FDDP Classical Mechanics - Assignment

22 November, 2020

Exercises (60 points)

1. For a particle of mass m moving under a conservative potential $V(x)$, show that:
$$t - t_0 = \pm \int_{x_0}^{x} \frac{dx'}{\sqrt{2(E - V(x'))/m}} \tag{1}$$

2. Consider a pendulum with length l and mass m moving in a constant gravitational field of strength g without resistance. Write down its equation of motion in terms of the angle of oscillation, θ, and show that it can be approximated as a simple harmonic oscillator when θ is small.

3. A particle moves in a fixed plane with a position vector $\vec{x}(t)$. Let (r, θ) be plane polar coordinates and let $\hat{r}$ and $\hat{\theta}$ be unit vectors in the direction of increasing r and θ respectively. Show that:
$$\dot{\vec{x}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{2}$$

4. For a particle subjected to a central force, show that:
   - The angular momentum is conserved in time.
   - The motion of this particle is restricted to a plane.

5. Consider a system of particles m_i, with a center of mass $\vec{R} = \frac{1}{M}\sum m_i \vec{x}_i$. Writing $\vec{x}_i = \vec{R} + \vec{y}_i$ so that $\vec{y}_i$ is the position of each particle relative to the center of mass, show that:
   - $\sum m_i \vec{y}_i = 0$
   - The total kinetic energy of the system of particles can be written as $T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum m_i \dot{\vec{y}}^2$

Problem 1 (20 points)

The simple harmonic oscillator is a simple, but important model that oftentimes acts as a starting point for important physical problems. In this problem, we consider the problem when two such oscillators are interacting with one another.

Consider first a system of two masses M, attached to walls with spring constants κ and to each other with spring constant κ_{12} (See Fig. 1). Let x_1 and x_2 be the displacement of the first and second masses from their equilibirum positions. To apply Newton's law, note that the force on m_1 when it is displaced from its equilibirum position, while m_2 is held fixed, is given by:

$$F_1 = -\kappa x_1 + \kappa_{12}(x_2 - x_1) = -(\kappa + \kappa_{12})x_1 + \kappa_{12}x_2 \tag{3}$$

Similarly for m_2, we have:

$$F_2 = -\kappa x_1 + \kappa_{12}(x_1 - x_2) = -(\kappa + \kappa_{12})x_2 + \kappa_{12}x_1 \tag{4}$$

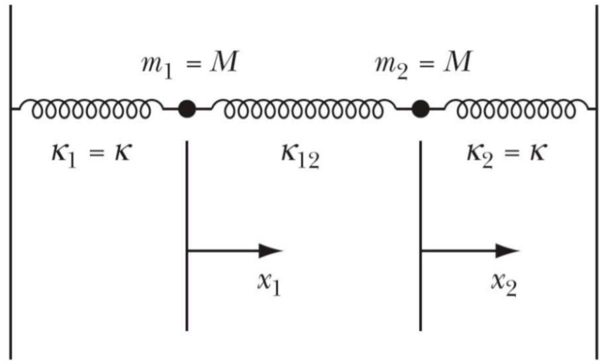

Figure 1: Two coupled harmonic oscillators.

1. Apply Newton's law to each mass and decouple the coupled differential equations by taking the sum and difference of the two equations, and introducing $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.
2. Solve the two differential equations in y_1 and y_2, and show that the solutions can be written in the form:

$$x_1 - x_2 = A_f \cos(\omega_f t + \phi_f) \tag{5}$$
$$x_1 + x_2 = A_s \cos(\omega_s t + \phi_s), \tag{6}$$

where

$$\omega_s = \sqrt{\kappa/m} \tag{7}$$
$$\omega_f = \sqrt{(\kappa + 2\kappa_{12})/m}. \tag{8}$$

3. Finally, write down the explicit forms of $x_1(t)$ and $x_2(t)$.
4. What are the initial conditions, i.e values of A and ϕ at $t = 0$, such that $x_1(t) = x_2(t)$ at all times? This is called the symmetric oscillation mode, where both masses always move in unison, in the same directions.
5. What are the initial conditions, i.e values of A and ϕ at $t = 0$, such that $x_1(t) = -x_2(t)$ at all times? This is called the antisymmetric oscillation mode, where both masses always move in an unsynced manner, in opposite directions.

Problem 2 (20 points)

In the following orbital questions, the particles move in the gravitational potential given by:

$$V = -\frac{km}{r}. \tag{9}$$

1. Show that the radius R of the orbit of a geostationary satellite in the equatorial plane (i.e a satellite that appears fixed in the sky when viewed from the ground) is approximately $(28)^{-3/2} R_{moon}$, where R_{moon} is the radius of the moon's orbit around the Earth.
2. A particle moves with angular momentum per unit mass l in an ellipse, for which the distances from the focus to the periapsis (closest point to focus) and apoapsis (furthest point) are p and q respectively. Show that

$$l^2(1/p + 1/q) = 2k. \tag{10}$$

[Hint : Use the geometrical properties of an ellipse, and the relation between l and the various parameters of an ellipse.]