Homework 3

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Exercise 1. Controllability and Observability

For the given state equation:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \qquad n = 3$$

To find eigen values for A:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -3 & -3-\lambda \end{bmatrix}\right) = -(\lambda+1)^3$$

Set $\det(A - \lambda I) = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

To use PBH test:

$$\lambda I - A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

i)

For controllability test:

$$\begin{bmatrix} \lambda I - A & \vdots & B \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

Utilize Gaussian Elimination:

$$\xRightarrow{R3+R1} \begin{bmatrix} -1 & -1 & 0 & 1 & \\ 0 & -1 & -1 & 0 & \\ 0 & 2 & 2 & & 1 \end{bmatrix}$$
$$\xRightarrow{R3+2R2} \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\text{rank}(\begin{bmatrix} \lambda I - A \vdots B \end{bmatrix}) = 3 = n$,

[ANSWER]: Given system is controllable.

ii)

For observability test:

$$\begin{bmatrix} \lambda I - A \\ \cdots \\ C \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Utilize Gaussian Elimination:

$$\overset{R3+R1,R4+R1}{\Longrightarrow} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\overset{R3+2R2,R4+R2}{\Longrightarrow} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\operatorname{rank}(\begin{bmatrix} \lambda I - A \\ \cdots \\ C \end{bmatrix}) = 2 < n = 3$,

[ANSWER]: Given system is NOT observable.

Exercise 2. Jordan form test

For the given Jordan-form state equation:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For $\lambda = 2$:

$$\hat{B}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \overset{R1-R2,R3-3R2}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \overset{R3+R2,R2-R1}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\hat{C}^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \overset{R1-R2,R2-R3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \overset{R2-R1}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \overset{C3-C1-C2}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

For $\lambda = 1$:

$$\hat{B}^1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \overset{R1-R2}{\Longrightarrow} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\hat{C}^1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \overset{R1+R3}{\Longrightarrow} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Since $\text{rank}_{row}(\hat{B}^2) = 3$, $\text{rank}_{row}(\hat{B}^1) = 2$, both $\hat{B}^1$ and $\hat{B}^2$ has full row rank,

[ANSWER]: Given system is controllable.

Since $\text{rank}_{column}(\hat{C}^2) = 2 < 3$, $\text{rank}_{column}(\hat{C}^1) = 2$, $\hat{C}^2$ doesn't have full column rank,

[ANSWER]: Given system is NOT observable.

Exercise 3. Controllability

i)

The system from last week in state space is:

$$A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad n = 2$$

For this system, the controllability matrix P :

$$P = \begin{bmatrix} B & \vdots & AB \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix} \overset{R1+R2}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

Since $\alpha \neq 0, \text{rank}(P) = 2 = n$,

[ANSWER]: According to the rank test result, given system is controllable.

ii)

Assume $x = [x_1, x_2]^T$, the differential equation can be written in state space:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The current system in state space is:

$$A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad n = 2$$

For this system, the controllability matrix P :

$$P = \begin{bmatrix} B & \vdots & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & \beta \end{bmatrix} \overset{R2-\beta R1}{\Longrightarrow} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Since $\text{rank}(P) = 1 < n = 2$,

[ANSWER]: According to the rank test result, given system is NOT controllable.

Exercise 4. Gauss Elimination and LU Decomposition

1.

For equation a), it can be written as:

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 0\\ 3\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1 & \vdots & 3\\ 1 & 2 & 3 & \vdots & 0\\ 1 & 3 & 2 & \vdots & 3\end{bmatrix}\xrightarrow{R2-R1,R3-R1}\begin{bmatrix}1 & 1 & 1 & \vdots & 3\\ 0 & 1 & 2 & \vdots & -3\\ 0 & 2 & 1 & \vdots & 0\end{bmatrix}\xrightarrow{R1-R2,R3-2R2}\begin{bmatrix}1 & 0 & -1 & \vdots & 6\\ 0 & 1 & 2 & \vdots & -3\\ 0 & 0 & -3 & \vdots & 6\end{bmatrix}$$

$$\begin{cases}-3z=6\\ y+2z=-3\\ x-z=6\end{cases}\Rightarrow\begin{cases}z=-2\\ y=1\\ x=4\end{cases}$$

[ANSWER]: $x=4, y=1, z=-2$.

For equation b), it can be written as:

$$\begin{bmatrix}1 & 2 & -1\\ 2 & 5 & -1\\ 1 & 3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 3\\ 6\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & -1 & \vdots & 1\\ 2 & 5 & -1 & \vdots & 3\\ 1 & 3 & 2 & \vdots & 6\end{bmatrix}\xrightarrow{R2-2R1,R3-R1}\begin{bmatrix}1 & 2 & -1 & \vdots & 1\\ 0 & 1 & 1 & \vdots & 1\\ 0 & 1 & 3 & \vdots & 5\end{bmatrix}\xrightarrow{R1-2R2,R3-R2}\begin{bmatrix}1 & 0 & -3 & \vdots & -1\\ 0 & 1 & 1 & \vdots & 1\\ 0 & 0 & 2 & \vdots & 4\end{bmatrix}$$

$$\begin{cases}2z=4\\ y+z=1\\ x-3z=-1\end{cases}\Rightarrow\begin{cases}z=2\\ y=-1\\ x=5\end{cases}$$

[ANSWER]: $x=5, y=-1, z=2$.

For equation c), it can be written as:

$$\begin{bmatrix}1 & 1 & -1 & 1\\ 2 & 3 & 1 & 0\\ 3 & 5 & 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}1\\ 4\\ 5\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -1 & 1 & \vdots & 1\\ 2 & 3 & 1 & 0 & \vdots & 4\\ 3 & 5 & 3 & -1 & \vdots & 5\end{bmatrix}\xrightarrow{R2-2R1,R3-3R1}\begin{bmatrix}1 & 1 & -1 & 1 & \vdots & 1\\ 0 & 1 & 3 & -2 & \vdots & 2\\ 0 & 2 & 6 & -4 & \vdots & 2\end{bmatrix}\xrightarrow{R3-2R2}\begin{bmatrix}1 & 1 & -1 & 1 & \vdots & 1\\ 0 & 1 & 3 & -2 & \vdots & 2\\ 0 & 0 & 0 & 0 & \vdots & -2\end{bmatrix}$$

$$\operatorname{rank}(A)=\operatorname{rank}(\begin{bmatrix}1&1&-1&1\\2&3&1&0\\3&5&3&-1\end{bmatrix})=\operatorname{rank}(\begin{bmatrix}1&1&-1&1\\0&1&3&-2\\0&0&0&0\end{bmatrix})=2$$

$$\operatorname{rank}(W)=\operatorname{rank}(\begin{bmatrix}1&1&-1&1&\vdots&1\\2&3&1&0&\vdots&4\\3&5&3&-1&\vdots&5\end{bmatrix})=\operatorname{rank}(\begin{bmatrix}1&1&-1&1&\vdots&1\\0&1&3&-2&\vdots&2\\0&0&0&0&\vdots&-2\end{bmatrix})=3$$

Since $\operatorname{rank}(A)<\operatorname{rank}(W)$,

[ANSWER]: Solution does NOT exist.

2)

The equation can be written as:

$$\begin{bmatrix}1&2&4\\3&8&14\\2&6&13\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3\\13\\4\end{bmatrix}$$

Assume the product of L and U:

$$LU=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}1&2&4\\3&8&14\\2&6&13\end{bmatrix}$$

$$u_{11}=1,u_{12}=2,u_{13}=4$$
$$l_{21}u_{11}=3\Rightarrow l_{21}=3$$
$$l_{21}u_{12}+u_{22}=8\Rightarrow u_{22}=2$$
$$l_{21}u_{13}+u_{23}=14\Rightarrow u_{23}=2$$
$$l_{31}u_{11}=2\Rightarrow l_{31}=2$$
$$l_{31}u_{12}+l_{32}u_{22}=6\Rightarrow l_{32}=1$$
$$l_{31}u_{13}+l_{32}u_{23}+u_{33}=13\Rightarrow u_{33}=3$$

The result of LU decomposition is:

$$L=\begin{bmatrix}1&0&0\\3&1&0\\2&1&1\end{bmatrix}\qquad U=\begin{bmatrix}1&2&4\\0&2&2\\0&0&3\end{bmatrix}$$

To solve the equation, assume $L\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}=\begin{bmatrix}3\\13\\4\end{bmatrix}$, $\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}=U\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.

$$\begin{cases}z_1=3\\3z_1+z_2=13\\2z_1+z_2+z_3=4\end{cases}\Rightarrow\begin{cases}z_1=3\\z_2=4\\z_3=-6\end{cases}\begin{cases}3x_3=-6\\2x_2+2x_3=4\\x_1+2x_2+4x_3=3\end{cases}\Rightarrow\begin{cases}x_3=-2\\x_2=4\\x_1=3\end{cases}$$

[ANSWER]: $x_1=3,x_2=4,x_3=-2$.

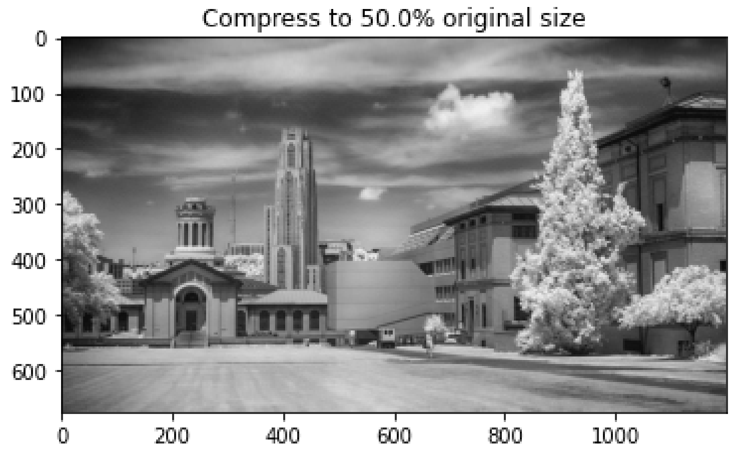
Exercise 5. SVD

Assume the compressed image storage requirement w.r.t r (# u_i and v_i) is:

$$sizeRatio = \frac{m \times r + n \times r + r}{m \times n}$$

i)

Compress the image to 50% of the original storage required:

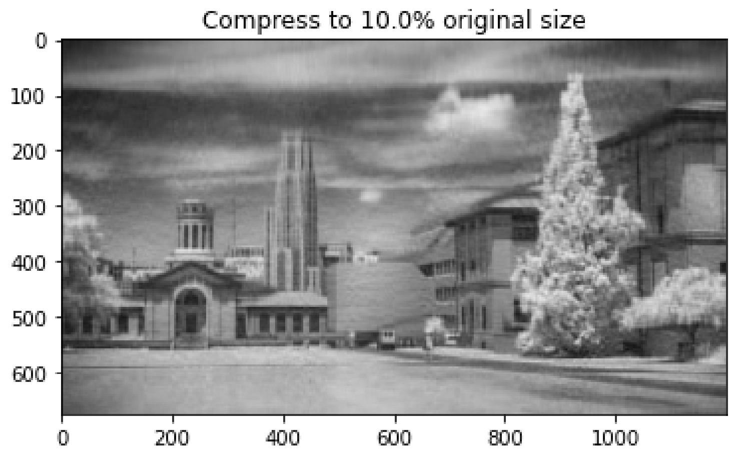


$$r = \frac{m * n * sizeRatio}{m + n + 1} = \frac{675 \times 1200 \times 50\%}{675 + 1200 + 1} \approx 216$$

[ANSWER]: The number of singular values in this compression level is 216.

ii)

Compress the image to 10% of the original storage required:

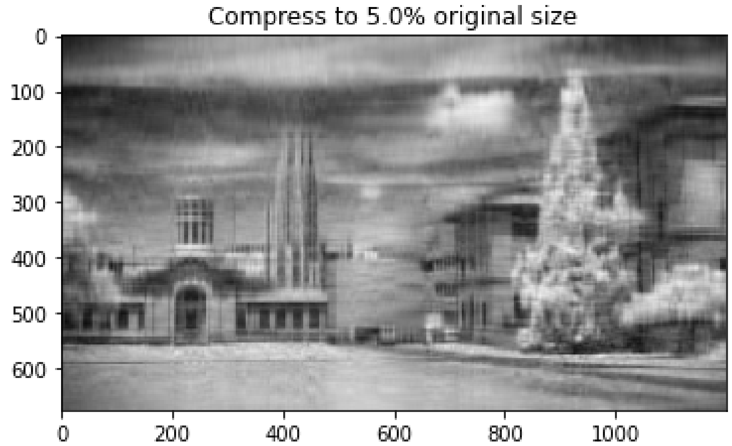


$$r = \frac{m * n * sizeRatio}{m + n + 1} = \frac{675 \times 1200 \times 10\%}{675 + 1200 + 1} \approx 43$$

[ANSWER]: The number of singular values in this compression level is 43.

iii)

Compress the image to 5% of the original storage required:



$$r = \frac{m * n * sizeRatio}{m + n + 1} = \frac{675 \times 1200 \times 10\%}{675 + 1200 + 1} \approx 22$$

[ANSWER]: The number of singular values in this compression level is 22.

Exercise 6. Design for Controllability and Observability

For given LTI system in state space:

$$A = \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad n = 2$$

The controllability matrix P:

$$P = \begin{bmatrix} B & \vdots & AB \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix}$$

The observability matrix Q:

$$Q = \begin{bmatrix} C \\ \cdots \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \gamma - 3 & -1 \end{bmatrix}$$

1.

In order to make the system controllable, P must have full rank.

$\det(P) = \gamma$, if and only if $\gamma \neq 0$, $\mathrm{rank}(P) = n = 2$.

In order to make the system unobservable, $\mathrm{rank}(Q) \neq n$.

$\det(Q) = 2 - \gamma$, if and only if $\gamma = 2 \neq 0$, $rank(Q) = 1 < 2 = n$.

[ANSWER]: When $\gamma = 2$, the system is controllable but not observable.

2.

In order to make the system observable, Q must have full rank.

$\det(Q) = 2 - \gamma$, if and only if $\gamma \neq 2$, $\mathrm{rank}(Q) = n = 2$.

In order to make the system uncontrollable, $\mathrm{rank}(P) \neq n$.

$\det(P) = \gamma$, if and only if $\gamma = 0 \neq 2$, $rank(P) = 1 < 2 = n$.

[ANSWER]: When $\gamma = 0$, the system is observable but not controllable .

Exercise 7. State Space Representation, Controllability

1.

Assume the brightness of the left-most LED is $x_1(k)$, and the brightness of right-most LED is $x_5(k)$.

The discrete system can be represented as:

$$\begin{cases} x_1(k+1) = u(k) \\ x_2(k+1) = x_1(k) \\ x_3(k+1) = x_2(k) \\ x_4(k+1) = x_3(k) \\ x_5(k+1) = x_4(k) \end{cases}$$

Choosing the brightness of five LEDs as the state,

[ANSWER]: This system in the state space is:

$$x(k+1) = Ax(k) + Bu(k)$$
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2.

Since the A matrix is lower-triangle matrix, the eigen value of A is $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.

Using PBH controllability test,

$$\begin{bmatrix} \lambda I - A & \vdots & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$\mathrm{rank}(\begin{bmatrix} \lambda I - A & \vdots & B \end{bmatrix}) = 5 = n$. Therefore,the LED system is controllable.

[ANSWER]: Intuitively, if a system is controllable, then it means that the control input will have impact on all the state through finite amount of time. In this system, the input brightness signal on the left-most LED will eventually be pass down to all of the LEDs, which indicates that the input signal has impact on all states.

Code for Ex.5

```
# 24677_2021_HW3_Ex5_SVD
import numpy as np
from scipy import linalg
import matplotlib.pyplot as plt
import matplotlib.image as mpimg

# please make sure 'CMU_Grayscale.png' is in the same direction
img_original = mpimg.imread('CMU_Grayscale.png')
# imgplot = plt.imshow(img_original,cmap='gray')
[u,s,v] = linalg.svd(img_original)

sizeU = np.size(u,0)
sizeV = np.size(v,0)

for CR in [0.5,0.1,0.05]:
  idx = round(CR*sizeU*sizeV/(sizeU+sizeV+1)) # calculate numbers of eigen vectors
  print(sizeU,sizeV,idx)
  u_CR = np.matrix(u[:,0:idx])
  v_CR = np.matrix(v[0:idx,:])
  s_CR = np.diag(s[0:idx])

  img_CR = u_CR*s_CR*v_CR
  imgplot = plt.imshow(img_CR,cmap='gray')
  plt.title(f"Compress to {CR*100}% original size")
  plt.show()
```