Homework 5

Name: Shaobo Wang

Exercise 1. Asymptotic stability and Lyapunov stability

(a)

The given system is DT linear system, check eigen values of A.

Assume

$$A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$$

Since A is in lower triangular form, the eigen values are

$$\lambda_1 = 1, \lambda_2 = 0.5$$

$\|\lambda_1\| = 1$, but λ_1 is not defective, i.e.

$$\exists \lambda_1, r_1 = 1, m = 0$$

[ANSWER]: system (a) is stable i.s.L., but not asymptotic stable.

(b)

The given system is CT linear system, check eigen values of A.

Assume

$$A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

Find the eigen values of A, set $\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{bmatrix}\right) = \lambda^3 + 9\lambda^2 + 23\lambda + 15 = 0$$

The eigen values are

$$\lambda_1 = -5, \lambda_2 = -3, \lambda_3 = -1$$

All eigen values are negative real number.

[ANSWER]: system (b) is not only stable i.s.L., but also asymptotic stable.

Exercise 2. Stabilizability

i)

The controllable matrix P

$$P = \left[B \vdots AB \vdots A^2 B\right] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathscr{R}(P) = span\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\}, n_c = \text{rank}(P) = 2$$

Construct basis matrix M as

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

[ANSWER]: Use M to do the controllable decomposition

$$\hat{A} = M^{-1}AM = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad A_c = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\hat{C} = CM = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix} \qquad C_c = \begin{bmatrix} 0 & 3 \end{bmatrix}$$

The reduced state equation

$$\begin{cases} \dot{x} = A_c x + B_c u \\ y = C_c x \end{cases}$$

ii)

Check the observability of reduced system.

Since A_c is in upper triangular form, it's eigen values are $\lambda_1 = 0, \lambda_2 = -1$.

Use similarity transformation, Jordan form $\hat{A}_c$

$$M_{A_c} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad M_{A_c}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \hat{A}_c = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{B}_c = M_{A_c}^{-1} B_c = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \hat{C}_c = CM_{A_c} = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

The column according to $\lambda_2 = -1$ does not have full column rank.

[ANSWER]: the reduced state equation is NOT observable.

iii)

Since the reduced state equation is in controllable form, no uncontrollable mode exists.

[ANSWER]: the reduced state equation is stabilizable.

iv)

According to the Jordan form $\hat{A}_c$, the unobservable mode of the reduced system is

$$(\hat{A}_c)_{\bar{o}} = -1$$

The “eigen value” of $(\hat{A}_c)_{\bar{o}}$ is negative real number, i.e. the unobservable mode is asymptotic stable.

[ANSWER]: the reduced state equation is detectable.

Exercise 3. Stability

The none linear system in state space is

$$\dot{z} = \begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} f_1(z,u) \\ f_2(z,u) \\ f_3(z,u) \\ f_4(z,u) \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \frac{-u_1 \sin\theta + \epsilon u_2 \cos\theta}{m} \\ \frac{u_1 \cos\theta + \epsilon u_2 \sin\theta - mg}{m} \\ \frac{u_2}{J} \end{bmatrix}$$

Linearize the given system in the neighborhood of the equilibrium point $\tilde{z}, \tilde{u}$

$$\dot{z} = \begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(z,u)}{\partial z}|_{\tilde{z}} \\ \frac{\partial f_2(z,u)}{\partial z}|_{\tilde{z}} \\ \frac{\partial f_3(z,u)}{\partial z}|_{\tilde{z}} \\ \frac{\partial f_4(z,u)}{\partial z}|_{\tilde{z}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{-\tilde{u}_1 \cos\tilde{\theta} - \epsilon \tilde{u}_2 \sin\tilde{\theta}}{m} & 0 & 0 & 0 \\ \frac{-\tilde{u}_1 \sin\tilde{\theta} + \epsilon \tilde{u}_2 \cos\tilde{\theta}}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ \frac{-\sin\tilde{\theta}}{m} & \frac{\epsilon\cos\tilde{\theta}}{m} \\ \frac{\cos\tilde{\theta}}{m} & \frac{\epsilon \sin\tilde{\theta}}{m} \\ 0 & \frac{1}{J} \end{bmatrix} u$$

Substitute $\tilde{z} = \begin{bmatrix} 0 & 0 & 0 & \dot{\theta} \end{bmatrix}^T, \tilde{u} = [mg \quad 0]^T$

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix} u$$

Assume A

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \det(A - \lambda) = \lambda^4$$

The eigen values of A , $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Eigen values have zero real part but are defective.

[ANSWER]: The linearized model around the equilibrium solution is unstable. But the nonlinear system’s local stability may rely on the nonlinear components.

Exercise 4. Lyapunov's direct method

The given positive definite Lyapunov function $V(x)$ and the partial derivatives are continuous.

The derivative $\dot{V}(x)$ is

$$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

Substitute $\dot{x}_1 = ax_1, \dot{x}_2 = x_1 - x_2$,

$$\dot{V} = 2ax_1^2 + 2x_1x_2 - 2x_2^2$$

In order to make the system asymptotically stable, $\forall x_1, x_2 \neq 0$, $\dot{V} < 0$.

[ANSWER]: The range of variable a should be:

$$a < \frac{x_2^2 - x_1x_2}{x_1^2}$$

$x_2 = kx_1$

$$= \frac{k^2x_1^2 - kx_1^2}{x_1^2}$$

$$= (k^2 - k + \tfrac{1}{4}) - \tfrac{1}{4}$$

$$\geq -\tfrac{1}{4}$$

Exercise 5. Stability of Non-Linear Systems

(a)

In order to find the equilibrium point of the given nonlinear system,

Let $\dot{x} = 0$,

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) = x_2 - x_1x_2^2 = 0 \\ \dot{x}_2 = f_2(x_1, x_2) = -x_1^3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

The linearized system around the equilibrium point $\tilde{x} = [0 \quad 0]^T$is:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x}|_{\tilde{x}} \\ \frac{\partial f_2}{\partial x}|_{\tilde{x}} \end{bmatrix} = \begin{bmatrix} -\tilde{x}_2{}^2 & 1 - 2\tilde{x}_1\tilde{x}_2 \\ -3\tilde{x}_1^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since A is in upper triangular form, the eigen values are $\lambda_1 = \lambda_2 = 0$.

Both eigen values are zero real number and they are defective.

[ANSWER]: Based on Lyapunov's Indirect method, the linearized system is not stable.

(b)

The derivative $\dot{V}(x)$ is

$$\dot{V} = 4x_1^3\dot{x}_1 + 4x_2\dot{x}_2$$

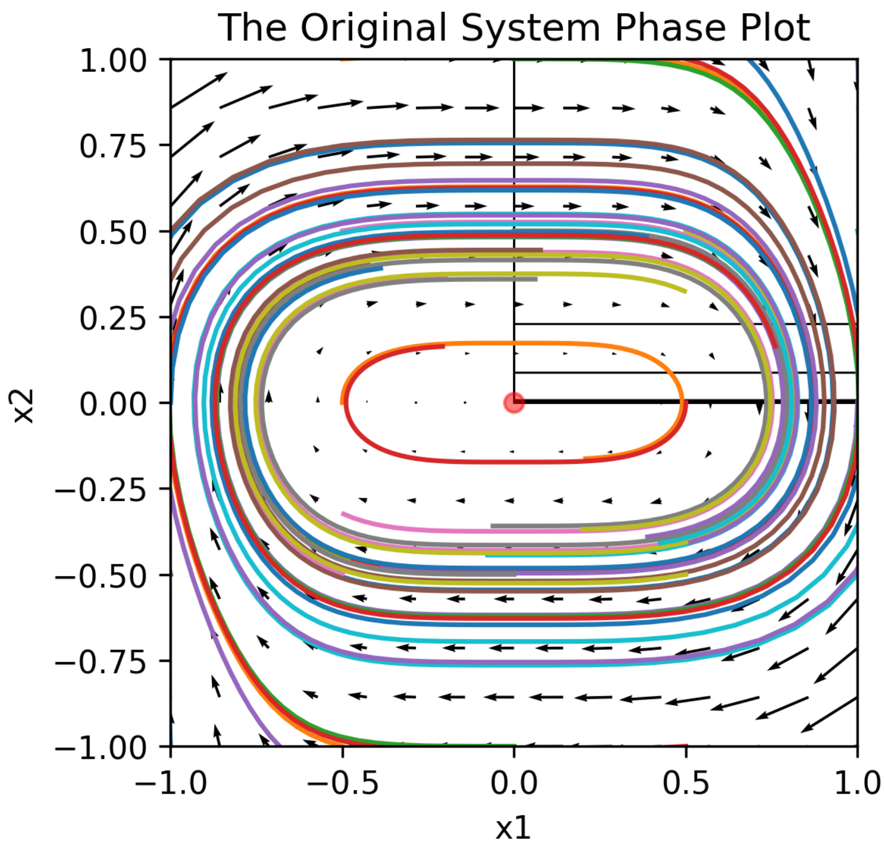
Substitute $\dot{x}_1 = x_2 - x_1x_2^2, \dot{x}_2 = -x_1^3$,

$$\dot{V} = 4x_1^3(x2 - x_1x_2^2) + 4x_2(-x_1^3) = -4x_1^4x_2^2 < 0, \forall x \neq 0$$

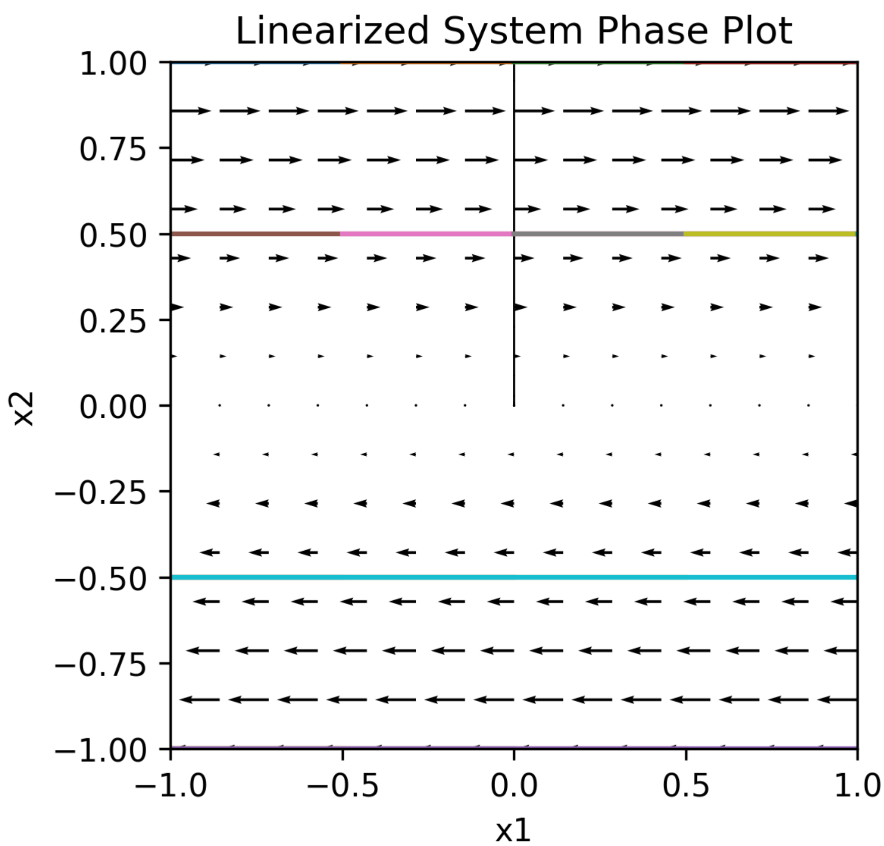
[ANSWER]: Based on Lyapunov's Direct method, the linearized system is stable.

(c)

[ANSWER]: The original system:

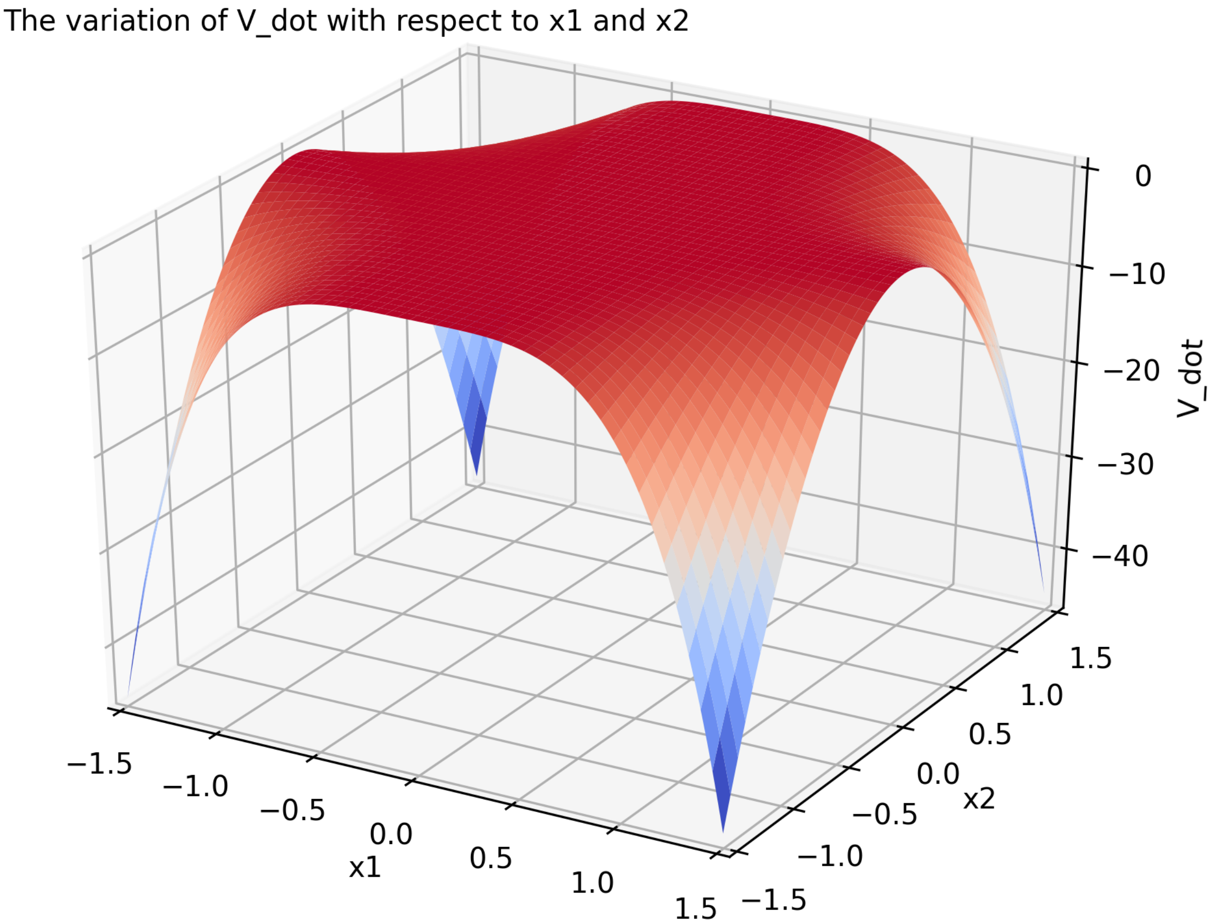


[ANSWER]: The linearized system:



(d)

[ANSWER]: the variation of $\dot{V}$ w.r.t x_1 and x_2



[The code for (c) and (d) is included in the end.]

Exercise 6. BIBO Stability

(a)

Use the given SS equation to generate the transfer function for the DT system,

$$zI - A = \begin{bmatrix} z-1 & 0 \\ 0.5 & z-0.5 \end{bmatrix}$$

$$(zI - A)^{-1} = \frac{1}{(z-1)(z-0.5)} \begin{bmatrix} z-0.5 & 0 \\ -0.5 & z-1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 5 \end{bmatrix}, \quad D = 0$$

$$G_D(s) = C(zI - A)^{-1}B + D = \frac{1}{(z-1)(z-0.5)} \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} z-0.5 & 0 \\ -0.5 & z-1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$\forall u(k), k \geq k_0, y(k) \equiv 0 \leq 0$

[ANSWER]: The system is BIBO stable.

(b)

Use the given SS equation to generate the transfer function for the CT system,

$$sI - A = \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^3 + 9s^2 + 23s + 15} \begin{bmatrix} s^2 + 2s - 7 & -2s - 10 & 6s + 22 \\ 2s + 2 & s^2 + 6s + 5 & -2s - 2 \\ -2s - 10 & -2s - 10 & s^2 + 10s + 25 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad D = 0$$

$$G_C(s) = C(sI - A)^{-1}B + D = \frac{\begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s^2 + 2s - 7 & -2s - 10 & 6s + 22 \\ 2s + 2 & s^2 + 6s + 5 & -2s - 2 \\ -2s - 10 & -2s - 10 & s^2 + 10s + 25 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}}{(s+1)(s+3)(s+5)}$$

$$= \frac{1}{(s+1)(s+3)(s+5)} \begin{bmatrix} 0 & 0 \\ (s+1)(s+5) & 0 \end{bmatrix} = \frac{1}{s+3} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The pole of $G_C(s)$ is -3, which is negative real number.

[ANSWER]: The system is BIBO stable.

Exercise 7. BIBO Stability

(1)

To write the SS equation of the given system,

choose state variables $x = [T_C \quad T_H]^T$

the input $u = [T_{Ci} \quad T_{Hi}]^T$

$$\dot{x} = \begin{bmatrix} -\frac{f_c+\beta}{V_C} & \frac{\beta}{V_C} \\ \frac{\beta}{V_C} & -\frac{f_H+\beta}{V_C} \end{bmatrix} x + \begin{bmatrix} f_C & 0 \\ 0 & f_H \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

[ANSWER]: Substitute $f_C = f_H = 0.1, \beta = 0.2, V_H = V_C = 1$

$$\dot{x} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} x + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

(2)

In the absence of any input, the solution of the given system can be get by integration of the system.

$$x(t) = e^{\begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}(t-t_0)} x(t_0)$$

Let $\det(A - \lambda I) = 0$,

$$\det\left(\begin{bmatrix} -0.3-\lambda & 0.2 \\ 0.2 & -0.3-\lambda \end{bmatrix}\right) = (\lambda+0.1)(\lambda+0.5) = 0$$

The eigen values of A are $\lambda_1 = -0.1, \lambda_2 = -0.5$. The eigen vectors are $v_1 = [1 \quad 1]^T$, $v_2 = [1 \quad -1]^T$.

The basis matrix M is

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

Use M to find the Jordan form of A

$$J = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.5 \end{bmatrix}$$

The matrix exponential

$$e^{A(t-t_0)} = Me^{J(t-t_0)}M^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-0.1(t-t_0)} & 0 \\ 0 & e^{-0.5(t-t_0)} \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

$$e^{A(t-t_0)} = \begin{bmatrix} \frac{1}{2}(e^{-0.1(t-t_0)} + e^{-0.5(t-t_0)}) & \frac{1}{2}(e^{-0.1(t-t_0)} - e^{-0.5(t-t_0)}) \\ \frac{1}{2}(e^{-0.1(t-t_0)} - e^{-0.5(t-t_0)}) & \frac{1}{2}(e^{-0.1(t-t_0)} + e^{-0.5(t-t_0)}) \end{bmatrix}$$

The output $y(t) = x(t)$

$$x(t) = \begin{bmatrix} \frac{1}{2}(e^{-0.1(t-t_0)} + e^{-0.5(t-t_0)}) & \frac{1}{2}(e^{-0.1(t-t_0)} - e^{-0.5(t-t_0)}) \\ \frac{1}{2}(e^{-0.1(t-t_0)} - e^{-0.5(t-t_0)}) & \frac{1}{2}(e^{-0.1(t-t_0)} + e^{-0.5(t-t_0)}) \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^{-0.1(t-t_0)} + e^{-0.5(t-t_0)})x_1(t_0) + \frac{1}{2}(e^{-0.1(t-t_0)} - e^{-0.5(t-t_0)})x_2(t_0) \\ \frac{1}{2}(e^{-0.1(t-t_0)} - e^{-0.5(t-t_0)})x_1(t_0) + \frac{1}{2}(e^{-0.1(t-t_0)} + e^{-0.5(t-t_0)})x_2(t_0) \end{bmatrix}$$

[ANSWER]: Assume $t_0 = 0$,

$$\begin{cases} y_1(t) = \frac{e^{-0.1t}+e^{-0.5t}}{2}x_1(0) + \frac{e^{-0.1t}-e^{-0.5t}}{2}x_2(0) \\ y_2(t) = \frac{e^{-0.1t}-e^{-0.5t}}{2}x_1(0) + \frac{e^{-0.1t}+e^{-0.5t}}{2}x_2(0) \end{cases}$$

(3)

Use the given SS equation to generate the transfer function for the CT system,

$$sI - A = \begin{bmatrix} s+0.3 & -0.2 \\ -0.2 & s+0.3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s+0.1)(s+0.5)} \begin{bmatrix} s+0.3 & 0.2 \\ 0.2 & s+0.3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0$$

$$G_C(s) = C(sI-A)^{-1}B + D = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+0.3 & 0.2 \\ 0.2 & s+0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}}{(s+0.1)(s+0.5)}$$

$$= \frac{0.1}{(s+0.1)(s+0.5)} \begin{bmatrix} s+0.3 & 0.2 \\ 0.2 & s+0.3 \end{bmatrix}$$

The poles of $G_C(s)$ are $-0.1, -0.5$, which are negative real numbers.

[ANSWER]: The system is BIBO stable.

Code for Ex.5 (c)

```
! pip install control  # need control library

import numpy as np
import matplotlib.pyplot as plt
from control.phaseplot import phase_plot

# Clear out any figures that are present
plt.close('all')

# Define the ODEs
def NLTI_ode(x, t):
    return [x[1] - x[0]*x[1]**2, -x[0]**3]

def LTI_ode(x, t):
    return [x[1], 0]

x = np.linspace(-1, 1, 5)
y = np.linspace(-1, 1, 5)
xv, yv = np.meshgrid(x, y)
X0 = np.hstack((xv.reshape(-1,1), yv.reshape(-1,1)))

# the original system
plt.figure(1, dpi = 300)
plt.gca().set_aspect('equal', adjustable='box')
plt.axis([-1, 1, -1, 1])
plt.title('The Original System Phase Plot')

phase_plot(
    NLTI_ode,
    X0 = X0,
    X = (-1, 1, 15),
    Y = (-1, 1, 15),
    T = 10
)
plt.scatter(x = 0,
            y = 0,
            c='r',
            alpha=0.5)
plt.show()

# linearized system
plt.figure(2, dpi = 300)
plt.gca().set_aspect('equal', adjustable='box')
plt.axis([-1, 1, -1, 1])
plt.title('Linearized System Phase Plot')

phase_plot(
    LTI_ode,
    X0 = X0,
    X = (-1, 1, 15),
    Y = (-1, 1, 15),
    T = 10
)
plt.show()
```

Code for Ex.5 (d)

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import cm
import mpl_toolkits.mplot3d

fig = plt.figure(figsize=(8, 6),dpi = 300)
ax = fig.add_subplot(111, projection='3d')
# ax = plt.subplots(subplot_kw={"projection": "3d"})

x1 = np.arange(-1.5, 1.5, 0.005)
x2 = np.arange(-1.5, 1.5, 0.005)
x1, x2 = np.meshgrid(x1, x2)
V_dot = -4*np.power(x1, 4)*np.power(x2,2)

ax.text2D(0.05, 0.95, "The variation of V_dot with respect to x1 and x2",
transform=ax.transAxes)
surf = ax.plot_surface(x1, x2, V_dot, cmap=cm.coolwarm)
ax.set_xlim(-1.5, 1.5)
ax.set_ylim(-1.5, 1.5)
ax.set_xlabel('x1')
ax.set_ylabel('x2')
ax.set_zlabel('V_dot')
plt.show()
```