Homework 4

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Exercise 1. Canonical forms

The given transfer function is in proper rational form, i.e. the TF is realizable.

The coefficient of the numerator denominator are:

$$\begin{cases} a_0 = 2, a_1 = 3 \\ b_0 = 3, b_1 = 1, b_2 = 0 \end{cases}$$

[ANSWER]: The controllable canonical form state space representation is given by:

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} b_0 & b_1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \end{bmatrix} \qquad D = b_2 = 0$$

Exercise 2. Realization matrix form of realizable MIMO system

Each elements of the given transfer function matrix is in proper rational form, i.e. the $\hat{G}_1(s)$ is realizable.

To find the strictly proper TF matrix, $\lim s \to \infty$:

$$\hat{G}_1(\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

The strictly proper TF matrix $\hat{G}_{1sp}(s)$:

$$\hat{G}_{1sp}(s) = \begin{bmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{1}{s+3} & \frac{-1}{s+1} \end{bmatrix}$$

The least common denominator is $\Delta(s) = s(s+1)(s+3) = s^3 + 4s^2 + 3s$.

Rewrite $\hat{G}_{1sp}(s)$ as:

$$\hat{G}_{1sp}(s) = \frac{1}{\Delta(s)}(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} s^2 + \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix} s + \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix})$$

Therefore,

$$N_1 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \quad N_2 = \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix} \quad N_3 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\alpha_1 = 4 \qquad \alpha_2 = 3 \qquad \alpha_3 = 0$$

[ANSWER]: The realization matrix form of realizable MIMO system is given by:

$$A = \begin{bmatrix} -\alpha_1 I_2 & -\alpha_2 I_2 & -\alpha_3 I_3 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} \qquad D = \hat{G}_1(\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise 3. Minimum Realizations

i)

for the first SS equation:

$$sI - A = \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}$$
$$\det(sI - A) = \Delta(s) = s^2 - 3s + 2$$
$$(sI - a)^{-1} = \frac{Adj(sI-A)}{\Delta(s)} = \frac{1}{\Delta(s)} \begin{bmatrix} s-1 & 1 \\ 0 & s-2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$
$$C(sI-A)^{-1}B + D = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{s-2}$$

To check controllability and observability:

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \mathrm{rank}(P) < 2 = n$$
$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \qquad \mathrm{rank}(Q) < 2 = n$$

ii)

for the second SS equation:

$$sI - A = \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}$$
$$\det(sI - A) = \Delta(s) = s^2 - s - 2$$
$$(sI - a)^{-1} = \frac{Adj(sI-A)}{\Delta(s)} = \frac{1}{\Delta(s)} \begin{bmatrix} s+1 & 0 \\ -1 & s-2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-2} & 0 \\ -\frac{1}{(s+1)(s-2)} & \frac{1}{s+1} \end{bmatrix}$$
$$C(sI-A)^{-1}B + D = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} & 0 \\ -\frac{1}{(s+1)(s-2)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{s-2}$$

To check controllability and observability:

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \qquad \mathrm{rank}(P) = 2 = n$$
$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} \qquad \mathrm{rank}(Q) < 2 = n$$

[ANSWER]: These two SS equations are equivalent for they have the same transfer function. But none of them are minimal realizations, given the first equation is neither controllable nor observable, and the second equation is only controllable but not observable.

Exercise 4. Realization

The given transfer function is in proper rational form, i.e. the TF is realizable.

The coefficient of the numerator denominator are:

$$\begin{cases} a_0 = 6, a_1 = -7, a_2 = 0 \\ b_0 = -4, b_1 = 2, b_2 = 0, b_3 = 0 \end{cases}$$

(a)

the standard controllable realization:

[ANSWER]: The controllable canonical form state space representation is given by:

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & 7 & 0\end{bmatrix}\qquad B=\begin{bmatrix}0\\0\\1\end{bmatrix}$$
$$C=\begin{bmatrix}b_0 & b_1 & b_2\end{bmatrix}=\begin{bmatrix}-4 & 2 & 0\end{bmatrix}\qquad D=b_3=0$$

(b)

the standard observable realization:

[ANSWER]: The observable canonical form state space representation is given by:

$$A=\begin{bmatrix}-a_2 & 1 & 0\\ -a_1 & 0 & 1\\ -a_0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 7 & 0 & 1\\ -6 & 0 & 0\end{bmatrix}\qquad B=\begin{bmatrix}b_2\\b_1\\b_0\end{bmatrix}=\begin{bmatrix}0\\2\\-4\end{bmatrix}$$
$$C=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\qquad D=b_3=0$$

(c)

the minimal realization:

the given transfer function can be further derived as:

$$g(s)=\frac{2s-4}{s^3-7s+6}=\frac{2}{s^2+2s-3}$$

The coefficient of the numerator denominator are:

$$\begin{cases}a_0=-3, a_1=2\\ b_0=2, b_1=0, b_2=0\end{cases}$$

The controllable canonical form state space representation is:

$$A=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 3 & -2\end{bmatrix}\qquad B=\begin{bmatrix}0\\1\end{bmatrix}$$
$$C=\begin{bmatrix}b_0 & b_1\end{bmatrix}=\begin{bmatrix}2 & 0\end{bmatrix}\qquad D=b_2=0$$

Check the observability:

$$Q=\begin{bmatrix}C\\CA\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\qquad \mathrm{rank}(Q)=2=n$$

[ANSWER]: The minimal realization is same as the controllable canonical form:

$$A=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 3 & -2\end{bmatrix}\qquad B=\begin{bmatrix}0\\1\end{bmatrix}$$
$$C=\begin{bmatrix}b_0 & b_1\end{bmatrix}=\begin{bmatrix}2 & 0\end{bmatrix}\qquad D=b_2=0$$

Exercise 5. Controllable decomposition

i)

The controllable matrix P:

$$P = \begin{bmatrix} B & \vdots & AB \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \qquad n_c = \operatorname{rank}(P) = 1 < n = 2 \qquad \mathscr{R}(P) = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$$

Construct basis matrix M as:

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Use M to do the controllable decomposition:

$$\hat{A} = M^{-1}AM = \begin{bmatrix} -1 & 4 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{C} = CM = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

[ANSWER]: The reduced state equation is:

$$A_c = 3, B_c = 1, C_c = 2, D = 0$$

$$\begin{cases} \dot{\hat{x}}_c = 3\hat{x}_c + u \\ y = 2\hat{x}_c \end{cases}$$

ii)

[ANSWER]: The reduced state equation is observable, since $\operatorname{rank}(Q) = 1 = n_c$

Exercise 6. Kalman decomposition

In order to calculate the controllable and observable matrix, first calculate the power of A matrix.

Since A matrix is in Jordan form, calculated the power of each Jordan blocks:

$$A^2 = \begin{bmatrix} \lambda_1^2 & 2\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^2 & 2\lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2^2 & 2\lambda_2 \\ 0 & 0 & 0 & 0 & \lambda_2^2 \end{bmatrix} \quad A^3 = \begin{bmatrix} \lambda_1^3 & 3\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1^3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^3 & 3\lambda_2^2 & 3\lambda_2 \\ 0 & 0 & 0 & \lambda_2^3 & 3\lambda_2^2 \\ 0 & 0 & 0 & 0 & \lambda_2^3 \end{bmatrix} \quad A^4 = \begin{bmatrix} \lambda_1^4 & 4\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1^4 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^4 & 4\lambda_2^3 & 6\lambda_2^2 \\ 0 & 0 & 0 & \lambda_2^4 & 4\lambda_2^3 \\ 0 & 0 & 0 & 0 & \lambda_2^4 \end{bmatrix}$$

The controllable matrix is:

$$P = \begin{bmatrix} B & \vdots AB & \vdots A^2B & \vdots A^3B & \vdots A^4B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\operatorname{rank}(P) = 3 \qquad \mathscr{R}(P) = span\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\}$$

The observable matrix is:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & \lambda_1 & \lambda_2 & 1 & \lambda_2 \\ 0 & \lambda_1^2 & \lambda_2^2 & 2\lambda_2 & \lambda_2^2+1 \\ 0 & \lambda_1^3 & \lambda_2^3 & 3\lambda_2^2 & \lambda_2^3+3\lambda_2 \\ 0 & \lambda_1^4 & \lambda_2^4 & 4\lambda_2^3 & \lambda_2^4+6\lambda_2^2 \end{bmatrix}$$

$$\text{rank}(Q) = 4 \qquad \mathscr{N}(Q) = span\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\}$$

Construct basis matrix M as:

$$M = [m_1 \quad m_2 \quad m_3 \quad m_4] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

m_2 is the intersect of $\mathscr{R}(P)$ and $\mathscr{N}(Q)$, $m_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0]^T$

m_1 is subtracting m_2 from $\mathscr{R}(P)$, $m_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$

m_4 is subtracting m_2 from $\mathscr{N}(Q)$, $m_4 = \emptyset$

m_3 is subtracting m_1, m_2, m_4 from $\mathbb{R}^5$, $m_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$

Use M to do the full decomposition:

$$\hat{A} = M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix} \qquad \hat{C} = CM = [1 \quad 1 \quad \cdots]$$

[ANSWER]: The full decomposition is:

$$A_{co} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad B_{co} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C_{co} = [1 \quad 1]$$

Exercise 7. Controllable Canonical Form

The overall transfer function is:

$$\frac{\Theta(s)}{V(s)} = \frac{k_T}{(Ls+R)(Js^2+bs)} = \frac{\frac{k_T}{LJ}}{s^3 + \frac{RJ+Lb}{LJ}s^2 + \frac{Rb}{LJ}s}$$

The given transfer function is in proper rational form, i.e. the TF is realizable.

The coefficient of the numerator denominator are:

$$\begin{cases} a_0 = 0, a_1 = \frac{Rb}{LJ}, a_2 = \frac{RJ+Lb}{LJ} \\ b_0 = \frac{k_T}{LJ}, b_1 = 0, b_2 = 0, b_3 = 0 \end{cases}$$

[ANSWER]: The controllable canonical form state space representation is given by:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{Rb}{LJ} & -\frac{RJ+Lb}{LJ} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{k_T}{LJ} & 0 & 0 \end{bmatrix} \qquad D = b_3 = 0$$