

# Differentially Private (Gradient) Expectation Maximization Algorithm with Statistical Guarantees

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## Abstract

(Gradient) Expectation Maximization (EM) is a widely used algorithm for estimating the maximum likelihood of mixture models or incomplete data problems. A major challenge facing this popular technique is how to effectively preserve the privacy of sensitive data. Previous research on this problem has already led to the discovery of some Differentially Private (DP) algorithms for (Gradient) EM. However, unlike in the non-private case, existing techniques are not yet able to provide finite sample statistical guarantees. To address this issue, we propose in this paper the first DP version of (Gradient) EM algorithm with statistical guarantees. Moreover, we apply our general framework to three canonical models: Gaussian Mixture Model (GMM), Mixture of Regressions Model (MRM) and Linear Regression with Missing Covariates (RMC). Specifically, for GMM in the DP model, our estimation error is near optimal in some cases. For the other two models, we provide the first finite sample statistical guarantees. Our theory is supported by thorough numerical experiments.

## Introduction

As one of the most popular techniques for estimating the maximum likelihood of mixture models or incomplete data problems, Expectation Maximization (EM) algorithm has been widely applied to many areas such as genomics (Laird 2010), finance (Faria and Gonçalves 2013), and crowdsourcing (Dawid and Skene 1979). EM algorithm is well-known for its convergence to an empirically good local estimator (Wu et al. 1983). Recent studies have further revealed that it can also provide finite sample statistical guarantees (Balakrishnan et al. 2017b; Zhu et al. 2017; Wang et al. 2015; Yi and Caramanis 2015). Specifically, (Balakrishnan et al. 2017b) showed that classical EM and its gradient ascent variant (Gradient EM) are capable of achieving the first local convergence (theory) and finite sample statistical rate of convergence. They also provided a (near) optimal minimax rate for some canonical statistical models such as Gaussian mixture model (GMM), mixture of regressions model (MRM) and linear regression with missing covariates (RMC).

The wide applications of EM also present some new challenges to this method. Particularly, due to the existence of

sensitive data and their distributed nature in many applications like social science, biomedicine, and genomics, it is often challenging to preserve the privacy of such data as they are extremely difficult to aggregate and learn from. Consider a case where health records are scattered across multiple hospitals (or even countries), it is not possible to process the whole dataset in a central server due to privacy and ownership concerns. A better solution is to use some differentially private mechanisms to conduct the aggregation and learning tasks. Differential Privacy (DP) (Dwork et al. 2006) is a commonly-accepted criterion that provides provable protection against identification and is resilient to arbitrary auxiliary information that might be available to attackers.

Thus, to be able to use (Gradient) EM algorithm to learn from these sensitive data, it is urgent to design some DP versions of the (gradient) EM algorithm. (Park et al. 2017) proposed the first DP EM algorithm which mainly focuses on the practical behaviors of the method. Their algorithm needs quite a few assumptions on the model and the data, which make it difficult to extend to some canonical models mentioned above. Furthermore, unlike the aforementioned non-private case, their algorithm does not provide any finite sample statistical guarantee on the solution (see Related Work section for detailed comparison). Thus, it is still unknown **whether there exists any DP variant of the (gradient) EM algorithm that has finite sample statistical guarantees.**

To answer this question, we propose in this paper the first  $(\epsilon, \delta)$ -DP (Gradient) EM algorithm with finite sample statistical guarantees. Specifically,

- We first show that, given an appropriate initialization  $\beta^{\text{init}}$  (i.e.,  $\|\beta^{\text{init}} - \beta^*\|_2 \leq \kappa \|\beta^*\|_2$  for some constant  $\kappa \in (0, 1)$ ), if the model satisfies some additional assumptions and the number of sample  $n$  is large enough, the output  $\beta^{\text{priv}}$  of our DP (Gradient) EM algorithm is guaranteed to have a bounded estimation error,  $\|\beta^{\text{priv}} - \beta^*\|_2 \leq \tilde{O}\left(\frac{d\sqrt{\tau}}{\sqrt{n}\epsilon}\right)$ , with high probability, where  $d$  is the dimensionality and  $\tau$  is an upper bound of the second-order moment of each coordinate of the gradient function.
- We then apply our general framework to the three canonical models: GMM, MRM and RMC. Our private estimator achieves an estimation error that is upper bounded by  $\tilde{O}\left(\frac{d}{\sqrt{n}\epsilon}\right)$ ,  $\tilde{O}\left(\frac{d^{\frac{3}{2}}}{\sqrt{n}\epsilon}\right)$  and  $\tilde{O}\left(\frac{d^{\frac{3}{2}}}{\sqrt{n}\epsilon}\right)$  for GMM, MRM and

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RMC, respectively. We note that they are the first statistical guarantees for MRM and RMC in the Differential Privacy model, and the error bound for GMM is near optimal in some cases. We also conduct thorough experiments on the these three models. Experimental results on these models are consistent with our theoretical analysis.

Due to the space limit, some additional background and experiments and omitted proofs are included in the Appendix of Supplementary Material. The source code of experiments can be found in the Supplementary Material or at <https://www.dropbox.com/s/d78ep22xtceu8mt/DPEM.zip?dl=0>.

## Related Work

As we mentioned previously, designing DP version of EM algorithm is still not well studied. To our best knowledge, the only work on DP EM algorithm is given by (Park et al. 2017). However, their result is incomparable with ours for the following reasons. Firstly, our work aims to achieve finite sample statistical guarantees for the DP EM algorithm, while (Park et al. 2017) mainly focuses on designing practical DP EM algorithm that does not provide any statistical guarantees. Particularly, (Park et al. 2017) assumed that datasets are pre-processed such that the  $\ell_2$ -norm of each data record is less than 1. This means that their algorithm will likely introduce additional bias on the statistical guarantees. Secondly, (Park et al. 2017) studied only the exponential family so that noise can be directly added to the sufficient statistics. However, most of the latent variable models do not satisfy such an assumption. This includes the MRM and RMC models to be considered in this paper.

In this paper, we implement our general framework on three specific models, and DP GMM is the only one that has been studied previously. Specifically, (Nissim, Raskhadnikova, and Smith 2007) provided the first result for the general  $k$ -GMM based on the sample-and-aggregate framework. Later on, (Kamath et al. 2019) improved the result by a factor of  $\sqrt{d}/\epsilon$ , and also claimed that their sample complexity is near optimal. Compared with their result, our proposed algorithm ensures that when the error  $\alpha$  is some constant, it has the same sample complexity. Also, although their algorithm has polynomial time complexity, it is actually not very practical and thus no practical study has been conducted. Moreover, their algorithm is heavily dependent on a previous clustering algorithm; it is unclear whether it can be extended to other mixture models. From these two perspectives, our framework is more general and practical.

## Preliminaries

Let  $Y$  and  $Z$  be two random variables taking values in the sample spaces  $\mathcal{Y}$  and  $\mathcal{Z}$ , respectively. Suppose that the pair  $(Y, Z)$  has a joint density function  $f_{\beta^*}$  that belongs to some parameterized family  $\{f_{\beta^*} | \beta^* \in \Omega\}$ . Rather than considering the whole pair of  $(Y, Z)$ , we observe only component  $Y$ . Thus, component  $Z$  can be viewed as the missing or latent structure. We assume that the term  $h_\beta(y)$  is the margin distribution over the latent variable  $Z$ , i.e.,  $h_\beta(y) = \int_{\mathcal{Z}} f_\beta(y, z) dz$ . Let  $k_\beta(z|y)$  be the density of  $Z$  conditional on the observed variable  $Y = y$ , that is,  $k_\beta(z|y) = \frac{f_\beta(y, z)}{h_\beta(y)}$ .

Given  $n$  observations  $y_1, y_2, \dots, y_n$  of  $Y$ , the EM algorithm is to maximize the log-likelihood  $\max_{\beta \in \Omega} \ell_n(\beta) = \sum_{i=1}^n \log h_\beta(y_i)$ . Due to the unobserved latent variable  $Z$ , it is often difficult to directly evaluate  $\ell_n(\beta)$ . Thus, we consider the lower bound of  $\ell_n(\beta)$ . By Jensen's inequality, we have

$$\begin{aligned} \frac{1}{n} [\ell_n(\beta) - \ell_n(\beta')] &\geq \frac{1}{n} \sum_{i=1}^n \int_{\mathcal{Z}} k_{\beta'}(z|y_i) \log f_\beta(y_i, z) dz \\ &\quad - \frac{1}{n} \sum_{i=1}^n \int_{\mathcal{Z}} k_{\beta'}(z|y_i) \log f_{\beta'}(y_i, z) dz. \end{aligned} \quad (1)$$

Let  $Q_n(\beta; \beta') = \frac{1}{n} \sum_{i=1}^n q_i(\beta; \beta')$ , where

$$q_i(\beta; \beta') = \int_{\mathcal{Z}} k_{\beta'}(z|y_i) \log f_\beta(y_i, z) dz. \quad (2)$$

Also, it is convenient to let  $Q(\beta; \beta')$  denote the expectation of  $Q_n(\beta; \beta')$  w.r.t  $\{y_i\}_{i=1}^n$ , that is,

$$Q(\beta; \beta') = \mathbb{E}_{y \sim h_{\beta^*}} \int_{\mathcal{Z}} k_{\beta'}(z|y) \log f_\beta(y, z) dz. \quad (3)$$

We can see that the second term on the right hand side of (1) is independent on  $\beta$ . Thus, given some fixed  $\beta'$ , we can maximize the lower bound function  $Q_n(\beta; \beta')$  over  $\beta$  to obtain sufficiently large  $\ell_n(\beta) - \ell_n(\beta')$ . Thus, in the  $t$ -th iteration of the standard EM algorithm, we can evaluate  $Q_n(\cdot; \beta^t)$  at the E-step and then perform the operation of  $\beta^{t+1} = \max_{\beta \in \Omega} Q_n(\beta; \beta^t)$  at the M-step. See (McLachlan and Krishnan 2007) for more details.

In addition to the exact maximization implementation of the M-step, we add a gradient ascent implementation of the M-step, which performs an approximate maximization via a gradient descent step.

**Gradient EM Algorithm (Balakrishnan et al. 2017b)**  
When  $Q_n(\cdot; \beta^t)$  is differentiable, the update of  $\beta^t$  to  $\beta^{t+1}$  consists of the following two steps.

- E-step: Evaluate the functions in (2) to compute  $Q_n(\cdot; \beta^t)$ .
- M-step: Update  $\beta^{t+1} = \beta^t + \eta \nabla Q_n(\beta^t; \beta^t)$ , where  $\nabla$  is the derivative of  $Q_n$  w.r.t the first component and  $\eta$  is the step size.

Next, we give some examples that use the gradient EM algorithm. Note that they are the typical examples for studying the statistical property of EM algorithm (Wang et al. 2015; Balakrishnan et al. 2017b; Yi and Caramanis 2015; Zhu et al. 2017). See Appendix for their specified  $\nabla q_i(\beta; \beta')$  in (2).

**Gaussian Mixture Model (GMM)** Let  $y_1, \dots, y_n$  be  $n$  i.i.d samples from  $Y \in \mathbb{R}^d$  with

$$Y = Z \cdot \beta^* + V, \quad (4)$$

where  $Z$  is a Rademacher random variable (i.e.,  $\mathbb{P}(Z = +1) = \mathbb{P}(Z = -1) = \frac{1}{2}$ ), and  $V \sim \mathcal{N}(0, \sigma^2 I_d)$  is independent of  $Z$  for some known standard deviation  $\sigma$ .

**Mixture of (Linear) Regressions Model (MRM)** Let  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  be  $n$  samples i.i.d sampled from  $Y \in \mathbb{R}$  and  $X \in \mathbb{R}^d$  with

$$Y = Z \langle \beta^*, X \rangle + V, \quad (5)$$

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<sup>1</sup>We use  $q(\beta; \beta')$  for general sample  $y$ .

where  $X \sim \mathcal{N}(0, I_d)$ ,  $V \sim \mathcal{N}(0, \sigma^2)$ ,  $Z$  is a Rademacher random variable, and  $X, V, Z$  are independent.

**Linear Regression with Missing Covariates (RMC)** We assume that  $Y \in \mathbb{R}$  and  $X \in \mathbb{R}^d$  satisfy

$$Y = \langle X, \beta^* \rangle + V, \quad (6)$$

where  $X \sim \mathcal{N}(0, I_d)$  and  $V \sim \mathcal{N}(0, \sigma^2)$  are independent. Let  $x_1, x_2, \dots, x_n$  be  $n$  observations of  $X$  with each coordinate of  $x_i$  missing (unobserved) independently with probability  $p_m \in [0, 1]$ .

Next, we provide several definitions on the required properties of functions  $Q_n(\cdot; \cdot)$  and  $Q(\cdot; \cdot)$ . Note that some of them have been used in previous studies on the statistical guarantees of EM algorithm (Balakrishnan et al. 2017b; Wang et al. 2015; Zhu et al. 2017).

**Definition 1.** Function  $Q(\cdot; \beta^*)$  is self-consistent if  $\beta^* = \arg \max_{\beta \in \Omega} Q(\beta; \beta^*)$ . That is,  $\beta^*$  maximizes the lower bound of the log likelihood function.

**Definition 2** (Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ )).  $Q(\cdot; \cdot)$  is called Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ), if for the underlying parameter  $\beta^*$  and any  $\beta \in \mathcal{B}$  for some set  $\mathcal{B}$ , the following holds

$$\|\nabla Q(\beta; \beta^*) - \nabla Q(\beta; \beta)\|_2 \leq \gamma \|\beta - \beta^*\|_2. \quad (7)$$

We note that there are some differences between the definition of Lipschitz-Gradient-2 and the Lipschitz continuity condition in the convex optimization literature (Nesterov 2013). Firstly, in (7), the gradient is w.r.t the second component, while the Lipschitz continuity is w.r.t the first component. Secondly, the property holds only for fixed  $\beta^*$  and any  $\beta$ , while the Lipschitz continuity is for all  $\beta, \beta' \in \mathcal{B}$ .

**Definition 3** ( $\mu$ -smooth).  $Q(\cdot; \beta^*)$  is  $\mu$ -smooth, that is if for any  $\beta, \beta' \in \mathcal{B}$ ,  $Q(\beta; \beta^*) \geq Q(\beta'; \beta^*) + (\beta - \beta')^T \nabla Q(\beta'; \beta^*) - \frac{\mu}{2} \|\beta' - \beta\|_2^2$ .

**Definition 4** ( $v$ -strongly concave).  $Q(\cdot; \beta^*)$  is  $v$ -strongly concave, that is if for any  $\beta, \beta' \in \mathcal{B}$ ,  $Q(\beta; \beta^*) \leq Q(\beta'; \beta^*) + (\beta - \beta')^T \nabla Q(\beta'; \beta^*) - \frac{v}{2} \|\beta' - \beta\|_2^2$ .

In the following we will propose the assumptions that will be used throughout the whole paper. Note that these assumptions are commonly used in other works on statistical analysis of EM algorithm such as (Balakrishnan et al. 2017a; Zhu et al. 2017; Wang et al. 2015).

**Assumption 1.** We assume that function  $Q(\cdot; \cdot)$  in (3) is self-consistent, Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ),  $\mu$ -smooth,  $v$ -strongly concave over some set  $\mathcal{B}$ . Moreover, we assume that  $\forall j \in [d]$  and  $\beta \in \mathcal{B}$ , there is some known upper bound  $\tau$  on the second-order moment of the  $j$ -coordinate of  $\nabla q(\beta, \beta)$ , i.e.,  $\mathbb{E}_y (\nabla_j q(\beta, \beta))^2 \leq \tau$  and for each  $i \in [n]$ ,  $\nabla_j q_i(\beta, \beta)$  is independent with others.

## Differential Privacy

**Definition 5** (Differential Privacy (Dwork et al. 2006)). Given a data universe  $\mathcal{X}$ , we say that two datasets  $D, D' \subseteq \mathcal{X}$  are neighbors if they differ by only one entry, which is denoted as  $D \sim D'$ . A randomized algorithm  $\mathcal{A}$  is  $(\epsilon, \delta)$ -differentially private (DP) if for all neighboring datasets  $D, D'$  and for all events  $S$  in the output space of  $\mathcal{A}$ , we have  $\mathbb{P}(\mathcal{A}(D) \in S) \leq e^\epsilon \mathbb{P}(\mathcal{A}(D') \in S) + \delta$ .

**Definition 6** (Gaussian Mechanism). Given a function  $q : \mathcal{X}^n \rightarrow \mathbb{R}^p$ , the Gaussian Mechanism is defined as:  $\mathcal{M}_G(D, q, \epsilon) = q(D) + Y$ , where  $Y$  is drawn from a Gaussian Distribution  $\mathcal{N}(0, \sigma^2 I_p)$  with  $\sigma \geq \frac{\sqrt{2 \ln(1.25/\delta)} \Delta_2(q)}{\epsilon}$ .  $\Delta_2(q)$  is the  $\ell_2$ -sensitivity of the function  $q$ , i.e.,  $\Delta_2(q) = \sup_{D \sim D'} \|q(D) - q(D')\|_2$ . Gaussian Mechanism preserves  $(\epsilon, \delta)$ -differentially private.

Due to the similarity with the Gradient Descent algorithm and the simplicity of illustrating our idea compared with the original EM algorithm, in this paper, we will mainly focus on DP Gradient EM algorithm. See Appendix for the statistical guarantees of the DP EM algorithm.

## Main Method

### Main Difficulty

In the previous section, we introduced the Gradient EM algorithm, which updates the estimator via the gradient  $\nabla Q_n(\beta^t; \beta^t)$ . It is notable that this idea is quite similar to the Gradient Descent algorithm. Moreover, we know that there are several DP versions of the (Stochastic) Gradient Descent algorithm such as (Bassily, Smith, and Thakurta 2014; Wang, Ye, and Xu 2017; Song, Chaudhuri, and Sarwate 2013; Wang and Xu 2019; Lee and Kifer 2018). The key idea of DP Gradient Descent is adding some randomized noise such as Gaussian noise to preserve DP property in each iteration, and by the composition theorem of DP ((Dwork, Roth et al. 2014)), the whole algorithm will still be DP. Thus, motivated by this, to design a DP variant of Gradient EM algorithm, the most direct way is adding some Gaussian noise to the gradient  $\nabla Q_n(\beta^t; \beta^t)$  in each iteration and updating the parameter.

However, it is notable that we cannot add Gaussian noise directly to the gradient in the Gradient EM algorithm. The main reason is that all previous DP Gradient Descent algorithms need to assume that each component of the gradient (which correspond to the function  $\nabla q_i$  in (2)) is bounded, or the loss function is  $O(1)$ -Lipschitz, such as Logistic Regression, so that its  $\ell_2$ -norm sensitivity is bounded and thus the Gaussian mechanism can be used. However, in the Gradient EM algorithm, each component  $(\nabla q_i(\beta^t; \beta^t))$  in (2)) is unbounded in most of the cases. For example, we can easily show the following fact.

**Theorem 1.** Consider the GMM in (4), there is a case with fixed  $\beta$ , such that for each constant  $c$ , with **positive probability** w.r.t  $y$  we have  $\|\nabla q(\beta; \beta)\|_2 \geq c$ .

Thus, to design a DP (Gradient) EM algorithm, the major difficulty lies in how to process the gradient to make its sensitivity bounded. Two main approaches are used in previous work: (1) (Park et al. 2017) assumed that datasets are pre-processed such that the  $\ell_2$  norm of each sample is bounded by 1. However, as mentioned previously, our goal is to achieve the statistical guarantees for the DP (Gradient) EM algorithm. If a similar approach is adopted in our algorithm, the (manual) normalization can easily destroy many statistical properties of the data and force the private estimator to introduce additional bias, making it inconsistent.<sup>2</sup> (2) Instead

<sup>2</sup>An estimator  $\beta_n$  is consistent if  $\lim_{n \rightarrow \infty} \|\beta_n - \beta^*\|_2 = 0$ .

of normalizing the datasets, (Abadi et al. 2016) first clipped the gradient to ensure that the  $\ell_2$ -norm of each component of the gradient is bounded by the threshold  $C$ , and then added Gaussian noise (see Algorithm 1 for more details). However, such an approach may cause two issues. First, in general clipping gradient could introduce additional bias even in statistical estimation, which has also been pointed out in (Song, Thakkar, and Thakurta 2020). Second, the threshold  $C$  heavily affects the convergence speed and selecting the best  $C$  is quite difficult (see Experimental section for more details). Due to these two reasons, it is hard to study the statistical guarantees of Algorithm 1. Thus, we need a new approach to pre-process the gradient to ensure that it has not only bounded  $\ell_2$ -norm but also consistent statistical guarantee.

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**Algorithm 1** Clipped DP Gradient EM

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**Input:**  $D = \{y_i\}_{i=1}^n \subset \mathbb{R}^d$ , privacy parameters  $\epsilon, \delta; Q_n(\cdot; \cdot)$  and its  $g(\cdot; \cdot)$ , initial parameter  $\beta^0$ , gradient norm  $C$ , step size  $\eta$  and the number of iterations  $T$ .

- 1: **for**  $t = 1, 2, \dots, T$  **do**
  - 2:   For each  $i \in [n]$ , evaluate the function in (2) to compute  $q_i(\beta; \beta^{t-1})$ .
  - 3:   Clip gradient:
- $$\nabla \bar{q}_i(\beta^{t-1}; \beta^{t-1}) = \frac{\nabla q_i(\beta^{t-1}; \beta^{t-1})}{\max\{1, \frac{\|\nabla q_i(\beta^{t-1}; \beta^{t-1})\|_2}{C}\}}.$$
- 4:   Update  $\beta^t = \beta^{t-1} + \eta(\nabla \bar{Q}_n(\beta^{t-1}; \beta^{t-1}) + \mathcal{N}(0, C^2 \sigma^2 I_d))$ , where  $\nabla \bar{Q}_n(\beta^{t-1}; \beta^{t-1}) = \frac{1}{n} \sum_{i=1}^n \nabla \bar{q}_i(\beta^{t-1}; \beta^{t-1})$  and  $\sigma^2 = c \frac{T \log \frac{1}{\delta}}{n^2 \epsilon^2}$  for some constant  $c$ .
  - 5: **end for**
  - 6: Return  $\beta^T$
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## Our Method

In this section, we will propose our method to overcome the aforementioned difficulties.

Our method is motivated by a robust and private mean estimator for heavy-tailed distributions, which was given in (Wang et al. 2020), and it is derived from the robust mean estimator in (Holland 2019). To be self-contained, we first review their estimator. Now, we consider a 1-dimensional random variable  $x$  and assume that  $x_1, x_2, \dots, x_n$  are i.i.d. sampled from  $x$ . The estimator consists of three steps:

**Scaling and Truncation** For each sample  $x_i$ , we first re-scale it by dividing  $s$  (which will be specified later). Then, we apply the re-scaled one to some soft truncation function  $\phi$ . Finally, we put the truncated mean back to the original scale. That is,

$$\frac{s}{n} \sum_{i=1}^n \phi\left(\frac{x_i}{s}\right) \approx \mathbb{E}X. \quad (8)$$

Here, we use the function given in (Catoni and Giulini 2017),

$$\phi(x) = \begin{cases} x - \frac{x^3}{6}, & -\sqrt{2} \leq x \leq \sqrt{2} \\ \frac{2\sqrt{2}}{3}, & x > \sqrt{2} \\ -\frac{2\sqrt{2}}{3}, & x < -\sqrt{2}. \end{cases} \quad (9)$$

Note that a key property for  $\phi$  is that  $\phi$  is bounded, that is,  $|\phi(x)| \leq \frac{2\sqrt{2}}{3}$ .

**Noise Multiplication** Let  $\eta_1, \eta_2, \dots, \eta_n$  be random noise generated from a common distribution  $\eta \sim \chi$  with  $\mathbb{E}\eta = 0$ . We multiply each data  $x_i$  by a factor of  $1 + \eta_i$ , and then perform the scaling and truncation step on the term  $x_i(1 + \eta_i)$ . That is,

$$\tilde{x}(\eta) = \frac{s}{n} \sum_{i=1}^n \phi\left(\frac{x_i + \eta_i x_i}{s}\right). \quad (10)$$

**Noise Smoothing** In this final step, we smooth the multiplicative noise by taking the expectation w.r.t. the distributions. That is,

$$\hat{x} = \mathbb{E}\tilde{x}(\eta) = \frac{s}{n} \sum_{i=1}^n \int \phi\left(\frac{x_i + \eta_i x_i}{s}\right) d\chi(\eta_i). \quad (11)$$

Computing the explicit form of each integral in (11) depends on the function  $\phi(\cdot)$  and the distribution  $\chi$ . Fortunately, (Catoni and Giulini 2017) showed that when  $\phi$  is in (9) and  $\chi \sim \mathcal{N}(0, \frac{1}{\beta})$  (where  $\beta$  will be specified later), we have for any  $a$  and  $b > 0$

$$\mathbb{E}_\eta \phi(a + b\sqrt{\beta}\eta) = a(1 - \frac{b^2}{2}) - \frac{a^3}{6} + C(a, b), \quad (12)$$

where  $C(a, b)$  is a correction term which is easy to implement and its explicit form will be given in the Appendix.

(Holland 2019) showed the following estimation error for the mean estimator  $\hat{x}$  after these three steps.

**Lemma 1** (Lemma 5 in (Holland 2019)). Let  $x_1, x_2, \dots, x_n$  be i.i.d. samples from distribution  $x \sim \mu$ . Assume that there is some known upper bound on the second-order moment, i.e.,  $\mathbb{E}_\mu x^2 \leq \tau$ . For a given failure probability  $\zeta$ , if set  $\beta = 2 \log \frac{1}{\zeta}$  and  $s = \sqrt{\frac{n\tau}{2 \log \frac{1}{\zeta}}}$ , then with probability at least  $1 - \zeta$  the following holds

$$|\hat{x} - \mathbb{E}x| \leq O\left(\sqrt{\frac{\tau \log \frac{1}{\zeta}}{n}}\right). \quad (13)$$

To obtain an  $(\epsilon, \delta)$ -DP estimator, the key observation is that the bounded function  $\phi$  in (9) also makes the integral form of (11) bounded by  $\frac{2\sqrt{2}}{3}$ . Thus, we know that the  $\ell_2$ -norm sensitivity is  $\frac{s}{n} \frac{4\sqrt{2}}{3}$ . Hence, the query

$$\mathcal{A}(D) = \hat{x} + Z, Z \sim \mathcal{N}(0, \sigma^2), \sigma^2 = O\left(\frac{s^2 \log \frac{1}{\delta}}{\epsilon^2 n^2}\right) \quad (14)$$

will be  $(\epsilon, \delta)$ -DP, which leads to the following result.

**Lemma 2** (Theorem 6 in (Wang et al. 2020)). Under the assumptions in Lemma 1, with probability at least  $1 - \zeta$  the following holds

$$|\mathcal{A}(D) - \mathbb{E}(x)| \leq O\left(\sqrt{\frac{v \log \frac{1}{\delta} \log \frac{1}{\zeta}}{n \epsilon^2}}\right). \quad (15)$$

It is notable that in Lemma 2 we just need to assume that  $x$  has bounded second order moment, instead of bounded norm. However, since we need weaker assumptions here, the error bound in (15) is larger than it for the bounded distributions (Bun, Ullman, and Vadhan 2018).

Inspired by the previous private 1-dimensional mean estimation, we propose our method (Algorithm 2). In Algorithm 2, the key idea is that, in the  $t$ -th iteration of Gradient EM algorithm, we first apply the previous private estimator to each coordinate of the gradient  $\nabla Q_n(\beta^{t-1}; \beta^{t-1})$ , and then perform the M-step. We can easily show that Algorithm 2 is  $(\epsilon, \delta)$ -DP.

**Theorem 2** (Privacy guarantee). For any  $0 < \epsilon, \delta < 1$ , Algorithm 2 is  $(\epsilon, \delta)$ -DP.

In the following, we will show the statistical guarantee for the models under Assumption 1, if the initial parameter  $\beta^0$  is closed to the underlying parameter  $\beta^*$  enough.

**Theorem 3** (Statistical guarantee of Algorithm 2). Let the parameter set  $\mathcal{B} = \{\beta : \|\beta - \beta^*\|_2 \leq R\}$  for  $R = \kappa \|\beta^*\|_2$  for some constant  $\kappa \in (0, 1)$ . Assume that Assumption 1 holds for parameters  $\gamma, \mathcal{B}, \mu, v, \tau$  satisfying the condition of  $1 - 2\frac{v-\gamma}{v+\mu} \in (0, 1)$ . Also, assume that  $\|\beta^0 - \beta^*\|_2 \leq \frac{R}{2}$ ,  $n$  is large enough so that

$$\tilde{\Omega}\left(\left(\frac{1}{v-\gamma}\right)^2 \frac{d^2 \tau T \log \frac{1}{\delta} \log \frac{1}{\zeta}}{\epsilon^2 R^2}\right) \leq n. \quad (17)$$

Then, with probability at least  $1 - 2T\zeta$ , we have, for all  $t \in [T]$ ,  $\beta^t \in \mathcal{B}$ . If it holds and if taking  $T = O\left(\frac{\mu+v}{v-\gamma} \log n\right)$  and  $\eta = \frac{2}{\mu+v}$ , we have

$$\|\beta^T - \beta^*\|_2 \leq \tilde{O}\left(R \sqrt{\frac{v+\mu}{(v-\gamma)^3} \frac{d \log \frac{1}{\delta} \log \frac{1}{\zeta} \sqrt{\tau}}{\sqrt{n\epsilon^2}}}\right), \quad (18)$$

where the  $\tilde{O}$ -term and  $\tilde{\Omega}$ -term omit  $\log d, \log n$  and other factors (see Appendix for the explicit form of the result).

**Remark 1.** There are several points that need to note. Firstly, the assumptions of the parameter set  $\beta$  and the initial parameter  $\beta^0$  are commonly used in other papers on statistical guarantees of (Gradient) EM algorithm such as (Balakrishnan et al. 2017a; Zhu et al. 2017; Wang et al. 2015). Even though Theorem 3 requires that the initial estimator be close enough to the optimal one, our experiments show that the algorithm actually performs quite well for any random initialization. Secondly, in (17) we need to assume that  $n \propto \frac{1}{R^2}$ , where  $R$  is the radius of  $\mathcal{B}$ . This is due to that in Algorithm 2, we need to keep each  $\beta^t \in \mathcal{B}$  under perturbation. When  $R$  is small, we have to let the noise be small enough, which means that  $n$  should be large enough. Finally, for specific models,  $R, v, \mu, \gamma$  are constants, this means that the error in (18) is  $\tilde{O}\left(\frac{d\sqrt{\tau}}{\sqrt{n\epsilon}}\right)$ . However, here  $\tau$  depends on the model, which may also depend on  $d$  and  $\|\beta^*\|_2$ .

### Implications for Some Specific Models

In this section, we apply our framework (*i.e.*, Algorithm 2) to the models mentioned in the Preliminaries section. To obtain results for these models, we only need to find the corresponding  $\mathcal{B}, \gamma, k, R, v, \mu, \tau$  to ensure that Assumption 1 and the assumptions in Theorem 3 hold. Due to the space limit, theoretical results of RMC are included in Appendix.

### Gaussian Mixture Model

The following lemma ensures the properties of Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ), smoothness, strongly concave and self-consistency for model (4).

**Lemma 3** ((Balakrishnan et al. 2017b; Yi and Caramanis 2015)). If  $\frac{\|\beta^*\|_2}{\sigma} \geq r$ , where  $r$  is a sufficiently large constant denoting the minimum signal-to-noise ratio (SNR), then there exists an absolute constant  $C > 0$  such that the properties of self-consistent, Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ),  $\mu$ -smoothness and  $v$ -strongly concave hold for function  $Q(\cdot, \cdot)$  with  $\gamma = \exp(-Cr^2)$ ,  $\mu = v = 1$ ,  $R = k\|\beta^*\|_2$ ,  $k = \frac{1}{4}$ , and  $\mathcal{B} = \{\beta : \|\beta - \beta^*\|_2 \leq R\}$ .

We can show the following second-order moment bound for  $\nabla_j q(\beta, \beta)$ .

**Lemma 4.** With the same notations as in Lemma 3, for each  $\beta \in \mathcal{B}$ , the  $j$ -the coordinate of  $\nabla q(\beta; \beta)$  (*i.e.*,  $\nabla_j q(\beta; \beta)$ ) satisfies the following inequality

$$\mathbb{E}_y (\nabla_j q(\beta; \beta))^2 \leq O((\|\beta^*\|_\infty^2 + \sigma^2)).$$

Also, for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n]$ , is independent with others.

Combining with Lemma 3, 4 and Theorem 3 we have the following statistical guarantee for GMM.

**Theorem 4.** With the same notations as in Lemma 3, in Algorithm 2 assume that  $\|\beta^0 - \beta^*\|_2 \leq \frac{1}{8} \|\beta^*\|_2$  and  $n$  is large enough so that

$$\tilde{\Omega}\left(\frac{d^2 \sqrt{\|\beta^*\|_\infty^2 + \sigma^2} \log \frac{1}{\delta} \log \frac{1}{\zeta}}{\epsilon^2 \|\beta^*\|_2^2}\right) \leq n. \quad (19)$$

Moreover, if take  $T = O(\log n)$  and  $\eta = O(1)$ , then we have with probability at least  $1 - 2T\zeta$

$$\|\beta^T - \beta^*\|_2 \leq \tilde{O}\left(\|\beta^*\|_2 \frac{d \log \frac{1}{\delta} \log \frac{1}{\zeta} \sqrt{\|\beta^*\|_\infty^2 + \sigma^2}}{\sqrt{n\epsilon^2}}\right), \quad (20)$$

where the  $\tilde{O}, \tilde{\Omega}$  terms omit logarithmic and other factors.

**Remark 2.** Note that if we assume that  $\sigma, \|\beta^*\|_2 = O(1)$ , then the error in (20) is upper bounded by  $\tilde{O}\left(\frac{d}{\sqrt{n\epsilon}}\right)$ . This means that to achieve the error of  $\alpha \in (0, 1)$ , the sample complexity is  $\tilde{O}\left(\frac{d^2}{\alpha^2 \zeta}\right)$ . It is notable that for GMM, the near optimal rate is  $\tilde{O}(d^2(\frac{1}{\alpha^2} + \frac{1}{\alpha\epsilon}))$  (Kamath et al. 2019).<sup>3</sup> Thus when  $\epsilon$  is some constant, our result matches their near optimal rate. However, as mentioned in previous section, their algorithm is too complicated to be practical and it is difficult to extend their method to other Mixture models.

### Mixture of Regressions Model

**Lemma 5** ((Balakrishnan et al. 2017b; Yi and Caramanis 2015)). If  $\frac{\|\beta^*\|_2}{\sigma} \geq r$ , where  $r$  is a sufficiently large constant denoting the required minimal signal-to-noise ratio (SNR), then function  $Q(\cdot, \cdot)$  of the Mixture of Regressions Model has the properties of self-consistent, Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ),  $\mu$ -smoothness, and  $v$ -strongly with  $\gamma \in (0, \frac{1}{4})$ ,  $\mu = v = 1$ ,  $\mathcal{B} = \{\beta : \|\beta - \beta^*\|_2 \leq R\}$ ,  $R = k\|\beta^*\|_2$ , and  $k = \frac{1}{32}$ .

<sup>3</sup>Note that although (Kamath et al. 2019) used TV distance, while we use the Euclidean distance, we can easily transfer our result to a result based on TV distance via Pinsker's inequality and the KL diatance between two Gaussian distributions.

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**Algorithm 2** DP Gradient EM Algorithm

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**Input:**  $D = \{y_i\}_{i=1}^n \subset \mathbb{R}^d$ , privacy parameters  $\epsilon, \delta, Q(\cdot; \cdot)$  and its  $q_i(\cdot; \cdot)$ , initial parameter  $\beta^0 \in \mathcal{B}$ ,  $\tau$  which satisfies Assumption 1, the number of iterations  $T$  (to be specified later), step size  $\eta$  and failure probability  $\zeta > 0$ .

1: Let  $\tilde{\epsilon} = \sqrt{\log \frac{1}{\delta} + \epsilon} - \sqrt{\log \frac{1}{\delta}}$ ,  $s = \sqrt{\frac{n\tau}{2\log \frac{d}{\zeta}}}$ ,  $\beta = \log \frac{d}{\zeta}$ .

2: **for**  $t = 1, 2, \dots, T$  **do**

3:     For each  $j \in [d]$ , calculate the robust gradient by (8)-(12) and add Gaussian noise, that is

$$g_j^{t-1}(\beta^{t-1}) = \frac{1}{n} \sum_{i=1}^n \left( \nabla_j q_i(\beta^{t-1}, \beta^{t-1}) \left( 1 - \frac{\nabla_j^2 q_i(\beta^{t-1}, \beta^{t-1})}{2s^2 \beta} \right) - \frac{\nabla_j^3 q_i(\beta^{t-1}, \beta^{t-1})}{6s^2} \right) + \frac{s}{n} \sum_{i=1}^n C \left( \frac{\nabla_j q_i(\beta^{t-1}, \beta^{t-1})}{s}, \frac{|\nabla_j q_i(\beta^{t-1}, \beta^{t-1})|}{s\sqrt{\beta}} \right) + Z_j^{t-1}, \quad (16)$$

where  $Z_j^{t-1} \sim \mathcal{N}(0, \sigma^2)$  with  $\sigma^2 = \frac{8\tau d T}{9\beta n \epsilon^2}$ .

4:     Let vector  $\tilde{\nabla}Q_n(\beta^{t-1}) \in \mathbb{R}^d$  denote  $\tilde{\nabla}Q_n(\beta^{t-1}) = (g_1^{t-1}(\beta^{t-1}), g_2^{t-1}(\beta^{t-1}), \dots, g_d^{t-1}(\beta^{t-1}))$ .

5:     Update  $\beta^t = \beta^{t-1} + \eta \tilde{\nabla}Q_n(\beta^{t-1})$ .

6: **end for**

---

**Lemma 6.** With the same notations as in Lemma 5, for each  $\beta \in \mathcal{B}$ , the  $j$ -the coordinate of  $\nabla q_i(\beta; \beta)$ , i.e.,  $\nabla_j q_i(\beta; \beta)$  satisfies the following inequality

$$\mathbb{E}_y (\nabla_j q_i(\beta; \beta))^2 \leq O(\max\{(\|\beta^*\|_2^2 + \sigma^2)^2, d\|\beta^*\|_2^2\}).$$

Also, for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$  is independent with others for  $i \in [n]$ .

**Theorem 5.** With the same notations as in Lemma 5, in Algorithm 2 assume that  $\|\beta^0 - \beta^*\|_2 \leq \frac{1}{64}\|\beta^*\|_2$  and  $n$  is large enough so that

$$\tilde{\Omega}\left(\frac{d^2 \max\{(\|\beta^*\|_2^2 + \sigma^2)^2, d\|\beta^*\|_2^2\} \log \frac{1}{\delta} \log \frac{1}{\zeta}}{\epsilon^2 \|\beta^*\|_2^2}\right) \leq n.$$

Moreover, if take  $T = O(\log n)$  and  $\eta = O(1)$ , then we have, with probability at least  $1 - 2T\zeta$ ,

$$\|\beta^T - \beta^*\|_2 \leq \tilde{O}\left(\frac{d\|\beta^*\|_2 \log \frac{1}{\delta} \sqrt{\max\{(\|\beta^*\|_2^2 + \sigma^2)^2, d\|\beta^*\|_2^2\}}}{\sqrt{n\epsilon^2}}\right), \quad (21)$$

where the  $\tilde{O}$ -term and  $\tilde{\Omega}$ -term omit logarithmic factors.

**Remark 3.** If we assume that  $\|\beta^*\|$  and  $\sigma = O(1)$ , then the error in (21) is upper bounded by  $\tilde{O}(\frac{d^{\frac{3}{2}}}{\sqrt{n\epsilon}})$ , which has an additional factor of  $\sqrt{d}$  compared with the bound in (20) for GMM. We note that this is the first statistical result for MRM in the DP model.

## Experiments

In this section, we evaluate the performance of Algorithm 2 on three canonical models: GMM, MRM, and RMC. Since in the paper we mainly focus on the statistical setting and its theoretical behaviors, we only evaluate our algorithm on the synthetic data. Note that previous papers on the statistical guarantees of EM algorithm all evaluating their algorithms on synthetic data only such as (Balakrishnan et al. 2017b; Yi and Caramanis 2015; Zhu et al. 2017). Thus, evaluating experiments on synthetic data only is sufficient and reasonable

for the paper. Note that due to space limit, we refer readers to Appendix for more experimental results.

**Baseline Methods** We compare our approach against two baseline algorithms. One is the Gradient EM algorithm (Balakrishnan et al. 2017b), namely, EM, as our non-private baseline method. The other is clipped DP Gradient EM (Algorithm 1), namely, clipped, as our private baseline method.

**Experimental Settings** For each of these models, we generate synthesized datasets according to the underlying distribution. We also utilize  $\|\beta - \beta^*\|_2$  to measure the estimation error. Instead of choosing the initial parameter  $\beta^0$  that is close to the optimal one, we consider random initialization. As we will see later, even if we select random initial parameter, the performance of our private estimator is good enough. We set signal-to-noise ratio  $\frac{\|\beta^*\|_2}{\sigma} = 3$ . For the privacy parameters, we choose  $\epsilon = \{0.2, 0.5, 1\}$  and  $\delta = \frac{1}{n}$ .

**Experimental Results** Firstly, we will show that the performance of Algorithm 1 is heavily affected by the clipping threshold  $C$ . As shown in Figure 1, we conduct the algorithm on three canonical models with fixed data size  $n$ , dimension data  $d$ , and privacy budget  $\epsilon$ . If  $C$  is set to be a small value (e.g., 0.1), it significantly reduces the adding noise in each iteration but at the same time it leads much information loss in gradient estimation. Conversely, if  $C$  is set too high (e.g., 5 or 10), the noise variance becomes high, resulting in introducing too much noise to the estimation. Thus, selecting the optimal  $C$  is quite difficult since too large or too small values of  $C$  has a negative effect on the performance of Algorithm 1. Even for  $C = 1$  that achieves lowest estimation error among other threshold values, the estimation error does not decay as the number of iterations increases, whereas under the same privacy guarantee, our proposed algorithm achieves the same convergence behavior as EM, and thoroughly outperforms Algorithm 1. For fair comparison, we thus fixed  $C = 1$  for Algorithm 1 in the following experiments.

In Figure 2, 3 and 4, we test how the privacy budget  $\epsilon$ , data dimension  $d$  and data size  $n$  affect the estimation error

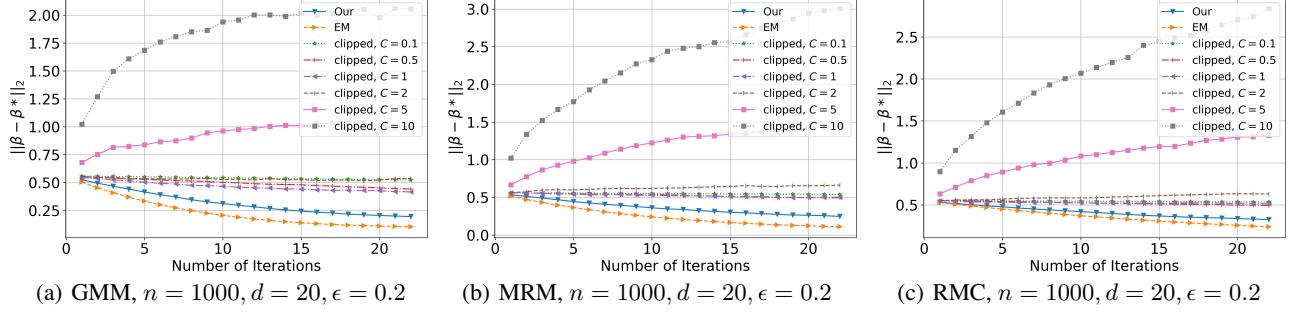


Figure 1: Estimation error of Algorithm 1 (clipped) v.s. iteration  $t$  under different clipping threshold  $C$

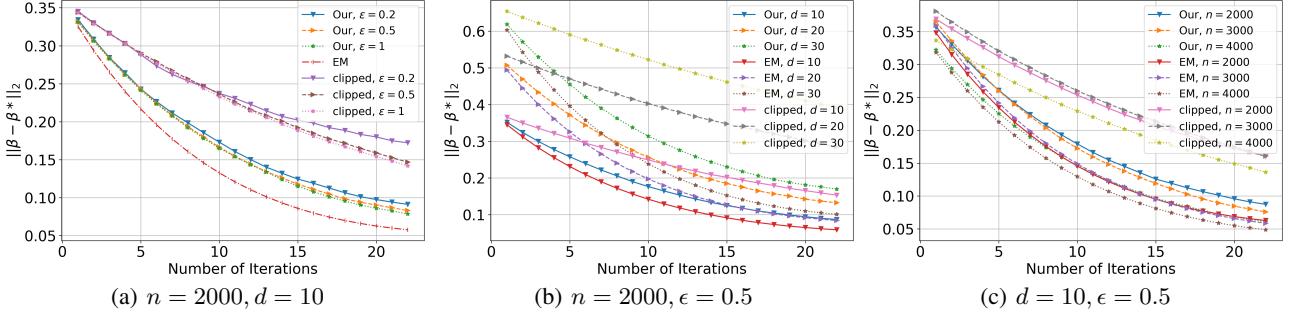


Figure 2: Estimation error of GMM w.r.t privacy budget  $\epsilon$ , data dimension  $d$ , data size  $n$  and iteration  $t$

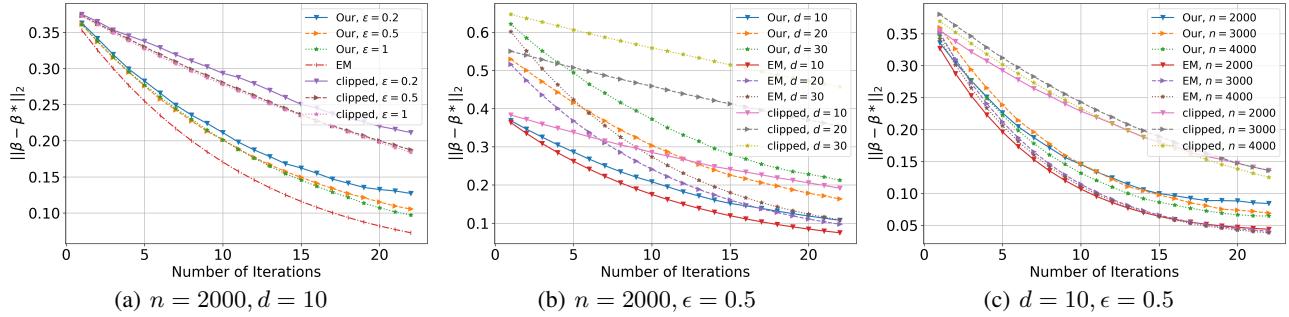


Figure 3: Estimation error of MRM w.r.t privacy budget  $\epsilon$ , data dimension  $d$ , data size  $n$  and iteration  $t$

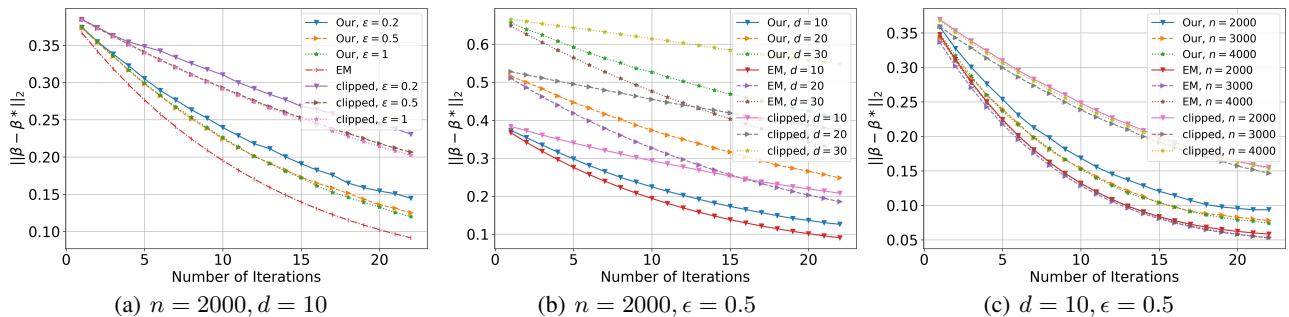


Figure 4: Estimation error of RMC w.r.t privacy budget  $\epsilon$ , data dimension  $d$ , data size  $n$  and iteration  $t$

$\|\beta - \beta^*\|_2$  of all algorithms on three canonical models over iteration  $t$ . We can see that the estimation error of our proposed algorithm in each of the three models decreases when  $\epsilon$  increases,  $d$  decreases or  $n$  increases, which are consistent with our theoretical results. In these figures, our algorithm exhibits nearly the same convergence behavior as the non-private baseline method and outperforms Algorithm 1.

## References

- Abadi, M.; Chu, A.; Goodfellow, I.; McMahan, H. B.; Mironov, I.; Talwar, K.; and Zhang, L. 2016. Deep learning with differential privacy. In *CCS*, 308–318.
- Balakrishnan, S.; Du, S. S.; Li, J.; and Singh, A. 2017a. Computationally efficient robust sparse estimation in high dimensions. In *Conference on Learning Theory*, 169–212.
- Balakrishnan, S.; Wainwright, M. J.; Yu, B.; et al. 2017b. Statistical guarantees for the EM algorithm: From population to sample-based analysis. *The Annals of Statistics* 45(1): 77–120.
- Bassily, R.; Smith, A.; and Thakurta, A. 2014. Private empirical risk minimization: Efficient algorithms and tight error bounds. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, 464–473. IEEE.
- Boucheron, S.; Lugosi, G.; and Massart, P. 2013. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press.
- Bun, M.; and Steinke, T. 2016. Concentrated differential privacy: Simplifications, extensions, and lower bounds. In *Theory of Cryptography Conference*, 635–658. Springer.
- Bun, M.; Ullman, J.; and Vadhan, S. 2018. Fingerprinting codes and the price of approximate differential privacy. *SIAM Journal on Computing* 47(5): 1888–1938.
- Catoni, O.; and Giulini, I. 2017. Dimension-free PAC-Bayesian bounds for matrices, vectors, and linear least squares regression. *arXiv preprint arXiv:1712.02747*.
- Dawid, A. P.; and Skene, A. M. 1979. Maximum likelihood estimation of observer error-rates using the EM algorithm. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 28(1): 20–28.
- Dwork, C.; McSherry, F.; Nissim, K.; and Smith, A. 2006. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography conference*, 265–284. Springer.
- Dwork, C.; Roth, A.; et al. 2014. The algorithmic foundations of differential privacy. *Foundations and Trends in Theoretical Computer Science* 9(3-4): 211–407.
- Faria, S.; and Gonçalves, F. 2013. Financial data modeling by Poisson mixture regression. *Journal of Applied Statistics* 40(10): 2150–2162.
- Holland, M. J. 2019. Robust descent using smoothed multiplicative noise. In *22nd International Conference on Artificial Intelligence and Statistics (AISTATS)*, volume 89 of *Proceedings of Machine Learning Research*, 703–711.
- Kamath, G.; Sheffet, O.; Singhal, V.; and Ullman, J. 2019. Differentially private algorithms for learning mixtures of separated gaussians. In *Advances in Neural Information Processing Systems*, 168–180.
- Laird, N. M. 2010. The EM algorithm in genetics, genomics and public health. *Statistical Science* 450–457.
- Lee, J.; and Kifer, D. 2018. Concentrated differentially private gradient descent with adaptive per-iteration privacy budget. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, 1656–1665.
- Loh, P.-L.; and Wainwright, M. J. 2011. High-dimensional regression with noisy and missing data: Provable guarantees with non-convexity. In *Advances in Neural Information Processing Systems*, 2726–2734.
- McLachlan, G.; and Krishnan, T. 2007. *The EM algorithm and extensions*, volume 382. John Wiley & Sons.
- Nesterov, Y. 2013. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media.
- Nissim, K.; Raskhodnikova, S.; and Smith, A. 2007. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, 75–84.
- Park, M.; Foulds, J.; Choudhary, K.; and Welling, M. 2017. DP-EM: Differentially Private Expectation Maximization. In *Artificial Intelligence and Statistics*, 896–904.
- Song, S.; Chaudhuri, K.; and Sarwate, A. D. 2013. Stochastic gradient descent with differentially private updates. In *2013 IEEE Global Conference on Signal and Information Processing*, 245–248. IEEE.
- Song, S.; Thakkar, O.; and Thakurta, A. 2020. Characterizing Private Clipped Gradient Descent on Convex Generalized Linear Problems. *arXiv preprint arXiv:2006.06783*.
- Vershynin, R. 2010. Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*.
- Wang, D.; Xiao, H.; Devadas, S.; and Xu, J. 2020. On Differentially Private Stochastic Optimization with Heavy-tailed Data. In *Proceedings of the 37th International Conference on Machine Learning, ICML 2020, 12-18 July 2020, Virtual Conference*.
- Wang, D.; and Xu, J. 2019. Differentially private empirical risk minimization with smooth non-convex loss functions: A non-stationary view. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, 1182–1189.
- Wang, D.; Ye, M.; and Xu, J. 2017. Differentially private empirical risk minimization revisited: Faster and more general. In *Advances in Neural Information Processing Systems*, 2722–2731.
- Wang, Z.; Gu, Q.; Ning, Y.; and Liu, H. 2015. High dimensional em algorithm: Statistical optimization and asymptotic normality. In *Advances in neural information processing systems*, 2521–2529.
- Wu, C. J.; et al. 1983. On the convergence properties of the EM algorithm. *The Annals of statistics* 11(1): 95–103.

Yi, X.; and Caramanis, C. 2015. Regularized em algorithms: A unified framework and statistical guarantees. In *Advances in Neural Information Processing Systems*, 1567–1575.

Zhu, R.; Wang, L.; Zhai, C.; and Gu, Q. 2017. High-dimensional variance-reduced stochastic gradient expectation-maximization algorithm. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, 4180–4188. JMLR.org.

## Appendix

### Implication to Linear Regression with Missing Covariates

**Lemma 7** ((Balakrishnan et al. 2017b; Yi and Caramanis 2015)). If  $\frac{\|\beta^*\|_2}{\sigma} \leq r$  and  $p_m < \frac{1}{1+2b+2b^2}$ , where  $r$  is a constant denoting the required maximum signal-to-noise ratio (SNR) and  $b = r^2(1+k)^2$  for some constant  $k \in (0, 1)$ , then function  $Q(\cdot; \cdot)$  of the linear regression with missing covariates has the properties of self-consistent, Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ),  $\mu$ -smoothness and  $v$ -strongly with

$$\gamma = \frac{b + p_m(1 + 2b + 2b^2)}{1 + b} < 1, \mu = v = 1,$$

$$\mathcal{B} = \{\beta : \|\beta - \beta^*\|_2 \leq R\}, \text{ where } R = k\|\beta^*\|_2.$$

We can show the following the second-order moment bound for  $\nabla_j q(\beta, \beta)$ .

**Lemma 8.** With the same assumptions as in Lemma 7, for each  $\beta \in \mathcal{B}$  and  $j \in [d]$ ,  $\nabla_j q(\beta; \beta)$  satisfies

$$\mathbb{E}(\nabla_j q(\beta; \beta))^2 \leq O((\sqrt{d}\|\beta^*\|_2 + \sigma^2 + \|\beta^*\|_2^2)^2). \quad (22)$$

Also, for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n]$ , is independent with others.

Combining with Lemma 7, 8 and Theorem 3 we have the following statistical guarantee for RMC.

**Theorem 6.** With the same notations as in Lemma 7, in Algorithm 2 assume  $\|\beta^0 - \beta^*\|_2 \leq \frac{k}{2}\|\beta^*\|_2$ ,  $n$  is large enough so that

$$\tilde{\Omega}\left(\frac{d^2(\sqrt{d}\|\beta^*\|_2 + \sigma^2 + \|\beta^*\|_2^2)^2 \log \frac{1}{\delta} \log \frac{1}{\zeta}}{\epsilon^2 \|\beta^*\|_2^2}\right) \leq n.$$

Moreover, if we take  $T = O(\log n)$  and  $\eta = O(1)$ , then we have, with probability at least  $1 - 2T\zeta$ ,

$$\|\beta^T - \beta^*\|_2 \leq \tilde{O}\left(\frac{d \log \frac{1}{\delta} \log \frac{1}{\zeta} \|\beta^*\|_2 (\sqrt{d}\|\beta^*\|_2 + \sigma^2 + \|\beta^*\|_2^2)}{\sqrt{n\epsilon^2}}\right),$$

where the  $\tilde{O}, \tilde{\Omega}$  terms omit logarithmic and other factors.

Note that unlike the previous two models, we assume here that SNR is upper bounded by some constant which is unavoidable as pointed out in (Loh and Wainwright 2011).

## Background

First, we will recall some definitions and lemmas on the sub-exponential and sub-Gaussian random variables. See (Vershynin 2010) for details.

**Definition 7.** For a sub-exponential random vector  $X$ , its sub-exponential norm  $\|X\|_{\psi_1}$  is defined as

$$\|X\|_{\psi_1} = \sup_{p \geq 1} p^{-1} (\mathbb{E}|X|^p)^{\frac{1}{p}}.$$

**Definition 8** ( $\xi$ -sub-exponential). A random variable  $X$  with mean  $\mathbb{E}(X)$  is  $\xi$ -sub-exponential for  $\xi > 0$  if for all  $|t| < \frac{1}{\xi}$ ,  $\mathbb{E}\{\exp(t[X - \mathbb{E}(X)])\} \leq \exp(\frac{\xi^2 t^2}{2})$ .

**Lemma 9.** Let  $X$  be a sub-exponential random variable, then there are absolute constants  $C, c > 0$ , such that when  $|t| \leq \frac{c}{\|X\|_{\psi_1}}$ ,

$$\mathbb{E}[\exp(tX)] \leq \exp(Ct^2\|X\|_{\psi_1}^2).$$

**Lemma 10.** From Definition 7, 8 we can see that for a zero-mean sub-exponential random variable  $X$ , its second-order moment is bounded, i.e.,  $\mathbb{E}X^2 \leq O(\|X\|_{\psi_1}^2)$ .

**Lemma 11** (Bernstein's inequality). Let  $X_1, \dots, X_n$  be  $n$  i.i.d realizations of  $v$ -sub-exponential random variable  $X$  with mean  $\mu$ . Then,

$$\Pr\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq t\right) \leq 2 \exp\left(-n \min\left(-\frac{t^2}{v^2}, \frac{t}{2v}\right)\right).$$

**Definition 9.** A random variable  $X$  is sub-Gaussian with variance  $\sigma^2$  if for all  $t > 0$ , the following holds

$$\Pr(|X - \mathbb{E}X| \geq t) \leq 2 \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

**Definition 10.** For a sub-Gaussian random variable  $X$ , its sub-Gaussian norm  $\|X\|_{\psi_2}$  is defined as

$$\|X\|_{\psi_2} = \sup_{p \geq 1} p^{-\frac{1}{2}} (\mathbb{E}|X|^p)^{\frac{1}{p}}.$$

**Lemma 12.** If  $X$  is sub-Gaussian or sub-exponential, then  $\|X - \mathbb{E}X\|_{\psi_2} \leq 2\|X\|_{\psi_2}$  or  $\|X - \mathbb{E}X\|_{\psi_1} \leq 2\|X\|_{\psi_1}$  holds, respectively.

**Lemma 13.** For two sub-Gaussian random variables  $X_1, X_2, X_1 \cdot X_2$  is a sub-exponential random variable with

$$\|X_1 \cdot X_2\|_{\psi_1} \leq C \max\{\|X_1\|_{\psi_2}^2, \|X_2\|_{\psi_2}^2\}.$$

**Lemma 14.** Let  $X_1, X_2, \dots, X_k$  be  $k$  independent zero-mean sub-Gaussian random variables, and  $X = \sum_{j=1}^k X_j$ . Then,  $X$  is sub-Gaussian with  $\|X\|_{\psi_2}^2 \leq C \sum_{j=1}^k \|X_j\|_{\psi_2}^2$  for some absolute constant  $C > 0$ .

Next, we provide some symmetrization results of random variables, which will be used in our proofs. See (Boucheron, Lugosi, and Massart 2013) for details.

**Lemma 15.** Let  $y_1, y_2, \dots, y_n$  be the  $n$  independent realizations of the random vector  $Y \in \mathcal{Y}$ , and  $\mathcal{F}$  be a function class defined on  $\mathcal{Y}$ . For any increasing convex function  $\phi(\cdot)$ , the following holds

$$\mathbb{E}\{\phi[\sup_{f \in \mathcal{F}} |\sum_{i=1}^n f(y_i) - \mathbb{E}(f(Y))|]\} \leq \mathbb{E}\{\phi[\sup_{f \in \mathcal{F}} |\sum_{i=1}^n \epsilon_i f(y_i)|]\},$$

where  $\epsilon_1, \dots, \epsilon_n$  are i.i.d Rademacher random variables that are independent of  $y_1, \dots, y_n$ .

**Lemma 16.** Let  $y_1, \dots, y_n$  be  $n$  independent realization of the random vector  $Z \in \mathcal{Z}$  and  $\mathcal{F}$  be a function class defined on  $\mathcal{Z}$ . If Lipschitz functions  $\{\phi_i(\cdot)\}_{i=1}^n$  satisfy the following for all  $v, v' \in \mathbb{R}$

$$|\phi_i(v) - \phi_i(v')| \leq L|v - v'|$$

and  $\phi_i(0) = 0$ , then for any increasing convex function  $\phi(\cdot)$ , the following holds

$$\mathbb{E}\{\phi[\sup_{f \in \mathcal{F}} \sum_{i=1}^n \epsilon_i \phi_i(f(y_i))]\} \leq \mathbb{E}\{\phi[2L \sup_{f \in \mathcal{F}} \sum_{i=1}^n \epsilon_i f(y_i)]\},$$

where  $\epsilon_1, \dots, \epsilon_n$  are i.i.d Rademacher random variables that are independent of  $y_1, \dots, y_n$ .

## Implication for Latent Variable Models

**Gaussian Mixture Model** We have

$$\nabla q(\beta; \beta) = [2w_\beta(y) - 1] \cdot y - \beta, \quad (23)$$

where  $w_\beta(y) = \frac{1}{1 + \exp(-\langle \beta, y \rangle / \sigma^2)}$ .

**Mixture of Regression Model** In this case, we have

$$\nabla q(\beta; \beta) = (2w_\beta(x, y) - 1) \cdot y \cdot x - xx^T \cdot \beta, \quad (24)$$

where  $w_\beta(x, y) = \frac{1}{1 + \exp(-y\langle \beta, x \rangle / \sigma^2)}$ .

**Linear Regression with Missing Covariates** In this case, we have

$$\nabla q(\beta; \beta) = y \cdot m_\beta(x^{\text{obs}}, y) - K_\beta(x^{\text{obs}}, y)\beta, \quad (25)$$

where the functions  $m_\beta(x^{\text{obs}}, y) \in \mathbb{R}^d$  and  $K_\beta(x^{\text{obs}}, y) \in \mathbb{R}^{d \times d}$  are defined as:

$$m_\beta(x^{\text{obs}}, y) = z \odot x + \frac{y - \langle \beta, z \odot x \rangle}{\sigma^2 + \|(1-z) \odot \beta\|_2^2} (1-z) \odot \beta \quad (26)$$

and

$$K_\beta(x^{\text{obs}}, y) = \text{diag}(1-z) + m_\beta(x^{\text{obs}}, y) \cdot [m_\beta(x^{\text{obs}}, y)]^T - [(1-z) \odot m_\beta(x^{\text{obs}}, y)] \cdot [(1-z) \odot m_\beta(x^{\text{obs}}, y)]^T, \quad (27)$$

where vector  $z \in \mathbb{R}^d$  is defined as  $z_j = 1$  if  $x_j$  is observed and  $z_j = 0$  is  $x_j$  is missing, and  $\odot$  denotes the Hadamard product of matrices.

## Explicit Form of $C(a, b)$ in (9)

We first define the following notations:

$$\begin{aligned} V_- &:= \frac{\sqrt{2}-a}{b}, V_+ = \frac{\sqrt{2}+a}{b} \\ F_- &:= \Phi(-V_-), F_+ := \Phi(-V_+) \\ E_- &:= \exp\left(-\frac{V_-^2}{2}\right), E_+ := \exp\left(-\frac{V_+^2}{2}\right), \end{aligned}$$

where  $\Phi$  denotes the CDF of the standard Gaussian distribution. Then

$$C(a, b) = T_1 + T_2 + \cdots + T_5,$$

where

$$\begin{aligned} T_1 &:= \frac{2\sqrt{2}}{3}(F_- - F_+) \\ T_2 &:= -(a - \frac{a^3}{6})(F_- + F_+) \\ T_3 &:= \frac{b}{\sqrt{2\pi}}(1 - \frac{a^2}{2})(E_+ - E_-) \\ T_4 &:= \frac{ab^2}{2} \left( F_+ + F_- + \frac{1}{\sqrt{2\pi}}(V_+ E_+ + V_- E_-) \right) \\ T_5 &:= \frac{b^3}{6\sqrt{2\pi}} ((2 + V_-^2)E_- - (2 + V_+^2)E_+). \end{aligned}$$

## Proof of Theorem 1

**Proof of Theorem 1.** Note that by (23), we have

$$\nabla q(\beta; \beta) = [\frac{2}{1 + \exp(-\langle \beta, y \rangle / \sigma^2)} - 1] \cdot y - \beta.$$

W.l.o.g, we assume that  $\beta = (1, 0, \dots, 0)^T$  and  $\sigma = 1$  in the GMM model. Then, we can see that for each constant  $c \geq 0$ , if

$$\begin{aligned} \|\frac{y}{3}\|_2 &\geq c + \|\beta\|_2 \\ \langle \beta, y \rangle &\geq \ln 2 \\ y &\geq 0 \end{aligned}$$

and denote the set of  $y$  satisfying the above assumptions as  $\mathcal{S}$ , we have

$$\|\nabla q(\beta; \beta)\|_2 \geq \|\frac{y}{3}\|_2 - \|\beta\|_2 \geq c.$$

The above assumptions hold if  $y = (\ln 2 + 1, 3s, a_3, a_4, \dots, a_d)$ , where  $s \geq c$  and  $a_3, \dots, a_d \geq 0$ . We can easily see that  $\mathbb{P}[y \in \mathcal{S}] > 0$  since  $y$  follows a mixture of Gaussian distributions.  $\square$

## Proof of Theorem 2

**Proof of Theorem 2.** We first give the definition of zCDP in (Bun and Steinke 2016).

**Definition 11.** A randomized algorithm  $\mathcal{A} : \mathcal{X}^n \mapsto \mathcal{Y}$  is  $\rho$ -zero Concentrated Differentially Private (zCDP) if for all neighboring datasets  $D \sim D'$  and all  $\alpha \in (1, \infty)$ ,

$$D_\alpha(\mathcal{A}(D) \| \mathcal{A}(D')) \leq \rho\alpha,$$

where  $D_\alpha(P \| Q) = \frac{1}{\alpha-1} \log \mathbb{E}_{X \sim P}[(\frac{P(X)}{Q(X)})^{\alpha-1}]$  denotes the Rényi divergence of order  $\alpha$ .

We first convert  $(\epsilon, \delta)$ -DP to  $\rho$ -zCDP by using the following lemma

**Lemma 17** ((Bun and Steinke 2016)). Let  $M : \mathcal{X}^n \mapsto \mathcal{Y}$  be a randomized algorithm. If  $M$  is  $\rho$ -zCDP, it is  $(\rho + 2\sqrt{\rho \log \frac{1}{\delta}}, \delta)$ -DP for all  $\delta > 0$ .

Thus, it suffices to show that Algorithm 2 is  $\tilde{\epsilon}^2 = (\sqrt{\epsilon + \log \frac{1}{\delta}} - \sqrt{\log \frac{1}{\delta}})^2$ -zCDP. The following lemma shows that adding some Gaussian noise will preserve zCDP.

**Lemma 18.** Given a function  $q : \mathcal{X}^n \rightarrow \mathbb{R}^p$ , the Gaussian Mechanism is defined as:  $\mathcal{M}_G(D, q, \epsilon) = q(D) + Y$ , where  $Y$  is drawn from a Gaussian Distribution  $\mathcal{N}(0, \sigma^2 I_p)$  is  $\frac{\Delta_2(q)}{2\sigma^2}$ -zCDP.  $\Delta_2(q)$  is the  $\ell_2$ -sensitivity of the function  $q$ , i.e.,  $\Delta_2(q) = \sup_{D \sim D'} \|q(D) - q(D')\|_2$ .

By Lemma 2 we know  $\Delta_2(g_j^{t-1}(\beta^{t-1})) = \frac{4\sqrt{2}}{3} \frac{s}{n}$ . By simple calculation we can show that in each iteration and each coordinate, outputting  $g_j^{t-1}(\beta^{t-1})$  will be  $\frac{\tilde{\epsilon}^2}{dT}$ -zCDP. Thus by the composition property of zCDP, we know that it is  $\tilde{\epsilon}^2$ -zCDP.  $\square$

### Proof of Theorem 3

**Proof of Theorem 3.** Consider  $t$ -th iteration, under the assumption that  $\beta^{t-1} \in \mathcal{B}$  we have

$$\begin{aligned} \|\beta^t - \beta^*\|_2 &= \|\beta^{t-1} + \eta \tilde{\nabla} Q_n(\beta^{t-1}) - \beta^*\|_2 \\ &\leq \|\beta^{t-1} + \eta \nabla Q(\beta^{t-1}; \beta^{t-1}) - \beta^*\|_2 + \eta \|\tilde{\nabla} Q_n(\beta^{t-1}) - \nabla Q(\beta^{t-1}; \beta^{t-1})\|_2 \end{aligned} \quad (28)$$

We first bound the first term of (28).

$$\|\beta^{t-1} + \eta \nabla Q(\beta^{t-1}; \beta^{t-1}) - \beta^*\|_2 \leq \|\beta^{t-1} + \eta \nabla Q(\beta^{t-1}; \beta^*) - \beta^*\|_2 + \eta \|\nabla Q(\beta^{t-1}; \beta^{t-1}) - \nabla Q(\beta^{t-1}; \beta^*)\|_2 \quad (29)$$

We then consider the first term of (29). We note that the self-consistent property in Definition 1 implies that

$$\beta^* = \arg \max_{\beta} Q(\beta; \beta^*), \quad (30)$$

which means that  $\beta^*$  is a maximizer of  $Q(\beta; \beta^*)$ . Thus, the proof follows from the convergence rate of the strongly convex and smooth functions  $Q(\beta; \beta^*)$  in (Nesterov 2013). For the step size  $\eta = \frac{2}{\mu+v}$ , we have

$$\|\beta^{t-1} + \eta \nabla Q(\beta^{t-1}; \beta^*) - \beta^*\|_2 \leq \left( \frac{\mu - v}{\mu + v} \right) \|\beta^{t-1} - \beta^*\|_2. \quad (31)$$

Thus, by the Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ) condition, we get the following of (29)

$$\begin{aligned} \|\beta^{t-1} + \eta \nabla Q(\beta^{t-1}; \beta^{t-1}) - \beta^*\|_2 &\leq \|\beta^{t-1} + \eta \nabla Q(\beta^{t-1}; \beta^*) - \beta^*\|_2 + \eta \|\nabla Q(\beta^{t-1}; \beta^{t-1}) - \nabla Q(\beta^{t-1}; \beta^*)\|_2 \\ &\leq \left( \frac{\mu - v}{\mu + v} \right) \|\beta^{t-1} - \beta^*\|_2 + \eta \gamma \|\beta^{t-1} - \beta^*\|_2 \\ &= (1 - 2 \frac{v - \gamma}{\mu + v}) \|\beta^{t-1} - \beta^*\|_2 \end{aligned} \quad (32)$$

where the the last inequality is due to taking  $\eta = \frac{2}{\mu+v}$ .

Next we bound the second term of (28). For convenience we denote the first sum of (16) (i.e., the robust mean estimator ) as  $\tilde{g}_j^{t-1}(\beta^{t-1})$ . So we have

$$\|\tilde{\nabla} Q_n(\beta^{t-1}) - \nabla Q(\beta^{t-1}; \beta^{t-1})\|_2^2 = \sum_{j=1}^d (g_j^{t-1}(\beta^{t-1}) - \mathbb{E} \nabla_j q(\beta^{t-1}; \beta^{t-1}))^2 \quad (33)$$

$$\leq \sum_{j=1}^d (\tilde{g}_j^{t-1}(\beta^{t-1}) - \mathbb{E} \nabla_j q(\beta^{t-1}; \beta^{t-1}))^2 + \sum_{j=1}^d |Z_j^{t-1}|^2 \quad (34)$$

The first equality is due to Assumption 1. For the second term of (34), by the high probability concentration bound of Gaussian random variable we have for fixed  $j$  with probability at least  $1 - \frac{\zeta}{d}$ ,  $|Z_j^{t-1}|^2 \leq \frac{8\tau d T \log \frac{d}{\zeta}}{9\beta n \tilde{\epsilon}^2}$ . Thus with probability at least  $1 - \zeta$  we have

$$\sum_{j=1}^d |Z_j^{t-1}|^2 \leq \frac{8\tau d^2 T \log \frac{d}{\zeta}}{9\beta n \tilde{\epsilon}^2}.$$

For the first term of (34), by Lemma 1 and taking  $\zeta = \frac{\zeta}{d}$ , we have for a fixed  $j \in [d]$ ,  $(\tilde{g}_j^{t-1}(\beta^{t-1}) - \mathbb{E} \nabla_j q(\beta^{t-1}; \beta^{t-1}))^2 \leq O\left(\frac{\tau \log \frac{d}{\zeta}}{n}\right)$ . Thus, with probability at least  $1 - \zeta$ , we have

$$\sum_{j=1}^d (\tilde{g}_j^{t-1}(\beta^{t-1}) - \mathbb{E} \nabla_j q(\beta^{t-1}; \beta^{t-1}))^2 \leq O\left(\frac{d\tau \log \frac{d}{\zeta}}{n}\right).$$

Hence, we have, with probability at least  $1 - 2\zeta$ , for some constant  $C_2$

$$\|\tilde{\nabla}Q_n(\beta^{t-1}) - \nabla Q(\beta^{t-1}; \beta^{t-1})\|_2 \leq C_2 \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}. \quad (35)$$

Plugging (35) and (32) into (28), we have, with probability  $1 - 2\zeta$  and for some constant  $C_3$ ,

$$\|\beta^t - \beta^*\|_2 \leq (1 - 2\frac{v-\gamma}{\mu+v})\|\beta^{t-1} - \beta^*\|_2 + C_3 \frac{2}{\mu+v} \cdot \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}} \quad (36)$$

Next, we will show that when  $n$  is large enough, if  $\|\beta^0 - \beta^*\|_2 \leq \frac{R}{2}$  then  $\|\beta^t - \beta^*\|_2 \leq \frac{R}{2}$  holds (and thus  $\beta \in \mathcal{B}$ ) for all  $t \in [T]$  if (36) holds for all  $t \in [T]$  (and this hold with probability at least  $1 - 2T\zeta$ ).

We will use induction. When  $t = 1$ , by (36) we have

$$\begin{aligned} \|\beta^1 - \beta^*\|_2 &\leq (1 - 2\frac{v-\gamma}{\mu+v})\|\beta^0 - \beta^*\|_2 + C_3 \frac{2}{\mu+v} \cdot \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}} \\ &\leq (1 - 2\frac{v-\gamma}{\mu+v})\frac{R}{2} + C_3 \frac{2}{\mu+v} \cdot \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}. \end{aligned}$$

If  $C_3 \frac{2}{\mu+v} \cdot \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}} \leq 2\frac{v-\gamma}{\mu+v} \cdot \frac{R}{2}$ , then we can see that  $\|\beta^1 - \beta^*\|_2 \leq \frac{R}{2}$ . This holds if

$$C_4 \left( \frac{1}{v-\gamma} \right)^2 \frac{d^2 \tau T \log \frac{d}{\zeta}}{R^2 \beta \tilde{\epsilon}^2} \leq n$$

for some constant  $C_4$ .

Next, we will assume that (36) holds for all  $t \in [T]$  and  $\beta \in \mathcal{B}$  for all  $t \in [T]$ . For convenience, we denote  $\iota = 1 - 2\frac{v-\gamma}{\mu+v}$ . By (36), we have

$$\begin{aligned} \|\beta^T - \beta^*\|_2 &\leq (1 - 2\frac{v-\gamma}{\mu+v})^T \|\beta^0 - \beta^*\|_2 + C_3(1 + \iota + \iota^2 + \dots) \frac{2}{\mu+v} \cdot \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}} \\ &\leq (1 - 2\frac{v-\gamma}{\mu+v})^T \frac{R}{2} + C_3 \frac{1}{1-\iota} \cdot \frac{2}{\mu+v} \cdot \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}} \\ &= (1 - 2\frac{v-\gamma}{\mu+v})^T \frac{R}{2} + O\left(\frac{1}{v-\gamma} \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}\right). \end{aligned}$$

Taking  $T = O(\frac{\mu+v}{v-\gamma} \log \frac{n\tilde{\epsilon}}{d})$ , we have, with probability at least  $1 - 2T\zeta$ ,

$$\|\beta^T - \beta^*\|_2 \leq \tilde{O}\left(R \sqrt{\frac{\mu+v}{(v-\gamma)^3}} \frac{d\sqrt{\tau \log n \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}\right).$$

Since  $\tilde{\epsilon} = \sqrt{\log \frac{1}{\delta} + \epsilon} - \sqrt{\log \frac{1}{\delta}}$ , by using the Taylor series of the function  $\sqrt{x+1} - \sqrt{x}$ , we have  $\tilde{\epsilon} = O(\frac{\epsilon}{\sqrt{\log \frac{1}{\delta}}})$ . Thus, we have the proof.  $\square$

## Proof of Lemma 4

**Proof of Lemma 4.** To prove Lemma 4, we need a stronger lemma.

**Lemma 19.** The  $j$ -the coordinate of  $\nabla q(\beta; \beta)$  is  $\xi$ -sub-exponential with

$$\xi = C_1 \sqrt{\|\beta^*\|_\infty^2 + \sigma^2}, \quad (37)$$

where  $C_1$  is some absolute constant. Also, for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n]$ , is independent with others.

If Lemma 19 holds, then by Lemma 10 we can get Lemma 4.  $\square$

*Proof of Lemma 19.* From (23) it is oblivious that each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n], j \in [d]$ , is independent with others. Next, we prove the property of sub-exponential for each coordinate.

Note that

$$\nabla_j q(\beta; \beta) = [2w_\beta(y) - 1]y_j - \beta_j,$$

and

$$\mathbb{E}_Y \nabla_j q(\beta; \beta) = \mathbb{E}_Y (2w_\beta(Y)Y_j - Y_j) - \beta_j.$$

By the symmetrization lemma in Lemma 15, we have the following for any  $t > 0$

$$\mathbb{E}\{\exp(t|\nabla_j q(\beta; \beta) - \mathbb{E}\nabla_j q(\beta; \beta)|)\} \leq \mathbb{E}\{\exp(t|\epsilon[2w_\beta(y) - 1]y_j|)\}, \quad (38)$$

where  $\epsilon$  is a Rademacher random variable.

Next, we use Lemma 16 with  $f(y_j) = y_j$ ,  $\mathcal{F} = \{f\}$ ,  $\phi(v) = [2w_\beta(y) - 1]v$  and  $\phi(v) = \exp(u \cdot v)$ . It is easy to see that  $\phi$  is 1-Lipschitz. Thus, by Lemma 16 we have

$$\mathbb{E}\{\exp(t|\epsilon[2w_\beta(y) - 1]y_j|)\} \leq \mathbb{E}\{\exp[2t|\epsilon y_j|]\}. \quad (39)$$

By the formulation of the model, we have  $y_j = z\beta_j^* + v_j$ , where  $z$  is a Rademacher random variable and  $v_j \sim \mathcal{N}(0, \sigma^2)$ . It is easy to see that  $y_j$  is sub-Gaussian and

$$\|y_j\|_{\psi_2} = \|z \cdot \beta_j^* + v_j\|_{\psi_2} \leq C \cdot \sqrt{\|z \cdot \beta_j^*\|_{\psi_2}^2 + \|v_j\|_{\psi_2}^2} \leq C' \sqrt{|\beta_j^*|^2 + \sigma^2}, \quad (40)$$

for some absolute constants  $C, C'$ , where the last inequality is due to the facts that  $\|z_j \beta_j^*\|_{\psi_2} \leq |\beta_j^*|$  and  $\|v_{i,j}\|_{\psi_2} \leq C'' \sigma^2$  for some  $C'' > 0$ .

Since  $|\epsilon y_j| = |y_j|$ ,  $\|\epsilon y_j\|_{\psi_2} = \|y_j\|_{\psi_2}$  and  $\mathbb{E}(\epsilon y_j) = 0$ , by Lemma 5.5 in (Vershynin 2010) we have that for any  $u'$  there exists a constant  $C^{(4)} > 0$  such that

$$\mathbb{E}\{\exp(u' \cdot \epsilon \cdot y_j)\} \leq \exp(u'^2 \cdot C^{(4)} \cdot (|\beta_j^*|^2 + \sigma^2)). \quad (41)$$

Thus, for any  $t > 0$  we get

$$\mathbb{E}\{\exp(2t \cdot |\epsilon \cdot y_j|)\} \leq 2 \exp(t^2 \cdot C^{(5)} \cdot (|\beta_j^*|^2 + \sigma^2)) \quad (42)$$

for some constant  $C^{(5)}$ . Therefore, in total we have the following for some constant  $C^{(6)} > 0$

$$\mathbb{E}\{\exp(t|\nabla_j q(\beta; \beta) - \mathbb{E}\nabla_j q(\beta; \beta)|)\} \leq \exp(t^2 \cdot C^{(6)} \cdot (|\beta_j^*|^2 + \sigma^2)) \leq \exp(t^2 \cdot C^{(6)} \cdot (\|\beta^*\|_\infty^2 + \sigma^2)). \quad (43)$$

Combining this with Lemma 12 and the definition, we know that  $\nabla_j q(\beta; \beta)$  is  $O(\sqrt{\|\beta^*\|_\infty^2 + \sigma^2})$ -sub-exponential.  $\square$

## Proof of Lemma 6

**Proof of Lemma 6.** Just as in the proof of Lemma 4, we will show that  $\nabla_j q(\beta; \beta)$  is sub-exponential instead.

**Lemma 20.** For each  $\beta \in \mathcal{B}$ , the  $j$ -the coordinate of  $\nabla q(\beta; \beta)$  is  $\xi$ -sub-exponential with

$$\xi = C \max\{\|\beta^*\|_2^2 + \sigma^2, 1, \sqrt{d}\|\beta^*\|_2\}, \quad (44)$$

where  $C > 0$  is some absolute constant. Also, for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n]$ , is independent with others.  $\square$

*Proof of Lemma 20.* From (24) it is oblivious that for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n]$ , is independent with others. Next, we prove the property of sub-exponential.

Note that  $\mathbb{E}\nabla_j q(\beta; \beta) = \mathbb{E}2w_\beta(x, y)y \cdot x_j - \beta_j$ . Thus, we have

$$\nabla_j q(\beta; \beta) - \mathbb{E}\nabla_j q(\beta; \beta) = \underbrace{2w_\beta(x, y)yx_j - \mathbb{E}[2w_\beta(x, y)yx_j]}_A + \underbrace{[xx^T \beta - \beta]_j - \underbrace{yx_j}_C. \quad (45)}$$

For term A and any  $t > 0$ , we have

$$\mathbb{E}\{\exp(t|A|)\} \leq \mathbb{E}\{\exp[t|2\epsilon w_\beta(x, y)yx_j|]\}. \quad (46)$$

Using Lemma 16 on  $f(yx_j) = yx_j$ ,  $\mathcal{F} = f$ ,  $\phi_i(v) = 2w_\beta(x, y)v$  and  $\phi(v) = \exp(uv)$ , we have

$$\mathbb{E}\{\exp[t|2\epsilon w_\beta(x, y)yx_j|]\} \leq \mathbb{E}\{\exp[4t|\epsilon yx_j|]\}. \quad (47)$$

Note that since  $y = z\langle \beta^*, x \rangle + v$  and  $\|z\langle \beta^*, x \rangle\|_{\psi_2} = \|\langle \beta^*, x \rangle\|_{\psi_2} \leq C\|\beta^*\|_2$  and  $\|v\|_{\psi_2} \leq C'\sigma$  for some constants  $C, C' > 0$ , by Lemma 14 we know that there exists a constant  $C'' > 0$  such that

$$\|y\|_{\psi_2} \leq C'' \sqrt{\|\beta^*\|_2^2 + \sigma^2}. \quad (48)$$

Thus, by Lemma 13 we have

$$\|yx_j\|_{\psi_1} \leq \max\{C''^2(\|\beta^*\|_2^2 + \sigma^2), C'''\} \leq C_4 \max\{\|\beta^*\|_2^2 + \sigma^2, 1\}. \quad (49)$$

For term B, we have

$$\mathbb{E}\{\exp[t|B|]\} = \mathbb{E}\{\exp[t|\sum_{k=1}^d x_j x_k \beta_k - \beta_j]|\}, \quad (50)$$

where  $x_j, x_k \sim \mathcal{N}(0, 1)$ . Now, by Lemma 13 we have  $\|x_j x_k \beta_k\|_{\psi_1} \leq |\beta_k| C^{(5)}$  for some constant  $C^{(5)} > 0$ . Thus, we get  $\|\sum_{k=1}^d x_j x_k \beta_k\|_{\psi_1} \leq C^{(5)} \|\beta\|_1$ .

Also, we know that  $\|\beta\|_1 \leq \sqrt{d}\|\beta\|_2$ . Furthermore, we have  $\|\beta\|_2 \leq \|\beta^*\|_2 + \|\beta^* - \beta\|_2 \leq O(\|\beta^*\|_2)$ , since  $\beta \in \mathcal{B}$  (by assumption). From Lemma 13, we get  $\|B\|_{\psi_1} \leq C^{(6)} \sqrt{d}\|\beta^*\|_2$  with some constant  $C^{(6)} > 0$ .

Thus, we know that there exist some constants  $C^{(7)} > 0$  and  $C^{(8)} > 0$  such that

$$\begin{aligned} \|\nabla_j q(\beta; \beta) - \mathbb{E}\nabla_j q(\beta; \beta)\|_{\psi_1} &\leq C^{(7)} \max\{\|\beta^*\|_2^2 + \sigma^2, 1\} + C^{(8)} \sqrt{d}\|\beta^*\|_2 \\ &\leq C^{(9)} \max\{\|\beta^*\|_2^2 + \sigma^2, 1, \sqrt{d}\|\beta^*\|_2\}. \end{aligned}$$

This means that  $\nabla_j q(\beta; \beta)$  is  $O(\max\{\|\beta^*\|_2^2 + \sigma^2, 1, \sqrt{d}\|\beta^*\|_2\})$ -sub-exponential.  $\square$

## Proof of Lemma 8

**Proof of Lemma 8.** Just as in the proof of Lemma 4, we will show that  $\nabla_j q(\beta; \beta)$  is sub-exponential instead.

**Lemma 21.** For each  $\beta \in \mathcal{B}$  and  $j \in [d]$ ,  $\nabla_j q(\beta; \beta)$  is  $\xi$ -sub-exponential with

$$\xi = C[(1+k)(1+kr)^2 \sqrt{d}\|\beta^*\|_2 + \max\{(1+kr)^2, \sigma^2 + \|\beta^*\|_2^2\}] = O(\sqrt{d}\|\beta^*\|_2 + \sigma^2 + \|\beta^*\|_2^2) \quad (51)$$

for some constant  $C > 0$ . Also, for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n]$ , is independent with others.  $\square$

*Proof of Lemma 21.* From (25) it is oblivious that for fixed  $j \in [d]$ , each  $\nabla_j q_i(\beta; \beta)$ , where  $i \in [n]$ , is independent with others. Next, we prove the property of sub-exponential.

For simplicity, we use notations  $\bar{m} = m_\beta(x^{\text{obs}}, y)$ ,  $\bar{m}_j = \beta(x^{\text{obs}}, y)$ ,  $\bar{K} = K_\beta(x^{\text{obs}}, y)$ , and  $\bar{K}_j = K_\beta(x^{\text{obs}}, y)$ . Then, we have

$$\nabla q(\beta; \beta) - \mathbb{E}\nabla q(\beta; \beta) = \underbrace{m_\beta(x^{\text{obs}}, y)y}_{A} - \mathbb{E}[m_\beta(x^{\text{obs}}, y)y] + \overbrace{(\bar{K}_\beta(x^{\text{obs}}, y) - \mathbb{E}\bar{K}_\beta(x^{\text{obs}}, y))\beta}^B. \quad (52)$$

For the  $j$ -th coordinate of  $A$ , we have

$$A_j = \bar{m}_j y - \mathbb{E}[\bar{m}_j y]. \quad (53)$$

We note that  $\bar{m}_j$  is a zero-mean sub-Gaussian random variable with  $\|\bar{m}_j\|_{\psi_2} \leq C(1+kr)$  (see Lemma B.3 in (Wang et al. 2015))

**Lemma 22.** Under the assumption of Lemma 6, for each  $j \in [d]$ ,  $\bar{m}_j$  is sub-Gaussian with mean zero and  $\|\bar{m}_j\|_{\psi_2} \leq C(1+kr)$ .

Thus, by Lemma 13 we have

$$\|\bar{m}_j y\|_{\psi_1} \leq C \max\{\|\bar{m}_j\|_{\psi_2}^2, \|y\|_{\psi_2}^2\} \leq C' \max\{(1+kr)^2, \sigma^2 + \|\beta^*\|_2^2\}, \quad (54)$$

where the last inequality is due to the fact that  $y = \langle \beta^*, x \rangle + v$ . Thus,  $\|y\|_{\psi_2}^2 \leq C_3(\|\langle \beta^*, x \rangle\|_{\psi_2}^2 + \|v\|_{\psi_2}^2)$  for some  $C_3$ .

For term B, we have

$$\bar{K}_j = \underbrace{(1-z_j)\beta_j}_C + \underbrace{\sum_{k=1}^d \bar{m}_j \bar{m}_k \beta_k}_D - \underbrace{\sum_{k=1}^d [(1-z_j)\bar{m}_j][(1-z_k)\bar{m}_k] \beta_k}_E. \quad (55)$$

For term C, we have the following (by Example 5.8 in (Vershynin 2010))

$$\|(1 - z_j)\beta_j\|_{\psi_2} \leq |\beta_j| \leq \|\beta\|_\infty \leq (1 + k)\sqrt{s}\|\beta^*\|_2. \quad (56)$$

For term D, by Lemma 22 and 13 we have

$$\left\| \sum_{k=1}^d \bar{m}_j \bar{m}_k \beta_k \right\|_{\psi_1} \leq \sum_{k=1}^d |\beta_k| \|\bar{m}_j \bar{m}_k\|_{\psi_1} \leq \sum_{k=1}^d |\beta_k| C^2 (1 + kr)^2 \leq C_4 (1 + kr)^2 \|\beta\|_1. \quad (57)$$

Since  $\beta \in \mathcal{B}$ , we get  $\|\beta\|_1 \leq \sqrt{d}\|\beta\|_2 \leq (1 + k)\sqrt{d}\|\beta^*\|_2$ . Thus, we have

$$\left\| \sum_{k=1}^d \bar{m}_j \bar{m}_k \beta_k \right\|_{\psi_1} \leq C_4 \sqrt{s} (1 + kr)^2 \|\beta^*\|_2. \quad (58)$$

For term E, since  $1 - z \in [0, 1]$ , we have  $\|(1 - z_j)\bar{m}_j\|_{\psi_2} \leq \|\bar{m}_j\|_{\psi_2} \leq C(1 + kr)$ . Hence, by Lemma 13 we get

$$\begin{aligned} \left\| \sum_{k=1}^d [(1 - z_j)\bar{m}_j][(1 - z_k)\bar{m}_k] \beta_k \right\|_{\psi_1} &\leq \sum_{k=1}^d |\beta_k| \left\| [(1 - z_j)\bar{m}_j][(1 - z_k)\bar{m}_k] \right\|_{\psi_1} \\ &\leq \sum_{k=1}^d |\beta_k| C(1 + kr)^2 \leq C_6 (1 + kr)^2 \sqrt{s} \|\beta^*\|_2. \end{aligned} \quad (59)$$

This gives us

$$\|\bar{K}_j\|_{\psi_1} \leq C_7 \sqrt{s} (1 + k) (1 + kr)^2 \|\beta^*\|_2. \quad (60)$$

By Lemma 12, we get

$$\|\nabla_j q(\beta; \beta) - \mathbb{E} \nabla_j q(\beta; \beta)\|_{\psi_1} \leq 2 \|\nabla_j q(\beta; \beta)\|_{\psi_1} \leq C_8 [(1 + k)(1 + kr)^2 \sqrt{s} \|\beta^*\|_2 + \max\{(1 + kr)^2, \sigma^2 + \|\beta^*\|_2^2\}]. \quad (61)$$

□

## Statistical Guarantees of DP Expectation Maximization Algorithm

Motivated by idea of the Differentially Private version of Gradient EM algorithm in the previous section, in this section, we will propose a DP variant of EM algorithm.

Recall that compared with the Gradient EM algorithm, the main difference in EM algorithm is that, in each iteration, we will update the parameter as  $\beta^{t+1} = \arg \max_{\beta \in \Omega} Q_n(\beta; \beta^t)$ , where the  $Q_n$ -function is in (2). Thus, to design a DP variant, we need to post-process the parameter  $\beta^{t+1}$  via the private 1-dimensional mean estimation of heavy-tailed distribution. Just as the way we post-process the Gradient in Algorithm 2, we wish to post-process each coordinate of  $\beta^{t+1}$  to make it DP. However, unlike the Gradient EM algorithm where the  $\nabla Q_n(\beta; \beta')$  can be written as a sum of  $n$  independent components  $\frac{1}{n} \sum_{i=1}^n \nabla q_i(\beta; \beta')$ ,  $\beta^{t+1}$  in the EM algorithm may not be written as  $n$  independent components (see the Examples below), or even there is no explicit form of  $\beta^{t+1}$ . Thus, compared with the Assumption 1, we need addition assumptions on the form of  $\beta^{t+1} = \arg \max_{\beta \in \Omega} Q_n(\beta; \beta^t)$ , which may not hold for some canonical models.

**Assumption 2.** We assume that for a fixed  $\beta' \in \mathcal{B}$ , the optimal solution  $M_n(\beta') = \arg \max_{\beta \in \Omega} Q_n(\beta; \beta')$  satisfies  $M_n(\beta') = \frac{1}{n} \sum_{i=1}^n f_i(\beta')$ , where  $f_i(\cdot)$  is a function of  $y_i$ . Moreover, we assume that for each pair  $i \neq i'$ ,  $f_i(\beta')$ ,  $f_{i'}(\beta')$  are independent. For any fixed  $j \in d$ , the  $j$ -th coordinate of  $f(\beta)$ <sup>4</sup> has bounded second order moment, i.e.,  $\mathbb{E}(f_j(\beta))^2 \leq \tau$ . We also assume that function  $Q(\cdot; \cdot)$  in (3) is self-consistent, Lipschitz-Gradient-2( $\gamma, \mathcal{B}$ ),  $v$ -strongly concave over some set  $\mathcal{B}$ .

Note that compared with Assumption 1, Assumption 2 does not need  $Q$  to be smooth. However, it needs some unnatural assumptions in the form of  $M_n(\beta')$ . To show that these assumptions are strong (especially the condition that  $f_i, f_{i'}$  are independent for each pair  $i \neq i'$ ), in the following, we will check the three canonical models in the previous section to see whether Assumption 2 holds.

**Gaussian Mixture Model** For GMM in (4), the  $Q$  function can be written as

$$Q_n(\beta; \beta') = -\frac{1}{2n} \sum_{i=1}^n (w_{\beta'}(y_i) \|y_i - \beta\|_2^2 + [1 - w_{\beta'}(y_i)] \|y_i + \beta\|_2^2).$$

---

<sup>4</sup>We denote function  $f(\cdot)$  as the function for general  $y$ .

where  $w_\beta(y) = \frac{1}{1+\exp(-\langle \beta, y \rangle / \sigma^2)}$ . Thus, for  $M_n(\beta') = \arg \max_{\beta \in \mathbb{R}^d} Q_n(\beta; \beta')$  we have

$$M_n(\beta') = \frac{2}{n} \sum_{i=1}^n w_{\beta'}(y_i) y_i - \frac{1}{n} \sum_{i=1}^n y_i,$$

Thus

$$M_n(\beta') = \frac{1}{n} \sum_{i=1}^n f_i(\beta')$$

for  $f_i(\beta') = 2w_{\beta'}(y_i)y_i - y_i$  and for each  $i \in [n]$ ,  $f_i$  is independent with others. Later, combining with Lemma 3 we will show GMM satisfies Assumption 2.

**Mixture of Regressions Model** For MRM in (5), the  $Q_n$  function can be written as

$$Q_n(\beta; \beta') = \frac{1}{2n} \left( -w_{\beta'}(x_i, y_i)(y - \langle x_i, \beta \rangle)^2 + [1 - w_{\beta'}(x_i, y_i)](y + \langle x_i, \beta \rangle)^2 \right),$$

where  $w_\beta(x, y) = \frac{1}{1+\exp(-y\langle \beta, x \rangle / \sigma^2)}$ . Thus, for  $M_n(\beta') = \arg \max_{\beta \in \mathbb{R}^d} Q_n(\beta; \beta')$  we have

$$M_n(\beta') = \left( \frac{1}{n} \sum_{i=1}^n x_i x_i^T \right)^{-1} \cdot \left( \frac{1}{n} \sum_{i=1}^n [2w_{\beta'}(x_i, y_i) - 1] y_i x_i \right).$$

Thus  $M_n(\beta') = \frac{1}{n} \sum_{i=1}^n f_i(\beta')$  for  $f_i(\beta') = (\frac{1}{n} \sum_{i=1}^n x_i x_i^T)^{-1} \cdot [2w_{\beta'}(x_i, y_i) - 1] y_i x_i$ . However, we can see that due to the term of  $(\frac{1}{n} \sum_{i=1}^n x_i x_i^T)^{-1}$ , for each  $i \in [n]$ ,  $f_i$  is dependent with others. Thus, MRM **does not satisfy** Assumption 2.

**Linear Regression with Missing Covariates** For RMC in (6), the  $Q_n$  function can be written as

$$Q_n(\beta; \beta') = \frac{1}{n} \sum_{i=1}^n y_i \beta^T m_{\beta'}(x_i^{\text{obs}}, y_i) - \frac{1}{2n} \sum_{i=1}^n \beta^T K_{\beta'}(x_i^{\text{obs}}, y_i) \beta',$$

where the functions  $m_{\beta'}(x^{\text{obs}}, y)$ ,  $K_{\beta'}(x^{\text{obs}}, y)$  are in (26) and (27), respectively. Thus, for  $M_n(\beta') = \arg \max_{\beta \in \mathbb{R}^d} Q_n(\beta; \beta')$  we have

$$M_n(\beta') = \left( \frac{1}{n} \sum_{i=1}^n K_{\beta'}(x_i^{\text{obs}}, y_i) \right)^{-1} \cdot \left( \frac{1}{n} \sum_{i=1}^n y_i m_{\beta'}(x_i^{\text{obs}}, y_i) \right).$$

Thus  $M_n(\beta') = \frac{1}{n} \sum_{i=1}^n f_i(\beta')$  for  $f_i(\beta') = (\frac{1}{n} \sum_{i=1}^n K_{\beta'}(x_i^{\text{obs}}, y_i))^{-1} (y_i m_{\beta'}(x_i^{\text{obs}}, y_i))$ . However, we can see that due to the term of  $(\frac{1}{n} \sum_{i=1}^n K_{\beta'}(x_i^{\text{obs}}, y_i))^{-1}$ , for each  $i \in [n]$ ,  $f_i$  is dependent with others. Thus, RMC **does not satisfy** Assumption 2.

From the previous models, we can see that two of them do not satisfy the condition of  $f_i$  is independent with others. We note that this assumption is necessary for our analysis of statistical guarantees, since we will use the private 1-dimensional mean estimator, which needs the i.i.d assumption on the samples. Thus, from this point of view, we can see that our DP Gradient EM algorithm needs to be presented before the DP EM algorithm.

## DP EM Algorithm

Next we will detail our DP EM algorithm and provide its statistical guarantee under Assumption 2, see Algorithm 3 for details. The key idea is that in each iteration, instead of post-processing the  $j$ -th coordinate of the gradient  $\nabla q_i(\beta^{t-1}, \beta^{t-1})$ , we will post-process  $j$ -th coordinate of the term  $f_i(\beta^{t-1})$ , i.e.,  $f_{i,j}(\beta^{t-1})$  via the previous private 1-dimension mean estimator. We can easily show Algorithm 3 is  $(\epsilon, \delta)$ -DP.

**Theorem 7** (Privacy guarantee). For any  $0 < \epsilon, \delta < 1$ , Algorithm 2 is  $(\epsilon, \delta)$ -DP.

*Proof.* The proof is almost the same as that of Theorem 2; we thus omit it here.  $\square$

As in Theorem 3, in the following, we will show the statistical guarantee for the models under the Assumption 2, if the initial parameter  $\beta^0$  is close enough to the underlying parameter  $\beta^*$ .

**Theorem 8** (Statistical guarantee of Algorithm 3). Let the parameter set  $\mathcal{B} = \{\beta : \|\beta - \beta^*\|_2 \leq R\}$  for  $R = \kappa \|\beta^*\|_2$  for some constant  $\kappa \in (0, 1)$ . Assume that Assumption 2 holds for parameters  $\gamma, \mathcal{B}, v, \tau$  satisfying the condition of  $1 - 2 \frac{v-\gamma}{v+\mu} \in (0, 1)$ . Also, assume that  $\|\beta^0 - \beta^*\|_2 \leq \frac{R}{2}$ ,  $n$  is large enough so that

$$\tilde{\Omega}\left(\left(\frac{v}{v-\gamma}\right)^2 \frac{d^2 \tau T \log \frac{1}{\delta} \log \frac{1}{\zeta}}{\epsilon^2 R^2}\right) \leq n. \quad (63)$$

---

**Algorithm 3** DP EM Algorithm

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**Input:**  $D = \{y_i\}_{i=1}^n \subset \mathbb{R}^d$ , privacy parameters  $\epsilon, \delta, Q(\cdot; \cdot)$  and its  $f_i(\cdot)$  in Assumption 2, initial parameter  $\beta^0 \in \mathcal{B}$ ,  $\tau$  which satisfies Assumption 2, the number of iterations  $T$  (to be specified later), and failure probability  $\zeta$ .

1: Let  $\tilde{\epsilon} = \sqrt{\log \frac{1}{\delta} + \epsilon} - \sqrt{\log \frac{1}{\delta}}$ ,  $s = \sqrt{\frac{n\tau}{2\log \frac{d}{\zeta}}}$ ,  $\beta = \log \frac{d}{\zeta}$ .

2: **for**  $t = 1, 2, \dots, T$  **do**

3:     For each  $j \in [d]$ , calculate the robust estimator by (8)-(12) and add Gaussian noise, that is

$$\begin{aligned} g_j^{t-1}(\beta^{t-1}) &= \frac{1}{n} \sum_{i=1}^n \left( f_{i,j}(\beta^{t-1}) \left( 1 - \frac{f_{i,j}^2(\beta^{t-1})}{2s^2\beta} \right) - \frac{f_{i,j}^3(\beta^{t-1})}{6s^2} \right) \\ &\quad + \frac{s}{n} \sum_{i=1}^n C \left( \frac{f_{i,j}(\beta^{t-1})}{s}, \frac{|f_{i,j}(\beta^{t-1})|}{s\sqrt{\beta}} \right) + Z_j^{t-1}, \end{aligned} \quad (62)$$

where  $f_{i,j}(\beta^{t-1})$  is the  $j$ -th coordinate of  $f_i(\beta^{t-1})$  and  $Z_j^{t-1} \sim \mathcal{N}(0, \sigma^2)$  with  $\sigma^2 = \frac{8\tau d T}{9\beta n \tilde{\epsilon}^2}$ .

4:     Let vector  $\tilde{f}(\beta^{t-1}) \in \mathbb{R}^d$  to denote  $\tilde{f}(\beta^{t-1}) = (g_1^{t-1}(\beta^{t-1}), g_2^{t-1}(\beta^{t-1}), \dots, g_d^{t-1}(\beta^{t-1}))$ .

5:     Update  $\beta^t = \tilde{f}(\beta^{t-1})$ .

6: **end for**

---

Then with probability at least  $1 - 2T\zeta$ , we have for all  $t \in [T]$ ,  $\beta^t \in \mathcal{B}$ . If it holds and if we take  $T = O(\frac{v}{v-\gamma} \log n)$ , then we have

$$\|\beta^T - \beta^*\|_2 \leq \tilde{O}\left(R \sqrt{\frac{v}{(v-\gamma)^3} \frac{d \log \frac{1}{\delta} \log \frac{1}{\zeta} \sqrt{\tau}}{\sqrt{n\epsilon^2}}}\right), \quad (64)$$

where the  $\tilde{O}$ -term and  $\tilde{\Omega}$ -term omit  $\log d, \log n$  and other factors (see Appendix for the explicit form of the result).

**Proof of Theorem 8.** For each iteration we denote  $M(\beta^{t-1}) = \arg \max Q(\beta; \beta^{t-1})$ , by the strongly concavity of  $Q(\beta; \beta^*)$  we have

$$\langle \nabla Q(M(\beta^{t-1}); \beta^*) - \nabla Q(\beta^*; \beta^*), M(\beta^{t-1}) - \beta^* \rangle \geq v \|M(\beta^{t-1}) - \beta^*\|_2^2.$$

On the other hand, by the Lipschitz-Gradient condition and the assumption of  $M(\beta^{t-1}), \beta^{t-1} \in \mathcal{B}$ , we have

$$\langle \nabla Q(M(\beta^{t-1}); \beta^*) - \nabla Q(M(\beta^{t-1}); \beta^{t-1}), \beta^* - M(\beta^{t-1}) \rangle \leq \gamma \|\beta^{t-1} - \beta^*\|_2 \|\beta^* - M(\beta^{t-1})\|_2.$$

Also by the optimality of  $M(\beta^{t-1})$  we have

$$\langle \nabla Q(M(\beta^{t-1}); \beta^*) - \nabla Q(\beta^*; \beta^*), M(\beta^{t-1}) - \beta^* \rangle \leq \langle \nabla Q(M(\beta^{t-1}); \beta^*) - \nabla Q(M(\beta^{t-1}); \beta^{t-1}), \beta^* - M(\beta^{t-1}) \rangle.$$

Thus, we have

$$v \|M(\beta^{t-1}) - \beta^*\|_2^2 \leq \gamma \|\beta^{t-1} - \beta^*\|_2 \|\beta^* - M(\beta^{t-1})\|_2.$$

That is,  $\|M(\beta^{t-1}) - \beta^*\|_2 \leq \frac{\gamma}{v} \|\beta^{t-1} - \beta^*\|_2$ . Next, we will bound the term of  $\|\beta^t - M(\beta^{t-1})\|_2$ .

Under the assumption that  $f_i$  is independent with others, just as almost the same as in (33)-(35) via Lemma 2, we have that with probability at least  $1 - \zeta$ ,

$$\|\beta^t - M(\beta^{t-1})\|_2 = \|\tilde{f}(\beta^{t-1}) - M(\beta^{t-1})\|_2 \leq O\left(\frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}\right).$$

Thus, we have with probability at least  $1 - \zeta$

$$\|\beta^t - \beta^*\|_2 \leq \frac{\gamma}{v} \|\beta^{t-1} - \beta^*\|_2 + O\left(\frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}\right).$$

Since we need to make  $\beta^t \in \mathcal{B}$ , this will be true under the assumption that

$$O\left(\frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}\right) \leq \frac{v-\gamma}{v} R.$$

If this holds, then we have with probability at least  $1 - T\zeta$

$$\|\beta^t - \beta^*\|_2 \leq \left(\frac{\gamma}{v}\right)^T R + O\left(\frac{v}{v-\gamma} \frac{d\sqrt{\tau T \log \frac{d}{\zeta}}}{\sqrt{\beta n \tilde{\epsilon}^2}}\right),$$

Taking  $T = O(\frac{v}{v-\gamma} \log n)$  and  $\tilde{\epsilon} = O(\frac{\epsilon}{\sqrt{\log \frac{1}{\delta}}})$ , we have the result.  $\square$

Comparing with Theorem 8 and Theorem 3, if we omit other factors instead of  $n, d, \epsilon, \delta$ , we can see that the two error bounds are asymptotically the same.

In the following we will apply our general framework to the GMM model in (4). Just the same as in Theorem 4, we will first show that  $f_j(\beta)$  has a bounded second order moment.

**Lemma 23.** Consider the function  $f(\cdot)$  in GMM. Then, for each  $j \in [d]$  we have

$$\mathbb{E}f_j^2(\beta) \leq O(\|\beta^*\|_\infty^2 + \sigma^2).$$

**Proof of Lemma 23.** To prove Lemma 23, we need a stronger lemma.

**Lemma 24.** The  $j$ -the coordinate of  $f(\beta)$  is  $\xi$ -sub-exponential with

$$\xi = C_1 \sqrt{\|\beta^*\|_\infty^2 + \sigma^2}, \quad (65)$$

where  $C_1$  is some absolute constant. Also, for fixed  $j \in [d]$ , each  $f_{i,j}(\beta)$ , where  $i \in [n]$ , is independent with others.

If Lemma 24 holds, then by Lemma 10 we can get Lemma 23.  $\square$

*Proof of Lemma 24.* The proof is almost the same as that of Lemma 4.

It is oblivious that each  $f_i(\beta)$ , where  $i \in [n]$ , is independent with others. Next, we prove the property of sub-exponential for each coordinate.

Note that

$$f_j(\beta) = [2w_\beta(y) - 1]y_j,$$

and

$$\mathbb{E}_Y f_i(\beta) = \mathbb{E}_Y (2w_\beta(Y)Y_j - Y_j).$$

By the symmetrization lemma in Lemma 15, we have the following for any  $t > 0$

$$\mathbb{E}\{\exp(t|[f_j(\beta) - \mathbb{E}f_j(\beta)]|)\} \leq \mathbb{E}\{\exp(t|\epsilon[2w_\beta(y) - 1]y_j|)\}, \quad (66)$$

where  $\epsilon$  is a Rademacher random variable.

Next, we use Lemma 16 with  $f(y_j) = y_j$ ,  $\mathcal{F} = \{f\}$ ,  $\phi(v) = [2w_\beta(y) - 1]v$  and  $\phi(v) = \exp(u \cdot v)$ . It is easy to see that  $\phi$  is 1-Lipschitz. Thus, by Lemma 16 we have

$$\mathbb{E}\{\exp(t|\epsilon[2w_\beta(y) - 1]y_j|)\} \leq \mathbb{E}\{\exp[2t|\epsilon y_j|]\}. \quad (67)$$

By the formulation of the model, we have  $y_j = z\beta_j^* + v_j$ , where  $z$  is a Rademacher random variable and  $v_j \sim \mathcal{N}(0, \sigma^2)$ . It is easy to see that  $y_j$  is sub-Gaussian and

$$\|y_j\|_{\psi_2} = \|z \cdot \beta_j^* + v_j\|_{\psi_2} \leq C \cdot \sqrt{\|z \cdot \beta_j^*\|_{\psi_2}^2 + \|v_j\|_{\psi_2}^2} \leq C' \sqrt{|\beta_j^*|^2 + \sigma^2} \quad (68)$$

for some absolute constants  $C, C'$ , where the last inequality is due to the facts that  $\|z_j \beta_j^*\|_{\psi_2} \leq |\beta_j^*|$  and  $\|v_{i,j}\|_{\psi_2} \leq C'' \sigma^2$  for some  $C'' > 0$ .

Since  $|\epsilon y_j| = |y_j|$ ,  $\|\epsilon y_j\|_{\psi_2} = \|y_j\|_{\psi_2}$  and  $\mathbb{E}(\epsilon y_j) = 0$ , by Lemma 5.5 in (Vershynin 2010) we have that for any  $u'$  there exists a constant  $C^{(4)} > 0$  such that

$$\mathbb{E}\{\exp(u' \cdot \epsilon \cdot y_j)\} \leq \exp(u'^2 \cdot C^{(4)} \cdot (|\beta_j^*|^2 + \sigma^2)). \quad (69)$$

Thus, for any  $t > 0$  we get

$$\mathbb{E}\{\exp(2t \cdot |\epsilon \cdot y_j|)\} \leq 2 \exp(t^2 \cdot C^{(5)} \cdot (|\beta_j^*|^2 + \sigma^2)) \quad (70)$$

for some constant  $C^{(5)}$ . Therefore, in total we have the following for some constant  $C^{(6)} > 0$

$$\mathbb{E}\{\exp(t|[f_j(\beta) - \mathbb{E}f_j(\beta)]|)\} \leq \exp(t^2 \cdot C^{(6)} \cdot (|\beta_j^*|^2 + \sigma^2)) \leq \exp(t^2 \cdot C^{(6)} \cdot (\|\beta^*\|_\infty^2 + \sigma^2)). \quad (71)$$

Combining this with Lemma 12 and the definition, we know that  $f_j(\beta)$  is  $O(\sqrt{\|\beta^*\|_\infty^2 + \sigma^2})$ -sub-exponential.  $\square$

Thus, combining with Lemma 3, Lemma 23 and Theorem 8 we have asymptotically the same result as in Theorem 4. We omit the details here.

## Additional Experiments

In Figure 5, 6 and 7, we set  $T = 22$  and compute the estimation error on  $\beta := \beta^T$ . We plot  $\|\beta - \beta^*\|_2$  of all algorithm on three canonical models over data size  $n$ , data dimension  $d$  and privacy budget  $\epsilon$ . As we can see from these figures, our proposed algorithm (Algorithm 2) on the three canonical models significantly outperforms the clipped algorithm (Algorithm 1).

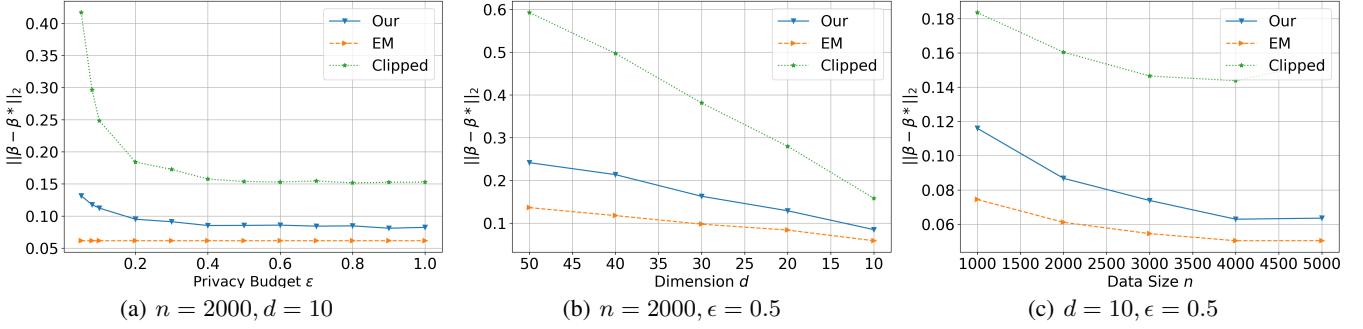


Figure 5: Estimation error of GMM w.r.t privacy budget  $\epsilon$ , data dimension  $d$  and data size  $n$  (we set  $\beta := \beta^T$  with  $T = 22$ )

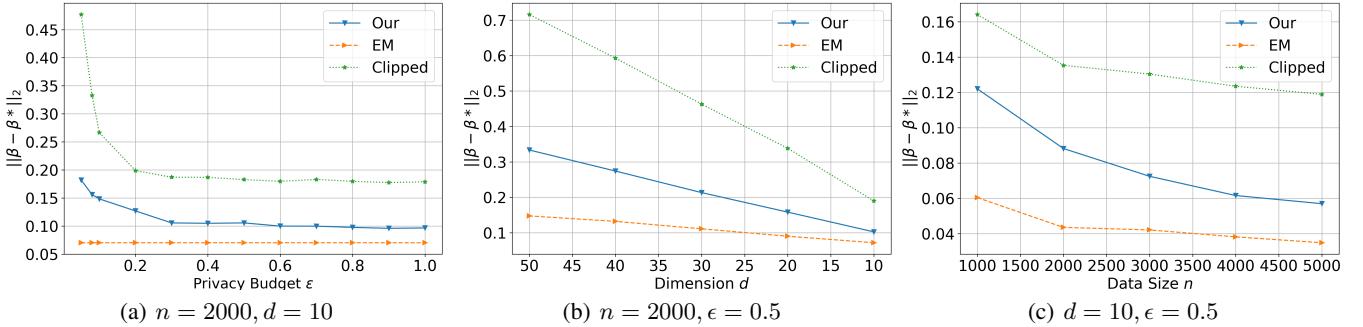


Figure 6: Estimation error of MRM w.r.t privacy budget  $\epsilon$ , data dimension  $d$  and data size  $n$  (we set  $\beta := \beta^T$  with  $T = 22$ )

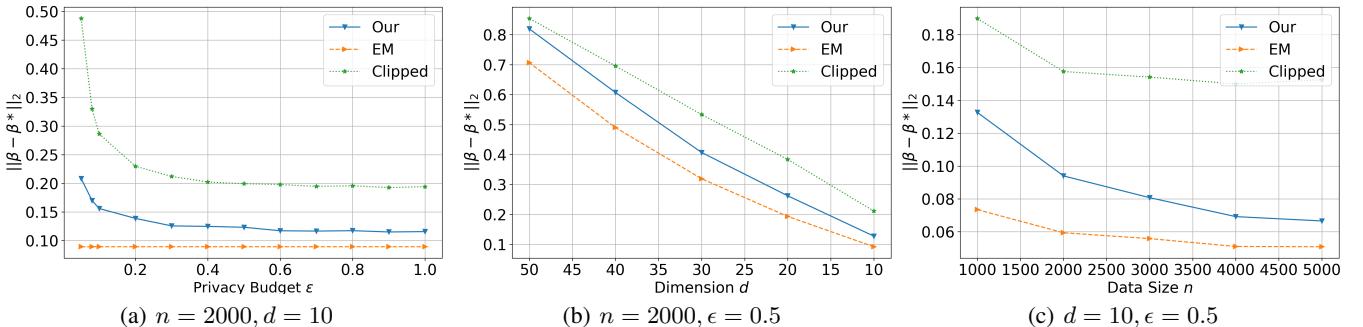


Figure 7: Estimation error of RMC w.r.t privacy budget  $\epsilon$ , data dimension  $d$  and data size  $n$  (we set  $\beta := \beta^T$  with  $T = 22$ )

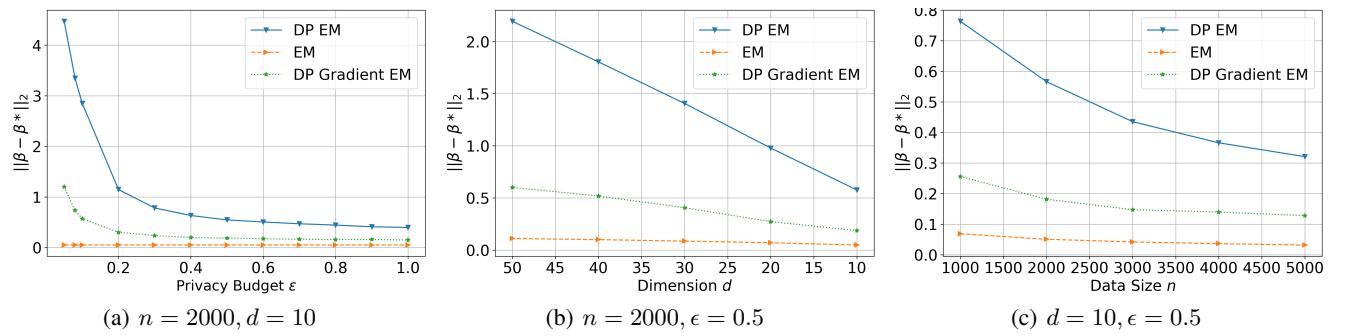


Figure 8: (Alg3) Estimation error of GMM w.r.t privacy budget  $\epsilon$ , data dimension  $d$  and data size  $n$  (we set  $\beta := \beta^T$  with  $T = 5$ )

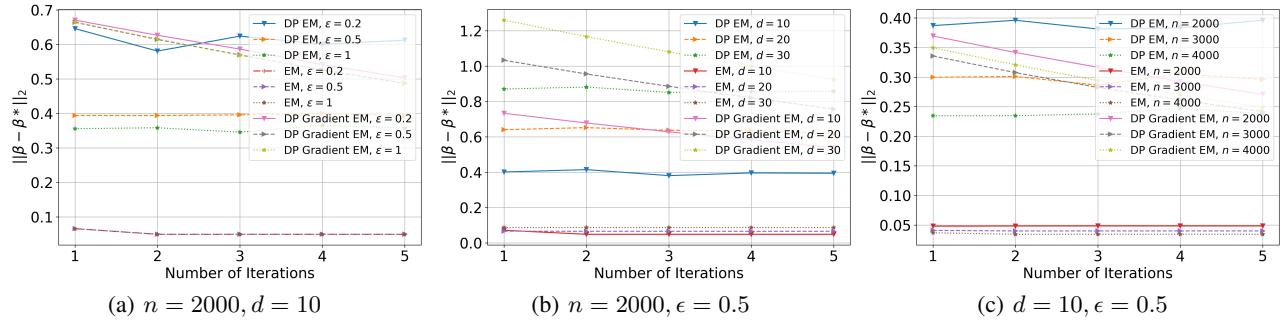


Figure 9: (Alg3) Estimation error of GMM w.r.t privacy budget  $\epsilon$ , data dimension  $d$ , data size  $n$  and iteration  $t$