Homework 4

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Problem 1

```
clear;clc
```

Q(a)

$$A=\begin{bmatrix}0 & 1\\ 3 & -2\end{bmatrix}, B=\begin{bmatrix}0 & 1\\ 3 & 0\end{bmatrix}$$

```
A = [0,1;3,-2];
B = [0,1;3,0];
```

Spectral radius is the largest magnitute of eigenvalues:

Let $\det(A-\lambda I)=\det(\begin{bmatrix}-\lambda & 1\\ 3 & -2-\lambda\end{bmatrix})=0\Rightarrow\begin{cases}\lambda_1=-3\\ \lambda_2=1\end{cases}\Rightarrow\rho(A)=3$

Let $\det(B-\lambda I)=\det(\begin{bmatrix}-\lambda & 1\\ 3 & -\lambda\end{bmatrix})=0\Rightarrow\begin{cases}\lambda_1=-\sqrt{3}\\ \lambda_2=\sqrt{3}\end{cases}\Rightarrow\rho(B)=\sqrt{3}$

Check with MATLAB commands:

```
sr_A = max(abs(eig(A)))
```

```
sr_A = 3
```

```
sr_B = max(abs(eig(B)))
```

```
sr_B = 1.7321
```

Frobenius norm:

$$\|A\|_F=\sqrt{\sum_{i,j}|a_{ij}|^2}=\sqrt{0+1^2+3^2+2^2}=\sqrt{14}$$

$$\|B\|_F=\sqrt{\sum_{i,j}|b_{ij}|^2}=\sqrt{0+1^2+3^2+0}=\sqrt{10}$$

Check with MATLAB commands:

```
Nfro_A = norm(A,'fro')
```

```
Nfro_A = 3.7417
```

```
Nfro_B = norm(B,'fro')
```

```
Nfro_B = 3.1623
```

The 1-norm:

$$||A||_1 = \max_j (\sum_i |a_{ij}|) = \max(3,3) = 3$$

$$||B||_1 = \max_j (\sum_i |b_{ij}|) = \max(3,1) = 3$$

Check with MATLAB commands:

```
N1_A = norm(A,1)
```

```
N1_A = 3
```

```
N1_B = norm(B,1)
```

```
N1_B = 3
```

The 2-norm:

$||A||_2 = \overline{\sigma}(A)$:

```
sigbar_A = max(svd(A))
```

```
sigbar_A = 3.6503
```

```
sigbar_A_ = sqrt(max(abs(eig(conj(A)'*A))))
```

```
sigbar_A_ = 3.6503
```

$||B||_2 = \overline{\sigma}(B)$:

```
sigbar_B = max(svd(B))
```

```
sigbar_B = 3
```

```
sigbar_B_ = sqrt(max(abs(eig(conj(B)'*B))))
```

```
sigbar_B_ = 3
```

Check with MATLAB commands:

```
N2_A = norm(A)
```

```
N2_A = 3.6503
```

```
N2_B = norm(B)
```

```
N2_B = 3
```

The ∞-norm:

$$||A||_\infty = \max_i (\sum_j |a_{ij}|) = \max(1,5) = 5$$

$\|B\|_\infty = \max_i (\sum_j |b_{ij}|) = \max(1,3) = 3$

Check with MATLAB commands:

```
Ninf_A = norm(A,'inf')
```

```
Ninf_A = 5
```

```
Ninf_B = norm(B,'inf')
```

```
Ninf_B = 3
```

Q(b)

Proof $\|A\|_{max} = \max_{i,j} |a_{ij}|$ is a generalized matrix norm:

Property 1: $\forall i, j \;\; \|A\|_{max} \geq |a_{ij}| \geq 0$

Property 2: $\begin{cases} A = 0 \Rightarrow \forall i, j \;\; |a_{ij}| = 0 \Rightarrow \|A\|_{max} = 0 \\ \|A\|_{max} = 0 \Rightarrow \forall i, j \;\; |a_{ij}| \leq 0 \Rightarrow \forall i, j \;\; |a_{ij}| = 0 \Rightarrow A = 0 \end{cases}$

Property 3: $\|\alpha A\|_{max} = \max_{i,j} |\alpha \cdot a_{ij}| = |\alpha| \cdot \max_{i,j} |a_{ij}| = |\alpha| \|A\|_{max}$

Property 4: $\|A + B\|_{max} = \max_{i,j} |a_{ij} + b_{ij}| \leq \max_{ij} |a_{ij}| + \max_{ij} |b_{ij}| = \|A\|_{max} + \|B\|_{max}$

Counterexamples:

$\|A\|_{max} < \rho(A)$: e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \|A\|_{max} = 1, \rho(A) = \sqrt{2}$

$\|AB\|_{max} > \|A\|_{max}\|B\|_{max}$: e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \|AB\|_{max} = 2, \|A\|_{max}\|B\|_{max} = 1$

Q(c)

For a SISO system, the transfer function G is a scalar:

$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega}$ and $\|G\|_\infty = \max_\omega |G(j\omega)|$.

For $Q = e^{-sT}$ or $Q = \frac{s-a}{s+a}$, $|Q(j\omega)| = 1$, $|QG(j\omega)| = |Q(j\omega)||G(j\omega)| = |G(j\omega)|$.

Therefore,

$\|G\|_2 = \|QG\|_2$ and $\|G\|_\infty = \|QG\|_\infty$

Test with numerical example:

```
s = tf('s');
T = 10;
G = (s+1)/(s+2)/(s+3);
```

```
Q1 = exp(-s*T);
Q2 = (s-3)/(s+3);
```

∞-norm:

```
Ninf_G = [norm(G,'inf');norm(Q1*G,'inf');norm(Q2*G,'inf')]
```

```
Ninf_G = 3×1
    0.2193
    0.2193
    0.2193
```

2-norm:

```
N2_G =[norm(G,2);norm(Q1*G,2);norm(Q2*G,2)]
```

```
N2_G = 3×1
    0.3416
    0.3416
    0.3416
```

Problem 2

```
clear;clc
```

DIDO systetransfer function:

```
s = tf('s');
G = [10*(s+2)/(s^2+0.2*s+100), 1/(s+5);
    (s+2)/(s^2+0.1*s+10), 5*(s+1)/(s+2)/(s+3)];
```

Q(a)

Check if G has any RHP poles:

```
isStable = isstable(G)
```

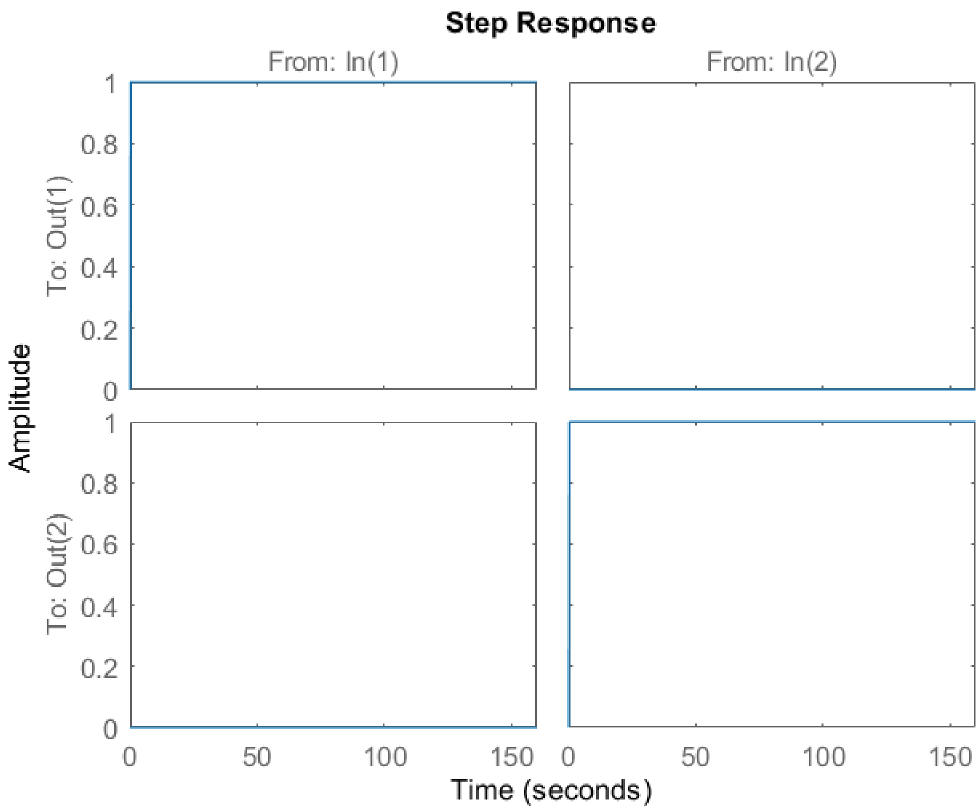
```
isStable = logical
   1
```

G is stable, then find the proper approximation of G^{-1}:

```
Ginv = G\(1000/(s+1000)*eye(2));
K_inv = Ginv*[100/s,0;0,100/s];
L_inv = minreal(G*K_inv, 0.1);
T_inv = minreal(feedback(L_inv,eye(2)),0.1);
S_inv = eye(2)-T_inv;
```

The closed loop system response after more aggresive zero-pole cancellations:

```
step(T_inv)
```

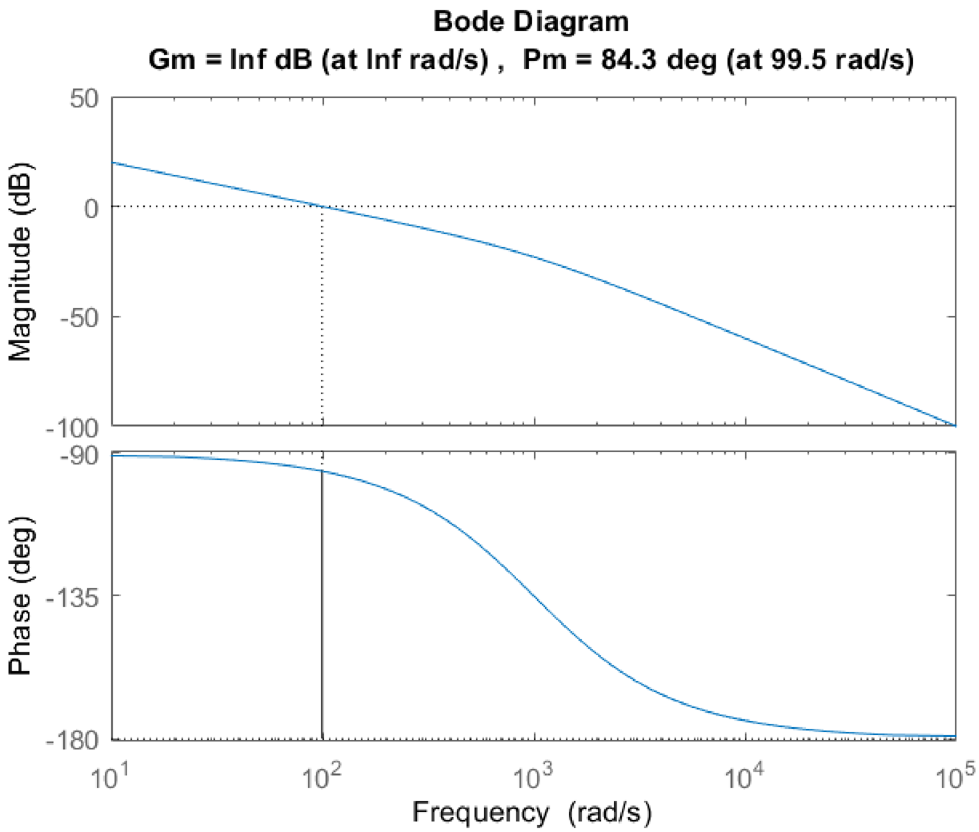



```
isstable(T_inv)
```

```
ans = logical
   1
```

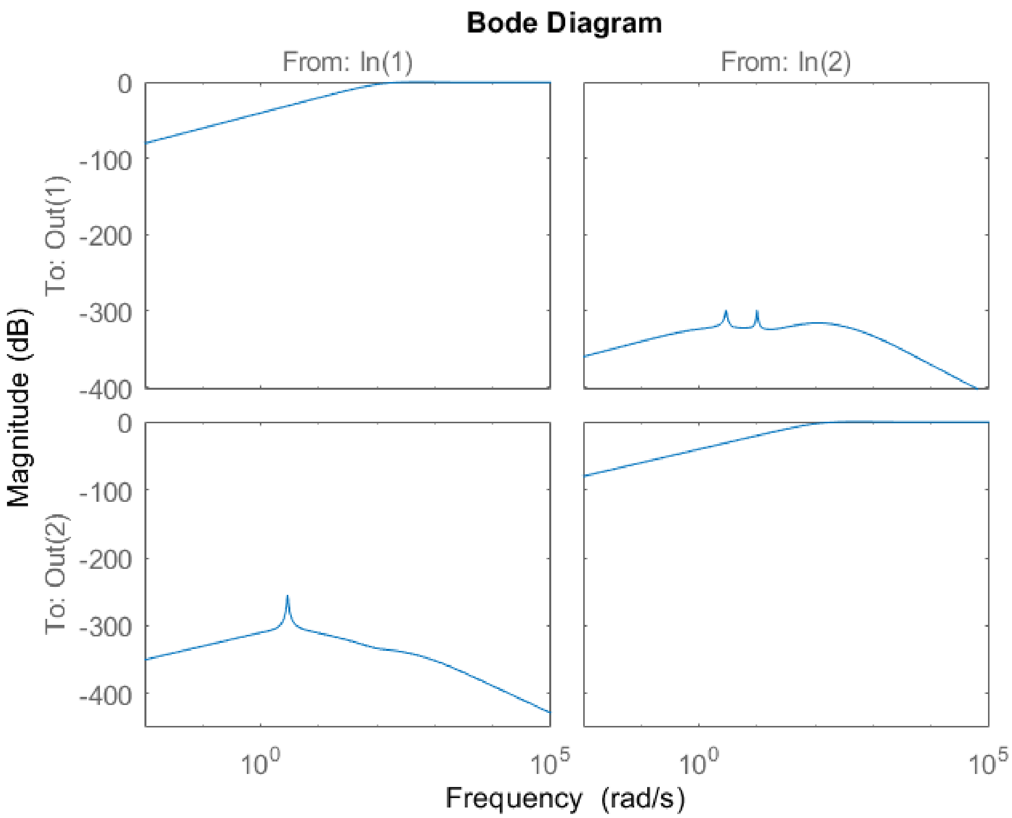
Use margin to check performance of $L_{1,1}$:

```
margin(L_inv(1,1));
```



The bode magnitude of sensitivity function:

```
bodemag(S_inv)
```



Q(b)

```
M = 2;
Mt = 1.5;
A = 1/1e3;  % reject disturbance by 1000
At = 1/1e3;

Wu = tf(1/100,1)*eye(2);
BW_right = 10^8*2*pi;
BW_left = 0;
Wp = zeros(2);
Wt = zeros(2);
GAM_ms = 0;
while true
    GAM_old = GAM_ms;
    BW = (BW_left+BW_right)/2;
    BWt = 3*BW;
    Wp = makeweight(1/A,BW,1/M)*eye(2);  % 1st order
    Wt = makeweight(1/Mt,BWt,1/At)*eye(2);
    [K_ms,CL_ms,GAM_ms] = mixsyn(G,Wp,Wu,Wt);
    if abs(GAM_ms-GAM_old)<1e-6
        break
    elseif GAM_ms<1
        BW_left = BW;
    else
        BW_right = BW;
    end
end
```

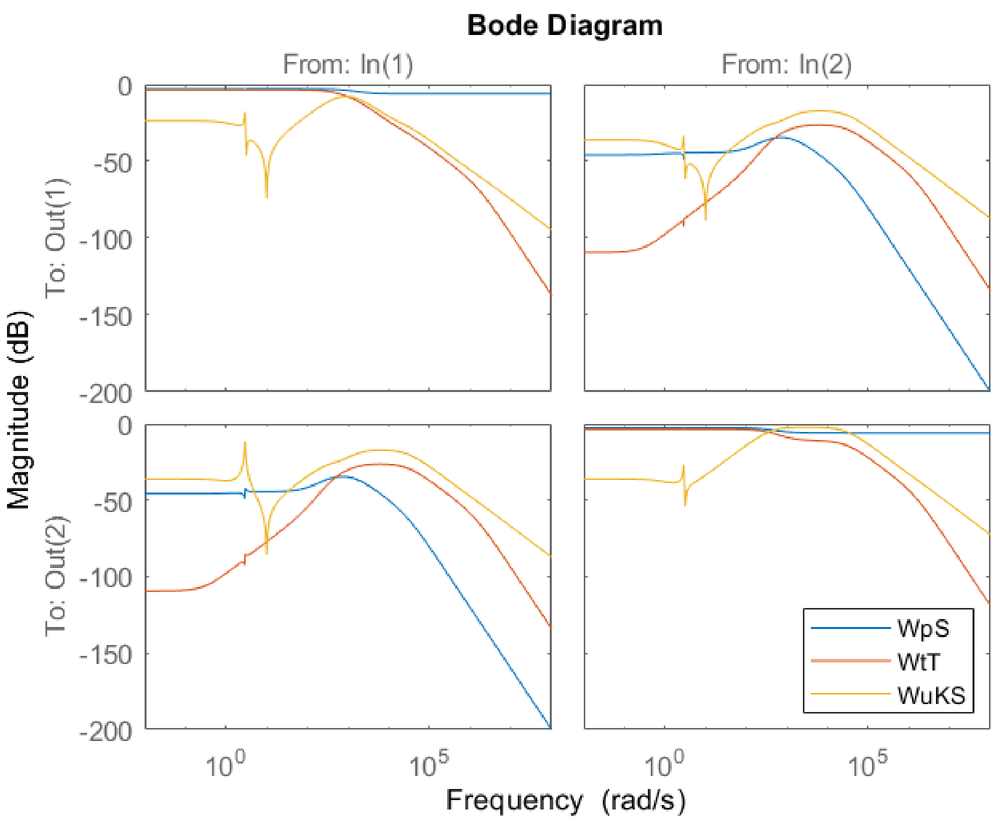
```
L_ms = G*K_ms;
T_ms = feedback(L_ms,eye(2));
S_ms = eye(2)-T_ms;
```

Save the iterative result:

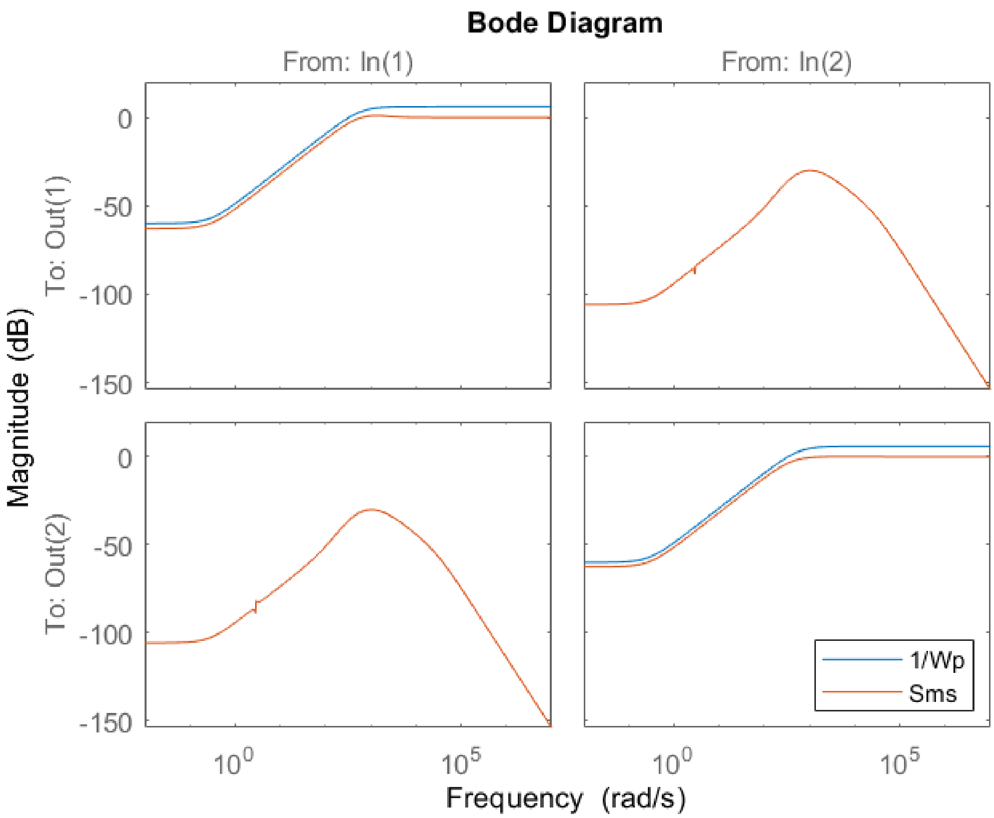
```
save('P2Qb_results','Wp','Wt','Wu','K_ms','L_ms','T_ms','S_ms','GAM_ms');
```

Plot the magnitudes of W_pS, W_tT, and W_uKS:

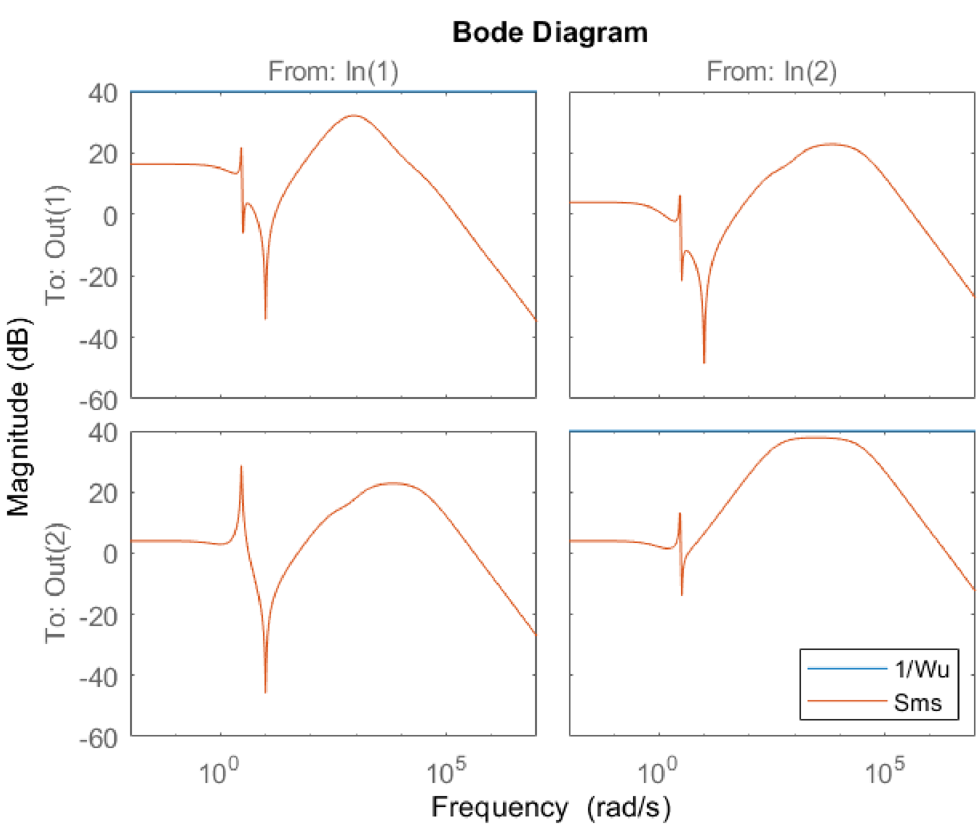
```
bodemag(Wp*S_ms, Wt*T_ms, Wu*K_ms*S_ms);
legend('WpS','WtT','WuKS','location','southeast')
```



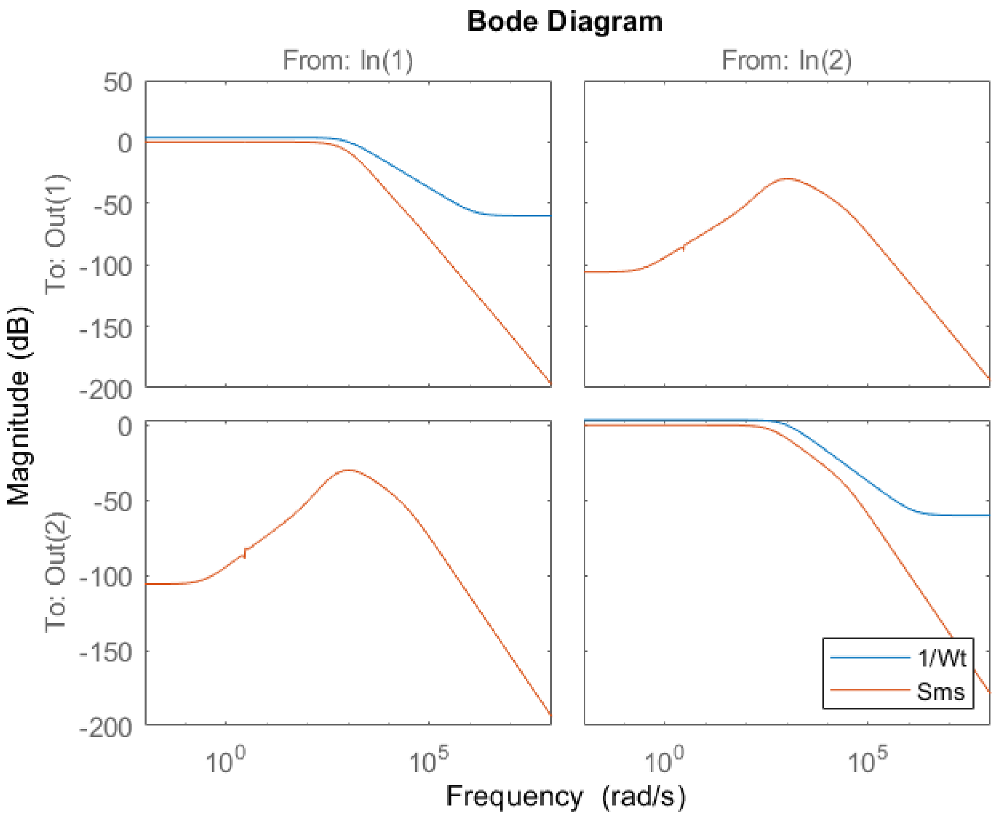

```
bodemag(eye/Wp,S_ms);
legend('1/Wp','Sms','location','southeast')
```




```
bodemag(eye/Wu,K_ms*S_ms);
legend('1/Wu','Sms','location','southeast')
```



```
bodemag(eye/Wt,T_ms);
legend('1/Wt','Sms','location','southeast')
```



Problem 3

```
clear;clc
```

Q(a)

Assume the given system inside the dash line is G^*.

Let the output of the generalized plant be:

$$\begin{cases} z_1 = e_y = W_p(G^*u - Mr) \\ z_2 = e_u = W_u u \end{cases}$$

Let the input of the generalied plant be:

$$\begin{cases} w_1 = r \\ w_2 = n \end{cases}$$

Let the output of the controller be u

Let the output of the controller be:

$$\begin{cases} v_1 = r \\ v_2 = -G^* u - W_n n \end{cases}$$

The generalized system $\begin{bmatrix} z \\ v \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix}$:

$$\begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -W_p M & 0 & W_p G^* \\ 0 & 0 & W_u \\ I & 0 & 0 \\ 0 & -W_u & -G^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} \Rightarrow P = \begin{bmatrix} -W_p M & 0 & W_p G^* \\ 0 & 0 & W_u \\ I & 0 & 0 \\ 0 & -W_u & -G^* \end{bmatrix}$$

Use linear fractional transformation to find N:

$$N = F_l(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

Where $P_{11} = \begin{bmatrix} -W_p M & 0 \\ 0 & 0 \end{bmatrix}$, $P_{12} = \begin{bmatrix} W_p G^* \\ 0 \end{bmatrix}$, $P_{21} = \begin{bmatrix} I & 0 \\ 0 & -W_u \end{bmatrix}$, $P_{22} = \begin{bmatrix} 0 \\ -G^* \end{bmatrix}$

Assume all signals are single input or single output:

```
syms Wp M Gs Wu
P11 = [-Wp*M,0;0,0];
P12 = [Wp*Gs;0];
P21 = [1,0;0,-Wu];
P22 = [0;-Gs];
K = sym('K', [1,2]);
N = P11+P12*K/(1-P22*K)*P21;
```

$$N = \begin{bmatrix} -W_p M - W_p & -W_p W_u \\ 0 & 0 \end{bmatrix}$$

Q(b)

The DIDO system in Problem2:

```
s = tf('s');
G = [10*(s+2)/(s^2+0.2*s+100), 1/(s+5);
    (s+2)/(s^2+0.1*s+10), 5*(s+1)/(s+2)/(s+3)];
```

Load previous results:

```
load('P2Qb_results')
```

Use sysic to generate generalized plant:

```
systemnames = 'G Wp Wu Wt';
inputvar = '[r{2};u{2}]';
outputvar = '[Wp;Wu;Wt;r-G]';
input_to_G = '[u]';
```

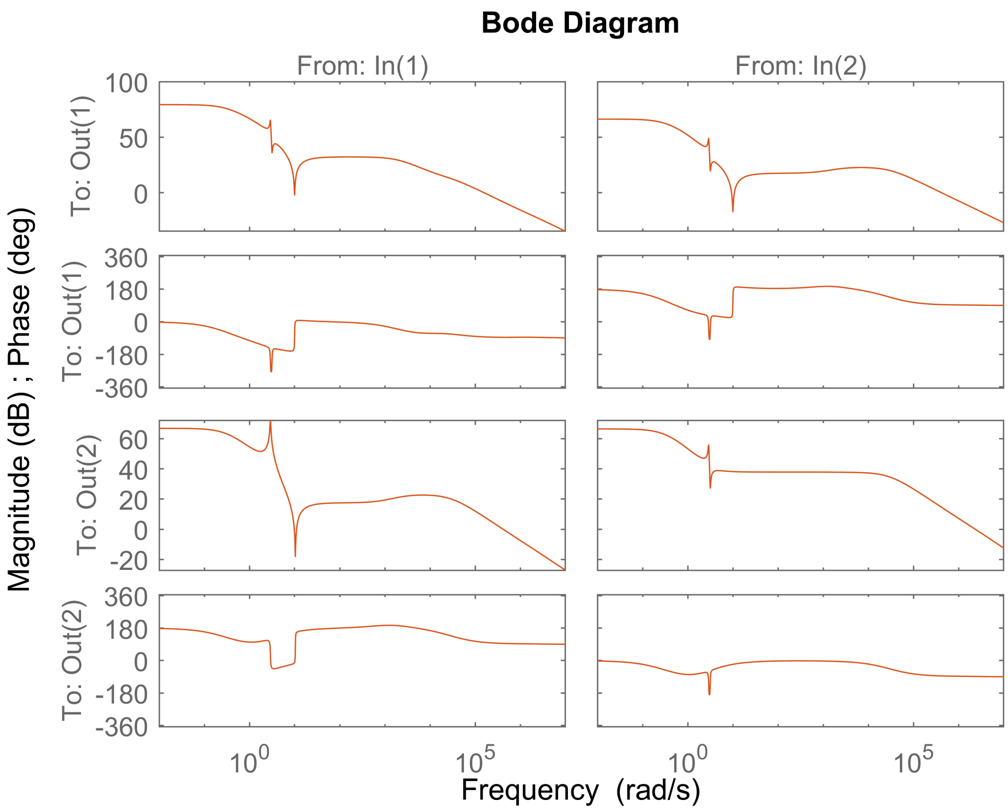
```
input_to_Wp = '[r-G]';
input_to_Wu = '[u]';
input_to_Wt = '[G]';
cleanupsysic = 'yes';  %This drops all the useless variables from workspace
P = sysic;
```

Design controller based on the generalized mixed sensitivity plant and H_∞ synthesis:

```
[K_hinf,CL_hinf,GAM_hinf] = hinfsyn(P,2,2);
```

Compare designed controller with mixsyn results:

```
bode(K_hinf,K_ms);
```



Q(c)

The linear fraction transformation:

$$F_l(H, \frac{1}{s}) = H_{11} + H_{12}\frac{1}{s}(I - H_{22}\frac{1}{s})^{-1}H_{21} = H_{11} + \frac{H_{12}}{s}(sI - H_{22})^{-1}\frac{H_{21}}{s}$$

Let $F_l(H, \frac{1}{s}) = C(sI - A)^{-1}B + D$:

$$\begin{cases} H_{11} = D \\ H_{12} = C \\ H_{22} = A \\ H_{21} = B \end{cases} \Rightarrow H = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$