FDDP Classical Mechanics - Tutorial 2

21 October, 2020

Chapter summary

1) Motivation : suppose a particle in one dimension is oscillating about an equilibrium point, so that it experiences a restoring force when displaced from this equilibrium point (set at $x = 0$). Expanding the force experienced by it in a Taylor series about $x = 0$,

$$F(x) = F_0 + x\left(\frac{dF}{dx}\right)_0 + \frac{1}{2!}x^2\left(\frac{d^2F}{dx^2}\right)_0 + \cdots \tag{1}$$

If we consider small displacements and retain only the first order term, we have:

$$F(x) = -kx \tag{2}$$

where $\left(\frac{dF}{dx}\right)_0 = -k$ and F_0 vanishes because by definition the force is zero at the equilibrium point. The dynamics of small oscillations is thus that of the simple harmonic oscillator (SHO).

2) Using Newton's equation $F = ma$ and defining $w_0^2 \equiv k/m$, the equation of motion is thus:

$$\boxed{\ddot{x} + w_0^2 x = 0} \tag{3}$$

The general solution of this equation is:

$$x(t) = A\sin(\omega_0 t - \delta). \tag{4}$$

3) Resistive/damping forces can be modelled as $F_{res} = -b\dot{x}$, where b is a positive constant. A damped harmonic oscillator experiencing a total force of $F = F_{SHO} + F_{res}$ thus has the equation of motion:

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}, \tag{5}$$

where $\beta = b/m$ is the damping parameter, and ω_0 is the SHO angular frequency. This is a homogenous second order linear differential equation.

Depending on the strength of the damping parameter, the solution can be separated into 3 classes (See Fig.1).

(a) Underdamping : $\omega_0^2 > \beta^2$

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta), \tag{6}$$

where $\omega_1^2 \equiv \omega_0^2 - \beta^2$. The system oscillates with an exponentially decaying envelope.

(b) Critical damping : $\omega_0^2 = \beta^2$

$$x(t) = e^{-\beta t}(A + Bt). \tag{7}$$

The systems approaches equilibrium without oscillating, and in the least amount of time.

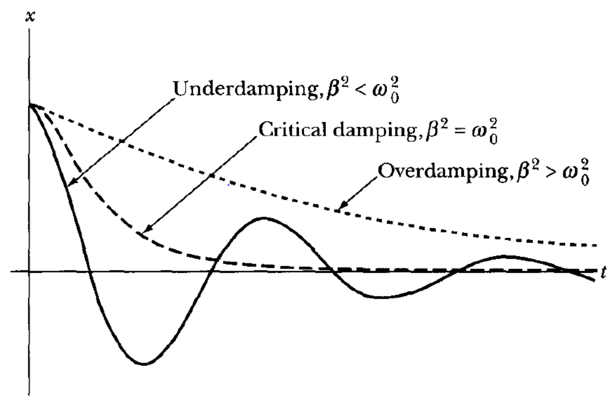

Figure 1: Time evolution of the different classes of damped SHOs.

(c) Overdamping : $\omega_0^2 < \beta^2$

$$x(t) = e^{-\beta t}(Ae^{\omega_2 t} + Be^{\omega_2 t}), \tag{8}$$

where $\omega_2^2 \equiv \beta^2 - \omega_0^2$. Its form is sensitive to initial conditions.

4) Driving forces are external forces/inhomogenity that can be introduced to "drive" an existing oscillator. They are most commonly periodic:

$$F_{drive} = A\cos(\omega t). \tag{9}$$

The equation of motion then becomes:

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\omega t)} \tag{10}$$

This is now an inhomogenous second order linear differential equation, whose solution is the sum of a complementary function and a particular integral, $x(t) = x_c(t) + x_p(t)$, where $x_c(t)$ is the solution of the homogenous part of the differential equation, and:

$$x_p(t) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}\cos(\omega t - \delta), \tag{11}$$

with $\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$.

$x_c(t)$ is sometimes referred to as the transient part of the solution, as it dies off after $t = 1/\beta$, the characteristic timescale of the system. Thereafter, the motion will be dominated by $x_c(t)$, i.e: $x(t \gg 1/\beta) \approx x_c(t)$.

5) Resonance between the damped SHO and the driving force occurs at the resonance frequency ω_R, at which the amplitude of $x_p(t)$ is maximum, i.e

$$\left(\frac{dD}{d\omega}\right)_{\omega=\omega_R} = 0. \tag{12}$$

We thus find:

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}. \tag{13}$$

(See Fig.2) ω_R is lowered as β is increased, until $\beta > \omega/2$, whereby no resonance occurs as ω_R becomes imaginary. The degree of damping of a system can be described by defining the quality factor, $Q \equiv \omega_R/2\beta$. Little damping means that Q is large, and the shape of the resonance curve becomes sharply peaked, approaching that of an undamped oscillator.

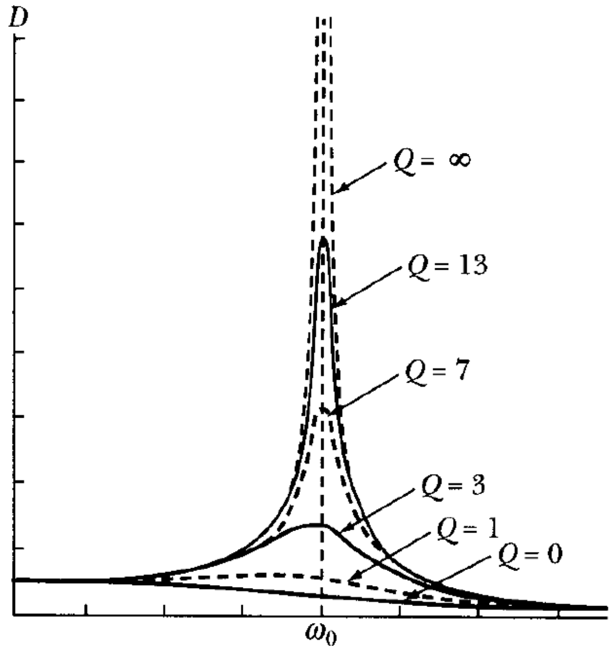

Figure 2: Amplitude against frequency for different quality factors Q.

Exercises

1. Deduce the motion of the damped driven harmonic oscillator for the case when $\beta^2 \neq \omega_0^2$ by solving

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\omega t). \tag{14}$$

2. (3-7.) A body of uniform cross-sectional area $A = 1cm^2$ and of mass density $\rho = 0.8g/cm^3$ floats in a liquid of density $\rho_0 = 1g/cm^3$ and at equilibrium displaces a volume $V = 0.8cm^3$. Show that the period of small oscillations about the equilibrium position is given by:

$$\tau = 2\pi\sqrt{V/gA}, \tag{15}$$

where g is the gravitational field strength. Determine the value of τ.

3. (3-44.) Consider an underdamped harmonic oscillator, with frequency ω_1. After four cycles the amplitude has dropped to $1/e$ of its initial value. Find the frequency of the damped oscillator to its natural frequency.

4. (3-19.) For a lightly damped oscillator, show that $Q \approx \omega_0/\Delta\omega$, where $\Delta\omega$ is the frequency interval between the points on the amplitude resonance curve that are $1/\sqrt{2} \approx 0.707$ of the maximum amplitude. Interpret this result.