Homework 2

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Exercise 1. Cayley-Hamilton Theorem

Given that A is 3×3 upper-triangle matrix, it has three eigen values on the diagonal: $\lambda_1 = \lambda_3 = 1, \lambda_2 = 0$.

The polynomial of A, $P(A) = \beta_2 A^2 + \beta_1 A + \beta_0 I$.

Since A has two equal eigen value, $P'(A) = 2\beta_2 A + \beta_1$ will be needed.

1)

For A^{10}, the eigen values satisfy:

$$\begin{cases} \lambda^{10} = \beta_2\lambda^2 + \beta_1\lambda + \beta_0 \ for \lambda_1, \lambda_2 \\ 10\lambda^9 = 2\beta_2\lambda + \beta_1 \ for \lambda_3 \end{cases}$$

Solve for $\beta = [\beta_2, \beta_1, \beta_0]^T = [-8, 9, 0]^T$

[ANSWER] Therefore, A^{10} can be calculated as:

$$A^{10} = -8A^2 + 9A = A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2)

For e^{At}, the eigen values satisfy:

$$\begin{cases} e^{\lambda t} = \beta_2\lambda^2 + \beta_1\lambda + \beta_0 \ for \lambda_1, \lambda_2 \\ te^{\lambda t} = 2\beta_2\lambda + \beta_1 \ for \lambda_3 \end{cases}$$

Solve for $\beta = [\beta_2, \beta_1, \beta_0]^T = [(t-1)e^t + 1, (2-t)e^t - 2, 1]^T$

[ANSWER] Therefore, e^{At} can be calculated as:

$$e^{At} = [(t-1)e^t + 1]A^2 + [(2-t)e^t - 2]A + I = (e^t - 1)A + I = \begin{bmatrix} e^t & e^t - 1 & 0 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

Exercise 2. Linear dynamics solution

Assume $x = [x_1, x_2]^T$, the differential equation can be written in state space:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Where $A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The equation can be solved as:

$$x(t) = e^{A(t)}x(0) + \int_0^t e^{A(t-\tau)}Bu \cdot d\tau$$

A is lower triangle matrix, which has two eigen values on the diagonal: $\lambda_1 = -\alpha, \lambda_2 = -\beta$.

Given $\alpha = 0.1 \neq \beta = 0.2$, $e^{A(t)} = k_1 A(t) + k_0 I$.

Solve for $k = [k_1, k_0]^T = \begin{bmatrix} \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \\ \frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\beta - \alpha} \end{bmatrix} = \begin{bmatrix} 10e^{-0.1t} - 10e^{-0.2t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix}$

Therefore,

$$e^{A(t)}x(0) = \begin{bmatrix} k_0 - \alpha k_1 & 0 \\ \alpha k_1 & k_0 - \beta k_1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2k_0 - 2\alpha k_1 \\ 2\alpha k_1 + k_0 - \beta k_1 \end{bmatrix} = \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix}$$

Similarly,

$$e^{A(t-\tau)}Bu = \begin{bmatrix} k_0(t-\tau) - \alpha k_1(t-\tau) & 0 \\ \alpha k_1(t-\tau) & k_0(t-\tau) - \beta k_1(t-\tau) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k_0(t-\tau) - \alpha k_1(t-\tau) \\ \alpha k_1(t-\tau) \end{bmatrix}$$

In order to compute $\int e^{A(t-\tau)}Bu \cdot d\tau$, first compute:

$$\begin{aligned} I_1 &= \int_0^t k_1(t-\tau)d\tau = 10(\frac{e^{-0.1(t-\tau)}}{0.1} - \frac{e^{-0.2(t-\tau)}}{0.2})|_0^t \\ &= -100e^{-0.1t} + 50e^{-0.2t} + 50 \\ I_0 &= \int_0^t k_0(t-\tau)d\tau = (2\frac{e^{-0.1(t-\tau)}}{0.1} - \frac{e^{-0.2(t-\tau)}}{0.2})|_0^t \\ &= -20e^{-0.1t} + 5e^{-0.2t} + 15 \end{aligned}$$

The original integer can be compute as:

$$\int_0^t e^{A(t-\tau)}Bu \cdot d\tau = \begin{bmatrix} I_0 - \alpha I_1 \\ \alpha I_1 \end{bmatrix} = \begin{bmatrix} -10e^{-0.1t} + 10 \\ -10e^{-0.1t} + 5e^{-0.2t} + 5 \end{bmatrix}$$

Accordingly, the solution is:

$$x(t) = \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \begin{bmatrix} -10e^{-0.1t} + 10 \\ -10e^{-0.1} + 5e^{-0.2t} + 5 \end{bmatrix} = \begin{bmatrix} -8e^{-0.1t} + 10 \\ -8e^{-0.1t} + 4e^{-0.2t} + 5 \end{bmatrix}$$

[ANSWER] The water level in both tanks after $5s$ is:

$$x(5) = \begin{bmatrix} -8e^{-0.5} + 10 \\ -8e^{-0.5} + 4e^{-1} + 5 \end{bmatrix} = \begin{bmatrix} 5.1478 \\ 1.6193 \end{bmatrix}$$

Exercise 3. Jordan form, decomposition

1)

For A_1, A_1 is 3×3 upper-triangle matrix, it has three eigen values on the diagonal: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

The eigen vectors can be: $v_1 = [1, 0, 0]^T$, $v_2 = [4, 1, 0]^T$, $v_3 = [4, 0, 1]^T$.

$p = 3, m_1 = m_2 = m_3 = 1$,$q_1 = q_2 = q_3 = 1$.

Therefore, no generalized eigenvectors are required.

[ANSWER] The Jordan-form is:

$$J_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$M = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2)

For A_2, assume $\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix}\right) = 0$,

i.e. $\lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$.

Solve for $\lambda_1 = -1, \lambda_2 = -1 + i, \lambda_3 = -1 - i$.

The eigen vectors can be: $v_1 = [1, -1, 1]^T$, $v_2 = [1, i - 1, -2i]^T$, $v_3 = [1, -i - 1, 2i]^T$.

$p = 3, m_1 = m_2 = m_3 = 1$, $q_1 = q_2 = q_3 = 1$.

Therefore, no generalized eigenvectors are required.

[ANSWER] The Jordan-form is:

$$J_2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix}$$

$$M = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & i-1 & -i-1 \\ 1 & -2i & 2i \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ -0.5-0.5i & -1-0.5i & -0.5 \\ -0.5+0.5i & -1+0.5i & -0.5 \end{bmatrix}$$

3)

For A_3 is 3×3 upper-triangle matrix, it has three eigen values on the diagonal: $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$.

The eigen vectors can be: $v_1 = [1,0,0]^T$, $v_2 = [0,1,0]^T$, $v_3 = [1,0,-1]^T$.

$p = 2, m_1 = 2, m_2 = 1, q_1 = 2, q_2 = 1$.

Therefore, no generalized eigenvectors are required.

[ANSWER] The Jordan-form is:

$$J_3 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$M = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4)

For A_4, Assume $\det\left(\begin{bmatrix} -\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{bmatrix}\right) = 0$,

i.e. $\lambda^3 = 0$. Solve for $\lambda_1 = \lambda_2 = \lambda_3 = 0$. $p = 1, m_1 = 3$, $q_1 = 1$.

Therefore, two generalized eigenvectors are required.

For $(A_4 - \lambda I)v_1 = 0$, $v_1 = [1,0,0]^T$

Let $(A_4 - \lambda I)v_2 = v_1$, $\begin{bmatrix} 4 & 3 \\ 20 & 16 \end{bmatrix} \begin{bmatrix} v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = [0,4,-5]^T$

Let $(A_4 - \lambda I)v_3 = v_2$, $\begin{bmatrix} 4 & 3 \\ 20 & 16 \end{bmatrix} \begin{bmatrix} v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $v_3 = [0,-3,4]^T$

[ANSWER] The Jordan-form is:

$$J_4 = M^{-1}A_4M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 5 & 4 \end{bmatrix}$$

Exercise 4. CT and DT dynamics

i)

Assume $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = [2, 3]$, $D = 0$.

Let $\det{(A - \lambda I)} = 0$, the eigen value of A is $\lambda_1 = -1 + i, \lambda_2 = -1 - i$.

Assume $e^{At} = \beta_1 A + \beta_0 I$, then the eigen values follow:

$$\begin{cases} e^{\lambda_1} = \beta_1 \lambda_1 + \beta_0 \\ e^{\lambda_2} = \beta_1 \lambda_2 + \beta_0 \end{cases}$$

Solve for

$$\beta = [\beta_1, \beta_0]^T = \begin{bmatrix} \frac{e^{(-1+i)t} - e^{(-1-i)t}}{2i} \\ \frac{(-1+i)e^{(-1-i)t} - (-1-i)e^{(-1+i)t}}{2i} \end{bmatrix}$$

Given $x(0) = 0$, $D = 0$, the solution can be written as:

$$y(t) = C \int_0^t e^{A(t-\tau)} Bu \cdot d\tau$$

Where $Ce^{A(t-\tau)}Bu$ is:

$$Ce^{A(t-\tau)}Bu = \beta_1(t-\tau)[2,3] \begin{bmatrix} 1 \\ -4 \end{bmatrix} + \beta_0(t-\tau)[2,3] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -10\beta_1(t-\tau) + 5\beta_0(t-\tau)$$

The integer can be calculated in these terms:

$$\int_0^t \beta_1(t-\tau) = \int_0^t \frac{e^{(-1+i)(t-\tau)} - e^{(-1-i)(t-\tau)}}{2i} d\tau = \frac{1}{2} - \frac{1+i}{4i} e^{(-1+i)t} + \frac{1-i}{4i} e^{(-1-i)t}$$

$$\int_0^t \beta_0(t-\tau) = \int_0^t \frac{(-1+i)e^{(-1-i)(t-\tau)} - (-1-i)e^{(-1+i)(t-\tau)}}{2i} d\tau = 1 - \frac{1}{2} e^{(-1+i)t} + \frac{1}{2} e^{(-1-i)t}$$

The solution for $y(t)$ is:

$$y(t) = -5 + \frac{5 \cdot (1+i)}{2 \cdot i} e^{(-1+i)t} - \frac{5 \cdot (1-i)}{2 \cdot i} e^{(-1-i)t} + 5 - \frac{5}{2} e^{(-1+i)t} - \frac{5}{2} e^{(-1-i)t}$$
$$i.e. \quad y(t) = 5e^{-t} \cdot \frac{e^{it} - e^{-it}}{2i} = 5e^{-t} \sin{(t)}$$

[ANSWER] Therefore, $y(5) = 5e^{-5} \sin{(5)} = -0.0323$.

ii)

The discrete state space representation follows:

$$\begin{aligned} A_d &= e^{AT} & B_d &= A^{-1}(A_d - I)B \\ C_d &= C & D_d &= D \end{aligned}$$

A_d can be expend with C-H therom:

$$A_d = e^{AT} = \beta_1(T)A + \beta_0 I$$

Where $\beta(T)$ is:

$$[\beta_1(T), \beta_0(T)]^T = \begin{bmatrix} \frac{e^{(-1+i)T} - e^{(-1-i)T}}{2i} \\ \frac{(-1+i)e^{(-1-i)T} - (-1-i)e^{(-1+i)T}}{2i} \end{bmatrix} = \begin{bmatrix} e^{-T}\sin(T) \\ e^{-T}(\sin(T) + \cos(T)) \end{bmatrix}$$

Therefore,

$$A_d = e^{-T}\begin{bmatrix} \sin(T) + \cos(T) & \sin(T) \\ -2\sin(T) & \cos(T) - \sin(T) \end{bmatrix}$$

$$B_d = \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-T}(\sin(T) + \cos(T)) - 1 & e^{-T}\sin(T) \\ -2e^{-T}\sin(T) & e^{-T}(\cos(T) - \sin(T)) - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-T}(\frac{1}{2}\sin(T) + \frac{3}{2}\cos(T)) + \frac{3}{2} \\ e^{-T}(2\sin(T) + \cos(T)) - 1 \end{bmatrix}$$

[ANSWER] Given time $T = 1$, the discretized state space representation is:

$$A_d = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} \qquad B_d = \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix}$$
$$C_d = [2, 3] \qquad D_d = 0$$

iii)

Given $x(0) = 0, D_d = 0$, the solution of the discrete time system is:

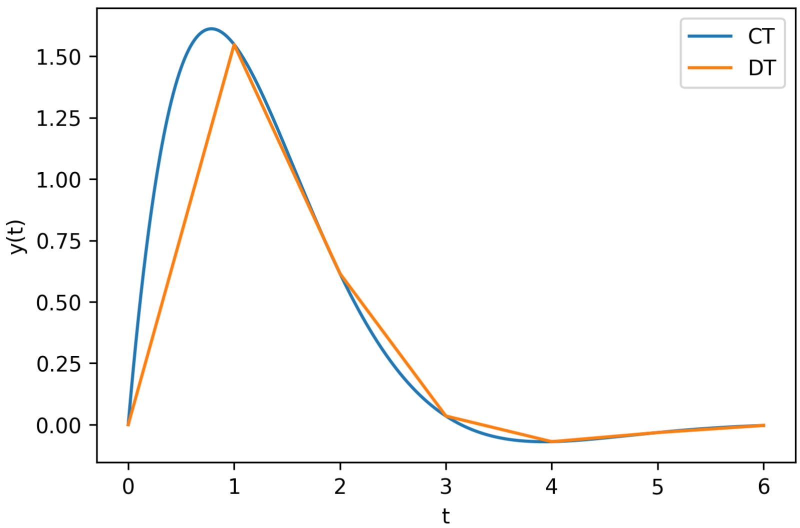
$$y(k) = \sum_{m=0}^{k-1} C_d A_d^{k-m-1} B_d u(m) = [2, 3] \sum_{m=0}^{k-1} \begin{bmatrix} 0.5083^{k-m-1} & 0.3096^{k-m-1} \\ -0.6191^{k-m-1} & -0.1108^{k-m-1} \end{bmatrix} \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix}$$

[ANSWER] For timestep $k = 5$:

$$y(5) = [2, 3] \sum_{m=0}^{4} \begin{bmatrix} 0.5083^{4-m} & 0.3096^{4-m} \\ -0.6191^{4-m} & -0.1108^{4-m} \end{bmatrix} \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix} = -0.0323$$

(The summation above is calculated through code. See Ex.4, part iii), "by calculation".)

[ANSWER] The plot of both CT and DT are shown below:



Exercise 5. Diagonalization

Assume $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$,

Therefore $x(k+1) = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} x_2(k) \\ x_1(k) + x_2(k) \end{bmatrix}$.

Using discrete state space representation:

$$A_d = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$C_d = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D_d = 0$$

Given $B_d = [0, 0]^T$, $D_d = 0$, the solution of the DT system can be written as:

$$y(k) = C_d \cdot (A_d)^k \cdot x(0)$$

Where $x(0) = [0, 1]^T$, $A_d = M(\hat{A})^k M^{-1}$.

Let $\det(A_d - \lambda I) = 0$, the eigen values are: $\lambda_1 = \frac{1-\sqrt{5}}{2}$, $\lambda_2 = \frac{1+\sqrt{5}}{2}$.

The eigen vectors can be: $v_1 = [1, \frac{1-\sqrt{5}}{2}]^T$, $v_2 = [1, \frac{1+\sqrt{5}}{2}]^T$.

The similarity transformation is:

$$A_d = M(\hat{A})M^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 0 \\ 0 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{5+\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ \frac{5-\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \end{bmatrix}$$

The solution can be rewritten as:

$$y(k) = C_d \cdot M(\hat{A})^k M^{-1} \cdot x(0)$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} (\frac{1-\sqrt{5}}{2})^k & 0 \\ 0 & (\frac{1+\sqrt{5}}{2})^k \end{bmatrix} \begin{bmatrix} \frac{5+\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ \frac{5-\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

[ANSWER] The 20th Fibonacci number $y(20) = 6765$.

(The matrix multiplication is done by the code. See Ex.5, "by calculation")

Python Code for Ex.4

```
''' HW2_Exercise4_CT_and_DT_Dynamics.py '''
import numpy as np
from scipy.signal import StateSpace, lsim, dlsim
import matplotlib.pyplot as plt

''' Using python programming to solve Exercise 4. CT and DT dynamics '''
# define control parameters
A = np.asarray([[0., 1.],
                [-2., -2.]])
B = np.asarray([[1.],
                [1.]])
C = np.asarray([[2., 3.]])
D = np.asarray([[0.]])
T = 1
```

```
# define control system
ctSystem = StateSpace(A, B, C, D)
dtSystem = ctSystem.to_discrete(T)

# simulate
t_max=6.
t_ct = np.arange(0, t_max, 1e-3)
t_dt = np.arange(0, t_max+T, T)

u_ct = np.ones(t_ct.size)
u_dt = np.ones(t_dt.size)

_, y_ct, x_ct = lsim(ctSystem, u_ct, t_ct, [0.,0.])
t_dt, y_dt, x_dt = dlsim(dtSystem, u_dt, t_dt,  [0.,0.])

''' i) Find y(5) for CT system '''
print("y(5) = \n{}\n".format(y_ct[t_ct==5]))

''' ii) discretized state space representation '''
print("Ad = \n{}".format(dtSystem.A))
print("Bd = \n{}".format(dtSystem.B))
print("Cd = \n{}".format(dtSystem.C))
print("Dd = \n{}\n".format(dtSystem.D))

''' iii) Find y(5) for DT system '''
# by simulation
print("By simulation, y(5) = \n{}".format(y_dt[t_dt==5]))

# by calculation
Ad = np.asmatrix([[np.exp(-T)*(np.sin(T)+np.cos(T)),np.exp(-T)*np.sin(T)],
                  [-2*np.exp(-T)*np.sin(T),np.exp(-T)*(np.cos(T)-np.sin(T))]])
Bd = np.asmatrix([[-np.exp(-T)*(.5*np.sin(T)+1.5*np.cos(T))+1.5],
                  [np.exp(-T)*(2*np.sin(T)+np.cos(T))-1]])
Cd = np.asmatrix([[2.,3.]])
y_cal=0.
for m in range(0,5):
  M = Cd @ (Ad**(4-m)) @ Bd
  y_cal = y_cal + M
print("By calculation, y(5) = \n{}\n".format(y_cal))

# plot y(t) for both CT and DT
plt.figure(dpi=300)
plt.plot(t_ct, y_ct,label='CT')
plt.plot(t_dt, y_dt,label='DT')
plt.legend()
plt.ylabel('y(t)')
plt.xlabel('t')
plt.show()
```

Python Code for Ex.5

```
''' HW2_Exercise5_Diagonalization '''
import numpy as np
from scipy.signal import dlti

''' Using python programming to solve Exercise 5. Diagonalization '''
# define control parameters
Ad = np.asarray([[0., 1.],
```

```
                [1., 1.]])
Bd = np.asarray([[0.],
                [0.]])
Cd = np.asarray([[1., 0.]])
Dd = np.asarray([[0.]])
T = 1
x0 = np.asarray([[0.],
                 [1.]])

# define control system
dtSystem = dlti(Ad,Bd,Cd,Dd)

# simulate
t_max=20.
t_dt = np.arange(0, t_max+T, T)
u_dt = np.zeros(t_dt.size)

# by simulation
t_dt, y_dt, x_dt = dlsim(dtSystem, u_dt, t_dt,  x0.T)
print("F(20) = \n{}".format(y_dt[t_dt==20]))

# by calculation
M = np.asmatrix([[1,1],[(1-np.sqrt(5))/2,(1+np.sqrt(5))/2]])
MI = M.I
A_hat = np.asmatrix([[((1-np.sqrt(5))/2)**20,0],[0,((1+np.sqrt(5))/2)**20]])
F_20 = Cd@M@A_hat@MI@x0
print("F(20) = \n{}\n".format(F_20))
```