Comparison of Option Pricing Models

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Option pricing is a key problem in the quantitative finance world. The most famous model for option pricing is the Black-Scholes model. It has been successfully applied to option pricing for a few decades. The Black-Scholes equation has an analytical solution for European option known as the Black-Scholes formula. Many numerical methods have been developed for option pricing too. For example, the binomial model and the Monte Carlo model are two numerical models widely used in practice. Both models are converged to the Black Scholes formula if the size of the simulation is large enough. In this project, I wrote computer codes to compute option price and compare the results of the Black-Scholes formula, the binomial model and the Monte Carlo model. Advantage and disadvantage of these methods will be discussed.

1. Black Scholes Formula

Based on the arbitrage-free principle and some other assumptions to the underlying stock and the market, the Black-Scholes equation (BSE) can be derived by stochastic calculus techniques,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad , \qquad (1)$$

where *S* is the price of the underlying stock, *t* is time, *V* is the price of the option as a function of *t* and *S*, *r* is the continuously-compounded risk-free interest rate, and *σ* is the standard deviation (volatility) of the stock's returns. This is a partial differential equation. Considering European option pricing, which means the option can be exercised only at the maturity time, there is an analytical solution of this equation for a call option,

$$\begin{aligned} C(S_t, t) &= N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned} \quad , \qquad (2)$$

where *C* is the call option price, S_t is the spot price of the underlying stock, *K* is the strike price, *T - t* is the time to maturity, and *N(x)* is the cumulative distribution function (CDF) of the standard normal distribution,

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\, dz \quad , \qquad (3)$$

Equation (1) is known as the Black-Scholes Formula. The price of a corresponding put option based on put-call parity is:

$$
\begin{aligned}
P(S_t,t) &= Ke^{-r(T-t)} - S_t + C(S_t,t) \\
&= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t
\end{aligned}
\tag{4}
$$

There is no analytical solution to the BSE for American option pricing. But many numerical methods have been developed for both European and American option pricing. In the following, we will discuss two of them: the binomial model and the Monte Carlo model.

2. Binomial Model: First-order Finite Difference Method

The derivatives in the BSE can be approximated by finite differences. Here we only consider the first order finite difference, which means only two adjacent (the left and the right) grid points are taken into account. This is equivalent to the so called binomial model. The binomial value (first order finite-difference value) for a call option is,

$$
C_{t-\Delta t,i} = e^{-r\Delta t}(pC_{t,i+1} + (1-p)C_{t,i-1}) \,, \tag{5}
$$

where $C_{t,i}$ is the option's value for the i^{th} node at time t, p is the probability that the stock price goes up the amount u, while 1-p is the probability that the stock price goes down the amount d. p is chosen such that the related binomial distribution simulates the geometric Brownian motion of the underlying stock,

$$
p = \frac{e^{(r-q)\Delta t} - d}{u-d} \,, \tag{6}
$$

where q is the dividend and is assumed to be zero here. Based on the condition that the variance of the log of the stock price is $\sigma^2 t$, we have:

$$
u = e^{\sigma\sqrt{t}} \,, \tag{7}
$$

and,

$$
d = e^{-\sigma\sqrt{t}} = \frac{1}{u}. \tag{8}
$$

A binomial tree is created to calculate the stock prices from time zero to maturity time, then the option prices is propagated backward using Eq. (5). If the number of nodes (steps) in the tree is large enough, the binomial result is converged to that of the Black-Sholes formula.

The corresponding put price can be calculated using the call-put parity. The binomial model can be applied to both European and American option pricing. Especially, the price of an American option at each node is the maximum of the binomial value and the exercise value.

3. Monte Carlo Model

Assume that the variation of the underlying stock price S follows a geometric Brownian motion W_t with constant drift (interest rate) r, and volatility σ, so,

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{9}$$

since W_t is a Brownian motion, W_t is distributed as a normal distribution with mean zero and variance T,

$$W_T = \sqrt{T}N(0,1), \tag{10}$$

Equation (9) has a solution as following,

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}N(0,1)}, \tag{11}$$

where S_0 is the spot price at time zero. The price of the corresponding option at time T is equal to the maximum of the stock price at time T and the strike price K, i.e.

$$f(S) = (S-K)_+ \tag{12}$$

The central idea of Monte Carlo method is to take into account as many random stock prices as possible and then average their effects. One simple implementation is to do these steps: 1) Generate a large number of random stock prices at time T using Eq. (11); 2) Compute the corresponding pay off for each random stock price using Eq. (12); 3) Compute the average (expectation) value of all pay off; 4) Discount the average value to time zero. So the option price at time zero is equal to

$$e^{-rT}\mathbb{E}\left(f\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}N(0,1)}\right)\right), \tag{13}$$

If the number of pricing paths is large enough, the Monte Carlo result is converged to that of Black-Scholes formula.

Up to here, the call option price of a European option is calculated. The corresponding put price can be calculated using the call-put parity. To price an American option, one needs to create a price tree and then apply the above Monte Carlo pricing method on each node of the tree.

4. Results and discussions

Computation results are discussed in this session. The algorithm for the binomial model is referenced to [2]. The code is in the file named *binomial.cpp*. The algorithm for the Monte Carlo method is referenced to [5]. The codes are in the files named *SimpleMC.cpp, Random1.cpp* and *Random1.h*. All computer codes are written in C++.

I compute European option price using the three methods: Black Scholes formula, binomial model and Monte Carlo model. The spot price is 10. The strike price is 11. The expiry time is 2 years. The annual interest rate is 1%. The volatility is 40%.

The convergence of binomial and Monte Carlo models is shown in Table 1. As the number of steps increases in the binomial model, the result is converged to the analytical result. As the number of paths increases in the Monte Carlo model, the result is converged to the analytical result too.

Binomial			Monte Carlo		
Steps	Call Price	Difference	Paths	Call Price	Difference
6	1.97084	1.435%	10000	1.88597	-2.933%
12	1.96379	1.072%	80000	1.92790	-0.775%
24	1.95352	0.544%	400000	1.94382	0.044%
240	1.94296	0.000%	2000000	1.94421	0.064%
1000	1.94306	0.005%	10000000	1.94437	0.073%
5000	1.94301	0.002%	50000000	1.94316	0.010%

Table 1. Convergence of binomial and Monte Carlo models. Difference is compared to the Black Scholes analytical result.

	Black Scholes (a)	Binomial (b)	Binomial (c)	Monte Carlo (d)	Monte Carlo (e)
Call Price	1.94296	1.94301	1.94296	1.94497	1.94316
Put Price	2.72515	2.72519	2.72515	2.72716	2.72534
Difference	-	0.003%	0.000%	0.103%	0.010%
Run Time (s)	-	0.241	22.204	0.988	8.359

Table 2. Computation accuracy and efficiency of binomial and Monte Carlo models. (a) Black Scholes formula (analytical solution); (b) Binomial model: using 5,000 steps; (c) Binomial model: using 50,000 steps; (d) Monte Carlo model: using 5,000,000 paths; (e) Monte Carlo model: using 50,000,000 paths. Difference is compared to the Black Scholes analytical result.

Computation accuracy and efficiency of binomial and Monte Carlo models is shown in Table 2. The binomial model is known to be accurate and efficient for simple cases. As a result, the binomial results [column (b) and (c)] are closer to the analytical result [column (a)] than the Monte Carlo results [column (d) and (e)].

The run time of binomial model is only 0.241 seconds at a relatively small number of steps (i.e. 5,000). In theory, the run time of binomial model is scaled in the power of two, i.e. $O(2^n)$. The result confirms it, since the run time is almost 100 times larger (22.204 seconds) when the number of steps is 10 times larger (i.e. 50,000). So the computational efficiency is a concern if the number of steps is large for option pricing in real world. The Monte Carlo simulation [column (d)] is slower than the binomial one [column (b)] in the relative small simulations. However, the scaling of Monte Carlo model is much better: the run time increases only around 8.5 times (from 0.988 seconds to 8.359 seconds), even the number of paths increases 10 times. This shows that the Monte Carlo simulations generally have a polynomial time complexity, which is better than that of the binomial model.

It is very convenient to apply the binomial module to American option, because a price tree is used and the option price is calculated at sampling times between zero and the expiry. In contrast, applying Monte Carlo model to price American option is much more complicated and much more expensive in computational cost. But Monte Carlo model is good for options with complicated features. For example, pricing Asian option requires path dependent information. It is therefore natural to use the Monte Carlo model. Monte Carlo model is also good for options with several sources of uncertainty. For example, for bond options the underlying is a bond, but the source of uncertainty is the annualized interest rate.

5. Conclusion

Option prices are computed by three different methods: the Black-Scholes formula, the binomial model and the Monte Carlo model. The analytical Black-Scholes formula is straightforward to price European option, but numerical methods are necessary for other options. The binomial model is accurate and efficient for simple cases with small number of steps in simulation, and it is also convenient to be applied to American option. However, the scaling of Monte Carlo model is much better and it is good for larger cases. Monte Carlo model is also good for more complicated options, such as Asian option or bond option.

References:

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[5] M. S. Joshi, *C++ Design Patterns and Derivatives Pricing*, 2nd edition, 2008 by Cambridge University Press