Comparison of Option Pricing Models

Shaohao Chen
Boston University

Outline

- Introduction
- Black Scholes Formula (analytical)
- Binomial Model (numerical)
- Monte Carlo Model (numerical)
- Results and discussions

Introduction

- Option pricing is a key problem in the quantitative finance world.
- The most famous model for option pricing is the Black-Scholes model. It has been successfully applied to option pricing for a few decades.
- The Black-Scholes equation has an analytical solution for European option known as the Black-Scholes formula. Many numerical methods have been developed for option pricing too.
- For example, the binomial model and the Monte Carlo model are two numerical models widely used in practice. Both models are converged to the Black Scholes formula if the size of the simulation is large enough.
- In this project, I wrote computer codes to compute option price and compare the results of the **Black-Scholes formula, the binomial model** and **the Monte Carlo model.**
- Advantage and disadvantage of these methods will be discussed.

Black Scholes Formula

- Black Scholes Equation (Based on the arbitrage-free principle, derived by stochastic calculus techniques)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Black Scholes Formula (analytical solution for European call option)

$$\begin{aligned} C(S_t, t) &= N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

- cumulative distribution function (CDF)

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\, dz$$

Binomial Model (1)

- Binomial model is equivalent to solving the Black-Scholes equation using first-order finite difference method.

- The binomial value for a call option is

$$C_{t-\Delta t,i} = e^{-r\Delta t}(pC_{t,i+1} + (1-p)C_{t,i-1})$$

- Probability of stock price going up

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

where,

$$u = e^{\sigma\sqrt{t}} \qquad d = e^{-\sigma\sqrt{t}} = \frac{1}{u}.$$

Binomial Model (2)

- A binomial tree is created to calculate the stock prices from time zero to maturity time, then the option prices is propagated backward.

- If the number of nodes (steps) in the tree is large enough, the binomial result is converged to that of the Black-Sholes formula.

- The binomial model can be applied to both European and American option pricing. Especially, the price of an American option at each node is the maximum of the binomial value and the exercise value.

Binomial Model (3)

- A C++ code to price European option using binomial model

```
double discount = exp( - interestRate * dt);
double p1 = ( exp(interestRate*dt) - d) / (u - d);
double p2 = 1.-p1;
for(int i=0; i<numPrice; i++){
    double stockPrice = spotStockPrice * pow(u, numStepsToMaturity - 2*i); // stock price tree
    if(stockPrice>strike){          // pay off at maturity time
      callPrice[i] = stockPrice - strike;
    }else{
      callPrice[i]=0.;
    }
}
for(int j = numStepsToMaturity - 1; j>=0; j--)      // propagate option price backward
   for(int i=0; i<=j; i++){
     callPrice[i] = (p1 * callPrice[i] + p2 * callPrice[i+1]) * discount;
}
```

Monte Carlo Model (1)

- The Variation of the underlying stock price follows a geometric Brownian motion,

$$dS_t = rS_t dt + \sigma S_t dW_t$$

- Brownian motion is distributed as a normal distribution,

$$W_T = \sqrt{T} N(0, 1)$$

- The stock price at maturity time T is

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} N(0,1)}$$

- The corresponding option price at maturity time T is

$$f(S) = (S - K)_+$$

Monte Carlo Model (2)

- The central idea of Monte Carlo method is to take into account as many random stock prices as possible and then average their effects.

  1. Generate a large number of random stock prices at time *T*;
  2. Compute the corresponding pay off for each random stock price;
  3. Compute the average (expectation) value of all possible pay off;
  4. Discount the average value to time zero.

- Then the corresponding option price at time zero is

$$e^{-rT}\mathbb{E}\Big(f\big(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}N(0,1)}\big)\Big)$$

- To price an American option, one needs to create a price tree and then apply the above Monte Carlo pricing method on each node of the tree.

Monte Carlo Model (3)

- A C++ code to price European option using Monte Carlo model

```
double variance = Vol*Vol*Expiry;
double rootVariance = sqrt(variance);
double itoCorrection = -0.5*variance;
double movedSpot = Spot*exp(r*Expiry + itoCorrection);
for (unsigned long i=0; i < NumberOfPaths; i++){      // loop of possible paths
    double thisGaussian = GetOneGaussianByBoxMuller();   // get random normal distributed numbers
    thisSpot = movedSpot*exp( rootVariance*thisGaussian);   // stock price at expiry
    double thisPayoff = thisSpot - Strike;
   thisPayoff = thisPayoff >0 ? thisPayoff : 0; // pay off
    runningSum += thisPayoff;    // sum of all pay off
 }
 double mean = runningSum / NumberOfPaths;   // average of all pay off
 mean *= exp(-r*Expiry);      // discount to time zero. This is the option price
```

Results: convergence

- European option price
- The spot price is 10.
- The strike price is 11.
- The expiry time is 2 years.
- The annual interest rate is 1%.
- The volatility is 40%.

- The Black-Scholes analytical result is 1.94296.

- In both models, as the size of simulation increases, the numerical result is converged to the analytical result.

	Binomial			Monte Carlo	
Steps	Call Price	Difference	Paths	Call Price	Difference
6	1.97084	1.435%	10000	1.88597	-2.933%
12	1.96379	1.072%	80000	1.92790	-0.775%
24	1.95352	0.544%	400000	1.94382	0.044%
240	1.94296	0.000%	2000000	1.94421	0.064%
1000	1.94306	0.005%	10000000	1.94437	0.073%
5000	1.94301	0.002%	50000000	1.94316	0.010%

Results: accuracy and efficiency

	Black Scholes (a)	Binomial (b)	Binomial (c)	Monte Carlo (d)	Monte Carlo (e)
Call Price	1.94296	1.94301	1.94296	1.94497	1.94316
Put Price	2.72515	2.72519	2.72515	2.72716	2.72534
Difference	-	0.003%	0.000%	0.103%	0.010%
Run Time (s)	-	0.241	22.204	0.988	8.359

(a) Black Scholes formula (analytical solution);

(b) Binomial model: 5,000 steps; (c) Binomial model: 50,000 steps;

(d) Monte Carlo model: 5,000,000 paths; (e) Monte Carlo model: 50,000,000 paths.

- The binomial model is known to be accurate and efficient for simple cases. As a result, the binomial results [column (b) and (c)] are closer to the analytical result [column (a)] than the Monte Carlo results [column (d) and (e)].

Discussion: computation efficiency

- The run time of binomial model is only 0.241 seconds at a relatively small number of steps (i.e. 5,000). Very efficient!
- In theory, the run time of binomial model is scaled in the power of two, i.e. $O(2^n)$. The result confirms it, since the run time is almost 100 times larger (22.204 seconds) when the number of steps is 10 times larger (i.e. 50,000). So the computational efficiency is a concern if the number of steps is large for option pricing in real world.

- The Monte Carlo simulation [column (d)] is slower than the binomial one [column (b)] in the relative small simulations.
- However, the scaling of Monte Carlo model is much better: the run time increases only around 8.5 times (from 0.988 seconds to 8.359 seconds), even the number of paths increases 10 times. This shows that the Monte Carlo simulations generally have a polynomial time complexity, which is better than that of the binomial model.

Discussion: other options

- It is very convenient to apply the binomial module to American option, because a price tree is used and the option price is calculated at sampling times between zero and the expiry. In contrast, applying Monte Carlo model to price American option is much more complicated and much more expensive in computational cost.
- But Monte Carlo model is good for options with complicated features. For example, pricing Asian option requires path dependent information. It is therefore natural to use the Monte Carlo model.
- Monte Carlo model is also good for options with several sources of uncertainty. For example, for bond options the underlying is a bond, but the source of uncertainty is the annualized interest rate.

References

- Wikipedia
- Books: