epsilon$ because $|y| < \epsilon$. We have $\left|\frac{\partial}{\partial y}\chi_\epsilon\right| \leq C'\frac{1}{\epsilon^{n+1}}$. Hence

$$\left(\left(\frac{\partial}{\partial x}\chi_\epsilon\right) * (au) - a\left(\frac{\partial}{\partial x}\chi_\epsilon * u\right)\right)(x) \leq C''\frac{1}{\epsilon^n}\int_{|y|<\epsilon}|u(x-y)|dy$$

and the L^2 norm of $\left(\frac{\partial}{\partial x}\chi_\epsilon\right) * (au) - a\left(\frac{\partial}{\partial x}\chi_\epsilon * u\right)$ is bounded by

$$C''\frac{1}{\epsilon^n}\int_{|y|<\epsilon}\left(\int |u(x-y)|^2 dx\right)dy$$

which is bounded if the L^2 norm of u is bounded.

We want to show that the image of $\bar\partial : \mathcal{L}^2(p-1) \to \mathcal{L}^2(p)$ is closed and the kernel of $\bar\partial : \mathcal{L}^2(p) \to \mathcal{L}^2(p+1)$ quotiented by the image of $\bar\partial : \mathcal{L}^2(p-1) \to \mathcal{L}^2(p)$ is isomorphic to the set of all $\bar\partial$-closed smooth V-valued $(0,p)$-forms on M modulo the set of all $\bar\partial$-closed smooth V-valued $(0,p)$-forms on M. First we handle the question of the closedness of the image of $\bar\partial : \mathcal{L}^2(p-1) \to \mathcal{L}^2(p)$. This means that if we have $f_\nu \in \mathcal{L}^2(p-1)$ so that $\bar\partial f_\nu$ converges to some g_∞ in $\mathcal{L}^2(p)$, then we want to show that $g_\infty = \bar\partial f_\infty$ for some $f_\infty \in \mathcal{L}^2(p-1)$. Let $g_\nu = \bar\partial f_\nu$. If we are able to solve the equation $\bar\partial h_\nu = g_\nu$ so that h_ν or a subsequence of it converges to some h_∞ in $\mathcal{L}^2(p-1)$, then we are done.

We are going to consider the operator $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. First we would like to discuss the motivation for this operator $\Box$. We look at the equation $\bar\partial h = g$ where g is in $\mathcal{L}^2(p)$ and $\bar\partial g = 0$. We would like to solve it so that h has a good bound when g is bounded. The equation $\bar\partial h = g$ is equivalent to $(\bar\partial h, \varphi) = (g, \varphi)$ for all φ in $\mathcal{L}^2(p)$, *i.e.* $(h, \bar\partial^*\varphi) = (g, \varphi)$. Now h is the unknown. To know h is the same as knowing the linear functional $\psi \to (h, \psi)$. Now we know this linear functional when $\psi = \bar\partial^*\varphi$ for some φ, because in that case the linear functional is $\psi \to (g, \varphi)$ and g is known. The set of all $\bar\partial^*\varphi$ is a subset of $\mathcal{L}^2(p-1)$. If we can extend this known linear functional from the set $\operatorname{Im}\bar\partial^*$ to a bounded linear functional on all of $\mathcal{L}^2(p-1)$, then we know h whose norm is bounded by that of the bounded linear functional. Such an extension is possible if we have an estimate $|(g, \varphi)| \leq C\|\bar\partial^*\varphi\|$. We write $\varphi = \varphi_1 + \varphi_2$ so that $\varphi_1 \in \operatorname{Ker}\bar\partial$ and $\varphi_2 \perp Ker\bar\partial$. Since $g \in \operatorname{Ker}\bar\partial$, we have $(g, \varphi) = (g, \varphi_1)$. From $\varphi_2 \perp \operatorname{Ker}\bar\partial$ we have $\varphi_2 \perp \operatorname{Im}\bar\partial$ and $\bar\partial^*\varphi_2 = 0$. Hence $|(g, \varphi)| \leq C\|\bar\partial^*\varphi\|$ is equivalent to $|(g, \varphi_1)| \leq C\|\bar\partial^*\varphi_1\|$. We can assume for that inequality that $\varphi \in \operatorname{Ker}\bar\partial$. By the Schwarz inequality it suffices to have an estimate $\|\varphi\| \leq C\|\bar\partial^*\varphi\|$. Since $\varphi \in \operatorname{Ker}\bar\partial$, it suffices to have the inequality $\|\bar\partial^*\varphi\|^2 + \|\bar\partial\varphi\|^2 \geq C\|\varphi\|^2$. This inequality can be rewritten as $(\Box\varphi, \varphi) \geq C\|\varphi\|^2$. This shows that the operator $\Box$ is closely related to the solution of the equation $\bar\partial h = g$. If we do have the inequality $(\Box\varphi, \varphi) \geq C\|\varphi\|^2$, then our preceding discussion shows that the equation $\bar\partial h = g$ admits a solution h with $\|h\| \leq C'\|g\|$. We do not expect in general to have the inequality $(\Box\varphi, \varphi) \geq C\|\varphi\|^2$, because this inequality would imply that $H^p(M, \mathcal{O}(V))$ vanishes. However, we have a related inequality that can serve our purpose of proving the closedness of the image of $\bar\partial$. This related inequality is the Gårding's inequality. It says that $(\Box\varphi, \varphi) + (\varphi, \varphi) \geq C\|\varphi\|_1^2$, where the norm $\|\varphi\|_1$ means a norm that is equivalent to the sum of the L^2 norm of φ and the L^2 norm of the first derivative of φ.

Before we prove the Gårding's inequality, let us look at its consequences. Rewrite the Gårding's inequality in the form $((\square + 1)\varphi, \varphi) \geq C\|\varphi\|_1^2$. We know that the operator $\square + 1$ admits an inverse whose norm $is \leq 1$, because the operator $\square$ is nonnegative. Replacing φ by $(\square + 1)^{-1}\varphi$ in the Gårding's inequality, we get

$$((\square + 1)^{-1}\varphi, \varphi) \geq C\|(\square + 1)^{-1}\varphi\|_1^2.$$

Since

$$((\square + 1)^{-1}\varphi, \varphi) \leq \|(\square + 1)^{-1}\varphi\|^2\|\varphi\|^2 \leq \|(\square + 1)^{-1}\varphi\|_1^2\|\varphi\|^2,$$

it follows that $C\|(\square + 1)^{-1}\varphi\|_1 \leq \|\varphi\|$. This means that the map $(\square + 1)^{-1}$ from $\mathcal{L}^2(p)$ to $\mathcal{L}_1^2(p)$ is continuous, where $\mathcal{L}_1^2(p)$ means the Hilbert space of all L^2V-valued $(0, p)$-forms whose first derivatives are also L^2. Since by Rellich's lemma the inclusion map $\mathcal{L}_1^2(p) \to \mathcal{L}^2(p)$ is a compact operator, it follows that the map $(\square+1)^{-1}$ from $\mathcal{L}^2(p)$ to $\mathcal{L}^2(p)$ is a compact operator. A compact operator means that it maps a bounded set of the domain space to a relatively compact subset of the target space. We will prove Rellich's lemma later. By the spectral theorem for self-adjoint compact operators, we know that the eigenvalues μ_i of the operator $(\square + 1)^{-1}$ from $\mathcal{L}^2(p)$ to $\mathcal{L}^2(p)$ are in (0,1] and the only possible accumulation point of $\{\mu_i\}$ is 0 and the eigenspace for each μ_i is finite-dimensional and the eigenfunctions of $(\square + 1)^{-1}$ span $\mathcal{L}^2(p)$. The eigenvalues μ_i are limited to (0,1], because $(\square + 1)^{-1}$ is positive and its norm $is \leq 1$. An equation $(\square + 1)^{-1}f_i = \mu_i f_i$ is equivalent to $\square f_i = \lambda_i f_i$ with $\lambda_i = \frac{1-\mu_i}{\mu_i}$. So we conclude that the eigenvalues λ_i of $\square$ are in $[0, \infty)$ and the eigenspace for each λ_i is finite-dimensional and the eigenfunctions f_i of $\square$ span $\mathcal{L}^2(p)$. If 0 is an eigenvalue, we call it λ_0. By allowing some positive eigenvalues λ_i $(1 \leq i < \infty)$ to be counted more than once, we can assume that the eigenspace for each positive eigenvalue λ_i is only 1-dimensional and the totality of the eigenfunction f_i for $\lambda_i(1 \leq i < \infty)$ is an orthonormal basis of the orthogonal complement $(\mathrm{Ker}\,\square)^\perp$ of $\mathrm{Ker}\,\square$ in $\mathcal{L}^2(p)$. We know that $\mathrm{Ker}\,\square$ is finite-dimensional. On $(\mathrm{Ker}\,\square)^\perp$ we can define the inverse of $\square$ by

$$G(\sum_{i=1}^{\infty} \alpha_i f_i) = \sum_{i=1}^{\infty} \frac{\alpha_i}{\lambda_i} f_i.$$

We extend G to all of $\mathcal{L}^2(p)$ by defining G to be zero on $\operatorname{Ker}\square$. We call G the Green's operator. The reason for the name is that G is the inverse of the Laplace operator $\square$. Let H be the orthogonal projection from $\mathcal{L}^2(p)$ onto $\operatorname{Ker}\square$. Then we have the identity $1 - H = G\square = \square G$.

Now we are ready to prove the closedness of $\operatorname{Im}\bar\partial$. We go back to our earlier notation of $g_\nu \to g_\infty$ in $\mathcal{L}^2(p)$ so that $g_\nu \in \operatorname{Im}\bar\partial$. Suppose $g \in \mathcal{L}^2(p)$ with $\bar\partial g = 0$. We have

$$g - Hg = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)Gg.$$

By applying $\bar\partial$ to both sides, we get $0 = \bar\partial\bar\partial^*\bar\partial Gg$. Taking the inner product with Gg, we obtain

$$0 = (\bar\partial\bar\partial^*\bar\partial Gg, \bar\partial Gg) = \|\bar\partial^*\bar\partial Gg\|^2.$$

Hence $\bar\partial Gg = 0$ and $g - Hg = \bar\partial\bar\partial^* Gg$. This shows that though in general we cannot solve the equation $g = \bar\partial h$ because $H^p(M, \mathcal{O}(V))$ may not vanish, yet we can always solve the equation $g - Hg = \bar\partial h$ with $h = \bar\partial^* Gg$. From $\bar\partial h = g - Hg$ and $\bar\partial^* h = 0$ and the Gårding's inequality we conclude that

$$\|g - Hg\|^2 \geq C\|h\|_1^2 \quad \text{and} \quad \|g\|^2 \geq C'\|h\|_1^2,$$

because $\|g - Hg\| \leq C''\|g\|$. We go back to our earlier notation of $g_\nu \to g_\infty$ in $\mathcal{L}^2(p)$ so that $g_\nu \in \operatorname{Im}\bar\partial$. Since $g_\nu \in \operatorname{Ker}\bar\partial$, we have

$$g_\nu - Hg_\nu = \bar\partial h_\nu \quad \text{with} \quad \|g_\nu\|^2 \geq C'\|h_\nu\|_1^2.$$

An element f of $\mathcal{L}^2(p)$ belongs to $\operatorname{Ker}\square$ if and only $\bar\partial f = 0$ and $\bar\partial^* f = 0$, because $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and

$$(\square f, f) = \|\bar\partial f\|^2 + \|\bar\partial^* f\|^2.$$

This implies every element of $\operatorname{Ker}\square$ is perpendicular to both $\operatorname{Im}\bar\partial$ and $\operatorname{Im}\bar\partial^*$. Since $Hg_\nu = g_\nu - \bar\partial h_\nu$ belongs to $\operatorname{Im}\bar\partial$, it follows from $Hg_\nu \in \operatorname{Ker}\square$ that $Hg_\nu = 0$. Hence $g_\nu = \bar\partial h_\nu$. Since $\|h_\nu\|_1^2$ is bounded independent of ν, by Rellich's lemma we can select a subsequence h_{ν_j} of h_ν converging to some h_∞ in $\mathcal{L}^2(p)$. Then $\bar\partial h_\infty = g_\infty$, because $h_{\nu_j} \to h_\infty$ and $\bar\partial h_{\nu_j} = g_{\nu_j} \to g_\infty$. Hence the image of $\bar\partial : \mathcal{L}^2(p-1) \to \mathcal{L}^2(p)$.

Now we want to show that the cohomology group $H^p(M, \mathcal{O}(V))$ can be calculated by using L^2V-valued forms instead of smooth V-valued forms.

For this we need a regularity result for the equation $\Box f = g$ in $\mathcal{L}^2(p)$. More precisely we want to show that if the norm $\|g\|_k$ which is equivalent to the sum of all the L^2 norms of derivatives of g of order $\leq k$ is finite, then $\|f\|_{k+2}$ is finite. Note that we assume already that the L^2 norms of both f and g are finite. First we observe that if $\varphi \in \mathcal{L}^2(p)$ and $\bar{\partial}\varphi \in \mathcal{L}^2_k(p+1)$ and $\bar{\partial}\varphi \in \mathcal{L}^2_k(p-1)$ for some $k \geq 1$, then $\varphi \in \mathcal{L}^2_{k+1}(p)$. The case $k=0$ is given right away by Gårding's inequality. For the general case we take any vector field X and apply the argument to $\nabla_X \varphi$ and use induction on k. Now we come back to the equation $\Box f = g$ in $\mathcal{L}^2(p)$. Again we look at first the case $k=0$. By Gårding's inequality

$$\begin{aligned}
\|\bar{\partial} f\|_1^2 &\leq C(\Box \bar{\partial} f, \bar{\partial} f) + C\|\bar{\partial} f\|^2 \\
&= C(\bar{\partial}^* \partial f, \bar{\partial}^* \bar{\partial} f) + C\|\bar{\partial} f\|^2 \\
&= C(\Box f, \Box f) + C\|\bar{\partial} f\|^2 \quad (\text{because } (\bar{\partial}^* \partial f, \bar{\partial}\bar{\partial}^* f) = 0) \\
&\leq C(\Box f, \Box f) + C(\Box f, f)^2 = C\|g\|^2 + C(g, f).
\end{aligned}$$

Likewise $\|\bar{\partial}^* f\|_1^2 \leq C\|g\|^2 + C(g, f)$. Hence by the above observation $\|f\|_2$ is finite. For the case of a general k, again we take any vector field X and apply the argument to $\nabla_X \varphi$ and use induction on k. In these arguments we have been quite sloppy above justifying such things as integration by parts though our forms are not smooth. The rigorous way to justify such arguments is to smooth out the forms first *in the graph norm* of the first-order operators by the result of Friedrichs and get estimates on the approximating smooth forms using constants dependent only on our original nonsmooth form and then take limit at the end. A form is in $\mathcal{L}^2_k(p)$ if and only if it can be approximated by smooth forms in the norm of $\mathcal{L}^2_k(p)$.

Now that we have the regularity result for $\Box$, we consider the map Ψ from the space

$$\mathrm{Ker}(\bar{\partial} : \mathcal{C}^\infty(p) \to \mathcal{C}^\infty(p+1)) / \mathrm{Im}(\bar{\partial} : \mathcal{C}^\infty(p-1) \to \mathcal{C}^\infty(p))$$

to the space

$$\mathrm{Ker}\,(\bar{\partial} : \mathcal{L}^2(p) \to \mathcal{L}^2(p+1)) / \mathrm{Im}(\bar{\partial} : \mathcal{L}^2(p-1) \to \mathcal{L}^2(p)).$$

We want to show that Ψ is an isomorphism. First we show that it is surjective. Take $\varphi \in \mathcal{L}^2(p)$ with $\bar{\partial}\varphi = 0$. We have

$$\varphi = H\varphi + \bar{\partial}^* \bar{\partial} G\varphi + \bar{\partial}\bar{\partial}^* G\varphi.$$

As we argued before, when we apply $\bar\partial$ to both sides and take the inner product with $G\varphi$, we conclude that $\bar\partial^*\bar\partial G\varphi = 0$. From $\Box H\varphi = 0$ and the regularity of $\Box$ we know that $H\varphi$ is smooth. Since φ differs from the $\bar\partial$-closed smooth form $H\varphi$ by the $\bar\partial$-exact form $\bar\partial^* G\varphi$, we conclude that Ψ is surjective. Now we look at the injectivity of Ψ. Suppose $\varphi \in \mathcal{C}^\infty(p)$ and $\varphi = \bar\partial\psi$ for some $\psi \in \mathcal{L}^2(p-1)$. Take the decomposition

$$\varphi = H\varphi + \bar\partial^*\bar\partial G\varphi + \bar\partial\bar\partial^* G\varphi.$$

It is clear that the three spaces $\operatorname{Ker}\Box$, $\operatorname{Im}\bar\partial$, $\operatorname{Im}\bar\partial^*$ are mutually orthogonal, because $\operatorname{Ker}\Box = (\operatorname{Ker}\bar\partial) \cap (\operatorname{Ker}\bar\partial^*)$. From

$$0 = H\varphi + \bar\partial^*\bar\partial G\varphi + \bar\partial(\bar\partial^* G\varphi - \psi)$$

we conclude that $H\varphi = 0$ and $\bar\partial^*\bar\partial G\varphi = 0$. So $\varphi = \bar\partial\bar\partial^* G\varphi$. Since φ is smooth, by the regularity of $\Box$ and $\Box G\varphi = \varphi - H\varphi$ we know that $G\varphi$ is smooth and $\bar\partial^* G\varphi$ is smooth. Since φ is the $\bar\partial$ of the smooth form $\bar\partial^* G\varphi$, we conclude that Ψ is injective. From the above argument we also see that both spaces

$$\operatorname{Ker}(\bar\partial : \mathcal{C}^\infty(p) \to \mathcal{C}^\infty(p+1))/\operatorname{Im}(\bar\partial : \mathcal{C}^\infty(p-1) \to \mathcal{C}^\infty(p))$$

and

$$\operatorname{Ker}(\bar\partial : \mathcal{L}^2(p) \to \mathcal{L}^2(p+1))/\operatorname{Im}(\bar\partial : \mathcal{L}^2(p-1) \to \mathcal{L}^2(p))$$

are isomorphic to $\operatorname{Ker}\Box$. An element of $\operatorname{Ker}\Box$ is called a *harmonic form.* So we have proved that the cohomology group $H^p(M, \mathcal{O}(V))$ is isomorphic to the space of all harmonic V-valued $(0,p)$-forms. Since $\operatorname{Ker}\Box$ is finite dimensional from the spectral theorem, we know that $H^p(M, \mathcal{O}(V))$ is always finite dimensional for a compact manifold M.

§2. *Gårding's inequality.*

Let us now verify the Gårding's inequality that we used earlier. We have a Hermitian metric $g_{i\bar j}$ for M and a Hermitian metric $h_{\alpha\bar\beta}$ for V. Recall that if we have a metric on a finite dimensional vector space E, we have a natural metric on $\wedge^p E$. The natural metric is given as follows. We choose an orthonormal basis $e_1, \cdots, e_n$ for E and then defining the natural metric on $\wedge^p E$ by saying that the set of all $e_{i_1} \wedge \cdots \wedge e_{i_p}$ for $i_1 < \cdots < i_p$ is an orthonormal basis. In order to facility the computation of inner product in $\wedge^p E$ we should represent each element φ in $\wedge^p E$ in the form

$$\varphi = \sum_{i_1 < \cdots < i_p} A_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$$

so that we can see right away that the inner product (φ, ψ) of φ with

$$\psi = \sum_{i_1 < \cdots < i_p} B_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$$

is $\sum_{i_1 < \cdots < i_p} A_{i_1 \cdots i_p} \overline{B_{i_1 \cdots i_p}}$. When we do this to the case $E = T^*_M$ and consider the exterior derivative of differential forms, it is difficult to keep track of the condition $i_1 < \cdots < i_p$. So we drop this condition by requiring that $A_{i_1 \cdots i_p}$ be skew-symmetric in $i_1, \cdots, i_p$. When we sum over all $i_1, \cdots, i_p$ without the requirement $i_1 < \cdots < i_p$, we have instead

$$\varphi = \frac{1}{p!} \sum_{i_1, \cdots, i_p} A_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$$

and

$$(\varphi, \psi) = \frac{1}{p!} \sum_{i_1, \cdots, i_p} A_{i_1 \cdots i_p} \overline{B_{i_1 \cdots i_p}}.$$

We want to derive the formulae for $\bar{\partial}$ and $\bar{\partial}^*$ and $\Box$. Take

$$\varphi = \frac{1}{p!} \sum_{i_1, \cdots, i_p} \varphi_{\bar{i}_1 \cdots \bar{i}_p} d\overline{z^{i_1}} \wedge \cdots \wedge d\overline{z^{i_p}}.$$

Then

$$\bar{\partial}\varphi = \frac{1}{p!} \sum_{i_0, i_1, \cdots, i_p} \partial_{\overline{i_0}} \varphi_{\bar{i}_1 \cdots \bar{i}_p} d\overline{z^{i_0}} \wedge d\overline{z^{i_1}} \wedge \cdots \wedge d\overline{z^{i_p}}.$$

Write

$$\bar{\partial}\varphi = \frac{1}{(p+1)!} \sum_{i_0, i_1, \cdots, i_p} (\bar{\partial}\varphi)_{\bar{i}_0 \bar{i}_1 \cdots \bar{i}_p} d\overline{z^{i_0}} \wedge d\overline{z^{i_1}} \wedge \cdots \wedge d\overline{z^{i_p}}.$$

To get the coefficient $(\bar{\partial}\varphi)_{\bar{i}_0 \bar{i}_1 \cdots \bar{i}_p}$ we have to skew-symmetrize $\partial_{\overline{i_0}} \varphi_{\bar{i}_1 \cdots \bar{i}_p}$ in the indices $i_0, i_1, \cdots, i_p$. Since $(\bar{\partial}\varphi)_{\bar{i}_0 \bar{i}_1 \cdots \bar{i}_p}$ is already skew-symmetric in the indices $i_1, \cdots, i_p$, we have

$$(\bar{\partial}\varphi)_{\bar{i}_0 \bar{i}_1 \cdots \bar{i}_p} = \sum_{\nu=0}^{p} (-1)^\nu \partial_{\overline{i_\nu}} \varphi_{\bar{i}_0 \cdots \widehat{\bar{i}_\nu} \cdots \bar{i}_p},$$

where $\widehat{\bar{i}_\nu}$ means that the index $\bar{i}_\nu$ is removed.

We now derive $\bar\partial^*\psi$ when

$$\psi = \frac{1}{(p+1)!}\sum_{i_0,\cdots,i_p}\psi_{\overline{i_0}\cdots\overline{i_p}}d\overline{z^{i_0}}\wedge\cdots\wedge d\overline{z^{i_p}}.$$

The definition of $\bar\partial^*\psi$ is given by $(\bar\partial^*\psi,\varphi)=(\psi,\bar\partial\varphi)$. That is,

$$\begin{aligned}&\frac{1}{p!}\int_M h_{\alpha\overline{\beta}}(\bar\partial^*\psi)^{\alpha i_1\cdots i_p}\overline{\varphi^{\beta}_{\overline{i_1}\cdots\overline{i_p}}}\det(g_{i\bar{j}})\\ =&\frac{1}{(p+1)!}\int_M h_{\alpha\overline{\beta}}\psi^{\alpha i_0 i_1\cdots i_p}\overline{(\overline{\partial}\varphi)^{\beta}_{\overline{i_0}\,\overline{i_1}\cdots\overline{i_p}}}\det(g_{i\bar{j}}),\end{aligned}$$

where $\det(g_{i\bar{j}})$ is the volume element for the metric $g_{i\bar{j}}$. We will drop the volume element from the integrals if there is no risk of confusion. In order to avoid putting the expression $g_{i_\nu\overline{j_\nu}}$ in the above equation we have raised the indices of $\bar\partial^*\psi$ and ψ. When we derived the formula for $(\bar\partial\varphi)_{\overline{i_0 i_1}\cdots\overline{i_p}}$, we skew-symmetrize $\partial_{\overline{i_0}}\varphi_{\overline{i_1}\cdots\overline{i_p}}$. However, the inner product of an element with a second skew-symmetric element is the same as the inner product of its skew-symmetrization with the second skew-symmetric element. So in the above equation we can replace $(\bar\partial\varphi)_{\overline{i_0 i_1}\cdots\overline{i_p}}$ by $(p+1)\partial_{\overline{i_0}}\varphi_{\overline{i_1}\cdots\overline{i_p}}$ and get

$$\begin{aligned}&\frac{1}{p!}\int_M h_{\alpha\overline{\beta}}(\bar\partial^*\psi)^{\alpha i_1\cdots i_p}\overline{\varphi^{\beta}_{\overline{i_1}\cdots\overline{i_p}}}\det(g_{i\bar{j}})\\ =&\frac{1}{p!}\int_M h_{\alpha\overline{\beta}}\psi^{\alpha i_0 i_1\cdots i_p}\overline{\partial_{\overline{i_0}}\varphi_{\overline{i_1}\cdots\overline{i_p}}}\det(g_{i\bar{j}}).\end{aligned}$$

We now perform integration by parts. We can write both ψ and φ as finite sums of forms with supports in coordinate charts and after we derive the formula for $\bar\partial^*$ for the summands of ψ with supports in coordinate charts we can recover $\bar\partial^*\psi$ by taking the sum. So without loss of generality we can assume that both φ and ψ are supported in a single coordinate chart and we can perform integration by parts there. Assume first that the metric $g_{i\bar{j}}$ is Kähler. At the point where differentiation is performed we assume that a holomorphic normal coordinate is chosen at that point so that the first derivative of $g_{i\bar{j}}$ vanishes at that point. Also we can assume that a local holomorphic basis of V is chosen near that point so that the first derivative of $h_{\alpha\overline{\beta}}$ vanishes at that point. Now integration by parts yields

$$\frac{1}{p!}\int_M h_{\alpha\overline{\beta}}(\bar\partial^*\psi)^{\alpha i_1\cdots i_p}\overline{\varphi^{\beta}_{\overline{i_1}\cdots\overline{i_p}}}\det(g_{i\bar{j}})$$

$$= -\frac{1}{p!}\int_M h_{\alpha\bar\beta}\left(\nabla_{i_0}\psi^{\alpha i_0 i_1\cdots i_p}\right)\overline{\varphi_{\overline{i_1}\cdots\overline{i_p}}}\det(g_{i\bar j}).$$

where the covariant differential operator is used to make the expression independent of the choice of coordinates of M and local frames for V. Hence

$$(\bar\partial^*\psi)^{i_1\cdots i_p} = -\nabla_{i_0}\psi^{i_0 i_1\cdots i_p}.$$

When we lower the indices, we get

$$(\bar\partial^*\psi)_{\overline{i_1}\cdots\overline{i_p}} = -g^{\lambda\bar\mu}\nabla_\lambda\psi_{\overline{\mu}\overline{i_1}\cdots\overline{i_p}}.$$

When we do not have a Kähler metric, we have to worry about differentiating $g^{i_\nu\bar j_\nu}$ and $\det(g_{i\bar j})$, but these only contribute to terms of lower order. So in general we have

$$(\bar\partial^*\psi)_{\overline{i_1}\cdots\overline{i_p}} = -g^{\lambda\bar\mu}\partial_\lambda\psi_{\mu\overline{i_1}\cdots\overline{i_p}} + \text{lower order terms}.$$

We will use $(\ell.o.)$ to denote lower order terms.

Now we can compute $\square\varphi$. We have

$$\begin{aligned}
(\bar\partial\bar\partial^*\varphi)_{\overline{i_1}\cdots\overline{i_p}} &= \sum_{\nu=1}^{p}(-1)^{\nu-1}\partial_{\overline{i_\nu}}(\bar\partial^*\varphi)_{\overline{i_1}\cdots\widehat{\overline{i_\nu}}\cdots\overline{i_p}} + (\ell.o) \\
&= -\sum_{\nu=1}^{p}(-1)^{\nu-1}\partial_{\overline{i_\nu}}\left(g^{\lambda\overline{\nu}}\partial_\lambda\varphi_{\overline{\lambda}\overline{i_1}\cdots\widehat{\overline{i_\nu}}\cdots\overline{i_p}}\right) + (\ell.o) \\
&= \sum_{\nu=1}^{p}(-1)^{\nu}g^{\lambda\overline{\nu}}\partial_{\overline{i_\nu}}\partial_\lambda\varphi_{\overline{\lambda}\overline{i_1}\cdots\widehat{\overline{i_\nu}}\cdots\overline{i_p}} + (\ell.o)
\end{aligned}$$

$$\begin{aligned}
(\bar\partial^*\bar\partial\varphi)_{\overline{i_1}\cdots\overline{i_p}} &= -g^{\lambda\bar\mu}\partial_\lambda(\bar\partial\varphi)_{\overline{\mu}\overline{i_1}\cdots\overline{i_p}} + (\ell.o) \\
&= -g^{\lambda\bar\mu}\partial_\lambda\partial_{\overline{\mu}}\varphi_{\overline{\mu}\overline{i_1}\cdots\overline{i_p}} + (\ell.o) \\
&+ \sum_{\nu=1}^{p}(-1)^{\nu-1}g^{\lambda\bar\mu}\partial_{\overline{i_\nu}}\partial_\lambda\varphi_{\overline{\lambda}\overline{i_1}\cdots\widehat{\overline{i_\nu}}\cdots\overline{i_p}} + (\ell.o).
\end{aligned}$$

Hence

$$(\square\varphi)_{\overline{i_1}\cdots\overline{i_p}} = -g^{\lambda\bar\mu}\partial_\lambda\partial_{\overline{\mu}}\varphi_{\overline{i_1}\cdots\overline{i_p}} + (\ell.o).$$

We can also write it as

$$(\square\varphi)_{\overline{i_1}\cdots\overline{i_p}} = -g^{\lambda\bar\mu}\partial_{\overline{\mu}}\partial_\lambda\varphi_{\overline{i_1}\cdots\overline{i_p}} + (\ell.o).$$

We take the inner product with φ and after integration by parts we get from both expressions of $\Box\varphi$ above

$$2(\Box\varphi, \varphi) = \|\varphi\|_1^2 + O(\|\varphi\|_1\|\varphi\|).$$

Now we use the inequality $2ab = 2(\sqrt{\epsilon}a)(\frac{1}{\sqrt{\epsilon}}b) \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$. So the error term $O(\|\varphi\|_1\|\varphi\|)$ is dominated in absolute value by $\epsilon\|\varphi\|_1^2 + C_\epsilon\|\varphi\|^2$, where C_ϵ is a constant dependent on ϵ. Thus

$$C_\epsilon\|\varphi\|^2 + 2(\Box\varphi, \varphi) \geq (1-\epsilon)\|\varphi\|_1^2.$$

Since $(\Box\varphi, \varphi)$ is nonnegative, we have (after we assume without loss of generality that $C_\epsilon \geq 2$)

$$\|\varphi\|^2 + (\Box\varphi, \varphi) \geq C\|\varphi\|_1^2,$$

where $C = \frac{1-\epsilon}{C_\epsilon}$.

§3. *Rellich's Lemma.*

The Rellich's lemma that we used is that the inclusion map $\mathcal{L}_1^2(p) \to \mathcal{L}^2(p)$ is compact in the sense that from every bounded sequence in $\mathcal{L}_1^2(p)$ we can select a subsequence which converges in $\mathcal{L}^2(p)$. We can use a partition of unity to assume that the support of every member in the sequence is contained in a single coordinate chart over which the vector bundle is trivial. So the question is reduced to the case of a sequence of functions with compact support on $[0,1]^n \subset \mathbf{R}^n$. Since these functions have compact support in $[0,1]^n$, we can extend them to all of $\mathbf{R}^n$ so that they are periodic in all the variables of $\mathbf{R}^n$ with unit period. So we can consider the Fourier series of these functions instead of the functions themselves. When we differentiate a function with respect to x^i, it is equivalent to multiplying the $(\alpha_1, \cdots, \alpha_n)$-th Fourier coefficient by α_i. Let H_s be the set of all sequences $\xi = (\xi_\alpha)$ with $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $-\infty < \alpha_1, \cdots, \alpha_n < \infty$ such that

$$\|\xi\|_s = \left(\sum_{\alpha_1, \cdots, \alpha_n} (1 + |\alpha|^2)^s |\xi_\alpha|^2\right)^{1/2}$$

is finite, where $|\alpha|^2 = \sum_{\nu=1}^n |\alpha_\nu|^2$. We want to show that for $s > r$ from a sequence $\xi^{(j)}$ bounded in $\|\cdot\|_s$ norm we can select a subsequence converging in the $\|\cdot\|_r$ norm. Since clearly for each fixed α the set $\{\xi_\alpha^{(j)}\}_j$ is bounded, we can select a subsequence of $\xi^{(j)}$ (which we can assume to be $\xi^{(j)}$ itself) so that for each fixed α the sequence $\xi_\alpha^{(j)}$ of numbers converges in $\mathbf{C}$. We claim that the sequence $\xi^{(j)}$ is now Cauchy in the $\|\cdot\|_r$ norm. Given any $\epsilon > 0$. We have

$$\sum_{|\alpha|>A}(1+|\alpha|^2)^r|\xi_\alpha^{(j)}-\xi_\alpha^{(k)}|^2 = \sum_{|\alpha|>A}\left(\frac{(1+|\alpha|^2)^r}{(1+|\alpha|^2)^s}\right)(1+|\alpha|^2)^s|\xi_\alpha^{(j)}-\xi_\alpha^{(k)}|^2$$
$$\leq \epsilon\sum_{|\alpha|>A}(1+|\alpha|^2)^s|\xi_\alpha^{(j)}-\xi_\alpha^{(k)}|^2 \leq \epsilon\left(\left\|\xi^{(j)}\right\|_s^2+\left\|\xi^{(k)}\right\|_s^2\right)$$

for A sufficiently large. We can choose N large enough so that

$$\sum_{|\alpha|\leq A}(1+|\alpha|^2)^r|\xi_\alpha^{(j)}-\xi_\alpha^{(k)}|^2 \leq \epsilon$$

for $j, k \geq N$. Hence

$$\left\|\xi^{(j)}-\xi^{(k)}\right\|_r^2 \leq \epsilon\left(1+\left\|\xi^{(j)}\right\|_s^2+\left\|\xi^{(k)}\right\|_s^2\right)$$

for $j, k \geq N$. This concludes the proof of the Rellich's lemma. We would like to point out that for the Rellich's lemma it is not necessary that s and r be integers.

§4. *Serre-Kodaira Duality.*

One application of the Hodge decomposition is the duality theorem of Serre-Kodaira. This duality theorem is the complex version of the duality theorem of Poincaré which says that for an orientable compact topological manifold M of real dimension n the cohomology group $H^p(M, \mathbf{R})$ is dual to the cohomology group $H^{n-p}(M, \mathbf{R})$. This kind of duality has its origin in the duality between the p-fold exterior product $\wedge^p E$ of an $\mathbf{R}$-vector space E of finite real dimension n and the $(n-p)$-fold exterior product. Let us discuss this more precisely. We assume that there is given an $\mathbf{R}$-isomorphism from $\wedge^n E$ to $\mathbf{R}$. This is the same as saying that we have a "volume element" on E. For example, when E is the space of all 1-forms at one point of a real manifold of real dimension n, if the manifold has a volume form, then dividing an element of $\wedge^n E$ by the volume form gives an $\mathbf{R}$-isomorphism from $\wedge^n E$ to $\mathbf{R}$. Now we have a pairing $\wedge^p E \times \wedge^{n-p} E \to \wedge^n E$ defined as exterior product and since we assume that we have an isomorphism from $\wedge^n E$ to $\mathbf{R}$, we have a pairing $\wedge^p E \times \wedge^{n-p} E \to \mathbf{R}$. This pairing gives us an isomorphism Ψ from $\wedge^p E$ to the dual $(\wedge^{n-p} E)^*$ of $\wedge^{n-p} E$. We now assume

that there is an inner product on E. The inner product on E induces an inner product on $\wedge^{n-p}E$ so that if $e_1, \cdots, e_n$ is an orthonormal basis of E then the set of all $e_{i_1} \wedge \cdots \wedge e_{i_{n-p}}$ with $i_1 < \cdots < i_{n-p}$ is an orthonormal basis of $\wedge^{n-p}E$. The inner product $\wedge^{n-p}E \times \wedge^{n-p}E \to \mathbf{R}$ establishes an isomorphism Φ from $(\wedge^{n-p}E)^*$ to $\wedge^{n-p}E$. The isomorphism $\Phi\,\Psi$ from $\wedge^p E$ to $\wedge^{n-p}E$ (which depends on the volume element and the inner product) is called the *star operator* denoted by $*$. We suppose from now on that the volume form of E is induced by the inner product of E together with a chosen orientation in the sense that if $e_1, \cdots, e_n$ is an *oriented* orthonormal basis of E, then the volume form is such that in the isomorphism $\wedge^n E$ the element $e_1 \wedge \cdots \wedge e_n$. The star operator is characterized by the property that if $e_1, \cdots, e_n$ is an oriented orthonormal basis of E, then
$*(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots \wedge e_n$. Clearly $** = (-1)^{p(n-p)}$, because

$$e_1 \wedge \cdots \wedge e_p = (-1)^{p(n-p)} e_{p+1} \wedge \cdots \wedge e_n \wedge e_1 \cdots \wedge e_p.$$

Another obvious property of the star operator is that $\varphi \wedge *\psi = \langle \varphi, \psi \rangle \, (*1)$ for any φ, $\psi \in \Lambda^p E$, where $\langle \cdot, \cdot \rangle$ is the inner product and $*1$ is simply the volume form $e_1, \cdots, e_n$ according to our definition of the star operator for the case of 0-forms.

Suppose that E has a complex structure, *i.e.* there exists an $\mathbf{R}$-endomorphism J of E so that $J^2 = 1$. Assume also that the inner product on E is Hermitian in the sense that $\langle e, e' \rangle = \langle Je, Je' \rangle$. When we consider the complexification $E \otimes \mathbf{C}$ of E, we can extend the star operator by $\mathbf{C}$-linearity to $\wedge^p(E \otimes \mathbf{C})$. We have a Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ on $E \otimes \mathbf{C}$ given by $\langle e, e' \rangle_{\mathbf{C}} = \left\langle e, \overline{e'} \right\rangle_{\mathbf{R}}$ after we extend the inner product $\langle \cdot, \cdot \rangle$ of E by $\mathbf{C}$-bilinearity to a $\mathbf{C}$-bilinear functional $\langle \cdot, \cdot \rangle_{\mathbf{R}}$ on $E \otimes \mathbf{C}$. The inner product $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ on $E \otimes \mathbf{C}$ induces an inner product $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ on $\wedge^p(E \otimes \mathbf{C})$. Then $\varphi \wedge *\bar{\psi} = \langle \varphi, \psi \rangle_{\mathbf{C}} \, (*1)$ for φ, $\psi \in \wedge^p(E \otimes \mathbf{C})$. The complex conjugate sign over ψ on the left-hand side simply is an acknowledgement that the inner product $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ is constructed from $\langle \cdot, \cdot \rangle_{\mathbf{R}}$ with a complex conjugation on the second argument.

When we have a Riemannian manifold M of real dimension n with a given orientation (*i.e.,* a given volume form induced by the Riemannian metric, we defined the star operator $*$ on M at a point P of M by setting the vector space E equal to the space of all 1-forms of M at P. If the Riemannian metric of M is (g_{ij}) with the orientation so that the positive square root of the determinant of (g_{ij}) is the volume form, then for any p-form

$$\varphi = \frac{1}{p!} \sum \varphi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

the star $*\varphi$ of φ is given by

$$*\varphi = \frac{1}{(n-p)!} \sum \psi_{i_1 \cdots i_{n-p}} dx^{i_1} \wedge \cdots \wedge dx^{i_{n-p}}$$

with

$$\psi_{i_1 \cdots i_{n-p}} = (\det(g_{ij}))^{-1/2} g_{1 \cdots n, j_1 \cdots j_p i_1 \cdots i_{n-p}} \varphi^{j_1 \cdots j_p}$$

where $g_{1 \cdots n, i_1 \cdots i_n}$ is the determinant whose (μ, ν)-th entry is $g_{\mu i_\nu}$ and $\varphi^{j_1 \cdots j_p}$ is obtained from $\varphi_{i_1 \cdots i_p}$ by raising its indices, as one can easily verify by checking the special cases $\varphi = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ with $i_1 < \cdots < i_p$ when $dx^1, \cdots, dx^n$ is an oriented orthonormal basis. The factor $(\det(g_{ij}))^{-1/2}$ is needed to take care of the covariant indices $1, 2, \cdots, n$ in a coordinate change. The expression

$$(\det(g_{ij}))^{-1/2} g_{1 \cdots n, j_1 \cdots j_p i_1 \cdots i_{n-p}}$$

is simply the volume form with the orientation $j_1, \cdots, j_p, i_1, \cdots, i_{n-p}$. Another way to write it is $(\det(g_{ij}))^{1/2}$ times the signature of the permutation changing $1, \cdots, n$ to $j_1, \cdots, j_p, i_1, \cdots, i_{n-p}$. We choose to write it in such a clumsy way just to make the invariance under coordinate change more transparent. Note the isomorphism Φ from $(\wedge^{n-p} E)^*$ to $\wedge^{n-p} E$ used in the definition of the star operator is used in raising the indices of $\varphi_{i_1 \cdots i_p}$. Actually we are using the equivalent isomorphism from $(\wedge^p E)^*$ to $\wedge^p E$ instead.

Suppose we have a complex manifold M of complex dimension m with $n = 2m$ and suppose that M carries a Hermitian metric. We have a star operator $*$ by regarding M as a Riemannian manifold. Then $\varphi \wedge *\bar{\psi} = \langle \varphi, \psi \rangle (*1)$ for any forms φ, ψ of the same degree on M at any point.. We want to say something about the effect of * on the type of a form. The star operator is characterized by the equation $\varphi \wedge *\bar{\psi} = \langle \varphi, \psi \rangle (*1)$. If ψ is a (p, q)-form and if we choose an $(p+q)$-form φ that is not of type (p, q), then $\langle \varphi, \psi \rangle (*1)$ clearly vanishes and as a consequence $\varphi \wedge *\bar{\psi}$ vanishes. Since we know that $*\bar{\psi}$ must be of degree $(p+q)$, we conclude that $*\bar{\psi}$ must be of type $(m-p, m-q)$. Since the star operator is a real operator, we know that the star of any (p,q)-form is of type $(m-q, m-p)$. We can also see this explicitly from the formula

$$(*\varphi)_{i_1\cdots i_{m-q}\overline{j_1}\cdots\overline{j_{m-p}}} = (-1)^{n(n-1)/2}\left(\det(g_{i\bar{j}})\right)^{-1} \times$$
$$g_{1\cdots m,\overline{k_1}\cdots\overline{k_p}\,\overline{j_1}\cdots\overline{j_{m-p}}}\overline{g_{1\cdots m,\overline{\ell_1}\cdots\overline{\ell_q}\,\overline{i_1}\cdots\overline{i_{m-q}}}}\varphi^{\overline{k_1}\cdots\overline{k_p}\ell_1\cdots\ell_q},$$

where $g_{1\cdots m,\overline{i_1}\cdots\overline{i_m}}$ is the determinant whose (μ,ν)-th entry is $g_{\mu\overline{i_\nu}}$. The sign $(-1)^{n(n-1)/2}$ on the right-hand side of the above equation is used, because when we use $g_{1\cdots m,\overline{i_1}\cdots\overline{i_m}}$ the orientation is $1,\cdots,m,\bar{1},\cdots,\bar{m}$ instead of the standard $1,\bar{1},\cdots,m,\bar{m}$. This formula is simply a rewriting of the formula for the Riemannian case in terms of complex coordinates.

Since the star operator is so tied up with the inner product, we can use express an adjoint operator in the language of the star operator. Let us compute $\bar{\partial}^*$ for $(p,q+1)$-forms ψ on a compact complex manifold M of complex dimension m. According to the definition of the adjoint operator $\bar{\partial}^*$ of $\bar{\partial}$ we have $\int_M \left\langle \bar{\partial}\varphi, \psi \right\rangle = \int_M \left\langle \varphi, \bar{\partial}^*\psi \right\rangle$, where $\langle\cdot,\cdot\rangle$ means the pointwise inner product. We can replace the above equation by the following equivalent equation using the star operator $\int_M (\bar{\partial}\varphi)\wedge *\bar{\psi} = \int_M \varphi \wedge *\bar{\partial}^*\psi$. Let $k = p+q$. Now

$$\int_M (\bar{\partial}\varphi)\wedge *\bar{\psi} = (-1)^{k+1}\int_M \varphi\wedge\bar{\partial}\bar{*}\bar{\psi}$$
$$\text{(because } d(\varphi\wedge *\bar{\psi}) = (\bar{\partial}\varphi)\wedge *\bar{\psi} + (-1)^k\varphi\wedge\bar{\partial}\bar{*}\bar{\psi} \text{ by type considerations)}$$
$$= (-1)^{k(2m-k)+k+1}\int_M \varphi\wedge *\bar{*}\bar{\partial}\bar{*}\bar{\psi} = -\int_M \langle\varphi, *\partial * \psi\rangle .$$

Hence we have the simple formula $\bar{\partial}^* = -*\partial*$.

Suppose we have a holomorphic vector bundle V of rank r over M and V is endowed with a Hermitian metric $h_{\alpha\overline{\beta}}$ along its fibers. We can naturally extend the definition of the star operator to V-valued (p,q)-forms. The extension is simply defined as the tensor product of the star operator for scalar-valued forms with the identity map of V. The formula $\bar{\partial}^* = -*\partial*$ continues to hold for V-valued (p,q)-forms when the exterior differential operator ∂ is interpreted as covariant differentiation with respect to the complex metric connection of V (note that there is no need to use the connection of the Hermitian metric of M because so far the variables of M are concerned the operator ∂ is simply the (1,0)-exterior differentiation and no connection for M is needed for its definition). The reason that $\bar{\partial}^* = -*\partial*$ continues to hold is that we can modify its derivation given above in the following way. We can assume by writing the V-valued (p,q)-form φ and the V-valued $(p,q+1)$-form ψ as sums of forms whose supports are inside open subsets of M where V is trivial. We have then

$$\int_M \left\langle \bar\partial\varphi, \psi \right\rangle = \int_M h_{\alpha\overline{\beta}}(\bar\partial\varphi^\alpha) \wedge *\overline{\psi^\beta} = (-1)^{k+1}\int_M h_{\alpha\overline{\beta}}\varphi^\alpha \wedge \overline{\partial^*\psi^\beta}$$
$$= (-1)^{2m(m-k)+k+1}\int_M h_{\alpha\overline{\beta}}\varphi^\alpha \wedge **\overline{\partial * \psi^\beta} = -\int_M \langle \varphi, *\partial * \psi\rangle .$$

Again the Stokes' theorem is applied once to the exact form

$$d(h_{\alpha\overline{\beta}}\varphi^\alpha \wedge *\overline{\psi^\beta}) = h_{\alpha\overline{\beta}}(\bar\partial\varphi^\alpha) \wedge *\overline{\psi^\beta} + (-1)^k h_{\alpha\overline{\beta}}\varphi^\alpha \wedge \overline{\partial * \psi^\beta}.$$

We are now ready to look at the duality theorem of Serre-Kodaira. A V-valued (p,q)-form φ is harmonic if and only if $\bar\partial\varphi = 0$ and $\bar\partial^*\varphi = 0$. We can rewrite $\bar\partial^*\varphi = 0$ as $\partial * \varphi = 0$. Taking complex conjugates, we conclude that φ is harmonic if and only if $\partial\bar\varphi = 0$ and $\bar\partial * \bar\varphi = 0$ which is equivalent to $*\partial(* * \bar\varphi) = 0$ and $\bar\partial * \bar\varphi = 0$. Now $*\bar\varphi$ is a $\bar V$-valued $(m-p, m-q)$-form. When we take $\bar\partial$ of $*\bar\varphi$, since $\bar V$ is anti-holomorphic vector bundle instead of a holomorphic vector bundle, we have to use the complex metric connection of $h_{\alpha,\overline{\beta}}$. Let us denote by $h * \bar\varphi$ the V^*-valued $(m-p, m-q)$-form given by $(h * \bar\varphi)_\alpha = h_{\alpha,\overline{\beta}}(*\overline{\varphi^\beta})$, where V^* is the dual bundle of V.

The harmonicity of φ is now equivalent to $*\partial(*(h*\bar\varphi)) = 0$ and $\bar\partial(h*\bar\varphi) = 0$ which is in turn equivalent to $\bar\partial^*(h * \bar\varphi) = 0$ and $\bar\partial(h * \bar\varphi) = 0$. So φ is a harmonic V-valued (p,q)-form if and only if $h * \bar\varphi$ is a harmonic V^*-valued $(m-p, m-q)$-form. Thus we have the duality of $H^q(M, V \otimes \Omega^p_M)$ and $H^{m-q}(M, V^* \otimes \Omega^{m-p}_M)$ which is the duality of Serre and Kodaira.

In the special case where $p = 0$ we have the duality between $H^q(M,V)$ and $H^{m-q}(M, V^* \otimes K_M)$, where $K_M = \Omega^m_M$ is the canonical line bundle of M. The duality for the case of a general p is now more general than the special case of $p = 0$, because the exterior product map $\Omega^p_M \times \Omega^{m-p}_M \to \Omega^m_M = K_M$ gives the isomorphism between $(V \otimes \Omega^p_M)^* \otimes K_M$ and $V^* \otimes \Omega^{m-p}_M$ and one can apply the special case to the bundle $V \otimes \Omega^p_M$ instead of V.

This proof of the Serre-Kodaira duality by harmonic forms is Kodaira's proof. Serre's proof uses the resolution of the sheaf $\mathcal{O}_M(V^* \otimes K_M)$ by sheaf of germs of (m,q)-currents and use the duality of this resolution with the resolution of the sheaf $\mathcal{O}_M(V)$ by the sheaf of germs of smooth $(0,q)$-forms.

§5. *Hodge Structure.*

We have shown that cohomology classes are represented by harmonic forms. Now we want to discuss the relation between the deRham cohomology and the Dolbeault cohomology. They are represented respectively by harmonic forms in the real case and the complex case. The former is given by the kernel of $\Delta = dd^* + d^*d$ and the latter by the kernel of $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. In the Kähler case Δ turns out to be $2\Box$ and harmonicity in both cases agree. Since $\Box$ respects the (p,q) type of a differential form, we have a natural decomposition in the case of a compact Kähler manifold of $H^k(M,\mathbf{C})$ into a direct sum of $H^q(M,\Omega_M^p)$ with $p+q=k$ given by harmonic forms of type (p,q). This is known as the Hodge structure. Note that even though the metric is Hermitian, if it is not Kähler, in general the real Laplacian Δ does not respect the (p,q) type of a differential form. We would like to point out that for our discussion of Hodge structure we do not use any holomorphic vector bundle. There is no such theory when a holomorphic vector bundle is used.

Let us now verify the identity $\Delta = 2\Box$ in the case of a Kähler manifold. Let $\overline{\Box} = \partial\partial^* + \partial^*\partial$ be the complex conjugate of $\Box$. Since the operator Δ is real, it would follow from $\Delta = 2\Box$ that we have also $\Box = \overline{\Box}$. Suppose φ is a (p,q)-form. We can regard it as an Ω_M^p-valued $(0,q)$-form. When we take $\bar\partial$ of φ, there are two meanings. Let us use $\bar\partial$ to denote the (0,1)-exterior differentiation of the (p,q)-form φ and use $\bar\partial_\Omega$ to denote the (0,1)-exterior differentiation of the Ω_M^p-valued $(0,q)$-form φ. Write

$$\varphi = \frac{1}{p!q!}\sum \varphi_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge dz^{\overline{j_1}}\wedge\cdots\wedge dz^{\overline{j_q}}.$$

Then

$$\bar\partial\varphi = \frac{(-1)^p}{p!q!}\sum \partial_{\overline{j_0}}\varphi_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge dz^{\bar j_0}\wedge ddz^{\overline{j_0}}\wedge dz^{\overline{j_1}}\wedge\cdots\wedge dz^{\overline{j_q}}.$$

On the other hand we have

$$\bar\partial_\Omega\varphi = \frac{1}{p!q!}\sum \partial_{\overline{j_0}}\varphi_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge dz^{\bar j_0}\wedge ddz^{\overline{j_0}}\wedge dz^{\overline{j_1}}\wedge\cdots\wedge dz^{\overline{j_q}},$$

because in this case we regard $dz^{i_1}\wedge\cdots\wedge dz^{i_p}$ as an element of the bundle Ω_M^p and $dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge dz^{\overline{j_1}}\wedge\cdots\wedge dz^{\overline{j_q}}$ actually means $(dz^{i_1}\wedge\cdots\wedge dz^{i_p})\wedge$

$\otimes \left(dz^{\overline{j_1}} \wedge \cdots \wedge dz^{\overline{j_q}}\right)$. So we have $\bar\partial\varphi = (-1)^p \bar\partial_\Omega \varphi$. This difference in sign is essential to the proof of $\Delta = 2\Box$.

We want to compare $\Delta = dd^* + d^*d$ with $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. Expanding Δ in terms of $\bar\partial$, $\bar\partial^*$, ∂, ∂^*, we get

$$\begin{aligned}\Delta &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\ &= \Box + \overline{\Box} + (\partial\bar\partial^* + \bar\partial^*\partial) + (\partial^*\bar\partial + \bar\partial\partial^*).\end{aligned}$$

We claim that the mixed terms $\partial\bar\partial^* + \bar\partial^*\partial$ and $\partial^*\bar\partial + \bar\partial\partial^*$ vanish. We verify only the vanishing of $\partial\bar\partial^* + \bar\partial^*\partial$, because the other one is simply its complex conjugate. In the Kähler case earlier we proved that

$$(\bar\partial^*_\Omega\varphi)_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_{q-1}}} = -g^{\lambda\bar\mu}\nabla_\lambda \varphi_{i_1\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_{q-1}}}.$$

Hence

$$(\bar\partial^*\varphi)_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_{q-1}}} = (-1)^{p+1} g^{\lambda\bar\mu}\nabla_\lambda \varphi_{i_1\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_{q-1}}}.$$

Now

$$(\partial\varphi)_{i_0 i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}} = \sum_{\nu=0}^{p}(-1)^\nu \partial_{i_\nu}\varphi_{i_0\cdots\widehat{i_\nu}\cdots i_p\overline{j_1}\cdots\overline{j_q}}.$$

So

$$\begin{aligned}(\partial\bar\partial^*\varphi)_{i_0\cdots i_p\overline{j_1}\cdots\overline{j_{q-1}}} &= \sum_{\nu=0}^{p}(-1)^\nu \partial_{i_\nu}(\bar\partial^*\varphi)_{i_0\cdots\widehat{i_\nu}\cdots i_p\overline{j_1}\cdots\overline{j_{q-1}}} \\ &= \sum_{\nu=0}^{p}(-1)^\nu \partial_{i_\nu}\left((-1)^{p+1} g^{\lambda\bar\mu}\nabla_\lambda\varphi_{i_0\cdots\widehat{i_\nu}\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_q}}\right) \\ &= (-1)^{p+1} g^{\lambda\bar\mu}\sum_{\nu=0}^{p}(-1)^\nu \nabla_{i_\nu}\nabla_\lambda\varphi_{i_0\cdots\widehat{i_\nu}\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_q}}.\end{aligned}$$

On the other hand

$$\begin{aligned}(\bar\partial^*\partial\varphi)_{i_0\cdots i_p\overline{j_1}\cdots\overline{j_{q-1}}} &= (-1)^p g^{\lambda\bar\mu}\nabla_\lambda(\partial\varphi)_{i_0\cdots\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_{q-1}}} \\ &= (-1)^p g^{\lambda\bar\mu}\nabla_\lambda\sum_{\nu=0}^{p}(-1)^\nu\partial_{i_\nu}\varphi_{i_0\cdots\widehat{i_\nu}\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_q}} \\ &= (-1)^p g^{\lambda\bar\mu}\sum_{\nu=0}^{p}(-1)^\nu\nabla_\lambda\nabla_{i_\nu}\varphi_{i_0\cdots\widehat{i_\nu}\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_q}}.\end{aligned}$$

Since the component $R_{ijk\ell}$ of the curvature tensor of the Kähler metric vanishes when i and j equal respectively the indices λ and i_ν of type (1,0), it follows that $\nabla_\lambda \nabla_{i_\mu} = \nabla_{i_\mu} \nabla_\lambda$. Hence $\partial\bar\partial^* + \bar\partial^*\partial$ vanishes. Note that $\partial\bar\partial^* + \bar\partial^*\partial$ maps a form of type (p,q) to a form of type $(p+1, q-1)$. So if $\partial\bar\partial^* + \bar\partial^*\partial$ does not vanish, then the operator Δ does not map a form of type (p,q) to a form of type (p,q). In the above verification of the vanishing of $\partial\bar\partial^* + \bar\partial^*\partial$ the Kähler property of the metric was used twice, once in the formula for $\bar\partial^*$ of φ otherwise there would be another lower order term from the torsion tensor of the metric, and another time in the commutativity of the two operators ∇_λ and ∇_{i_ν}.

Now we have to verify that $\square = \overline{\square}$. We compute $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. We did this before in the proof of the Gårding's inequality, but there we ignore the lower order terms. Now we redo it and keep the lower order terms.

$$
\begin{aligned}
(\bar\partial\bar\partial^*\varphi)_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}} &= (-1)^p \sum_{\nu=1}^{q} (-1)^{\nu-1} \partial_{\overline{j_\nu}} (\bar\partial^*\varphi)_{i_1\cdots i_p\overline{j_1}\cdots\widehat{\overline{j_\nu}}\cdots\overline{j_q}} \\
&= -\sum_{\nu=1}^{q} (-1)^{\nu-1} \partial_{\overline{j_\nu}} \left((-1)^{p+1} g^{\lambda\bar\mu} \nabla_\lambda \varphi_{i_1\cdots i_p\overline{\mu}\overline{j_1}\cdots\widehat{\overline{j_\nu}}\cdots\overline{j_q}} \right) \\
&= g^{\lambda\bar\mu} \sum_{\nu=1}^{q} (-1)^{\nu} \nabla_{\overline{j_\nu}} \nabla_\lambda \varphi_{i_1\cdots i_p\overline{\mu}\overline{j_1}\cdots\widehat{\overline{j_\nu}}\cdots\overline{j_q}}.
\end{aligned}
$$

$$
\begin{aligned}
(\bar\partial^*\bar\partial\varphi)_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}} &= (-1)^{p+1} g^{\lambda\bar\mu} \nabla_\lambda (\bar\partial\varphi)_{i_1\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_q}} \\
= -g^{\lambda\bar\mu}\nabla_\lambda\nabla_{\overline{\mu}}\varphi_{i_1\cdots i_p\overline{\mu}\overline{j_1}\cdots\overline{j_q}} &+ \sum_{\nu=1}^{q} (-1)^{\nu-1} g^{\lambda\bar\mu} \nabla_\lambda \nabla_{\overline{j_\nu}} \varphi_{i_1\cdots i_p\overline{\mu}\overline{j_1}\cdots\widehat{\overline{j_\nu}}\cdots\overline{j_q}}.
\end{aligned}
$$

$$
(\square\varphi)_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}} = -g^{\lambda\bar\mu}\nabla_\lambda\nabla_{\overline{\mu}}\varphi_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}} - \sum_{\nu=1}^{q} g^{\lambda\bar\mu}[\nabla_{\overline{j_\nu}}, \nabla_\lambda]\varphi_{i_1\cdots i_p\overline{j_1}\cdots(\widehat{\overline{\nu}})\cdots\overline{j_q}},
$$

where $(\bar\mu)_\nu$ means that the barred index in the ν^{th} position is replaced by $\bar\mu$. The factor $(-1)^{\nu-1}$ disappeared, because we have moved the index $\bar\mu$ passed the $(\nu-1)$ indices $\bar j_1, \cdots, \bar j_{\nu-1}$ to put it in the place which was earlier occupied by $\bar j_\nu$. Let us now take care of the zero-order term

$$
-\sum_{\nu=1}^{p} g^{\lambda\bar\mu}[\nabla_{\bar j_\nu}, \nabla_\lambda]\varphi_{i_1\cdots i_p\overline{j_1}\cdots(\bar\mu)_\nu\cdots\cdots\bar j_q}.
$$

Recall the complex metric connection is for the Kähler metric is given by definition by $\nabla_{\bar j}\xi^\alpha = \partial_{\bar j}\xi^\alpha$ and $\nabla_\lambda\xi^\alpha = \partial_\lambda\xi^\alpha + \xi^\beta(\partial_\lambda h_{\beta\bar\gamma})h^{\bar\gamma\alpha}$ for a vector field ξ^α of type (1,0). Hence $\nabla_{\bar j}\eta_\alpha = \partial_{\bar j}\eta_\alpha$ and $\nabla_\lambda \eta_\alpha = \partial_\lambda \eta_\alpha - \eta_\beta(\partial_\lambda h_{\alpha\bar\gamma})h^{\bar\gamma\beta}$. Using normal coordinates for our computation we have

$$[\nabla_{\bar j}, \nabla_\lambda]\eta_\alpha = \partial_{\bar j}\partial_\lambda \eta_\alpha - \eta_\beta \partial_{\bar j}((\partial_\lambda h_{\alpha\bar\gamma})h^{\bar\gamma\beta}) - \partial_\lambda\partial_{\bar j}\eta_\alpha$$

$$= -\eta_\beta \partial_{\bar j}((\partial_\lambda h_{\alpha\bar\gamma})h^{\bar\gamma\beta}) = g^{\bar\beta\gamma}R_{\lambda\bar j\alpha\bar\beta}\eta_\gamma,$$

where $R_{\lambda\bar j\alpha\bar\beta} = -\partial_\lambda\partial_{\bar j}h_{\alpha\bar\beta} + h^{\bar\gamma\delta}\partial_\lambda h_{\alpha\bar\gamma}\partial_{\bar j}h_{\delta\bar\beta}$. Taking complex conjugates we get

$$[\nabla_{\bar i}, \nabla_\lambda]\eta_{\bar\alpha} = -g^{\bar\beta\gamma}R_{\lambda\bar i\beta\bar\alpha}\eta_{\bar\gamma}.$$

Since a (p,q)-form is a linear combination of the products of p $(1,0)$-forms and q $(0,1)$-forms, we have

$$[\nabla_{\bar j_\nu}, \nabla_\lambda]\varphi_{i_1\cdots i_p\bar j_1\cdots(\bar\mu)_\nu\cdots\bar j_q} = \sum_{\sigma=1}^{p} R_{\lambda\bar j_\nu i_\sigma}{}^{\kappa}\varphi_{i_1\cdots(\kappa)_\sigma\cdots i_p\bar j_1\cdots(\bar\mu)_\nu\cdots\bar j_q}$$

$$-\sum_{\tau=1,\tau\neq\nu}^{q} R_{\lambda\bar j_\nu}{}^{\bar\theta}{}_{\bar j_\tau}\varphi_{i_1\cdots i_p\bar j_1\cdots(\bar\theta)_\tau\cdots(\bar\mu)_\nu\cdots\bar j_q} - R_{\lambda\bar j_\nu}{}^{\bar\theta}{}_{\bar\mu}\varphi_{i_1\cdots i_p\bar j_1\cdots(\bar\theta)_\nu\cdots\bar j_q}.$$

Hence

$$(\Box\varphi)_{i_1\cdots i_p\bar j_1\cdots\bar j_q} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

where

$$\mathrm{I} = -g^{\lambda\bar\mu}\nabla_\lambda\nabla_{\bar\mu}\varphi_{i_1\cdots i_p\bar j_1\cdots\bar j_q},$$
$$\mathrm{II} = -\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar\mu}_{\bar j_\nu i_\sigma}{}^{\kappa}{}_{\nu^i\sigma}\varphi_{i_1\cdots(\kappa)_\sigma\cdots i_p\bar j_1\cdots(\bar\mu)_\nu\cdots\bar j_q},$$
$$\mathrm{III} = \sum_{\nu=1}^{q}\sum_{\tau=1,\tau\neq\nu}^{q} R^{\bar\mu}_{\bar j_\nu}{}^{\bar\theta}{}_{\bar j_\tau}\varphi_{i_1\cdots i_p\bar j_1\cdots(\bar\theta)_\tau\cdots(\bar\mu)_\nu\cdots\bar j_q},$$
$$\mathrm{IV} = \sum_{\nu=1}^{q} R_{\bar j_\nu}{}^{\bar\theta}\varphi_{i_1\cdots i_p\bar j_1\cdots(\bar\theta)_\nu\cdots\bar j_q},$$

where $R_{\bar j_\nu}{}^{\bar\theta}$ is the Ricci curvature tensor $R_{\bar\theta\alpha} = g^{\lambda\bar\mu}R_{\lambda\bar\beta\alpha\bar\mu}$ with the last covariant index raised. Now III vanishes always because $R^{\bar\mu}{}_{\bar i_\nu}{}^{\bar\theta}{}_{\bar i_\sigma}$ is symmetric in $\bar\mu$ and $\bar\theta$ yet $\varphi_{i_1\cdots i_p\bar j_1\cdots(\bar\theta)_\tau\cdots(\bar\mu)_\nu\cdots\bar j_q}$ is skew-symmetric in $\bar\mu$ and $\bar\theta$. So the final expression is

$$(\Box\varphi)_{i_1\cdots i_p\bar j_1\cdots\bar j_q} = -g^{\lambda\bar\mu}\nabla_\lambda\nabla_{\bar\mu}\varphi_{i_1\cdots i_p\bar j_1\cdots\bar j_q}$$

$$-\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar\mu}{}_{\bar j_\nu i_\sigma}{}^{\kappa}\varphi_{i_1\cdots(\kappa)_\sigma\cdots i_p\bar j_1\cdots(\bar\mu)_\nu\cdots\bar j_q}$$

$$+\sum_{\nu=1}^{q} R_{\bar j_\nu}{}^{\bar\theta}\varphi_{i_1\cdots i_p\bar j_1\cdots(\bar\theta)_\nu\cdots\bar j_q}.$$

In invariant formulation

$$\Box\varphi = -(\mathrm{Tr}\,\nabla\bar\nabla)\varphi - R\,\varphi + \mathrm{Ric}\,\varphi,$$

where the action $R\,\varphi$ of the curvature tensor R on φ and the action $\mathrm{Ric}\,\varphi$ of the Ricci tensor Ric on φ should be interpreted as follows. The curvature tensor R acts on a (1,1)-form $\psi_{\alpha,\bar\beta}$ naturally to give a (1,1)-form $R^{\bar\gamma\delta}_{\alpha\bar\beta}\psi_{\gamma\bar\delta}$. Another way of looking at it is that R is an element of $T^*_M\otimes\overline{T^*_M}\otimes T^*_M\otimes\overline{T^*_M}$ which is isomorphic to $T^*_M\otimes\overline{T^*_M}\otimes\overline{T_M}\otimes T_M = \mathrm{End}(T^*_M\otimes\overline{T^*_M})$ because the isomorphism of T_M and $\overline{T^*_M}$ given by the Kähler metric. The action of R on the space $T^*_M\otimes\overline{T^*_M}$ of (1,1)-forms is simply the natural one when it is regarded as an element of $\mathrm{End}(T^*_M\otimes\overline{T^*_M})$. We suppress all the covariant indices of the (p,q)-form φ except two to make it a (1,1)-form (which is the same as evaluating φ at $p-1$ $(1,0)$-vectors and $q-1$ $(0,1)$-vectors) and the action of R on φ is the action on this (1,1)-form with summation over all the pq ways of suppressing indices to make φ a (1,1)-form. Likewise the Ricci curvature Ric can be regarded as an element of $\mathrm{End}(\overline{T_M})$ and so it acts on (0,1)-forms to give (0,1)-forms. We suppress all the covariant indices of the (p,q)-form φ except one to make it a (0,1)-form (which is the same as evaluating φ at $p(1,0)$-vectors and $q-1(0,1)$-vectors) and the action of Ric on φ is the action on this $(0,1)$-form with summation over all the q ways of suppressing indices to make φ a (0,1)-form. We could have done all the computations in invariant form with the same steps. The indices were used simply to keep track of how a tensor acts on another tensor of different covariant and contravariant ranks.

We now compute $\bar\Box\varphi$. We can either go through an analogous computation or we can simply take complex conjugates. Let us take the second route which is easier. We have $\bar\Box\varphi = \overline{(\Box\overline{\varphi})}$. Let $\psi=\bar\varphi$. Then ψ is a (q,p)-form and

$$\psi_{j_1\cdots j_q,\overline{i_1}\cdots\overline{i_p}} = (-1)^{pq}\overline{\varphi_{i_1\cdots i_p,\overline{j_1}\cdots\overline{j_q}}}.$$

$$(\Box\psi)_{j_1\cdots j_q,\overline{i_1}\cdots\overline{i_p}} = -g^{\bar\lambda\mu}\nabla_\mu\nabla_{\bar\lambda}\psi_{j_1\cdots j_q,\overline{i_1}\cdots\overline{i_p}}$$

$$-\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar\kappa}{}_{\overline{i_\sigma} j_\nu}{}^{\mu}\psi_{j_1\cdots(\mu)_\nu\cdots j_q\overline{i_1}\cdots(\overline{\kappa})_\sigma\cdots\overline{i_p}}$$

$$+R_{\bar{i}_\nu}{}^{\overline{\kappa}}\psi_{j_1\cdots j_q\overline{i_1}\cdots(\overline{\kappa})_\sigma\cdots\overline{i_p}}.$$

Taking complex conjugates, we get

$$(\overline{\Box}\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} = -g^{\lambda\bar\mu}\nabla_{\bar\mu}\nabla_\lambda\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$$

$$-\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar\mu}{}_{\bar{j}_\nu i_\sigma}{}^{\kappa}\varphi_{i_1\cdots(\kappa)_\sigma\cdots i_p\bar{j}_1\cdots(\bar\mu)_\nu\cdots\bar{j}_q}$$

$$+R_{i_\nu}{}^{\theta}\varphi_{i_1\cdots(\theta)_\nu\cdots i_p\bar{j}_1\cdots\bar{j}_q}.$$

The net effect is that the terms I and IV above are now changed, but we have the same term II in the expression for $\overline{\Box}\varphi$, the reason being that the action of R on φ is real action. The new term I changes the order of the action of the operators $\nabla_{\bar\mu}$ and ∇_λ. The new term IV has Ric acting on the unbarred indices of φ instead of the barred indices of φ. The two Laplacians $\Box$ and $\overline{\Box}$ will be shown to be equal by verifying that the commutation formula for $\nabla_{\bar\mu}$ and ∇_λ precisely compensates for the change of the action of the Ricci curvature from the barred indices the unbarred ones. We have the commutation formula

$$[\nabla_{\bar\mu},\nabla_\lambda]\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$$

$$=\sum_{\sigma=1}^{p} R_{\lambda\bar\mu i_\sigma}{}^{\kappa}\varphi_{i_1\cdots(\kappa)_\sigma\cdots i_p\bar{j}_1\cdots\bar{j}_q}$$

$$-\sum_{\tau=1}^{q} R_{\lambda\bar\mu}{}^{\bar\theta}{}_{\bar{j}_\tau}\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar\theta)_\tau\cdots\bar{j}_q}.$$

Hence

$$-g^{\lambda\bar\mu}[\nabla_{\bar\mu},\nabla_\lambda]\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} = -\sum_{\sigma=1}^{p} R_{i_\sigma}{}^{\kappa}\varphi_{i_1\cdots(\kappa)_\sigma\cdots i_p\bar{j}_1\cdots\bar{j}_q}$$

$$+\sum_{\tau=1}^{q} R^{\bar\theta}{}_{\bar{j}_\tau}\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar\theta)_\tau\cdots\bar{j}_q}$$

and

$$(\overline{\Box}\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} = -g^{\lambda\bar\mu}\nabla_\lambda\nabla_{\bar\mu}\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$$

$$-\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar\mu}{}_{\bar{j}_\nu i_\sigma}{}^{\kappa}\varphi_{i_1\cdots(\kappa)_\sigma\cdots i_p\bar{j}_1\cdots(\bar\mu)_\nu\cdots\bar{j}_q}$$

$$+\sum_{\tau=1}^{q} R_{\bar{j}_\tau}^{\bar{\theta}} \varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar{\theta})_\tau\cdots\bar{j}_q}$$

$$= (\square\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}.$$

This concludes the proof that $\Delta = 2\square = 2\overline{\square}$ for a Kähler manifold. We have the natural Hodge structure for compact Kähler manifolds

$$H^k(M, \mathbf{C}) = \oplus_{p+q=k} H^q(M, \Omega_M^p)$$

$$\overline{H^q(M, \Omega_M^p)} = H^p(M, \Omega_M^q),$$

because harmonicity in the sense of Δ agrees with harmonicity in the sense of $\square$ and also with harmonicity in the sense of $\overline{\square}$ and because both Δ and $\square$ are real operators. Moreover, the operator Δ respects types. So a k-form is harmonic if and only if all its (p,q)-components are harmonic.

There is another consequence from $\Delta = \square$. It is the statement that every holomorphic $(p,0)$-form on a compact Kähler manifold is closed. The reason is as follows. A $(p,0)$-form φ is always in the kernel of $\bar{\partial}^*$ because of type considerations. So a $(p,0)$-form φ is harmonic in the sense of $\square$ if and only if it is holomorphic (*i.e.* all coefficients are holomorphic functions) which is equivalent to $\bar{\partial}\varphi = 0$. On the other hand harmonicity in the sense of $\square$ is identically to harmonicity in the sense of Δ which is equivalent to being d-closed and d^*-closed.

§6. *Obstructions to Existence of Kähler Metrics, Calabi-Eckmann and Iwasawa Manifolds.*

The properties derived from $\Delta = 2\square$ for a Kähler yield obstructions to the existence of Kähler metrics for compact complex manifolds. The first obvious obstruction to the existence of Kähler metrics for compact complex manifolds which has nothing to do with the property $\Delta = 2\square$ is that the cohomology class in $H^2(M, \mathbf{C})$ defined by the Kähler form is nonzero. The reason is the following. Let $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge \overline{dz^j}$ be the Kähler form of a compact complex manifold M of complex dimension m. We know that ω is closed. If the cohomology class in $H^2(M, \mathbf{C})$ defined by the Kähler form is zero, then ω is d-exact and $\omega = d\alpha$ for some 1-form α. Hence $\omega^m = d(\alpha \wedge \omega^{m-1})$ and $\int_M \omega^m = 0$, which contradicts the fact that $\omega^m = m! \det(g_{i\bar{j}}) \Pi_{k=1}^m (\sqrt{-1} dz^k \wedge \overline{dz^k})$ is a positive top degree form on M. The same argument shows that the p-fold cup-product of the cohomology class in $H^2(M, \mathbf{C})$ defined by the Kähler form is also nonzero for $1 \leq p \leq m$.

A simple example of a compact complex manifold no admitting a Kähler metric is one whose second Betti number is zero. Consider the quotient M of $\mathbf{C}^2 - 0$ by the equivalence relation $(z^1, z^2) \approx (w^1, w^2)$ if and only if $(z^1, z^2) = 2^k(w^1, w^2)$ for some integer k. The manifold M is called a Hopf manifold. Topologically M can be obtained by taking the shell bounded by the 3-sphere in $\mathbf{C}^2$ of unit radius and the 3-sphere of radius 2 and identifying the two points on the two spheres collinear with the origin. The manifold M is topologically the same as the product of the 3-sphere S^3 and the circle S^1 with the 3-sphere in $\mathbf{C}^2$ of unit radius contributes to the factor S^3 and the line segment with end-points on the two 3-spheres in $\mathbf{C}^2$ collinear with the origin contributing to the factor S^1. Recall that the Poincaré polynomial $P_X(x)$ of a topological manifold X is defined as $\sum_{i=0}^{\infty} b_i(X)x^i$ with $b_i(X)$ equal to the i^{th} Betti number of X. The Poincaré polynomial of a product manifold is equal to the product of the Poincaré polynomials of its factors because of the Künneth formula. The Poincaré polynomial of S^3 is $1 + x^3$ and that of S^1 is $1 + x$. Hence the Poincaré polynomial of M is $1 + x + x^3 + x^4$ and the second Betti number of M is zero.

Calabi and Eckmann generalized this construction of Hopf manifold to the case of the product of two odd-dimensional spheres and get complex structures on such products. These complex manifolds are called Calabi-Eckmann manifolds. Before we discuss them, let us look once again at the Hopf manifold we introduced above. We have another equivalence relation on $\mathbf{C}^2 - 0$ with bigger equivalence classes. For this equivalence relation two points (z^1, z^2) and (w^1, w^2) are equivalent if and only if $(z^1, z^2) = \lambda(w^1, w^2)$ for some nonzero complex number λ. The quotient is the Riemann sphere $\mathbf{P}_1$. When we look at only the 3-sphere S^3 of unit radius 1 in $\mathbf{C}^2$. An equivalence class is the circle S^1 which is the intersection of S^3 with some complex line $\mathbf{C}$ in $\mathbf{C}^2$ through the origin. So S^3 is a bundle over $\mathbf{P}_1$ with S^1 as fibers. The Hopf manifold is the product of S^3 with S^1. Since the equivalence classes of $\mathbf{C}^2 - 0$ that give rise to $\mathbf{P}_1$ are bigger than those giving rise to the Hopf manifold M, there is a quotient map from the Hopf manifold M to $\mathbf{P}_1$ and the fibers of this quotient map are S^1 times S^1. That means that the fibers are torus. Since the quotient map is clearly holomorphic, the Hopf manifold M is a holomorphic family of complex tori over $\mathbf{P}_1$. Now when we have the product of an odd-dimensional sphere S^{2p+1} with another odd-dimensional sphere S^{2q+1}, we can consider the two quotient maps $S^{2p+1} \to \mathbf{P}_p$ and $S^{2q+1} \to \mathbf{P}_q$, each one with fiber S^1. The quotient map $S^{2p+1} \times S^{2q+1} \to \mathbf{P}_p \times \mathbf{P}_q$ has

$S^1 \times S^1$ as fibers. The idea of the Calabi-Eckmann manifolds is to give suitable complex structures to the fibers $S^1 \times S^1$ of $S^{2p+1} \times S^{2q+1} \to \mathbf{P}_p \times \mathbf{P}_q$ to make $S^{2p+1} \times S^{2q+1}$ a holomorphic family of complex tori over $\mathbf{P}_p \times \mathbf{P}_q$. We describe S^{2p+1} as the set of all points $z = (z_0, \cdots, z_p)$ in $\mathbf{C}^{p+1}$ with $\sum_{\nu=0}^{p} z_\nu \overline{z_\nu} = 1$ and describe S^{2p+1} as the set of all points $w = (w_0, \cdots, w_q)$ in $\mathbf{C}^{q+1}$ with $\sum_{\nu=0}^{q} w_\nu \overline{w_\nu} = 1$. The coordinates of $\mathbf{P}_p$ and $\mathbf{P}_q$ are given in coordinate patches by quotients of $z_0, \cdots, z_p$ and quotients of $w_0, \cdots, w_q$ respectively. We are going to give $S^{2p+1} \times S^{2q+1}$ by giving it holomorphic coordinate charts which correspond to the standard coordinate charts of $\mathbf{P}_p$ and $\mathbf{P}_q$. We let U_{ij} be the set of all (z, w) in $S^{2p+1} \times S^{2q+1}$ with $z_i \neq 0$ and $w_j \neq 0$. So U_{ij} is over the product of the i^{th} standard coordinate chart Z_i of $\mathbf{P}_p$ and the j^{th} standard coordinate chart W_j of $\mathbf{P}_q$. The coordinates on those standard coordinate charts of $\mathbf{P}_p$ and $\mathbf{P}_q$ are respectively given by $z_k^{(i)} = \frac{z_k}{z_i}$ and $w_\ell^{(j)} = \frac{w_\ell}{w_j}$. The points on the fiber $S^1 \times S^1$ over the point of $\mathbf{P}_p \times \mathbf{P}_q$ with homogeneous coordinates $[z_0, \cdots, z_p]$ and $[w_0, \cdots, w_q]$ are given by points of $\mathbf{C}^p \times \mathbf{C}^q$ with coordinates $e^{i\varphi}(z_0, \cdots, z_p)$ and $e^{i\psi}(w_0, \cdots, w_p)$. So we are going to use (φ, ψ) as the two extra real coordinates for the coordinate chart U_{ij} of $S^{2p+1} \times S^{2q+1}$ in addition to the coordinates on the standard coordinate charts of $\mathbf{P}_p$ and $\mathbf{P}_q$. We combine the two real functions φ and ψ together to form a complex-valued function. Of course the most obvious way is to form $\varphi + \sqrt{-1}\psi$. This we can do, but the argument works in a more general setting. So we choose a complex number $\tau \in \mathbf{C}$ whose imaginary part $\operatorname{Im} \tau$ is nonzero and form the complex-valued function $\varphi + \tau\, \psi$. To be precise, we introduce the *multi-valued* complex-valued function

$$t_{ij} = \frac{1}{2\pi\sqrt{-1}} \left(\log z_i + \tau \, \log w_j \right)$$

on the coordinate chart U_{ij} of $S^{2p+1} \times S^{2q+1}$. This multi-valued function t_{ij} defines a *single-valued* map from U_{ij} to the torus $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$. We denote the torus $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ by T. So we have a bijective map Ψ_{ij} from U_{ij} to the open subset $Z_i \times W_j \times T$ of $\mathbf{P}_p \times \mathbf{P}_q \times T$. We give U_{ij} the complex structure induced from the product $Z_i \times W_j \times T$. So strictly speaking U_{ij} is not a coordinate chart because one of the coordinate function is not a function but a map to a torus. We have to show that at the intersection of U_{ij} with $U_{k\ell}$ the two complex structures from U_{ij} and $U_{k\ell}$ are compatible. This follows from the fact that since

$$t_{k\ell} = t_{ij} + \frac{1}{2\pi\sqrt{-1}} \left(\log z_k^{(i)} + \tau \, \log w_\ell^{(j)} \right),$$

the map $\Psi_{k\ell}(\Psi_{ij})^{-1}$ over a fixed point P of $Z_i \times W_j$ with coordinates

$$(z_0^{(i)}, \cdots, z_{i-1}^{(i)}, z_{i+1}^{(i)}, \cdots, z_p^{(i)})$$

and

$$(w_0^{(j)}, \cdots, w_{j-1}^{(j)}, w_{j+1}^{(j)}, \cdots, w_p^{(j)})$$

is simply a translation of T by the number $\frac{1}{2\pi\sqrt{-1}}\left(\log z_k^{(i)} + \tau \log w_\ell^{(j)}\right)$ and this translation clearly depends holomorphically on the point P. This concludes the construction of a complex structure on the Calabi-Eckmann manifold which is topologically the same as $S^{2p+1} \times S^{2q+1}$. The Calabi-Eckmann manifolds are actually holomorphic bundles over $\mathbf{P}_p \times \mathbf{P}_q$ with a complex torus as fibers. The complex torus in the fiber can be arbitrarily prescribed. By computation using the Poincaré polynomials it is clear that the second Betti number of any Calabi Eckmann manifold is zero if one of p and q is greater than 1. So those Calabi Eckmann manifolds admit no Kähler metric.

One consequence of the Hodge structure

$$H^k(M, \mathbf{C}) = \oplus_{p+q=k} H^q(M, \Omega_M^p)$$

$$\overline{H^q(M, \Omega_M^p)} = H^p(M, \Omega_M^q)$$

is that the odd-dimensional Betti numbers are even. The first Betti number of our Hopf manifold described above is 1 which is not an even number. This also shows that the Hopf manifold cannot admit a Kähler metric.

On a general compact complex manifold not every holomorphic $(p, 0)$-form (usually referred to simply as a holomorphic p-form) is d-closed. This is also an obstruction to the existence of a Kähler metric. Let us give as an example the Iwasawa manifold. Let G be the complex Lie group consisting of all square matrices of order 3 of the form

$$g = \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}$$

where u, v, w are complex numbers, with matrix multiplication as the group operation. Let Γ denote the subgroup of G consisting of all g with u, v, w belonging to the set of Gaussian integers $\mathbf{Z}+\sqrt{-1}\mathbf{Z}$. The quotient $M = G/$ of left cosets is the Iwasawa manifold we are interested in. We want to produce a holomorphic form on M that is not closed. For this kind of homogeneous manifold there is a standard way of producing holomorphic forms. We have global holomorphic coordinates u, v, w on G. So the entries of the matrix-valued 1-form dg are holomorphic 1-forms (most the entries being zero except three). However, these 1-forms do not descend to 1-forms on M. They descend precisely when they are invariant under the right action by elements of Γ. On the complex Lie group G the way to produce 1-forms which are invariant under right action is to look at the entries of the matrix-valued 1-form dg g^{-1}. These entries are clearly invariant under right action by all elements of G and in particular by right action by all elements of Γ. They are called the Cartan-Maurer forms of the Lie group G. The inverse matrix g^{-1} is given by

$$g^{-1} = \begin{pmatrix} 1 & -u & uw - v \\ 0 & 1 & -w \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$dg\ g^{-1} = \begin{pmatrix} 0 & du & -wdu + dv \\ 0 & 0 & dw \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega^1 & \omega^2 \\ 0 & 0 & \omega^3 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\omega^1 = du$, $\omega^2 = -wdu + dv$, and $\omega^3 = dw$. All three forms are holomorphic 1-forms. Clearly $d\omega^2 = \omega^1 \wedge \omega^3$ is nonzero. Hence the Iwasawa manifold does not admit a Kähler metric. Note that the dual frame for ω^1, ω^2, ω^3 is a basis of the space of all right invariant vector fields X_1, X_2, X_3. The Lie algebra of the group G is given by $[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma$. The constants $c^\gamma_{\alpha\beta}$ are the structure constants of the Lie algebra of G. By evaluating $d\omega^\gamma$ at (X_α, X_β), we obtain $d\omega^\gamma = c^\gamma_{\alpha\beta}\omega^\alpha \wedge \omega^\beta$. The fact that there exists a $non-d$-closed holomorphic form simply means that the Lie algebra of G is nonabelian. So if we start with a complex Lie group whose Lie algebra is nonabelian and form a compact quotient by a discrete subgroup, then we obtain a compact complex manifold which does not admit a Kähler metric. The Iwasawa manifold is simply an easy example of this kind.

There is another property of compact Kähler manifolds that is very useful. Namely, any d-exact smooth (1,1)-form α on a compact Kähler manifold M can be written as $\partial\bar{\partial} f$ for some smooth function f. The key point of the proof is that $H^1(M, \mathcal{O}_M)$ is the complex conjugate of $\Gamma(M, \Omega^1_M)$. So any harmonic (0,1)-form is of the form $\bar{\varphi}$ for some holomorphic 1-form φ. By breaking α up into each real and imaginary parts $\frac{1}{2}(\alpha + \bar{\alpha})$ and $\frac{1}{2\sqrt{-1}}(\alpha - \bar{\alpha})$, we can assume that α is real. Since α is d-exact, we can write $\alpha = d(\beta + \bar{\beta})$, where β is a (1,0)-form. Now $\alpha = \partial\beta + (\bar{\partial}\beta + \partial\bar{\beta}) + \bar{\partial}\bar{\beta}$. Since α is of type (1,1), we have $\bar{\partial}\bar{\beta} = 0$ and $\alpha = \bar{\partial}\beta + \partial\bar{\beta}$. There is a unique harmonic (0,1)-form defining the same class in $H^1(M, \mathcal{O}_M)$ as $\bar{\beta}$. This harmonic (0,1)-form, as we observed above, is of the form $\bar{\varphi}$ for some holomorphic 1-form φ. Hence $\bar{\beta} = \bar{\varphi} + \bar{\partial}\gamma$ for some smooth function γ. So $\partial\bar{\beta} = \partial\bar{\varphi} + \partial\bar{\partial}\gamma = \partial\bar{\partial}\gamma$ because φ is holomorphic. Finally $\alpha = \bar{\partial}\beta + \partial\bar{\beta} = \bar{\partial}\partial\bar{\gamma} + \partial\bar{\partial}\gamma = \partial\bar{\partial}(\gamma - \bar{\gamma})$.

CHAPTER 4. VANISHING THEOREMS AND THEIR APPLICATIONS

§1. *Vanishing Theorem of Kodaira.*

(1.1) The vanishing theorem of Kodaira says that the cohomology groups $H^q(M, L \otimes K_M^{-1})$ vanish for $q \geq 1$ when L is a holomoprhic line bundle over a compact complex manifold M and L admits a Hermitian metric whose curvature form is positive definite. Here K_M denotes the canonical line bundle of M.

Before we prove the vanishing theorem of Kodaira, we would like to intuitively discuss why such a theorem should hold. We have seen that a cohomology class can be uniquely expressed by a harmonic form. To show that a certain cohomology group vanishes it is equivalent to show that every harmonic form for that cohomology group is identically zero. How does one show that a harmonic form vanishes? Let us look at the case of a real-valued harmonic function. We know that on a compact manifold every real-valued harmonic function is constant. There are two ways of doing it. One is to use the maximum principle. Another way is to use integration by parts. If instead of considering a harmonic function we consider a real-valued function f which satisfies the equation $\Delta f = c\, f$ for some positive constant c. Such a function f must vanish identically. One use either the maximum principle or integration by parts. For the maximum principle we consider a point where the maximum of f is achieved. At that point Δf is nonpositive. Because of the positivity of c it forces f to be nonpositive. So f is nonpositive everywhere. Applying the same argument $to - f$ instead of f we conclude that f is nonnegative everywhere. Hence f is identically zero. For integration by parts we simply multiply $\Delta f = c\, f$ by f and integrate over M and get $-(df, df)_M = c(f, f)_M$ which forces f to be identically zero by the positivity of c. The argument of integration by parts is the same as saying the operator $-\Delta + c$ is positive definite. Let us go back to the consideration of harmonic forms φ. Instead of Δ we have the operator $\square = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$. When we compute $\square\varphi$, we will get $\square\varphi = -(\operatorname{tr}\nabla\bar\nabla\varphi) + A\,\varphi$, where A is a zero-order linear operator. The term $-(\operatorname{tr}\nabla\bar\nabla\varphi)$ corresponds to $-\Delta\, f$ and the term $A\,\varphi$ corresponds to $c\,\varphi$. When the curvature form of L is positive, the operator A is a positive operator and it corresponds to the positivity of c and we get the vanishing of the harmonic form φ.

(1.2) We now rigorously prove the vanishing theorem of Kodaira by deriving the formula for $\Box\varphi$. Let M be a compact complex manifold of complex dimension n. We assume that M is endowed with a Kähler metric $g_{i\bar{j}}$. Recall that the curvature tensor $R_{i\bar{j}k\bar{\ell}}$ is given by

$$R_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}g_{k\bar{\ell}} + g^{\bar{s}t}\partial_i g_{k\bar{s}}\partial_{\bar{j}}g_{t\bar{\ell}}.$$

Let E be a holomorphic vector bundle of $\mathbf{C}$-rank r over M with a Hermitian metric $h_{\alpha\bar{\beta}}$ along its fibers. The curvature tensor

$$\Omega_{\alpha\bar{\beta}i\bar{j}} = -\partial_i\partial_{\bar{j}}h_{\alpha\bar{\beta}} + h^{\bar{\gamma}\delta}\partial_i h_{\alpha\bar{\gamma}}\partial_{\bar{j}}h_{\delta\bar{\beta}}.$$

We now compute $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. We did this twice before. The first time is in the proof of the Gårding's inequality, where we ignore the lower order terms. The second time is in the proof of the Hodge decomposition where no holomorphic vector bundle is used. Now we redo it with a holomorphic vector bundle and keep the lower order terms. Let

$$\varphi = \frac{1}{p!q!}\sum \varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge dz^{\bar{j}_1}\wedge\cdots\wedge dz^{\bar{j}_q}$$

be an E-valued (p,q)-form on M. At a point where we do the computation we assume that both $g_{i\bar{j}}$ and $h_{\alpha\bar{\beta}}$ are identity matrices and their first derivatives vanish.

$$(\bar{\partial}^*\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_{q-1}} = (-1)^{p+1}g^{s\bar{t}}\nabla_s\varphi_{i_1\cdots i_p\bar{t}\bar{j}_1\cdots\bar{j}_{q-1}}.$$

$$\begin{aligned}(\bar{\partial}\bar{\partial}^*\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} &= (-1)^p\sum_{\nu=1}^{q}(-1)^{\nu-1}\partial_{\bar{j}_\nu}(\bar{\partial}^*\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\widehat{\bar{j}_\nu}\cdots\bar{j}_{q-1}}\\
&= -\sum_{\nu=1}^{q}(-1)^{\nu-1}\partial_{\bar{j}_\nu}\left(g^{s\bar{t}}\nabla_s\varphi_{i_1\cdots i_p\bar{t}\bar{j}_1\cdots\widehat{\bar{j}_\nu}\cdots\bar{j}_{q-1}}\right)\\
&= g^{s\bar{t}}\sum_{\nu=1}^{q}(-1)^{\nu}\nabla_{\bar{j}_\nu}\nabla_s\varphi_{i_1\cdots i_p\bar{t}\bar{j}_1\cdots\widehat{\bar{j}_\nu}\cdots\bar{j}_{q-1}}.\end{aligned}$$

$$(\bar{\partial}^*\bar{\partial}\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} = (-1)^{p+1}g^{s\bar{t}}\nabla_s(\bar{\partial}\varphi)_{i_1\cdots i_p\bar{t}\bar{j}_1\cdots\widehat{\bar{j}_\nu}\cdots\bar{j}_{q-1}}$$

$$= -g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} + \sum_{\nu=1}^{q}(-1)^{\nu-1}g^{s\bar{t}}\nabla_s\nabla_{\bar{j}_\nu}\varphi_{i_1\cdots i_p\bar{t}\bar{j}_1\cdots\widehat{\bar{j}_\nu}\cdots\bar{j}_{q-1}}.$$

$$\begin{aligned}(\Box\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} &= -g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\\
&\quad -\sum_{\nu=1}^{p}g^{s\bar{t}}[\nabla_{\bar{j}_\nu},\nabla_s]\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu,\cdots\bar{j}_q},\end{aligned}$$

where $(\bar{t})_\nu$ means that the barred index in the ν^{th} position is replaced by $\bar{t}$. The factor $(-1)^{\nu-1}$ disappeared, because we have moved the index $\bar{t}$ passed the $(\nu-1)$ indices $\bar{j}_1, \cdots, \bar{j}_{\nu-1}$ to put it in the place which was earlier occupied by $\bar{j}_\nu$. Let us now take care of the zero-order term

$$-\sum_{\nu=1}^{p} g^{s\bar{t}}[\nabla_{\bar{j}_\nu}, \nabla_s]\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu,\cdots\bar{j}_q}.$$

Recall the definitions for the complex metric connections of the Kähler metric $g_{i\bar{j}}$ and the Hermitian metric $h_{\alpha\bar{\beta}}$. For a section θ^α of E we have $\nabla_{\bar{j}}\theta^{i\alpha} = \partial_{\bar{j}}\theta^{i\alpha}$ and $\nabla_j\theta^\alpha = \partial_j\theta^\alpha + \theta^\beta(\partial_j h_{\beta\bar{\gamma}})h^{\bar{\gamma}\alpha}$. Hence

$$\begin{aligned}[\nabla_{\bar{j}}, \nabla_s]\theta^\alpha &= \partial_{\bar{j}}\partial_s\theta^\alpha + \theta^\beta(\partial_{\bar{j}}\partial_s h_{\beta\bar{\gamma}})h^{\bar{\gamma}\alpha} - \partial_s\partial_{\bar{j}}\theta^\alpha \\ &= \theta^\beta(\partial_{\bar{j}}\partial_s h_{\beta\bar{\gamma}})h^{\bar{\gamma}\alpha} = -\eta_i^\beta\Omega_{\beta\bar{\gamma}s\bar{j}}h^{\bar{\gamma}\alpha}.\end{aligned}$$

For a vector field ξ^i of type (1,0) we have $\nabla_{\bar{j}}\xi^i = \partial_{\bar{j}}\xi^i$ and

$$\nabla_j\xi^i = \partial_j\xi^i + \xi^k(\partial_j g_{k\bar{\ell}})g^{\bar{\ell}i}$$

Hence $\nabla_{\bar{j}}\eta_i = \partial_{\bar{j}}\eta_i$ and $\nabla_j\eta_i = \partial_j\eta_i - \eta_k(\partial_j g_{i\bar{\ell}})g^{\bar{\ell}k}$ for an E-valued (1,0)-form. Hence

$$\begin{aligned}[\nabla_{\bar{j}}, \nabla_s]\eta_i &= \partial_{\bar{j}}\partial_s\eta_i - \eta_k(\partial_{\bar{j}}\partial_s g_{i\bar{\ell}})g^{\bar{\ell}k} - \partial_s\partial_{\bar{j}}\eta_i^\alpha \\ &= -\eta_k^\alpha(\partial_{\bar{j}}\partial_s g_{i\bar{\ell}})g^{\bar{\ell}k} = \eta_k^\alpha R_{i\bar{j}s\bar{\ell}}g^{\bar{\ell}k}.\end{aligned}$$

Taking complex conjugates, we get $[\nabla_{\bar{i}}, \nabla_s]\eta_{\bar{j}} = -g^{\bar{\ell}k}R_{s\bar{i}k\bar{j}}\eta_{\bar{\ell}}$ for any (0,1)-form $\eta_{\bar{j}}$. Since an (p,q)-form is a linear combination of the products of p $(1,0)$-forms and q $(0,1)$-forms and a section of E, we have

$$\begin{aligned}&[\nabla_{\bar{j}_\nu}, \nabla_s]\varphi^\alpha_{i_1\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q} \\ &= \sum_{\sigma=1}^{p} R_{s\bar{j}_\nu i_\sigma}{}^{a}\varphi^\alpha_{i_1\cdots(a)_\sigma\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q} \\ &\quad - \sum_{\tau=1,\tau\neq\nu}^{q} R_{s\bar{j}_\nu}{}^{\bar{b}}{}_{\bar{j}_\tau}\varphi^\alpha_{i_1\cdots i_p\bar{j}_1\cdots(\bar{b})_\tau\cdots(\bar{t})_\nu\cdots\bar{j}_q} \\ &\quad - R_{s\bar{j}_\nu}{}^{\bar{b}}{}_{\bar{t}}\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q} \\ &\quad - \Omega_\beta{}^\alpha{}_{s\bar{j}_\nu}\varphi^\beta_{i_1\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q}.\end{aligned}$$

Hence

$$(\Box\varphi)_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} = I + II + III + IV + V,$$

where

$$I = -g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\varphi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q},$$

$$II = -\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar{t}}{}_{\bar{j}_\nu i\sigma}{}^{a}\varphi_{i_1\cdots(a)_\sigma\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q},$$

$$III = \sum_{\nu=1}^{q}\sum_{\tau=1,\tau\neq\nu}^{q} R^{\bar{t}}{}_{\bar{j}_\nu}{}^{\bar{b}}{}_{\bar{j}_\tau}\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar{b})_\tau\cdots(\bar{t})_\nu\cdots\bar{j}_q},$$

$$IV = \sum_{\nu=1}^{q} R_{\bar{j}}{}^{\bar{b}}\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar{b})_\nu\cdots\bar{j}_q},$$

$$V = \sum_{\nu=1}^{q}\Omega_{\beta}{}^{\alpha\bar{t}}{}_{\bar{j}_\nu}\varphi^{\beta}_{i_1\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q},$$

where $R_{\bar{j}_\nu}{}^{\bar{b}}$ is the Ricci curvature tensor $R_{\bar{\ell}k} = g^{s\bar{t}}R_{s\bar{\ell}k\bar{t}}$ with the last covariant index raised. Now III vanishes always because $R^{\bar{t}}{}_{\bar{i}_\nu}{}^{\bar{b}}{}_{i_\sigma}$ is symmetric in $\bar{t}$ and $\bar{b}$ yet $\varphi_{i_1\cdots i_p\bar{j}_1\cdots(\bar{b})_\tau\cdots(\bar{t})_\nu\cdots\bar{j}_q}$ is skew-symmetric in $\bar{t}$ and $\bar{b}$.

(Besides going back to the Riemannian case, one can also see the symmetric of $R^{\bar{t}}{}_{\bar{i}_\nu}{}^{\bar{b}}{}_{i_\sigma}$ in $\bar{t}$ and $\bar{b}$ as follows. At a point with respect to holomorphic normal coordinates,

$$R_{\alpha\,\bar{\beta}\,\gamma\,\bar{\delta}} = \partial_\alpha\left(\partial_{\bar{\beta}}h_{\gamma\,\bar{\delta}}\right) = \partial_\alpha\left(\partial_{\bar{\delta}}h_{\gamma\,\bar{\beta}}\right) = R_{\alpha\,\bar{\delta}\,\gamma\,\bar{\beta}},$$

giving us the symmetry.)

So the final expression is

$$\begin{aligned}(\Box\varphi)^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} &= -g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\\ &\quad-\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar{t}}{}_{\bar{j}_\nu i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q}\\ &\quad+\sum_{\nu=1}^{q} R_{\bar{j}_\nu}{}^{\bar{b}}\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots(\bar{b})_\nu\cdots\bar{j}_q}\\ &\quad+\sum_{\nu=1}^{q}\Omega_{\beta}{}^{\alpha\bar{t}}{}_{\bar{j}_\nu}\varphi^{\beta}_{i_1\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q}.\end{aligned}$$

In invariant formulation

$$\Box\varphi = -(\operatorname{Tr}\nabla\bar{\nabla})\varphi - R\,\varphi + \operatorname{Ric}\varphi + \Omega\,\varphi,$$

where the action $R\varphi$ of the curvature tensor R on φ and the action $\mathrm{Ric}\,\varphi$ of the Ricci tensor Ric on φ should be interpreted as follows. The curvature tensor R acts on a (1,1)-form $\psi_{\alpha\bar{\beta}}$ naturally to give a (1,1)-form $R_{\alpha\bar{\beta}}{}^{\bar{\gamma}\delta}\psi_{\gamma\bar{\delta}}$. Another way of looking at it is that R is an element of $T_M^* \otimes \overline{T_M^*} \otimes T_M^* \otimes \overline{T_M^*}$ which is isomorphic to $T_M^* \otimes \overline{T_M^*} \otimes \overline{T_M} \otimes T_M = \mathrm{End}(T_M^* \otimes \overline{T_M^*})$ because the isomorphism of T_M and $\overline{T_M^*}$ given by the Kähler metric. The action of R on the space $T_M^* \otimes \overline{T_M^*}$ of (1,1)-forms is simply the natural one when it is regarded as an element of $\mathrm{End}(T_M^* \otimes \overline{T_M^*})$. This action of R on $\mathrm{End}(T_M^* \otimes \overline{T_M^*})$ induces an action of R on $\mathrm{End}(E \otimes T_M^* \otimes \overline{T_M^*})$ after composition with the identity map of E. We suppress all the covariant indices of the (p,q)-form φ except two to make it an E-valued (1,1)-form (which is the same as evaluating φ at $p-1(1,0)$-vectors and $q-1(0,1)$-vectors) and the action of R on φ is the action on this E-valued (1,1)-form with summation over all the pq ways of suppressing indices to make φ an E-valued (1,1)-form. Likewise the Ricci curvature Ric can be regarded as an element of $\mathrm{End}(\overline{T_M})$ and so it acts on (0,1)-forms to give (0,1)-forms. Together with the identity map of E we can regard Ric as an element of $\mathrm{End}(E \otimes \overline{T_M})$. We suppress all the covariant indices of the (p,q)-form φ except one to make it an E-valued (0,1)-form (which is the same as evaluating φ at $p(1,0)$-vectors and $q-1(0,1)$-vectors) and the action of Ric on φ is the action on this E-valued (0,1)-form with summation over all the q ways of suppressing indices to make φ an E-valued (0,1)-form. The curvature tensor Ω of E acts on an E-valued (0,1)-form $\theta^{\alpha}_{\bar{j}}$ naturally to give an E-valued (0,1)-form $\Omega_{\beta}{}^{\alpha\bar{k}}{}_{\bar{j}}\theta^{\beta}{}_{\bar{k}}$. Another way of looking at it is that Ω is an element of $\mathrm{End}(E) \otimes T_M^* \otimes \overline{T_M^*}$ which is isomorphic to $\mathrm{End}\left(E \otimes \overline{T_M^*}\right)$. The action of Ω on the space $E \otimes \overline{T_M^*}$ of E-valued (0,1)-forms is simply the natural one when it is regarded as an element of $\mathrm{End}(E \otimes \overline{T_M^*})$. We suppress all the covariant indices of the (p,q)-form φ except one to make it an E-valued (0,1)-form (which is the same as evaluating φ at p $(1,0)$-vectors and $q-1$ $(0,1)$-vectors) and the action of Ω on φ is the action on this E-valued (0,1)-form with summation over all the q ways of suppressing indices to make φ an E-valued (0,1)-form. We could have done all the computations in invariant form with the same steps. The indices were used simply to keep track of how a tensor acts on another tensor of different covariant and contravariant ranks.

Consider the special case when E is a line bundle and $p=0$. Then

$$(\Box\varphi)^{\alpha}{}_{\bar{j}_1\cdots\bar{j}_q} = -g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\varphi^{\alpha}{}_{\bar{j}_1\cdots\bar{j}_q}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{q} R_{\bar{j}_\nu}{}^{\bar{b}} \varphi^{\alpha}{}_{\bar{j}_1\cdots(\bar{b})_\nu\cdots\bar{j}_q}\\
&+\sum_{\nu=1}^{q} \Omega^{\bar{t}}{}_{\bar{j}_\nu} \varphi^{\beta}{}_{\bar{j}_1\cdots(\bar{b})_\nu\cdots\bar{j}_q}.
\end{aligned}
$$

Multiplying both sides by $\overline{\varphi_{\bar{\alpha}}{}^{j_1\cdots j_q}}$ and integrating over M, we get

$$
(\Box\varphi,\varphi)_M = \|\bar{\nabla}\varphi\|_M^2 + (\mathrm{Ric}\,\varphi,\varphi)_M + (\Omega\varphi,\varphi)_M,
$$

where

$$
\begin{aligned}
\|\bar{\nabla}\varphi\|_M^2 &= \int_M g^{s\bar{t}} \nabla_{\bar{t}} \varphi^{\alpha}{}_{\bar{j}_1\cdots\bar{j}_q} \overline{\nabla_{\bar{s}}\varphi_{\bar{\alpha}}{}^{j_1\cdots j_q}},\\
(\mathrm{Ric}\,\varphi,\varphi)_M &= \int_M \sum_{\nu=1}^{q} R_{\bar{j}_\nu}{}^{\bar{b}} \varphi^{\alpha}{}_{\bar{j}_1\cdots(\bar{b})_\nu\cdots j_q} \overline{\varphi_{\bar{\alpha}}{}^{j_1\cdots j_q}}\\
(\Omega\varphi,\varphi)_M &= \int_M \sum_{\nu=1}^{q} \Omega^{\bar{b}}{}_{\bar{j}_\nu} \varphi^{\alpha}{}_{\bar{j}_1\cdots(\bar{b})_\nu\cdots\bar{j}_q} \overline{\varphi_{\bar{\alpha}}{}^{j_1\cdots j_q}}.
\end{aligned}
$$

Suppose the (1,1)-form $\Omega_{i\bar{j}} + R_{i\bar{j}}$ is positive definite at every point. Then we conclude that $(\mathrm{Ric}\,\varphi,\varphi)_M + (\Omega\varphi,\varphi)_M$ is positive definite unless φ is identically zero. If φ is harmonic, then $\Box\varphi = 0$ and $(\Box\varphi,\varphi)_M = 0$. Hence we conclude that φ must be identically zero. We now have the following vanishing theorem of Kodaira. *Suppose L is a holomorphic line bundle over a compact complex manifold M and L admits a Hermitian metric along its fibers whose curvature form is positive definite. Then $H^q(M, L\otimes K_M^{-1})$ vanishes for $q \geq 1$ where K_M is the canonical line bundle of M.* For the proof of the vanishing theorem of Kodaira we apply our previous argument to the case $E = L \otimes K_M^{-1}$. We use the positive definite curvature form of L as a Kähler form and give M a Kähler metric. We give K_M^{-1} the metric induced from the Kähler metric so that its curvature form is $-R_{i\bar{j}}$. We now give E the metric induced from those of L and K_M^{-1}. Then the sum of the curvature form $\Omega_{i\bar{j}}$ of E and $R_{i\bar{j}}$ equals the curvature form of L and is positive definite. Thus any harmonic E-valued $(0,q)$-form on M vanishes identically and $H^q(M,E) = H^q(M, L\otimes K_M^{-1}) = 0$ for $q \geq 1$.

By using the duality of Kodaira-Serre, we have the following form of Kodaira's vanishing theorem. *Suppose L is a holomorphic line bundle over a compact complex manifold M of complex dimension n and L admits a*

Hermitian metric along its fibers whose curvature form is negative definite. Then $H^q(M, L)$ vanishes for $q < n$.

(1.3) When we have a holomorphic vector bundle E instead of a line bundle, the above argument gives the following generalization of Kodaira's vanishing theorem due to Nakano. Suppose E is a holomorphic vector bundle over a compact complex manifold M and E admits a Hermitian metric along its fibers whose curvature form Ω as a Hermitian form on $E \otimes T_M^{1,0}$ is positive definite. Then $H^q(M, E \otimes K_M^{-1})$ vanishes for $q \geq 1$.

Here the Hermitian form on $E \otimes T_M^{1,0}$ defined by Ω is the following. For an element $(\xi^{\alpha i})$ of $E \otimes T_M^{1,0}$ the inner product of $(\xi^{\alpha i})$ with itself is $\Omega_{\alpha\bar{\beta}}{}^{\bar{j}i}\xi_i^{\alpha}\overline{\xi_j^{\beta}}$.

Using the duality theorem of Kodaira-Serre, one has the following form of Nakano's generalization of Kodaira's vanishing theorem. Suppose E is a holomorphic vector bundle over a compact complex manifold M of complex dimension n and E admits a Hermitian metric along its fibers whose curvature form Ω as a Hermitian form on $E \otimes (T_M^{1,0})^*$ is negative definite. Then $H^q(M, E)$ vanishes for $q < n$.

Here the Hermitian form on $E \otimes (T_M^{1,0})^*$ defined by Ω is the following. For an element (ξ_i^{α}) of $E \otimes (T_M^{1,0})^*$ the inner product of (ξ_i^{α}) with itself is $\Omega_{\alpha\bar{\beta}}{}^{\bar{j}i}\xi_i^{\alpha}\overline{\xi_j^{\beta}}$.

The reason for considering the Hermitian form on $E \otimes (T_M^{1,0})^*$ instead of on $E \otimes T_M^{1,0}$ in the negative case is the following. Consider the dual bundle E^* of E and represent an element of E by (ξ^{α}) and an element of E^* by (η_{α}). Let $h_{\alpha\bar{\beta}}$ be the Hermitian metric along the fibers of E with curvature form $\Omega_{\alpha}{}^{\beta}{}_{i\bar{j}}$. Then $h^{\alpha\bar{\beta}}$ is the corresponding Hermitian metric along the fibers of E^*. Let the curvature form of the metric $h^{\alpha\bar{\beta}}$ of E^* be $(\Omega^*)^{\alpha}{}_{\beta i\bar{j}}$. Then $(\Omega^*)^{\alpha}_{\beta i\bar{j}} = -h^{\alpha\bar{\gamma}}h_{\beta\bar{\delta}}\Omega_{\alpha}{}^{\beta}{}_{i\bar{j}}$. The positivity of the Hermitian form $(\xi^{\alpha i}) \to \Omega_{\alpha\bar{\beta}i\bar{j}}\xi^{\alpha i}\overline{\xi^{\beta j}}$ on $E \otimes T_M^{1,0}$ is equivalent to the negativity of the Hermitian form $(\eta_{\alpha i}) \to (\Omega^*)^{\alpha\bar{\beta}\bar{i}j}\eta_{\alpha i}\overline{\eta_{\beta j}}$ on $E \otimes (T_M^{1,0})^*$, where $\eta_{\alpha i} = h_{\alpha\bar{\beta}}g_{i\bar{j}}\overline{\xi^{\beta j}}$.

§2. *Akizuki-Nakano's Generalization.*

The applicability of the above generalizations by Nakano of Kodaira's vanishing theorem is rather limited, because the positivity of the Hermitian form $(\xi^{\alpha i}) \to \Omega_{\alpha\bar{\beta}i\bar{j}}\xi^{\alpha i}\overline{\xi^{\beta j}}$ on $E \otimes T_M^{1,0}$ is a very strong condition. However, there is another generalization of Kodaira's theorem by Akizuki and Nakano that is very useful. The statement is the following. Suppose L is a holomorphic vector bundle over a compact complex manifold M of complex dimension n and L admits a Hermitian metric along its fibers whose curvature form is negative definite. Then $H^q(M, L \otimes \Omega^p_M)$ vanishes for $p + q < n$, where Ω^p_M is the vector bundle of $(p, 0)$-forms on M.

This generalization by Akizuki and Nakano requires a method different from the one used in proving Kodaira's vanishing theorem. The motivation of this generalization is the following. In the proof of Kodaira's vanishing theorem, we use integration by parts and get

$$(\Box\varphi, \varphi)_M = \|\bar{\nabla}\varphi\|^2_M + (\operatorname{Ric}\varphi, \varphi)_M + (\Omega\varphi, \varphi)_M$$

from which the vanishing of φ follows because of the positivity of the Hermitian form $\varphi \to (\operatorname{Ric}\varphi, \varphi)_M + (\Omega\varphi, \varphi)$. We have not used the term $\|\bar{\nabla}\varphi\|^2_M$ which is also nonnegative. In a way the term $\|\bar{\nabla}\varphi\|^2_M$ is wasted. The generalization by Akizuki and Nakano makes use of this term. The main idea is to calculate $(\overline{\Box}\varphi, \varphi)_M$ and obtain

$$\|\partial\varphi\|^2_M + \|\partial^*\varphi\|^2_M = (\overline{\Box}\varphi, \varphi) = \|\bar{\nabla}\varphi\|^2_M + (A\,\varphi, \varphi),$$

where A is some zero-order linear operator. Hence for a harmonic form φ one has

$$0 = (\Box\varphi, \varphi)_M = \|\partial\varphi\|^2_M + \|\partial^*\varphi\|^2_M + (A\,\varphi, \varphi)_M + (\operatorname{Ric}\varphi, \varphi)_M + (\Omega\varphi, \varphi)_M$$

from which follows the identical vanishing of φ if the Hermitian form

$$\varphi \to (A\,\varphi, \varphi)_M + (\operatorname{Ric}\varphi, \varphi)_M + (\Omega\varphi, \varphi)_M$$

is positive definite.

We now rigorously prove the generalization of Kodaira's vanishing theorem by Akizukia and Nakano. Let E be a holomorphic vector bundle of $\mathbf{C}$-rank r over a Kähler manifold M of complex dimension n with Kähler metric $g_{i\bar{j}}$. We first compute $\overline{\Box}\varphi = (\partial^*\partial + \partial\partial^*)\varphi$. One can derive a formula for $\overline{\Box}\varphi$ in the same way as one derives the formula for $\Box\varphi$. Another way is to use the Hermitian metric $h_{\alpha\bar{\beta}}$ of E to transform the E-valued (p, q)-form φ to an E^*-valued (q, p)-form ψ defined as follows.

$$\psi_{\alpha j_1 \cdots j_q \bar{i}_1 \cdots \bar{i}_p} = h_{\alpha\bar{\beta}} \overline{\varphi^{\beta}{}_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}}.$$

Now use the formula for $\Box\varphi$ we derived before and replace φ by ψ. We get

$$\begin{aligned}(\Box\psi)_{\alpha j_1\cdots j_q\bar{i}_1\cdots\bar{i}_p} &= -g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\psi_{\alpha j_1\cdots j_q\bar{i}_1\cdots\bar{i}_p} \\ &- \sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar{a}}{}_{\bar{i}_\sigma j_\nu}{}^{t}\psi_{\alpha j_1\cdots(t)_\nu\cdots j_q\bar{i}_1\cdots(\bar{a})_\sigma\bar{i}_p} \\ &+ \sum_{\sigma=1}^{p} R_{\bar{i}_\sigma}{}^{\bar{a}}\psi_{\alpha j_1\cdots j_p\bar{i}_1\cdots(\bar{a})_\sigma\bar{i}_p} \\ &+ \sum_{\sigma=1}^{p}(\Omega^*)^{\beta}{}_{\alpha}{}^{\bar{a}}{}_{\bar{i}_\sigma}\psi_{\beta j_1\cdots j_p\bar{i}_1\cdots(\bar{a})_\sigma\bar{i}_p}.\end{aligned}$$

Taking complex conjugates of both sides, we get

$$\begin{aligned}(\overline{\Box}\varphi)^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} &= -g^{s\bar{t}}\nabla_{\bar{t}}\nabla_s\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} \\ &- \sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar{t}}{}_{\bar{j}i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar{j}_1\cdots(\bar{t})_\nu\bar{j}_q} \\ &+ \sum_{\sigma=1}^{p} R_{i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar{j}_1\cdots\bar{j}_q} \\ &- \sum_{\sigma=1}^{p}\Omega_{\alpha}{}^{\beta}{}_{i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar{j}_1\cdots\bar{j}_q}.\end{aligned}$$

Since we would like to have an equation of the form $(\overline{\Box}\varphi,\varphi) = \|\bar{\nabla}\varphi\|^2_M + (A\,\varphi,\varphi)$, we have to transform $g^{s\bar{t}}\nabla_{\bar{t}}\nabla_s\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$ to $g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$. For this we have to compute $[\nabla_{\bar{t}},\nabla_s]\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$. We have done this before. We have

$$\begin{aligned}&[\nabla_{\bar{t}},\nabla_s]\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} \\ &= \sum_{\sigma=1}^{p} R_{s\bar{t}i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar{j}_1\cdots\bar{j}_q} \\ &- \sum_{\tau=1}^{q} R_{s\bar{t}}{}^{\bar{b}}{}_{\bar{j}_\tau}\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots(\bar{b})_\tau\cdots\bar{j}_q} \\ &-\Omega_{\beta}{}^{\alpha}{}_{s\bar{t}}\varphi^{\beta}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\end{aligned}$$

and

$$(\overline{\Box}\varphi)^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} = -g^{s\bar{t}}\nabla_s\nabla_{\bar{t}}\varphi^{\alpha}_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$$

$$
\begin{aligned}
&+\sum_{\tau=1}^{q} R^{\bar b}{}_{\bar j_\tau}\varphi^{\alpha}_{i_1\cdots i_p\bar j_1\cdots(\bar b)_\tau\cdots\bar j_q}\\
&+\Omega_\beta{}^\alpha\varphi^{\beta}_{i_1\cdots i_p\bar j_1\cdots\bar j_q}\\
&-\sum_{\sigma=1}^{p}\sum_{\nu=1}^{q} R^{\bar t}{}_{\bar j_\nu i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar j_1\cdots(\bar t)_\nu\cdots\bar j_q}\\
&-\sum_{\sigma=1}^{p}\Omega_\alpha{}^\beta{}_{i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar j_1\cdots\bar j_q}
\end{aligned}
$$

where $\Omega^\alpha_\beta = g^{s\bar t}\Omega_\beta{}^\alpha{}_{s\bar t}$. Finally

$$
\begin{aligned}
((\square-\overline{\square})\varphi)^{\alpha}_{i_1\cdots i_p\bar j_1\cdots\bar j_q} &= \sum_{\sigma=1}^{p}\Omega_\alpha{}^\beta{}_{i_\sigma}{}^{a}\varphi^{\alpha}_{i_1\cdots(a)_\sigma\cdots i_p\bar j_1\cdots\bar j_q}\\
&+\sum_{\nu=1}^{q}\Omega_\beta{}^{\alpha\bar t}{}_{\bar j_\nu}\varphi^{\beta}_{i_1\cdots i_p\bar j_1\cdots(\bar t)_\nu\cdots\bar j_q}\\
&-\Omega_\beta{}^\alpha\varphi^{\beta}_{i_1\cdots i_p\bar j_1\cdots\bar j_q}.
\end{aligned}
$$

This computation is the same as the earlier verification that $\square = \overline{\square}$ for a Kähler manifold, except that here we have a holomorphic vector bundle E. When the holomorphic vector bundle E is trivial, the above equation precisely says that $\square = \overline{\square}$. So with a holomorphic vector bundle E the above equation expresses quantitatively the failure of the equation $\square = \overline{\square}$. The generalization of Kodaira's vanishing theorem by Akizuki and Nakano is precisely a consequence of the failure of the equation $\square = \overline{\square}$ when a holomorphic vector bundle E is used. Suppose E is a holomorphic line bundle with a Hermitian metric whose curvature form is negative definite. We use as the Kähler form on M the negative of the curvature form of E. Then $(\square-\overline{\square})\varphi = (n-p-q)\varphi$. Taking the global inner product with φ and using the harmonicity of φ, we get $-\|\partial\varphi\|^2_M - \|\partial^*\varphi\|^2_M = (n-p-q)\|\varphi\|^2_M$. Hence φ vanishes identically when $p+q<n$ and we have the generalization by Akizuki and Nakano of Kodaira's vanishing theorem.

§3. *Blowing up.*

As an application of the vanishing theorem of Kodaira, we would like to discuss the embedding theorem of Kodaira, which says that a compact complex manifold admitting a positive holomorphic line bundle can be embedded as a complex submanifold of some complex projective space. The embedding will be done by using global holomorphic sections of a sufficiently high power of the positive line bundle as homogeneous coordinates of a map into a complex projective space. The key point is to produce enough such global holomorphic sections. The first cohomology group gives the obstruction to the existence of global holomorphic sections. So the vanishing of the first cohomology group guaranteed by the vanishing theorem of Kodaira will yield sufficiently many global holomorphic sections to give us the embedding we want. However, when we go from the vanishing of the first cohomology group to the existence of sufficiently many global holomorphic sections, there is a technical point we have to deal with. Let us discuss what this technical point is.

When we introduced sheaf cohomology groups, we used the example of producing a nonconstant meromorphic function on a Riemann surface as an example. We cover the Riemann surface M by a covering $\mathcal{U} = \{U_\alpha\}$ of coordinate charts and pick a point P in U_0 not belonging to any other U_α. We can find a meromorphic function f_0 having P as its only pole which is of order one and find a holomorphic function f_α on other U_α. Then we form the 1-cocycle $\{g_{\alpha\beta}\}$ in $Z^1(\mathcal{U}, \mathcal{O}_M)$ defined by $g_{\alpha\beta} = f_\beta - f_\alpha$ on $U_\alpha \cap U_\beta$, where $\mathcal{O}_M$ is the sheaf of germs of all local holomorphic functions on M. If $H^1(\mathcal{U}, \mathcal{O}_M)$ vanishes, then we can write $g_{\alpha\beta} = h_\beta - h_\alpha$ on $U_\alpha \cap U_\beta$ for h_α holomorphic on U_α and the meromorphic function which is equal to $f_\alpha - h_\alpha$ on U_α is a global meromorphic function on M having P as its only pole whose order is one. Even after we have the vanishing of the first cohomology group $H^1(\mathcal{U}, \mathcal{O}_M)$, it is important that f_0 has a pole at P whereas h_0 is holomorphic at P, otherwise the difference $f_\alpha - h_\alpha$ may simply be identically zero and we do not end up with any nonconstant function. We can work with a zero instead of a pole. We can choose f_0 so that f_0 has a nonzero derivative at P. We let L be the line bundle $[P]^{-2}$ which is the line bundle having a meromorphic section s without zero and with P as the only pole which is of order two. Let $g_{\alpha\beta} = s(f_\beta - f_\alpha)$ on $U_\alpha \cap U_\beta$. Then $\{g_{\alpha\beta}\}$ is a 1-cocycle in $Z^1(\mathcal{U}, \mathcal{O}_M(L))$, where $\mathcal{O}_M(L)$ is the sheaf of germs of all local holomorphic sections of L. If $H^1(\mathcal{U}, \mathcal{O}_M(L))$ vanishes, then $g_{\alpha\beta} = h_\beta - h_\alpha$ for some $h_\alpha \in \Gamma(U_\alpha, \mathcal{O}_M(L))$. Then the function which is equal to $f_\alpha - s^{-1}h_\alpha$ on U_α is a global holomorphic function on M whose derivative at P is nonzero. Here we use zeroes of order two at P to make sure that the final function $f_\alpha - s^{-1}h_\alpha$ has a nonzero derivative at P. The sheaf $\mathcal{O}_M(L)$ is isomorphic to the sheaf $\mathbf{m}_P^2$ of germs of all local holomorphic functions vanishing at least to order two at P. The isomorphism $\mathcal{O}_M(L) \to \mathbf{m}_P^2$ is defined by division by the meromorphic section s of L. So instead of using the line bundle L we could also argue with the sheaf $\mathbf{m}_P^2$ and use as condition the vanishing of $H^1(\mathcal{U}, \mathbf{m}_P^2)$.

For a higher dimensional complex manifold instead of a Riemann surface, we can use the same method. A pole-set in the higher dimensional case is a complex hypersurface. While it is easy to find a point P in a Riemann surface, it is nontrivial to find a complex hypersurface in a higher dimensional complex manifold. We can also work on a zero of order two in the higher dimensional case. However, in the higher dimensional case the sheaf of germs of all local holomorphic functions vanishing (to some fixed finite order) at a point is not the sheaf of germs of all local holomorphic sections of a line bundle. Our vanishing theorems can be applied only to line bundles (or vector bundles). Using the sheaf of germs of all local holomorphic functions vanishing (to some fixed finite order) at a point is not going to help even though in this case one needs no complex hypersurface. Let us pick a point P in a higher dimensional complex manifold M and consider the sheaf $\mathbf{m}_P$ of germs of all local holomorphic functions vanishing at P. We want to examine why $\mathbf{m}_P$ fails to be the sheaf of germs of all local holomorphic sections of some holomorphic line bundle. The failure is at P and to consider $\mathbf{m}_P$ near P it is the same as looking at the case where the manifold is $\mathbf{C}^n$ and P is the origin. For a sheaf to be the sheaf of germs of all local holomorphic sections of some holomorphic line bundle, it is necessary and sufficient that the sheaf be generated locally by a *single* continuous section of the sheaf. For the sheaf $\mathbf{m}_0$ we know that it is generated at the origin by the coordinate functions $z_1, \cdots, z_n$, but no single one of $z_1, \cdots, z_n$ can do the generation all by itself. Take a holomorphic function f defined on some open neighborhood of the origin which vanishes at the origin. Then we have the expansion $f = \sum_{\nu=1}^n f_\nu z_\nu$, where f_ν is a holomorphic function defined on some open neighborhood of the origin. If we want to say that f can be generates by z_μ, then we should consider

$$f = \left(f_\mu + \sum_{\nu \neq \mu} f_\nu \frac{z_\nu}{z_\mu} \right) z_\mu$$

and the difficulty is that $\frac{z_\nu}{z_\mu}$ is not a holomorphic function in some open neighborhood of the origin. However, $\frac{z_\nu}{z_\mu}$ is holomorphic on the set $D_\mu = \{z_\mu \neq 0\}$ and the union of D_μ $(1 \leq \mu \leq n)$ is a punctured neighborhood of the origin. To overcome the difficulty we change the space so that D_μ together with the origin corresponds to an open subset in the new space. Since we want $\frac{z_\nu}{z_\mu}(\nu \neq \mu)$ and z_μ to be a holomorphic function on our new space, the simplest way to construct our space is to use coordinate charts $U_\nu = \mathbf{C}^n$ on which z_μ, $\zeta_\nu^{(\mu)}$ $(\nu \neq \mu\ )$ are global coordinates, where $\zeta_\nu^{(\mu)}$ corresponds to $\frac{z_\nu}{z_\mu}(\nu \neq \mu)$. Two different coordinate charts U_λ and U_μ are glued together on $U_\lambda \cap \{\zeta_\mu^{(\lambda)} \neq 0\}$ and $U_\mu = \{\zeta_\lambda^{(\mu)} \neq 0\}$ by

$$z_\mu = \zeta_\mu^{(\lambda)} z_\lambda \quad \text{and} \quad \zeta_\nu^{(\mu)} = \frac{\zeta_\nu^{(\lambda)}}{\zeta_\mu^{(\lambda)}} \quad (\nu \neq \lambda, \mu) \quad \text{and} \quad \zeta_\lambda^{(\mu)} = \frac{1}{\zeta_\mu^{(\lambda)}}.$$

The gluing is motivated by the correspondence between $\zeta_\nu^{(\mu)}$ and $\frac{z_\nu}{z_\mu}(\nu \neq \mu)$. We denote by $\widetilde{\mathbf{C}^n}$ the space obtained by gluing together U_λ and U_μ $(1 \leq \lambda, \mu \leq n,\ \lambda \neq \mu)$. There is a projection $p : \widetilde{\mathbf{C}^n} \to \mathbf{C}^n$ which sends the point

$$z_\mu, \quad \zeta_\nu^{(\mu)} \quad (\nu \neq \mu) \quad \text{in} \quad U_\nu$$

to the point

$$(z_1, \cdots, z_n) \quad \text{with} \quad z_\nu = \zeta_\nu^{(\mu)} z_\mu \quad (\nu \neq \mu).$$

The map p gives a biholomorphism between $p^{-1}(\mathbf{C}^n - 0)$ and $\mathbf{C}^n - 0$. It maps U_μ to $D_\mu \cup \{0\}$. The fiber $p^{-1}(0)$ is the union of $U_\mu \cap \{z_\mu = 0\}$ $(1 \leq \mu \leq n)$. We can regard $U_\mu \cap \{z_\mu = 0\}$ as a coordinate chart with global coordinates $\zeta_\nu^{(\mu)}$ $(\nu \neq \mu\ )$ and two of these coordinate charts $U_\lambda \cap \{z_\lambda = 0\}$ and $U_\mu \cap \{z_\mu = 0\}$ of $p^{-1}(0)$ are glued together by

$$\zeta_\nu^{(\mu)} = \frac{\zeta_\nu^{(\lambda)}}{\zeta_\mu^{(\lambda)}} \quad (\nu \neq \lambda, \mu) \quad \text{and} \quad \zeta_\lambda^{(\mu)} = \frac{1}{\zeta_\mu^{(\lambda)}}.$$

This means that $p^{-1}(0)$ is biholomorphic to the complex projective space $\mathbf{P}_{n-1}$ of complex dimension $n-1$. The sheaf $\mathbf{m}_0$ over $\mathbf{C}^n$ corresponds via p to the sheaf of germs of all local holomorphic functions of $\widetilde{\mathbf{C}^n}$ which vanish on $p^{-1}(0)$. This sheaf over $\widetilde{\mathbf{C}^n}$ is generated over U_μ by the single holomorphic function z_μ on U_μ. It is the sheaf of germs of all local holomorphic functions of the line bundle L associated to $p^{-1}(0)$. This means that L is defined by the transition function $\frac{z_\lambda}{z_\mu}$ from U_μ to U_λ and $\{z_\mu\}$ defines a holomorphic section of L whose zero set is the complex hypersurface $p^{-1}(0)$ with multiplicity one. The space $\widetilde{\mathbf{C}^n}$ is said to be obtained by blowing up the origin of $\mathbf{C}^n$.

The process of getting $\widetilde{\mathbf{C}^n}$ from blowing up the origin of $\mathbf{C}^n$ can be regarded as the complex analog of the change from the Cartesian coordinate system of $\mathbf{R}^2$ to the polar coordinate system of $\mathbf{R}^2$. The polar coordinate associates to a point with Cartesian coordinates $(x.y)$ the distance r from the origin and the angle θ made with the x-axis by the vector from the origin to the point. To pass to the complex case we have to make some changes. Instead of using r to measure the distance from the origin to that point, we use the abscissa x which is the projection of the radius vector onto the x-axis. Instead of using the angle θ we use the slope $\frac{y}{x}$. So we use $(x, \frac{y}{x})$ as our modified polar coordinate system. This modified polar coordinate system is good only for points whose Cartesian coordinates (x, y) satisfies $x \neq 0$. For those points whose Cartesian coordinates (x, y) satisfies $y \neq 0$, we have to use the modified polar coordinate $(y, \frac{x}{y})$. That leaves only the origin without any modified polar coordinates. This is what we expect from a polar coordinate system. There is no way to assign a *unique* set of polar coordinates to the origin. In the usual polar coordinates the origin corresponds to (r, θ) with $r = 0$ and $\theta \in \mathbf{R}$. In the complex case $\mathbf{C}^n$ we have the modified polar coordinates

$$\left(z_\mu, \frac{z_1}{z_\mu}, \cdots, \frac{z_{\mu-1}}{z_\mu}, \frac{z_{\mu+1}}{z_\mu}, \cdots, \frac{z_n}{z_\mu}\right)$$

on the set $\{z_\mu \neq 0\}$. The origin corresponds to not just one set of modified polar coordinates but a collection of modified polar coordinates parametrized by $\mathbf{P}_{n-1}$. Our space $\widetilde{\mathbf{C}^n}$ is constructed so that every one of its points corresponds to one set of modified polar coordinates for $\mathbf{C}^n$. So a point of $\mathbf{C}^n - 0$ corresponds to a point of $\mathbf{C}^n$, but the origin of $\mathbf{C}^n$ corresponds to a subset of $\widetilde{\mathbf{C}^n}$ which can be identified with $\mathbf{P}_{n-1}$. In the modified polar coordinates

$$\left(z_\mu, \frac{z_1}{z_\mu}, \cdots, \frac{z_{\mu-1}}{z_\mu}, \frac{z_{\mu+1}}{z_\mu}, \cdots, \frac{z_n}{z_\mu}\right),$$

the part

$$\left(\frac{z_1}{z_\mu}, \cdots, \frac{z_{\mu-1}}{z_\mu}, \frac{z_{\mu+1}}{z_\mu}, \cdots, \frac{z_n}{z_\mu}\right)$$

is simply the inhomogeneous coordinates of the projective space $\mathbf{P}_{n-1}$ of all complex lines of $\mathbf{C}^n$ passing through the origin. So we can also interpret the blow-up $p : \widetilde{\mathbf{C}^n} \to \mathbf{C}^n$ as follows by considering the natural map $\mathbf{C}^n - 0 \to \mathbf{P}_{n-1}$.

Let $z_1, \cdots, z_n$ be the coordinates of $\mathbf{C}^n$. Consider the complex projective space $\mathbf{P}_{n-1}$ of complex dimension $n-1$. Let $[w_1, \cdots, w_n]$ be the homogeneous coordinates of $\mathbf{P}_{n-1}$. Consider the map from $\mathbf{C}^n - 0$ to $\mathbf{P}_{n-1}$ which maps $(z_1, \cdots, z_n)$ to $[z_1, \cdots, z_n]$ and let Γ be its graph in $(\mathbf{C}^n - 0) \times \mathbf{P}_{n-1}$. Consider the topological closure $\bar{\Gamma}$ of Γ in $\mathbf{C}^n \times \mathbf{P}_{n-1}$. We claim that $\bar{\Gamma}$ is equal to the closed subset Z of $\mathbf{C}^n \times \mathbf{P}_{n-1}$ consisting of all points

$$((z_1, \cdots, z_n), [w_1, \cdots, w_n])$$

in $\mathbf{C}^n \times \mathbf{P}_{n-1}$ with $z_i w_j = z_j w_i$ for all $1 \leq i, j \leq n$. Clearly the intersection of Z with $(\mathbf{C}^n - 0) \times \mathbf{P}_{n-1}$ agrees with Γ and the set Z contains $0 \times \mathbf{P}_{n-1}$. To verify the claim it suffices to show that $\bar{\Gamma}$ contains the subset $0 \times \mathbf{P}_{n-1}$ of $\mathbf{C}^n \times \mathbf{P}_{n-1}$. This is clear, because any point $(0, [w_1, \cdots, w_n])$ of $0 \times \mathbf{P}_{n-1}$ is the limit of the point

$$((tw_1, \cdots, tw_n), [w_1, \cdots, w_n])$$

of Γ for $t \in \mathbf{C} - 0$ as $t \to 0$.

Next we observe that the set $\bar{\Gamma}$ is a submanifold of $\mathbf{C}^n \times \mathbf{P}_{n-1}$. We cover $\mathbf{P}_{n-1}$ by open subsets U_i defined by $w_i \neq 0$. The set U_i is biholomorphic to $\mathbf{C}^{n-1}$ and has global coordinates $\zeta_j^{(i)} = \frac{w_j}{w_i}$ for $j \neq i$. Let $\tilde{U}_i = \mathbf{C}^n \times U_i$. Then

$$z_1, \cdots, z_n, \zeta_1^{(i)}, \cdots, \zeta_{i-1}^{(i)}, \zeta_{i+1}^{(i)}, \cdots, \zeta_n^{(i)}$$

are global coordinates of $\mathbf{C}^n \times U_i$. The set $\bar{\Gamma} \cap \mathbf{C}^n \times U_i$ is given by the equations $z_j = \zeta_j^{(i)} z_i$ for $j \neq i$ and is therefore the graph of the holomorphic map from $\mathbf{C} \times U_i$ to $\mathbf{C}^{n-1}$ given by

$$(z_i, \zeta_1^{(i)}, \cdots, \zeta_{i-1}^{(i)}, \zeta_{i+1}^{(i)}, \cdots, \zeta_n^{(i)}) \to (\zeta_1^{(i)} z_i, \cdots, \zeta_{i-1}^{(i)} z_i, \zeta_{i+1}^{(i)} z_i, \cdots, \zeta_n^{(i)} z_i)$$

and is a complex submanifold of $\mathbf{C}^n \times U_i$. Another way to see that $\bar{\Gamma} \cap \mathbf{C}^n \times U_i$ is a complex submanifold of $\mathbf{C}^n \times U_i$ is to observe that the Jacobian matrix of the defining functions

$$F_i = z_j - \zeta_j^{(i)} z_i \quad (j \neq i)$$

of $\bar{\Gamma} \cap \mathbf{C}^n \times U_i$ with respect to z_j $(j \neq i)$ when $\zeta_j^{(i)}$ $(j \neq i)$ are fixed is simply the identity matrix and is nonsingular. Let $U_i^* = \bar{\Gamma} \cap \mathbf{C}^n \times U_i$. Then U_i^* is biholomorphic to $\mathbf{C}^n$ with global coordinates

$$z_i, \zeta_1^{(i)}, \cdots, \zeta_{i-1}^{(i)}, \zeta_{i+1}^{(i)}, \cdots, \zeta_n^{(i)}.$$

The manifold $\bar{\Gamma}$ is said to be obtained from $\mathbf{C}^n$ by blowing up the origin. We denote $\bar{\Gamma}$ by $\widetilde{\mathbf{C}^n}$ and denote by p the map $\widetilde{\mathbf{C}^n} \to \mathbf{C}^n$ induced by the natural projection $\mathbf{C}^n \times \mathbf{P}_{n-1} \to \mathbf{C}^n$. The map p is holomorphic and is a biholomorphism between $p^{-1}(\mathbf{C}^n - 0)$ and $\mathbf{C}^n - 0$. The fiber $p^{-1}(0)$ is simply $0 \times \mathbf{P}_{n-1}$.

Now consider a complex manifold M of complex dimension n. Let P be a point of M. An open neighborhood G of P is biholomorphic to an open neighborhood W of the origin in $\mathbf{C}^n$ under a map $\varphi : G \to U$. Let $\tilde{M}$ be the manifold obtained by identifying $G - P$ with $p^{-1}(U - 0)$ in the disjoint union of M and $p^{-1}(U)$ through the map $(p^{-1})\varphi$. Then $\tilde{M}$ is a complex manifold of complex dimension n and we have a holomorphic map $\pi : \tilde{M} \to M$ induced by the identity map of M and the projection $p : p^{-1}(U) \to U$. The map π is a biholomorphism between $\pi^{-1}(M - P)$ and $M - P$ and the fiber $\pi^{-1}(P)$ is biholomorphic to $\mathbf{P}_{n-1}$. We say that the manifold $\tilde{M}$ is obtained from M by blowing up the point P.

Since we intend to use the vanishing theorem of Kodaira, we need to relate the canonical line bundle K_M of M to the canonical line bundle $K_{\tilde{M}}$ of $\tilde{M}$. Let us go back for a moment to the blow-up $p : \widetilde{\mathbf{C}^n} \to \mathbf{C}^n$. Take the holomorphic n-form $dz_1 \wedge \cdots \wedge dz_n$ on $\mathbf{C}^n$ and consider its pullback $p^*(dz_1 \wedge \cdots \wedge dz_n)$. On U_i^* we express $p^*(dz_1 \wedge \cdots \wedge dz_n)$ in terms of the global coordinates

$$z_i, \zeta_1^{(i)}, \cdots, \zeta_{i-1}^{(i)}, \zeta_{i+1}^{(i)}, \cdots, \zeta_n^{(i)}$$

and get

$$\begin{aligned} dz_1 \wedge \cdots \wedge dz_n &= (d\zeta_1^{(i)} z_i + \zeta_1^{(i)} dz_i) \wedge \cdots \wedge (d\zeta_{i-1}^{(i)} z_i + \zeta_{i-1}^{(i)} dz_i) \\ &\quad \wedge (d\zeta_{i+1}^{(i)} z_i + \zeta_{i+1}^{(i)} dz_i) \wedge \cdots \wedge (d\zeta_n^{(i)} z_i + \zeta_n^{(i)} dz_i) \\ &= z_i^{n-1} d\zeta_1^{(i)} \wedge \cdots \wedge d\zeta_{i-1}^{(i)} \wedge d\zeta_{i+1}^{(i)} \wedge \cdots \wedge d\zeta_n^{(i)}. \end{aligned}$$

So we know that $p^*(dz_1 \wedge \cdots \wedge dz_n)$ precisely vanishes to order $n-1$ on the zero-set $z_i = 0$ in U_i^* which is equal to $U_i^* \cap p^{-1}(0)$. Coming back to $\pi : \tilde{M} \to M$, we conclude that for any nowhere zero holomorphic n-form ω_0 on some open coordinate chart G of M containing P, the pullback $\pi^*\omega_0$ vanishes precisely to order $n-1$ on $\pi^{-1}(P)$. Consider the line bundle $[\pi^{-1}(P)]$ associated to the hypersurface $\pi^{-1}(P)$. The line bundle $[\pi^{-1}(P)]$ admits a holomorphic section s called the canonical section whose zero-set is $\pi^{-1}(P)$ with multiplicity one. We cover M by open coordinate charts W_α so that

$W_0 = G$ and P is not contained in any other W_α. We choose a nowhere zero holomorphic n-form ω_α on W_α. The transition functions $g_{\alpha\beta}$ of K_M are given by $\omega_\alpha g_{\alpha\beta} = \omega_\beta$. Now $s^{-(n-1)}\pi^*\omega_\alpha$ is a nowhere zero holomorphic section of $K_{\tilde{M}} \otimes [\pi^{-1}(P)]^{-(n-1)}$ on $\pi^{-1}(W_\alpha)$. It follows from

$$(s^{-(n-1)}\pi^*\omega_\alpha)\pi^* g_{\alpha\beta} = s^{-(n-1)}\pi^*\omega_\beta$$

that the transition functions for $K_{\tilde{M}} \otimes [\pi^{-1}(P)]^{-(n-1)}$ are $\pi^* g_{\alpha\beta}$ and $K_{\tilde{M}} \otimes [\pi^{-1}(P)]^{-(n-1)}$ is isomorphic to $\pi^* K_M$. Hence $K_{\tilde{M}} = \pi^* K_M \otimes [\pi^{-1}(P)]^{n-1}$.

We would like to understand more about the line bundle $[\pi^{-1}(P)]$ because of its appearance in the formula for $K_{\tilde{M}}$. For our discussion we go back to the blow-up $p : \widetilde{\mathbf{C}^n} \to \mathbf{C}^n$. On the coordinate chart U_i^* of $\widetilde{\mathbf{C}^n}$ with global coordinates

$$z_i, \zeta_1^{(i)}, \cdots, \zeta_{i-1}^{(i)}, \zeta_{i+1}^{(i)}, \cdots, \zeta_n^{(i)},$$

the set $p^{-1}(0)$ is defined by $z_i = 0$. So the transition function for the line bundle $[p^{-1}(0)]$ from U_j^* to U_i^* is $\frac{z_i}{z_j}$ which on $\widetilde{\mathbf{C}^n}$ is equal to $\frac{w_i}{w_j}$. Consider a hyperplane H in $\mathbf{P}_{n-1}$ defined by a linear equation $\ell(w_1, \cdots, w_n) = 0$ of the homogeneous coordinates. On U_i the hyperplane is the zero set of the holomorphic function $\frac{\ell(w)}{w_i}$. The transition function for the line bundle $[H]$ associated to H from U_j to U_i is

$$\left(\frac{\ell(w)}{w_i}\right)\left(\frac{\ell(w)}{w_j}\right)^{-1} = \frac{w_j}{w_i}.$$

Since its transition functions are independent of the linear function $\ell(w)$ of the homogeneous coordinates, it follows that the line bundle $[H]$ is independent of the choice of H and it is simply referred to as the hyperplane section line bundle of $\mathbf{P}_{n-1}$. By looking at the transition functions we conclude that $[p^{-1}(0)]$ is isomorphic to the pullback of $[H]$ via the projection $\widetilde{\mathbf{C}^n} \to \mathbf{P}_{n-1}$ induced by $\mathbf{C}^n \times \mathbf{P}_{n-1} \to \mathbf{P}_{n-1}$.

§4. *The embedding theorem of Kodaira.*

Let M be a compact complex manifold of complex dimension n and L be a holomorphic line bundle over M with a Hermitian metric along its fibers whose curvature form is positive definite. Take a point P of M and consider the map $\pi : \tilde{M} \to M$ obtained by blowing up the point P. Let $D = \pi^{-1}(P)$ and let $\tilde{L}$ be the pullback $\pi^* L$ of L via π. Then $\tilde{L}^k K_{\tilde{M}}^{-1} = \pi^*(L^k K_M^{-1})[D]^{-(n-1)}$. We want to get the vanishing of $H^1(\tilde{M}, \tilde{L})$

for k sufficiently large from Kodaira's vanishing theorem. So we want the positivity of $\tilde{L}^k K_{\tilde{M}}^{-1} = \pi^*(L^k K_M^{-1})[D]^{-(n-1)}$ for k sufficiently large. We know that since L is positive, we have the positivity of $L^k K_M^{-1}$ for k sufficiently large. However, we cannot get from it the positivity of $\pi^*(L^k K_M^{-1})$. Clearly we have the positivity of $\pi^*(L^k K_M^{-1})$ on $\tilde{M} - D$ and the its semipositivity on $\tilde{M}$, but the line bundle $\pi^*(L^k K_M^{-1})$ cannot be positive in a neighborhood of D. Since the line bundle $\pi^*(L^k K_M^{-1})$ is trivial on an open neighborhood of D, a Hermitian metric for $\pi^*(L^k K_M^{-1})$ on that neighborhood is simply a positive function h and the positivity of $\pi^*(L^k K_M^{-1})$ on that neighborhood means that $h = e^{-\varphi}$ for some strictly plurisubharmonic function φ. This would contradict the maximum principle when one considers the restriction of φ to the positive-dimensional complex manifold D. To get the positivity of $\tilde{L}^k K_{\tilde{M}}^{-1}$ we need the help of $[D]^{-(n-1)}$. We claim that on some open neighborhood of D the line bundle $[D]^{-1}$ admits a Hermitian metric along its fibers whose curvature tensor is positive. To prove the claim, it suffices to show that in the blow-up $p : \widetilde{\mathbf{C}^n} \to \mathbf{C}^n$ the line bundle $[p^{-1}(0)]$ on $\widetilde{\mathbf{C}^n}$ carries a Hermitian metric with positive curvature. Let H be a hyperplane in $\mathbf{P}_{n-1}$. We have seen that the line bundle $[p^{-1}(0)]$ is isomorphic to the pullback of $[H]$ via the projection $\sigma : \widetilde{\mathbf{C}^n} \to \mathbf{P}_{n-1}$ induced by $\mathbf{C}^n \times \mathbf{P}_{n-1} \to \mathbf{P}_{n-1}$. It suffices to show that $[H]$ carries a Hermitian metric with positive curvature, because we can regard $[p^{-1}(0)]$ as the tensor product of $\sigma^*[H]$ and the pullback p^*1 via p of the trivial bundle 1 over $\mathbf{C}^n$ and the trivial bundle over $\mathbf{C}^n$ given the metric $e^{-|z|^2}$ has positive curvature.

Before we show that $[H]$ carries a Hermitian metric with positive curvature, let us first make some general remarks about constructing Hermitian metrics along the fibers of a line bundle. Suppose we have a smooth local section s of a holomorphic line bundle. A Hermitian metric along the fibers of the line bundle means a certain way of assigning to s a positive-value function. Suppose we have another smooth local section s_1 which is nowhere zero. Then we can assign to s the positive-valued function $\frac{|s|^2}{|s_1|^2}$. Here $|\cdot|$ means the absolute value of a function that represents the section in some local trivialization of the line bundle. We have then a Hermitian metric for the line bundle on the domain of definition of s_1. The reason for dividing $|s|^2$ by $|s_1|^2$ is to get a function that does not depend on the local trivialization. The independence comes from the fact that $|s|^2$ by $|s_1|^2$ change in the same way when a different local trivialization is used. We can achieve the same purpose by using a finite number of smooth sections $s_1, \cdots, s_k$ instead of just one smooth section s_1. We can assign to s the positive-valued function $|s|^2\left(\sum_{i=1}^k |s_i|^2\right)^{-1}$ and get a Hermitian metric along the fibers of the line bundle. The advantage of using a finite number of smooth sections $s_1, \cdots, s_k$ is that when we want a Hermitian metric for the entire line bundle with positive curvature we will have to use global *holomorphic* sections $s_1, \cdots, s_k$ and, while it is possible to get a globally nowhere zero $\sum_{i=1}^k |s_i|^2$, it is impossible to get a globally nowhere zero $|s_1|^2$ unless the line bundle is globally trivial. The curvature form of the Hermitian metric $s \to |s|^2\left(\sum_{i=1}^k |s_i|^2\right)^{-1}$ is given by $\partial\bar\partial \log\left(\sum_{i=1}^k |s_i|^2\right)$. To see when this curvature form is positive definite, we calculate $\partial\bar\partial\log\left(\sum_i |f_i|^2\right)$ for holomorphic functions f_i defined on some open neighborhood of origin in $\mathbf{C}^n$. We have

$$\partial\bar\partial\log\left(\sum_i |f_i|^2\right) = \partial\left[\left(\sum_i |f_i|^2\right)^{-1}\sum_i f_i d\bar f_i\right]$$

$$= \left(\sum_k |f_k|^2\right)^{-1}\sum_i df_i\wedge d\bar f_i - \left(\sum_k |f_k|^2\right)^{-1}\left(\sum_i \bar f_i df_i\right)\wedge\left(\sum_j f_j d\bar f_j\right)$$

$$= \left(\sum_k |f_k|^2\right)^{-2}\left[\left(\sum_j |f_j|^2\right)\left(\sum_i df_i\wedge d\bar f_i\right) - \left(\sum_i \bar f_i df_i\right)\wedge\left(\sum_j f_j d\bar f_j\right)\right]$$

$$= \left(\sum_k |f_k|^2\right)^{-2}\sum_{i,j}\left(f_j\bar f_j df_i\wedge d\bar f_i - \bar f_i f_j df_i\wedge d\bar f_j\right)$$

$$= \frac{1}{2}\left(\sum_k |f_k|^2\right)^{-2}\sum_{i,j}\left(f_i df_j - f_j df_i\right)\wedge\overline{\left(f_i df_j - f_j df_i\right)}.$$

The last two steps is motivated by the identity

$$\left(\sum_i a_i\overline{a_i}\right)\left(\sum_i b_i\overline{b_i}\right) = \left(\sum_i a_i\bar b_i\right)^2 + \left|\sum_{i\neq j}(a_i b_j - a_j b_i)\right|^2$$

which is the basis for the Schwarz's inequality and is the generalization of the trigonometric identity $1 = \cos^2\theta + \sin^2\theta$ written in vector form $|\vec u|^2|\vec v|^2 = |\vec u\cdot\vec v|^2 + |\vec u\times\vec v|^2$. Clearly $\partial\bar\partial\log\left(\sum_i |f_i|^2\right)$ is always semipositive. Suppose its value at some vector X of type (1,0) at some point Q is zero. Then one must have the vanishing of $f_i df_j - f_j df_i$ at X for all i and j. Since $f_i^{-2}(f_i df_j - f_j df_i) = d\left(\frac{f_j}{f_i}\right)$, this can happen for some such X if and only if the (1,0)-forms $d\left(\frac{f_j}{f_i}\right)$ for all i and j with $f_i(Q) \neq 0$ do not span the space of all (1,0)-forms at Q.

Let us return to the question of constructing a Hermitian metric along the fibers of $[H]$ with positive curvature. To use the scheme described above we should use global holomorphic sections s_i of $[H]$. We take a linear function $\ell(w)$ of the homogeneous coordinates $w_1, \cdots, w_n$. The line bundle associated with the zero-set of $\ell(w)$ is isomorphic to $[H]$. So its canonical section is a holomorphic section of $[H]$. By selecting different such linear functions $\ell(w)$ of the homogeneous coordinates $w_1, \cdots, w_n$, we can get a collection of global holomorphic sections of $[H]$. With respect to the trivialization of $[H]$ on $U_i = \{w_i \neq 0\}$, the global holomorphic section given by $\ell(w)$ is described by the holomorphic function $\frac{\ell(w)}{w_i}$ on U_i. So if we have a finite number of such linear functions $\ell_j(w)$, the Hermitian metric constructed from them would be given by $|w_i|^2 \left(\sum_j |\ell_j(w)|^2\right)^{-2}$ with respect to the trivialization of $[H]$ on U_i. To make sure that we have a Hermitian metric with positive curvature we should choose the linear functions so that at every point Q of $\mathbf{P}_{n-1}$ the (1,0)-forms $d\left(\frac{\ell_j(w)}{\ell_i(w)}\right)$ for all i and j with $\ell_i(Q) \neq 0$ span the space of all (1,0)-forms at Q. The simplest choice that fits the bill is $\ell_j(w) = w_j$ for $1 \leq j \leq n$. Then the Hermitian metric along the fibers of $[H]$ is $|w_i|^2 \left(\sum_j |w_j|^2\right)^{-2}$ with respect to the trivialization of $[H]$ on U_i and the curvature form on U_i in terms of the inhomogeneous coordinates is $\partial\bar\partial \log\left(1 + \sum_{j \neq i} |\frac{w_j}{w_i}|^2\right)$ which can also be written in the more symmetric form $\partial\bar\partial \log\left(\sum_{j=1}^n |w_j|^2\right)$ because $\partial\bar\partial \log |w_i|^2$ is identically zero. This positive definite (1,1)-form $\partial\bar\partial \log\left(\sum_{j=1}^n |w_j|^2\right)$ defines a Kähler metric on $\mathbf{P}_{n-1}$ known as the Fubini-Study metric. With the construction of a Hermitian metric for $[H]$ with positive curvature we know that the line bundle $\tilde{L}^k K_{\tilde{M}}^{-1} = \pi^*(L^k K_M^{-1})[D]^{-(n-1)}$ on $\tilde{M}$ admits a Hermitian metric with positive curvature for k sufficiently large. As a matter of fact the argument shows that for any fixed $\nu \geq 1$ the line bundle $\pi^*(L^k K_M^{-1})[D]^{-\nu}$ on $\tilde{M}$ admits a Hermitian metric with positive curvature for k sufficiently large.

We now return to the question of embedding M into a complex projective space. Consider the exact sequence

$$(*) \qquad 0 \to \mathcal{O}(\tilde{L}^k[D]^{-2}) \xrightarrow{\psi} \mathcal{O}(\tilde{L}^k) \to \mathcal{Q} \to 0$$

over $\tilde{M}$, where ψ is defined by multiplication by s_D^2 with s_D being the canonical section of $[D]$ and $\mathcal{Q}$ is the cokernel of ψ. Clearly the sheaf $\mathcal{Q}$ is supported on D. Since $\tilde{L}^k[D]^{-2}K_{\tilde{M}}^{-1} = \pi^*(L^kK_M^{-1})[D]^{-(n+1)}$ carries a Hermitian metric with positive curvature for k sufficiently large, it follows from the vanishing theorem of Kodaira that $H^1(\tilde{M}, \tilde{L}^k[D]^{-2})$ vanishes for k sufficiently large. From the long cohomology exact sequence of (*) it follows that $\Gamma(\tilde{M}, \mathcal{O}(\tilde{L}^k)) \to \Gamma(\tilde{M}, \mathcal{Q})$ is surjective.

Choose local coordinates $z_1, \cdots, z_n$ of M centered at P defined on some open neighborhood W of P in M. We can assume that L is trivial on W. Then the functions $1, z_1, \cdots, z_n$ on W can be regarded as elements of $\Gamma(W, \mathcal{O}(L^k))$ and their pullbacks $g_0, g_1, \cdots, g_n$ via π are elements of $\Gamma(\pi^{-1}(W), \mathcal{O}(\tilde{L}^k))$. By the surjectivity of $\Gamma(\tilde{M}, \mathcal{O}(\tilde{L}^k)) \to \Gamma(\tilde{M}, \mathcal{Q})$ and the fact that the sheaf $\mathcal{Q}$ is supported on $\pi^{-1}(P)$, we can find elements $f_0, f_1, \cdots, f_n$ of $\Gamma(\tilde{M}, \mathcal{O}(\tilde{L}^k))$ whose images under $\Gamma(\tilde{M}, \mathcal{O}(\tilde{L}^k)) \to \Gamma(\tilde{M}, \mathcal{Q})$ agree with the images of $g_0, g_1, \cdots, g_n$ under $\Gamma(\pi^{-1}(W), \mathcal{O}(\tilde{L}^k)) \to \Gamma(\tilde{M}, \mathcal{Q})$. Since $\tilde{L}$ is trivial on $\pi^{-1}(W)$, there exist elements $h_0, h_1, \cdots, h_n$ of $\Gamma(M, \mathcal{O}(L^k))$ whose pullbacks via π agree with $f_0, f_1, \cdots, f_n$. Then $h_0, h_1, \cdots, h_n$ agree with $1, z_1, \cdots, z_n$ to second order at P. So we conclude that given any P for k sufficiently large there exist elements $h_0, h_1, \cdots, h_n$ of $\Gamma(M, \mathcal{O}(L^k))$ so that h_0 is nonzero at P and $\frac{h_1}{h_0}, \cdots, \frac{h_1}{h_0}$ form a coordinate system of some open neighborhood W of P in M. To emphasize the dependence of k, W, and $h_0, h_1, \cdots, h_n$ on P we denote them respectively by $k(P)$, $W(P)$, and $h_{P,0}, h_{P,1}, \cdots, h_{P,n}$. We do this for every point P of M and by the compactness of M we can select a finite number of points $P_1, \cdots, P_\ell$ so that $W(P_1), \cdots, W(P_\ell)$ cover M. Let k_1 be the product $k(P_1) \cdots k(P_\ell)$ and let $J_1 = (n+1)^\ell$. Let $F_j (1 \leq j \leq J_1)$ denote the elements $(h_{P_1,i_1}) \cdots (h_{P_\ell,i_\ell})$, $0 \leq i_1, \cdots, i_\ell \leq n$, of $\Gamma(M, \mathcal{O}(L^{k_1}))$. Then the collection $F_j (1 \leq j \leq J_1)$ of elements of $\Gamma(M, \mathcal{O}(L^{k_1}))$ has the property that for $1 \leq i \leq \ell$ we can select a subset $F_{j_0}, F_{j_1}, \cdots, F_{j_n}$ such that F_{j_0} is nonzero on $W(P_i)$ and $\frac{F_{j_1}}{F_{j_0}}, \cdots, \frac{F_{j_1}}{F_{j_0}}$ form a coordinate system on $W(P_i)$.

Take two distinct points P and P' of M. We can blow up both P and P' at the same time or successively and get a new complex manifold $\hat{M}$ with a holomorphic projection $\Pi : \hat{M} \to M$ so that Π maps $\Pi^{-1}(M - P - P')$ biholomorphically onto $M - P - P'$ and each of the two fibers $\Pi^{-1}(P)$ and $\Pi^{-1}(P')$ is a hypersurface of $\hat{M}$ biholomorphic to $\mathbf{P}_{n-1}$. Let $D = \Pi^{-1}(P)$ and $D' = \Pi^{-1}(P')$. Let $\hat{L} = \Pi^* L$. Our preceding discussion applied to this successive blow-up yields the following. $\hat{L}^k K_{\hat{M}}^{-1} = \Pi^*(L^k K_M^{-1})[D]^{-(n-1)}[D']^{-(n-1)}$ on $\hat{M}$. For any fixed $\nu \geq 1$ the line bundle $\Pi^*(L^k K_M^{-1})[D]^{-\nu}[D']^{-\nu}$ on $\hat{M}$ admits a Hermitian metric with positive curvature for k sufficiently large.

Consider the exact sequence

$$(\#) \qquad 0 \to \mathcal{O}(\hat{L}^k[D]^{-1}[D']^{-1}) \xrightarrow{\hat{\psi}} \mathcal{O}(\hat{L}^k) \to \hat{\mathcal{Q}} \to 0$$

over $\hat{M}$, where $\hat{\psi}$ is defined by multiplication by $s_D s_{D'}$ with s_D and $s_{D'}$ being respectively the canonical sections of $[D]$ and $[D']$ and $\hat{\mathcal{Q}}$ is the cokernel of $\hat{\psi}$. Clearly the sheaf $\hat{\mathcal{Q}}$ is supported on $D \cup D'$. Since $\hat{L}^k[D]^{-1}[D]^{-1}K_{\tilde{M}}^{-1} = \Pi^*(L^k K_M^{-1})[D]^{-n}[D']^{-n}$ carries a Hermitian metric with positive curvature for k sufficiently large, it follows from the vanishing theorem of Kodaira that $H^1(\tilde{M}, \hat{L}^k[D]^{-1}[D]^{-1})$ vanishes for k sufficiently large. From the long cohomology exact sequence of $(\#)$ it follows that $\Gamma(\hat{M}, \mathcal{O}(\hat{L}^k)) \to \Gamma(\hat{M}, \hat{\mathcal{Q}})$ is surjective.

Choose in M disjoint open neighborhoods W of P and W' of P' so that L is trivial on $W \cup W'$. Then the function which is identically 1 on W and identically 0 on W' can be regarded as an element of $\Gamma(W \cup W',\, \mathcal{O}(L^k))$ and its pullback g via Π is an element of $\Gamma(\Pi^{-1}(W \cup W'),\, \mathcal{O}(\hat{L}^k))$. By the surjectivity of $\Gamma(\hat{M}, \mathcal{O}(\hat{L}^k)) \to \Gamma(\hat{M}, \hat{\mathcal{Q}})$ and the fact that the sheaf $\mathcal{Q}$ is supported on $\Pi^{-1}(P) \cup \Pi^{-1}(P')$, we can find an element f of $\Gamma(\hat{M}, \mathcal{O}(\hat{L}^k))$ whose image under $\Gamma(\tilde{M}, \mathcal{O}(\tilde{L}^k)) \to \Gamma(\tilde{M}, \mathcal{Q})$ agrees with the image of g under $\Gamma(\Pi^{-1}(W \cup W'),\, \mathcal{O}(\hat{L}^k)) \to \Gamma(\hat{M}, \hat{\mathcal{Q}})$. Since $\hat{L}$ is trivial on $\Pi^{-1}(W \cup W')$, there exist an element $h^{(0)}$ of $\Gamma(M,\, \mathcal{O}(L^k))$ whose pullbacks via Π agree with f. Then $h^{(0)}$ is nonzero at P and zero at P'. Reversing the roles of P and P', we find an element $h^{(1)}$ of $\Gamma(M,\, \mathcal{O}(L^k))$ which is zero at P and nonzero at P'. So we conclude that given any pair of points P and P' in M for k sufficiently large there exist elements $h^{(0)}$ and $h^{(1)}$ of $\Gamma(M,\, \mathcal{O}(L^k))$ so that the meromorphic function $\frac{h^{(1)}}{h^{(0)}}$ assumes different values at P and P'. For some open neighborhood V and V' of P and P' respectively in M, the meromorphic function $\frac{h^{(1)}}{h^{(0)}}$ assumes different values at Q and Q' for $Q \in V$ and $Q' \in V'$. To emphasize the dependence of k, $h^{(0)}$, $h^{(1)}$, V and V' on P and P' we denote them respectively by $k(P, P')$, $h^{(0)}_{P,P'}$, $h^{(1)}_{P,P'}$, $V(P, P')$, and $V'(P, P')$. We do this for every pair of points P and P' of M not lying in the same $W(P_i)$ for any $1 \leq i \leq \ell$. By the compactness of $M \times M - \cup_{i=1}^{\ell} W(P_i) \times W(P_i)$ we can select a finite number of pairs of points $(Q_1, Q_1'), \cdots, (Q_m, Q_m')$ so that $V(Q_1, Q_1') \times V'(Q_1, Q_1'), \cdots, V(Q_m, Q_m') \times V'(Q_m, Q_m')$ cover $M \times M - \cup_{i=1}^{\ell} W(P_i) \times W(P_i)$. Let k_2 be the product $k(Q_1, Q_1') \cdots k(Q_m, Q_m')$ and let $J_2 = 2^m$. Let $G_j (1 \leq j \leq J_2)$ denote the elements $(h^{(\nu_1)}_{Q_1, Q_1'}) \cdots (h^{(\nu_m)}_{Q_m, Q_m'})$, $0 \leq \nu_1, \cdots, \nu_m \leq 1$, of $\Gamma(M,\, \mathcal{O}(L^{k_2}))$. Then the collection $G_j (1 \leq j \leq J_2)$ of elements of $\Gamma(M, \mathcal{O}(L^{k_2}))$ has the property that for any two points P and P' of M not lying in the same $W(P_i)$ for any $1 \leq i \leq \ell$ we can select a subset G_{j_0}, G_{j_1} such that the meromorphic function $\frac{G_{j_1}}{G_{j_0}}$ assumes different values at P and P'.

Finally let $k = k_1 k_2$. Let $s_0, \cdots, s_N$ be a basis of $\Gamma(M, \mathcal{O}(L^k))$ over $\mathbf{C}$. We claim that the map from M to $\mathbf{P}_N$ defined by $s_0, \cdots, s_N$ as homogeneous coordinates maps M biholomorphically onto a complex submanifold of $\mathbf{P}_N$. The reason is the following. For $1 \leq i \leq \ell$ we have a subset $F_{j_0}, F_{j_1}, \cdots, F_{j_n}$ such that F_{j_0} is nonzero on $W(P_i)$ and $\frac{F_{j_1}}{F_{j_0}}, \cdots, \frac{F_{j_1}}{F_{j_0}}$ form a coordinate system on $W(P_i)$. Now $F_{j_0}^{k_2}$ and $F_{j_0}^{k_2-1} F_{j_\nu}$ are elements of $\Gamma(M, \mathcal{O}(L^k))$ for $1 \leq \nu \leq n$. The section $F_{j_0}^{k_2}$ is nowhere zero on $W(P_i)$ and the quotients of $F_{j_0}^{k_2-1} F_{j_\nu}$ by $F_{j_0}^{k_2}$ for $1 \leq \nu \leq n$ form a coordinate system on $W(P_i)$. For any two points P and P' of M not lying in the same $W(P_i)$ for any $1 \leq i \leq \ell$ we have a subset G_{j_0}, G_{j_1} such that the meromorphic function $\frac{G_{j_1}}{G_{j_0}}$ assumes different values at P and P'. Now $G_{j_1}^{k_1}$ and $G_{j_2}^{k_1}$ are elements of $\Gamma(M, \mathcal{O}(L^k))$ and their quotient is a meromorphic function assuming different values at P and P'. Since $F_{j_0}^{k_2}$, $F_{j_0}^{k_2-1} F_{j_\nu}$, $G_{j_1}^{k_1}$, and $G_{j_2}^{k_1}$ are $\mathbf{C}$-linear combinations of $s_0, \cdots, s_N$, follows that for any point P of M we can select $s_{j_0}, \cdots, s_{j_n}$ so that s_{j_0} is nonzero at P and $\frac{s_{j_1}}{s_{j_0}}, \cdots, \frac{s_{j_1}}{s_{j_0}}$ form a local coordinate system at P and for any pair of distinct points P and P' there exist s_{ℓ_0} and s_{ℓ_1} so that the meromorphic function $\frac{s_{\ell_1}}{s_{\ell_0}}$ assumes different values at P and P'. This concludes the proof of the embedding theorem of Kodaira.

§5. *Compact quotients of bounded domains.*

As an example of the applications of the embedding theorem of Kodaira we are going to prove that a compact quotient of a bounded Euclidean domain is biholomorphic to a complex submanifold of a complex projective space. Before we prove this statement, let us discuss a little bit about its motivation. In the case of a compact Riemann surface we produce directly meromorphic functions on it by using the Dirichlet principle or the theorem of Riemann-Roch. By the uniformization theorem a compact Riemann surface of genus at least two is the compact quotient of the open unit disc Δ by a discrete group. We can produce meromorphic functions on the Riemann surface by taking quotients of holomorphic 1-forms invariant under the discrete group. One way that can possibly produce an invariant holomorphic

1-form is to take any holomorphic 1-form and then sum over the action of the discrete group on the holomorphic 1-form just like the construction of the Weierstrass $\mathcal{P}$ functions. There is a problem of convergence. However, when instead of holomorphic 1-forms one uses sections of the k^{th} power K_Δ^k of the canonical line bundle K_Δ of Δ for $k \geq 2$, this procedure produces convergent series when one starts with a *bounded* section of K_Δ^k(*i.e.* with bounded coefficients when expressed in terms of $(dz)^k$ with the global coordinate z of $\mathbf{C}$). The reason for the convergence is that an element of the discrete group eventually maps a fundamental domain in Δ arbitrarily close to the boundary of Δ and the Jacobian determinant of an element of the discrete group eventually becomes arbitrary small on the fundamental domain. Since the k^{th} power of the Jacobian determinant is involved in the action of the discrete group on sections of K_Δ^k, the higher k is the better the convergence of the series. It turns out that k at least two would guarantee convergence. Such convergnet series are called Poincaré series. It is by means obvious and takes some work to prove that the Poincaré series for a sufficiently large k would produce enough meromorphic functions to embed the Riemann surface as a complex curve of some complex projective space. The application of Kodaira's theorem to compact quotients of bounded Euclidean domain is to answer the question whether there are enough invariant holomorphic sections of high powers of the canonical bundle of the bounded Euclidean domain to embed the compact quotient as a complex submanifold of some complex projective space.

Let Ω be a bounded domain in $\mathbf{C}^n$ and it is the covering space of some compact complex manifold M. We would like to produce sufficiently many elements of $\Gamma(M,\, K_M^k)$ for some large k to embed M as a complex submanifold of some complex projective space. By Kodaira's embedding theorem it suffices to show that K_M admits a Hermitian metric with positive curvature. It is the same as the existence of a Hermitian metric of K_Ω invariant under the fundamental group of M whose curvature is positive. We actually will produce such a Hermitian metric invariant under the full automorphism group of Ω. Recall that one way to produce a Hermitian metric of a line bundle is to use $\left(\sum_i |f_i|^2\right)^{-1}$ where f_i is a holomorphic section of the line bundle. We are going to use this procedure, but we will use an infinite number of sections f_i.

For two elements f and g of $\Gamma(M,\, K_M)$ we can form the global inner product $(f, g) = (\sqrt{-1})^{n(n+2)} \int_\Omega f \wedge \bar{g}$. The set of all elements of $\Gamma(M,\, K_M)$

with finite number with respect to this inner product form a Hilbert space and we take an orthonormal basis $f_1, f_2, \cdots$ of this Hilbert space. Let $K(z, \bar{w}) = \sum_{i=1}^{\infty} f_i(z)\overline{f_i(w)}$. Our Hermitian metric for K_M will be $K(z, \bar{z})^{-1}$. First we must prove convergence. The partial sum $\sum_{i=p}^{q} f_i(z)\overline{f_i(w)}$ is holomorphic on $\Omega \times \bar{\Omega}$, where $\bar{\Omega}$ means the complex manifold which is the complex conjugate of Ω. By the Schwarz's inequality

$$\left|\sum_{i=p}^{q} f_i(z)\overline{f_i(w)}\right|^2 \leq \left(\sum_{i=p}^{q} f_i(z)\overline{f_i(z)}\right)\left(\sum_{i=p}^{q} f_i(w)\overline{f_i(w)}\right)$$

it suffices to show that $\sum_{i=p}^{q} f_i(z)\overline{f_i(z)}$ is arbitrarily small uniformly on compact subsets of Ω when p and q are sufficiently large. So one needs only verify that $\sum_{i=1}^{\infty} f_i(z)\overline{f_i(z)}$ is bounded uniformly on compact subsets of Ω. The key point is to use the orthonormality property of $f_1, f_2, \cdots$ and the sub mean value property of $|f|^2$ for holomorphic functions. For the proof of the convergence we temporarily identify f_i with its coefficient of $dz_1 \wedge \cdots \wedge dz_n$. Take a compact subset A of Ω and let the distance from A to the boundary of Ω be $r > 0$. Let $a_1, \cdots, a_q$ be complex numbers and let $f = \sum_{i=1}^{q} a_i f_i$. By the sub mean value property of the subharmonic function $|f|^2$, the value of $|f|^2$ at a point P of A is dominated by the average of the value of $|f|^2$ over the ball of radius r centered at P. So

$$|\sum_{i=1}^{q} a_i f_i(P)|^2 = |f(P)|^2 \leq n!(\pi r^2)^{-n} \int_{\Omega} |f|^2 = n!(\pi r^2)^{-n} \left(\sum_{i=1}^{q} |a_i|^2\right).$$

The trick is to set $a_i = \overline{f_i(P)}$. Thus $\sum_{i=1}^{q} |f_i(P)|^2 \leq n!(\pi r^2)^{-n}$. By letting $q \to \infty$ we conclude that $\sum_{i=1}^{\infty} f_i(z)\overline{f_i(z)}$ is bounded uniformly on compact subsets of Ω.

We want to show that $K(z, \bar{w})$ is independent of the choice of the orthonormal basis $f_1, f_2, \cdots$. Clearly $f(z) = (-1)^n \int_{w \in \Omega} K(z, \bar{w}) \wedge f(w)$ for any square integral holomorphic n-form f on Ω. Suppose we have chosen another orthonormal basis and get another $\tilde{K}(z, \bar{w})$. Then $\int_{w \in \Omega}(K(z, \bar{w}) - \tilde{K}(z, \bar{w})) \wedge f(w)$ vanishes for all square integral holomorphic n-form f on Ω. From the convergence we have proved we know that for any fixed z both $\overline{K(z, \bar{w})}$ and $\overline{\tilde{K}(z, \bar{w})}$ are square integral holomorphic n-forms on Ω for the variable w. Thus the vanishing of $\int_{w \in \Omega}(K(z, \bar{w}) - \tilde{K}(z, \bar{w})) \wedge f(w)$ vanishes for all square integral holomorphic n-form f on Ω implies that $K(z, \bar{w}) - \tilde{K}(z, \bar{w})$ is identically zero in w and $K(z, \bar{w})$ is independent of the choice of the orthonormal basis. The form $K(z, \bar{w})$ on $\Omega \times \bar{\Omega}$ is know as the Bergman kernel. Since $K(z, \bar{w})$ is independent of the choice of the orthonormal basis, we know that it is invariant under the action of the automorphism group of Ω on $\Omega \times \bar{\Omega}$. Hence $K(z, \bar{z})^{-1}$ defines a Hermitian metric along the fibers of K_M. To show that the curvature of the Hermitian metric $K(z, \bar{z})^{-1}$ is suffices to show that $\partial\bar{\partial} \log \left(\sum_i f_i \overline{f_i}\right)$ is positive definite. We have done this computation before for a finite sum. Since we have convergence, the same computation works for the infinite sum as well. To show positive definiteness it suffices to show that for any point P of Ω there exist $f_{j_0}, f_{j_1}, \cdots, f_{j_n}$ so that f_{j_0} is nonzero at P and $\frac{f_{j_1}}{f_{j_0}}, \cdots, \frac{f_{j_1}}{f_{j_0}}$ is a local coordinate system at P. When we select our orthonormal basis we start with the holomorphic n-forms $dz_1 \wedge \cdots \wedge dz_n$, $z_\nu dz_1 \wedge \cdots \wedge dz_n (1 \leq \nu \leq n)$. These are square integrable because Ω is bounded. We now use the Gramm-Schmidt process and get $f_\mu = (c_{\mu 0} + \sum_{\nu=1}^{\mu} c_{\mu\nu} z_\nu) dz_1 \wedge \cdots \wedge dz_n (0 \leq \mu \leq n)$ for some constants with $c_{\mu\mu} \neq 0$. Then we can use $j_\mu = \mu$ for $0 \leq \mu \leq n$ and get the positivity of the curvature form. The metric on Ω whose Kähler form is $\partial\bar{\partial} \log \left(\sum_i f_i \overline{f_i}\right)$ is called the Bergman metric of Ω.

§6. *Lefschetz theorem of hyperplane sections.*

As another application of vanishing theorems, we now apply the vanishing theorem of Akizuki-Nakano to get the Lefschetz theorem for hyperplane sections. Suppose M is a compact complex manifold of complex dimension n and H is an ample divisor. So $[H]$ is an ample line bundle. By the vanishing theorem of Akizuki-Nakano we have $H^q(M, \Omega^p \otimes [H]^{-1}) = 0$ for $p + q < n$. Consider the exact sequence

$$0 \to \mathcal{O}_M(\Omega_M^p \otimes [H]^{-1}) \xrightarrow{\varphi} \mathcal{O}_M(\Omega_M^p) \to \mathcal{O}_H(\Omega_M^p | H) \to 0$$

where φ is defined by multiplication by the canonical section of $[H]$. Then from the long cohomology exact sequence we have the exact sequence

$$H^q(M, \Omega_M^p \otimes [H]^{-1}) \to H^q(M, \Omega_M^p) \xrightarrow{\psi} H^q(H, \Omega_M^p | H) \to H^{q+1}(M, \Omega_M^p \otimes [H]^{-1}).$$

Thus ψ is injective for $p + q < n$ and bijective for $p + q < n - 1$. Now we would like to relate $H^q(H, \Omega_M^p | H)$ to $H^q(H, \Omega_H^p)$. We have on H the exact sequence

$$0 \to T_H \to T_M \to N_{H,M} \to 0$$

where T_H, T_M, $N_{H,M}$ are respectively the tangent bundle of H, of M, and the normal bundle of H in M. Taking the duals of the bundles in the above sequence, we get

(*)
$$0 \to N^*_{H,M} \to \Omega^1_M \to \Omega^1_H \to 0.$$

Let us first identify the conormal bundle $N^*_{H,M}$ of H in M. It consists of all (1,0)-forms of M on H whose pullback to H under the inclusion map $H \subset M$ yield the zero (1,0)-form. Choose local holomorphic coordinates $z_1^{(\nu)}, \cdots, z_n^{(\nu)}$ on the coordinate chart U_ν of M so that H is defined by $z_n^{(\nu)} = 0$. Every (1,0)-form of M on H is given by $\sum_{i=1}^n f_i^{(\nu)} dz_i^{(\nu)}$ with $f_i^{(\nu)}$ defined on H. Its pullback to H under the inclusion map $H \subset M$ gives the zero (1,0)-form if and only if $f_i^{(\nu)} = 0$ on H for $1 \leq i \leq n-1$. So the local sections of $N^*_{H,M}$ is given by the function $f_n^{(\nu)}$. The transition function from $U_\nu \cap H$ to $U_\mu \cap H$ is given by $\frac{f_n^{(\mu)}}{f_n^{(\nu)}}$. We can expand $z_n^{(\mu)}$ as a power series in $z_n^{(\nu)}$ and get $z_n^{(\mu)} = F_{\mu\nu} z_n^{(\nu)}$ for some holomorphic function $F_{\mu\nu}$ in the variables in $z_1^{(\mu)}, \cdots, z_n^{(\mu)}$. Then $dz_n^{(\mu)} = F_{\mu\nu} dz_n^{(\nu)}$ on H and $\frac{f_n^{(\nu)}}{f_n^{(\mu)}} = F_{\mu\nu} = \frac{z_n^{(\mu)}}{z_n^{(\nu)}}$. This means that the line bundle $N^*_{H,M}$ agrees with the line bundle $[H]^{-1}$. We now obtain the following exact sequence by taking exterior powers in $(*)$:

$$(\dagger) \qquad 0 \to N^*_{H,M} \otimes \Omega^{p-1}_M \to \Omega^p_M \to \Omega^p_H \to 0.$$

To see the exactness of (†), we simplify notations and consider the exact sequence
$$0 \to L \to E \xrightarrow{\alpha} F \to 0$$
of vector bundles so that L is a line bundle. The map $\wedge^p E \to \wedge^p F$ is clearly well-defined. We want to define the map $L \otimes \wedge^{p-1} F$. Take an element $\sum f_{i_1} \wedge \cdots \wedge f_{i_{p-1}}$ of $\wedge^{p-1} F$. We lift it up to $\sum e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}$ with $\alpha(e_{i_\nu}) = f_{i_\nu}$. Clearly the difference of two different liftings would give us an element of the form $L \wedge (\wedge^{p-1} E)$ in E. Since L is rank one, the difference wedged with an element of L would give us a zero element in $\wedge^p E$. So the map from $L \otimes \wedge^{p-1} F$ to $\wedge^p E$ defined by mapping $\ell \otimes (\sum f_{i_1} \wedge \cdots \wedge f_{i_{p-1}})$ to $\ell \wedge \sum e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}$ is well-defined.

$$0 \to L \otimes \wedge^{p-1} F \to \wedge^p E \to \wedge^p F \to 0.$$

The exactness is clear from using a splitting $E_x = L_x \oplus F_x$ at fibers over a point x. From the exact sequence (†) we have the exactness of

$$H^q(H, N^*_{H,M} \otimes \Omega^{p-1}_M) \to H^q(H, \Omega^p_M) \to H^q(H, \Omega^p_H) \to H^{q+1}(H, N^*_{H,M} \otimes \Omega^{p-1}_M).$$

Since the line bundle $N^*_{H,M} = [H]^{-1}$ is negative, it follows from the vanishing theorem of Akizuki-Nakano that $H^q(H, \Omega^p_M) \to H^q(H, \Omega^p_H)$ is injective for $p + q - 1 < n - 1$ and bijective for $p + q < n - 1$. Combining this with our previous statement concerning ψ we have the bijectivity of $H^q(M, \Omega^p_M) \to H^q(H, \Omega^p_H)$ for $p+q < n-1$ and the injectivity of $H^q(M, \Omega^p_M) \to H^q(H, \Omega^p_H)$ for $p + q < n$. By the Hodge decomposition we have the bijectivity of $H^k(M, \mathbf{C}) \to H^k(H, \mathbf{C})$ for $k < n - 1$ and the injectivity of $H^k(M, \mathbf{C}) \to H^k(H, \mathbf{C})$ for $k < n$. This is a rather weak form of the theorem of Lefschetz. It relates the topology of the n-dimensional compact complex manifold M to the topology of the manifold H which is one complex dimension lower.

Originally this was used for a complex submanifold M of some complex projective space $\mathbf{P}_N$ and H is the hypersurface of M cut out by a hyperplane of $\mathbf{P}_N$. There is an alternative proof of the Lefschetz theorem using Morse theory and one gets the corresponding conclusion for the map $H^k(M, \mathbf{Z}) \to H^k(H, \mathbf{Z})$ instead of the map $H^k(M, \mathbf{C}) \to H^k(H, \mathbf{C})$. Let us quickly sketch this proof by Morse theory. The part $M - H$ is a closed complex submanifold of the complex Euclidean space $\mathbf{C}^N$. On $\mathbf{C}^N$ we take the distance square function $|z|^2$ from the origin. Let φ be the function on $M - H$ which is the restriction of the function $-|z|^2$. By a good choice of the origin the function φ is a Morse function in the sense that its Hessian is nondegenerate at its critical points. Let us calculate the number of negative eigenvalues of the Hessian of φ. We claim that the number is at least n. Since the Hessian at a critical point is nondegenerate, the Hessian at a sufficiently close neighboring point has the same number of negative eigenvalues. So in our computation we can assume that the first derivative of the function $-\varphi$ is nonzero at a point P of $M - H$. We claim that we can choose a local holomorphic coordinate system $\zeta_1, \cdots, \zeta_n$ of M centered at P so that $\frac{\partial^2 \varphi}{\partial \zeta_\mu \partial \zeta_\nu}$ vanishes at P for all $1 \leq \mu, \nu \leq n$. We first choose a local holomorphic coordinate system $w_1, \cdots, w_n$ of M centered at P so that $\partial \varphi = dw_1$. We define a new local holomorphic coordinate system $\zeta_1, \cdots, \zeta_n$ so that $w_1 = \zeta_1 - \frac{1}{2} \left(\frac{\partial^2 \varphi}{\partial w_\mu \partial w_\nu} \right)_P \zeta_\mu \zeta_\nu$

and $w_\lambda = \zeta_\lambda$ for $1 < \lambda \leq n$. Then $\left(\frac{\partial^2 w_1}{\partial\zeta_\mu \partial\zeta_\nu}\right)_P = -\left(\frac{\partial^2 \varphi}{\partial w_\mu \partial w_\nu}\right)_P$. Then at P we have

$$\begin{aligned}\frac{\partial^2 \varphi}{\partial\zeta_\mu \partial\zeta_\nu} &= \sum_{\sigma,\tau} \frac{\partial^2 \varphi}{\partial w_\sigma \partial w_\tau} \frac{\partial w_\sigma}{\partial \zeta_\mu} \frac{\partial w_\tau}{\partial \zeta_\nu} + \sum_\lambda \frac{\partial \varphi}{\partial w_\lambda} \frac{\partial^2 w_\lambda}{\partial\zeta_\mu \partial\zeta_\nu} \\ &= \frac{\partial^2 \varphi}{\partial w_\mu \partial w_\nu} + \frac{\partial^2 w_1}{\partial\zeta_\mu \partial\zeta_\nu} = 0.\end{aligned}$$

The function $-\varphi$ is strictly plurisubharmonic, because it is the restriction of the function $|z|^2$. Fix a point P of $M - H$. After a complex linear transformation of the local holomorphic coordinates $\zeta_1, \cdots, \zeta_n$, we can assume in addition that the *complex* Hessian $of - \varphi$ at P is the identity matrix. On each coordinate line with variable $\zeta_\nu = \xi_\nu + \sqrt{-1}\eta_\nu$ we can multiply ζ_ν by a complex number of absolute value 1 and make the *real* Hessian with respect to $\xi_1, \eta_1, \cdots, \xi_n, \eta_n$ in diagonal form so that for each $1 \leq \nu \leq n$ the sum of the two eigenvalues of the real Hessian $of - \varphi$ in the direction of ξ_ν and in the direction of η_ν is 1. Thus we know that the number of positive eigenvalues of the real Hessian $of - \varphi$ is at least n. Hence at any critical point of φ the number of negative eigenvalues of the real Hessian of φ is at least n. Now we compare the homotopy type of $M_c = \{P \in M | \varphi(P) \leq c\}$ for different values of c. We start with the c which is the maximum of φ on M. When we decrease c and get pass a critical value c of φ we have to consider the attaching of a cell whose real dimension equals the number of negative eigenvalues of φ at that critical point. The real dimensions of these cells are at least n. When c is a very large negative number, H becomes a deformation retract of M_c. Thus $\pi_\nu(M, H)$ vanishes for $\nu < n$. Hence we have the surjectivity of $\pi_\nu(H) \to \pi_\nu(M)$ for $\nu < n$ and the bijectivity of $\pi_\nu(H) \to \pi_\nu(M)$ for $\nu < n - 1$. By the Hurewicz isomorphism theorem which says that the vanishing of $\pi_\nu(M, H)$ for $\nu < p$ implies the isomorphism of $\pi_p(M, H)$ and $H_p(M, H; \mathbf{Z})$, we conclude that $H_\nu(M, H; \mathbf{Z})$ vanishes for $\nu < p$ and $H_\nu(H, \mathbf{Z}) \to H_\nu(M, \mathbf{Z})$ is surjective for $\nu < n$ and bijective for $\nu < n - 1$.

§7. *Characterization of Hodge manifolds.*

Suppose $\frac{\sqrt{-1}}{2\pi}\varphi_{\alpha\bar\beta} dz^\alpha \wedge \overline{dz^\beta}$ is a positive closed (1,1)-form on a compact complex manifold M and $\frac{\sqrt{-1}}{2\pi}\varphi_{\alpha\bar\beta} dz^\alpha \wedge \overline{dz^\beta}$ is of integral type in the sense that the class defined by $\frac{\sqrt{-1}}{2\pi}\varphi_{\alpha\bar\beta} dz^\alpha \wedge \overline{dz^\beta}$ in $H^2(M, \mathbf{C})$ is in the image of $H^2(M, \mathbf{Z}) \to H^2(M, \mathbf{C})$. We want to show that M admits a positive line bundle whose Chern form is $\frac{\sqrt{-1}}{2\pi}\varphi_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\overline{z^{\beta}}$. On any member U_i of a simple open covering of M we can write $\varphi_{\alpha\bar{\beta}} = \partial_{\alpha}\partial_{\bar{\beta}}\log h_i$ with $\log h_i$ real plurisubharmonic. On $U_i \cap U_j$ we have $\log h_i - \log h_j = 2\mathrm{Re}\ g_{ij}$ with g_{ij} holomorphic on $U_i \cap U_j$. Then in the complex used to prove the isomorphism between the DeRham and the Cech cohomologies we have

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 & & & & \varphi_{\alpha\bar{\beta}} \\
 & & & & \downarrow \\
 & & -\partial\log h_i & \rightarrow & \partial\bar{\partial}\log h_i \\
 & & \bar{\partial}\downarrow & & \\
\log g_{ij} & \stackrel{d}{\rightarrow} & \partial\log g_{ij} & & \\
\delta\downarrow & & & & \\
c_{ijk} & & & & .
\end{array}
$$

Here $c_{ijk} = \log g_{ij} + \log g_{jk} + \log g_{ki}$ and $\frac{\sqrt{-1}}{2\pi}c_{ijk}$ defines an element of $H^2(M, \mathbf{Z})$. So we conclude that $\frac{\sqrt{-1}}{2\pi}(\log g_{ij} + \log g_{jk} + \log g_{ki})$ is an integer.

§9. *Morrey's Trick to Handle the Boundary Term*

We discussed the Bochner-Kodaira formula for $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, which is a way of rewriting $\square$ as a nonnegative differential operator and a zeroth-order term involving curvature. There are three ways of writing the term of a nonnegative differential operator.

(i) One is $-\mathrm{tr}\nabla\bar{\nabla}$, which is used for the proof of the vanishing theorem of Kodaira in the case of positive curvature.

(ii) The second one is $-\mathrm{tr}\bar{\nabla}\nabla$, which is used for the proof of the vanishing theorem of Kodaira in the case of negative curvature.

(iii) The third one is $\bar{\square}$, which is used for the proof of the vanishing theorem of Akizuki-Nakano (in the case of negative curvature).

All these three formulas can apply only to compact complex manifolds to get vanishing theorems, because integration by parts is needed for converting

$$(\square\varphi, \varphi)\,, \ \left(\left(-\mathrm{tr}\nabla\bar{\nabla}\right)\varphi, \varphi\right), \ \left(\left(-\mathrm{tr}\bar{\nabla}\nabla\right)\varphi, \varphi\right)$$

to

$$\left\|\bar{\partial}\varphi\right\|^2 + \left\|\bar{\partial}^*\varphi\right\|^2, \quad \|\nabla\varphi\|^2, \quad \|\nabla\varphi\|^2$$

respectively in order to get the final conclusions for the three kinds of vanishing theorems. For application to the solvability of the $\bar{\partial}$-equation on non-compact manifolds, we need to handle the boundary terms arising from the above integration by parts. We need to get a fixed sign for the boundary term. This is handled by what is known as the Morrey trick. Let us first discuss this Morrey trick in the simplest case of a domain in the complex Euclidean space and a $(0,1)$-form.

When we have a domain D in $\mathbf{C}^n$ defined by $r < 0$ so that dr is of unit length at every point of ∂D. First of all we would like to get the condition for a $(0,1)$-form φ smooth on D up to ∂D to be in the domain of the actual adjoint of $\bar{\partial}$. Let $\varphi = \sum_{j=1}^n \varphi_j dz^j$, where $z^1, \cdots, z^n$ are the coordinates of $\mathbf{C}^n$.

Take any smooth function ψ on D up to ∂D. In order for φ to be in the domain of $\bar{\partial}^*$ we need

$$(*) \qquad \left(\varphi, \bar{\partial}\psi\right)_D = \left(\bar{\partial}^*\varphi, \psi\right)_D$$

for some function $\bar{\partial}^*\varphi$, where $(\cdot,\cdot)_D$ means the inner product of the usual L^2 norm on D. When the support of ψ is in D, we conclude from the above condition that the function $\bar{\partial}^*\varphi$ must equal the image of φ under the formal adjoint of $\bar{\partial}$. When the support of ψ is not in D, the condition $(*)$ gives the vanishing of the boundary term

$$\int_{\partial D} \sum_j \left(\partial_j r\right) \varphi_j.$$

The boundary term is derived from the usual divergence theorem as follows. The usual divergence theorem gives

$$\int_D \frac{\partial}{\partial x_j} F = \int_{\partial D} \left(\frac{\partial r}{\partial x_j}\right) F$$

for real coordinates x_j and any smooth function F , which, over $\mathbf{C}$-linear combinations, gives

$$\int_D \partial_j F = \int_{\partial D} \left(\partial_j r\right) F,$$
$$\int_D \bar{\partial}_j F = \int_{\partial D} \left(\bar{\partial}_j r\right) F.$$

The vanishing of

$$\int_{\partial D} \sum_j (\partial_j r) \, \varphi_j$$

for all smooth ψ smooth on D up to ∂D means that

$$(\&) \qquad \sum_j (\partial_j r) \, \varphi_j = 0 \quad \text{on} \quad \partial D.$$

This is the condition for a $(0,1)$-form φ smooth on D up to ∂D to belong to the domain of $\bar\partial^*$.

To use the Bochner formula for $\Box\varphi$ involving $-\mathrm{tr}\nabla\bar\nabla$ to solve the $\bar\partial$-equation on D for $(0,1)$-forms, we have to handle the boundary terms from the process of integration by parts to transform

$$(\bar\partial^*\varphi, \bar\partial^*\varphi)_D + (\bar\partial\varphi, \bar\partial\varphi)_D - (\bar\nabla\varphi, \bar\nabla\varphi)_D$$

to

$$\left((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\varphi, \varphi\right)_D + \left(\left(\mathrm{tr}\,\nabla\bar\nabla\right)\varphi, \varphi\right)_D.$$

Since the term

$$(\bar\partial^*\varphi, \bar\partial^*\varphi)_D$$

does not yield any boundary term when we apply Stokes' theorem, we need only consider the transformation from

$$\left(\bar\partial\varphi, \bar\partial\varphi\right)_D - \left(\bar\nabla\varphi, \bar\nabla\varphi\right)_D$$

to

$$\left(\bar\partial^*\bar\partial\varphi, \varphi\right)_D + \left(\left(\mathrm{tr}\,\nabla\bar\nabla\right)\varphi, \varphi\right)_D.$$

We have

$$(\bar\partial\varphi, \bar\partial\varphi)_D = \int_D \sum_{i<j} |\bar\partial_i\varphi_j - \bar\partial_j\varphi_i|^2 = \int_D \sum_{i,j} \left(|\bar\partial_i\varphi_j|^2 - (\bar\partial_i\varphi_j)(\partial_j\bar\varphi_i)\right)$$

or

$$(\bar\partial\varphi, \bar\partial\varphi)_D - (\bar\nabla\varphi, \bar\nabla\varphi)_D = -\int_D \sum_{i,j} (\bar\partial_i\varphi_j)(\partial_j\bar\varphi_i).$$

The boundary term is

$$-\int_{\partial D} \sum_{i,j} (\bar\partial_i r)\varphi_j\partial_j\bar\varphi_i.$$

Thus

$$\int_D \bar{\partial}_k \left(\sum_{i,j} \varphi_j \partial_j \bar{\varphi}_i \right) = \int_{\partial D} \sum_{i,j} (\bar{\partial}_i r) \left(\varphi_j \partial_j \bar{\varphi}_i \right).$$

Now we apply the Morrey trick to the boundary term

$$-\int_{\partial D} \sum_{i,j} (\bar{\partial}_i r) \varphi_j \partial_j \bar{\varphi}_i.$$

The Morrey trick will perform the task of integration by parts for only the two factors $(\bar{\partial}_i r)\partial_j \bar{\varphi}_i$ so that the result will be $-\left(\partial_j \bar{\partial}_i r\right) \bar{\varphi}_i$. Now integration by parts simply comes from applying the differentiation to the product of the two factors. In our case it means applying ∂_j to $\left(\bar{\partial}_i r\right) \bar{\varphi}_i$. We will actually do it to the expression with summation over i. So we apply ∂_j to $\sum_i \left(\bar{\partial}_i r\right) \bar{\varphi}_i$. We will use the condition $\sum_i \left(\bar{\partial}_i r\right) \bar{\varphi}_i \equiv 0$ on ∂D due to $\varphi \in \operatorname{Dom} \bar{\partial}^*$ from (&). Since we have to apply ∂_j which may not be tangential to ∂D, we transform the condition $\sum_i \left(\bar{\partial}_i r\right) \bar{\varphi}_i \equiv 0$ on ∂D to a condition on some neighborhood of ∂D. We rewrite it as

$$\sum_i (\bar{\partial}_i r) \bar{\varphi}_i = \lambda r$$

for some smooth function λ on some open neighborhood of ∂D. Now apply ∂_j to get

$$\partial_j \sum_i (\bar{\partial}_i r) \bar{\varphi}_i = (\partial_j \lambda)\, r + \lambda \partial_j r$$

on an open neighborhood of ∂D, which on ∂D gives

$$(\dagger) \qquad \partial_j \sum_i (\bar{\partial}_i r) \bar{\varphi}_i = \lambda \partial_j r$$

because of the vanishing of r on ∂D. Now we use the condition $\varphi \in \operatorname{Dom} \bar{\partial}^*$ once again by applying $\sum_j \varphi_j$ to $(\dagger)$ to get

$$\sum_j \varphi_j \left(\partial_j \sum_i (\bar{\partial}_i r) \bar{\varphi}_i \right) = \lambda \left(\sum_j \varphi_j \partial_j r \right) = 0$$

on ∂D, because $\sum_j \varphi_j \partial_j r = 0$ on ∂D from (&). This means that

$$\sum_{i,j} (\bar{\partial}_i r) \varphi_j \partial_j \bar{\varphi}_i = -\sum_{i,j} (\partial_j \bar{\partial}_i r) \varphi_j \bar{\varphi}_i$$

on ∂D. Thus the boundary term is

$$\int_{\partial D} \sum_{i,j} (\partial_j \bar{\partial}_i r) \varphi_j \bar{\varphi}_i.$$

The condition $\varphi \in \operatorname{Dom} \bar{\partial}^*$ means that the component of φ which is normal to ∂D must be zero on ∂D. It means precisely $\sum_j \varphi_j \partial_j r = 0$ on ∂D. In order to make sure that the boundary term

$$\int_{\partial D} \sum_{i,j} (\partial_j \bar{\partial}_i r) \varphi_j \bar{\varphi}_i$$

is nonnegative, we introduce the condition that $\partial_j \bar{\partial}_i r$ is nonnegative as a Hermitian quadratic form on $T^{(1,0)}_{\partial D}$. This condition is called the *pseudoconvexity* of ∂D.

For the case of a domain D in $\mathbf{C}^n$, in our computation of the boundary term, since no commutation of covariant differentiation is used, the curvature of the background manifold does not occur. We can put in the Kähler metric and repeat the same computation. Recall that in the case of a compact Kähler manifold M we have

$$(\Box \varphi, \varphi)_M = \|\bar{\nabla} \varphi\|^2_M + (\operatorname{Ric} \varphi, \varphi)_M + (\Omega \varphi, \varphi)_M,$$

where

$$\begin{aligned}
\|\bar{\nabla} \varphi\|^2_M &= \int_M g^{s\bar{t}} \nabla_{\bar{t}} \varphi^{\alpha}{}_{\bar{j}_1 \cdots \bar{j}_q} \overline{\nabla_{\bar{s}} \varphi_{\bar{\alpha}}{}^{j_1 \cdots j_q}}, \\
(\operatorname{Ric} \varphi, \varphi)_M &= \int_M \sum_{\nu=1}^{q} R_{\bar{j}_\nu}{}^{\bar{b}} \varphi^{\alpha}{}_{\bar{j}_1 \cdots (\bar{b})_\nu \cdots j_q} \overline{\varphi_{\bar{\alpha}}{}^{j_1 \cdots j_q}} \\
(\Omega \varphi, \varphi)_M &= \int_M \sum_{\nu=1}^{q} \Omega^{\bar{b}}{}_{\bar{j}_\nu} \varphi^{\alpha}{}_{\bar{j}_1 \cdots (\bar{b})_\nu \cdots \bar{j}_q} \overline{\varphi_{\bar{\alpha}}{}^{j_1 \cdots j_q}}.
\end{aligned}$$

When we have a domain D in M defined by $r < 0$ so that dr is of unit length at every point of ∂D, then

$$(\bar{\partial} \varphi, \bar{\partial} \varphi)_D + (\bar{\partial}^* \varphi, \bar{\partial}^* \varphi)_D = \int_{\partial D} \langle \operatorname{Levi}(r) \varphi, \varphi \rangle + \|\bar{\nabla} \varphi\|^2_D + (\operatorname{Ric} \varphi, \varphi)_D + (\Omega \varphi, \varphi)_D,$$

where $\operatorname{Levi}(r) = \partial \bar{\partial} r$ and

$$\langle \text{Levi}_r\varphi, \varphi\rangle = \sum_{\nu=1}^{q} g^{s\bar{t}} \left(\partial_s \partial_{\bar{j}_\nu} r\right) \varphi^{\alpha}{}_{\bar{j}_1\cdots(\bar{t})_\nu\cdots\bar{j}_q} \overline{\varphi_{\bar{\alpha}}{}^{j_1\cdots j_q}}.$$

§10. *Hilbert Space Technique*

We use the standard technique of using functional analysis and Hilbert spaces to solve the $\overline{\partial}$ equation. Denote $\overline{\partial}$ by T and the $\overline{\partial}$ in the next step of the Dolbeault complex by S. Given g with $Sg = 0$ we would like to solve the equation $Tu = g$. The equation $Tu = g$ is equivalent to $(v, Tu) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$, which means $(T^*v, u) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. To get a solution u it suffices to prove that the map $T^*v \to (v, g)$ can be extended to a bounded linear functional, which means that $|(v, g)| \leq C\|T^*v\|$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. In that case we can solve the equation $Tu = g$ with $\|u\| \leq C$. We could also use the equivalent inequality

$$|(v, g)|^2 \leq C^2(\|T^*v\|^2 + \|Sv\|^2)$$

for all $v \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$. Suppose we have the following weaker inequality

$$\|v\|^2 \leq A^2 \left(\|T^*v\|^2 + \|Sv\|^2\right)$$

for some positive constant A. Then we can use the Schwarz inequality and get

$$|(v, g)|^2 \leq \|v\|^2 \, \|g\|^2 \leq A^2\|g\|^2(\|T^*v\|^2 + \|Sv\|^2).$$

We conclude that the equation $Tu = g$ can be solved with $\|u\| \leq A\|g\|$.

§11. *Density in the Graph Norm.*

We take a moving frame for $(0, 1)$-forms so that $\overline{\omega_1}, \cdots, \overline{\omega_{n-1}}$ are along the boundary of D and $\overline{\omega_n}$ is perpendicular to the boundary of D. These are smooth $(0, 1)$-forms.

We now locally straighten out the boundary of D so that D is defined by $x_{2n} < 0$, where $x_1, \cdots, x_{2n}$ are smooth local coordinates unrelated to the real and imaginary parts of holomorphic local coordinates.

We apply a partition of unity to φ so that the support of φ is contained in a compact subset of $U \cap \bar{D}$ for some small open neighborhood U in which ∂D has been straightened.

We are going to smooth out

$$\varphi = \sum_{\nu=1}^{n} \varphi_{\bar{\nu}} \overline{\omega_\nu}$$

coefficientwise. More precisely, for $\varphi_{\bar{n}}$ we are going to trivially extend $\varphi_{\bar{n}}$ to the full neighborhood U and then move the graph of the trivial extension along x_{2n} from the right to the left. We end up with a smoothing which is zero on some neighborhood of the boundary. This guarantees that the smoothing again belongs to the domain of $\bar{\partial}^*$. Since $\varphi_{\bar{n}}$ belongs to the domain of $\bar{\partial}^*$, we know that the $\bar{\partial}$ of the trivial extension is equal to the trivial extension of $\bar{\partial}$. This means that we have approximation in the graph norm.

For the other coefficients $\varphi_{\bar{1}}, \cdots, \varphi_{\overline{n-1}}$ we are going to smooth by moving their graphs from the left to the right, to make sure that the smoothing is an approximation in the graph norm. If we move the graphs of $\varphi_{\bar{1}}, \cdots, \varphi_{\overline{n-1}}$ from the right to the left, we may not have the approximation in the graph norm, because $\bar{\partial}$ of the trivial extension is not equal to the trivial extension of $\bar{\partial}$.

Chapter 5. The Fujita Conjecture and the Extension Theorem of Ohsawa-Takegoshi

§1. Introduction and Statement of Results

Let L be an ample line bundle over a compact complex manifold X of complex dimension n. We discuss here the most recent result of myself and Angehrn [AS94] on the conjecture of Fujita [F87] on freeness. Fujita's conjecture states that $(n+1)L + K_X$ is free. The conjecture of Fujita has a second part on very ampleness which states that $(n+2)L+K_X$ is very ample. We will confine ourselves to the freeness part of the Fujita conjecture. The case $n = 1$ is well-known and the case $n = 2$ was proved by Reider [R88]. Ein-Lazarsfeld [EL93] proved the freeness part for $n = 3$. My result with Angehrn [AS94] is the following.

Main Theorem. Let κ be a positive number. If $(L^d \cdot W)^{\frac{1}{d}} \geq \frac{1}{2}n(n+2r-1)+\kappa$ for any irreducible subvariety W of dimension $1 \leq d \leq n$ in X, then the global holomorphic sections of $L + K_X$ over X separate any set of r distinct points $P_1, \cdots, P_r$ of X. In other words, the restriction map $\Gamma(X, L) \to \oplus_{\nu=1}^{r} \mathcal{O}_X / \mathbf{m}_{P_\nu}$ is surjective, where $\mathbf{m}_{P_\nu}$ is the maximum ideal at P_ν.

Corollary. $mL + K_X$ is free for $m \geq \frac{1}{2}(n^2 + n + 2)$.

To avoid distracting technical details, we will discuss here the proof of the Corollary to the Main Theorem instead of the Main Theorem itself. Besides the routine part of the proof, there is one important new ingredient in the proof. The routine part of the proof uses the well-known techniques of the theorem of Riemann-Roch and the vanishing theorem of Nadel for multiplier ideal sheaves. The most important ingredient in the proof is the new technique of the semi-continuity of the multiplier ideal sheaf which is a consequence of the extension theorem of Ohsawa-Takegoshi for L^2 holomorphic functions [OT87]. This new technique of the semi-continuity of the multiplier ideal sheaf solves the difficulty that has been, until now, the only obstacle to obtaining, for general dimension, an effective bound m so that $mL + K_X$ is free. Ein and Lazarsfeld pointed out that there are already encountered insurmountable difficulties in the method to improve the quadratic bound even in the case of dimension 3 and a divisor occurring in the first step with quadratic conic singularities. This example of Ein and Lazarsfeld of a surface with quadratic conic singularities in dimension 3 will be given in §6 where we will discuss of the difficulty involved in improving the quadratic bound of m to the linear bound $m \geq n+1$ conjectured by the freeness part of the Fujita conjecture.

The extension theorem of Ohsawa-Takegoshi needed in the new technique requires heavy analysis. After learning of this new technique of the semi-continuity of the multiplier ideal sheaf, Kollar [K94] came up with an algebraic proof of the semi-continuity of the multiplier ideal sheaf by using the inversion of adjunction (instead of the extension theorem of Ohsawa-Takegoshi for L^2 holomorphic functions). Kollar's algebraic proof enabled him to generalize the Corollary to the Main Theorem to the case where X may have log terminal singularities.

The known proof of the extension theorem of Ohsawa-Takegoshi for L^2 holomorphic functions uses a long series of commutation identities in Kähler geometry and some specially constructed complete Kähler metrics. The usual L^2 estimates of $\overline{\partial}$ is not sufficient to give a proof of it. We present here a simpler and clearer proof of the theorem of Ohsawa-Takegoshi and explains the new methods needed and why the usual L^2 estimates of $\overline{\partial}$ is insufficient. We also explain, in terms of vanishing theorems in an analytic setting, Kollar's algebraic proof of the semi-continuity of the multiplier ideal sheaf by the inversion of adjunction. I am indebted to Kawamata for patiently explaining to me the meanings of a host of related terms, Kawamata log terminal, log canonical, etc. in order for me to understand Kollar's algebraic proof.

§2. Multiplier Ideal Sheaves and the Induction Argument

Fix P_0 in X. We now explain the induction argument needed to produce a singular metric for mL so that the vanishing theorem for its multiplier ideal sheaf would give the existence of a global holomorphic section of $mL + K_X$ over X which is nonzero at P_0. In order to make the explanation easier to understand, we do the first couple of steps before we give the induction statement.

First we fix some terminology and introduce Nadel's vanishing theorem for multiplier ideal sheaves ([N89],[D93]). For a positive rational number α, by a multivalued holomorphic section s of αL we mean that, for some positive integer p, $p\alpha$ is a positive integer and s^p is a global holomorphic section of $p\alpha L$. We say that the vanishing order of s at P_0 is q if the vanishing order of s^p at P_0 is pq. For a finite number of multivalued holomorphic sections $s_1, \cdots, s_k$ of L we have a (possibly) singular metric which, with respect to some local trivialization of L, is locally given by $(\sum_{\nu=1}^k |s_\nu|^2)^{-1}$.

The vanishing theorem of Nadel states the following. Suppose there is a (possibly singular) metric of L locally given by $e^{-\varphi}$ such that the curvature current $\sqrt{-1}\partial\overline{\partial}\varphi$ dominates a positive definite smooth $(1,1)$-form on X. Let $\mathcal{I}$ be the multiplier ideal sheaf (for the metric) which is defined as the ideal sheaf consisting of all holomorphic function germs f such that $|f|^2 e^{-\varphi}$ is locally integrable. Then $H^k(X, \mathcal{I}(L+K_X))$ vanishes for $k \geq 1$. The algebraic version of Nadel's theorem is the theorem of Kawamata-Viehweg [Ka82,V82], which for our purpose could be used instead of Nadel's theorem.

Let ϵ denote some sufficiently small positive rational number. We will use ϵ as a generic notation for a sufficiently small positive rational number so that even after a small change we keep the same notation ϵ. For the first step we choose by the theorem of Riemann-Roch a multivalued holomorphic section s of $(n+\epsilon)L$ which vanish to order at least n at P_0. There exists a positive rational number α (as one can see for example by resolving the singularity of the divisor of s) such that the zero-set $\tilde{X}_1$ of the multiplier ideal sheaf for the metric $|s|^{-2\alpha}$ contains P_0 but the zero-set of the multiplier ideal sheaf for $|s|^{-2\beta}$ does not contain P_0 for any $\beta < \alpha$. Since L is ample, we can find a finite number of multivalued holomorphic sections $t_1^{(1)}, \cdots, t_{k_1}^{(1)}$ of L over X whose common zero-set is one branch X_1 of $\tilde{X}_1$ containing P_0. Now we choose two suitable sufficiently small positive numbers σ, τ so that, with $s_j^{(1)} = s^{\alpha-\sigma}(t_j^{(1)})^\tau$ $(1 \leq j \leq k_1)$, the zero-set of the multiplier ideal sheaf for the metric $h_1 := (\sum_{j=1}^{k_1} |s_j^{(1)}|^2)^{-1}$ is X_1, and that, for any $\gamma < 1$, the zero-set of the multiplier ideal sheaf for the metric h_1^γ does not contain P_0. The metric h_1 is a metric for $(n+\epsilon_1)L$, where ϵ_1 is a sufficiently small positive rational number obtained by slightly changing ϵ.

Let d_1 be the dimension of X_1 at P_0. The induction process ends if $d_1 = 0$. So we assume that $d_1 > 0$. Now we do the second step which will construct a metric h_2 for $(d_1 + n + \epsilon_2)L$ (for some sufficiently small positive number ϵ_2) so that precisely one branch with dimension $< d_1$ of the zero-set of the multiplier ideal sheaf for h_2 contains P_0. The obstacle mentioned in the introduction is the difficulty encountered in the application of the theorem of Riemann-Roch when X_1 is singular at P_0. Let us first set aside this obstacle by considering the case where P_0 is a regular point of X_1. Then by the theorem of Riemann-Roch we can find a multivalued holomorphic section s of $(d_1+\epsilon)L$ over X_1 which vanishes to order at least d_1 at P_0. We can extend s to a multivalued holomorphic section $\hat{s}$ of $(d_1+\epsilon)L$ over all of X. However, the vanishing order of $\hat{s}$ along the normal direction of X_1 at P_0 may be a very small positive rational number. To make up for the very small vanishing order along the normal direction of X_1 at P_0, for some sufficiently small η we use the metric $\tilde{h}_2 := h_1^{1-\eta}|\hat{s}|^{-2}$ so that the zero-set of its multiplier ideal sheaf contains P_0. Since $\eta > 0$, by the choice of h_1 we know that near P_0 the zero-set $\tilde{X}_2$ of the multiplier ideal sheaf of the metric $\tilde{h}_2$ must be contained in the zero-set of $\hat{s}$.

Now we look at the case when X_1 is singular at P_0. Because we have no control over the multiplicity of X_1 at P_0 and over the nature of the singularity there, we are unable to say that we need only $(d_1 + \epsilon)L$ and not mL for some large m not effectively determinable in order to get a multivalued holomorphic section s over X_1 to construct our singular metric with the desired property for its multiplied ideal sheaf. It is until now an insurmountable obstacle. To overcome this obstacle, we take a local complex curve Δ in X_1 passing through P_0 so that Δ intersects the singular set of X_1 only at P_0. By using the theorem of Riemann-Roch with Δ as the parameter space, we can find a multivalued holomorphic section s_P of $(d_1 + \epsilon)L$ over X_1 depending holomorphically on $P \in \Delta$ such that for $P \in \Delta - P_0$ the vanishing order of s_P at P is at least d_1. We extend each s_P to a multivalued holomorphic section $\hat{s}_P$ of $(d_1 + \epsilon)L$ over all of X so that $\hat{s}_P$ is holomorphic in $P \in \Delta$. Then we get a family of singular metrics $(\tilde{h_2})_P$ $(P \in \Delta)$ so that the zero-set of its multiplier ideal sheaf contains P for $P \in \Delta - P_0$. Here is the key step of the new technique to overcome the obstacle. The holomorphic family of multiplier ideal sheaves parametrized by $P \in \Delta$ enjoys the following semicontinuity property. From the fact that P belongs to the zero-set of the multiplier ideal sheaf of $(\tilde{h_2})_P$ for $P \in \Delta - P_0$, it follows that P_0 belongs to the zero-set of the multiplier ideal sheaf of $(\tilde{h_2})_{P_0}$. We will discuss this semicontinuity property of multiplier ideal sheaves in a later section. With the obstacle overcome, we continue our discussion of the second step in the induction argument with $\hat{s} = \hat{s}_{P_0}$.

At this point, it is still too early to conclude that the dimension of $\tilde{X}_2$ at P_0 is $< d_1$, because the extra vanishing order from $\hat{s}$ may add to some of the pole order of h_1 which earlier was not high enough to contribute to X_1 and thus may prevent us from concluding that $\hat{X}_2$ is contained in X_1. One way to avoid this is to use locally the metric $h_1^{1-\eta}(h_1^{-\rho} + |\hat{s}|^2)^{-1}$ for some sufficiently small positive rational number ρ (and we may have to replace η by a smaller positive rational number). To order to get a globally defined metric we have to multiply $h_1^{-\rho}$ by a factor so that after the multiplication the sum of $h_1^{-\rho}$ and $|\hat{s}|^2$ are globally meaningful. To do this, we let $\theta_1, \cdots, \theta_\ell$

be multivalued holomorphic sections of L without common zeroes and we replace $h_1^{-\rho}$ by $(\sum_{j=1}^{k_1} |s_j^{(1)}|^{2\rho})(\sum_{i=1}^{\ell} |\theta_i|^{2\delta})$, where $\delta = d_1 + \epsilon - (n+\epsilon_1)\rho$. So we get a globally defined metric for $(d_1 + n + \epsilon)L$ (after slightly changing ϵ) so that the zero-set of its multiplier ideal sheaf contains P_0 and has dimension $< d_1$ at P_0. As in the first step where we use the multivalued holomorphic sections $t_1^{(1)}, \cdots, t_{k_1}^{(1)}$, before we go on to the next step we modify the metric so that precisely one branch of the zero-set of the multiplier ideal sheaf for the new metric of $(d_1 + n + \epsilon)L$ contains P_0. After these two steps it is clear how we should go on in our argument to get the following induction statement on ν for some integers $0 = d_r < d_{r-1} < \cdots < d_1 < d_0 = n$.

There exist a finite number of multivalued holomorphic sections $s_1^{(\nu)}, \cdots, s_{k_\nu}^{(\nu)}$ of $(\epsilon_\nu + \sum_{\lambda=0}^{\nu-1} d_\lambda)L$ (for some sufficiently small positive rational number ϵ_ν) so that precisely one branch of dimension d_ν of the zero-set of the multiplier ideal sheaf of the metric $h_\nu := (\sum_{j=1}^{k_\nu} |s_j^{(\nu)}|^2)^{-1}$ of $(\epsilon_\nu + \sum_{\lambda=0}^{\nu-1} d_\lambda)L$ contains P_0.

Finally we take any smooth metric of L with positive definite curvature form and with it, for any $m \geq 1 + (1 + 2 + \cdots + n)$, we construct from h_r a singular metric of mL whose multiplier ideal sheaf has isolated zero at P_0. From Nadel's vanishing theorem it follows that there exists a global holomorphic section of $mL + K_X$ which is nonzero at P_0, finishing the proof of the Corollary to the Main Theorem.

§3. Semicontinuity of Multiplier Ideal Sheaves

We now state and prove the semicontinuity property of multiplier ideal sheaves.

Lemma on the semicontinuity of multiplier ideal sheaves. Let X be a compact complex manifold of complex dimension n and L be an ample line bundle over X. Let P_0 be a point of X and U' be a local holomorphic curve in X passing through P_0 with P_0 as the only singularity and U be the open unit disk in $\mathbf{C}$ and $\sigma : U \to U'$ be the normalization of U' so that $\sigma(0) = P_0$. Let β be a positive rational number. Let $s_1, \cdots, s_k$ be multivalued holomorphic sections of $pr_1^*(\beta L)$ over $X \times U$. Suppose that for almost all $u \in U - 0$ (in the sense that the statement is true up to a subset of measure zero) the point $\sigma(u) \times u$ belongs to the zero-set of the multiplier ideal sheaf of the singular metric $(\sum_{\nu=1}^{k} |s_\nu|^2)^{-1} | X \times u$ of $\beta L = pr_1^*(\beta L) | X \times u$ (*i.e.*, the function $(\sum_{\nu=1}^{k} |s_\nu|^2)^{-1}(\cdot, u)$ is not locally integrable at $\sigma(u)$). Then $P_0 \times 0$ belongs to the zero-set of the multiplier ideal sheaf of the singular metric $(\sum_{\nu=1}^k |s_\nu|^2)^{-1}|X \times 0$ of $\beta L = pr_1^*(\beta L)|X \times 0$. (*i.e.*, the function $(\sum_{\nu=1}^k |s_\nu|^2)^{-1}(\cdot, 0)$ is not locally integrable at P_0).

The proof depends on the following extension theorem of Ohsawa-Takegoshi.

Theorem of Ohsawa-Takegoshi. Let Ω be a bounded smooth pseudoconvex domain in $\mathbf{C}^{n+1}$ with coordinates $z_1, \cdots, z_n, w$. Let H be defined by $w = 0$. Let φ be a smooth plurisubharmonic function on Ω. There exists a constant C_Ω depending only on Ω such that for any holomorphic function f on $\Omega \cap H$ with $\int_{H\cap\Omega} |f|^2 e^{-\varphi} < \infty$ there exists a holomorphic function F on Ω extending f with the property that $\int_\Omega |F|^2 e^{-\varphi} \leq C_\Omega \int_{H\cap\Omega} |f|^2 e^{-\varphi}$. Moreover, C_Ω can be chosen to be $\frac{64}{9}\pi A^2(1 + \frac{1}{4e})^{1/2}$, where A is any positive number with $\Omega \subset \{|w| < A\}$.

Proof of the lemma on the semicontinuity of multiplier ideal sheaves. Assume the contrary. Then for some open neighborhood D of P_0 in X the function $(\sum_{\nu=1}^k |s_\nu|^2)^{-1}(\cdot, 0)$ is integrable on D. We can assume without loss of generality that $pr_1^* L|D \times U$ is holomorphically trivial and D is biholomorphic to a bounded pseudoconvex domain in $\mathbf{C}^n$. We apply Theorem (5.1) to the domain $\Omega = D \times U$ and the hyperplane $H = \mathbf{C}^n \times 0$. For the plurisubharmonic function we use $\varphi = \log(\sum_{\nu=1}^k |s_\nu|^2)$ and for the function to be extended we use $f \equiv 1$. Let F be the holomorphic function on $D \times U$ such that $\int_{D\times U} |F|^2(\sum_{\nu=1}^k |s_\nu|^2)^{-1} < \infty$ and $F(\cdot, 0) = f$ on D. There exist an open neighborhood D' of P_0 in D and an open neighborhood W of 0 in U such that $|F|$ is bounded from below on $D' \times W$ by some positive number. There is a set E of measure zero in W such that $\int_{D'\times\{u\}}(\sum_{\nu=1}^k |s_\nu|^2)^{-1}$ is finite for $u \in W - E$, contradicting the assumption that $(\sum_{\nu=1}^k |s_\nu|^2)^{-1}(\cdot, u)$ is not locally integrable at $\sigma(u)$ for almost all $u \in U - 0$. Q.E.D.

§4. Proof of the Extension Theorem of Ohsawa-Takegoshi

Before we prove the theorem, we would like to make some remarks about its proof.

First we recall the standard technique of using functional analysis and Hilbert spaces to solve the $\overline{\partial}$ equation. Denote $\overline{\partial}$ by T and the $\overline{\partial}$ in the next step of the Dolbeault complex by S. Given g with $Sg = 0$ we would like to solve the equation $Tu = g$. The equation $Tu = g$ is equivalent to $(v, Tu) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$, which means $(T^*v, u) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. To get a solution u it suffices to prove that the map $T^*v \to (v, g)$ can be extended to a bounded linear functional, which means that $|(v, g)| \leq C\|T^* v\|$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. In that case we can solve the equation $Tu = g$ with $\|u\| \leq C$. We could also use the equivalent inequality

$$|(v, g)|^2 \leq C^2(\|T^* v\|^2 + \|Sv\|^2)$$

for all $v \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$. Suppose L is a semipositive operator. Suppose we have the following inequality

$$|(Lv, v)| \leq \|T^* v\|^2 + \|Sv\|^2.$$

Then we can use the Schwarz inequality and get

$$|(v, g)|^2 \leq |(Lv, v)||(L^{-1}g, g)| \leq |(L^{-1}g, g)|(\|T^* v\|^2 + \|Sv\|^2),$$

where $|(L^{-1}g, g)|$ means ∞ if g is not in the domain of L^{-1}. This means that we could choose C to be equal to $|(L^{-1}g, g)|^{\frac{1}{2}}$ and conclude that the equation $Tu = g$ can be solved with $\|u\| \leq |(L^{-1}g, g)|^{\frac{1}{2}}$. In the usual application L is the operator defined by multiplication by a positive number. In our application where g is a $(0, 1)$-form $\sum g_{\overline{\alpha}} dz^{\overline{\alpha}}$, the semipositive operator L is defined by $Lg = \sum_\alpha (\sum_\beta \gamma_{\overline{\alpha}\beta} g_{\overline{\beta}}) dz^{\overline{\alpha}}$ where $(\gamma_{\overline{\alpha}\beta})$ is a semipositive matrix of scalar functions. An equivalent way of achieving the same effect as using the semipositive operator L is to use the (limit) Kähler metric defined by $(\gamma_{\overline{\alpha}\beta})$.

Since the constant C_Ω is to be independent of φ, it suffices to prove the theorem for a smooth φ, we can write φ as the limit of smooth plurisubharmonic functions φ_ν and then for each ν get a holomorphic function F_ν extending f such that $\int_\Omega |F_\nu|^2 e^{-\varphi_\nu} \leq C_\Omega \int_{H \cap \Omega} |f|^2 e^{-\varphi_\nu}$ and then we can pass to limit as $\nu \to \infty$ to get F from F_ν. By the same reasoning we can assume without loss of generality that the boundary of Ω is smooth.

The natural approach clearly is the following. We extend f to some holomorphic function on Ω without any condition on the bound of the extension and we denote the extension also by f. Then we take $0 < \lambda < 1$ and take a cut-off function χ so that $\chi(\xi)$ is identically 1 on $\xi \leq \lambda$ and ξ is supported in $\xi \leq 1$. Let $\chi_\epsilon(w) = \chi(\frac{|w|^2}{\epsilon^2})$. Consider $v_\epsilon = \frac{1}{w}(\overline{\partial}(\chi_\epsilon f)) = \frac{1}{w}(\overline{\partial}\chi_\epsilon)f$. We will solve the $\overline{\partial}$-equation $\overline{\partial} u_\epsilon = v_\epsilon$ and set $F = \chi_\epsilon f - w u_\epsilon$ so that $\overline{\partial} F = 0$ and F agrees with f on $w = 0$. The difficulty is to keep track of the estimates.

Let $U_\epsilon = \Omega \cap \{\chi_\epsilon \neq 0\}$. Then

$$\int_{U_\epsilon} \frac{|\overline{\partial}\chi_\epsilon|^2}{|w|^2} |f|^2 e^{-\varphi}$$

is of the order $\frac{1}{\epsilon^2}$. To offset this order, we can introduce the weight function $\log(|w|^2+\epsilon^2)$ so that $\partial_w\partial_{\overline{w}}\log(|w|^2+\epsilon^2)=\frac{\epsilon^2}{(|w|^2+\epsilon^2)^2}$ can be used to define L and L^{-1} gives the reciprocal $\frac{(|w|^2+\epsilon^2)^2}{\epsilon^2}$ of $\partial_w\partial_{\overline{w}}\log(|w|^2+\epsilon^2)$ which is of order ϵ^2 precisely cancelling the unwanted order $\frac{1}{\epsilon^2}$. However, the use of the weight function $\log(|w|^2+\epsilon^2)$ necessitates the introduction of the metric $\frac{1}{|w|^2+\epsilon^2}$ which contributes back the unwanted factor $\frac{1}{\epsilon^2}$. The ideal situation is to be able to get the contribution of the curvature term $\frac{\epsilon^2}{(|w|^2+\epsilon^2)^2}$ without the contribution of the metric $\frac{1}{|w|^2+\epsilon^2}$. Such an ideal situation is clearly impossible by the usual Bochner-Kodaira type formula. Ohsawa-Takegoshi introduced a way of producing the curvature term without the contribution of the metric. In their proof and also the later generalization by Manivel [M93] to the case of a general manifold, the actual underlying reason for the argument to work is obscured by a long series of commutation identities in Kähler geometry and the introduction of a specially constructed complete Kähler metric. Actually the key point for the argument to work is the replacement, in the $\overline{\partial}$-equation, of $\overline{\partial}$ by $\overline{\partial}$ composed with a scalar function on the right. The square of this scalar function is broken down as the sum of two scalar functions. The first summand is used in the usual Bochner-Kodaira formula to produce a term similar to the desired curvature term. The second summand is used to insure that an inequality holds so that the standard technique of functional analysis and Hilbert spaces can be applied. The first summand used to produce the curvature term is $\log\frac{A^2}{|w|^2+\epsilon^2}$ so that the curvature term it produces is $\frac{\epsilon^2}{(|w|^2+\epsilon^2)^2}$. In the proof below the scalar function is $\sqrt{\eta+\gamma}$ with $\eta=\log\frac{A^2}{|w|^2+\epsilon^2}$. In this easier way of looking at the proof of Ohsawa-Takegoshi, there is no need to introduce any specially constructed complete Kähler metric and the estimate obtained is far sharper than those produced in the original proofs of Ohsawa-Takegoshi and the subsequent generalization by Manivel.

Proof of the extension theorem of Ohsawa-Takegoshi. For the proof we use the notations introduced in the preceding remarks and most of the time use the summation convention of repeated indices. In order to separate the new arguments in the approach of Ohsawa-Takegoshi from the standard basic estimates of the Bochner-Kodaira formula, we divide the proof into two parts. Part I is simply a reproduction of the usual Bochner-Kodaira formula with an introduction of a scalar factor η. Part II consists of the new arguments.

Part I. Let u be in the domain of $\overline{\partial}^*$ on Ω. We consider the integration by parts for

$$\int_\Omega < \eta\overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u > e^{-\psi} + \int_\Omega < \eta\overline{\partial} u, \overline{\partial} u > e^{-\psi}$$

$$= \int_\Omega < \overline{\partial}(\eta\overline{\partial}^*_\psi u), u > e^{-\psi} + \int_\Omega < \overline{\partial}^*_\psi(\eta\overline{\partial} u), u > e^{-\psi}$$

$$+ \int_{\partial\Omega} \eta(\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\beta}} u_{\overline{\alpha}})(\partial_\alpha \rho)\overline{u_{\overline{\beta}}} e^{-\psi}.$$

Here for the integration by parts for $\int_\Omega < \eta\overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u > e^{-\psi}$ there is no boundary term because $u \in \mathrm{Dom}\,\overline{\partial}^*$. Though $\overline{\partial} u = \frac{1}{2}(\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\beta}} u_{\overline{\alpha}}) dz^{\overline{\alpha}} \wedge dz^{\overline{\beta}}$, we do not have a factor $\frac{1}{2}$ in the boundary term because the last factor in that integral is only $\overline{u_{\overline{\beta}}}$. More precisely,

$$< \overline{\partial} u, \overline{\partial} u >= \sum_{\alpha<\beta} (\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\alpha}} u_{\overline{\beta}})\overline{(\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\alpha}} u_{\overline{\beta}})}$$

$$= \frac{1}{2}\sum_{\alpha,\beta} (\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\alpha}} u_{\overline{\beta}})\overline{(\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\alpha}} u_{\overline{\beta}})}$$

$$= \sum_{\alpha,\beta} (\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\alpha}} u_{\overline{\beta}})\overline{(\partial_{\overline{\alpha}} u_{\overline{\beta}})}.$$

The sign for the boundary term on the right-hand side is checked against the usual divergence theorem

$$\int_\Omega (\partial_j u_j) \cdot 1 = -\int_\Omega u_j \cdot (\partial_j 1) + \int_{\partial\Omega} (\partial_j \rho) u_j$$

where ρ is the defining function for Ω in the sense that $\Omega = \{\rho < 0\}$ and $|d\rho| \equiv 1$ on $\partial\Omega$. Note that the rule for the boundary term is that we replace the differentiation ∂u_j of u_j by $\partial \rho_j$.

On the right-hand side of the Bochner-Kodaira formula we have the term $\int_\Omega < \eta\overline{\nabla} u, \overline{\nabla} u > e^{-\psi}$ which is transformed by

$$\int_\Omega < \eta\overline{\nabla} u, \overline{\nabla} u > e^{-\psi} = \int_\Omega \eta\partial_{\overline{\alpha}} u_{\overline{\beta}} \overline{\partial_{\overline{\alpha}} u_{\overline{\beta}}} e^{-\psi}$$

$$= -\int_\Omega e^{\psi}\partial_\alpha(e^{-\psi}\eta\partial_{\overline{\alpha}} u_{\overline{\beta}})\overline{u_{\overline{\beta}}} e^{-\psi} + \int_{\partial\Omega} \eta\partial_{\overline{\alpha}} u_{\overline{\beta}}(\partial_\alpha\rho)\overline{u_{\overline{\beta}}} e^{-\psi}.$$

We use the boundary term to cancel with part of the boundary term $\int_{\partial\Omega} \eta(\partial_{\overline{\alpha}} u_{\overline{\beta}} - \partial_{\overline{\beta}} u_{\overline{\alpha}})(\partial_\alpha\rho)\overline{u_{\overline{\beta}}} e^{-\psi}$ of $\int_\Omega < \eta\overline{\partial}^* u, \overline{\partial}^* u > + \int_\Omega < \eta\overline{\partial} u, \overline{\partial} u >$ and we are left with the other half $-\int_{\partial\Omega}\eta\partial_{\overline{\beta}}u_{\overline{\alpha}}(\partial_\alpha\rho)\overline{u_{\overline{\beta}}}e^{-\psi}$ of boundary term. We use the trick of C.B. Morrey to handle it. Since $u\in\operatorname{Dom}\overline{\partial}^*$, it follows that $u_{\overline{\alpha}}\partial_\alpha\rho=0$ on $\partial\Omega$. In other words, $u_{\overline{\alpha}}\partial_\alpha\rho=\theta\rho$ for some smooth function θ. Differentiation yields

$$\partial_{\overline{\beta}}u_{\overline{\alpha}}\partial_\alpha\rho+u_{\overline{\alpha}}\partial_{\overline{\beta}}\partial_\alpha\rho=\rho\partial_{\overline{\beta}}\theta+\theta\partial_{\overline{\beta}}\rho=\theta\partial_{\overline{\beta}}\rho$$

on $\partial\Omega$ and

$$\overline{u_{\overline{\beta}}}\partial_{\overline{\beta}}u_{\overline{\alpha}}\partial_\alpha\rho+\overline{u_{\overline{\beta}}}u_{\overline{\alpha}}\partial_{\overline{\beta}}\partial_\alpha\rho=\overline{u_{\overline{\beta}}}\theta\partial_{\overline{\beta}}\rho=0.$$

Hence

$$-\int_{\partial\Omega}\eta\partial_{\overline{\beta}}u_{\overline{\alpha}}(\partial_\alpha\rho)\overline{u_{\overline{\beta}}}e^{-\psi}=\int_{\partial\Omega}\eta\overline{u_{\overline{\beta}}}u_{\overline{\alpha}}(\partial_{\overline{\beta}}\partial_\alpha\rho)e^{-\psi}.$$

We have the following error term $E=E_1+E_2+E_3$, where

$$\begin{aligned}
E_1&=\int_\Omega<\overline{\partial}(\eta\overline{\partial}^*_\psi u),u>e^{-\psi}-\int_\Omega<\eta\overline{\partial}\overline{\partial}^*_\psi u,u>e^{-\psi}\\
&=\int_\Omega(\partial_{\overline{\alpha}}\eta)(\overline{\partial}^*_\psi u)\overline{u_{\overline{\alpha}}}e^{-\psi},\\
E_2&=\int_\Omega<\overline{\partial}^*_\psi(\eta\overline{\partial}u),u>e^{-\psi}-\int_\Omega<\eta\overline{\partial}^*_\psi\overline{\partial}u,u>e^{-\psi}\\
&=-\int_\Omega(\partial_\alpha\eta)(\partial_{\overline{\alpha}}u_{\overline{\beta}}-\partial_{\overline{\beta}}u_{\overline{\alpha}})\overline{u_{\overline{\beta}}}e^{-\psi},\\
E_3&=\int_\Omega<\nabla_\psi(\eta\overline{\nabla}u),u>e^{-\psi}-\int_\Omega<\eta\nabla_\psi\overline{\nabla}u,u>e^{-\psi}\\
&=\int_\Omega(\partial_\alpha\eta)(\partial_{\overline{\alpha}}u_{\overline{\beta}})\overline{u_{\overline{\beta}}}e^{-\psi}.
\end{aligned}$$

With this error term we need up with

$$\int_\Omega<\eta\overline{\partial}^*_\psi u,\overline{\partial}^*_\psi u>e^{-\psi}+\int_\Omega<\eta\overline{\partial}u,\overline{\partial}u>e^{-\psi}$$

$$=\int_{\partial\Omega}\eta\overline{u_{\overline{\beta}}}u_{\overline{\alpha}}(\partial_{\overline{\beta}}\partial_\alpha\rho)e^{-\psi}+\int_\Omega<\eta\overline{\nabla}u,\overline{\nabla}u>e^{-\psi}+\int_\Omega\eta\overline{u_{\overline{\beta}}}u_{\overline{\alpha}}(\partial_{\overline{\beta}}\partial_\alpha\psi)e^{-\psi}+E.$$

Part II. We now introduce the main new arguments which transform the error term to yield a curvature term. We leave E_1 alone and combine E_2 and E_3 together and transform the sum. We get

$$E_2+E_3=\int_\Omega(\partial_\alpha\eta)(\partial_{\overline{\beta}}u_{\overline{\alpha}})\overline{u_{\overline{\beta}}}e^{-\psi}$$

$$= \int_{\partial\Omega} (\partial_\alpha \eta)(\partial_{\overline{\beta}} \rho) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi} - \int_\Omega (\partial_{\overline{\beta}} \partial_\alpha \eta) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi} + \int_\Omega (\partial_\alpha \eta) u_{\overline{\alpha}} \overline{\overline{\partial}^*_\psi u} e^{-\psi}$$

$$= - \int_\Omega (\partial_{\overline{\beta}} \partial_\alpha \eta) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi} + \int_\Omega (\partial_\alpha \eta) u_{\overline{\alpha}} \overline{(\overline{\partial}^*_\psi u)} e^{-\psi},$$

where the vanishing of the boundary term $\int_{\partial\Omega} (\partial_\alpha \eta)(\partial_{\overline{\beta}} \rho) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi}$ from $u_{\overline{\beta}} \partial_\beta \rho = 0$ on $\partial\Omega$ because $u \in \operatorname{Dom} \overline{\partial}^*$. By putting together the three error terms, we get

$$E_1 + E_2 + E_3 = - \int_\Omega (\partial_{\overline{\beta}} \partial_\alpha \eta) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi} + 2\mathrm{Re} \int_\Omega (\partial_\alpha \eta) u_{\overline{\alpha}} \overline{(\overline{\partial}^*_\psi u)} e^{-\psi}.$$

Thus we have

$$\int_\Omega < \eta \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u > e^{-\psi} + \int_\Omega < \eta \overline{\partial} u, \overline{\partial} u > e^{-\psi}$$

$$= \int_{\partial\Omega} \eta \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}} \partial_\alpha \rho) e^{-\psi} + \int_\Omega < \eta \overline{\nabla} u, \overline{\nabla} u > e^{-\psi}$$

$$+ \int_\Omega \eta \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}} \partial_\alpha \psi) e^{-\psi} - \int_\Omega (\partial_{\overline{\beta}} \partial_\alpha \eta) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi} + 2\mathrm{Re} \int_\Omega (\partial_\alpha \eta) u_{\overline{\alpha}} \overline{(\overline{\partial}^*_\psi u)} e^{-\psi}.$$

Recall that $\Omega \subset \{|w| < A\}$. Let

$$\eta = \log \frac{A^2}{|w|^2 + \epsilon^2},$$

$$\gamma = \frac{1}{|w|^2 + \epsilon^2}.$$

Then

$$-\partial_w \partial_{\overline{w}} \eta = \frac{\epsilon^2}{(|w|^2 + \epsilon^2)^2},$$

$$\partial_w \eta = -\frac{\overline{w}}{|w|^2 + \epsilon^2},$$

$$\partial_{\overline{w}} \eta = -\frac{w}{|w|^2 + \epsilon^2}.$$

The function η is introduced to give us the desired the curvature term while γ is introduced to help us take care of the error term by Schwarz inequality. We use the estimate

$$|2 \text{ Re } \int_\Omega (\partial_\alpha \eta) u_{\overline{\alpha}} (\overline{\partial}^*_\psi u) e^{-\psi}| \leq 2 \int_\Omega \frac{|w|}{|w|^2 + \epsilon^2} |u| |\overline{\partial}^*_\psi u| e^{-\psi}$$

$$\leq \int_\Omega \frac{|w|^2}{|w|^2+\epsilon^2}|u|^2 e^{-\psi} + \int_\Omega \frac{1}{|w|^2+\epsilon^2}|\overline{\partial}_\psi^* u|^2 e^{-\psi}.$$

Assume that $\eta(\partial_\alpha \partial_{\overline{\beta}} \psi) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} \geq \frac{|w|^2}{|w|^2+\epsilon^2}|u|^2$ and that Ω is pseudoconvex. Then

$$\int_\Omega < (\eta+\gamma)\overline{\partial}_\psi^* u, \overline{\partial}_\psi^* u > e^{-\psi} + \int_\Omega < (\eta+\gamma)\overline{\partial} u, \overline{\partial} u > e^{-\psi}$$

$$\geq \int_\Omega (\partial_{\overline{\beta}} \partial_\alpha \log(|w|^2+\epsilon^2)) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi}.$$

We now consider the operator T defined by $Tu = \overline{\partial}(\sqrt{\eta+\gamma} u)$ and the operator S defined by $Su = \sqrt{\eta+\gamma}\,\overline{\partial} u$. Then $ST = 0$ and

$$\|T^* u\|^2 + \|Su\|^2 \geq \int_\Omega (\partial_{\overline{\beta}} \partial_\alpha \log(|w|^2+\epsilon^2)) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi}.$$

Let $U_\epsilon = \Omega \cap \{\chi_\epsilon \neq 0\} \subset \Omega \cap \{|w|^2 < \lambda\epsilon^2\}$. We can solve $\overline{\partial}(\sqrt{\eta+\gamma} u_\epsilon) = v_\epsilon$ with

$$\int_\Omega |u_\epsilon|^2 e^{-\psi} \leq \int_{U_\epsilon} e^{-\psi} \frac{(|w|^2+\epsilon^2)^2}{\epsilon^2} \frac{|\overline{\partial}\chi_\epsilon|^2}{|w|^2}|f|^2.$$

The limit of the right-hand side as $\epsilon \to 0$ becomes $2\frac{\lambda^2(1+\lambda)^2\pi}{(1-\lambda)^2}\int_{\Omega\cap H}|f|^2 e^{-\psi}$, because

$$\left|\overline{\partial}\chi_\epsilon\right|^2 = \left|\chi'\left(\frac{|w|^2}{\epsilon^2}\right)\right|^2 \frac{|w|^2}{\epsilon^4}|dw|^2 \leq \left(\frac{1}{1-\lambda}\right)^2 \frac{\lambda\epsilon^2}{\epsilon^4}|dw|^2 = \frac{\lambda}{(1-\lambda)^2}\frac{1}{\epsilon^2}|dw|^2$$

and the integral of $|dw|^2$ over $\{|w|^2 < \lambda\epsilon^2\}$ is $2\pi\lambda\epsilon^2$. Recall that $F = \chi_\epsilon f - w\sqrt{\eta+\gamma} u_\epsilon$. Observe that the L^2 norm of $\chi_\epsilon f$ vanishes as $\epsilon \to 0$, because f is L^2 and the support of $\chi_\epsilon f$ approaches a set of measure zero. The L^2 norm of $w\sqrt{\eta+\gamma} u_\epsilon$ is dominated by

$$\sup_\Omega |w\sqrt{\eta+\gamma}| \int_\Omega |u_\epsilon|^2 e^{-\psi} \leq \sup_\Omega |w\sqrt{\eta+\gamma}| 2\pi \frac{\lambda^2(1+\lambda)^2}{(1-\lambda)^2}\int_{\Omega\cap H}|f|^2 e^{-\psi}.$$

Finally we take a plurisubharmonic function σ such that $(\partial_\alpha \partial_{\overline{\beta}} \sigma) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} \geq |u|^2$ and use $\psi = \varphi + \sigma$. Let

$$B = \frac{\sup_\Omega e^{-\sigma}}{\inf_\Omega e^{-\sigma}}$$

The constant C_Ω can be taken to be any positive number which dominates

$$B\left(1 + \sup_\Omega |w|^2 \log\frac{A^2}{|w|^2}\right)^{\frac{1}{2}} 2\frac{\lambda^2(1+\lambda)^2}{(1-\lambda)^2}$$

for some $0 < \lambda < 1$.

We now choose σ and λ to get a good explicit numerical value for the constant C_Ω. We first find the critical value of $\frac{\lambda^2(1+\lambda)^2}{(1-\lambda)^2}$ which is the same as the critical value $\frac{4}{9}$ for $\log\lambda + \log(1+\lambda) - \log(1-\lambda)$ achieved at $\lambda = \frac{1}{3}$ where its derivative vanishes. For the choice of σ we consider first the case where Ω is contained in $\{|w| < \frac{1}{2}\}$ and for such a case we choose $\sigma = \log\left(|w|^2 + (\frac{1}{2})^2\right)$. Then $\partial_w \overline{\partial}_{\overline{w}}\sigma = \frac{\left(\frac{1}{2}\right)^2}{|w|^2+\left(\frac{1}{2}\right)^2} > 1$ on Ω. We have

$$B = \frac{\sup_\Omega e^{-\sigma}}{\inf_\Omega e^{-\sigma}} = \frac{\sup_\Omega \frac{1}{|w|^2+(\frac{1}{2})^2}}{\inf_\Omega \frac{1}{|w|^2+(\frac{1}{2})^2}} \leq 2.$$

Finally we estimate $\sup_\Omega |w|^2 \log \frac{1}{4|w|^2}$ by finding its critical value as a function of $|w|^2$ which is $\frac{1}{4e}$ achieved at $|w|^2 = \frac{1}{4e}$. Thus we can choose C_Ω to be $\frac{16}{9}\pi\left(1 + \frac{1}{4e}\right)^{1/2}$ when $\Omega \subset \{|w| < \frac{1}{2}\}$. In general, we do a change of scale in the variable w and conclude that we can choose C_Ω to be $\frac{64}{9}\pi A^2\left(1 + \frac{1}{4e}\right)^{1/2}$. Thus the theorem of Ohsawa-Takegoshi is proved.

§5. Alternative to the Use of the Extension Theorem of Ohsawa-Takegoshi

We now present, in an analytic setting, the replacement by Kollar of the use of the extension theorem of Ohsawa-Takegoshi by the inversion of adjunction. The extension theorem of Ohsawa-Takegoshi was used only to show that, for local holomorphic functions $F_j(z_1, \cdots, z_n, w)$ at 0 $(1 \leq j \leq k)$ and for any positive rational number α, if $\frac{1}{(\sum_{j=1}^k |f_j|)^{2\alpha}}$ is locally integrable at 0 as a function on $\mathbf{C}^n$, then $\frac{1}{(\sum_{j=1}^k |F_j|)^{2\alpha}}$ is locally integrable at 0 as a function on $\mathbf{C}^{n+1}$, where $f_j(z_1, \cdots, z_n) = F_j(z_1, \cdots, z_n, 0)$. Since the techniques of the proof can already be seen in the case of $k = 1$, we present here only the case $k = 1$.

Lemma on the inheritance of non-integrability by restriction. Let $F(z_1, \cdots, z_n, w)$ be a holomorphic function defined on some Stein open neighborhood $U \times W$ of 0 in $\mathbf{C}^n \times \mathbf{C}$ and let α be a positive rational number. Let $f(z_1, \cdots, z_n) = F(z_1, \cdots, z_n, 0)$. If $\frac{1}{|f|^{2\alpha}}$ is locally integrable at 0 as a function on $\mathbf{C}^n$, then $\frac{1}{|F|^{2\alpha}}$ is locally integrable at 0 as a function on $\mathbf{C}^{n+1}$.

Proof. By shrinking U and W we can assume without loss of generality that $\frac{1}{|F|^{2\alpha}}$ is integrable on $U \times W$. Let $D = \alpha \operatorname{div} F$ and $S = U \times 0 \subset U \times W$ be $\mathbf{Q}$-divisors in $U \times W$. We use $D|S$ to denote $\alpha \operatorname{div} f$. Let $\pi : \tilde{U} \to U \times W$ be a resolution of singularities for the $\mathbf{Q}$-divisor $D + S$ obtained by successive monoidal transformations with nonsingular centers so that for some collection $\{E_\mu\}_{\mu=0}^{\ell}$ of nonsingular hypersurfaces in $\tilde{U}$ in normal crossing the following three conditions hold.
(i) $\pi^*(D+S) = E_0 + \sum_{\mu=1}^{\ell} a_\mu E_\mu$ with a_μ being nonnegative rational numbers,.
(ii) $K_{\tilde{U}} - \pi^* K_{U\times W} = \sum_{\mu=1}^{\ell} b_\mu E_\mu$ with b_μ being nonnegative integers.
(iii) the restriction of π to E_0 is a modification of S so that from (i) and (ii) it follows that

$$\begin{gathered}\pi^*(D|S) - K_{E_0} + (\pi|E_0)^* K_S \\ = \pi^*(D + S|S) - \pi^*(S|S) - (K_{\tilde{U}} + E_0)|E_0 + \pi^*(K_{U\times W} + S)|S \\ = \sum_{\mu\in J} (a_\mu - b_\mu)(E_\mu \cap E_0),\end{gathered}$$

where J is the set of all $1 \leq \mu \leq \ell$ with $E_\mu \cap E_0 \neq \emptyset$.

Note that one needs to consider $\pi^*(D + S)$ instead of $\pi^* D$ in (i) in order for the formula for $K_{E_0} - (\pi|E_0)^* K_S$ in (iii) to hold, because of the identity $K_{E_0} = (K_{\tilde{U}} + E_0)|E_0$ and the corresponding identity $K_S = (K_{U\times W} + S)|S$, compatible with the map π, for the trivial line bundles $K_S, K_{U\times W}|S, S|S$.

Since $\frac{1}{|f|^{2\alpha}}$ is integrable on $U\times W$, it follows that $a_\mu - b_\mu < 1$ for $\mu \in J$. We can conclude that $\frac{1}{|F|^{2\alpha}}$ is locally integrable at every point of S if $a_\mu - b_\mu < 1$ for $1 \leq \mu \leq \ell$ whenever $\pi(E_\mu) \cap S \neq \emptyset$. Thus to finish the proof it suffices to show that there is no $1 \leq \mu \leq \ell$ with $E_\mu \cap E_0 = \emptyset$ and $\pi(E_\mu) \cap S \neq \emptyset$. Suppose there is such an index μ. We are going to derive a contradiction.

Since $\pi : \tilde{U} \to U \times W$ is obtained by a finite number of monoidal transformations, we can find sufficiently small positive rational numbers δ_μ ($1 \leq \mu \leq \ell$) such that the $\mathbf{Q}$-bundle $-\sum_{\mu=1}^{\ell} \delta_\mu E_\mu$ admits a Hermitian metric h_0 along its fibers whose curvature form is positive definite at every point of $\tilde{U}$. Moreover, δ_μ ($1 \leq \mu \leq \ell$) are chosen so small that $a_\mu - b_\mu + \delta_\mu < 1$ whenever $a_\mu - b_\mu < 1$. Let s_0 be the canonical section of the $\mathbf{Q}$-bundle $\sum_{\mu=1}^{\ell} \delta_\mu E_\mu$ so that the divisor of s_0 is $\sum_{\mu=1}^{\ell} \delta_\mu E_\mu$. Let ω be the holomorphic $(n+1)$-form $\pi^*(dz_1 \wedge \cdots \wedge dz_n \wedge w dw)$ on $\tilde{U}$. Let h be the (singular) Hermitian metric of along the fibers of $-K_{\tilde{U}}$ defined by $h_0 |s_0|^{-2} |\omega|^2 |F|^{-2\alpha}$. The multiplier ideal sheaf of h is the ideal sheaf defined by the $\mathbf{Q}$-divisor $E_0 + \sum_{\mu=1}^{\ell} \lfloor a_\mu - b_\mu + \delta_\mu \rfloor E_\mu$, where $\lfloor a_\mu - b_\mu + \delta_\mu \rfloor$ is the rounddown (*i.e.* the integral part) of $a_\mu - b_\mu + \delta_\mu$. One concludes the vanishing of $H^1(\tilde{U}, E_0 + \sum_{\mu=1}^{\ell} \lfloor a_\mu - b_\mu + \delta_\mu \rfloor E_\mu)$. Though the vanishing theorem for multiplier ideal sheaves is usually for compact manifolds, a simple modification of its proof also gives its extension to the case of a complex manifold admitting a proper holomorphic map onto a Stein manifold. Since $a_\mu - b_\mu < 1$ for every $E_\mu \cap E_0 = \emptyset$, the support of the divisor $\sum_{\mu=1}^{\ell} \lfloor a_\mu - b_\mu + \delta_\mu \rfloor E_\mu$ is disjoint from E_0. There exists a holomorphic function G over $\tilde{U}$ which is identically 0 on E_0 and is identically 1 on the support of $\sum_{\mu=1}^{\ell} \lfloor a_\mu - b_\mu + \delta_\mu \rfloor E_\mu)$ in $\tilde{U}$. The holomorphic function G on $\tilde{U}$ descends to a holomorphic function g on $U \times W$ which is at the same time identically 0 on S due to its identically zero value on E_0 and identically 1 on the image of the support of $\sum_{\mu=1}^{\ell} \lfloor a_\mu - b_\mu + \delta_\mu \rfloor E_\mu$, yielding a contradiction, because the image of the support of $\sum_{\mu=1}^{\ell} \lfloor a_\mu - b_\mu + \delta_\mu \rfloor E_\mu$ intersects S. Q.E.D.

§6. Difficulty in Improving the Quadratic Bound to the Conjectured Linear Bound

We would like to discuss the difficulty in improving the quadratic bound to the linear bound conjectured by the Fujita conjecture on freeness. By using the theorem of Riemann-Roch for any $0 < \epsilon < 1$ we could get a multivalued holomorphic section s of L over X vanishing at P_0 to order at least $1 - \epsilon$. We choose a positive number q (as small as possible) such that $|s|^{-2q}$ is locally non-integrable at P_0 and let X_1 be the subvariety of X consisting of all the points of X where $|s|^{-2q}$ is locally non-integrable. To be able to use iterative notations, we let $a(X) = q, V(X) = X_1, b(X) = \dim_{P_0} X_1$ and inductively define $V^{(\nu)}(X) = V(V^{(\nu-1)}(X))$ with $V^{(0)}(X) = X$. For this inductive definition we have to handle the case of X being singular at P_0. In such a case, as discussed above, we consider a 1-parameter holomorphic family of multivalued holomorphic sections s_P with the holomorphic parameter P being a point in a local holomorphic curve Δ in X containing P_0 so that s is equal to s_P when $P = P_0$ and that X is regular at P for $P \neq P_0$. The vanishing order of s at P_0 is replaced by the lim sup at P_0 of the vanishing order of s_P at $P \in \Delta - \{P_0\}$ as P approaches P_0. The property of local non-integrability of $|s|^{-2q}$ at P_0 is replaced by the existence of a sequence of $P_j \in \Delta - \{P_0\}$ of Δ with limit point P_0 such that $|s_{P_j}|^{-2q}$ is not locally integrable at every P_j. Let ℓ be the smallest ν such that $b(V^{(\nu)}(X)) = 0$. With such notations we have the freeness of $mL + K_X$ for $m > \sum_{\nu=0}^{\ell} a(V^{(\nu)}(X))$.

To get a small m for the freeness of $mL+K_X$, we would like to have a small

$a(V^{(\nu)}(X))$ and at the same time a rapid decrease of $b(V^{(\nu)}(X))$ as a function of ν. The reason why we have to use a quadratic bound is that we may encounter the worst situation of $a(X) = \dim_{P_0} X$ and $b(X) = \dim_{P_0} X - 1$ and the corresponding worst situation at every subsequent step. So we end up with $m = 1 + (1 + 2 + \cdots + n)$ for the freeness of $mL + K_X$ if the worst situation is assumed in every step. One way to try to get a small m closer to the conjectured linear bound is to use at any given step all the vanishing orders unused in the previous steps, so to speak. However, even such savings of the unused vanishing orders are in general not enough to give us the conjectured linear bound by this method.

The following example from Ein and Lazarsfeld illustrates in the case $n = 3$ how a possible situation prevents the construction, by this method, of a singular metric for $(n+\epsilon)L$ whose multiplier ideal sheaf has an isolated zero at a prescribed point. Again ϵ denotes a generic sufficiently small positive number. When one uses a multivalued holomorphic section s of $(1+\epsilon)L$ to get vanishing order at least 1 at the point P_0, the divisor of s is locally $\frac{1}{2}$ times the divisor of $\sum_{j=1}^3 z_j^2$. So in order for the zero-set of the multiplier ideal sheaf to contain P_0 we have to use the metric $|s^2|^{-2}$. The zero set of the multiplier ideal sheaf locally is the zero-set X_1 of $\sum_{j=1}^3 z_j^2$. For the next step we have to find a section of L over X_1 vanishing at P_0. By the theorem of Riemann-Roch, the best one can do is to get a multivalued holomorphic section s_1 of $\sqrt{2}(1+\epsilon)L$ over X_1 whose divisor is locally $z_1 + z_2$. The number $\sqrt{2}$ comes from the formula of Riemann-Roch when one tries to get a section vanishing at two points in the regular part of X_1. By blowing up the single point P_0, near the inverse image of P_0, the pullbacks of the divisors of s^2 and s_1 are expressed in terms of regular surfaces in normal crossing. One can see that, in order for the zero-set of the multiplier ideal sheaf of the singular metric constructed from $s^{2(1-\epsilon)}$ and s_1^α to contain P_0 for some sufficiently small ϵ, one must have $\alpha \geq 1$. So for these two steps one is already forced to use $2+\sqrt{2}+\epsilon$ times L. It turns out that the Fujita conjecture on freeness is true for $n = 3$ as was proved in [EL93] or from the method of the semicontinuity of multiplier ideal sheaves, simply because $2 + \sqrt{2} < 3 + 1$. In this example of a surface singularity given by the divisor of $\sum_{j=1}^3 z_j^2$, for the final step we need only use up an additional ϵL because some very small additional pole order at P_0 in the singular metric would already produce an isolated zero at P_0 for the singular metric.

In the higher dimensional case, analogous but far more complicated situations of singularities occur, which makes it impossible to simply refine this method to improve the quadratic bound to the linear bound conjectured by the Fujita conjecture.

§7. Remarks on Very Ampleness

One can get very ampleness from freeness by using the following lemma which is proved by simply using the global holomorphic sections of the free ample line bundle to get a holomorphic map into a complex projective space. The map must have finite fibers by the ampleness of the line bundles. Then one uses the pullbacks of suitable hyperplane sections of the projective space to construct a singular metric with the desired multiplier ideal sheaf to get very ampleness.

Lemma to conclude very ampleness from freeness. Let L be an ample line bundle over a compact complex manifold X of complex dimension n such that L is free. Let A be an ample line bundle. Then $(n+1)L + A + K_X$ is very ample.

By using this lemma and the Corollary to the Main Theorem we conclude that, for any ample line bundle L over a compact complex manifold X of complex dimension n, the line bundle $(n+1)(mL+K_X)+L+K_X$ is very ample for $m \geq \frac{1}{2}(n^2+n+2)$. In particular, for a compact complex manifold X with ample K_X, the line bundle mK_X is very ample for $m \geq \frac{1}{2}(n^3+2n^2+5n+8)$.

Chapter 6. Invariance of Plurigenera

In this chapter we give a proof of the following long conjectured result on the invariance of the plurigenera.

Main Theorem. Let $\pi : X \to \Delta$ be a smooth projective family of compact complex manifolds parametrized by the open unit 1-disk Δ. Assume that the fibers $X_t = \pi^{-1}(t)$, $t \in \Delta$, are of general type. Then for every positive integer m the plurigenus $\dim_{\mathbf{C}} \Gamma(X_t, mK_{X_t})$ is independent of $t \in \Delta$, where K_{X_t} is the canonical line bundle of X_t.

Notations and Terminology. The canonical line bundle of a complex manifold Y is denoted by K_Y. A compact manifold Y is said to be of general type if there is a point $P \in Y$ with the property that one can find elements $s_0, s_1, \ldots, s_n \in \Gamma(Y, mK_Y)$ such that s_0 is nonzero at P and $\frac{s_1}{s_0}, \ldots, \frac{s_n}{s_0}$ form a local coordinate system of Y at P, where $\dim Y = n$.

By the family $\pi : X \to \Delta$ being projective we mean that there exists a positive holomorphic line bundle on the total space X of the family. Let n be the complex dimension of each X_t for $t \in \Delta$. Let $K_{X,\pi}$ be the line bundle on X whose restriction to X_t is K_{X_t} for each $t \in \Delta$. Since the normal bundle of X_t in X is trivial, the two line bundles K_X and $K_{X,\pi}$ are naturally isomorphic. Under this natural isomorphism a local section s of $K_{X,\pi}$ corresponds to the local section $s \wedge \pi^*(dt)$ of K_X. Unless there is some risk of confusion, in this paper we will, without any further explicit mention, always identify $K_{X,\pi}$ with K_X by this natural isomorphism. The dimension of $\Gamma(X_t, mK_t)$ is an upper semi-continuous function of t. However, the dimension of $\Gamma(X_t, mK_{X_t})$ might *a priori* jump at points $t \in \Delta$ where there are elements of $\Gamma(X_t, mK_{X_t})$ which do not extend to elements of $\Gamma(X, mK_X)$. The Main Theorem is therefore equivalent to the statement that for every $t \in \Delta$ and every positive integer m every element of $\Gamma(X_t, mK_{X_t})$ can be extended to an element of $\Gamma(X, mK_X)$.

The Hermitian metrics of holomorphic line bundles in this paper are allowed to have singularities and may not be smooth. For a Hermitian metric $e^{-\varphi_0}$ of a holomorphic line bundle L_0 over X_0 we denote by $\mathcal{I}_{\varphi_0}$ its multiplier ideal sheaf on X_0 which by definition means the ideal sheaf on X_0 of all local holomorphic function germs f on X_0 such that $|f|^2 e^{-\varphi_0}$ is integrable. For the proof of the Main Theorem, only multiplier ideal sheaves on X_0 are considered and no multiplier ideal sheaves on X are used. In the case of a Hermitian metric $e^{-\varphi}$ of a holomorphic line bundle L over X, for notational simplicity we simply use the notation $\mathcal{I}_\varphi$ to mean the multiplier ideal sheaf for the Hermitian metric $e^{-\varphi}|X_0$ of the holomorphic line bundle $L|X_0$ over X_0 and suppress the notation for restriction to X_0.

The stalk of a sheaf $\mathcal{F}$ at a point P is denoted by $\mathcal{F}_P$. The structure sheaf of a complex manifold Y is denoted by $\mathcal{O}_Y$. For a holomorphic line bundle L over Y the sheaf of germs of holomorphic sections of L is denoted by $\mathcal{O}_Y(L)$. If s is a global holomorphic section of L over Y and if $\mathcal{I}$ is an ideal sheaf on Y, we say that the germ of s at a point P belongs to $\mathcal{I}_P$ if the holomorphic function germ at P which corresponds to the germ of s at P with respect to some local trivialization of L belongs to $\mathcal{I}_P$. A $(1,1)$-current Θ on a complex manifold is said to dominate some smooth positive $(1,1)$-form ω if $\Theta \geq \omega$ as $(1,1)$-currents.

History and Sketch of the Proof of the Main Theorem. Iitaka [I69-71] proved the special case of the invariance of the plurigenera in a family of surfaces. His method works only for surfaces because it uses much of the information from the classification of surfaces. Levine [L83,L85] proved that for every positive integer m every element of $\Gamma(X_0, mK_{X_0})$ can be extended to the fiber of X over the double point of Δ at $t = 0$. So far there is no way to continue the process to get an extension to the fiber of X over a point of Δ at $t = 0$ of any finite order. Nakayama [Nak86] pointed out that if the relative case of the minimal model program can be carried out for a certain dimension, the conjecture of the invariance of the plurigenera for that dimension would follow directly from it. Thus the invariance of the plurigenera for threefolds is a consequence of the completion of the relative case of the minimal model program for the case of threefolds by Kollar and Mori [KM92].

For the proof of the Main Theorem here we use a strategy completely different from those used by the others in the past. We now sketch our strategy and leave out the less essential technical details. There are some unavoidable technical inaccuracies in the sketch due to the suppression of precise details. There are three ingredients in our proof: Nadel's multiplier ideal sheaves [Nad89], Skoda's result on the generation of ideals with L^2 estimates with respect to a plurisubharmonic weight [Sk72], and the extension theorem of Ohsawa-Takegoshi-Manivel for holomorphic top-degree forms which are L^2 with respect to a plurisubharmonic weight [OT87,M93]. The extension theorem of Ohsawa-Takegoshi-Manivel is for the setting of a Stein domain or manifold and a global plurisubharmonic function as weight. Here we adapt it to the case of a projective family of compact complex manifolds and a

Hermitian metric of a line bundle with nonnegative curvature current. The adaptation is done by restricting to a Stein Zariski open subset on which the line bundle is globally trivial, because L^2 bounds automatically extend the domain of definition from the Zariski open subset to the family of compact manifolds.

We take the m-th roots of basis elements of $\Gamma(X_0, mK_{X_0})$ for every positive integer m to use them in an infinite series to construct a Hermitian metric $e^{-\varphi_0}$ for K_{X_0}. We also take the m-th roots of basis elements of $\Gamma(X, mK_X)|X_0$ for every positive integer m to use them in an infinite series to construct a Hermitian metric $e^{-\varphi}$ for K_{X_0}. By construction, we have $\varphi \leq \varphi_0$ and φ might have more poles than φ_0, since there might be elements of sections in $\Gamma(X_0, mK_{X_0})$ which do not extend to elements of $\Gamma(X, mK_X)$. However, Skoda's result can be used in combination with the extension theorem of Ohsawa-Takegoshi-Manivel to show that the singularity of φ on X_0 is not much worse than the singularity of φ_0. This is made by comparing the multiplier ideal sheaves $\mathcal{I}_{\ell\varphi+\psi}$, $\mathcal{I}_{\ell\varphi_0+\psi_0}$, with small corrections ψ, ψ_0 on φ, φ_0 to make sure that the curvature currents of the Hermitian metrics $e^{-(\ell\varphi+\psi)}$ and $e^{-(\ell\varphi_0+\psi_0)}$ dominate some smooth positive $(1,1)$-form. At this point, an induction on ℓ is used. Finally, we take ℓ-th roots and use Hölder's inequality in the L^2 estimates to make the contribution of the extra terms ψ and ψ_0 negligible. A new application of the Ohsawa-Takegoshi-Manivel extension theorem then shows that every holomorphic section of mK_{X_0} over X_0 extends to a holomorphic section of mK_X over X.

Preliminary Tools. Curvature properties of K_X in the sense of currents and of related line bundles will be exploited through Propositions 1 and 2 below. Proposition 1 establishes a uniform global generation property of multiplier ideal sheaves. Proposition 2 will allow us to extend pluricanonical sections from X_0 to X under suitable conditions.

Proposition 1. Let L be a holomorphic line bundle over an n-dimensional compact complex manifold Y with a Hermitian metric which is locally of the form $e^{-\xi}$ with ξ plurisubharmonic. Let $\mathcal{I}_\xi$ be the multiplier ideal sheaf of the Hermitian metric $e^{-\xi}$. Let E be an ample holomorphic line bundle over Y such that for every point P of Y there are a finite number of elements of $\Gamma(Y, E)$ which all vanish to order at least $n+1$ at P and which do not simultaneously vanish outside P. Then $\Gamma(Y, \mathcal{I}_\xi \otimes (L + E + K_Y))$ generates $\mathcal{I}_\xi \otimes (L + E + K_Y)$ at every point of Y.

Proof. The key ingredient is the following result of Skoda [Sk72, Th.1, pp.555-556].

Let Ω be a pseudoconvex domain in $\mathbf{C}^n$ and ψ be a plurisubharmonic function on Ω. Let $g_1, \ldots, g_p$ be holomorphic functions on Ω. Let $\alpha > 1$ and $q = \inf(n, p-1)$. Then for every holomorphic function f on Ω such that

$$\int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda < \infty,$$

there exist holomorphic functions $h_1, \ldots, h_p$ on Ω such that

$$f = \sum_{j=1}^p g_j h_j$$

and

$$\int_\Omega |h|^2 |g|^{-2\alpha q} e^{-\psi} d\lambda \leq \frac{\alpha}{\alpha - 1} \int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda,$$

where

$$|g| = \left(\sum_{j=1}^p |g_j|^2 \right)^{\frac{1}{2}}, \qquad |h| = \left(\sum_{j=1}^p |h_j|^2 \right)^{\frac{1}{2}},$$

and $d\lambda$ is the Euclidean volume element of $\mathbf{C}^n$.

Fix arbitrarily $P \in Y$. Take an arbitrary element s of $(\mathcal{I}_\xi)_P$. Let $z = (z_1, \ldots, z_n)$ be a local coordinate system on some open neighborhood U of P with $z(P) = 0$ such that $L|U$ is trivial. Let ρ be a cut-off function centered at P so that ρ is a smooth nonnegative-valued function with compact support in U which is identically 1 on some Stein open neighborhood Ω of P. Choose $u_1, \ldots, u_N \in \Gamma(Y, E)$ whose common zero-set consists of the single point P and which all vanish to order at least $n+1$ at P. Let h_E be a smooth Hermitian metric of E whose curvature form is strictly positive at every point of Y. Let $0 < \eta < \frac{1}{n+1}$. By the standard techniques of L^2 estimates of $\bar\partial$, we can solve the equation

$$\bar\partial \sigma = s \bar\partial \rho$$

for a smooth section σ of $L + E + K_Y$ over Y which is L^2 with respect to the Hermitian metric

$$\frac{e^{-\xi} (h_E)^\eta}{\left(\sum_{j=1}^N |u_j|^2 \right)^{1-\eta}}$$

of $L+E$. Then $\rho s - \sigma$ is an element of $\Gamma(Y, \mathcal{I}_\xi \otimes (L+E+K_Y))$. Since $s\bar\partial\rho$ is identically zero on Ω, it follows that σ is holomorphic on Ω. We now apply Skoda's result to the case $g_j = z_j$ $(1 \leq j \leq n)$ with $q = n-1$ and $\alpha = \frac{(1-\eta)(n+1)-1}{n-1} > 1$ and $\psi = \xi$. (For the case $n = 1$ we simply choose α be any number greater than 1, because in that case αq is always zero.) Let $|z| = \left(\sum_{j=1}^n |z_j|^2\right)^{\frac{1}{2}}$. Since $u_1, \ldots, u_N$ all vanish to order at least $n+1$ at P, it follows that

$$\int_\Omega |\sigma|^2 e^{-\xi} |z|^{-2\alpha q - 2} = \int_\Omega |\sigma|^2 e^{-\xi} |z|^{-2(1-\eta)(n+1)} < \infty.$$

By Skoda's result

$$\sigma = \sum_{j=1} \tau_j z_j$$

locally at P for some $\tau_1, \ldots, \tau_n \in (\mathcal{I}_\xi)_P$.

Let J be the ideal at P generated by elements of $\Gamma(Y, \mathcal{I}_\xi \otimes (L+E+K_Y))$ over $(\mathcal{O}_Y)_P$. It follows from

$$\rho s - \sigma \in \Gamma(Y, \mathcal{I}_\xi \otimes (L+E+K_Y))$$

that

$$s \in J + \mathbf{m}_P (\mathcal{I}_\xi)_P,$$

where $\mathbf{m}_P$ is the maximum ideal of Y at P. Since s is an arbitrary element of $(\mathcal{I}_\xi)_P$, it follows that

$$(\mathcal{I}_\xi)_P \subset J + \mathbf{m}_P (\mathcal{I}_\xi)_P.$$

Clearly we have $J \subset (\mathcal{I}_\xi)_P$. Thus

$$(\mathcal{I}_\xi)_P / J \subset \mathbf{m}_P \left((\mathcal{I}_\xi)_P / J\right).$$

By Nakayama's lemma,

$$(\mathcal{I}_\xi)_P / J = 0$$

and $J = (\mathcal{I}_\xi)_P$. Q.E.D.

The next Proposition is an extension statement which is an adaptation of the extension theorem of Ohsawa-Takegoshi-Manivel.

Proposition 2. Let $\gamma : Y \to \Delta$ be a projective family of compact complex manifolds parametrized by the open unit 1-disk Δ. Let $Y_0 = \gamma^{-1}(0)$ and let

n be the complex dimension of Y_0. Let L be a holomorphic line bundle with a Hermitian metric which locally is represented by $e^{-\chi}$ such that $\sqrt{-1}\partial\bar{\partial}\chi \geq \omega$ in the sense of currents for some smooth positive $(1,1)$-form ω on Y. Let $0 < r < 1$ and $\Delta_r = \{\, t \in \Delta \mid |t| < r \,\}$. Then there exists a positive constant A_r with the following property. For any holomorphic L-valued n-form f on Y_0 with

$$\int_{Y_0} |f|^2 e^{-\chi} < \infty,$$

there exists a holomorphic L-valued $(n+1)$-form $\tilde{f}$ on $\gamma^{-1}(\Delta_r)$ such that $\tilde{f}|_{Y_0} = f \wedge \gamma^*(dt)$ at points of Y_0 and

$$\int_{Y} |\tilde{f}|^2 e^{-\chi} \leq A_r \int_{Y_0} |f|^2 e^{-\chi}.$$

Here no metrics of the tangent bundles of Y_0 and Y are needed to define the integrals of the absolute-value squares of top-degree holomorphic forms f and $\tilde{f}$ respectively on Y_0 and Y.

Proof. The proof can be easily adapted in the following way from the techniques given in [Si96] for the alternative proof there of the theorem of Ohsawa-Takegoshi. (Proofs can also be obtained by modifying those in [OT83, M93].) Let v be a meromorphic section of L over Y so that neither the pole-set nor the zero-set of v contains Y_0. Choose a complex hypersurface Z in Y containing the zero-set and the pole-set of v such that Z does not contain Y_0 and $Y - Z$ is Stein. For every positive integer ν let Ω_ν be a relatively compact Stein open subset of $Y - Z$ with smooth strictly pseudoconvex boundary such that $\cup_{\nu=1}^{\infty}\Omega_\nu = Y - Z$ and Ω_ν is relatively compact in $\Omega_{\nu+1}$. On $Y - Z$ under the isomorphism defined by division by v the line bundle $L|(Y - Z)$ is globally trivial. We let $\tilde{\chi}$ be the plurisubharmonic function $-\log(|v|^2 e^{-\chi})$ on $Y - Z$.

We now apply the techniques in [Si96] of the alternative proof of the theorem of Ohsawa-Takegoshi to extend, after multiplication by $\gamma^*(dt)$, the top-degree holomorphic form $\frac{f}{v}$ on $\Omega_\nu \cap Y_0$ which is L^2 on $\Omega_\nu \cap Y_0$ with respect to $e^{-\tilde{\chi}}$ to a top-degree holomorphic form G_ν on $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$ whose L^2 norm on $\gamma^{-1}(\Delta_r) \cap \Omega_\nu$ with respect to $e^{-\tilde{\chi}}$ is bounded by a finite constant independent of ν. When we apply the techniques of the alternative proof of the theorem of Ohsawa-Takegoshi, we have to use holomorphic tangent vector fields of the Stein manifold $\Omega_{\nu+1}$ to get a sequence of smooth plurisubharmonic functions on Ω_ν which approach the plurisubharmonic function $\tilde{\chi}$ on

Ω_ν. The extension $\tilde{f}$ is obtained as the limit of $G_\nu v$ as ν goes to infinity. The smooth positive $(1,1)$-form ω in the assumption is needed for the ν-independent *a priori* estimates for the solution of the modified $\bar\partial$ equation on $\gamma^{-1}(\Delta_r)\cap\Omega_\nu$ in the techniques of the alternative proof of the theorem of Ohsawa-Takegoshi. Q.E.D.

Construction of Hermitian metrics on K_{X_0}. One of the technical details is that the Hermitian metric on K_X has to be chosen to make sure that its curvature current dominates some smooth positive $(1,1)$-form in order to apply the L^2 existence and extension theorems of Skoda and Ohsawa-Takegoshi-Manivel. For that modification the Kodaira technique of writing some high multiple of a big line bundle as an effective divisor plus an ample line bundle is used.

Lemma 3. There is a positive integer a such that $aK_X = D + F$, where D is an effective divisor on X not containing X_0 and F is a positive line bundle on X.

Proof. Choose a positive line bundle F over X such that the divisor Y of some $u \in \Gamma(X, F)$ intersects X_0 normally at every point of $X_0 \cap Y$. For every positive integer a the zeroth direct image $R^0\pi_*(\mathcal{O}_X(aK_X))$ of $\mathcal{O}_X(aK_X)$ under π and the zeroth direct image $R^0(\pi|Y)_*(\mathcal{O}_Y(aK_X|Y))$ of $\mathcal{O}_Y(aK_X|Y)$ under $\pi|Y$ are both locally free at the point $t = 0$ of Δ. Since each X_t is of general type for $t \in \Delta$, the rank of $R^0\pi_*(\mathcal{O}_X(aK_X))$ at $t = 0$ is at least $c_1 a^n$ for some positive constant c_1 independent of a. The rank of $R^0(\pi|Y)_*(\mathcal{O}_Y(aK_X|Y))$ at $t = 0$ is at most $c_2 a^{n-1}$ for some positive constant c_2 independent of a. From the exact sequence

$$R^0\pi_*(\mathcal{O}_X(aK_X - F)) \xrightarrow{\alpha} R^0\pi_*(\mathcal{O}_X(aK_X)) \to R^0(\pi|Y)_*(\mathcal{O}_Y(aK_X|Y))$$

with the map α defined by multiplication by u it follows that, for a sufficiently large, there exists a nonzero element of the stalk of $R^0\pi_*(\mathcal{O}_X(aK_X - F))$ at $t = 0$. Since $R^0\pi_*(\mathcal{O}_X(aK_X - F))$ is coherent on Δ and Δ is Stein, there exists a nonzero element $s \in \Gamma(X, aK_X - F)$ for a sufficiently large. There exists a unique nonnegative integer p such that $\frac{s}{t^p}$ is a holomorphic section of $aK_X - F$ whose restriction to X_0 is not identically zero. It suffices to set D to be the divisor of $\frac{s}{t^p}$ to get $aK_X = D + F$. Q.E.D.

The next step is to construct Hermitian metrics on K_{X_0} and K_X, respectively, and to compare their related multiplier ideal sheaves. For every positive integer m we choose a basis

$$s_{0,1}^{(m)}, \cdots, s_{0,q_{m,0}}^{(m)} \in \Gamma\left(X_0, mK_{X_0}\right)$$

and we choose

$$s_1^{(m)}, \cdots, s_{q_m}^{(m)} \in \Gamma\left(X, mK_X\right)$$

so that

$$s_1^{(m)}|X_0, \cdots, s_{q_m}^{(m)}|X_0$$

is a basis of $\Gamma\left(X, mK_X\right)|X_0$ and $s_\nu^{(m)} = s_{0,\nu}^{(m)}$ for $1 \leq \nu \leq q_m$. We choose a sequence of positive numbers θ_m so that

$$\sum_{m=1}^{\infty} \theta_m \left(\sum_{\nu=1}^{q_{m,0}} \left| s_{0,\nu}^{(m)} \right|^{\frac{2}{m}} \right)$$

converges uniformly on compact subsets of X_0 to a Hermitian metric of $-K_{X_0}$ and

$$\sum_{m=1}^{\infty} \theta_m \left(\sum_{\nu=1}^{q_m} \left| s_\nu^{(m)} \right|^{\frac{2}{m}} \right)$$

converges uniformly on compact subsets of X to a Hermitian metric of $-K_X$. Locally on X_0 we define

$$\varphi_0 = \log \sum_{m=1}^{\infty} \theta_m \left(\sum_{\nu=1}^{q_{m,0}} \left| s_{0,\nu}^{(m)} \right|^{\frac{2}{m}} \right)$$

so that $e^{-\varphi_0}$ is a Hermitian metric of K_{X_0}. Locally on X we define

$$\varphi = \log \sum_{m=1}^{\infty} \theta_m \left(\sum_{\nu=1}^{q_m} \left| s_\nu^{(m)} \right|^{\frac{2}{m}} \right)$$

so that $e^{-\varphi}$ is a Hermitian metric of K_X. By construction, we have $\varphi \leq \varphi_0$ on X_0.

After possibly passing to a multiple in Lemma 3, we can choose an integer $a \geq 2$ such that $aK_X = D + F$, where D is an effective divisor on X not containing X_0 and F is such a high multiple of a positive line bundle on X that

(i) for every point $P \in X_0$, there exist a finite number of elements of $\Gamma(X, F - 2K_X)|X_0$ whose common zero-set consists only of the single point P and which all vanish to order at least $n + 1$ at P and

(ii) a basis of $\Gamma(X, F)|X_0$ embeds X_0 as a complex submanifold of some complex projective space.

Let s_D be the canonical section of the holomorphic line bundle D so that the divisor of s_D is D. Let

$$u_1, \ldots, u_N \in \Gamma(X, F)$$

such that

$$u_1 \mid X_0, \ldots, u_N \mid X_0$$

form a basis of $\Gamma(X, F)|X_0$. Since $s_D u_j \in \Gamma(X, aK_X)$ $(1 \leq j \leq N)$, we get a Hermitian metric

$$e^{-\psi} = \left(\frac{1}{|s_D|^2 \sum_{j=1}^N |u_j|^2}\right)^{\frac{1}{a}}$$

for the line bundle K_X. We can multiply the sections u_j by small coefficients in such a way that the sections $s_D u_j|X_0$ are linear combinations with small coefficients of the basis sections $s_1^{(a)}|X_0, \cdots, s_{q_a}^{(a)}|X_0$. Then we can assume without loss of generality that $\psi \leq \varphi$ over X, possibly after shrinking Δ if necessary. Moreover

$$\psi = \frac{1}{a}\left(\log |s_D|^2 + \chi\right)$$

where $\chi = \log(\sum_{j=1}^N |u_j|^2)$ defines a smooth Hermitian metric of F with smooth positive curvature form on X, after shrinking Δ if necessary. Thus we have the following lemma.

Lemma 4. The curvature current of the Hermitian metric $e^{-\psi}$ dominates some smooth positive $(1,1)$-form on X and $\psi \leq \varphi \leq \varphi_0$ on X_0.

Our crucial argument is the following comparison result for multiplier ideal sheaves over X_0. As stated at the beginning of the paper, in the statement below and for the rest of the paper, the notation $\mathcal{I}_{(\ell+a-\epsilon)\varphi+\epsilon\psi}$ denotes an ideal sheaf on X_0 and not an ideal sheaf on X.

Proposition 5. Let $0 < \epsilon < 1$ be sufficiently small so that $e^{-\epsilon\psi}$ is locally integrable on X (after shrinking Δ if necessary) and $e^{-\epsilon\psi}|X_0$ is locally integrable on X_0. Then

$$\mathcal{I}_{(\ell-\epsilon)\varphi_0+(a+\epsilon)\psi} \subset \mathcal{I}_{(\ell-1+a-\epsilon)\varphi+\epsilon\psi}$$

as ideal sheaves on X_0 for all integers $\ell \geq 1$.

Proof. We prove by induction on ℓ. For $\ell = 1$, Lemma 4 implies that

$$(1-\epsilon)\varphi_0 + (a+\epsilon)\psi \leq (1-\epsilon)\varphi_0 + a\varphi + \epsilon\psi \leq C + (a-\epsilon)\varphi + \epsilon\psi,$$

where C is a constant which is a local upper bound of the local plurisubharmonic function $(1-\epsilon)\varphi_0 + \epsilon\varphi$. We thus get

$$\mathcal{I}_{(1-\epsilon)\varphi_0+(a+\epsilon)\psi} \subset \mathcal{I}_{(a-\epsilon)\varphi+\epsilon\psi}$$

as required.

Now, assume that the required inclusion has been proved for Step ℓ. For the proof of Step $\ell+1$, we take a function germ f in the ideal sheaf $\mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi}$ and we are going to prove that the function germ f belongs to the ideal sheaf $\mathcal{I}_{(\ell+a-\epsilon)\varphi+\epsilon\psi}$.

Let $E = (F - 2K_X)\,|X_0$. From $aK_{X_0} = (D+F)|X_0$ it follows that

$$\begin{aligned}(\ell+a)K_{X_0} &= (\ell+1)K_{X_0} + (a-2)K_{X_0} + K_{X_0} \\ &= ((\ell+1)K_{X_0} + (D|X_0)) + E + K_{X_0}.\end{aligned}$$

We give $(\ell+1)K_{X_0} + (D|X_0)$ the metric

$$e^{-\xi} = \frac{e^{-(\ell+1-\epsilon)\varphi_0-\epsilon\psi}}{|s_D|^2}.$$

Since

$$\xi = (\ell+1-\epsilon)\varphi_0 + (a+\epsilon)\psi - a\chi$$

and χ is smooth, it follows that

$$\mathcal{I}_\xi = \mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi}$$

as ideal sheaves on X_0. By Proposition 1 applied to $Y = X_0$ and $L = (\ell+1)K_{X_0} + (D|X_0)$ with the metric $e^{-\xi}|X_0$, it follows that the multiplier ideal sheaf $\mathcal{I}_\xi$ on X_0 is generated by

$$\begin{aligned}&\Gamma\left(X_0, \mathcal{I}_\xi \otimes ((\ell+1)K_{X_0} + (D|X_0)) + E + K_{X_0}\right) \\ &\quad = \Gamma\left(X_0, \mathcal{I}_\xi \otimes ((\ell+a)K_{X_0})\right).\end{aligned}$$

Hence the multiplier ideal sheaf $\mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi}$ on X_0 is generated by

$$\Gamma\left(X_0, \mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi} \otimes ((\ell+a)K_{X_0})\right).$$

To prove that the function germ f belongs to the ideal sheaf $\mathcal{I}_{(\ell+a)\varphi+\epsilon\psi}$, we can now assume without loss of generality that f is the germ of an element

$$F \in \Gamma\left(X_0, \mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi} \otimes ((\ell+a)K_{X_0})\right).$$

Since φ_0 is locally bounded from above on X_0, we have

$$\mathcal{I}_{(\ell+1-\epsilon)\varphi_0+(a+\epsilon)\psi} \subset \mathcal{I}_{(\ell-\epsilon)\varphi_0+(a+\epsilon)\psi}.$$

By induction assumption,

$$\mathcal{I}_{(\ell-\epsilon)\varphi_0+(a+\epsilon)\psi} \subset \mathcal{I}_{(\ell-1+a-\epsilon)\varphi+\epsilon\psi}.$$

Hence the germ of F at any point of X_0 belongs to the ideal sheaf $\mathcal{I}_{(\ell-1+a-\epsilon)\varphi+\epsilon\psi}$ and

$$F \in \Gamma\left(X_0, \mathcal{I}_{(\ell-1+a-\epsilon)\varphi+\epsilon\psi} \otimes ((\ell+a)K_{X_0})\right).$$

Since $(\ell+a)K_{X_0} = ((\ell-1+a)K_X)|X_0+K_{X_0}$ and the weight $(\ell-1+a-\epsilon)\varphi+\epsilon\psi$ defines a Hermitian metric of $(\ell-1+a)K_X$ whose curvature current dominates some smooth positive $(1,1)$-form on X, it follows from Proposition 2 that F can be extended to an element

$$\tilde{F} \in \Gamma\left(X, (\ell+a)K_X\right).$$

Since the definition of φ implies that $|F|^2 e^{-(\ell+a)\varphi}$ is uniformly bounded on X_0, it follows from the local integrability of $e^{-\epsilon\psi}$ that the germ of F at any point of X_0 belongs to the ideal sheaf $\mathcal{I}_{(\ell+a-\epsilon)\varphi+\epsilon\psi}$. Thus Step $\ell+1$ of the induction is proved. Q.E.D

Proof of the Main Theorem. Let m be any positive integer. Take a section $s \in \Gamma(X_0, mK_{X_0})$. The definition of φ_0 implies that $|s|^2 e^{-m\varphi_0}$ is uniformly bounded on X_0. Thus $s^\ell s_D$ is locally L^2 with respect to the weight $e^{-\ell m\varphi_0-(a+\epsilon)\psi}$, because locally the difference between $a\psi$ and $\log|s_D|^2$ is a smooth function and $e^{-\epsilon\psi}$ is locally integrable on X_0. Hence the germ of $s^\ell s_D$ at any point of X_0 belongs to the ideal sheaf $\mathcal{I}_{(\ell m-\epsilon)\varphi_0+(a+\epsilon)\psi}$. By Proposition 5 the germ of $s^\ell s_D$ at any point of X_0 also belongs to $\mathcal{I}_{(\ell m-1-\epsilon)\varphi+\epsilon\psi}$. Thus

$$\int_U |s^\ell s_D|^2 e^{-(\ell m-1-\epsilon)\varphi-\epsilon\psi} < +\infty$$

on every sufficiently small open set U on which both K_{X_0} and the line bundle defined by the divisor D are trivial. It follows from the definition of ψ that

$$\int_U |s|^{2\ell} e^{-(\ell m-1-\epsilon)\varphi+(a-\epsilon)\psi} < \infty.$$

Take ℓ so large that $\frac{a}{\ell-1} \leq \epsilon$. Then $e^{\varphi - \frac{a}{\ell-1}\psi}$ is integrable over U, because, as a plurisubharmonic function, φ is locally bounded from above. Let $\delta = \frac{\epsilon}{\ell}$. By Hölder's inequality with conjugate exponents ℓ, $\ell' = \frac{\ell}{\ell-1}$, we conclude that

$$
\begin{aligned}
&\int_U |s|^2 e^{-(m-1-\delta)\varphi - \delta\psi} = \int_U |s|^2 e^{-(m-\frac{1}{\ell}-\frac{\epsilon}{\ell})\varphi + (\frac{a}{\ell}-\frac{\epsilon}{\ell})\psi} e^{(1-\frac{1}{\ell})\varphi - \frac{a}{\ell}\psi} \\
&\leq \left(\int_U |s|^{2\ell} e^{-(\ell m-1-\epsilon)\varphi + (a-\epsilon)\psi}\right)^{1/\ell} \left(\int_U e^{\varphi - \frac{a}{\ell-1}\psi}\right)^{(\ell-1)/\ell} < \infty.
\end{aligned}
$$

We now regard s as a section of $K_{X_0} + (L|X_0)$ with $L = (m-1)K_X$. The weight $(m-1-\delta)\varphi - \delta\psi$ defines a Hermitian metric of L whose curvature current dominates some smooth positive $(1,1)$-form on X. By Proposition 2 we can therefore extend s to $\tilde{s} \in \Gamma(X, K_X + L) = \Gamma(X, mK_X)$, possibly after shrinking Δ. This concludes the proof of the Main Theorem.

Chapter 7. An Effective Matsusaka Big Theorem

§0. *Introduction*

In this paper we will prove an effective form of the following Matsusaka Big Theorem ([M1, M2, LM]). Let $P(k)$ be a polynomial whose coefficients are rational numbers and whose values are integers at integral values of k. Then there is a positive integer k_0 depending on $P(k)$ such that, for every compact projective algebraic manifold X of complex dimension n and every ample line bundle L over X with $\sum_{\nu=0}^{n}(-1)^\nu \dim H^\nu(X,\, k\ L) = P(k)$ for every k, the line bundle $k\ L$ is very ample for $k \geq k_0$. Here ampleness means that the holomorphic line bundle admits a smooth Hermitian metric whose curvature form is positive definite everywhere. Very ampleness means that global holomorphic sections separate points and give local homogeneous coordinates at every point. By a result of Kollar and Matsusaka on Riemann-Roch type inequalities [KM], the positive integer k_0 can be made to depend only on the coefficients of k^n and k^{n-1} in the polynomial $P(k)$ of degree n. The known proofs of Matsusaka's Big Theorem depend on the boundedness of numbers calculated for some varieties and divisors in a bounded family and thus the positive integer k_0 from such proofs cannot be effectively computed from $P(k)$. To state our effective Matsusaka big theorem, we use the following standard notation. For holomorphic line bundles $L_1, \cdots, L_\ell$ over a compact complex manifold X of complex dimension n with Chern classes $c(L_1), \cdots, c(L_\ell)$ and for positive integers $k_1, \cdots, k_\ell$ with $k_1 + \cdots + k_\ell = n$, we denote the Chern number $c(L_1)^{k_1} \cdots c(L_\ell)^{k_\ell}$ by $L_1^{k_1} \cdots L_\ell^{k_\ell}$.

Our effective version of Matsusaka's Big Theorem is the following.

Theorem (*0.1*). Let L be an ample holomorphic line bundle over a compact complex manifold X of complex dimension n with canonical line bundle K_X. Let $C_n = 3(3n-2)^n$ and $C_n' = 2C_n - (n+1)$. Then $m\ L$ is very ample for $m \geq (C_n')^{-((2n-1)3^{n-d}-1)/2}(2^{2n-1}5n)^{3^{n-1}}((L^n)^{-(n-1)}((C_nL+K_X)\cdot L^{n-1})^n)^{2\cdot 3^{n-1}}$.

Theorem (0.1) is a consequence of the following Theorem (0.2) when the numerically effective holomorphic line bundle B in Theorem (0.2) is specialized to the trivial line bundle. Here the numerical effectiveness of B means that the value of the first Chern class of the line bundle B evaluated at any complex curve is nonnegative.

Theorem (*0.2*). Let X be a compact complex manifold of complex dimension n and L be an ample line bundle over X and B be a numerically effective holomorphic line bundle over X. Let $C_n = 3(3n-2)^n$ and $H = 2(K_X + C_n L)$ and $C'_n = 2C_n - (n+1)$. Then $mL - B$ is very ample for $m \geq (C'_n)^{-((2n-1)3^{n-d}-1)/2}(n\ H^n(H^{n-1}\cdot B + \frac{5}{2}H^n))^{3^{n-1}}$.

The result of Demailly [D2] that $12\ n^n L + 2\ K_X$ is very ample for any ample line bundle L over X can also be regarded as an effective version of Matsusaka's big theorem. The difference between Demailly's result and Theorem (0.2) is that in Theorem (0.2) $2\ K_X$ is no longer needed and in its place $-B$ can be used for any ample line bundle B. On the other hand the coefficient $12\ n^n$ of L in Demailly's result depends only on n and is far sharper, whereas in Theorem (0.2) the coefficient for L depends on L^n, $L^{n-1}\cdot K_X$, $L^{n-1}\cdot B$, and $L^{n-2}\cdot B\cdot K_X$ as is expected. (The condition in Theorem (0.2) is expressible in terms of L^n, $L^{n-1}\cdot K_X$, $L^{n-1}\cdot B$, and $L^{n-2}\cdot B\cdot K_X$ by using inequalities of Chern numbers of numerically effective line bundles (see [D2, Prop.5.2(b)] and the end of §4.) Kollar [K] gave an algebraic geometric proof of a result similar to but weaker than Demailly's criterion for very ampleness. For example, Kollar's result gives the very amplenss of $(n+3)!2(n+1)((n+2)L+K_X)+K_X$ for an ample line bundle L over X (by replacing L by $(n+2)L+K_X$ and setting $a=1$ in Theorem (1.1) and setting $N = K_X$ in Lemma (1.2) in [K]). We would like to mention also the following result in this area due to Ein and Lazarsfeld [EL]: For a big and numerically effective divisor L on a smooth complex projective threefold X, if $L\cdot C \geq 3$ and $L^2\cdot S \geq 7$ for any curve C in X and any surface S in X, then $L + K_X$ is generated by global holomorphic sections on X.

An earlier version of this paper gave the following bound for Theorem (0.1): $m\ L$ is very ample for $m \geq (24n^n C(1+C)^n)^{n(6n^3)^n}$ with $C = ((n+2)L+K_X)L^{n-1}$, which is by several order of magnitude not as sharp as the bound stated in the present Theorem (0.1). The earlier version of Theorem (0.2) gave the very ampleness of $m\ L - B$ for $m \geq (24n^n C(1+C)^n)^{n(6n^3)^n}$ with $C = ((n+2)L+B+K_X)L^{n-1}$. After the earlier version was circulated, Demailly told me a way to simplify the original proof and the simplification yielded the much sharper bound for Theorem (0.1). The simplification is to verify directly the numerical effectiveness of some line bundle by using its holomorphic sections on subvarieties of decreasing dimensions and to bypass the step, in the earlier version, of extending first those sections to the ambient manifold. I would like to thank Demailly for the simplification of the proof and the sharpening of the bound. One lemma in the proof of our effective Matsusaka big theorem uses the strong Morse inequality of Demailly which

Demailly obtained by analytic methods. Lawrence Ein and Robert Lazarsfeld told me a simple algebraic proof of that lemma which avoids the use of the strong Morse inequality of Demailly. F. Catanese also has a simple algebraic proof of that lemma similar to the proof of Ein and Lazarsfeld. Both simple algebraic proofs of that lemma are reproduced here. I would like to than Ein, Lazarsfeld, and Catanese for their simple algebraic proofs of that lemma.

The method of proof of Theorem (0.2) uses the strong Morse inequality and the numerical criterion of very ampleness of Demailly [D1, D2]. The earlier version used also Nadel's vanishing theorem [N] in order to extend some holomorphic sections of a line bundle from a subvariety to the ambient manifold. Kollar's result (whose proof is algebraic geometric) can be used instead of the very ampleness criterion of Demailly [D2] whose proof uses analysis. With the use of Kollar's result and the simple algebraic proofs of the lemma mentioned above due to Ein-Lazarsfeld and Catanese, the proof of our effective Matsusaka big theorem in this paper can be done purely algebraically, though the bound of Kollar's result is weaker than that of Demailly and therefore results in a bound that is less sharp.

The main idea of the proof of the effective Matsusaka big theorem in this paper is the following. Because of the very ampleness criterion of Demailly and Kollar, it suffices to show that for a given numerically effective line bundle B and an ample line bundle L there is a effective lower bound m such that $m\ L - B$ is numerically effective. This is done in three steps. The first step is a lemma on the existence of nontrivial holomorphic sections of a multiple of the difference of two ample line bundles whose Chern classes satisfy a certain inequality. This is the lemma for which the strong Morse inequality of Demailly is used and for which Ein-Lazarsfeld and Catanese gave simple algebraic proofs. Such a nontrivial holomorphic section enables us to construct a closed positive current which is a curvature current of the line bundle and the curvature current will be used in an application of L^2 estimates of $\bar\partial$ to construct holomorphic sections. The second step is to produce, for any d-dimensional irreducible subvariety Y of X and any very ample line bundle H of X, a nontrivial holomorphic section over Y of the homomorphism sheaf from the sheaf of holomorphic d-forms of Y to $(3\lambda(\lambda-1)/2-d-1)H|Y$ for $\lambda \geq H^d \cdot Y$. The sheaf of holomorphic d-forms of Y is defined from the presheaf of holomorphic p-forms on the regular part of Y which are L^2. This second step is obtained by representing Y as a branched cover over a complex projective space and the section is obtained by constructing explicitly a global sheaf-homomorphism by solving linear equations by Cramer's rule. The key point is that since the canonical line bundle of the complex projective space of complex dimension d is equal *to* $-(d+1)$ times the hyperplane section line bundle, by representing a compact complex space as a branched cover of the complex projective space, one can relate its canonical line bundle to a very ample line bundle on it. The third step is to get the numerical effectiveness of $m\,L - B$. The earlier version of this paper used Nadel's vanishing theorem for multiplier ideal sheaves [N] to produce global holomorphic sections of high multiples of $m\,L - B$ so that the dimension of the common zero-sets of these sections is inductively reduced to zero. A new stratification of unreduced structure sheaves by multiplier ideal sheaves was introduced in the earlier version to carry out the induction process. Demailly's simplification verifies the numerically effectiveness of $m\,L - B$ by producing holomorphic sections of high multiples of $m\,L - B$ on subvarieties of decreasing dimensions. The bypassing of the extension of those holomorphic sections (called $\sigma_{d,j}$ in §4) to the ambient manifold makes it unnecessary to use Nadel's vanishing theorem for multiplier ideal sheaves and the new way of stratifying unreduced structure sheaves. Though not used here, such a new stratification of unreduced structure sheaves by multiplier ideal sheaves can be applied in other context, *e.g.* to get a Demailly type very ampleness criterion for $m\,L + 2\,K_X$ without using the analytic tools of the Monge-Ampère equation, but we will not discuss it here.

§1. *Effectiveness of Multiples of Differences of Ample Line Bundles.*

First we state the strong Morse inequality due to Demailly [D1]. Let X be a compact complex manifold of complex dimension n, E be a holomorphic line bundle over X with smooth Hermitian metric along its fibers whose curvature form is $\theta(E)$, $X(\leq q)$ be the open subset of X consisting of all points of X where the curvature form $\theta(E)$ has no more than q negative eigenvalues. Demailly proved the following strong Morse inequality [D1].

$$\Sigma_{j=0}^{q}(-1)^{q-j}\dim H^j(X,\,k\,E) \leq \frac{k^n}{n!}\int_{X(\leq q)}(-1)^q(\theta(E))^n + o(k^n).$$

Here $o(k^n)$ means the Landau symbol denoting a term whose quotient by k^n goes to 0 as $k \to \infty$. We now apply Demailly's strong Morse inequality to the case where $E = F - G$ with F and G being Hermitian holomorphic line bundles over a compact Kähler manifold X with semipositive curvature forms.

Lemma (1.1). Let F and G be holomorphic line bundles over a compact Kähler manifold X with Hermitian metrics so that their curvature forms are semipositive. Then for every $0 \leq q \leq \dim X$,

$$\sum_{0 \leq j \leq q} (-1)^q \dim H^j(X,\, k(F-G)) \leq \frac{k^n}{n!} \sum_{0 \leq j \leq q} (-1)^{q-j} \tbinom{n}{j} F^{n-j} G^j + o(k^n)$$

as $k \to \infty$.

Proof. Let $\theta(F)$ and $\theta(G)$ be respectively the curvature forms of F and G which are smooth and semipositive. Let ω be the Kähler form of X. For $\epsilon > 0$, let $\theta_\epsilon(F) = \theta(F) + \epsilon\,\omega$ and $\theta_\epsilon(G) = \theta(G) + \epsilon\,\omega$. Then $\theta(F-G) = \theta_\epsilon(F) - \theta_\epsilon(G)$. Let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be the eigenvalues of $\theta_\epsilon(G)$ with respect to $\theta_\epsilon(F)$. Then $X(\leq q)$ is precisely the set of all x in X such that $\lambda_{q+1}(x) < 1$. Demailly's strong Morse inequality [D1] now reads

(1.1.1)
$$\sum_{0 \leq j \leq q} (-1)^q \dim H^j(X, k(F-G)) \leq \frac{k^n}{n!} \int_{X(\leq q)} (-1)^q \prod_{1 \leq j \leq n} (1-\lambda_j) \theta_\epsilon(F)^n + o(k^n).$$

Let σ_k^j be the j^{th} elementary symmetric function in $\lambda_1, \cdots, \lambda_k$. We use the convention that $\sigma_k^j = 1$ when $j = 0$ and $\sigma_k^j = 0$ when $j < 0$. Then

$$\tbinom{n}{j} \theta_\epsilon(F)^{n-j} \theta_\epsilon(G)^j = \sigma_n^j \theta_\epsilon(F)^n. \tag{1.1.2}$$

We claim that

$$\sum_{0 \leq j \leq q} (-1)^{q-j} \sigma_n^j > (-1)^q \prod_{1 \leq j \leq n} (1 - \lambda_j)$$

for $\lambda_{q+1} < 1$. We verify the claim by induction on n. Assume that it is true when n is replaced by $n-1$. From

$$\sigma_n^j = \sigma_{n-1}^j + \sigma_{n-1}^{j-1} \lambda_n$$

it follows that

$$\sum_{0 \leq j \leq q} (-1)^{q-j} \sigma_n^j = \sum_{0 \leq j \leq q} (-1)^{q-j} (\sigma_{n-1}^j + \sigma_{n-1}^{j-1} \lambda_n)$$

$$
\begin{aligned}
&= \sum_{0\le j\le q}(-1)^{q-j}\sigma_{n-1}^{j} - \lambda_n \sum_{0\le j\le q-1}(-1)^{q-j}\sigma_{n-1}^{j} \\
&= (1-\lambda_n)\sum_{0\le j\le q}(-1)^{q-j}\sigma_{n-1}^{j} + \lambda_n\sigma_{n-1}^{q} \\
&\ge (1-\lambda_n)\sum_{0\le j\le q}(-1)^{q-j}\sigma_{n-1}^{j} > (-1)^q\prod_{1\le j\le n}(1-\lambda_j).
\end{aligned}
$$

This proves the claim. The lemma now follows from the claim and (1.1.1) and (1.1.2) when we let $\epsilon \to 0$. Q.E.D. In the earlier version of this paper the argument for the lemma was carried out only for $q = 1$ and was applied to get a nontrivial holomorphic section of $k(F - G)$ over X for k sufficiently large under the assumption

$$(F-G)^n - \sum_{q=1}^{[n/2]}\binom{n}{2q}F^{n-2q}G^{2q} > 0,$$

where the square bracket $[\cdot]$ means the integral part. There the same method of simultaneous diagonalization of curvature forms was used but with less precise analysis of the inequality concerning the eigenvalues $\lambda_1, \cdots, \lambda_n$. This more precise form for a general q was suggested to me by Demailly. In this paper the lemma will be applied only to the case $q = 1$ in the form of the following corollary.

Corollary (1.2). Let X be a compact projective algebraic manifold of complex dimension n and F and G be numerically effective line bundles over X such that $F^n > n\ F^{n-1}G$. Then for k sufficiently large there exists a nontrivial holomorphic section of $k(F - G)$ over X.

Ein-Lazarsfeld and Catanese independently obtained similar algebraic geometric proofs of this corollary. We present them here.

The proof of Ein-Lazarsfeld. For a coherent sheaf $\mathcal{F}$ over X, let $h^p(\mathcal{F})$ denote $\dim_{\mathbf{C}} H^p(X, \mathcal{F})$. By replacing F and G by $\ell\ F+L$ and $\ell\ G+L$ for some ample line bundle L over X and some sufficiently large ℓ we can assume without loss of generality that both F and G are ample. By multiplying both F and G by the same large positive number, we can assume without loss of generality that F and G are both very ample. Choose a smooth irreducible divisor D in the linear system $|G|$. Consider the exact sequence

$$0 \to \mathcal{O}_X(k(F-G)) \to \mathcal{O}_X(k\ F) \to \mathcal{O}_{kD}(k\ F) \to 0.$$

It is enough to show that for k sufficiently large $h^0(\mathcal{O}_X(k\ F)) > h^0(\mathcal{O}_{kD}(k\ F))$. There is a natural filtration of the sheaf $\mathcal{O}_{kD}(k\ F)$ whose quotients are sheaves of the form $\mathcal{O}_{kD}(k\ F - j\ D)$ for $0 \leq j \leq k-1$. So

$$h^0(\mathcal{O}_{kD}(k\ F)) \leq \sum_{j=0}^{k-1} h^0(\mathcal{O}_{kD}(k\ F - j\ G)).$$

On the other hand, since G is very ample, we can choose a second smooth irreducible divisor D' in the linear system $|G|$ which meets D in normal crossing. Since $\mathcal{O}_{kD}(k\ F - j\ G)$ is isomorphic to $\mathcal{O}_{kD}(k\ F - j\ D')$ which is a subsheaf of $\mathcal{O}_{kD}(k\ F)$, it follows that $h^0(\mathcal{O}_{kD}(k\ F - j\ G)) \leq h^0(\mathcal{O}_{kD}(k\ F))$. Thus $h^0(\mathcal{O}_{kD}(k\ F)) \leq k \cdot h^0(\mathcal{O}_D(k\ F))$. By Kodaira's vanishing theorem and the theorem of Riemann-Roch

$$k \cdot h^0(\mathcal{O}_D(k\ F)) = \frac{k \cdot (k\ F)^{n-1} \cdot D}{(n-1)!} + o(k^n) = \frac{k^n F^{n-1} \cdot G}{(n-1)!} + o(k^n),$$

whereas $h^0(\mathcal{O}_X(k\ F)) = \frac{k^n F^n}{n!} + o(k^n)$, from which we obtain the conclusion.

Catanese's Proof. For a line bundle H over X, let $h^p(X,\ H)$ denote $\dim_{\mathbf{C}} H^p(X,\ H)$. As in the proof of Ein-Lazarsfeld we can assume without loss of generality that F and G are both very ample. Let k be any positive integer. We select k smooth members G_j, $1 \leq j \leq k$ in the linear system $|G|$ and consider the exact sequence

$$(1.2.1)\quad 0 \to H^0(X,\ k\ F - \sum_j G_j) \to H^0(X,\ k\ F) \to \oplus_{j=1}^k H^0(G_j,\ k\ F|G_j).$$

By (1.2.1) and Kodaira's vanishing theorem and the theorem of Riemann-Roch

$$\begin{aligned} h^0(X, k(F-G)) &\geq \frac{k^n}{n!}F^n - o(k^n) - \sum_{j=1}^{k}\left(\frac{k^{n-1}}{(n-1)!}F^{n-1} \cdot G_j - o(k^{n-1})\right) \\ &\geq \frac{k^n}{n!}(F^n - n\ F^{n-1} \cdot G) - o(k^n). \end{aligned}$$

So for k sufficiently large there exists nontrivial holomorphic section of $k(F - G)$ over X.

What we need from Corollary (1.2) is a curvature current for the difference of the two numerically effective line bundles. Let L be a holomorphic line bundle on a compact complex manifold X of complex dimension n. Let X be covered by a finite open cover $\{U_j\}$ so that $L|U_j$ is trivial and let g_{jk} on $U_j \cap U_k$ be the transition function for L. A curvature current $\theta(L)$ for L is a closed positive (1,1)-current on X which is given by $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_j$ on U_j where φ_i is a plurisubharmoinc function on U_j such that $e^{-\varphi_j}|g_{jk}|^2 = e^{-\varphi_k}$ on $U_j \cap U_k$. The collection $\{e^{-\varphi_j}\}$ defines a (possibly nonsmooth) Hermitian metric along the fibers of L. A closed positive (1,1)-current θ means a current of type (1,1) which is closed and is positive in the sense that $\theta \wedge \prod_{j=1}^{n-1}(\sqrt{-1}\alpha_j \wedge \bar{\alpha}_j^-)$ is a nonnegative measure on X for any smooth (1,0)-forms $\alpha_j (1 \leq j \leq n-1)$ with compact support. A closed positive (1,1)-current is characterized by the fact that locally it is of the form $\sqrt{-1}\partial\bar{\partial}\psi$ for some plurisubharmonic function ψ. The Lelong number, at a point P, of a closed positive (1,1)-current θ on some open subset of $\mathbf{C}^n$ is defined as the limit of $(\pi r^2)^{1-n}\theta \wedge (\frac{\sqrt{-1}}{2}\partial\bar{\partial}|z|^2)^{n-1}$ over the ball of radius r centered at P as $r \to 0$, where $z = (z_1, \cdots, z_n)$ is the coordinate of $\mathbf{C}^n$. The normalization is such that the Lelong number of $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^2$ is 1 at 0. The Lelong number is independent of the choice of local coordinates. The function $e^{-\psi}$ is locally integrable at P if the closed positive (1,1)-current $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$ has Lelong number less than 1 at P. The function $e^{-\psi}$ is not locally integrable at P if the closed positive (1,1)-current $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$ has Lelong number at least n at P. See *e.g.* [L] and [S] for more detailed information on closed positive currents and Lelong numbers.

Corollary (*1.3*). Let X be a compact projective algebraic manifold of complex dimension n and F and G be numerically effective line bundles over X such that $F^n > n\, F^{n-1}G$. Then there exists a closed positive current which is the curvature current for $F - G$.

Proof. Let X be covered by a finite open cover $\{U_j\}$ so that $(F-G)|U_j$ is trivial and let $g_{j\ell}$ on $U_j \cap U_\ell$ be the transition function for $F - G$. Let s be a nontrivial holomorphic section for $k(F-G)$ for some sufficiently large k. The section s is given by a collection $\{s_j\}$ where s_j is a holomorphic function on U_j with $s_j = g_{j\ell}^k s_\ell$ on $U_j \cap U_\ell$. Define $\varphi_j = \frac{1}{k}\log|s_j|^2$. Then the closed positive current on X defined by $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_j$ on U_j is a curvature current for $F - G$. Q.E.D.

§*2*. *The Use of the Very Ampleness Criterion of Demailly.*

In §1 we obtain nontrivial sections for high multiples of the difference of ample line bundles, but we cannot control yet the effective bound for the multiple. In this section we are going to use closed positive currents and the very ampleness criterion of Demailly [D2] to get an effective bound for the multiple. The very ampleness criterion of Demailly [D2] states that for any ample line bundle L over a compact complex manifold X of complex dimension n, the line bundle $12n^nL + 2K_X$ is a very ample line bundle over X. We will use the more general result of Demailly on the global generation of s-jets to further improve on the bound. The argument works with the very ampleness of $12n^nL + 2K_X$ but with a slightly worse bound. The result of Demailly on the global generation of s-jets states that, if L is an ample line bundle over a compact complex manifold X of complex dimension n, the global holomorphic sections of $m\ L + 2K_X$ generate the s-jets at every point of X if $m \geq 6(n+s)^n$. Let X be a compact projective algebraic manifold and

Y be an irreducible subvariety of complex dimension d in X. Let ω_Y be the sheaf on Y defined by the presheaf which assigns to an open subset U of Y the space of all holomorphic d-forms on $U \cap Reg\ Y$ which are L^2 on $U \cap Reg\ Y$, where Reg Y is the set of all regular points of Y. The sheaf ω_Y is coherent and is equal to the zeroth direct image of the sheaf of holomorphic d-forms on a desingularization Y' of Y under the desingularization map $Y' \to Y$.

Proposition (*2.1*). Let L be an ample line bundle over a compact complex manifold X of complex dimension n and let B be a numerically effective line bundle over X. Let Y be an irreducible subvariety of complex dimension d in X. Let $C_n = 3(3n-2)^n$. Then for $m > d\frac{L^{d-1}\cdot B\cdot Y}{L^d\cdot Y}$ there exists a nontrivial holomorphic section of the sheaf $\omega_Y \otimes \mathcal{O}(mL - B + K_X + C_nL)|Y$ over Y.

Proof. Let $\pi : Y' \to Y$ be a desingularization of Y. Let E be an ample line bundle of Y'. We apply Corollary (1.3) to the numerically effective line bundles $F = k\,\pi^*(m\ L)$ and $G = E + k\,\pi^*B$ over Y' for some sufficiently large k. The choice of m implies that $F^d > d\ F^{d-1}\cdot G$ for k sufficiently large. Hence there exists a closed positive (1,1)-current θ on Y' which is a curvature current for $k(F-G) - E$. Choose a point P in Y' such that $\pi(P)$ is in the regular part of Y and the Lelong number of θ at P is zero. Since the global holomorphic sections of $p\ L + 2\ K_X$ over X generate the 2d-jets at every point of X, for $p \geq 6(n+2d)^n$, there exists a Hermitian metric along the fibers of $p\ L + 2\ K_X$ which is smooth on $X - \pi(P)$ and whose Lelong number at $\pi(P)$ is 2d. Thus for $p \geq 3(n+2d)^n$ there exists a Hermitian metric along the fibers of $p\ L + K_X$ which is smooth on $X - \pi(P)$ and whose Lelong number at $\pi(P)$ is d. Since $d \leq n-1$, we can use $p = C_n$.

We give E a smooth metric with positive definite curvature form on Y'. By putting together the nonsmooth Hermitian metrics of $k(C_nL + K_X)$ and $k(F - G) - E$ and the smooth Hermitian metric of E, we get a nonsmooth Hermitian metric h of $\pi^*(mL - B + K_X + C_nL)$ such that the curvature current θ of h has Lelong number d at P and has Lelong number zero at every point of $U - \{P\}$ for some open neighborhood U of P in Y'. Moreover, the curvature current θ is no less than $\frac{1}{k}$ times the positive definite curvature form of E on Y. We can assume without loss of generality that both line bundles $\pi^*(mL - B + K_X + C_nL)|U$ and $K_{Y'}|U$ are trivial. Let σ be a holomorphic section of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$ over U such that σ is nonzero at P. We give $K_{Y'}$ any smooth metric so that from h we have some nonsmooth metric h' of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$. We take a smooth function ρ with compact support on U so that ρ is identically 1 on some open neighborhood W of P in U. Since the $(\pi^*(mL - B + K_X + C_nL) + K_{Y'})$-valued $\bar\partial$-closed (0,1)-form $(\bar\partial\rho)\sigma$ on X which is supported on U is zero on W and since the Lelong number of the curvature current of the metric h' is 0 at every point of $U - \{P\}$, it follows that $(\bar\partial\rho)\sigma$ is L^2 with respect to the metric h' of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$. Since the curvature current θ of the line bundle $\pi^*(mL - B + K_X + C_nL)$ is no less than some positive definite smooth (1,1)-form on Y', by the L^2 estimates of $\bar\partial$ (see *e.g.,* [D2,p.332,Prop.4.1]) there exists some L^2 section τ of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$ over Y' such that $\bar\partial\tau = (\bar\partial\rho)\sigma$ on Y'. Since the Lelong number of θ is d at P, it follows that τ vanishes at P and $\rho\,\sigma - \tau$ is a global holomorphic section of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$ over Y' which is nonzero at P. The direct image of $\rho\,\sigma - \tau$ with respect to $\pi : Y' \to Y$ is a global holomorphic section of $\omega_Y \otimes \mathcal{O}(mL - B + K_X + C_nL)|Y$ over Y which is non identically zero. Q.E.D.

§3. Sections of the Sum of a Very Ample Line Bundle and the Anticanonical Bundle.

Because of the very nature of the very ampleness criterion of Demailly, in §2 we could only construct holomorphic sections of line bundles containing the canonical line bundle as a summand. In this section we are going to construct holomorphic sections of the sum of a very ample line bundle and the anticanonical line bundle which will later be used to obtain holomorphic sections of line bundles without the canonical line bundle as a summand. The idea is that since the canonical line bundle of the complex projective space of complex dimension d is equal $to-(d+1)$ times the hyperplane section line bundle, by representing a compact complex space as a branched cover of the complex projective space, one can relate its canonical line bundle to a very ample line bundle on it.

Let Y be a proper irreducible d-dimensional subvariety in $\mathbf{P}_N$ and H_N be the hyperplane section line bundle of $\mathbf{P}_N$. Let λ be a positive number no less than the degree $H_N^d \cdot Y$ of Y. We use the notation introduced in §2 for ω_Y which is the sheaf on Y from the presheaf of local holomorphic d-forms on the regular part of Y which are L^2 locally at the singular points of Y.

Let V be a linear subspace of $\mathbf{P}_N$ of complex dimension $N-d-1$ which is disjoint from Y and let W be a linear subspace of $\mathbf{P}_N$ of complex dimension d which is disjoint from V. Denote by π' the projection map from $\mathbf{P}_N - V$ to W defined by the linear subspaces V and W of $\mathbf{P}_N$, which means that for $x \in \mathbf{P}_N - V$ the point $\pi'(x)$ is the point of intersection of W and the linear span of x and V. We identify W with $\mathbf{P}_d$ and let $\pi : Y \to \mathbf{P}_d = W$ be the restriction of π' to Y. We denote the restriction of H_N to $\mathbf{P}_d = W$ by H_d so that H_d is the hyperplane section line bundle of $\mathbf{P}_d$. We denote the restriction of H_N to Y by H.

Lemma (3.1). There exists a nontrivial holomorphic section of the sheaf $\mathcal{H}om(\omega_Y, \mathcal{O}_Y((3\lambda(\lambda-1)/2 - d - 1)H)$ over Y.

Proof. Since there are nontrivial holomorphic sections of any positive multiple of H over Y, without loss of generality we can assume that λ is equal to the degree $H_N^d \cdot Y$ of Y. Let $Z \subset \mathbf{P}_d$ be the branching locus of $\pi : Y \to \mathbf{P}_d$. Let D be some hyperplane in $\mathbf{P}_d$ not contained entirely in Z. Let D^* be the hyperplane in $\mathbf{P}_N$ containing D and V so that D^* is the topological closure in $\mathbf{P}_N$ of $\pi'^{-1}(D)$. Let s_{D^*} be the holomorphic section of H_N over $\mathbf{P}_N$ whose divisor is D^*. Let f be a holomorphic section of H_N over $\mathbf{P}_N$ such that

$$h := \frac{f}{s_{D^*}}$$

assumes λ distinct values at the λ points of $\pi^{-1}(P_1)$ for some P_1 in $\mathbf{P}_d$.

Let s_D be the holomorphic section of H_d over $\mathbf{P}_d$ whose divisor is D. Let $z_1, \cdots, z_d$ be the inhomogeneous coordinates of $\mathbf{P}_d - D$. Then

$$t = (s_D)^{d+1} dz_1 \wedge \cdots \wedge dz_d$$

can be regarded as a nowhere zero $(d+1)[D]$-valued holomorphic d-form on $\mathbf{P}_d$.

The nontrivial holomorphic section of

$$\mathcal{H}om(\omega_Y, \mathcal{O}_Y(3(\lambda(\lambda-1)/2-d-1)H)$$

over Y will be defined by using h and t. We will define it in the following way. We take a holomorphic section θ of ω_Y over an open neighborhood U of some point P of Y and we will define a holomorphic section of $\mathcal{O}_Y((3\lambda(\lambda-1)/2-d-1)H$ over an open neighborhood of P in U. Without loss of generality we can assume that U is of the form $\pi^{-1}(U')$ for some open subset U' of $\mathbf{P}_d$. We consider the Vandermonde determinant defined for the λ values of $h|Y$ on the same fiber of $\pi : Y \to \mathbf{P}_d$. In other words, we take $P \in \mathbf{P}_d$ such that $\pi^{-1}(P)$ has λ distinct points $P^{(1)}, \cdots, P^{(\lambda)}$ and we form the determinant

$$h'_0 = \det((h(P^{(\mu)}))^{\nu-1})_{1\le\mu,\nu\le\lambda}.$$

Let $h' = (h'_0)^2$. Then h' is a meromorphic function on $\mathbf{P}_d$ with pole only along D. By considering the pole order of h along $\pi^{-1}(D)$, we conclude that the pole order of h' along D is

$$2\sum_{\mu=1}^{\lambda-1}\mu = \lambda(\lambda-1).$$

We write

$$\theta(P^{(j)}) = \sum_{\nu=0}^{\lambda-1} a_\nu(P)(h(P^{(j)}))^\nu (1 \le j \le \lambda) \tag{3.1.1}$$

for some d-form a_ν on $U'(0 \le \nu \le \lambda-1)$. We solve the system (3.1.1) of equations by Cramer's rule. Let $c^{(\nu)}_{\rho\mu} = h(P^{(\mu)})^{\rho-1}$ for $\rho \ne \nu$ and $c^{(\nu)}_{\nu\mu} = \theta(P^{(\mu)})$. Let b_ν be the determinant of the $\lambda \times \lambda$ matrix

$$\left(c^{(\nu)}_{\rho\mu}\right)_{1\le\rho,\mu\le\lambda}.$$

Then

$$a_\nu(P) = \frac{b_\nu}{h'_0} = \frac{b_\nu h'_0}{h'}.$$

We are going to use the removable singularity for L^2 holomorphic top degree forms so that an L^2 holomorphic d-form on $\Omega - Z$ for some open subset Ω of $\mathbf{P}_d$ extends to a holomorphic d-form on Ω. Thus, since θ is an L^2 holomorphic

d-form on $U \cap \operatorname{Reg} Y$, we conclude from the pole orders of h and h' that the restriction of

$$s_D^{3\lambda(\lambda-1)/2} h' \, a_\nu$$

to $U' - (Z \cap D)$ is a holomorphic $(3(\lambda(\lambda-1)/2)H_d$-valued d-form on $U' - (Z \cap D)$ and therefore

$$s_D^{3\lambda(\lambda-1)/2} h' \, a_\nu$$

is a holomorphic $(3(\lambda(\lambda-1)/2)H_d$-valued d-form on U' by removability of singularity of codimension two. We have

$$\theta = \sum_{\nu=0}^{\lambda-1} (a_\nu \circ \pi)(h^\nu | Y).$$

The section

$$\frac{\pi^*(s_D^{3\lambda(\lambda-1)/2} h')\theta}{\pi^*(t)}$$

of $(3\lambda(\lambda-1)/2 - d - 1)H$ over U is holomorphic, because

$$\frac{\pi^*(s_D^{3\lambda(\lambda-1)/2} h')\theta}{\pi^*(t)} = \sum_{\nu=0}^{\lambda-1} ((s_D^{3\lambda(\lambda-1)/2} h' \; a_\nu / t) \circ \pi)(h^\nu | Y).$$

The section of

$$\mathcal{H}om(\omega_Y, \mathcal{O}_Y((3\lambda(\lambda-1)/2 - d - 1)H)$$

over Y to be constructed is now given by the map which sends the holomorphic section θ of ω_Y over U to the holomorphic section

$$\frac{\pi^*(s_D^{3\lambda(\lambda-1)/2} h')\theta}{\pi^*(t)}$$

of $(3\lambda(\lambda-1)/2 - d - 1)H$ over U. Q.E.D.

Proposition (*3.2*). Let L be an ample line bundle over a compact complex manifold X of complex dimension n and let B be a numerically effective line bundle over X. Let Y be a proper irreducible subvariety of complex dimension d in X. Let H be a very ample line bundle of Y. Let $C_n = 3(3n-2)^n$. Then for

$$m > d \, \frac{L^{d-1} \cdot B \cdot Y}{L^d \cdot Y}$$

there exists a nontrivial holomorphic section of the holomorphic line bundle

$$(3\lambda(\lambda-1)/2-d-1)H+(mL-B+K_X+C_nL)|_Y$$

over Y.

Proof. The proposition follows from Lemma (3.1) and Proposition (2.1) after we embed $Y into$ a complex projective space by using the holomorphic sections of H over Y. Q.E.D.

Corollary (*3.3*). Let L be an ample line bundle over a compact complex manifold X of complex dimension n and let B be a numerically effective line bundle over X. Let Y be a proper irreducible subvariety of complex dimension d in X. Let $C_n = 3(3n-2)^n$ and $H = 2\ C_nL + 2\ K_X$. Then for

$$m > d\,\frac{L^{d-1}\cdot B\cdot Y}{L^d\cdot Y}$$

there exists a nontrivial holomorphic section of the holomorphic line bundle $((3\lambda(\lambda-1)/2)H + mL - B)|Y$ over Y.

Proof. The case $d=0$ is trivial. Hence we can assume that $d\geq 1$ and $n\geq 2$. In that case $2C_n = 3(3n-2)^n \geq 12\ n^n$ and by Demailly's very ampleness criterion the line bundle H is very ample over X. Since $(d+\frac{1}{2})H$ is very ample over X, the corollary follows from Proposition (3.2). Q.E.D.

We will need a statement similar to Corollary (3.3) for the case $Y = X$.

Proposition (*3.4*). Let L be an ample line bundle over a compact complex manifold X of complex dimension $n\geq 2$ and let B be a numerically effective line bundle over X. Let $C_n = 3(3n-2)^n$ and $H = 2\ C_nL + 2\ K_X$. Then for $m > n\frac{L^{n-1}\cdot B}{L^n}$ there exists a nontrivial holomorphic section of the sheaf $mL - B + H$ over X.

Proof. By Corollary (1.3) there exists a nonsmooth Hermitian metric h_1 for $m\ L - B$ whose curvature current θ_1 is a closed positive (1,1)-current. Take a point P in X at which the Lelong number of θ_1 is 0. By [D2] (see the proof of Theorem 11.6 on p.364 and the proof of Corollary 2 on p.369) there exists a nonsmooth Hermitian metric h_2 for $2C_nL + K_X$ such that

(i) the Lelong number of the curvature current θ_2 of h_2 is at least n at P,
(ii) the Lelong number of $\theta_2 is < 1$ at every point of $U - \{P\}$ for some open neighborhood U of P in X,

(iii) θ_2 is no less than some positive definite smooth (1,1)-form on X.

Now we put together the metrics h_1 and h_2 to form a nonsmooth metric h_3 for $m\,L - B + 2C_nL + K_X$. Without loss of generality we can assume that at every point of U the Lelong number of θ_1 is zero and that the holomorphic line bundles $(m\,L - B + 2C_nL)|U$ and $K_X|U$ are trivial. We give K_X any smooth metric so that from h we have some nonsmooth metric h' of $mL - B + H$. We take a smooth function ρ with compact support on U so that ρ is identically 1 on some open neighborhood W of P in U. Since the Lelong number of $\theta_1 + \theta_2 is < 1$ on $U - \{P\}$, it follows that the $(mL - B + H)$-valued $\bar\partial$-closed (0,1)-form $(\bar\partial\rho)\sigma$ on X is L^2. By applying the L^2 estimates of $\bar\partial$, we obtain an L^2 section τ of $mL - B + H$ over X such that $\bar\partial\tau = (\bar\partial\rho)\sigma$ on X. Since the Lelong number of $\theta_1 + \theta_2$ is at least n at P, it follows that τ vanishes at P and $\rho\,\sigma - \tau$ is a global holomorphic section of $mL - B + H$ over X which is nonzero at P. Q.E.D.

§4. *Final Step in the Proof of the Effective Matsusaka big theorem.*

Since the effective Matsusaka big theorem is obviously true for the case when X is of complex dimension of one, we can assume that the complex dimension n of X is at least 2. To get Matsusaka's big theorem, it is enough to get an effective bound on m for $m\,L - (B + 2K_X + pL)$ to be numerically effective for some $p \geq 2(n+1)$, because then by Demailly's very ampleness criterion [D2, p.370, Remark 12.7] the holomorphic line bundle $(12n^n + m - p)L - B = 12\,n^nL + 2\,K_X + m\,L - (B + 2K_X + pL)$ becomes very ample. We use $B + 2K_X + pL$ instead of just $B + 2K_X$, because we need Fujita's result [F] on the numerical effectiveness of $(n+1)L + K_X$.

The numerical effectiveness of $m\,L - (B + 2K_X + pL)$ is verified by evaluating it on an arbitrary compact curve. The main idea is to use Corollary (3.3). We can obtain nontrivial holomorphic sections of the line bundle of the form $p\,L - B - q\,K_X$ over subvarieties and inductively we apply the argument to subvarieties which are the zero sets of such nontrivial holomorphic sections. This way we will get a numerical criterion for the numerical effectiveness of $m\,L - B$ and then afterwards we will replace B by $B + 2K_X + pL$.

Let $C_n = 3(3n-2)^n$ and $H = 2C_nL + 2\,K_X$. As in the outline of the argument in the preceding paragraph, to obtain the numerical effectiveness of $m\,L - B$, we are going to use Corollary (3.1) to construct inductively a sequence of (not necessarily irreducible) algebraic subvarieties $X = Y_n \supset$

$Y_{n-1} \supset \cdots \supset Y_1 \supset Y_0$ and positive integers $m_d (1 \leq d \leq n)$ with the following two properties.

(i) Y_d is d-dimensional.

(ii) For every irreducible component $Y_{d,j}$ of Y_d, there exists a nontrivial holomorphic section $\sigma_{d,j}$ of $m_d L - B$ over $Y_{d,j}$ such that Y_{d-1} is the union of the zero-sets of $\sigma_{d,j}$ when j runs through the set indexing the branches $Y_{d,j}$ of Y_d.

To start the induction, by Proposition (3.4) we can use $m_n > n \frac{L^{n-1} \cdot (B+H)}{L^n}$ and get a nontrivial holomorphic section σ_n of $m_n L - B$ over X. Let Y_{n-1} be the zero-set of σ_n. Suppose we have constructed Y_p for $d \leq p \leq n$ and m_p for $d < p \leq n-1$. We are going to construct Y_{d-1} and m_d. Let $C'_n = 2(C_n - (n+1))$. Then $H - C'_n\ L = 2((n+1)L + K_X)$ is numerically effective over X by Fujita's result [F]. To choose m_d we have to compute $Y_{d,j} \cdot H^d$ and $Y_{d,j} L^{d-1} B$ and $Y_{d,j} L^d$ for every irreducible component $Y_{d,j}$ of Y_d. Now $Y_{d,j}$ is a branch of the zero-set of some nontrivial holomorphic section $\sigma_{d+1,k}$ of $m_{d+1} L - B$ over $Y_{d+1,k}$ for some branch $Y_{d+1,k}$ of Y_d. For any numerically effective line bundles $E_1, \cdots, E_d$ over X, we have

$$\begin{aligned} Y_{d,j} \cdot E_1 \cdots E_d &\leq Y_{d+1,k} (m_{d+1} L - B) \cdot E_1 \cdots E_d \\ &\leq Y_{d+1,k} m_{d+1} L \cdot E_1 \cdots E_d \\ &\leq Y_{d+1,k} \frac{m_{d+1}}{C'_n} H\ E_1 \cdots E_d \end{aligned}$$

(by the numerical effectiveness of $H - C'_n\ L$) which by induction on d yields

$$Y_{d,j} \cdot E_1 \cdots E_d \leq \frac{m_n}{C'_n} \cdots \frac{m_{d+1}}{C'_n} H^{n-d} E_1 \cdots E_d.$$

Hence

$$\begin{aligned} Y_{d,j} H^d &\leq \frac{m_n}{C'_n} \cdots \frac{m_{d+1}}{C'_n} H^n, \\ Y_{d,j} L^{d-1} B &\leq \frac{m_n}{C'_n} \cdots \frac{m_{d+1}}{C'_n} H^{n-d} L^{d-1} B. \end{aligned}$$

To apply Corollary (3.3) we impose on $m_d (1 \leq d < n)$ the condition that

$$m_d > d \left(L^d \cdot Y_{d,j}\right)^{-1} (L^{d-1} \cdot Y_{d,j} \cdot \left(B + \frac{3}{2} \lambda_d (\lambda_d - 1) H\right)$$

for some $\lambda_d \geq Y_{d,j} \cdot H^d$ for all j. Since $L^d \cdot Y_{p,j} \geq 1$ and $H - C'_n\, L$ is numerically effective, to get the condition we can let

$$\lambda_d \geq \frac{m_n}{C'_n} \cdots \frac{m_{d+1}}{C'_n} H^n$$

and set

$$m_d > \frac{d\lambda_d}{(C'_n)^{d-1}} (\frac{H^{n-1} \cdot B}{H^n} + \frac{3}{2}(\lambda_d - 1)).$$

In order to get a simpler closed expression later we replace the last inequality by the stronger inequality that m_d is no less than the integral part of

$$\frac{n\lambda_d^2}{(C'_n)^{d-1}} \left(\frac{H^{n-1} \cdot B}{H^n} + \frac{3}{2} \right).$$

Once these inequalities are satisfied, by Corollary (3.3) we can find a nontrivial holomorphic section $\sigma_{d,j}$ of $m_d L - B$ over $Y_{d,j}$. Our Y_{d-1} is now the union of the zero-sets of all $\sigma_{d,j}$ when j runs through the set indexing the branches $Y_{d,j}$ of Y_d. The above inequality for m_d $(1 \leq d < n)$ is satisfied if we inductively have m_d no less than the integral part of

$$\frac{n}{(C'_n)^{d-1}} \left(\frac{H^{n-1} \cdot B}{H^n} + \frac{3}{2} \right) \left(\frac{m_n}{C'_n} \cdots \frac{m_{d+1}}{C'_n} H^n \right)^2.$$

To get a closed formula, we consider for $1 \leq d \leq n$ the equations

$$q_d = \frac{n}{(C'_n)^{d-1}} \left(\frac{H^{n-1} \cdot B}{H^n} + \frac{3}{2} \right) \left(\frac{q_n}{C'_n} \cdots \frac{q_{d+1}}{C'_n} H^n \right)^2.$$

Then $q_d = (C'_n)^{-1} q_{d+1}^3$ and $q_d = (C'_n)^{-(3^{n-d}-1)/2} q_n^{3^{n-d}}$ for $1 \leq d < n$. Thus we can inductively define m_d to be the integral part of

$$\begin{aligned} &(C'_n)^{-(3^{n-d}-1)/2} (n (C'_n)^{-(n-1)} H^n (H^{n-1} \cdot B + \frac{3}{2} H^n))^{3^{n-d}} \\ &= (C'_n)^{-((2n-1)3^{n-d}-1)/2} (n\ H^n (H^{n-1} \cdot B + \frac{3}{2} H^n))^{3^{n-d}}. \end{aligned}$$

Clearly $m_d \leq m_1$ for $1 \leq d \leq n$.

We now verify that the line bundle $m_1 L - B$ is numerically effective over X. Let Γ be an irreducible complex curve in X. We have to verify that

$(m_1 L - B)\cdot\Gamma$ is nonnegative. There exists an integer $1 \le d \le n$ such that Γ is contained entirely in some branch $Y_{d,j}$ of Y_d but is not contained entirely in Y_{d-1}. There exists a nontrivial holomorphic section $\sigma_{d,j}$ of $m_d L - B$ over $Y_{d,j}$ whose zero-set is contained in Y_{d-1}. Thus $\sigma_{d,j}$ does not vanish identically on Γ and we conclude that $(m_d L - B)\cdot\Gamma$ is nonnegative. Since $(m_1 - m_d)L$ is ample, it follows that $(m_1 L - B)\cdot\Gamma$ is nonnegative. This completes the proof that $m_1 L - B$ is numerically effective.

Since $2C_n \ge 12n^n$ for $n \ge 2$, by [D2, p.370, Remark 12.7] from the numerical effectiveness of $m_1 L - B$ it follows that the line bundle $m_1 L - B + H$ is very ample. After replacing B by $B + H$, we conclude that $m\ L - B$ is very ample for

$$m \ge (C_n')^{-((2n-1)3^{n-d}-1)/2}\left(n\ H^n\left(H^{n-1}\cdot B + \frac{5}{2}H^n\right)\right)^{3^{n-1}}$$

and Theorem (0.2) is proved.

We now derive Theorem (0.1) from Theorem (0.2) by using the following inequalities of Chern numbers of numerically effective line bundles [D2, Prop. 5.2(b)]. If $L_1, \cdots, L_n$ are numerically effective holomorphic line bundles over a compact projective algebraic manifold X of complex dimension n and $k_1, \cdots, k_\ell$ are positive integers with $k_1 + \cdots + k_\ell = n$, then

$$L_1^{k_1}\cdots L_\ell^{k_\ell} \ge (L_1^n)^{k_1/n}\cdots(L_\ell^n)^{k_\ell/n}.$$

We now apply the inequality to the case of $L_1 = F + G$ and $L_2 = F$. Then

$$(F+G)F^{n-1} \ge ((F+G)^n)^{1/n}(F^n)^{(n-1)/n}$$

or

$$(F+G)^n \le \frac{((F+G)F^{n-1})^n}{(F^n)^{n-1}}.$$

Let $F = (C_n - (n+1))L$ and $G = (n+1)L + K_X$. Then $F + G = \frac{1}{2}H$ and

$$2^{-n}H^n \le (L^n)^{-(n-1)}((C_n L + K_X)\cdot L^{n-1})^n.$$

Thus

$$\left(n\frac{5}{2}(H^n)^2\right)^{3^{n-1}} \le (2^{2n-1}5n)^{3^{n-1}}((L^n)^{-(n-1)}((C_n L + K_X)\cdot L^{n-1})^n)^{2\cdot 3^{n-1}}.$$

In the case of $B = 0$ the line bundle $m\ L$ is very ample if

$$m \ge (C_n')^{-((2n-1)3^{n-d}-1)/2}(2^{2n-1}5n)^{3^{n-1}}((L^n)^{-(n-1)}((C_n L + K_X)\cdot L^{n-1})^n)^{2\cdot 3^{n-1}}.$$

Thus Theorem (0.1) is proved.

Chapter 8. Effective Very Ampleness

Fujita [F] conjectured the very ampleness of $(n+2)L+K_X$ and the freeness of $(n+1)L+K_X$ for an ample line bundle L over a compact complex manifold X of complex dimension n. The cases of $n \leq 2$ are known for the very ampleness part [R] and the cases of $n \leq 3$ are known for the freeness part [EL]. For a general n Demailly [D] used the method of the solution of the Monge-Ampère equation to obtain the very ampleness of $12\ n^n L + 2K_X$ and the generation of t-jets at every point of X by global holomorphic sections of $m\ L + 2\ K_X$ for $m \geq 6(n+t)^n$. Kollár [Ko] gave an algebraic geometric proof for the weaker result of the freeness of $2(n+2)!(n+1)((n+2)L+K_X)$ and the very ampleness of $2(n+2)!(n+1)(n+3)((n+2)L+K_X)+K_X$. A related result is the effective Matsusaka big theorem in [S]. Here ampleness means that the line bundle admits a smooth Hermitian metric whose curvature form is positive definite everywhere. Freeness means that global holomorphic sections generate the line bundle.

Recently Ein-Lazarsfeld-Nakamaye [ELN] obtained the following result by algebraic geometric methods. For an ample line bundle B and a numerically effective line bundle A over a compact complex manifold X of complex dimension n, if $B^{n-d} \cdot V > 2(2+n-d+\frac{n!}{d!})^d (n+1)^n$ for any subvariety V in X of codimension d for $1 \leq d \leq n-1$, then $2\ K_X + B + A$ is very ample. In particular, $2K_X + 2(n+1+(n!))^2(n+1)^n L$ is very ample for any ample line bundle L over X. The order n^{3n} of m for the very ampleness of $m\ L + 2\ K_X$ in the result of Ein-Lazarsfeld-Nakamaye is higher than the order n^n in Demailly's result. On the other hand, the method of Ein-Lazarsfeld-Nakamaye is algebraic geometric and does not need any of the heavy analytic machinery used in Demailly's result. Another significance of the results of Ein-Lazarsfeld-Nakamaye is that the very ample line bundle 2 $K_X + B + A$ does not involve a high multiple of the ample line bundle B but instead the Chern number of the restriction of B to any positive dimensional proper subvariety is assumed high enough.

In this note we present a simple argument which is algebraic geometric in nature and which for an ample line bundle L gives the very ampleness of $m\ L + 2\ K_X$ with the order of m no more than $(3e)^n$ which is better than Demailly's order of n^n. The main result is the following.

Theorem (0.1). Let L be an ample line bundle over a compact complex manifold X of complex dimension n. Let ℓ be a positive integer and $P_1, \cdots, P_\ell$

be distinct points in X and $t_1, \cdots, t_\ell$ be nonnegative integers. Then for m no less than $2\left(n+2+n\ \sum_{\nu=1}^{\ell}\binom{3n+2t_\nu-1}{n}\right)$ the global holomorphic sections of $m\,L+2,\ K_X$ generate simultaneously the t_ν-jets at $P_\nu(1\leq\nu\leq\ell)$ in the sense that the restriction map from $\Gamma(X,\ m\ L+2\ K_X)$ to

$$\bigoplus_{\nu=1}^{\ell}\left(\mathcal{O}_{X,P_\nu}/\,\mathbf{m}_{P_\nu}^{t_\nu+1}\right)$$

is surjective, where $\mathbf{m}_{P_\nu}$ is the maximum ideal at P_ν.

Corollary (*0.2*). Let L be an ample line bundle over a compact complex manifold X of complex dimension n. Let t be a nonnegative integer. Then global holomorphic sections of $m\ L+2\ K_X$ generate the t-jets of $m\,L+2\,K_X$ at every point of X for m no less than $2\left(n+2+n\binom{3n+2t-1}{n}\right)$.

Corollary (*0.3*). Let L be an ample line bundle over a compact complex manifold X of complex dimension n. Then $m\,L+K_X$ is very ample for m no less than $2\left(n+2+n\binom{3n+1}{n}\right)$.

By Stirling's formula $2\left(n+2+n\binom{3n+2t-1}{n}\right)$ is no more than

$$\left(\frac{2}{\pi}\right)^{1/2}(3n+2t-1)^n\,n^{-n+(1/2)}e^n.$$

As a result the number m needed for the very ampleness of $m\,L+2\,K_X$ in Corollary (0.3) is of an order no more than $(3e)^n$. Since the number $2\left(n+2+n\binom{3n+1}{n}\right)$ is less than $12\,n^n$ for $n\geq 5$, Corollary (0.3) implies immediately Demailly's result for $n\geq 5$. In the proof of Theorem (0.1) some rounding off is done to get a simpler expression for the lower bound of m. If the rounding off is not done, as shown in Remark (4.3) the proof of Theorem (0.1) gives Demailly's result also for $2\leq n<5$. By more carefully keeping track of the constants in the proof of Theorem (0.1), one can sharpen somewhat the lower bound for m. However, because of the nature of the method, any such sharpening would still end up with m of the order a^n for some constant a.

The method uses only the theorem of Riemann-Roch and the vanishing theorem for multiplier ideal sheaves and the arguments used are very simple. In the situation under consideration the vanishing theorem for multiplier ideal sheaves is equivalent to the vanishing theorem of Kawamata-Viehweg

[Ka, V]. So the proof is actually algebraic geometric though the presentation uses the language of multiplier ideal sheaves.

There are two new ingredients in the method presented here. The first one is to use, for the existence of holomorphic sections, the polynomial in m which is the arithmetic genus of a line bundle (twisted by a coherent sheaf) containing mL as a summand. When the positive dimensional cohomology groups vanish and m avoids the estimable number of values at which the polynomial assumes small values, the space of sections has high dimension. The second one is the filtration of the multiplier ideal sheaf for the curvature current T of an ample line bundle by the multiplier ideal sheaves for $\alpha\, T$ with $0 < \alpha < 1$ so that the branches of the zero set of the multiplier ideal sheaf for T can be handled one at a time.

§1. *Existence of Sections by Using the Arithmetic Genus.* We will need the following well-known formula of Hirzebruch-Riemann-Roch which is a consequence of the vector bundle case [H] and a projective locally free resolution of the coherent analytic sheaf.

Lemma (1.1). Let X be a compact projective algebraic complex manifold of complex dimension n and let $\mathcal{F}$ be a coherent analytic sheaf over X. Let L and F be holomorphic line bundles over X. For any integer m let $P(m)$ be $\sum_{\nu=0}^{n}(-1)^{\nu}\dim_{\mathbf{C}} H^{\nu}(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F})$. Then $P(m)$ is a polynomial of degree at most n in the variable m.

Lemma (1.2). Let X be a compact complex manifold of complex dimension n and let $\mathcal{F}$ be a coherent analytic sheaf over X. Let L and F be holomorphic line bundles over X. Suppose L is ample and the support of $\mathcal{F}$ is positive dimensional. Let m_0 be a nonnegative integer. If $H^{\nu}(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F})$ vanishes for $\nu \geq 1$ and $m \geq m_0$, then $\dim_{\mathbf{C}}\Gamma(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F})$ is a nonconstant polynomial in m of degree at most n for $m \geq m_0$. As a consequence, given any positive integer s there exists $m_0 \leq m \leq s\, n + m_0$ such that $\dim_{\mathbf{C}}\Gamma(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F}) \geq s$.

Proof. Since L is ample and the support of $\mathcal{F}$ is positive dimensional, we know that $\dim_{\mathbf{C}}\Gamma(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F})$ becomes infinite as m goes to infinity. From the vanishing of $H^{\nu}(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F})$ for $\nu \geq 1$ and $m \geq m_0$, it follows that for $m \geq m_0$, $\dim_{\mathbf{C}}\Gamma(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F}) = \sum_{\nu=0}^{n}(-1)^{\nu}\dim_{\mathbf{C}} H^{\nu}(X, \mathcal{O}_X(mL+F)\otimes\mathcal{F})$ is equal to a polynomial $P(m)$ of

m of degree at most n and at least 1. For each $0 \leq \mu < s$ the polynomial equation $P(m) = \mu$ in m has at most n roots. Since there are only s choices of μ in the inequality $0 \leq \mu < s$, there are at most $s\, n$ roots for the polynomial equation $P(m) = \mu$ for $0 \leq \mu < s$ and we can find some integer $m_0 \leq m \leq s\, n + m_0$ so that m is not a root of any of the s equations $P(m) = \mu$ with $0 \leq \mu < s$. However, we know that $P(m) = \dim_{\mathbf{C}} \Gamma(X, \mathcal{O}_X(mL+F) \otimes \mathcal{F})$ is nonnegative. Hence $P(m) = \dim_{\mathbf{C}} \Gamma(X, \mathcal{O}_X(mL + F) \otimes \mathcal{F}) \geq s$. Q.E.D.

Remark (*1.3*). In Lemma (1.2) if the dimension of the support of $\mathcal{F}$ is d, then there exists $m_0 \leq m \leq s\, d + m_0$ (instead of $m_0 \leq m \leq s\, n + m_0$) such that $\dim_{\mathbf{C}} \Gamma(X, \mathcal{O}_X(mL+F) \otimes \mathcal{F}) \geq s$, because the degree of the polynomial $\dim_{\mathbf{C}} \Gamma(X, \mathcal{O}_X(mL + F) \otimes \mathcal{F})$ in the variable m is d.

§2. *The Use of Multiplier Ideal Sheaves.* Suppose F is a holomorphic line bundle over a compact complex manifold X with a (possibly singular) metric which is locally of the form $e^{-\varphi}$. The curvature current T of the metric is defined as $\frac{\sqrt{-1}}{2\pi} \partial\bar\partial\varphi$. The multiplier ideal sheaf of T (or of the metric) is defined as the sheaf of the germs of all holomorphic functions f with $|f|^2 e^{-\varphi}$ locally integrable. The vanishing theorem for multiplier ideal sheaves states the following [N, D]. If F is a holomorphic line bundle on a compact complex manifold X with a metric whose curvature current T is positive and dominates a smooth positive (1,1)-form and if $\mathcal{I}$ is the multiplier ideal sheaf for the current T, then $H^\nu(X, \mathcal{O}_X(F + K_X) \otimes \mathcal{I})$ vanishes for $\nu \geq 1$. The application will be to the following situation. Assume that F is ample and θ is a smooth positive definite curvature form of F. For $0 \leq \alpha \leq 1$ let $\mathcal{I}_\alpha$ be the multiplier ideal sheaf for the curvature current $(1-\alpha)\theta + \alpha\, T$ of F. Then we have the vanishing of $H^\nu(X, \mathcal{O}_X(F + K_X) \otimes \mathcal{I}_\alpha)$ for $\nu \geq 1$. From the cohomology exact sequence of $0 \to \mathcal{I}_\beta \to \mathcal{I}_\alpha \to \mathcal{I}_\alpha/\mathcal{I}_\beta \to 0$ we have also the vanishing of $H^\nu(X, \mathcal{O}_X(F + K_X) \otimes (\mathcal{I}_\alpha/\mathcal{I}_\beta))$ for $\nu \geq 1$ and $0 \leq \alpha < \beta \leq 1$.

We are only interested in multiplier ideal sheaves of curvature currents from metrics constructed from multivalued holomorphic sections of line bundles. By a multivalued holomorphic section u of a holomorphic line bundle F we simply mean that u^p is a holomorphic section of the holomorphic line bundle $p\, F$. We say that the multivalued holomorphic section u vanishes at a point P to order at least q if u^p vanishes to order at least $p\, q$. We observe that, given a finite number of local holomorphic functions $f_1, \cdots, f_N$ and positive rational numbers q_ν, the function $\log(\sum_{\nu=1}^N |f_\nu|^{q_\nu})$ is plurisubharmonic. The simple reason of the observation is as follows. Clearly $\log(\sum_{\nu=1}^N |f_\nu|^{q_\nu})$

is plurisubharmonic outside the union of the zero-sets of $f_\nu (1 \leq \nu \leq N)$. On the other hand, it is also clear that $\log(\sum_{\nu=1}^N |f_\nu|^{q_\nu})$ is locally bounded from above and thus can be extended across the union of the zero-sets of $f_\nu (1 \leq \nu \leq N)$ as a plurisubharmonic function. We are going to use a metric for F of the form $(\sum_{\nu=1}^N |s_\nu|^2)^{-1}$ for multivalued holomorphic sections $s_1, \cdots, s_N$ of F. From the above observation the curvature current $T = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(\sum_{\nu=1}^N |s_\nu|^2)$ of this metric is a closed positive current.

When only multiplier ideal sheaves of curvature currents from metrics constructed from multivalued holomorphic sections of line bundles are used, the vanishing theorem for multiplier ideal sheaves is the same as the theorem of Kawamata-Viehweg. So everything done here can be done by using the theorem of Kawamata-Viehweg instead of the vanishing theorem for multiplier ideal sheaves. We choose to use the language of multiplier ideal sheaves just out of personal preference. We will describe the multiplier ideal sheaves in terms of blowup maps so that one can readily see the relation between the vanishing theorem for multiplier ideal sheaves and the theorem of Kawamata-Viehweg. Consider an ample line bundle L over a compact complex manifold X. Let the metric h of L be defined as $(\sum_{\nu=1}^N |s_\nu|^2)^{-1}$ for some multivalued holomorphic sections s_j of L over X. Let k be a positive integer such that $(s_j)^k$ is a holomorphic section of $k\ L$ for $1 \leq j \leq N$. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf on X generated locally by $(s_j)^k (1 \leq j \leq N)$. We use a blowup $f : Y \to X$ given by successive monoidal transformations with nonsingular centers so that there exist a finite number of nonsingular divisors E_μ in Y in normal crossing with the property that $K_Y - f^* K_X = \sum_\mu r_\mu E_\mu$ for some nonnegative integers r_μ and that the pullback of the ideal sheaf $\mathcal{J}$ by f is equal to the ideal sheaf of the divisor $\sum_\mu e_\mu^\# E_\mu$ for some nonnegative integers $e_\mu^\#$. Let $e_\mu = e_\mu^\# / k$. Let e'_μ be the largest integer not exceeding $e_\mu - r_\mu$. Then the zeroth direct image of the ideal sheaf of $\sum_\mu \max(e'_\mu, 0) E_\mu$ under f is the multiplier ideal sheaf of the metric h.

§3. Generation of Jets by Global Sections.

Lemma (3.1). Let L and F be ample line bundles over a compact complex manifold X of complex dimension n. Let η_F be a singular metric of F (defined by global multivalued holomorphic sections of F) with curvature current T. Let Y be the zero set of the multiplier ideal sheaf of T. Let ℓ be a positive integer and $P_1, \cdots, P_\ell$ be distinct points in X. Let $q_1, \cdots, q_\ell$ be nonnegative integers. Let m_0 be a positive integer and $s = 1 + \sum_{\nu=1}^\ell \binom{n+q_\nu-1}{n}$. Let $Y_j (1 \leq$

$j \leq J)$ be the positive dimensional branches of Y. Then for $1 \leq j \leq J$ there exists $m_0 \leq m_j \leq s\ n + m_0$ and an element $g^{(j)}$ of $\Gamma(X, m_j L + F + K_X)$ which vanishes to order at least q_ν at the point $P_\nu(1 \leq \nu \leq \ell)$ but does not vanish identically on Y_j. As a consequence, for $m_\# \geq s\, n + m_0$ there exists a multivalued holomorphic section of $m_\# L + F + K_X$ over X which vanishes to order at least q_ν at the point $P_\nu(1 \leq \nu \leq \ell)$ but does not vanish identically on any positive dimensional branch of Y.

Proof. Let h be a smooth metric for F (defined by global multivalued holomorphic sections of F) whose curvature form θ is positive definite at every point of X. Let $\mathcal{I}$ be the multiplier ideal sheaf for T. Let the global multivalued holomorphic sections of F over X which define the metric η_F for F be $s_j(1 \leq j \leq N)$. In other words, $\eta_F = (\sum_{j=1}^N |s_j|^2)^{-1}$. Let k be a positive integer such that $(s_j)^k$ is a holomorphic section of $k\ F$ for $1 \leq j \leq N$. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf on X generated locally by $(s_j)^k(1 \leq j \leq N)$. Let $\Psi : X^\# \to X$ be a blowup by successive monoidal transformations with nonsingular centers so that there exist in $X^\#$ a finite number of nonsingular divisors E_μ in normal crossing with the property that $K_{X^\#} = \Psi^* K_X + \sum_\mu r_\mu E_\mu$ for some nonnegative integers r_μ and that the pullback of the ideal sheaf $\mathcal{J}$ by f is equal to the ideal sheaf of the divisor $\sum_\mu e_\mu^\# E_\mu$ for some nonnegative integers $e_\mu^\#$. Let $e_\mu = e_\mu^\# / k$. By relabelling $\{E_\mu\}$, we can assume without loss of generality that $\Psi(E_j) = Y_j$ for $1 \leq j \leq J$. Since $\Psi(E_j) = Y_j$ is contained in the zero-set of the multiplier ideal sheaf of T, it follows that

$$e_j \geq r_j + 1 \text{ for } \quad 1 \leq j \leq J. \tag{3.1.1}$$

We choose positive rational numbers δ_μ and let p be a large multiple of the common denominator of δ_μ and consider $p(\Psi^* L - \sum_\mu \delta_\mu E_\mu)$ which is an ample line bundle over $X^\#$ for δ_μ sufficiently small. By replacing p by a large multiple of it, we can assume that $p(\Psi^* L - \sum_\mu \delta_\mu E_\mu)$ is generated by global holomorphic sections over $X^\#$. Those global holomorphic sections descend to holomorphic sections of $p\ L$ over X and their p^{th} roots are multivalued holomorphic sections of L over X and we use them to get a singular metric h_δ for L.

Let h_L be a smooth metric of L (defined by global multivalued holomorphic sections of L) whose curvature form θ_L is positive definite at every point. Choose $0 < \gamma < 1$. For any $0 < \alpha \leq 1$ we introduce the metric $h_L^{1-\gamma} h_\delta^\gamma h^{1-\alpha} \eta_F^\alpha$ for $L + F$. Let T'_α be the curvature current of $h_L^{1-\gamma} h_\delta^\gamma h^{1-\alpha} \eta_F^\alpha$. Note that T'_α dominates $(1-\gamma)\theta_L$. The multiplier ideal sheaf $\mathcal{I}_\alpha$ for T'_α is the zeroth direct image under Ψ of the ideal sheaf of $\sum_\mu \max(\lambda_\mu, 0)E_\mu$, where λ_μ is the largest integer not greater than $\alpha\ e_\mu - r_\mu + \gamma\ \delta_\mu$. Now we can set $\alpha_\mu = \frac{r_\mu + 1 - \gamma\delta_\mu}{e_\mu}$. We can choose δ_μ and γ so that all the numbers α_μ are distinct. After relabelling Y_j, we can assume without loss of generality that $\alpha_j > \alpha_{j+1}$ for $1 \leq j < J$. It follows from (3.1.1) and the positivity and sufficient smallness of δ_μ that $0 < \alpha_j < 1$ for $1 \leq j \leq J$. Choose $0 < \epsilon < \alpha_J$ to be any positive number which is less than $\alpha_j - \alpha_{j+1}$ for $1 \leq j < J$. Then the following three conditions are satisfied:

(i) $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}_{\alpha_j})$ contains $Y_j \cup \cdots \cup Y_J$ for $1 \leq j \leq J$,
(ii) $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}_{\alpha_j}) \cap Y_p$ is a proper subvariety of Y_p for $1 \leq p < j \leq J$, and
(iii) $\mathcal{I}_{Y_j} \cap \mathcal{I}_{\alpha_j - \epsilon} \subset \mathcal{I}_{\alpha_j}$ for $1 \leq j \leq J$,

where $\mathcal{I}_{Y_j}$ denotes the ideal sheaf of Y_j.

Fix $1 \leq j \leq J$. Then $H^\nu(X,\, \mathcal{O}_X(mL + F + K_X) \otimes (\mathcal{I}_{\alpha_j - \epsilon}/\, \mathcal{I}_{\alpha_j}))$ vanishes for $\nu \geq 1$ and $m \geq m_0$. Given any positive integer s, by Lemma (1.2) there exists some $m_0 \leq m_j \leq s\ n + m_0$ such that we can find linearly independent elements $f_1^{(j)}, \cdots, f_s^{(j)}$ of $\Gamma(X,\, \mathcal{O}_X(m_jL + F + K_X) \otimes (\mathcal{I}_{\alpha_j - \epsilon}/\, \mathcal{I}_{\alpha_j}))$. We can lift $f_1^{(j)}, \cdots, f_s^{(j)}$ up to $g_1^{(j)}, \cdots, g_s^{(j)}$ in $\Gamma(X,\, \mathcal{O}_X(m_jL + F + K_X) \otimes \mathcal{I}_{\alpha_j - \epsilon})$, because the cohomology group $H^1(X,\, \mathcal{O}_X(m_jL + F + K_X) \otimes \mathcal{I}_{\alpha_j})$ vanishes. For a holomorphic function germ on $\mathbf{C}^n$ to vanish at a point to order q, the number of linear equations to be satisfied by the coefficients of its power series at that point is $\binom{n+q-1}{n}$. We now set $s = \binom{n+q-1}{n} + 1$. We can find complex numbers $a_1, \cdots, a_s$ not all zero such that $g^{(j)} := \sum_{\mu=1}^s a_s g_s^{(j)}$ as an element of $\Gamma(X,\, m_jL + F + K_X)$ vanishes at P_0 to order at least q.

The element $g^{(j)}$ of $\Gamma(X,\, mL + F + K_X)$ cannot vanish identically on $\mathcal{O}_X/\mathcal{I}_{Y_j}$, otherwise the inclusion $\mathcal{I}_{Y_j} \cap \mathcal{I}_{\alpha_j - \epsilon} \subset \mathcal{I}_{\alpha_j}$ implies that $g^{(j)}$ vanishes identically on $\mathcal{O}_X/\mathcal{I}_{\alpha_j}$ and $\sum_{\mu=1}^s a_s f_s^{(j)}$ is identically zero as an element of $\Gamma(X,\, \mathcal{O}_X(mL + F + K_X) \otimes (\mathcal{I}_{\alpha_j - \epsilon}/\, \mathcal{I}_{\alpha_j}))$, contradicting the linear independence of the elements $f_1^{(j)}, \cdots, f_s^{(j)}$ of $\Gamma(X,\, \mathcal{O}_X(mL + F + K_X) \otimes (\mathcal{I}_{\alpha_j - \epsilon}/\, \mathcal{I}_{\alpha_j}))$.

For $1 \leq j \leq J$, since we can find a multivalued holomorphic section h of L over X which does not vanish identically on Y_j, it follows that we can multiply $g^{(j)}$ by $h^{m_\# - m_j}$ and get a multivalued holomorphic section $g_\#^{(j)}$ of $m_\# L + F + K_X$ which vanishes to order at least q_ν at P_ν ($1 \leq \nu \leq \ell$) and does not vanish identically on Y_j. Choose a positive number p such that $g_\#^{(j)}$ raised to the power p is a holomorphic section $G^{(j)}$ of $p(m_\# L + F + K_X)$. Consider the linear subspace E of $\Gamma(X,\, p(m_\# L + F + K_X))$ consisting of all elements which vanish to order at least $p\,q_\nu$ at the point P_ν $(1 \leq \nu \leq \ell)$. The existence of $G^{(j)}$ shows that the subspace E_j of all elements of E which vanish identically on Y_j is a proper subspace of E. Thus the union $\cup_{j=1}^{J} E_j$ is a proper subvariety of E and there exists an element G of $\Gamma(X,\, p(m_\# L + F + K_X))$ which vanishes to order at least pq_ν at the point P_ν $(1 \leq \nu \leq \ell\,)$ but does not vanish identically on any positive dimensional branch of Y. We consider the p^{th} root $g_\#$ of G which is a multivalued holomorphic section of $m_\# L + F + K_X$ over X with the property that it vanishes to order at least q_ν at the point $P_\nu (1 \leq \nu \leq \ell)$ but does not vanish identically on any positive dimensional branch of Y. Q.E.D.

Remark (3.2). When we use Remark (1.3), we can replace $m_0 \leq m_j \leq s\,n + m_0$ in Lemma (3.1) by $m_0 \leq m_j \leq s\, \dim Y + m_0$ and replace $m_\# \geq s\,n + m_0$ by $m_\# \geq s\, \dim Y + m_0$.

§4. *Proof of Theorem (0.1).*

Let $\{h_k\}$ be a finite set of multivalued holomorphic sections of L over X without common zeroes. Let $p_n = 0$. We are going to use descending induction on $0 \leq d \leq n-1$ to prove the following.

$(4.1)_d$ For $p_d \geq \frac{1}{2}\left(p_{d+1} + n + 1 + n\ \sum_{\nu=1}^{\ell} \binom{n+2q_\nu - 1}{n}\right)$ and $p_d \geq p_\mu$ $(d < \mu < n)$, there exists a multivalued holomorphic section f_d of $p_d L + K_X$ over X which vanishes to order at least q_ν at P_ν for $1 \leq \nu \leq \ell$ such that the zero set Y_d of the multiplier ideal sheaf of the metric of $p_d L + K_X$ defined by $\frac{1}{\sum_{d \leq j \leq n-1, k} \left| f_j h_k^{p_d - p_j}\right|^2}$ has dimension $\leq d$.

For $d = n-1$, by the theorem of Riemann-Roch we can simply use $p_{n-1} > 1 + \sum_{\nu=1}^{\ell}(q_\nu - 1)$ and also in this case we do not even need K_X but we use it anyway to make the induction process easier. Or we can more elegantly start out with the induction $p_n = 0$.

For the derivation of $(4.1)_{d-1}$ from $(4.1)_d$, we do not use from $(4.1)_d$ the fact that the holomorphic multivalued section f_d of $p_d L + K_X$ over X vanishes to order at least q_ν at P_ν for $1 \leq \nu \leq \ell$. What we use from $(4.1)_d$ is simply that the zero set Y_d of the multiplier ideal sheaf of the metric of $p_d L + K_X$ defined by

$$\frac{1}{\sum_{d \leq j \leq n-1, k} \left| f_j h_k^{p_d - p_j}\right|^2}$$

has dimension $\leq d$.

We now derive $(4.1)_{d-1}$ from $(4.1)_d$. We apply Lemma (3.1) to the case $F = p_d L + K_X$ and $m_0 = 1$ and use $2q_\nu$ instead of q_ν $(1 \leq \nu \leq \ell)$. We set

$$s = 1 + \sum_{\nu=1}^{\ell} \binom{n + 2q_\nu - 1}{n}$$

and let $p_\#$ be the smallest integer such that $p_\# \geq s\,n+1$ and $p_\# + p_d$ is even. Lemma (3.1) tells us that there exists a multivalued holomorphic section $g_\#$ of $p_\# L + F + K_X$ over X such that $g_\#$ vanishes at P_ν to order at least $2q_\nu$ $(1 \leq \nu \leq \ell)$ and $g_\#$ does not vanish identically on any positive dimensional branch of Y. We let $p_{d-1} = \frac{1}{2}(p_\# + p_d)$ and $f_{d-1} = g_\#^{1/2}$. Then f_{d-1} is a multivalued holomorphic section of $p_{d-1}L + K_X$ which vanishes at P_ν to order at least q_ν $(1 \leq \nu \leq \ell)$ and the zero set Y_{d-1} of the multiplier ideal sheaf of the metric

$$\frac{1}{\sum_{d-1 \leq j \leq n-1, k} \left| f_j h_k^{p_d - p_j} \right|^2}$$

has dimension $\leq d-1$. Clearly the conclusion still holds when p_{d-1} is replaced by a greater integer, because we can multiply f_{d-1} by the power of a multivalued holomorphic section of L not vanishing identically on any branch of Y_d. Thus we conclude the proof of $(4.1)_d$ by induction. Let us estimate a bound for the number p_d needed for $(4.1)_d$ in terms of n and q_ν $(1 \leq \nu \leq \ell)$. We know that

$$p_\# \leq n + 2 + n \sum_{\nu=1}^{\ell} \binom{n + 2q_\nu - 1}{n}$$

and we can choose

$$p_{d-1} = \frac{1}{2}(p_\# + p_d) \leq \frac{1}{2}\left(p_d + n + 2 + n \sum_{\nu=1}^{\ell} \binom{n + 2q_\nu - 1}{n}\right).$$

We define $p'_n = 0$ and inductively

$$p'_{d-1} = \frac{1}{2}\left(p'_d + n + 2 + n \sum_{\nu=1}^{\ell} \binom{n + 2q_\nu - 1}{n}\right).$$

Then we have

$$\begin{aligned} p'_{n-d} &= \left(\sum_{j=1}^{d} \frac{1}{2^j}\right)\left(n + 2 + n \sum_{\nu=1}^{\ell} \binom{n + 2q_\nu - 1}{n}\right) \\ &\leq 2\left(n + 2 + n \sum_{\nu=1}^{\ell} \binom{n + 2q_\nu - 1}{n}\right). \end{aligned}$$

We can choose p_d in $(4.1)_d$ so that $p_d \leq p'_d (0 \leq d < n)$. Let $q_\nu = n + t_\nu$ for $1 \leq \nu \leq \ell$. Let $m \geq 2\left(n+2+n\sum_{\nu=1}^{\ell}\binom{3n+2t_\nu-1}{n}\right)$. Let η be the metric for $m\,L + K_X$ defined by

$$\frac{1}{\sum_{0\leq j\leq n-1,k}|f_j h_k^{m-p_j}|^2}$$

and $\mathcal{I}$ be the multiplier ideal sheaf of the metric η of $m\,L+K_X$. By $(4.1)_0$ the dimension of the zero set of $\mathcal{I}$ is 0. Moreover, $\mathcal{I}$ is contained in $\prod_{\nu=1}^{\ell}\mathbf{m}_{P_\nu}^{t_\nu+1}$. By the vanishing theorem for multiplier ideal sheaves we have

$$H^1\left(X,\,\mathcal{O}_X(m\;L+2\;K_X)\otimes\left(\prod_{\nu=1}^{\ell}\mathbf{m}_{P_\nu}^{t_\nu+1}\right)\right)=0.$$

As a consequence the restriction map from $\Gamma(X,\,m\,L+2\,K_X)$ to

$$\bigoplus_{\nu=1}^{\ell}\left(\mathcal{O}_{X,P_\nu}/\,\mathbf{m}_{P_\nu}^{t_\nu+1}\right)$$

is surjective. Q.E.D.

Remark (4.2). In Theorem (0.1) and Corollaries (0.2) and (0.3) the conclusion holds also for $m\,L+A+2\,K_X$ (instead of $m\,L+2\,K_X$) for any numerically effective line bundle A over X.

Remark (4.3). In order to obtain Demailly's result for the very ampleness of $12\,n^n L + 2\,K_X$ for $2 \leq n < 5$, we have to avoid the process of rounding off in obtaining p_d in the proof of Theorem (0.1). A better choice of p_d is inductively defined as follows.

(i) $p_{n-1} = 2 + \sum_{\nu=1}^{\ell}(t_\nu + n - 1)$.
(ii) p_{d-1} is the smallest integer no less than $\frac{1}{2}\left(p_d + d\sum_{\nu=1}^{\ell}\binom{3n+2t_\nu-1}{n}+d+1\right)$ and no less than p_μ for $d \leq \mu \leq n-1$.

Then the global holomorphic sections of $p_0 L + 2\;K_X$ generate simultaneously the t_ν-jets at P_ν for $1 \leq \nu \leq \ell$.

For $\ell = 1$ and $q_1 = 1$ and $n = 2$, we have $p_1 = 4$ and $p_0 = 14$. For $\ell = 2$ and $q_1 = q_2 = 0$ and $n = 2$, we have clearly a smaller p_0. Thus we have the very ampleness of $14\;L + 2\;K_X$ when X is of dimension 2. On the other hand, $12\;n^n = 48$ is greater than 14.

For $\ell = 1$ and $q_1 = 1$ and $n = 3$, we have $p_2 = 5$, $p_1 = p_0 = 124$. For $\ell = 2$ and $q_1 = q_2 = 0$ and $n = 3$, we have clearly a smaller p_0. Thus we have the very ampleness of $124\,L + 2\,K_X$ when X is of dimension 3. On the other hand, $12\,n^n = 324$ is greater than 124.

For $\ell = 1$ and $q_1 = 1$ and $n = 4$, we have $p_3 = 6$, $p_2 = 1078$, $p_1 = p_0 = 1256$. For $\ell = 2$ and $q_1 = q_2 = 0$ and $n = 4$, we have clearly a smaller p_0. Thus we have the very ampleness of $1256\,L + 2\,K_X$ when X is of dimension 4. On the other hand, $12\,n^n = 3072$ is greater than 1256.

Skoda's Surjectivity Technique.

Functional Analysis. Consider the diagram of Hilbert spaces and linear operators

$$\begin{array}{ccc} H_0 & \xrightarrow{T_1} & H_1 \\ T_2 \downarrow & & \\ H_2 & & \end{array}$$

where T_1 is continuous and T_2 is closed and has dense domain. Let G_1 be a closed subspace of H_1. We would like to find conditions so that

$$T_1 (\operatorname{Ker} T_2) = G_1.$$

Lemma. For

$$T_1 (\operatorname{Ker} T_2) = G_1$$

to hold it is necessary and sufficient that there exists some $c > 0$ such that

$$\|T_1^* x_1 + T_2^* x_2\|_{H_0} \geq c\|x_1\|_{H_1}$$

for all $x_1 \in G_1$ and all $x_2 \in \operatorname{Dom} T_2^*$. When the equivalent condition is satisfied, then for every $x_1 \in G_1$ there exists $x_0 \in \operatorname{Ker} T_2$ such that

$$\begin{cases} T_1 x_0 = x_1, \\ \\ \|x_0\|_{H_0} \leq \frac{1}{c}\|x_1\|_{H_1}. \end{cases}$$

Proof. Let $G_0 = \operatorname{Ker} T_2$. The surjectivity $T_1(G_0) = G_1$ means that $(T_1|G_0)^* : G_1^* \to G_0^*$ is injective, which means that

$$(\flat) \qquad \|(T_1|G_1)^* x_1\|_{G_0^*} \geq c\|x_1\|_{G_1^*}$$

for some $c > 0$ and for all $x_1 \in G_1^*$. Take $x_1 \in G_1$. Then x_1 by inner product in H_1 defines an element of G_1^* and its norm $\|x_1\|_{G_1^*}$ in G_1^* is equal to $\|x_1\|_{H_1}$. We consider $T_1^* x_1$ which is a linear functional on H_0 and in particular a linear function on G_0. However, its norm as a linear functional on H_0 is different from its norm as a linear functional on G_0. Its norm as a linear functional on G_0 is equal to

$$\inf_{x_0 \in G_0^\perp} \|T_1^* x_1 + x_0\|_{H_0}.$$

Since G_0 is the kernel of T_2, we know that the image of T_2^* is dense in $G_0^\perp$. Hence

$$\|(T_1|G_1)^* x_1\|_{G_0^*} = \inf_{x_0 \in G_0^\perp} \|T_1^* x_1 + x_0\|_{H_0}$$

and the condition (♭) becomes

$$\|T_1^* x_1 + T_2^* x_2\|_{H_0} \geq c\|x_1\|_{H_1}$$

for all $x_2 \in \operatorname{Dom} T_2^*$. Q.E.D.

Inequality from Integration by Parts. Take a pseudoconvex domain Ω spread over $\mathbf{C}^n$ with Lebesgue volume form $d\lambda$. Let $g_1, \cdots, g_p$ be holomorphic functions on Ω. Let φ_0 and φ_1 are real-valued functions on Ω. Consider

$$\begin{array}{ccc} \left[L^2\left(\Omega, \varphi_0\right)\right]^p & \xrightarrow{T_1} & L^2\left(\Omega, \varphi_1\right) \\ T_2 \downarrow & & \\ \operatorname{Ker}\left(\left[L^2_{(0,1)}\left(\Omega, \varphi_0\right)\right]^p \xrightarrow{\bar\partial} \left[L^2_{(0,2)}\left(\Omega, \varphi_0\right)\right]^p\right) & & \end{array}$$

where

$$T_1\left(h_1, \cdots, h_p\right) = \sum_{j=1}^p g_j h_j$$

and

$$T_2\left(h_1, \cdots, h_p\right) = \left(\bar\partial h_1, \cdots, \bar\partial h_p\right).$$

Let $\varphi = \varphi_1 - \varphi_0$. Then

$$T_1^* u = \left(\overline{g_1} u e^{-\varphi}, \overline{g_2} u e^{-\varphi}, \cdots, \overline{g_p} u e^{-\varphi}\right)$$

for $u \in L^2\left(\Omega, \varphi_1\right)$ in the domain of T_1^*. For

$$v = \left(v_1, \cdots, v_p\right)$$

with $v_j \in L^2_{(0,1)}\left(\Omega, \varphi_0\right)$ in the domain of $\bar\partial^*$ and $\bar\partial v_j = 0$ for $1 \leq j \leq p$,

$$T_2^* v = \left(\bar\partial^* v_1, \cdots, \bar\partial^* v_p\right).$$

We would like to have

$$\sum_{j=1}^p \|\overline{g_j} u e^{-\varphi} + \bar\partial^* v_j\|^2_{L^2(\Omega,\varphi_0)} \geq C^2 \|u\|^2_{L^2(\Omega,\varphi_1)}$$

for $u \in L^2\left(\Omega, \varphi_1\right)$ holomorphic and for $v_j \in L^2_{(0,1)}\left(\Omega, \varphi_0\right)$ in the domain of $\bar\partial^*$ with $\bar\partial v_j = 0$ for $1 \leq j \leq p$.

We expand
(♯)
$$\textstyle\sum_{j=1}^{p}\|\overline{g_j}ue^{-\varphi}+\bar\partial^* v_j\|^2_{L^2(\Omega,\varphi_0)}=\sum_{j=1}^{p}\int_\Omega\left|\overline{g_j}ue^{-\varphi}\right|^2e^{-\varphi_0}d\lambda$$
$$\textstyle +2\mathrm{Re}\,\sum_{j=1}^{p}\int_\Omega \overline{g_j}ue^{-\varphi}\overline{\bar\partial^* v_j}e^{-\varphi_0}d\lambda+\sum_{j=1}^{p}\int_\Omega\left|\bar\partial^* v_j\right|^2e^{-\varphi_0}d\lambda.$$
Let
$$v_j=\sum_{k=}^{n}v_{j\bar k}d\overline{z^k}.$$
Since $v_j\in L^2_{(0,1)}(\Omega,\varphi_0)$ is in the domain of $\bar\partial^*$ for $1\le j\le p$, we can do integration by parts in the second term on the right-hand side of (♯) and from the holomorphicity of u we get
$$\sum_{j=1}^{p}\|\overline{g_j}ue^{-\varphi}+\bar\partial^* v_j\|^2_{L^2(\Omega,\varphi_0)}=\sum_{j=1}^{p}\int_\Omega\left|\overline{g_j}ue^{-\varphi}\right|^2e^{-\varphi_0}d\lambda$$
$$+2\mathrm{Re}\sum_{j=1}^{p}\sum_{k=1}^{n}\int_\Omega u\frac{\partial}{\partial\overline{z^k}}\left(\overline{g_j}e^{-\varphi}\right)\overline{v_{j\bar k}}e^{-\varphi_0}d\lambda+\sum_{j=1}^{p}\int_\Omega\left|\bar\partial^* v_j\right|^2e^{-\varphi_0}d\lambda.$$
Write
$$|g|^2=\sum_{j=1}^{p}|g_j|^2.$$
We apply
$$2ab\le\frac{1}{\alpha}\,a^2+\alpha\,b^2$$
(actually the negative
$$-2ab\ge-\frac{1}{\alpha}\,a^2-\alpha\,b^2$$
of the inequality) with
$$\begin{cases}a=|g\,u|e^{-\varphi}\\[2ex] b=|g|^{-1}\left|\sum_{j=1}^{p}\sum_{k=1}^{n}e^{\varphi}\frac{\partial}{\partial z^k}\left(g_je^{-\varphi}\right)v_{j\bar k}\right|\end{cases}$$
to the second term of the right-hand side to get
$$\sum_{j=1}^{p}\|\overline{g_j}ue^{-\varphi}+\bar\partial^* v_j\|^2_{L^2(\Omega,\varphi_0)}\ge\left(1-\frac{1}{\alpha}\right)\int_\Omega|g|^2|u|^2e^{-2\varphi-\varphi_0}d\lambda$$
$$-\alpha\int_\Omega|g|^{-2}\left|\sum_{j=1}^{p}\sum_{k=1}^{n}e^{\varphi}\frac{\partial}{\partial z^k}\left(g_je^{-\varphi}\right)v_{j\bar k}\right|^2e^{-\varphi_0}d\lambda+\sum_{j=1}^{p}\int_\Omega\left|\bar\partial^* v_j\right|^2e^{-\varphi_0}d\lambda.$$

We now make use of the last term on the right-hand side by adding

$$\sum_{j=1}^{p} \|\bar{\partial} v_j\|^2_{L^2(\Omega,\varphi_0)}$$

to it and invoke the Bochner-Kodaira formula for $L^2(\Omega, \varphi_0)$, which says that

$$\|\bar{\partial}^* v_j\|^2_{L^2(\Omega,\varphi_0)} + \|\bar{\partial} v_j\|^2_{L^2(\Omega,\varphi_0)} \geq \int_\Omega \sum_{k,\ell=1}^{n} \frac{\partial^2 \varphi_0}{\partial z^k \partial \overline{z^\ell}} v_{j,\bar{k}} \overline{v_{j,\bar{\ell}}} e^{-\varphi_0} d\lambda$$

for any $1 \leq j \leq p$. Hence

$$\sum_{j=1}^{p} \|\overline{g_j} u e^{-\varphi} + \bar{\partial}^* v_j\|^2_{L^2(\Omega,\varphi_0)} + \sum_{j=1}^{p} \|\bar{\partial} v_j\|^2_{L^2(\Omega,\varphi_0)} \geq \left(1 - \frac{1}{\alpha}\right) \int_\Omega |g|^2 |u|^2 e^{-2\varphi - \varphi_0} d\lambda$$
$$+ \int_\Omega \left[\sum_{j=1}^{p} \sum_{k,\ell=1}^{n} \frac{\partial^2 \varphi_0}{\partial z^k \partial \overline{z^\ell}} v_{j,\bar{k}} \overline{v_{j,\bar{\ell}}} e^{-\varphi_0} - \alpha |g|^{-2} \left| \sum_{j=1}^{p} \sum_{k=1}^{n} e^{\varphi} \frac{\partial}{\partial z^k} \left(g_j e^{-\varphi}\right) v_{j\bar{k}} \right|^2 \right] e^{-\varphi_0} d\lambda.$$

We are going to introduce explicitly φ_0 and φ_1. Their definition will be motivated by the following two formulas. The first one is simply the Fubini-Study metric (with the index j in $v_{j,\bar{k}}$ fixed).

$$\sum_{k,\ell=1}^{n} \frac{\partial^2}{\partial z^k \partial \overline{z^\ell}} \left(\log |g|^2\right) v_{j,\bar{k}} \overline{v_{j,\bar{\ell}}}$$
$$= \frac{1}{|g|^4} \sum_{1 \leq m < \ell \leq n} \left| \sum_{k=1}^{n} \left(g_m \frac{\partial g_\ell}{\partial z^k} - g_\ell \frac{\partial g_m}{\partial z^k} \right) v_{j,\bar{k}} \right|^2.$$

The second one is the formula for covariant differentiation of g_j with respect to the connection defined by the metric $\frac{1}{|g|^2}$. Let $\varphi = \log |g|^2$ so that $e^{-\varphi} = \frac{1}{|g|^2}$. Then

$$e^{\varphi} \frac{\partial}{\partial z^k} \left(g_j\, e^{-\varphi}\right) = \frac{\partial g_j}{\partial z^k} - \frac{\partial \varphi}{\partial z^k} g_j$$
$$= \frac{\partial g_j}{\partial z^k} - g_j |g|^{-2} \left(\sum_{\ell=1}^{p} \overline{g_\ell} \frac{\partial g_\ell}{\partial z^k} \right)$$
$$= |g|^{-2} \sum_{\ell=1}^{p} \overline{g_\ell} \left(g_\ell \frac{\partial g_j}{\partial z^k} - g_j \frac{\partial g_\ell}{\partial z^k} \right).$$

Thus

$$\begin{aligned}&\left|\sum_{j=1}^{p}\sum_{k=1}^{n} e^{\varphi}\frac{\partial}{\partial z^k}\left(g_j\, e^{-\varphi}\right) v_{j,\bar{k}}\right|^2\\ &= |g|^{-4}\left|\sum_{j,\ell=1}^{p}\sum_{k=1}^{n}\overline{g_\ell}\left(g_\ell\frac{\partial g_j}{\partial z^k}-g_j\frac{\partial g_\ell}{\partial z^k}\right)v_{j,\bar{k}}\right|^2.\end{aligned}$$

Now it comes to the comparison of the curvature tensor $\Theta_{\ell,\bar{k}}$ of $\frac{1}{|g|^2}$ and the tensor from the square of the covariant derivatives $A_{k,j} := \nabla_k g_j$ of g_j with respect to z^k. The checking is as follows. Since

$$\nabla_k g_j = e^{\varphi}\partial_k\left(e^{-\varphi}g_j\right) = \partial_k g_j - \left(\partial_k\varphi\right)g_j,$$

from $\varphi = \log|g|^2$ it follows that

$$\partial_k\varphi = \partial_k\log|g|^2 = \frac{1}{|g|^2}\sum_{\ell=1}^{p}\left(\partial_k g_\ell\right)\overline{g_\ell}$$

and

$$\begin{aligned}A_{k,j} = \nabla_k g_j &= \partial_k g_j - \frac{1}{|g|^2}\sum_{\ell=1}^{p}\left(\partial_k g_\ell\right)\overline{g_\ell}g_j\\ &= \frac{1}{|g|^2}\left(\left(\sum_{\ell=1}^{p}g_\ell\overline{g_\ell}\right)\partial_k g_j - \sum_{\ell=1}^{p}\left(\partial_k g_\ell\right)\overline{g_\ell}g_j\right)\\ &= \frac{1}{|g|^2}\sum_{\ell=1}^{p}\overline{g_\ell}\left(g_\ell\frac{\partial g_j}{\partial z^k}-g_j\frac{\partial g_\ell}{\partial z^k}\right).\end{aligned}$$

At a point where $|g|$ is normalized to be 1 by homogeneity, we would like to have

$$\sum_{j=1}^{p}\sum_{k,\ell=1}^{n}\Theta_{\ell,\bar{k}}\,v_{j,\bar{\ell}}\,\overline{v_{j,\bar{k}}} \geq \frac{1}{q}\left|\sum_{j=1}^{p}\sum_{k=1}^{n}A_{k,j}\,v_{j,\bar{k}}\right|^2,$$

where $q = \min(n, p-1)$. In other words,

$$(\dagger)\qquad \delta_{i,\bar{j}}\,\Theta_{\ell,\bar{k}} \geq \frac{1}{q}A_{\ell,i}\,\overline{A_{k,j}}$$

as quadratic forms on tensors with double index j,ℓ with $1 \leq j \leq p$ and $1 \leq \ell \leq n$, where $\delta_{i,\bar{j}}$ is the Kronecker delta.

On $\mathbf{C}^p$ with coordinates $w_1, \cdots, w_p$ we have a trivial line bundle with the metric $\frac{1}{\sum_{j=1}^p |w_j|^2}$. We can also interpret this as the hyperplane section line bundle over $\mathbf{P}_{p-1}$ with homogeneous coordinates $[w_1, \cdots, w_p]$ which is given the standard metric $\frac{1}{\sum_{j=1}^p |w_j|^2}$ whose curvature is the Fubini-Study metric of $\mathbf{P}_{p-1}$. Let s_k be the holomorphic section of the trivial line bundle over $\mathbf{C}^p$ defined by the coordinate function w_k for $1 \leq k \leq p$. We can also interpret s_k as a holomorphic section of the hyperplane section line bundle of $\mathbf{P}_{p-1}$ over $\mathbf{P}_{p-1}$ for $1 \leq k \leq p$. We have the curvature

$$\Theta = \partial\bar{\partial} \log \sum_{j=1}^p |w_j|^2$$

on $\mathbf{C}^p$ and the $(1,0)$-covariant differential ∇s_k of s_k with respect to the weight function $\frac{1}{\sum_{j=1}^p |w_j|^2}$. We consider ∇s_k as a $(1,0)$-form on $\mathbf{C}^p$ for $1 \leq k \leq p$. The inequality can be interpreted geometrically as follows. For any local complex submanifold of complex dimension n defined by $\Phi : U \to \mathbf{C}^p$ (where U is an open subset of $\mathbf{C}^n$, consider the pullback $\Phi^*\left(\mathcal{O}_{\mathbf{C}^p}^{\oplus p}\right)$ by Φ of the direct sum of p copies of the trivial line bundle over $\mathbf{C}^p$. This is the same as tensoring $\Phi^*\left(\mathcal{O}_{\mathbf{C}^p}\right)$ with $\mathcal{O}_{\mathbf{C}^n}^{\oplus p}$. We give $\mathcal{O}_{\mathbf{C}^n}^{\oplus p}$ the standard Hermitian form δ which represents the standard metric of the trivial vector bundle $\mathcal{O}_{\mathbf{C}^n}^{\oplus p}$ of rank p. We consider the quadratic form

$$(\Phi^*\Theta) \otimes \delta$$

on $T^*_{\mathbf{C}^n} \otimes \left(\mathcal{O}_{\mathbf{C}^n}^{\oplus p}\right)$. On the other hand, we have the quadratic form

$$\left((\Phi^*\nabla s_k)\,\overline{(\Phi^*\nabla s_\ell)}\right)_{1 \leq k,\ell \leq p}$$

on $T^*_{\mathbf{C}^n} \otimes \left(\mathcal{O}_{\mathbf{C}^n}^{\oplus p}\right)$, where the indices k, ℓ are for the fiber index of $\mathcal{O}_{\mathbf{C}^n}^{\oplus p}$. The inequality

$$(\ddagger) \qquad (\Phi^*\Theta) \otimes \delta \geq \frac{1}{\min(n, p-1)} \left((\Phi^*\nabla s_k)\,\overline{(\Phi^*\nabla s_\ell)}\right)_{1 \leq k,\ell \leq p}$$

is for any local complex manifold of dimension n in $\mathbf{C}^p$ given by $\Phi : U \to \mathbf{C}^p$.

We now prove the following lemma on multilinear algebra, which gives us the inequality (†) (or the inequality (‡) in a slightly different geometric interpretation).

Lemma. Let $q = \min(n, p-1)$. Then

$$\left|\sum_{j,k=1}^{p}\sum_{\ell=1}^{n}\overline{a_j}\,(a_j\,b_{k,\ell} - a_k\,b_{j,\ell})\,c_{k,\ell}\right|^2 \leq q|a|^2\sum_{\ell=1}^{p}\sum_{1\leq m<j\leq n}\left|\sum_{k=1}^{n}(a_m\,b_{j,k} - a_j\,b_{m,k})\,c_{\ell,k}\right|^2,$$

where $|a|^2 = \sum_{j=1}^{p}|a_j|^2$.

Proof of the multilinear algebra lemma. By the inequality of Cauchy-Schwarz (applied twice)

$$\begin{aligned}&\left|\sum_{j,\ell=1}^{p}\sum_{k=1}^{n}\overline{a_j}\,(a_j\,b_{\ell,k} - a_\ell\,b_{j,k})\,c_{\ell,k}\right|^2\\ &\leq |a|^2\sum_{j=1}p\left|\sum_{\ell=1}^{p}\sum_{k=1}^{n}(a_j\,b_{\ell,k} - a_\ell\,b_{j,k})\,c_{\ell,k}\right|^2\\ &\leq (p-1)|a|^2\sum_{j,\ell=1}^{p}\left|\sum_{k=1}^{n}(a_j\,b_{\ell,k} - a_\ell\,b_{j,k})\,c_{\ell,k}\right|^2,\end{aligned}$$

because in the second application of the inequality of Cauchy-Schwarz for fixed $1 \leq j \leq p$ the factor 1 is used only when $1 \leq \ell \leq p$ is not equal to j and so there are only $p-1$ such indices ℓ. Thus the inequality for the case $q = p-1$ is obtained. Moreover, the result is much weaker than what could be achieved in this inequality. For fixed ℓ which is allowed to be summed independently, we have the freedom of choosing the double index (j, ℓ) with the restriction $1 \leq j < m \leq p$. So we need only choose $m = \ell$ when $j < \ell$. When $j > \ell$, we relabel j as m and we relabel ℓ as j. It is not clear how one can refine the above argument to give better conclusions.

Now we look at the difficult case of $q = n$. For vectors $X = (x_1, \cdots, x_p)$ and $Y = (y_1, \cdots, y_p)$ we consider the Hermitian form

$$H(X, Y) = \sum_{1\leq m<j\leq p}(a_m x_j - a_j x_m)\,\overline{(a_m y_j - a_j y_m)}.$$

Let

$$B_k = (b_{1,k}, \cdots, b_{p,k})$$

for $1 \leq k \leq n$.

First we consider a linear change among the vectors $B_1, \cdots, B_n$. Suppose we have another set of vectors $B_1', \cdots, B_n'$ linearly equivalent to the set

$B_1, \cdots, B_n$. It means that we can linearly express one set in terms of the other. Let

$$B_k = \sum_{\ell=1}^{p} \alpha_{k,\ell} B'_\ell$$

for some complex numbers $(\alpha_{k,\ell})_{1 \leq k,\ell \leq p}$. Define

$$c'_{j,\ell} = \sum_{k=1}^{p} \alpha_{k,\ell} c_{j,k}.$$

Then

$$\sum_{k=1}^{p} b_{j,k} c_{\ell,k} = \sum_{k=1}^{p} \sum_{m=1}^{p} \alpha_{k,m} b'_{j,m} c_{\ell,k} = \sum_{k=1}^{p} \sum_{m=1}^{p} \alpha_{k,m} b'_{j,m} c_{\ell,k} = \sum_{m=1}^{p} b'_{j,m} c'_{\ell,m}.$$

This simply means that when we take an inner product of two vectors, the transformation on one vector and the inverse transpose of the same transformation on the other vector would not change the inner product. For this interpretation we are actually taking the inner product for the second index, which means in $\mathbf{C}^n$. We want his step, because in the inequality we seek, the summation between $b_{j,\ell}$ and $c_{k,\ell}$ is always over the last index of $b_{j,\ell}$ and the last index of $c_{k,\ell}$. This means that only the inner product between $b_{j,\ell}$ and $c_{k,\ell}$ in $\mathbf{C}^n$ for the last index matters.

This enables us to replace $B_1, \cdots, B_n$ by a $\mathbf{C}$-linear combination. However, when we choose a good $\mathbf{C}$-linear combination, we do it so that we end up with an orthonormal basis with respect to the Hermitian form $H(\cdot, \cdot)$. Thus we are considering two different $\mathbf{C}$-vector spaces, one is $\mathbf{C}^n$ and the other is $\mathbf{C}^p$ so that for the first one we let the second index of $b_{j,\ell}$ and $c_{k,\ell}$ to be the component index, and for the second one we let the first index of $b_{j,\ell}$ and $c_{k,\ell}$ to be the component index.

Now without loss of generality we can assume that $B_1, \cdots, B_n$ are orthonormal (possibly with a smaller n which is even better) with respect to $H(\cdot, \cdot)$.

We now transform both sides of the inequality to expressions in terms of the Hermitian form $H(\cdot, \cdot)$. For fixed $1 \leq \ell \leq p$ we rewrite the summand on the right-hand of the inequality we would like to prove as

$$\sum_{1 \leq m < j \leq p} \left| \sum_{k=1}^{n} (a_m\, b_{j,k} - a_j\, b_{m,k})\, c_{\ell,k} \right|^2$$

$$= \sum_{1 \le m < j \le p} \sum_{k,s=1}^{n} (a_m\, b_{j,k} - a_j\, b_{m,k})\, c_{\ell,k} \overline{(a_m\, b_{j,s} - a_j\, b_{m,s})\, c_{\ell,s}}$$

$$= \sum_{k,s=1}^{n} H\left(B_k, B_s\right) c_{\ell,k} \overline{c_{\ell,s}} = \sum_{k,s=1}^{n} H\left(B_k, B_s\right) c_{\ell,k} \overline{c_{\ell,s}} = \sum_{k=1}^{n} \left|c_{\ell,k}\right|^2 .$$

Now we look at the left-hand side of the inequality which we seek to prove.

$$\left| \sum_{j,\ell=1}^{p} \sum_{k=1}^{n} \overline{a_j}\, (a_j\, b_{\ell,k} - a_\ell\, b_{j,k})\, c_{\ell,k} \right|^2$$

$$\le n \left(\sum_{k=1}^{n} \left| \sum_{j,\ell=1}^{p} \overline{a_j}\, (a_j\, b_{\ell,k} - a_\ell\, b_{j,k})\, c_{\ell,k} \right|^2 \right).$$

So we need only show that for fixed $1 \le k \le n$ we have

$$\left| \sum_{j,\ell=1}^{p} \overline{a_j}\, (a_j\, b_{\ell,k} - a_\ell\, b_{j,k})\, c_{\ell,k} \right|^2 \le |a|^2 \sum_{\ell=1}^{p} \left|c_{\ell,k}\right|^2 .$$

Now

$$\left| \sum_{j,\ell=1}^{p} \overline{a_j}\, (a_j\, b_{\ell,k} - a_\ell\, b_{j,k})\, c_{\ell,k} \right|^2$$

$$= \left| \sum_{1 \le j < \ell \le p} (a_j\, b_{\ell,k} - a_\ell\, b_{j,k})\, (\overline{a_j}\, c_{\ell,k} - \overline{a_\ell}\, c_{j,k}) \right|^2$$

$$\le H(B_k, B_k) H\left(\overline{C_k}, \overline{C_k}\right) = H\left(\overline{C_k}, \overline{C_k}\right)$$

$$= |a|^2 \sum_{\ell=1}^{p} \left|c_{\ell,k}\right|^2 - \left| \sum_{\ell=1}^{p} a_\ell c_{\ell,k} \right|^2 \le |a|^2 \sum_{\ell=1}^{p} \left|c_{\ell,k}\right|^2 ,$$

where

$$C_k = \left(c_{1,k}, \cdots, c_{p,k}\right).$$

Putting all the steps together, we get our inequality which we seek to prove.

The two important ingredients of this proof are the following.

(i) The left side is a product

$$\sum_{j,k=1}^{p} \overline{a_j}\,(a_j\,b_{k,\ell} - a_k\,b_{j,\ell})\,c_{k,\ell}$$

of the factor $a_j\,b_{k,\ell} - a_k\,b_{j,\ell}$ which is skew-symmetric in j and ℓ with another factor $\overline{a_j}\,c_{k,\ell}$ which may not be skew symmetric in j and ℓ. The product remains the same if we replace the second factor by its skew-symmetrization.

(ii) On both the left-hand side and the right-hand side of the inequality to be proved, both $b_{k,\ell}$ and $c_{j,\ell}$ occur only in summations over the second index ℓ.

The first ingredient enables us to convert the right-hand side to an inner product with respect to an appropriately defined Hermitian form on $\mathbf{C}^p$. The second ingredient enables us to convert $b_{k,\ell}$ and $c_{j,\ell}$ at the same time so that we have orthonormality of $b_{k,\ell}$ when the first index is used as the component index, by using a linear transformation together with its transpose inverse on the two arguments of an inner product in $\mathbf{C}^n$. Q.E.D.

Now we choose our weight functions. There are two of them. One is φ_0 and the other is φ_1. We are going to choose φ_0 and $\varphi = \varphi_1 - \varphi_0$. We need

$$\sum_{j=1}^{p} \sum_{1 \le k,\ell \le n} \frac{\partial^2 \varphi_0}{\partial z^k \partial \overline{z^\ell}}\, v_{j,\bar{k}}\, \overline{v_{j,\bar{\ell}}} \ge \alpha |g|^{-2} \left| \sum_{j=1}^{p} \sum_{k=1}^{n} e^{\varphi} \frac{\partial}{\partial z^k} \left(g_j\, e^{-\varphi} \right) v_{j,\bar{k}} \right|^2 .$$

We can achieve this by applying our multilinear algebra lemma to the choices of

$$a_j = g_j, \quad b_{j,\ell} = \frac{\partial g_j}{\partial z_\ell}, \quad c_{j,\ell} = v_{j,\bar{\ell}}$$

and choosing the weight functions

$$\varphi = \log |g|^2.$$

and

$$\varphi_0 = \alpha q \log |g|^2 + \psi$$

for any plurisubharmonic function ψ. For such choices we end up with the inequality

$$\sum_{j=1}^{p} \|\overline{g_j} u e^{-\varphi} + \bar\partial^* v_j\|^2_{L^2(\Omega,\varphi_0)} + \sum_{j=1}^{p} \|\bar\partial v_j\|^2_{L^2(\Omega,\varphi_0)}$$
$$\geq \left(1 - \frac{1}{\alpha}\right) \int_\Omega |u|^2 e^{-\varphi_0} d\lambda + \int_\Omega \sum_{j=1}^{p} \sum_{k,\ell=1}^{n} \frac{\partial^2 \psi}{\partial z^k \partial \overline{z^\ell}} \, v_{j,\bar k} \, \overline{v_{j,\bar\ell}} \, e^{-\varphi_0}.$$

We thus have the following important theorem of Skoda.

Theorem (Skoda). Let Ω be a pseudoconvex domain in $\mathbf{C}^n$ and ψ be a plurisubharmonic function on Ω. Let $g_1, \ldots, g_p$ be holomorphic functions on Ω. Let $\alpha > 1$ and $q = \inf(n, p-1)$. Then for every holomorphic function f on Ω such that

$$\int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda < \infty,$$

there exist holomorphic functions $h_1, \ldots, h_p$ on Ω such that

$$f = \sum_{j=1}^{p} g_j h_j$$

and

$$\int_\Omega |h|^2 |g|^{-2\alpha q} e^{-\psi} d\lambda \leq \frac{\alpha}{\alpha - 1} \int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda,$$

where

$$|g| = \left(\sum_{j=1}^{p} |g_j|^2\right)^{\frac{1}{2}}, \qquad |h| = \left(\sum_{j=1}^{p} |h_j|^2\right)^{\frac{1}{2}},$$

and $d\lambda$ is the Euclidean volume element of $\mathbf{C}^n$.

Corollary. Let Ω be a bounded Stein open subset of $\mathbf{C}^n$. Let $g_1, \cdots, g_n, \rho$ be holomorphic functions on some open neighborhood $\tilde\Omega$ of the topological closure $\bar\Omega$ of Ω. Let Z be the common zero-set of $g_1, \cdots, g_n$ in $\tilde\Omega$. Assume that ρ vanishes on Z. Let J be the Jacobian determinant of $g_1, \cdots, g_n$. Then there exist holomorphic $h_1, \cdots, h_n$ on Ω such that $\rho J = \sum_{j=1}^{n} h_j g_j$. In particular, if the common zero-set of $g_1, \cdots, g_n$ is a single point P_0 in $\mathbf{C}^n$, then $\mathbf{m}_{\mathbf{C}^n, P_0} J \subset \sum_{j=1}^{n} \mathcal{O}_{\mathbf{C}^n, P_0} \, g_j$.

Proof. There exists $\alpha > 1$ such that

$$\int_\Omega \frac{|\rho\, J|^2}{\left(\sum_{j=1}^n |g_j|^2\right)^{\alpha n}} < \infty.$$

The finiteness is done by arguing with $g_1, \cdots, g_n$ as local coordinates after a local coordinate change. By Skoda's theorem

(i) with $p = n$ and $q = n - 1$,

(ii) Ω replaced by $\Omega - \{g = 0\}$,

(iii) with $\psi = -2 \log |g|$, and

(iv) with $f = \rho\, J$,

we obtain $h_1, \cdots, h_n$. Q.E.D.

Corollary. Let f be a holomorphic function germ on $\mathbf{C}^n$ at the origin which vanishes at the origin. Then f^{n+1} belongs to the ideal $\mathcal{I}$ generated by $\frac{\partial f}{\partial z_j}$ for $1 \leq j \leq n$ at the origin, where $z_1, \cdots, z_n$ are the coordinates of $\mathbf{C}^n$.

Proof. Take a resolution $\pi : \widetilde{\mathbf{C}^n} \to \mathbf{C}^n$ of simultaneously of the ideal $\mathcal{I}$ and the ideal $\mathcal{O}_{\mathbf{C}^n} f$ and we get hypersurfaces $\{E_\ell\}_\ell$ in normal crossing in $\widetilde{\mathbf{C}^n}$. We claim that

$$\frac{|f|^2}{\sum_{j=1}^n \left|\frac{\partial f}{\partial z_j}\right|^2}$$

is uniformly bounded in a neighborhood of the origin. Otherwise, when we write the divisor of $\pi^* f$ of f as $\sum_\ell a_\ell E_\ell$ and write $\pi^* \mathcal{I}$ as $\sum_\ell b_\ell E_\ell$, we have $b_\ell > a_\ell$ for some ℓ with $0 \in \pi\,(E_\ell)$ and we can find a local holomorphic curve $\tilde{\varphi} : U \to \widetilde{\mathbf{C}^n}$ with U being an open neighborhood of the origin in $\mathbf{C}$ and $\pi\tilde{\varphi}(0) = 0$ such that $\varphi(U)$ is transversal to E_ℓ. Then $d\,(f \circ \varphi)$ vanishes at 0 to an order higher than that $f \circ \varphi$, which is a contradiction, because $f \circ \varphi$ vanishes at 0. This argument actually gives a slightly higher vanishing order of $|f|^2$ than that of $\sum_{j=1}^n \left|\frac{\partial f}{\partial z_j}\right|^2$ along each E_ℓ when they are pulled back to $\widetilde{\mathbf{C}^n}$ so that

$$\int_W \frac{|f^{n+1}|^2}{\left(\sum_{j=1}^n \left|\frac{\partial f}{\partial z_j}\right|^2\right)^{\alpha(n+1)}} < \infty$$

for some $\alpha > 1$. The conclusion of the Corollary now follows from Skoda's theorem. Q.E.D.

Lemma ????3. Let $h_1, \cdots, h_n$ be holomorphic function germs on $\mathbf{C}^n$ at the origin so that the origin is their only common zero. Let $dh_1 \wedge \cdots \wedge dh_n = J\,(dz_1 \wedge \cdots \wedge dz_n)$. Then J does not belong to the ideal generated by $h_1, \cdots, h_n$.

Proof. Suppose the contrary. Then there exist holomorphic function germs $f_1, \cdots, f_n$ on $\mathbf{C}^n$ at the origin such that $J = \sum_{j=1}^n f_j h_j$. We let $\omega_j = f_j\,(dz_1 \wedge \cdots \wedge dz_n)$ for $1 \leq j \leq n$ so that

$$(\sharp) \qquad dh_1 \wedge \cdots \wedge dh_n = \sum_{j=1}^n h_j \omega_j.$$

Since the origin is the only common zero of $h_1, \cdots, h_n$, we can find connected open neighborhoods U and W of the origin in $\mathbf{C}^n$ so that the map $\pi : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$(z_1, \cdots, z_n) \mapsto (w_1, \cdots, w_n) = (h_1\,(z_1, \cdots, z_n)\,, \cdots, h_n\,(z_1, \cdots, z_n))$$

maps U properly and surjectively onto W and makes U a branched cover over W of λ sheets. By replacing U and W by relatively compact open neighborhoods U' and W' of the origin in U and W respectively, we can assume without loss of generality that $\int_U |\omega_j|^2 \leq C < \infty$ for $1 \leq j \leq n$. We take the direct image of the equation $(\sharp)$ under π. The left-hand side of $(\sharp)$ yields $\lambda\,(dw_1 \wedge \cdots \wedge dw_n)$, because the map π is defined by $w_j = h_j$ for $1 \leq j \leq n$. Let θ_j be the direct image of ω_j under π for $1 \leq j \leq n$. Let Z be the branching locus of π in W. For any simply connected open subset G of $W - Z$, $U \cap \pi^{-1}(G)$ is the disjoint union of λ open subsets $H_1, \cdots, H_\lambda$ of U and $\theta_j(Q) = \sum_{\ell=1}^\lambda \omega_j\left(\tilde{Q}_j\right)$, where $U \cap \pi^{-1}(Q) = \left\{\tilde{Q}_1, \cdots, \tilde{Q}_\lambda\right\}$ with $\tilde{Q}_j \in H_j$. Now

$$\int_G |\theta_j|^2 \leq \lambda \sum_{j=1}^\lambda \int_{H_j} |\omega_j|^2 \leq \lambda C.$$

Since $W - Z$ can be covered by a finite number of simply connected open subsets, it follows that

$$\int_G |\theta_j|^2 < \infty \quad \text{for } 1 \leq j \leq n.$$

Thus θ_j is a holomorphic n-form on G and

$$\lambda\, dz_1 \wedge \cdots \wedge dz_n = \sum_{j=1}^n z_j\, \theta_j$$

on G, which gives a contradiction, because the left-hand side does not vanish at the origin whereas the right-hand side does. Q.E.D.

Counter-Example for Sharpness of Exponent. The exponent used in the denominator of the assumption in Skoda's Theorem cannot be lowered even in the case of Riemann surface. Let X be the Riemann sphere $\mathbf{P}_1$. Consider the hyperplane section line bundle $H_{\mathbf{P}_1}$. Take two holomorphic sections g_1, g_2 of $H_{\mathbf{P}_1}$ without common zeroes. Take the holomorphic section f of $2\,H_{\mathbf{P}_1}+K_{\mathbf{P}_1}$ over $\mathbf{P}_1$ which corresponds to a constant function on $\mathbf{P}_1$ via the isomorphism between $K_{\mathbf{P}_1}$ and $-2\,H_{\mathbf{P}_1}$. If the exponent used in the denominator of the assumption in Skoda's Theorem cannot be lowered can be lowered so that $\alpha = 1$, then $p = 2$ and $n = 1$ and $q = \min(n, p-q) = 1$ and $\alpha q + 1 = 2$ and the assumption

$$\int_{\mathbf{P}_1} \frac{|f|^2}{\left(|g_1|^2 + |g_2|^2\right)^{\alpha q+1}} < \infty$$

is satisfied because g_1, g_2 have no common zeroes. Note that when $\alpha > 1$, the integrand of the above inequality makes no sense unless

$$f \in \Gamma\left(\mathbf{P}_1,\, m\, H_{\mathbf{P}_1} + K_{\mathbf{P}_1}\right)$$

for some $m > 2$. By Skoda's theorem we can write $f = h_1 g_1 + h_2 g_2$ with

$$h_1,\, h_2 \in \Gamma\left(\mathbf{P}_1,\, H_{\mathbf{P}_1} + K_{\mathbf{P}_1}\right)$$

which is impossible, because

$$\Gamma\left(\mathbf{P}_1,\, H_{\mathbf{P}_1} + K_{\mathbf{P}_1}\right) = 0$$

from the isomorphism between $K_{\mathbf{P}_1}$ and $-2\,H_{\mathbf{P}_1}$.

Theorem (1.1). Let L be a holomorphic line bundle over a compact algebraic manifold X of complex dimension n. Let E be a holomorphic line bundle over X with a (possibly singular) Hermitian metric $e^{-\psi}$ along its fibers with ψ plurisubharmonic. Let $S \in \Gamma(X, E)$. Let k be a positive integer. Let $G_1, \cdots, G_p \in \Gamma(X, L)$ and $|G|^2 = \sum_{j=1}^p |G_j|^2$ and $\mathcal{I} = \mathcal{I}_{(n+k+1)\log|G|^2+\varphi}$ and $\mathcal{J} = \mathcal{I}_{(n+k)\log|G|^2+\varphi}$. Then

$$\begin{aligned}&\Gamma(X, \mathcal{I} \otimes ((n+k+1)L + E + K_X)) \\ &= \sum_{j=1}^p G_j \Gamma(X, \mathcal{J} \otimes ((n+k)L + E + K_X)).\end{aligned}$$

Here $\mathcal{I}_\chi$ means the subsheaf of all $f \in \mathcal{O}_X$ such that $|f|^2 e^{-\chi}$ is locally integrable.

Proof of Theorem (1.1). We take a branched cover map $\pi : X \to \mathbf{P}_n$. Let Z_0 be a hypersurface in $\mathbf{P}_n$ which contains the infinity hyperplane of $\mathbf{P}_n$ and the branching locus of π in $\mathbf{P}_n$ such that $Z := \pi^{-1}(Z_0)$ contains the divisor of G_1. Let $\Omega = X - Z$. Let $g_j = \frac{G_j}{G_1}$ $(1 \leq j \leq p)$ and define f by

$$\frac{F}{G_1^{n+k+1} S} = f dz_1 \wedge \cdots \wedge dz_n,$$

where $z_1, \cdots, z_n$ are the affine coordinates of $\mathbf{C}^n$. Use $\alpha = \frac{n+k}{n}$. Let $\psi = \varphi - \log |S|^2$. It follows from $F \in \mathcal{I}_{(n+k+1)\log|G|^2+\varphi}$ that

$$\int_X \frac{|F|^2}{|G|^{2(n+k+1)}} e^{-\varphi} < \infty,$$

which implies that

$$\int_\Omega \frac{|f|^2}{|g|^{2(n+k+1)}} e^{-\psi} = \int_\Omega \frac{\left|\frac{F}{G_1^{n+k+1}S}\right|^2}{\left|\frac{G}{G_1}\right|^{2(n+k+1)}} e^{-\psi} = \int_\Omega \frac{|F|^2}{|G|^{2(n+k+1)}} e^{-\varphi} < \infty.$$

By the above proposition with $q = n$ (which we assume by adding some $F_{p+1} \equiv \cdots \equiv F_{n+1} \equiv 0$ if $p < n+1$) so that $2\alpha q + 2 = 2 \cdot \frac{n+k}{n} \cdot n + 2 = 2(n+k+1)$ and $\psi \equiv 0$. Thus there exist holomorphic functions $h_1, \cdots, h_p$ on Ω such that $f = \sum_{j=1}^p g_j h_j$ and

$$\sum_{j=1}^p \int_\Omega \frac{|h_j|^2}{|g|^{2(n+k)}} e^{-\psi} < \infty.$$

Define

$$H_j = G_1^{n+k} h_j S dz_1 \wedge \cdots \wedge dz_n.$$

Then

$$\int_\Omega \frac{|H_j|}{|G|^{2(n+k)}} e^{-\varphi} = \int_\Omega \frac{|h_j|}{|g|^{2(n+k)}} e^{-\psi} < \infty$$

so that H_j can be extended to an element of $\Gamma(X, (n+k)L + E + K_X)$. Q.E.D.

This theorem gives as an immediate consequence the following result on effective finite generation for the simple case of a numerically effective canonical line bundle, which is interesting in its own right, but not useful for the purpose of this article.

Corollary (1.3). Let F be a holomorphic line bundle over a compact projective algebraic manifold X of complex dimension n. Let $a > 1$ and $b \geq 0$ are integers such that aF and $bF - K_X$ are globally free over X. Then the ring $\cup_{m=1}^{\infty}\Gamma(X, mF)$ is generated by $\cup_{m=1}^{(n+2)a+b-1}\Gamma(X, mF)$.

Proof. For $0 \leq \ell < a$ let $E_\ell = (b+\ell)F - K_X$ and $L = aF$. Let $G_1, \cdots, G_p$ be a basis of $\Gamma(X, L) = \Gamma(X, aF)$. Let $H_1, \cdots, H_q$ be a basis of $\Gamma(X, bF - K_X)$. We give E_ℓ the metric

$$\frac{1}{(\sum_{j=1}^{p} |G_j|^{\frac{2\ell}{a}})(\sum_{j=1}^{q} |H_j|^2)}.$$

Since both $\mathcal{I}$ and $\mathcal{J}$ are unit ideal sheaves, it follows that

$$\Gamma(X, (n+k+1)L + E_\ell + K_X) = \sum_{j=1}^{p} G_j \Gamma(X, (n+k)L + E + K_X)$$

for $k \geq 1$ and $0 \leq \ell < a$, which means that

$$\Gamma(X, ((n+k+1)a + \ell + b)F) = \sum_{j=1}^{p} G_j \Gamma(X, ((n+k)a + \ell + b)F)$$

for $k \geq 1$ and $0 \leq \ell < a$. Thus $\cup_{m=1}^{(n+2)a+b-1}\Gamma(X, mF)$ generates the ring $\cup_{m=1}^{\infty}\Gamma(X, mF)$. Q.E.D.

I need the following more refined version of Theorem 1.

Theorem (1.4). Let L be a holomorphic line bundle over a compact algebraic manifold X of complex dimension n. Let $0 < \gamma < 1$. Let E be a holomorphic line bundle over X such that $(1-\gamma)L + E$ carries a (possibly singular) Hermitian metric $e^{-\varphi}$ along its fibers with φ plurisubharmonic. Let $S \in \Gamma(X, E)$. Let $G_1, \cdots, G_p \in \Gamma(X, L)$ and $|G|^2 = \sum_{j=1}^{p} |G_j|^2$ and $\mathcal{I} = \mathcal{I}_{(n+\gamma+1)\log|G|^2 + \varphi}$ and $\mathcal{J} = \mathcal{I}_{(n+\gamma)\log|G|^2 + \varphi}$. Then

$$\begin{aligned} &\Gamma(X, \mathcal{I} \otimes ((n+2)L + E + K_X)) \\ &= \sum_{j=1}^{p} G_j \Gamma(X, \mathcal{J} \otimes ((n+1)L + E + K_X)). \end{aligned}$$

Here $\mathcal{I}_\chi$ means the subsheaf of all $f \in \mathcal{O}_X$ such that $|f|^2 e^{-\chi}$ is locally integrable.

Proof of Theorem (1.4). We take a branched cover map $\pi : X \to \mathbf{P}_n$. Let Z_0 be a hypersurface in $\mathbf{P}_n$ which contains the infinity hyperplane of $\mathbf{P}_n$ and the branching locus of π in $\mathbf{P}_n$ such that $Z := \pi^{-1}(Z_0)$ contains the divisor of G_1. Let $\Omega = X - Z$. Let $g_j = \frac{G_j}{G_1}$ $(1 \le j \le p)$ and define f by

$$\frac{F}{G_1^{n+2} S} = f dz_1 \wedge \cdots \wedge dz_n,$$

where $z_1, \cdots, z_n$ are the affine coordinates of $\mathbf{C}^n$. Use $\alpha = \frac{n+\gamma}{n}$. Let

$$\psi = \varphi - \log |S|^2 - \log |G_1|^{2(1-\gamma)}.$$

It follows from $F \in \mathcal{I}_{(n+\gamma+1)\log|G|^2+\varphi}$ that

$$\int_X \frac{|F|^2}{|G|^{2(n+\gamma+1)}} e^{-\varphi} < \infty,$$

which implies that

$$\int_\Omega \frac{|f|^2}{|g|^{2(n+\gamma+1)}} e^{-\psi} = \int_\Omega \frac{\left|\frac{F}{G_1^{n+2}S}\right|^2 |G_1|^{2(1-\gamma)}|S|^2}{\left|\frac{G}{G_1}\right|^{2(n+\gamma+1)}} e^{-\psi}$$

$$= \int_\Omega \frac{\left|\frac{F}{G_1^{n+2}S}\right|^2 |G_1|^{2(1-\gamma)}|S|^2}{\left|\frac{G}{G_1}\right|^{2(n+\gamma+1)}} e^{-\varphi+\log|S|^2+\log|G_1|^{2(1-\gamma)}}$$

$$= \int_\Omega \frac{|F|^2}{|G|^{2(n+\gamma+1)}} e^{-\varphi} < \infty.$$

By the above proposition with $q = n$ (which we assume by adding some $F_{p+1} \equiv \cdots \equiv F_{n+1} \equiv 0$ if $p < n+1$) so that

$$2\alpha q + 2 = 2 \cdot \frac{n+\gamma}{n} \cdot n + 2 = 2(n+\gamma+1)$$

and $\psi \equiv 0$. Thus there exist holomorphic functions $h_1, \cdots, h_p$ on Ω such that $f = \sum_{j=1}^p g_j h_j$ and

$$\sum_{j=1}^p \int_\Omega \frac{|h_j|^2}{|g|^{2(n+\gamma)}} e^{-\psi} < \infty.$$

Define

$$H_j = G_1^{n+1} h_j S dz_1 \wedge \cdots \wedge dz_n.$$

Then

$$\int_\Omega \frac{|H_j|}{|G|^{2(n+\gamma)}} e^{-\varphi} = \int_\Omega \frac{|h_j|}{|g|^{2(n+\gamma)}} e^{-\psi} < \infty$$

so that H_j can be extended to an element of $\Gamma(X, (n+1)L+E+K_X)$. Q.E.D.

Skoda's Other Estimate

Skoda's estimate for the setup

$$\begin{array}{ccc} H_0 & \xrightarrow{T_1} & H_1 \\ T_2 \downarrow & & \\ H_2 & & \end{array}$$

is symmetric in $T_1 : H_0 \to H_1$ and $T_2 : H_0 \to H_2$, which is

$$\|T_1^* u + T_2^* v\|_{H_0} \geq \left(1 - \frac{1}{\alpha}\right) \int_\Omega |u|^2 \, e^{-\varphi_2} + \int_\Omega \sum_{j,k,\ell} \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_\ell} \, v_{j,\bar{k}} \overline{v_{j,\bar{\ell}}} \, e^{-\varphi_1},$$

where

$$\alpha \geq 1, \quad q = \inf(n, p-1),$$
$$\varphi_1 = \psi + \alpha q \log |g|^2, \quad \varphi_2 = \psi + (\alpha q + 1) \log |g|^2,$$
$$v_j \in \operatorname{Dom} T_2^* \text{ with } \bar\partial v = 0, \text{ and } u \in \operatorname{Dom} T_1^* \text{ with } \bar\partial u = 0.$$

When we focus on the question of whether T_2 restricted to the kernel of T_1 is surjective, we use the term

$$\int_\Omega \sum_{j,k,\ell} \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_\ell} \, v_{j,\bar{k}} \overline{v_{j,\bar{\ell}}} \, e^{-\varphi_1}$$

on the right-hand side instead of the term

$$\left(1 - \frac{1}{\alpha}\right) \int_\Omega |u|^2 \, e^{-\varphi_2}.$$

So we set $\alpha = 1$. The emphasis is on the choice of ψ. We go back to our original problem of writing f in the form $f = \sum_{j=1}^p h_j g_j$. We could always do this for smooth functions h_j' instead of holomorphic functions h_j by setting

$$h_j' = \frac{f \overline{g_j}}{|g|^2}.$$

Then $f = \sum_{j=1}^{p} h'_j g_j$. We take $\bar{\partial}$ of both sides to get

$$0 = \bar{\partial} f = \sum_{j=1}^{p} \left(\bar{\partial} h'_j \right) g_j$$

and then we would like to solve the $\bar{\partial}$-equation $\bar{\partial} w_j = \bar{\partial} h'_j$ under the constraint

$$\sum_{j=1}^{p} w_j g_j = 0.$$

The estimate

$$\|T_1^* u + T_2^* v\|_{H_0} \geq \int_\Omega \sum_{k,\ell=1}^{n} \sum_{j=1}^{p} \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_\ell} v_{j,\bar{k}} \overline{v_{j,\bar{\ell}}}\, e^{-\varphi_1}$$

precisely enables us to do it. Let us now compute the relevant L^2 norms. First of all we computer $w_j = \bar{\partial} h'_j$ and get

$$w_i = \frac{f}{|g|^4} \sum_{j=1}^{p} \sum_{k=1}^{n} g_j \overline{\left(g_j \frac{\partial g_i}{\partial z_k} - g_i \frac{\partial g_j}{\partial z_k} \right)} d\bar{z}_k.$$

The relevant L^2 norm for $w = (w_1, \cdots, w_p)$ is given by

$$\int_\Omega \frac{|w|^2 e^{-\psi}}{|g|^{2q}} \leq \int_\Omega \frac{|f|^2}{|g|^{2q+2}} \sum_{i,j=1}^{p} \sum_{k=1}^{n} \frac{\left| g_j \frac{\partial g_i}{\partial z_k} - g_i \frac{\partial g_j}{\partial z_k} \right|^2 e^{-\psi}}{|g|^4}.$$

We use

$$\frac{1}{4} \Delta \log |g|^2 = \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \bar{z}_k} \left(\log |g|^2 \right) = \frac{1}{2|g|^4} \sum_{i,j=1}^{p} \sum_{k=1}^{n} \left| g_j \frac{\partial g_i}{\partial z_k} - g_i \frac{\partial g_j}{\partial z_k} \right|^2$$

to rewrite the factor

$$\sum_{i,j=1}^{p} \sum_{k=1}^{n} \frac{\left| g_j \frac{\partial g_i}{\partial z_k} - g_i \frac{\partial g_j}{\partial z_k} \right|^2}{|g|^4}$$

to get

$$\int_\Omega \frac{|w|^2 e^{-\psi}}{|g|^{2q}} \leq \frac{1}{2} \int_\Omega \frac{|f|^2}{|g|^{2q+2}} \Delta \left(\log |g|^2 \right) e^{-\psi}.$$

We can solve the $\bar{\partial}$-equation and get h_j'' so that $\bar{\partial} h_j'' = w_j$ and

$$\int_\Omega \left(\sum_{j=1}^p g_j h_j'' \right) \bar{u}\, e^{-\varphi_2} = 0 \quad \forall\, u \in \mathrm{Dom}\, T_1^* \text{ and } \bar{\partial} u = 0$$

and

$$\int_\Omega \frac{|h''|^2}{|g|^{2q}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} \leq \int_\Omega \frac{|w|^2}{|g|^{2q}}\, e^{-\psi}.$$

Let $h_j = h_j' - h_j''$ and $h = (h_1, \cdots, h_p)$. Then

$$\bar{\partial} h_j = \bar{\partial} h_j' - \bar{\partial} h_j'' = w_j - \bar{\partial} h_j'' = 0 \text{ for } 1 \leq j \leq p.$$

So h is a p-tuple of holomorphic functions on Ω. The equation

$$\int_\Omega \left(\sum_{j=1}^p g_j h_j'' \right) \bar{u}\, e^{-\varphi_2} = 0 \quad \forall\, u \in \mathrm{Dom}\, T_1^* \text{ and } \bar{\partial} u = 0$$

can be rewritten as

$$(T_1\, h'',\, u)_{H_2} = 0 \quad \forall\, u \in \mathrm{Dom}\, T_1^* \text{ and } \bar{\partial} u = 0.$$

From

$$(T_1\, h,\, u)_{H_2} = (T_1\, h' - T_1\, h'',\, u)_{H_2} = (T_1\, h',\, u)_{H_2} = (f,\, u)_{H_2}$$

it follows that

$$(T_1\, h - f,\, u)_{H_2} = 0 \quad \forall\, u \in \mathrm{Dom}\, T_1^* \text{ and } \bar{\partial} u = 0.$$

By setting $u = T_1 h - f$, we conclude that $T_1 h = \sum_{j=1}^p g_j h_j = f$ and

$$\begin{aligned}
&\int_\Omega \frac{|h' - h''|^2}{|g|^{2q}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} \\
&\leq 2 \int_\Omega \frac{|h'|^2}{|g|^{2q}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} + 2 \int_\Omega \frac{|h''|^2}{|g|^{2q}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} \\
&\quad \leq 2 \int_\Omega \frac{\left| f \frac{\bar{g}}{|g|^2} \right|^2}{|g|^{2q}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} + 2 \int_\Omega \frac{|w|^2}{|g|^{2q}},\, e^{-\psi} \\
&\leq 2 \int_\Omega \frac{|f|^2}{|g|^{2q+2}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} + 2 \int_\Omega \frac{|f|^2}{|g|^{2q+2}}\, \Delta \log |g|^2\, e^{-\psi}.
\end{aligned}$$

Hence

$$\int_\Omega \frac{|h|^2}{|g|^{2q}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} \le 2 \int_\Omega \frac{|f|^2}{|g|^{2q+2}} \left(1+\Delta \log |g|^2\right) e^{-\psi}.$$

We can summarize the result in the following theorem of Skoda.

Theorem (Skoda). Let Ω be a pseudoconvex domain spread over $\mathbf{C}^n$, ψ be a plurisubharmonic function on Ω, $g_1, \cdots, g_p$ be holomorphic functions on Ω, $q = \inf(n, p-1)$, X be the common zero-set of $g_1, \cdots, g_p$, and f be a holomorphic function on Ω such that

$$\int_{\Omega - X} \frac{|f|^2}{|g|^{2q+2}} \left(1+\Delta \log |g|^2\right) e^{-\psi} < \infty,$$

where $|g|^2 = \sum_{j=1}^p |g_j|^2$. Then there exist holomorphic functions $h_1, \cdots, h_p$ on Ω such that $f = \sum_{j=1}^p g_j h_j$ and

$$\int_\Omega \frac{|h|^2}{|g|^{2q}} \frac{1}{(1+|z|^2)^2}\, e^{-\psi} \le 2 \int_{\Omega - X} \frac{|f|^2}{|g|^{2q+2}} \left(1+\Delta \log |g|^2\right) e^{-\psi}.$$

Formulation as Extension Problem. Before we go to the precise formulation, we start out with the heuristic consideration. For the surjectivity statement we consider the map $E \to Q$ defined by

$$(x, w_1, \cdots, w_p) \mapsto (x, \zeta)$$

with $\zeta = \sum_{j=1}^p g_j(x) w_j$. We go to the dual spaces $E^* \to Q^*$ with coordinates $(x, w_1^*, \cdots, w_1 *)$ and (x, ζ^*) and the embedding defined by $w_j^* = g_j \zeta^*$ for $1 \le j \le p$. When we have a function f, we assign to it the function $f(x)\zeta^*$ on E^*. The function now is a function on the subspace Q^* of E^* defined by $g_k(x) w_j^* = g_j(x) w_k^*$. The function f on Q^* is given by $f(x)\frac{w_j^*}{g_j}$ when Q^* is parametrized by the coordinate w_j^* (*i.e.* when the projection to the coordinate w_j^* is used as the parametrization of Q^*). We seek to extend it to a function $h_1 w_1^* + \cdots + h_p w_p^*$ such that the restriction to Q^* agrees with $f(x)\frac{w_j^*}{g_j}$ in the parametrization by w_j^*, which means that $h_1 w_1^* + \cdots + h_p w_p^*$ should be rewritten as

$$h_1 \frac{g_1}{g_j} w_j^* + \cdots + h_{j-1} \frac{g_{j-1}}{g_j} w_j^* + h_{j+1} \frac{g_{j+1}}{g_j} w_j^* + \cdots + h_p \frac{g_p}{g_j} w_j^*$$

which should agree with $f(x)\frac{w_j^*}{g_j}$. In other words, we should have

$$f(x) = \sum_{j=1}^{g} h_j g_j.$$

We now go to the precise formulation. The precise formulation is in temrs of the extension of $(n+1)$-forms. Let Ω be a pseudoconvex domain in $\mathbf{C}^n$ (or more generally spread over $\mathbf{C}^n$). Consider the product space $\Omega \times \mathbf{C}^g$. We use coordinates $z = (z_1, \cdots, z_n)$ for points of $\mathbf{C}^n$ and coordinates $w = (w_1, \cdots, w_g)$ for points of $\mathbf{C}^g$. Let $g_1(z), \cdots, g_p(z)$ be holomorphic functions on Ω and let X be the subvariety in $\Omega \times \mathbf{C}^g$ defined by $w_j g_k(z) = w_k g_j(z)$ for all $1 \leq j, k \leq p$. We can also consider the case of a submanifold X when excluding the common zero-set of $g_1, \cdots, g_n$ from Ω. Let $f(z)$ be a holomorphic n-form on Ω and consider the holomorphic $(n+1)$-form $\omega = f \wedge \frac{dw_j}{g_j(z)}$ on U_j, where U_j is the set of points of X, where $g_j(z)$ is nonzero. The $(n+1)$-form ω is well-defined, because

$$f \wedge \frac{dw_j}{g_j(z)} = f \wedge \frac{dw_k}{g_k(z)} \quad \text{on } U_j \cap U_k.$$

The problem is to extend ω to an $(n+1)$-form $\tilde{\omega}$ on all of $\Omega \times \mathbf{C}^g$. Write

$$\tilde{\omega} = \sum_{j=1}^{p} h_j(z) \wedge dw_j,$$

where $h_j(z)$ is a holomorphic n-form on Ω. The condition of extension is that the pullback of $\tilde{\omega}$ to X equals ω. On the open subset U_k of X where $g_k(z)$ is nonzero, the pullback of $\tilde{\omega}$ expressed in terms of $dz_1 \wedge \cdots \wedge dz_n \wedge dw_k$ becomes

$$\tilde{\omega} = \sum_{j=1}^{p} h_j(z) \wedge \frac{g_j}{g_k}\, dw_k,$$

because

$$dz_1 \wedge \cdots \wedge dz_n \wedge dw_j = dz_1 \wedge \cdots \wedge dz_n \wedge \frac{g_j}{g_k}\, dw_k.$$

The equation $\tilde{\omega} = \omega$ after pulling back to U_k means that

$$f \wedge \frac{dw_k}{g_k(z)} = \sum_{j=1}^{p} h_j(z) \wedge \frac{g_j}{g_k}\, dw_k,$$

which means that $f = \sum_{j=1}^{p} h_j g_j$. The question is what the natural conditions are for such an extension of $(n+1)$-form with L^2 estimates. It is not clear what kind of L^2 estimates to put in besides those motivated by Skoda's result.

We can also projectivize in the vertical direction and consider $\Omega \times \mathbf{P}_{g-1}$ with coordinates

$$((z_1, \cdots, z_n), [w_1, \cdots, w_g]).$$

The subvariety or submanifold Y of $\Omega \times \mathbf{P}_{g-1}$ is now defined by $w_j g_k(z) = w_k g_j(z)$ for $1 \leq j, k \leq g$.

Skoda's Integrability Criterion for Plurisubharmonic Weights

Notations.

$$|z|^2 = \sum_{\nu=1}^{n} |z_\nu|^2.$$

$$\alpha = \sqrt{-1}\,\partial\bar\partial \log |z|^2.$$

$$\beta = \sqrt{-1}\,\partial\bar\partial |z|^2 = \sqrt{-1}\,\sum_{\nu=1}^{n} dz_\nu \wedge d\overline{z_\nu}.$$

$$\gamma = \sqrt{-1}\,\partial|z|^2 \wedge \bar\partial|z|^2 = \sqrt{-1}\left(\sum_{\mu=1}^{n} \overline{z_\mu} dz_\mu\right) \wedge \left(\sum_{\nu=1}^{n} z_\nu d\overline{z_\nu}\right).$$

$$\alpha = \frac{\beta}{|z|^2} - \frac{\gamma}{|z|^4} \text{ by direct computation.}$$

$$\alpha^p = \frac{\beta^p}{|z|^{2p}} - p\frac{\beta^{p-1} \wedge \gamma}{|z|^{2p+2}} \text{ because } \gamma^\nu = 0 \text{ for } \nu > 0.$$

$$\alpha^p \wedge \gamma = \frac{\beta^p \wedge \gamma}{|z|^{2p}}.$$

For a positive current θ of type $(n-p, n-p)$ let

$$\sigma = \frac{1}{2^p p!}\,\theta \wedge \beta^p \text{ (Euclidean trace density)},$$

$$\nu = \frac{1}{(2\pi)^p}\theta \wedge \alpha^p \text{ (projective trace density)},$$

$$\sigma(r) = \int_{|z|<r} d\sigma(z) \text{ (total Euclidean trace measure in a ball of radius } r).$$

Here we use the same notation σ to denote two different kinds of objects. When σ is used by itself, we mean a measure on an open subset of $\mathbf{C}^n$. When $\sigma(r)$ is used, we mean a scalar function $\sigma(r)$ of a single real variable r. Let

$$\nu(r) = \frac{1}{\frac{\pi^p}{p!}\,r^{2p}}\sigma(r).$$

The Lelong number $\nu(0)$ of θ at the origin 0 is defined as

$$\nu(0) = \lim_{r\to 0^+} \nu(r) = \lim_{r\to 0^+} \frac{1}{\frac{\pi^p}{p!}\,r^{2p}}\sigma(r).$$

We would like to remark on the choice of the normalizing constant $\frac{1}{\frac{\pi^p}{p!}r^{2p}}$. The denominator $\frac{\pi^p}{p!}r^{2p}$ is the volume of the Euclidean ball of complex dimension p and radius r. The Lelong number is motivated by multiplicity of a subvariety of complex dimension p. When one measures the volume of the subvariety in a ball of radius r, the multiplicity measures the ratio of this volume to the volume of the Euclidean ball of complex dimension p and radius r as $r \to 0^+$.

The existence of the Lelong number comes from the following argument.

$$\int_{B(r)-B(s)} \theta \wedge \left(\partial\bar\partial \log|z|^2\right)^{n-p} = \int_{\partial B(r)-\partial B(s)} \theta \wedge \left(\bar\partial \log|z|^2\right) \wedge \left(\partial\bar\partial \log|z|^2\right)^{n-p-1}.$$

From

$$\bar\partial \log|z|^2 = \frac{\sum_j z_j d\overline{z_j}}{|z|^2}$$

it follows that

$$\partial\bar\partial \log|z|^2 = \frac{\sum_j dz_j \wedge d\overline{z_j}}{|z|^2} - \frac{\left(\sum_j \overline{z_j} dz_j\right) \wedge \left(\sum_j dz_j d\overline{z_j}\right)}{|z|^4}.$$

On $\partial B(r)$, $\sum_j |z_j|^2 = r^2$ and $d\sum_j |z_j|^2 = 0$ or $\sum_j \overline{z_j} dz_j + \sum_j dz_j \overline{z_j} = 0$. So $\left(\sum_j \overline{z_j} dz_j\right) \wedge \left(\sum_j dz_j \overline{z_j}\right) = 0$ and

$$\partial\bar\partial \log|z|^2 = \frac{\sum_j dz_j \wedge d\overline{z_j}}{|z|^2}$$

on $\partial B(r)$. Hence

$$\left(\bar\partial \log|z|^2\right) \wedge \left(\partial\bar\partial \log|z|^2\right)^{n-p-1} = \frac{\left(\bar\partial|z|^2\right) \wedge \left(\partial\bar\partial|z|^2\right)^{n-p-1}}{r^{2(n-p)}}$$

on $\partial B(r)$. So

$$\begin{aligned}
&\int_{\partial B(r)} \theta \wedge \left(\bar\partial \log|z|^2\right) \wedge \left(\partial\bar\partial \log|z|^2\right)^{n-p-1} \\
&= \frac{1}{r^{2(n-p)}} \int_{\partial B(r)} \theta \wedge \left(\bar\partial|z|^2\right) \wedge \left(\partial\bar\partial|z|^2\right)^{n-p-1} \\
&= \frac{1}{r^{2(n-p)}} \int_{B(r)} \theta \wedge \left(\partial\bar\partial|z|^2\right)^{n-p}
\end{aligned}$$

and

$$\int_{B(r)-B(s)} \theta \wedge \left(\partial\bar{\partial}\log|z|^2\right)^{n-p}$$

$$= \frac{1}{r^{2(n-p)}} \int_{B(r)} \theta \wedge \left(\partial\bar{\partial}|z|^2\right)^{n-p} - \frac{1}{s^{2(n-p)}} \int_{B(s)} \theta \wedge \left(\partial\bar{\partial}|z|^2\right)^{n-p}.$$

It implies that

$$r \mapsto \frac{1}{r^{2(n-p)}} \int_{B(r)} \theta \wedge \left(\partial\bar{\partial}|z|^2\right)^{n-p}$$

is a non-decreasing function of r. As a result the Lelong number

$$\nu(0) = \lim_{r \to 0^+} \frac{1}{\frac{\pi^p}{p!} r^{2p}} \sigma(r)$$

exits.

Consider

$$U_1(z) = -\frac{p!}{\pi^p} \int_{x \in \mathbf{C}^n} \frac{1}{|z-x|^{2p}} \eta(x) d\sigma(x),$$

where $0 \leq \eta \leq 1$ is a C^∞ cut-off function. The reason to consider the function is the following. In the case of $p = n-1$, we are interested in relating the integrability of $e^{-\varphi}$ to the Lelong number of $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. The Lelong number is defined from the trace of $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. So it is a matter of relating the integrability of $e^{-\varphi}$ to the Laplacian of $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. The kernel of the Laplacian is a positive constant times $\frac{-1}{|z-x|^{2(n-1)}}$ when $n \neq 1$.

Proposition (Skoda). Let Ω be the complement of the support of $1-\eta$ and let $c > 0$. Then $\exp\left(-\frac{n}{pc}U_1\right)$ is not integrable in a neighborhood of $z \in \Omega$ with $\nu(z) \geq c$ and is integrable in a neighborhood of $z \in \Omega$ with $\nu(z) < \left(1 - \frac{p}{n}\right) c$.

Remark. In the case of $p = n-1$ which is the most important case, $1 - \frac{p}{n} = \frac{1}{n}$. The condition for non-integrability $\nu(z) \geq c$ and the condition for integrability $\nu(z) < \frac{c}{n}$ differ by a factor of n.

Proof. To consider the situation near the origin, we can just drop the cut-off function η and consider

$$U_2(z) = -\frac{p!}{\pi^p} \int_{|x| \leq R} \frac{1}{|z-x|^{2p}} d\sigma(x).$$

We handle first the case of non-integrability. For that case we want to get a lower bound for $-U_2(z)$. To be generous, we replace $\frac{1}{|z-x|^{2p}}$ by $\frac{1}{(|z|+|x|)^{2p}}$ and consider

$$\frac{p!}{\pi^p}\int_{t=0}^{t=R}\frac{1}{(|z|+t)^{2p}}d\sigma(t)$$

which we get by integrating over the angular directions of x and end up with an integral over the absolute value $t=|x|$ of x. We have to link it up with the Lelong number which involves the $\sigma(r)$ instead of $d\sigma(x)$. So we use an integration by parts to convert $d\sigma(x)$ to $\sigma(r)$.

$$\int_{t=0}^{t=R}\frac{1}{(|z|+t)^{2p}}d\sigma(x)=\frac{\sigma(R)}{(|z|+R)^{2p}}+2p\int_{t=0}^{t=R}\frac{\sigma(t)dt}{(|z|+t)^{2p+1}}.$$

Since we are looking for a lower bound, to be generous we can drop the first term on the right-hand side and consider only

$$\int_{t=0}^{t=R}\frac{\sigma(t)dt}{(|z|+t)^{2p+1}}.$$

Again we have to relate it the Lelong number which is defined as the limit of $\frac{\sigma(r)}{r^{2p}}$ as $r\longrightarrow 0^+$ after normalization by a constant. So we write

$$\begin{aligned}\frac{1}{(|z|+t)^{2p+1}}&=\frac{1}{t^{2p}\left(|z|+t\right)}-\left(\frac{1}{t^{2p}}-\frac{1}{(|z|+t)^{2p}}\right)\frac{1}{|z|+t}\\&=\frac{1}{t^{2p}\left(|z|+t\right)}-\left(\frac{(|z|+t)^{2p}-t^{2p}}{t^{2p}\left(|z|+t\right)^{2p+1}}\right)\end{aligned}$$

which is no less than

$$\frac{1}{t^{2p}\left(|z|+t\right)}-\left(\frac{2p|z|}{t^{2p}\left(|z|+t\right)^2}\right),$$

because

$$(|z|+t)^{2p}-t^{2p}\leq 2p|z|\left(|z|+t\right)^{2p-1}$$

from

$$a^m-b^m=(a-b)\sum_{j=0}^{m-1}a^jb^{m-1-j}.$$

Thus

$$\int_{t=0}^{t=R}\frac{1}{(|z|+t)^{2p}}d\sigma(x)\geq 2p\int_{t=0}^{t=R}\frac{\sigma(t)dt}{t^{2p}\left(|z|+t\right)}-4p^2|z|\int_{t=0}^{t=R}\frac{\sigma(t)dt}{t^{2p}\left(|z|+t\right)^2}.$$

We now use the fact that $\frac{\sigma(t)}{t^{2p}}$ is a non-decreasing function of r and get

$$\frac{p!}{\pi^p}\frac{\sigma(R)}{R^{2p}} \geq \frac{p!}{\pi^p}\frac{\sigma(t)}{t^{2p}} \geq \nu(0) \geq c.$$

Hence

$$\begin{aligned}&\frac{p!}{\pi^p}\int_{t=0}^{t=R}\frac{d\sigma(t)}{(|z|+t)^{2p}}\\ \geq 2pc\int_{t=0}^{t=R}\frac{dt}{|z|+t} &- 4p^2\frac{p!}{\pi^p}\frac{\sigma(R)}{R^{2p}}|z|\int_{t=0}^{t=R}\frac{dt}{(|z|+t)^2}\\ &\geq 2pc\log\left(1+\frac{R}{|z|}\right) - 4p^2\frac{p!}{\pi^p}\frac{\sigma(R)}{R^{2p}}\end{aligned}$$

and

$$-U_2(z) \geq \frac{p!}{\pi^p}\int_{|x|\leq R}\frac{d\sigma(x)}{(|z|+|x|)^{2p}} \geq 2pc\log\left(1+\frac{R}{|z|}\right) - 4p^2\frac{p!}{\pi^p}\frac{\sigma(R)}{R^{2p}.}$$

Since

$$\exp\left(-\frac{n}{pc}U_2(z)\right) \geq C(n,p,c,R)\left(1+\frac{R}{|z|}\right)^{2n}$$

for some positive constant $C(n,p,c,R)$, it follows that $\exp\left(-\frac{n}{pc}U_2(z)\right)$ is not integrable at z when $\nu(z) \geq c$.

We now look at the other half of integrability. It suffices to consider the integrability at the origin. The main idea of its proof is to transform the kernel to integration along a complex line through the point in question and then average over all such complex lines. This means the transformation from the use of the Euclidean Kähler form β to the projective Kähler form α. After the transformation the singularity of the kernel on a complex line is of logarithmic order. We can then take the exponential and use the convexity of the exponential function. The pole order is now related to the Lelong number.

The relation between the Euclidean Kähler form β and the projective Kähler form α is

$$\frac{\beta^p}{|z-x|^{2p}} = \alpha^p(z-x) + pi\partial\log|z-x|^2 \wedge \bar{\partial}\log|z-x|^2 \wedge \alpha^{p-1}(z-x).$$

For our purpose of verifying integrability we can ignore terms which are bounded near the point in the question. In particular, we can use cut-off functions and ignore boundary integrals. We are going to do such things without explicit mention. We replace θ by $\theta' = \eta\theta$ for some cut-off function η. Consider

$$U(z) = -\int_{x\in\mathbf{C}^n} |z-x|^{2p}\beta^p \wedge \theta'$$

$$= -\int_{x\in\mathbf{C}^n} \alpha^p(z-x)\wedge\theta' - p\int_{x\in\mathbf{C}^n} i\partial\log|z-x|^2\wedge\bar\partial\log|z-x|^2\wedge\alpha^{p-1}(z-x)\wedge\theta'.$$

When we do integration by parts or apply Stokes's theorem, we have to worry about the singularities involved. When θ is smooth, as one can verify, the orders of the singularities are so mild that they do not matter. What happens in the general case of a closed positive current θ (which is the case we are interested)? Our goal for all the manipulation is to get integrability which means that we are interested in bounding the integral of the exponential of some function involving θ. We can derive our expressions first for the smooth case and then use Fatou's lemma to pass to limit in the final step. The main point is to make sure that the final bound is meaningful when θ is allowed to be a general closed positive current.

The term

$$\int_{x\in\mathbf{C}^n} \alpha^p(z-x)\wedge\theta'$$

we can ignore, because $\alpha(z-x) = i\partial\bar\partial\log|z-x|^2$ is exact and we can apply Stokes's theorem and end up only with some boundary term which is bounded. For the term

$$-p\int_{x\in\mathbf{C}^n} i\partial\log|z-x|^2\wedge\bar\partial\log|z-x|^2\wedge\alpha^{p-1}(z-x)\wedge\theta'$$

the part that matters after applying Stokes's theorem is

$$p\int_{x\in\mathbf{C}^n}\log|z-x|^2\, i\partial\bar\partial\log|z-x|^2\wedge\alpha^{p-1}(z-x)\wedge\theta'$$
$$= p\int_{x\in\mathbf{C}^n}\log|z-x|^2\alpha^p(z-x)\wedge\theta'.$$

After we renormalize, we need only consider

$$U_4(z) = \frac{1}{(2\pi)^p}\,p\int_{|x|\le R}\log|z-x|^2\alpha^p(z-x)\wedge\theta(x).$$

This is the kernel of logarithmic pole order on each complex line through z when $p = n - 1$ with averaging over all complex such lines. To relate this to the Lelong number and to prepare for the application of the convexity of the exponential function, we introduce

$$\mu(z) = \int_{|x| \leq R} \alpha^p(z - x) \wedge \theta(x),$$
$$\kappa(z) = \frac{1}{(2\pi)^p} \, \mu(z) \, \frac{n}{c}.$$

Then

$$-\frac{n}{pc} U_4(z) = \int_{|x| \leq R} \log \left(|z - x|^{-2\kappa(z)} \right) \, \frac{\alpha^p(z - x) \wedge \theta(x)}{\mu(z)}.$$

Since

$$\int_{|x| \leq R} \frac{\alpha^p(z - x) \wedge \theta(x)}{\mu(z)} = 1,$$

it follows from the convexity of the exponential function that

$$\exp \left(-\frac{n}{pc} U_4(z) \right) \leq \int_{|x| \leq R} \frac{1}{|z - x|^{2\kappa(z)}} \, \frac{\alpha^p(z - x) \wedge \theta(x)}{\mu(z)}$$

and

$$\int_{|z| \leq r} \exp \left(-\frac{n}{pc} U_4(z) \right) \leq \int_{|z| \leq r, \, |x| \leq R} \frac{1}{|z - x|^{2\kappa(z)}} \, \frac{\alpha^p(z - x) \wedge \theta(x)}{\mu(z)}.$$

We recall that, in deriving the estimates, we assume that θ is smooth and only after we get the estimates we use Fatou's lemma to pass to limit to get the general case of a closed positive current.

To finish the proof of the integrability statement, we need a lower bound for $\mu(z)$ and an upper bound for $\kappa(z)$ (which is the same as an upper bound for $\mu(z)$) for $|z| \leq r$ when $r > 0$ is sufficiently small. For the lower bound of $\mu(z)$, from the definition

$$\mu(z) = \int_{|x| \leq R} \alpha^p(z - x) \wedge \theta(x)$$

of $\mu(z)$ it follows that, for $|z| \leq r$,

$$\frac{1}{(2\pi)^p} \mu(z) \geq \frac{1}{(2\pi)^p} \int_{|z - x| \leq R - r} \alpha^p(x) \wedge \theta(x)$$

$$
\begin{aligned}
&= \frac{p!}{\pi^p}\frac{1}{(R-r)^p}\int_{|z-x|\le R-r} d\sigma(x) \\
&\ge \frac{p!}{\pi^p}\frac{1}{(R-r)^p}\int_{|x|\le R-2r} d\sigma(x) \\
&\ge \left(\frac{R-2r}{R-r}\right)^{2p}\nu(R-2r),
\end{aligned}
$$

where for the last inequality the non-decreasing property of the integral which defines the Lelong number is used.

The derivation of an upper bound for $\mu(z)$ is analogous. From the definition

$$
\mu(z) = \int_{|x|\le R} \alpha^p(z-x)\wedge\theta(x)
$$

of $\mu(z)$ it follows that, for $|z|\le r$,

$$
\begin{aligned}
\frac{1}{(2\pi)^p}\mu(z) &\ge \frac{1}{(2\pi)^p}\int_{|z-x|\le R+r} \alpha^p(x)\wedge\theta(x) \\
&= \frac{p!}{\pi^p}\frac{1}{(R+r)^p}\int_{|z-x|\le R+r} d\sigma(x) \\
&\ge \frac{p!}{\pi^p}\frac{1}{(R+r)^p}\int_{|x|\le R+2r} d\sigma(x) \\
&\ge \left(\frac{R+2r}{R+r}\right)^{2p}\nu(R+2r),
\end{aligned}
$$

where, again, for the last inequality the non-decreasing property of the integral which defines the Lelong number is used.

Since

$$
\mu(r) \to \nu(0) < c\left(1-\frac{p}{n}\right)
$$

as $r \to 0^+$, it follows that we can choose $\epsilon > 0$, R, and $\frac{r}{R}$ so small that

$$
\kappa(z) = \frac{1}{(2\pi)^p}\mu(z)\frac{n}{c} < \frac{n}{c}\,c\left(1-\frac{p}{n}-\frac{\epsilon}{n}\right) = n-p-\epsilon.
$$

Thus

$$
\int_{|z|\le r} \exp\left(-\frac{n}{pc}U_4(z)\right) \le \left(\frac{R-r}{R-2r}\right)^{2p}\frac{(2\pi)^p}{\nu(R-2r)}\int_{|z|\le r,\,|x|\le R}\frac{\alpha^p(z-x)\wedge\theta(x)}{|z-x|^{2(n-p-\epsilon)}}.
$$

To make sure that $\nu(R-2r)>0$ for $R-2r>0$, we could add at the beginning the p-th power of a strictly positive C^∞ $(1,1)$-form, for example, β^p. Since the coefficients of $\alpha^p(z-x)$ are dominated by

$$\frac{C(n,p)}{|z-x|^{2p}},$$

it follows that

$$\int_{|z|\leq r}\exp\left(-\frac{n}{pc}U_4(z)\right)\leq\left(\frac{R-r}{R-2r}\right)^{2p}\frac{C(n,p)(2\pi)^p}{\nu(R-2r)}\int_{|z|\leq r,\,|x|\leq R}\frac{d\sigma(x)dz}{|z-x|^{2(n-\epsilon)}}.$$

We can now integrate the right-hand side with respect to z first. From

$$\sup_{|x|\leq R}\int_{|z|\leq r}\frac{dz}{|z-x|^{2(n-\epsilon)}}<\infty$$

it follows that

$$\int_{|z|\leq r}\exp\left(-\frac{n}{pc}U_4(z)\right)<\infty.$$

Q.E.D.

Normalization Constants

We now discuss the normalization constants used above. What we are most concerned about is the verification that if $\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi=\theta$ for some closed positive $(1,1)$-current θ on some open subset of $\mathbf{C}^n$, then $e^{-\varphi}$ is locally integrable if the Lelong number of ω is less than 1 and $e^{-\varphi}$ is locally non-integrable if the Lelong number of θ is at least n.

First we check the constant for the fundamental solution of the Laplacian. On $\mathbf{R}^m$ (with $m>2$) let r be the distance from the origin and let $B(s)$ be the ball in $\mathbf{R}^m$ centered at the origin with radius s. We would like to determine the constant a_m such that $a_m\Delta\frac{1}{r^{m-2}}$ is the Dirac delta at the origin. The condition is that

$$\int_{B(\eta)}a_m\Delta\frac{1}{r^{m-2}}=1$$

for $\eta>0$ when $\Delta\frac{1}{r^{m-2}}$ is taken in the sense of distributions. In the sense of distributions

$$\int_{B(\eta)}\Delta\frac{1}{r^{m-2}}=\int_{\partial B(\eta)}\frac{\partial}{\partial r}\frac{1}{r^{m-2}}$$

which is equal to the volume of $\partial B(1)$ times $-(m-2)$. Hence $a_m = \frac{-1}{(m-2)\mathrm{Vol}(\partial B(1))}$. To compute the volume of $\partial B(1)$, we first compute the volume of $B(1)$.

Computation of Volume of Unit Complex n-Ball. From the intersection of the hyperplane section in $\mathbf{P}_{n-1}$ with itself $n-1$ it follows that

$$\int_{\mathbf{P}_{n-1}} \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^2\right)^{n-1} = 1.$$

From Fubini's theorem it follows that

$$\int_{B^n(1)} \left(\frac{\sqrt{-1}}{2}\partial\bar{\partial}|z|^2\right)\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^2\right)^{n-1} = \int_{\Delta(1)} \frac{\sqrt{-1}}{2}\partial\bar{\partial}|z|^2 = \pi.$$

Applying Stokes's theorem to the left-hand side we get

$$\begin{aligned}
&\int_{B^n(1)} \left(\frac{\sqrt{-1}}{2}\partial\bar{\partial}|z|^2\right)\left(\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log|z|^2\right)^{n-1} \\
&= \int_{\partial B^n(1)} \left(\frac{\sqrt{-1}}{2}\bar{\partial}|z|^2\right)\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^2\right)^{n-1} \\
&= \int_{\partial B^n(1)} \left(\frac{\sqrt{-1}}{2}\bar{\partial}|z|^2\right)\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}|z|^2\right)^{n-1} \\
&= \frac{1}{\pi^{n-1}}\int_{B^n(1)} \left(\frac{\sqrt{-1}}{2}\partial\bar{\partial}|z|^2\right)^{n} = \frac{n!}{\pi^{n-1}}\mathrm{Vol}\left(B^n(1)\right).
\end{aligned}$$

Hence

$$\mathrm{Vol}\left(B^n(1)\right) = \frac{\pi^n}{n!}.$$

We now return to the computation of a_m. We are only interested in the case $m = 2n$. According to our computation above, the volume of the Euclidean ball of complex dimension n and radius r is $\frac{\pi^n}{n!}r^{2n}$. Thus

$$\begin{aligned}
\frac{\pi^n}{n!}r^{2n} &= \int_{s=0}^{r} \mathrm{Vol}(\partial B(s))ds \\
&= \int_{s=0}^{r} \mathrm{Vol}(\partial B(1))s^{2n-1}ds = \mathrm{Vol}(\partial B(1))\frac{r^{2n}}{2n}
\end{aligned}$$

and

$$\mathrm{Vol}(\partial B(1)) = 2\pi \, \frac{\pi^{n-1}}{(n-1)!}.$$

When $m = 2n$, we have

$$a_{2n} = \frac{-1}{4\pi(n-1)\, \frac{\pi^{n-1}}{(n-1)!}}$$

and

$$\Delta \left(\frac{-1}{4\pi(n-1)\, \frac{\pi^{n-1}}{(n-1)!}} \, \frac{1}{|z|^{2n-2}} \right) = \text{Dirac delta at } 0 \text{ on } \mathbf{C}^n.$$

Hence for any function Φ we have

$$\Delta \left(\left(\frac{-1}{4\pi(n-1)\, \frac{\pi^{n-1}}{(n-1)!}} \, \frac{1}{|z|^{2n-2}} \right) * \Phi \right) = \Phi.$$

Suppose we have a closed positive $(1,1)$-current θ and $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi = \theta$. We take the wedge product with

$$\frac{1}{2^{n-1}(n-1)!}\beta^{n-1} = \frac{1}{2^{n-1}(n-1)!} \left(i\partial\bar{\partial}|z|^2 \right)^{n-1}$$

and get

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \wedge \frac{1}{2^{n-1}(n-1)!}\beta^{n-1} = \theta \wedge \frac{1}{2^{n-1}(n-1)!}\beta^{n-1} = \sigma.$$

It means that

$$\frac{1}{4\pi}\,(\Delta\phi)\,\text{Euclidean volume form} = \sigma$$

and, so far as the normalization constants are concerned,

$$\phi = \left(\frac{-1}{(n-1)\, \frac{\pi^{n-1}}{(n-1)!}} \, \frac{1}{|z|^{2n-2}} \right) * \sigma.$$

Since

$$U_1(z) = -\frac{p!}{\pi^p} \int_{x \in \mathbf{C}^n} \frac{1}{|z-x|^{2p}} \eta(x) d\sigma(x),$$

so far as the normalization constants are concerned when $p = n-1$,

$$\varphi = \frac{1}{n-1} U_1$$

and the non-integrability of $e^{-\frac{n}{pc}U_1}$ is the same as the non-integrability of $e^{-\frac{n}{c}\varphi}$ when $p = n-1$ and the Lelong number of θ is at least c.

Use of a Weight for the Norm Different from That to Define the Adjoint.

We now derive the formula for the basic estimates in which the weight for the norm is different from the weight used to define the adjoint of $\bar{\partial}$.

We introduce two weights $e^{-\psi}$ and $e^{-\chi}$. We first write down the usual formula for the weight $e^{-(\psi+\chi)}$, which states the following.

Let u be in the domain of $\overline{\partial}^*$ on Ω. We have

$$\int_\Omega \left\langle \overline{\partial}^*_{\psi+\chi} u, \overline{\partial}^*_{psi+\chi} u \right\rangle e^{-(\psi+\chi)} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u \right\rangle e^{-(\psi+\chi)}$$
$$= \int_{\partial\Omega} \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}} \partial_\alpha \rho) e^{-(\psi+\chi)} + \int_\Omega \left\langle \overline{\nabla} u, \overline{\nabla} u \right\rangle e^{-(\psi+\chi)}$$
$$+ \int_\Omega \overline{u_{\overline{\beta}}} u_{\bar{\alpha}} \left(\partial_{\overline{\beta}} \partial_\alpha (\psi+\chi) \right) e^{-(\psi+\chi)}.$$

The relation between the adjoints of $\bar{\partial}$ for different weights is as follows.

$$\bar{\partial}^*_{\psi+\chi} u = -e^{\psi+\chi} \partial_\alpha \left(e^{-(\psi+\chi)} u_{\bar{\alpha}} \right)$$
$$= -e^{\psi+\chi} \left(-(\partial \chi_\alpha) e^{-\chi} \left(e^{-\psi} u_{\bar{\alpha}} \right) + e^{-\chi} \bar{\partial}^*_\psi u \right) = (\partial_\alpha \chi) u_{\bar{\alpha}} + \bar{\partial}^*_\psi u.$$

Let $\eta = e^{-\chi}$. Then $\partial \chi = -\frac{\partial \eta}{\eta}$ and

$$\partial \bar{\partial} \chi - |\partial \chi|^2 = -\frac{\partial \bar{\partial} \eta}{\eta}.$$

Thus

$$\left| \bar{\partial}^*_{\psi+\chi} u \right|^2 e^{-(\psi+\chi)} = \left| (\partial \chi_\alpha) u_{\bar{\alpha}} + \bar{\partial}^*_\psi u \right|^2 e^{-(\psi+\chi)}$$
$$= |(\partial \chi_\alpha) u_{\bar{\alpha}}|^2 e^{-(\psi+\chi)} + 2 \operatorname{Re} \left((\partial \chi_\alpha) u_{\bar{\alpha}} \overline{\left(\bar{\partial}^*_\psi u \right)} \right) e^{-(\psi+\chi)} + \left| \bar{\partial}^*_\psi u \right|^2 e^{-(\psi+\chi)}$$
$$= \left(\partial_\alpha \partial_{\bar{\beta}} \chi \right) u_{\bar{\alpha}} \overline{u_{\overline{\beta}}} e^{-(\psi+\chi)} + \left(\partial_\alpha \partial_{\bar{\beta}} \eta \right) u_{\bar{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi}$$
$$+2 \operatorname{Re} \left((\partial \eta_\alpha) u_{\bar{\alpha}} \overline{\left(\bar{\partial}^*_\psi u \right)} \right) e^{-\psi} + \left| \bar{\partial}^*_\psi u \right|^2 e^{-(\psi+\chi)}.$$

Hence

$$\int_\Omega < \eta \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u > e^{-\psi} + \int_\Omega < \eta \overline{\partial} u, \overline{\partial} u > e^{-\psi}$$
$$= \int_{\partial\Omega} \eta \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}} \partial_\alpha \rho) e^{-\psi} + \int_\Omega < \eta \overline{\nabla} u, \overline{\nabla} u > e^{-\psi}$$
$$+ \int_\Omega \eta \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}} \partial_\alpha \psi) e^{-\psi} - \int_\Omega (\partial_{\overline{\beta}} \partial_\alpha \eta) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} e^{-\psi} + 2 \operatorname{Re} \int_\Omega (\partial_\alpha \eta) u_{\overline{\alpha}} \overline{(\overline{\partial}^*_\psi u)} e^{-\psi}.$$

We now apply the formula to the following special case. We have a compact complex projective algebraic manifold X and an ample line bundle L over X with smooth Hermitian metric $h = e^{-\varphi}$ with positive curvature. We have a holomorphic section s of L whose divisor is Y and is nonsingular. Assume that the pointwise norm of s_Y is less than 1 at every point of X. We are going to essentially let

$$\eta = \log \frac{1}{|s|^2 e^{-\varphi}} > 0.$$

We have to modify this a little bit, otherwise

$$\partial\bar{\partial}\eta = \partial\bar{\partial}\varphi - [Y],$$

where $[Y]$ is the current defined by integration over Y.

References.

[AS94] Angehrn, U., Siu, Y.-T.: Effective freeness and separation of points for adjoint bundles. Invent. math., to appear.

[D1] J.-P. Demailly, Champs magnétiques et inégalités de Morse pour la d" cohomologie, Compte-Rendus Acad. Sci, Série I, 301 (1985), 119-122 and Ann. Inst. Fourier 35,4 (1985), 189-229.

[D2] J.-P. Demailly, A numerical criterion for very ampleness, J. Diff. Geom 37 (1993), 323-374.

[EL93] Ein, L. and Lazarsfeld, R.: Global generation of pluricanonical and adjoint linear series on smooth projective threefolds, J. of the A.M.S., 6, 875-903 (1993).

[ELN] Ein, L., Lazarsfeld, R., Nakamaye, M.: private notes and Nakamaye's lecture in the Algebraic Geometry Seminar at Harvard, 1994.

[F87] Fujita, T.: On polarized manifolds whose adjoint bundles are not semipositive. In: Algebraic Geometry, Sendai, Advanced Studies in Pure Math., 10, 167-178 (1987).

[H] Hirzebruch, F : Topological Methods in Algebraic Geometry. Berlin Heidelberg New York: Springer 1966.

[I69-71] S. Iitaka, Deformations of compact complex surfaces I, II, and III. In: *Global Analysis*, papers in honor of K. Kodaira, Princeton University Press 1969, pp.267-272; J. Math. Soc. Japan 22 (1970), 247-261; *ibid* 23 (1971), 692-705.

[Ka] Kawamata, Y : A generalization of the Kodaira-Ramanujam's vanishing theorem, Math. Ann. 261, 43-46 (1982).

[Kol] Kollár, J.: Effective base point freeness, Math. Ann. 296, 595-605 (1993).

[KM] J. Kollar and T. Matsusaka, Riemann-Roch type inequalities, Amer. J. Math. 105 (1983), 229-252.

[K94] Kollár, J.: private e-mail communication, October, 1994.

[KM92] Kollar, J., Mori, S.: Classification of three dimensional flips, Journal of Amer. Math. Soc. 5 (1992), 533-702.

[L] P. Lelong, Plurisubharmonic functions and positive differential forms, Gordon and Breach, New York, 1969.

[L83] Levine, M.: Pluri-canonical divisors on Kähler manifolds, Invent. Math. 74 (1983), 293-903.

[L85] Levine, M.: Pluri-canonical divisors on Kähler manifolds II, Duke Math. J. 52 (1985), 61-65.

[LM] D. Lieberman and D. Mumford, Matsusaka's big theorem, (Algebraic Geometry, Arcata 1974) Proceedings of Symposia in Pure Math. 29 (1975), 513-530.
[M93] Manivel, L.: Un thórème de prolongement L^2 de sections holomorphes d'un fibré hermitien. Math. Zeitschr. 212, 107-122 (1993).
[M1] T. Matsusaka, On canonically polarized varieties II, Amer. J. Math. 92 (1970), 283-292.
[M2] T. Matsusaka, Polarized varieties with a given Hilbert polynomial, Amer. J. Math. 94 (1972), 1027-1077.
[Nad89] Nadel, A.: Multiplier ideal sheaves and the existence of Kähler-Einstein metrics of positive scalar curvature, Proc. Natl. Acad. Sci. USA, 86, 7299-7300 (1989), and Ann. of Math., 132, 549-596 (1989).
[Nak86] Nakayama, N.: Invariance of plurigenera of algebraic varieties under minimal model conjecture, Topology 25 (1986), 237-251.
[OT87] Ohsawa, T., Takegoshi, K.: On the extension of L^2 holomorphic functions, Math. Z., 195, 197-204 (1987).
[R88] Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math., 127, 309-316 (1988).
[R] Reider, I: Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math. 127, 309-316 (1988).
[S] Y.-T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, Invent. Math. 27 (1974), 53-156.
[S] Siu, Y.-T.: An effective Matsusaka big theorem, Ann. Inst. Fourier 43, 1199-1209 (1993).
[Si96] Siu, Y.-T., The Fujita conjecture and the extension theorem of Ohsawa-Takegoshi, in *Geometric Complex Analysis* ed. Junjiro Noguchi *et al*, World Scientific: Singapore, New Jersey, London, Hong Kong 1996, pp. 577-592.
[Sk72] H Skoda, Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids, Ann. Sci. Ec. Norm. Sup. 5 (1972), 548-580.
[V82] Viehweg, E.: Vanishing theorems, J. reine und angew. Math., 335, 1-8 (1982).

Extension Theorems of Ohsawa-Takegoshi Type from Usual Basic Estimates with Two Weight Functions

In this section we are going to state and derive the extension theorem of Ohsawa-Takegoshi type with estimates which we need for the effective version of the arguments of [Siu98]. Such extension theorems originated in a paper of Ohsawa-Takegoshi [OT87] and generalizations were made by Manivel [Man93] and a series of papers of Ohsawa [Ohs88], [Ohs94], [Ohs95], [Ohs01]. Ohsawa's series of papers [Ohs88], [Ohs94], [Ohs95], [Ohs01] contain more general results, which were proved from identities in Kähler geometry and specially constructed complete metrics. Here we use the simple approach of the usual basic estimates with two weight functions. We choose to derive here the extension theorem we need instead of just quoting from more general results, because the simple approach given here gives a clearer picture what and why additional techniques of solving the $\bar{\partial}$ equation other than the standard ones are required for the proof of the extension result. The derivation given here is essentially the same as the one given in [**?**] with the modifications needed for the present case of no strictly positive lower bound for the curvature current. The only modifications consist of the use of $|\langle u, dw\rangle|$ instead of $|u|$ in some inequalities between (**??**) and (**??**). The modification simply replaces the strictly positive lower bound of the curvature current in all directions by the strictly positive lower bound of the curvature just for the direction normal to the hypersuface from which the holomorphic section is extended. The precise statement which we need is the following.

Theorem(3.1) Let Y be a complex manifold of complex dimension n. Let w be a bounded holomorphic function on Y with nonsingular zero-set Z so that dw is nonzero at any point of Z. Let L be a holomorphic line bundle over Y with a (possibly singular) metric $e^{-\kappa}$ whose curvature current is semipositive. Assume that there exists a hypersurface V in Y such that $V \cap Z$ is a subvariety of codimension at least 1 in Z and $Y - V$ is the union of a sequence of Stein subdomains Ω_ν of smooth boundary and Ω_ν is relatively compact in $\Omega_{\nu+1}$. If f is an L-valued holomorphic $(n-1)$-form on Z with

$$\int_Z |f|^2 \, e^{-\kappa} < \infty \, ,$$

then $f dw$ can be extended to an L-valued holomorphic n-form F on Y such that

$$\int_Y |F|^2 \, e^{-\kappa} \leq 8\pi \, e \sqrt{2 + \frac{1}{e}} \left(\sup_Y |w|^2 \right) \int_Z |f|^2 \, e^{-\kappa} \, .$$

We will devote most of this section to the proof of Theorem(3.1). We will fix ν and solve the problem on Ω_ν, instead of on Y, and we will do it with the estimate on L^2 norms which is independent of ν and then we will take limit as $\nu \to \infty$. For notational simplicity, in the presentation of our argument, we will drop the index ν in Ω_ν and simply denote Ω_ν by Ω. After dividing w by the supremum of $|w|$ on Y, we can assume without loss of generality that the supremum norm of w on Y is no more than 1. Moreover, since $\Omega_{\nu+1}$ is Stein there exists a holomorphic L-valued holomorphic $(n-1)$-form $\tilde{f}$ on $\Omega_{\nu+1}$ such that $(f \wedge dw)\,|(\Omega_{\nu+1} \cap Z)$ is the restriction of $\tilde{f}$ to $\Omega_{\nu+1} \cap Z$. Of course, when such an extension $\tilde{f}$ is obtained simply by the Stein property of $\Omega_{\nu+1}$, we do not have any L^2 norm estimate on $\tilde{f}$ which is independent of ν.

(3.2) *Functional Analysis Preliminaries.* We recall the standard technique of using functional analysis and Hilbert spaces to solve the $\bar{\partial}$ equation. Consider an operator T which later will be an operator modified from $\bar{\partial}$. Let S be an operator such that $ST = 0$. The operator S later will be an operator modified from the $\bar{\partial}$ operator of the next step in the Dolbeault complex. Given g with $Sg = 0$ we would like to solve the equation $Tu = g$. The equation $Tu = g$ is equivalent to $(v, Tu) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$, which means $(T^*v, u) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. To get a solution u it suffices to prove that the map $T^*v \to (v, g)$ can be extended to a bounded linear functional, which means that there exists a positive constant C such that $|(v, g)| \leq C\|T^*v\|$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. In that case we can solve the equation $Tu = g$ with $\|u\| \leq C$. We could also use the equivalent inequality

$$|(v, g)|^2 \leq C^2 \left(\|T^*v\|^2 + \|Sv\|^2\right)$$

for all $v \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$.

(3.3) *Bochner-Kodaira Formula with Two Weights.* The crucial point of the argument is the use of two different weights. One weight is for the norm and the other is for the definition of the adjoint of $\bar{\partial}$. We now derive the formula for the Bochner-Kodaira formula in which the weight for the norm is different from the weight used to define the adjoint of $\bar{\partial}$. Formulas, of such a kind, for different weight functions were already given in the literature in the nineteen sixties by authors such as Hörmander (for example, [Hor66]). There is nothing particular new here, except that we need the statement in the form precisely stated below for our case at hand.

We start with a weight $e^{-\varphi}$ and use the usual Bochner-Kodaira formula for this particular weight. Let η be a positive-valued function and let $e^{-\psi} = \frac{e^{-\varphi}}{\eta}$. We will use the weight $e^{-\psi}$ for the definition of the adjoint of $\bar\partial$. We use $\bar\partial^*_\varphi$ (respectively $\bar\partial^*_\psi$) to denote the formal adjoint of $\bar\partial$ with respect to the weight function $e^{-\varphi}$ (respectively $e^{-\psi}$). We agree to use the summation convention that, when a lower-case Greek index appears twice in a term, once with a bar and once without a bar, we mean the contraction of the two indices by the Kähler metric tensor. An index without (respectively with) a bar inside the complex conjugation of a factor is counted as an index with (respectively without) a bar. We use $\langle \cdot, \cdot \rangle$ to denote the pointwise inner product. Let $\bar\nabla$ be the covariant differentiation in the $(0,1)$-direction. The formula we seek is the following.

Proposition (3.4) Let Ω be defined by $r < 0$ so that $|dr|$ with respect to the Kähler metric is identically 1 on the boundary $\partial\Omega$ of Ω. Let u be an $(n,1)$-form in the domain of the actual adjoint of $\overline{\partial}$ on Ω. Then

$$\begin{aligned}
&\int_\Omega \left\langle \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u \right\rangle e^{-\varphi} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u \right\rangle e^{-\varphi} \\
&\qquad = \int_{\partial\Omega} \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}} \partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \overline{\nabla} u, \overline{\nabla} u \right\rangle e^{-\varphi} \\
&\qquad\quad + \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}} \partial_\alpha \psi \right) e^{-\varphi} - \int_\Omega \left(\frac{\partial_{\bar\alpha} \partial_\beta \eta}{\eta} \right) u_\alpha \overline{u_\beta} e^{-\varphi} \\
&\qquad\quad + 2 \operatorname{Re} \int_\Omega \left(\frac{\partial_\alpha \eta}{\eta} \right) u_{\bar\alpha} \overline{\left(\bar\partial^*_\psi u \right)} e^{-\varphi}.
\end{aligned}$$

Proof. The usual Bochner-Kodaira formula for a domain with smooth boundary for the same weight (also known as the basic estimate) gives

$$\begin{aligned}
&\int_\Omega \left\langle \overline{\partial}^*_\varphi u, \overline{\partial}^*_\varphi u \right\rangle e^{-\varphi} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u \right\rangle e^{-\varphi} \\
&\qquad = \int_{\partial\Omega} \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}} \partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \overline{\nabla} u, \overline{\nabla} u \right\rangle e^{-\varphi} \\
&\qquad\quad + \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}} \partial_\alpha \varphi \right) e^{-\varphi} \,. \qquad (1)
\end{aligned}$$

The relation between the formal adjoints of $\bar\partial$ for different weights is as follows:

$$\bar\partial^*_\varphi u = -e^{\varphi} \partial_\alpha \left(e^{-\varphi} u_{\bar\alpha} \right) = -\frac{e^{\psi}}{\eta} \partial_\alpha \left(\eta \, e^{-\psi} u_{\bar\alpha} \right) = -\frac{\partial_\alpha \eta}{\eta} \, u_{\bar\alpha} + \bar\partial^*_\psi u \,.$$

Thus

$$\begin{aligned}\left|\bar\partial^*_\varphi u\right|^2 e^{-\varphi} &= \left|-\frac{\partial_\alpha\eta}{\eta}\, u_{\bar\alpha} + \bar\partial^*_\psi u\right|^2 e^{-\varphi}\\ &= \left|\frac{\partial_\alpha\eta}{\eta}\, u_{\bar\alpha}\right|^2 e^{-\varphi} - 2\,\mathrm{Re}\left(\frac{\partial_\alpha\eta}{\eta}\, u_{\bar\alpha}\, \overline{\left(\bar\partial^*_\psi u\right)}\right) e^{-\varphi} + \left|\bar\partial^*_\psi u\right|^2 e^{-\varphi}\ .\end{aligned}$$

We now rewrite (**??**) as

$$\begin{aligned}&\int_\Omega \left\langle \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u\right\rangle e^{-\varphi}\\ &\qquad = \int_{\partial\Omega} \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}}\partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \overline{\nabla} u, \overline{\nabla} u\right\rangle e^{-\varphi}\\ &\qquad\quad + \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}}\partial_\alpha\varphi\right) e^{-\varphi} - \int_\Omega \left|\frac{\partial_\alpha\eta}{\eta}\, u_{\bar\alpha}\right|^2 e^{-\varphi}\\ &\qquad\qquad + 2\,\mathrm{Re}\int_\Omega \frac{\partial_\alpha\eta}{\eta}\, u_{\bar\alpha}\, \overline{\left(\bar\partial^*_\psi u\right)} e^{-\varphi}\ .\end{aligned} \tag{2}$$

From $\varphi = \psi - \log\eta$ it follows that

$$\partial\bar\partial\varphi = \partial\bar\partial\psi - \frac{\partial\bar\partial\eta}{\eta} + \frac{\partial\eta\wedge\bar\partial\eta}{\eta^2}\ .$$

Hence we can rewrite (**??**) as

$$\begin{aligned}&\int_\Omega \left\langle \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u\right\rangle e^{-\varphi}\\ &\quad = \int_{\partial\Omega} \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}}\partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \overline{\nabla} u, \overline{\nabla} u\right\rangle e^{-\varphi}\\ &\qquad + \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}}\partial_\alpha\psi\right) e^{-\varphi} - \int_\Omega \left(\frac{\partial_{\bar\alpha}\partial_\beta\eta}{\eta}\right) u_\alpha \overline{u_\beta} e^{-\varphi}\\ &\qquad + 2\,\mathrm{Re}\int_\Omega \left(\frac{\partial_\alpha\eta}{\eta}\right) u_{\bar\alpha}\, \overline{\left(\bar\partial^*_\psi u\right)} e^{-\varphi}\ .\end{aligned}$$

(3.5) *Choice of Two Different Weights.* Since Ω is weakly pseudoconvex, the Levi form of r is semi-positive at every point of the boundary $\partial\Omega$ of Ω. The inequality in Proposition(3.4) becomes

$$\int_\Omega \left\langle \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u\right\rangle e^{-\varphi} \geq \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}}\partial_\alpha\psi\right) e^{-\varphi}$$

$$-\int_\Omega \left(\frac{\partial_{\bar\alpha}\partial_\beta \eta}{\eta}\right) u_\alpha \overline{u_\beta} e^{-\varphi} + \ 2\,\mathrm{Re}\int_\Omega \left(\frac{\partial_\alpha \eta}{\eta}\right) u_{\bar\alpha}\,\overline{\left(\bar\partial^*_\psi u\right)} e^{-\varphi} \ . \tag{3}$$

Take any positive number $A > e$, where e is the base of the natural logarithm. Let

$$\varepsilon_0 = \sqrt{\frac{A}{e} - 1}\ .$$

For any positive $\varepsilon < \varepsilon_0$, we let

$$\begin{aligned} \eta &= \log \frac{A}{|w|^2 + \varepsilon^2}\ , \\ \gamma &= \frac{1}{|w|^2 + \varepsilon^2}\ . \end{aligned}$$

Then $\eta > 1$ on Ω, because the supremum norm of w is no more than 1 on Ω.

$$\begin{aligned} -\partial_w \partial_{\bar w} \eta &= \frac{\varepsilon^2}{(|w|^2 + \varepsilon^2)^2}\ , \\ \partial_w \eta &= -\frac{\bar w}{|w|^2 + \varepsilon^2}\ , \\ \partial_{\bar w} \eta &= -\frac{w}{|w|^2 + \varepsilon^2}\ . \end{aligned}$$

We have the estimate

$$\begin{aligned} &\left| 2\mathrm{Re} \int_\Omega \frac{\partial_\alpha \eta}{\eta}\, u_{\bar\alpha} (\bar\partial^*_\psi u) e^{-\varphi} \right| \\ &\quad \le \ 2 \int_\Omega \frac{|w|}{|w|^2 + \varepsilon^2}\, |\langle u, dw\rangle|\, \left|\bar\partial^*_\psi u\right|\, e^{-\psi} \\ &\qquad \le \ \int_\Omega \frac{|w|^2}{|w|^2 + \varepsilon^2}\ |\langle u, dw\rangle|^2 e^{-\psi} + \int_\Omega \frac{1}{|w|^2 + \varepsilon^2} \left|\bar\partial^*_\psi u\right|^2 e^{-\psi} \\ &\qquad = \int_\Omega \frac{|w|^2}{|w|^2 + \varepsilon^2}\ |\langle u, dw\rangle|^2 e^{-\psi} + \int_\Omega \gamma\, |\bar\partial^*_\psi u|^2 e^{-\psi}\ . \end{aligned} \tag{4}$$

Choose $\psi = |w|^2 + \kappa$. Since

$$\eta(\partial_\alpha \partial_{\overline{\beta}} \psi) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}} \ge |\langle u, dw\rangle|^2 \ge \frac{|w|^2}{|w|^2 + \varepsilon^2}\, |\langle u, dw\rangle|^2\ ,$$

it follows that

$$\begin{aligned}
&\int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}}\partial_\alpha \psi\right) e^{-\varphi} + \int_\Omega \gamma\, |\overline{\partial}^*_\psi u|^2\, e^{-\psi} \\
&\qquad = \int_\Omega \eta (\partial_\alpha \partial_{\overline{\beta}}\psi) u_{\overline{\alpha}} \overline{u_{\overline{\beta}}}\, e^{-\psi} + \int_\Omega \gamma\, |\overline{\partial}^*_\psi u|^2\, e^{-\psi} \\
&\qquad \geq \int_\Omega \frac{|w|^2}{|w|^2+\varepsilon^2}\ \left|\langle u, dw\rangle\right|^2 e^{-\psi} + \int_\Omega \gamma\, \left|\bar{\partial}^*_\psi u\right|^2 e^{-\psi} \\
&\qquad \geq \left| 2\mathrm{Re} \int_\Omega \frac{\partial_\alpha \eta}{\eta}\, u_{\bar{\alpha}} (\overline{\partial}^*_\psi u) e^{-\varphi} \right| ,
\end{aligned}$$

where the last inequality is from (3.5.2). Adding $\int_\Omega \gamma\, |\overline{\partial}^*_\psi u|^2\, e^{-\psi}$ to both sides of (**??**), we obtain

$$\begin{aligned}
&\int_\Omega \left\langle (\eta+\gamma)\overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u \right\rangle e^{-\psi} + \int_\Omega \left\langle \eta\, \bar{\partial} u, \bar{\partial} u \right\rangle e^{-\psi} \\
&\qquad \geq \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}}\partial_\alpha \psi\right) e^{-\varphi} + \int_\Omega \gamma\, |\overline{\partial}^*_\psi u|^2\, e^{-\psi} \\
&\qquad\qquad - \int_\Omega \left(\frac{\partial_{\bar{\alpha}}\partial_\beta \eta}{\eta}\right) u_\alpha \overline{u_\beta} e^{-\varphi} + 2\,\mathrm{Re} \int_\Omega \left(\frac{\partial_\alpha \eta}{\eta}\right) u_{\bar{\alpha}}\, \overline{\left(\bar{\partial}^*_\psi u\right)} e^{-\varphi} \\
&\qquad\qquad \geq - \int_\Omega \left(\frac{\partial_{\bar{\alpha}}\partial_\beta \eta}{\eta}\right) u_\alpha \overline{u_\beta} e^{-\varphi} \\
&\qquad\qquad = \int_\Omega \frac{\varepsilon^2}{(|w|^2+\varepsilon^2)^2}\, \left|\langle u, dw\rangle\right|^2 e^{-\psi}\ . \qquad (5)
\end{aligned}$$

We now consider the operator T defined by $Tu = \bar{\partial}\,(\sqrt{\eta+\gamma}\,u)$ and the operator S defined by $Su = \sqrt{\eta}\,\bar{\partial} u$. Then $ST = 0$ and we can rewrite (3.5.3) as

$$\|T^* u\|^2_{\Omega,\psi} + \|Su\|^2_{\Omega,\psi} \geq \int_\Omega \frac{\varepsilon^2}{(|w|^2+\varepsilon^2)^2}\, \left|\langle u, dw\rangle\right|^2 e^{-\psi} \qquad (6)$$

Here $\|\cdot\|_{\Omega,\psi}$ means the L^2 norm over Ω with respect to the weight function $e^{-\psi}$.

(3.6) *Choice of Cut-Off Function.* Choose any positive number $\delta < 1$. Choose a C^∞ function $0 \leq \varrho(x) \leq 1$ of a single real variable x on $[0,\infty)$ so that the support of ϱ is in $[0,1]$ and $\varrho(x)$ is identically 1 on $[0, \frac{\delta}{2}]$ and the supremum norm of $\frac{\partial}{\partial x}\varrho(x)$ on $[0,1]$ is no more than $1+\delta$.

Let $\varrho_\varepsilon(w) = \varrho\left(\frac{|w|^2}{\varepsilon^2}\right)$ and let

$$g_\varepsilon = \frac{\left(\tilde f \wedge dw\right) \bar\partial \varrho_\varepsilon}{w} = \frac{\tilde f \wedge dw}{\varepsilon^2} \varrho'\left(\frac{|w|^2}{\varepsilon^2}\right) d\bar w \; .$$

We would like to solve the equation $T\, h_\varepsilon = g_\varepsilon$ for some $(n,0)$-form h_ε on Ω. For that we would like to verify the inequality

$$|(u, g_\varepsilon)_{\Omega,\psi}|^2 \le C^2 \left(\|T^* u\|^2_{\Omega,\psi} + \|S\, u\|^2_{\Omega,\psi} \right)$$

for some positive constant C and for all $u \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$. Here $(\cdot,\cdot)_{\Omega,\psi}$ means the inner product on Ω with respect to the weight function $e^{-\psi}$. We have

$$\begin{aligned}
|(u, g_\varepsilon)_{\Omega,\psi}|^2 &\le \left(\int_\Omega |\langle u, g_\varepsilon\rangle|\, e^{-\psi} \right)^2 \\
&= \left(\int_\Omega \left| \left\langle u, \frac{\tilde f \wedge dw}{\varepsilon^2} \varrho'\left(\frac{|w|^2}{\varepsilon^2}\right) d\bar w \right\rangle \right| e^{-\psi} \right)^2 \\
&\le \left(\int_\Omega \left| \frac{\tilde f \wedge dw}{\varepsilon^2} \varrho'\left(\frac{|w|^2}{\varepsilon^2}\right) \right|^2 \frac{(|w|^2+\varepsilon^2)^2}{\varepsilon^2}\, e^{-\psi} \right) \\
&\quad \cdot \left(\int_\Omega |\langle u, d\bar w\rangle|^2 \frac{\varepsilon^2}{(|w|^2+\varepsilon^2)^2}\, e^{-\psi} \right) \\
&\le C_\varepsilon (\|T^* u\|^2_{\Omega,\psi} + \|S\, u\|^2_{\Omega,\psi}) \; ,
\end{aligned}$$

where

$$C_\varepsilon = \int_\Omega \left| \frac{\tilde f \wedge dw}{\varepsilon^2} \varrho'\left(\frac{|w|^2}{\varepsilon^2}\right) \right|^2 \frac{(|w|^2+\varepsilon^2)^2}{\varepsilon^2} e^{-\psi}$$

and the last inequality is from (**??**). We can solve $\bar\partial(\sqrt{\eta+\gamma}\, h_\varepsilon) = g_\varepsilon$ with

$$\int_\Omega |h_\varepsilon|^2\, e^{-\psi} \le C_\varepsilon \; . \tag{7}$$

As $\varepsilon \to 0$, we have the following bound for the limit of C_ε.

$$\limsup_{\varepsilon\to 0} C_\varepsilon \le \left(\int_{\Omega\cap Y} |f|^2 e^{-\kappa}\right)\left(\limsup_{\varepsilon\to 0}\,(1+\delta)^2 \int_{|w|\le\varepsilon} \frac{(|w|^2+\varepsilon^2)^2}{\varepsilon^6}|dw|^2\right)$$
$$\le 8\pi\ (1+\delta)^2 \int_{\Omega\cap Y} |f|^2 e^{-\kappa}\ . \tag{8}$$

(3.7) *Final Step in the Proof of Theorem (3.1).* We now set

$$F_\varepsilon = \varrho_\varepsilon \tilde{f}\wedge dw - w\sqrt{\eta+\gamma}\, h_\varepsilon\ .$$

Then

$$\lim_{\varepsilon\to 0}\int_\Omega \left|\varrho_\varepsilon \tilde{f}\wedge dw\right|^2 e^{-\kappa} = 0\ ,$$

because $\tilde{f}\wedge dw$ is smooth in a relatively compact open neighborhood of $\bar{\Omega}$ in $Y-V$ and the support of $\varrho_\varepsilon \tilde{f}\wedge dw$ approaches a set of measure zero in Ω as $\varepsilon\to 0$. The supremum norm of $w\sqrt{\eta+\gamma}$ on $\Omega\subset\{|w|<1\}$ is no more than the square root of

$$\sup_{0<x\le 1} x^2\left(\log A + \log\frac{1}{x^2+\varepsilon^2} + \frac{1}{x^2+\varepsilon^2}\right) \le \log A + \frac{1}{e} + 1\ ,$$

because the maximum of $y\log\frac{1}{y}$ on $0<y\le 1$ occurs at $y=\frac{1}{e}$ where its value is $\frac{1}{e}$ as one can easily verify by checking the critical points of $y\log\frac{1}{y}$. Since A is any number greater than the base of natural logarithm e and δ is any positive number, when we take limit as $A\to e$ and $\delta\to 0$ and $\nu\to\infty$ and we use (**??**) and

$$\int_\Omega |h_\varepsilon|^2 e^{-\kappa} \le e\, C_\varepsilon$$

from (**??**) and $\sup_\Omega |w|\le 1$, it follows that the limit of F is an L-valued holomorphic n-form on Y whose restriction to Z is $f\wedge dw$ with the following estimate on its norm.

$$\int_Y |F|^2 e^{-\kappa} \le 8\pi\, e\sqrt{2+\frac{1}{e}}\int_Z |f|^2 e^{-\kappa}\ .$$

This finishes the proof of Theorem (3.1). The following version of extension from submanifolds of higher codimension follows from successive applications of Theorem (3.1).

Theorem(3.8) Let Y be a complex manifold of complex dimension n. Let $1 \leq k \leq n$ be an integer and $w_1, \cdots, w_k$ be bounded holomorphic functions on Y whose common zero-set is a complex submanifold Z of complex codimension k in Y (with multiplicity 1 at every point of it). Let L be a holomorphic line bundle over Y with a (possibly singular) metric $e^{-\kappa}$ whose curvature current is semipositive. Assume that there exists a hypersurface V in Y such that $V \cap Z$ is a subvariety of dimension $\leq n-k-1$ in Z and $Y - V$ is the union of a sequence of Stein subdomains Ω_ν of smooth boundary and Ω_ν is relatively compact in $\Omega_{\nu+1}$. If f is an L-valued holomorphic $(n-k)$-form on Z with

$$\int_Z |f|^2 \, e^{-\kappa} < \infty \, ,$$

then $f dw_1 \wedge \cdots \wedge dw_k$ can be extended to an L-valued holomorphic n-form F on Y such that

$$\int_Y |F|^2 \, e^{-\kappa} \leq \left(8\pi \, e \sqrt{2 + \frac{1}{e}} \right)^k \left(\sup_Y |w_1 \cdots w_k|^2 \right) \int_Z |f|^2 \, e^{-\kappa} \, .$$

Proof. We can find a hypersurface V_ν in Ω_ν such that $V_\nu \cap Z$ is of complex dimension $\leq n - k - 1$ in Z and $dw_1 \wedge \cdots \wedge dw_k$ is nowhere zero on $\Omega_\nu - V_\nu$. We can now apply Theorem **??** to the case with Y replaced

$$(\Omega_\nu - V_\nu) \cap \{w_\ell = \cdots = w_k = 0\}$$

and Z replaced by

$$(\Omega_\nu - V_\nu) \cap \{w_{\ell+1} = \cdots = w_k = 0\}$$

and w replaced by $w_{\ell+1}$ for $0 \leq \ell < k$ and use descending induction on ℓ. The theorem now follows by removable singularity for L^2 holomorphic functions and the independence of the constants on $\Omega_\nu - V_\nu$ so that one can pass to limit as $\nu \to \infty$.

Vanishing Theorem for Multiplier Ideal Sheaves on Compact Complex Projective Algebraic Manifolds

Vanishing Theorem of Kawamata-Viehweg. Let X be a compact complex projective algebraic manifold of complex dimension n. Let M be a holomorphic line bundle which is numerically effective (in the sense that $M \cdot C = c_1(M|_C) \geq 0$ for any compact complex curve C in X) and $c_1(M)^n > 0$, where $c_1(\cdot)$ means the first Chern class. Let $\{E_j\}_{1 \leq j \leq J}$ be a family of regular hypersurfaces in X in normal crossing. Let a_j be positive integers ($1 \leq j \leq J$) and N be a positive integer such that $M + \sum_{j=1}^{J} a_j E_j = N\tilde{L}$ for some holomorphic line bundle $\tilde{L}$. Let L be

$$\tilde{L} - \sum_{j=1}^{n} \left\lfloor \frac{a_j}{N} \right\rfloor E_j = \frac{1}{N} M + \sum_{j=1}^{n} \left(\frac{a_j}{N} - \left\lfloor \frac{a_j}{N} \right\rfloor \right) E_j,$$

where $\lfloor u \rfloor$ means the largest integer not exceeding u. Then $H^p(X,\, L + K_X) = 0$ for $p \geq 1$, where K_X is the canonical line bundle of X.

Vanishing Theorem of Nadel. Let X be a compact complex projective algebraic manifold of complex dimension n and L be a holomorphic line bundle over X with (a possibly singular) Hermitian metric $e^{-\chi}$ whose curvature current $\sqrt{-1}\,\bar{\partial}\bar{\partial}^* \chi$ dominates a smooth strictly positive $(1,1)$-form ω_0 on X in the sense of distributions. Let $\mathcal{I}_\chi$ be the multiplier ideal sheaf for $e^{-\chi}$ in the sense that it consists of all holomorphic function germs f on X with $|f|^2 e^{-\chi}$ integrable. Then

$$H^p(X,\, \mathcal{I}(L + K_X)) = 0 \quad \text{for } p \geq 1.$$

Theorem of Kawamata-Viehweg as Consequence of Nadel's Theorem. Let A be a positive line bundle over X with smooth metric $h_A = e^{-\varphi_A}$ such that

(i) φ_A is strictly plurisubharmonic,

(ii) $A - K_X$ is a positive holomorphic line bundle over X , and

(iii) $3A$ admits a global section over X whose divisor Y is nonsingular.

Since M is numerically effective, it follows that $mM + A$ is a positive holomorphic line bundle over X for any positive integer m. By Kodaira's embedding theorem, there exists a sufficiently large positive integer ℓ_m such that

$\ell_m (mM + A)$ is very ample in the sense that a basis of its global holomorphic sections define an embedding Φ of X into some complex projective space $\mathbb{P}_b$. We can find a global holomorphic section of $\ell_m (mM + A)$ over X (by choosing a hyperplane in the complex projective space $\mathbb{P}_b$ where X is embedded by Φ) such that its divisor D_m is nonsingular. Write $mM = \frac{1}{\ell_m} D_m - A$.

From the positivity of $A - K_X$ and the numerical effectiveness of M it follows from Kodaira's vanishing theorem that $H^p (X, mM + A) = 0$ for $p \geq 1$. Since $c_1 (mM + A)^n \geq m^n c_1 (M)^n > 0$, by the theorem of Riemann-Roch

$$\dim_{\mathbb{C}} \Gamma (X, mM + A) \geq \frac{m^n \, c_1 (M)^n}{n!} + O\left(m^{n-1}\right)$$

for m very large, where $O(\cdot)$ is Landau's symbol. From

$$\dim_{\mathbb{C}} \Gamma (Y, mM + A) = O\left(m^{n-1}\right)$$

it follows that for m_0 sufficiently large there exists some non identically zero holomorphic section of $m_0 M + A$ which vanishes on Y. We can write its divisor as $Y + V$ which is equal to $3A + V$ as a line bundle. Thus $m_0 M = 2A + V$ and from $mM = \frac{1}{\ell_m} D_m - A$ it follows that $(m + m_0) M = \frac{1}{\ell_m} D_m + V + A$. Hence

$$M = \frac{1}{(m + m_0)\, \ell_m} D_m + \frac{1}{m + m_0} V + \frac{1}{m + m_0} A.$$

Let $\gamma_j = \frac{a_j}{N} - \left\lfloor \frac{a_j}{N} \right\rfloor$ which is < 1 and nonnegative. Choose a real number $p_1 > 1$ large enough such that $p_1 \gamma_j < 1$ for $1 \leq j \leq J$. Since $\{E_j\}_{1 \leq j \leq J}$ is a family of nonsingular hypersurfaces in normal crossing and $p_1 \gamma_j < 1$ for $1 \leq j \leq J$, it follows that

$$\frac{1}{\left(\prod_{j=1}^{J} \left|s_{E_j}\right|^{2\gamma_j}\right)^{p_1}}$$

is locally integrable on X where s_{E_j} is the canonical section for the divisor E_j.

Choose two real numbers $p_2 > 1$ and $p_3 > 1$ such that $\sum_{j=1}^{3} \frac{1}{p_j} = 1$. Now choose m large enough such that

(i) $(m + m_0) N \ell_m > p_2$ and

(ii) the local function

$$\frac{1}{\left|s_V\right|^{\frac{2\,p_3}{(m+m_0)N}}}$$

is locally integrable on X, where s_V is the canonical section for the divisor V.

Since D_m is nonsingular and $(m+m_0)\,N\ell_m > p_2$, it follows that

$$\frac{1}{\left|s_{D_m}\right|^{\frac{2\,p_2}{(m+m_0)N\ell_m}}}$$

is locally integrable on X, where s_{D_m} is the canonical section for the divisor D_m. Since $\sum_{j=1}^{3}\frac{1}{p_j}=1$, by Hölder's inequality

$$(*)\qquad \int_U \frac{1}{\left(\prod_{j=1}^J\left|s_{E_j}\right|^{2\gamma_j}\right)\left|s_V\right|^{\frac{2}{(m+m_0)N}}\left|s_{D_m}\right|^{\frac{2}{(m+m_0)N\ell_m}}}$$

is dominated by

$$\left(\int_U\frac{1}{\left(\prod_{j=1}^J\left|s_{E_j}\right|\right)^{2\gamma_j p_1}}\right)^{\frac{1}{p_1}}\left(\int_U\frac{1}{\left|s_{D_m}\right|^{\frac{2\,p_2}{(m+m_0)N\ell_m}}}\right)^{\frac{1}{p_2}}\left(\int_U\frac{1}{\left|s_V\right|^{\frac{2\,p_3}{(m+m_0)N}}}\right)^{\frac{1}{p_3}}<\infty$$

for open subsets U of X where $s_{E_j}|_U$, $s_{D_m}|_U$ and $s_V|_U$ can be regarded as holomorphic functions on U. We now introduce the metric $e^{-\chi}$ of L given by

$$e^{-\chi}=\frac{h_A^{\frac{1}{(m+m_0)N}}}{\left(\prod_{j=1}^J\left|s_{E_j}\right|^{2\gamma_j}\right)\left|s_V\right|^{\frac{2}{(m+m_0)N}}\left|s_{D_m}\right|^{\frac{2}{(m+m_0)N\ell_m}}},$$

because

$$\begin{aligned} L &= \frac{1}{N}M+\sum_{j=1}^J\gamma_jE_j\\ &= \frac{1}{(m+m_0)\,N\ell_m}D_m+\frac{1}{(m+m_0)\,N}V+\frac{1}{(m+m_0)\,N}A+\sum_{j=1}^J\gamma_jE_j. \end{aligned}$$

The numerator
$$h_A^{\frac{1}{(m+m_0)N}}$$
in the quotient defining $e^{-\chi}$ guarantees that $\sqrt{-1}\,\partial\bar{\partial}\chi$ dominates some smooth strictly positive $(1,1)$-form on X in the sense of currents. By the finiteness of the integral in $(*)$, the multiplier ideal sheaf $\mathcal{I}_\chi$ is simply $\mathcal{O}_X$. Hence the vanishing theorem of Nadel implies the vanishing of $H^p\left(X, L+K_X\right)$ for $p \geq 1$.

Proof of Nadel's Theorem. Let X be embedded as a complex submanifold of some complex projective space $\mathbb{P}_N$. Choose some linear $\mathbb{P}_{N-n-1}$ in $\mathbb{P}_N$ which is disjoint from X. Choose some linear $\mathbb{P}_n$ in $\mathbb{P}_N$ which is disjoint from $\mathbb{P}_{N-n-1}$. Define $\pi : X \to \mathbb{P}_n$ as follows. For $x \in X$ the point $\pi(x)$ is the intersection in $\mathbb{P}_N$ between $\mathbb{P}_n$ and the linear span of x and $\mathbb{P}_{N-n-1}$. The map $\pi : X \to \mathbb{P}_n$ makes X an analytic cover over $\mathbb{P}_n$. Let s be a meromorphic section of L over X whose zero-set is Z and whose pole-set is W. Choose a complex hypersurface V in $\mathbb{P}_n$ such that

(i) V contains some infinity hyperplane $\mathbb{P}_{n-1}^{(\infty)}$ of $\mathbb{P}_n$,

(ii) $\Omega := X - \pi^{-1}(V)$ is mapped by π locally biholomorphically onto $\mathbb{P}_n - V$,

(iii) $\pi^{-1}(V)$ contains Z and W so that $L|_\Omega$ is globally trivial via the map $\mathcal{O}_X \to \mathcal{O}_X(L)$ defined by multiplication by s.

Let $z_1, \cdots, z_n$ be the affine coordinates of $\mathbb{C}^n := \mathbb{P}_n - \mathbb{P}_{n-1}^{(\infty)}$ and let $F\left(z_1, \cdots, z_n\right)$ be a polynomial which defines $V \cap \mathbb{C}^n$. For $\varepsilon > 0$ define

$$\Omega_\varepsilon := \left\{ P \in \Omega \;\middle|\; \frac{1}{\left|F(\pi(P))\right|^2} + \sum_{j=1}^n \left|z_j\left(\pi(P)\right)\right|^2 < \frac{1}{\varepsilon} \right\}.$$

Choose a strictly monotone sequence of positive numbers ε_ν decreasing to 0 as $\nu \to \infty$ such that Ω_{ε_ν} has smooth boundary for each ν. This is possible by applying Sard's theorem to the map $\Omega \to \mathbb{R}$ defined by the smooth function

$$P \mapsto \frac{1}{\left|F(\pi(P))\right|^2} + \sum_{j=1}^n \left|z_j\left(\pi(P)\right)\right|^2$$

on Ω. Let

$$\tilde{\chi} = -\log\left(e^{-\chi}\left|s\right|^2\right)$$

on Ω, which is strictly plurisubharmonic. Since the operation of convolution is simply averaging the translation with respect to a weight function and since the local biholomorphism π from Ω to $\mathbb{C}^n$ allows the lifting of vector fields $\frac{\partial}{\partial z_j}$, convolution can be defined on relatively open subsets of Ω. We use a smooth nonnegative radially symmetric function $\rho(z) = \rho(|z|)$ on $\mathbb{C}^n$ with compact support and unit L^1 norm and its rescalings $k^{2n}\rho(kz)$ (for $k \in \mathbb{N}$) as the weight functions to perform the convolution $\chi_k := \rho_k * \tilde{\chi}$ on Ω_{ε_ν} for $k \geq k_\nu$ (where k_ν is a sufficiently large positive integer depending on ν). The functions χ_k are smooth plurisubharmonic functions on Ω_{ε_ν} for $k \geq k_\nu$ and form a monotonically non-increasing sequence of functions approaching non-increasingly to $\tilde{\chi}$ on Ω_{ε_ν} as $k \to \infty$.

Fix an integer $1 \leq p \leq n$. Take an L-valued (n,p)-form f on X which is L^2 with respect to $e^{-\chi}$ with $\bar{\partial} f = 0$ on X in the sense of distributions. Let $\tilde{f} = \frac{f}{s}$ on Ω. From $\chi_k \geq \tilde{\chi}$ on Ω_{ε_ν} for $k \geq k_\nu$ it follows that

$$\int_{\Omega_{\varepsilon_\nu}} \left|\tilde{f}\right|^2 e^{-\chi_k} \leq \int_X |f|^2 e^{-\chi} < \infty.$$

We can now solve $\bar{\partial}\tilde{u}_{\nu,k} = \tilde{f}$ on Ω_{ε_ν} and get an $(n,p-1)$-form $u_{\nu,k}$ on Ω_{ε_ν} such that

$$\int_{\Omega_{\varepsilon_\nu}} |u_{\nu,k}|^2 e^{-\tilde{\chi}_k} \leq \frac{1}{c}\int_{\Omega_{\varepsilon_\nu}} \left|\tilde{f}\right|^2 e^{-\chi_k} \leq \frac{1}{c}\int_X |f|^2 e^{-\chi}.$$

Here c is a positive number determined from ω_0 and is independent of ν and k.

Because $\chi_k \geq \tilde{\chi}$, on compact subsets K of Ω contained in Ω_{ε_ν} for $k \geq k_\nu$, the L^2 norm of the $(n,p-1)$-form $u_{\nu,k}$ on K is uniformly bounded independently of k and ν. Thus we can select a sequence $u_{\nu,\hat{k}_\nu}$ for $\nu \in \mathbb{N}$ which *weakly* approaches some L^2 (n,p)-form u on Ω as $\nu \to \infty$. By Fatou's lemma,

$$\int_\Omega |u|^2 e^{-\tilde{\chi}} \leq \liminf_{\nu\to\infty} \int_{\Omega_{\varepsilon_\nu}} \left|u_{\nu,\hat{k}_\nu}\right|^2 e^{-\chi_k} \leq \frac{1}{c}\int_X |f|^2 e^{-\chi}.$$

In the sense of distributions $\bar{\partial} u = f$ on Ω.

We now verify that $\bar{\partial} u = f$ on X in the sense of distributions. When $p = 1$, we can just use the removable singularity theorem of Riemann as follows. Locally we take $P_0 \in \pi^{-1}(V) = X - \Omega$ and solve $\bar{\partial} v = f$ for some

L^2 function v on some open neighborhood G of P_0 in X. Since $u - v$ is holomorphic on $G - \pi^{-1}(V)$ and is L^2, it can be extended to a holomorphic function g on U by the removable singularity theorem of Riemann. Since $u = v + g$ and $\bar{\partial}(v+g) = f$ on G in the sense of distributions, it follows that $\bar{\partial} u = f$ on X in the sense of distributions.

For the case of a general $1 \leq p \leq n$, for $\delta > 0$ we use a smooth nonnegative cut-off function σ_δ with support in a tubular neighborhood $\mathcal{U}_\delta$ of $\pi^{-1}(V)$ in X so that

(i) the distance of a point of the boundary of $\mathcal{U}_\delta$ to $\pi^{-1}(V)$ is comparable to δ,

(ii) the cut-off function σ_δ is identically equal to 1 on some open neighborhood of $\pi^{-1}(V)$ in X,

(iii) the distance of any point in the support of $\partial\sigma_\delta$ to $\pi^{-1}(V)$ is comparable to δ, and

(iv) the supremum of $\left|\bar{\partial}\sigma_\delta\right|$ is comparable to $\frac{1}{\delta}$ so that the L^2 norm of $\bar{\partial}\sigma_\delta$ is uniformly bounded independently of $\delta > 0$.

To verify $\bar{\partial} u = f$ on X, take a test smooth (n, p)-form ψ on X to check that

$$\left(\bar{\partial} u,\, \psi\right)_{L^2(X)} = \left(f,\, \psi\right)_{L^2(X)},$$

where $(\cdot, \cdot)_{L^2(X)}$ is the L^2 inner product over X. We know that $\bar{\partial} u = f$ on Ω and the support of $(1-\sigma_\delta)\,\psi$ is in Ω. So

$$\left(\bar{\partial} u,\, (1-\sigma_\delta)\,\psi\right)_{L^2(X)} = \left(f,\, (1-\sigma_\delta)\,\psi\right)_{L^2(X)},$$

It suffices to verify that

$$\left(u,\, \bar{\partial}^*\left(\sigma_\delta\psi\right)\right)_{L^2(X)}$$

approaches 0 as $\delta \to 0$. By Hölder's inequality, its absolute value is dominated by some δ-independent positive constant times the product of the L^2 norm of u over $\mathcal{U}_\delta$ and the L^2 norm of $\partial\sigma_\delta$. The conclusion now follows from the fact that the L^2 norm of u over $\mathcal{U}_\delta$ approaches 0 as $\delta \to 0$, because u is L^2 on X and the intersection of all $\mathcal{U}_\delta$ for all $\delta > 0$ is $\pi^{-1}(V)$.

Shokurov's Nonvanishing Theorem

Theorem (Shokurov's Nonvanishing Theorem). Let X be a compact complex algebraic manifold, D a divisor of X and A a $\mathbb{Q}$-divisor of X. If D is numerically effective, $p_0D + A - K_X$ is numerically effective and big, and the round-up $\lceil A \rceil$ is effective, and the fractional part $\{A\}$ has support with normal crossing, then $\Gamma(X, pD + \lceil A \rceil)$ is nonzero for p sufficiently large.

Lemma. To prove Shokurov's nonvanishing theorem, it suffices to assume that $p_0D + A - K_X$ is ample.

Proof. Since the $\mathbb{Q}$-bundle $p_0D + A - K_X$ is numerically effective and big, we can write it as $p_0D + A - K_X = E + B$, where E is an effective $\mathbb{Q}$-divisor and B is an ample $\mathbb{Q}$-divisor. Pick any small positive rational number $\varepsilon < 1$ and write

$$\begin{aligned} &p_0D + (A - \varepsilon E) - K_X \\ &= (1-\varepsilon)(p_0D + A - K_X) + \varepsilon(p_0D + A - K_X) - \varepsilon E \\ &= (1-\varepsilon)(p_0D + A - K_X) + \varepsilon(E + B) - \varepsilon E \\ &= (1-\varepsilon)(p_0D + A - K_X) + \varepsilon B \end{aligned}$$

which is the sum of a numerically effective $\mathbb{Q}$-bundle $(1-\varepsilon)(p_0D + A - K_X)$ and an ample $\mathbb{Q}$-bundle εB and is therefore ample. We are going to replace A by $A - \epsilon E$ after using a resolution of singularities to make sure that the support of $A - \epsilon E$ becomes normal crossing. A suitable resolution $f : Y \to X$ of embedded singularities, whose exceptional divisor is supported by $F = \sum_j F_j$ in normal crossing, gives us the following.

(i) $K_Y = f^*K_X + \sum_j a_jF_j$ with $a_j \in \mathbb{Z}$ and $a_j \geq 0$,

(ii) $f^*(p_0D + (A - \epsilon E) - K_X) - \sum_j \delta F_j$ is ample for some $\delta_j \in \mathbb{Q}$ with $\delta_j > 0$,

(iii) the support of the proper transform of $E + A$ is $\sum_k E_k$ with $\sum_j F_j + \sum_k E_k$ in normal crossing,

(iv) $A' \overset{\text{def.}}{=} f^*(A - \epsilon E) + \sum_j (a_j - \delta_j) F_j = \sum_j b_jF_j + \sum_k c_kE_k$ for some $b_j, c_k \in \mathbb{Q}$,

(v) $\lceil A' \rceil$ is effective if δ_j and ϵ are sufficiently small.

We now replace X by Y, D by f^*D and A by A'. We then conclude that $\Gamma(Y, pf^*D + \lceil A' \rceil)$ is nonzero for p sufficiently large. When the positive rational number ϵ is sufficiently small, the difference between $\lceil A' \rceil$ and $f^*(\lceil A \rceil)$ is supported on $F = \sum_j F_j$. Since each F_j is exceptional, it follows that $\Gamma(Y, pf^*D + \lceil A' \rceil)$ agrees with the pullback of $\Gamma(X, pD + \lceil A \rceil)$. Q.E.D.

From this point on we assume without loss of generality that $p_0 D + A - K_X$ is ample. We will distinguish between the following two cases.

(a) $D \cdot C > 0$ for some compact complex curve C in X. For this case we use the notation $D \not\approx 0$ numerically. For this case

$$D\left(p_0 D + A - K_X\right)^{n-1} > 0,$$

because for m sufficiently large we can find

$$s_1, \cdots, s_{n-1} \in \Gamma\left(X, m\left(p_0 D + A - K_X\right)\right)$$

such that the subspace defined by $s_1, \cdots, s_{n-1}$ is the union of C and another compact complex curve C' and from the numerical effectiveness of D it follows that

$$D\left(p_0 D + A - K_X\right)^{n-1} = D \cdot C + D \cdot C' \geq D \cdot C > 0.$$

(The sections $s_1, \cdots, s_{n-1}$ we can get by using global holomorphic sections of some very ample multiple $m'\left(p_0 D + A - K_X\right)$ of $p_0 D + A - K_X$ to embed X into some $\mathbb{P}_N$ and considering a number of linear projections $\mathbb{P}_N - \mathbb{P}_{N-3} \to \mathbb{P}_2$ and the homogeneous polynomials which define the images of C in $\mathbb{P}_2$.)

(b) $D \cdot C = 0$ for every compact complex curve C in X. For this case we use the notation $D \approx 0$ numerically. For this case, $mD + \left(p_0 D + A - K_X\right)$ is ample for any $m \in \mathbb{Z}$ including the case when $m < 0$, as one can conclude from Kleiman's criterion by verifying that

$$\left(mD + \left(p_0 D + A - K_X\right)\right) \cdot C = \left(p_0 D + A - K_X\right) \cdot C > 0$$

for every compact complex curve C in X.

Lemma. Suppose $a \in \mathbb{Z}$ with $a > 0$ and aA an integral divisor. If $p_1 \geq p_0$ is sufficiently large and $D \not\approx 0$ numerically, then $\dim_{\mathbb{C}} |\, ak\,(p_1 D + A - K_X)| > \nu k^n$ for all k sufficiently large with $\nu > \frac{a^n (n+1)^n}{n!}$.

Proof. Since the $\mathbb{Q}$-bundle $p_1 D + A - K_X = (p_1 - p_0)\, D + (p_0 D + A - K_X)$ is ample, it follows that $H^i\,(X, ak\,(p_1 D + A - K_X)) = 0$ for $i > 0$ and k sufficiently large. Hence

$$\dim_{\mathbb{C}} |ak\,(p_1 D + A - K_X)| = \frac{a^n}{n!}\,(p_1 D + A - K_X)^n\, k^n + O\left(k^{n-1}\right)$$

for k sufficiently large. Since $p_0 D + A - K_X$ is ample and $D \not\approx 0$ numerically and D is numerically effective, it follows that $D \cdot (p_0 D + A - K_X) > 0$ and

$$\begin{aligned}(p_1 D + A - K_X)^n &= ((p_1 - p_0)\, D + (p_0 D + A - K_X))^n \\ &\geq n\,(p_1 - p_0)\, D \cdot (p_0 D + A - K_X)^{n-1} \geq (n+1)^n\end{aligned}$$

for p_1 sufficiently large. Q.E.D.

Lemma. Fix $x \in X$ not in the support of A. Let $f_1 : X_1 \to X$ be the blow-up of X at x with exceptional divisor E in X_1 so that $K_{X_1} = f_1^* K_X + (n-1)\, E$. If $D \not\approx 0$ numerically, then there exists $M \in |\, ak\,(p_1 D + A - K_X)|$ such that $f_1^* M = L_1 + eE$ for some effective divisor L_1 and $e \geq ak\,(n+1)$.

Proof. The proof is by counting. The conclusion simply says that there exists some nonzero holomorphic section of $ak\,(p_1 D + A - K_X)$ over X which vanishes at x to order at least $ak\,(n+1)$. Such an existence follows from $\dim_{\mathbb{C}} |ak\,(p_1 D + A - K_X)| > \nu k^n$ for all k sufficiently large with $\nu > \frac{a^n (n+1)^n}{n!}$ in the preceding lemma. Q.E.D.

Proof of Shokurov's Nonvanishing Theorem. We first deal with the case $D \not\approx 0$ numerically. Introduce another resolution of singularities $f_2 : Y \to X_1$ and let $f = f_2 \circ f_1$ so that the following properties hold with the divisor $F = \sum_j F_j$ in Y in normal crossing.

(i) $K_Y = f^* K_X + \sum_j a_j F_j$ for some $a_j \in \mathbb{Z}$ and $a_j \geq 0$,

(ii) $f^*\,(p_1 D + A - K_X) - (n+1) \sum_j \delta_j F_j$ is ample for some $\delta_j \in \mathbb{Q}$ and $0 < \delta_j < 1$,

(iii) $f^* A + \sum_j a_j F_j = \sum_j b_j F_j$ with $b_j \in \mathbb{Q}$,

(iv) $f^*M = \sum_j r_j F_j$ with $r_j \in \mathbb{Z}$ nonnegative.

Let
$$c = \min_j \left(\frac{b_j + 1 - \delta_j}{r_j} \right).$$
Let $F_1 = E'$ be the strict transform of E by f_2. We have $b_1 = n-1$ (because $a_1 = n-1$ and x is not in the support of E) and $r_1 \geq ak\,(n-1)$ and

$$(\sharp) \qquad cak < \frac{n}{n+1} = 1 - \frac{1}{n+1}$$

(because c is no more than $\frac{b_1+1-\delta_1}{r_1}$).

We choose $\{\delta_j\}_j$ so that for only one $j = j_0$ we have

$$(\dagger) \qquad -cr_{j_0} + b_{j_0} - \delta_{j_0} = -1$$

with all other $-cr_j + b_j - \delta_j > -1$ for $j \neq j_0$. Let
$$A' = \sum_{j \neq j_0} \left(-cr_j + b_j - \delta_j\right) F_j$$
and $B = F_{j_0}$. Then
$$\begin{aligned}
A' &= F_{j_0} + \sum_j \left(-cr_j + b_j - \delta_j\right) F_j \quad \text{(by } (\dagger)\text{)} \\
&= F_{j_0} - c\sum_j r_j F_j + \sum_j \left(b_j - \delta_j\right) F_j \\
&= F_{j_0} - cf^*M + \sum_j \left(b_j - \delta_j\right) F_j \quad \text{(by (iv))} \\
&\approx B - cak\, f^*\left(p_1 D + A - K_X\right) + \sum_j \left(b_j - \delta_j\right) F_j \\
&\quad \text{(because } M \in \left|ak\left(p_1 D + A - K_X\right)\right|\text{)},
\end{aligned}$$
where "$\approx$" means that the two divisors define the same line bundle. From (i) and (iii) we have
$$f^*A - f^*K_X = \sum_j b_j F_j - K_Y$$

and

$$(\ddagger) \qquad f^*(p_1 D + A - K_X) = p_1 f^* D + \sum_j b_j F_j - K_Y.$$

We now use the above expression for A' to rewrite $f^*(pD) + A' - B - K_Y$ as a line bundle instead of a divisor so that the above "$\approx$" is simply replaced by "$=$" in the following computation.

$$\begin{aligned}
f^*(pD) + A' - B - K_Y &= (p - p_1) f^* D + f^*(p_1 D) + A' - B - K_Y \\
&= (p - p_1) f^* D + f^*(p_1 D) + B \\
&\quad + \left(-cak\, f^*(p_1 D + A - K_X) + \sum_j (b_j - \delta_j) F_j\right) - B - K_Y \\
&= (p - p_1) f^* D + \left(f^*(p_1 D) + \sum_j b_j F_j - K_Y\right) \\
&\quad - \sum_j \delta_j F_j - cak\, f^*(p_1 D + A - K_X) \\
&= (p - p_1) f^* D - \sum_j \delta_j F_j + (1 - cak) f^*(p_1 D + A - K_X) \quad \text{(by } (\ddagger)) \\
&= (p - p_1) f^* D + \frac{1}{n+1}\left(f^*(p_1 D + A - K_X) - (n+1)\sum_j \delta_j F_j\right) \\
&\quad + \left(1 - cak - \frac{1}{n+1}\right) f^*(p_1 D + A - K_X)
\end{aligned}$$

is ample, because

$$f^*(p_1 D + A - K_X) - (n+1)\sum_j \delta_j F_j$$

is ample by (ii), and

$$1 - cak - \frac{1}{n+1} > 0$$

by ($\sharp$), and

$$p_1 D + A - K_X = (p_1 - p_0) D + (p_0 D + A - K_X)$$

is numerically effective by assumption (or even ample by reduction of the general case of numerical effectiveness to the special case of ampleness).

The vanishing theorem of Kawamata-Viehweg implies that

$$(*) \qquad H^1\left(Y, f^*(pD) + \lceil A' \rceil - B\right) = 0.$$

(One also get $(*)$ by applying Nadel's vanishing theorem to the product of the smooth positively curved metric of the ample line bundle

$$f^*(pD) + A' - B - K_Y$$

and the metric

$$\frac{1}{\left|s_{\lceil A' \rceil - A'}\right|^2}$$

of $\lceil A' \rceil - A'$, where $s_{\lceil A' \rceil - A'}$ is the canonical multivalued section of $\lceil A' \rceil - A'$.)

From $(*)$ it follows that

$$\Gamma\left(Y, f^*(pD) + \lceil A' \rceil\right) \to \Gamma\left(B, f^*(pD) + \lceil A' \rceil\right)$$

is surjective. Since

$$f^*(p_1 D + A')|_B - K_B = \left(f^*(p_1 D) + A' - B - K_Y\right)|_B$$

is ample, by induction on the dimension of X

$$\Gamma\left(B, f^*(pD) + \lceil A' \rceil\right) \neq 0$$

for p sufficiently large. Hence

$$\Gamma\left(Y, f^*(pD) + \lceil A' \rceil\right) \neq 0$$

and

$$\Gamma\left(X, pD + \lceil A \rceil\right) \neq 0.$$

Now we have to handle the case $D \approx 0$ numerically. Since $p_0 D + A - K_X$ is ample and $D \approx 0$ numerically, it follows that

$$pD + A - K_X = (p - p_0) D + (p_0 D + A - K_X)$$

is ample for all integers p (even when $p - p_0 < 0$). (See Case (b) presented above.) Hence $H^i\left(X, pD + \lceil A \rceil\right) = 0$ for $i > 0$ and all integers p. By the theorem of Riemann-Roch

$$\dim_{\mathbb{C}} \Gamma\left(X, pD + \lceil A \rceil\right) = \dim_{\mathbb{C}} \Gamma\left(X, \lceil A \rceil\right) \neq 0,$$

because $\lceil A \rceil$ is effective. Q.E.D.