FDP 2022 Classical and quantum mechanics - Finals

24 April, 2021

The duration of the test is **3 hours**, and it will account for 40% of your final grade. There are 2 exercises and 3 problems in total, with uniform weightage.

Exercises

1) Lagrangian and Newtonian formalisms
Write down the Lagrangians and recover Newton's equations (without solving them) for the following situations:

(a) 1D simple harmonic oscillator: $F = -kx$

(b) Charged particle in electromagnetic field: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, where:

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{1}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{2}$$

2) Properties of Gaussian wavepackets
Consider a 1D Gaussian wavepacket centered at the origin, with momentum p_0:

$$\psi(x) = e^{ip_0x'/\hbar}\frac{e^{(-x'^2/(2\Delta^2))}}{(\pi\Delta^2)^{1/4}}. \tag{3}$$

Using expressions for Gaussian integrals,

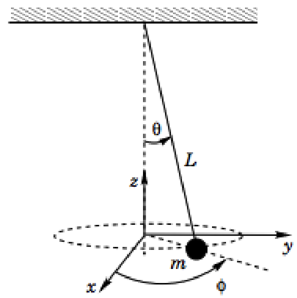
(a) Find the probability distribution $P_\psi(x)$ of the wavepacket, and show that it is normalized to unity.

(b) Show that the packet has a mean position $\langle X\rangle_\psi = 0$, mean momentum $\langle P\rangle_\psi = p_0$, position uncertainty $(\Delta X)_\psi = \Delta/2^{1/2}$, and momentum uncertainty $(\Delta P)_\psi = \hbar/(2^{1/2}\Delta)$.

(c) Show that this packet saturates the lower bound given by Heisenberg's uncertainty relation.

Problems

1) Spherical pendulum
A pendulum with mass m and a massless stick of fixed length L moves in three dimensions under the influence of a constant gravitational field, with gravitational acceleration g:



(a) Working with spherical coordinates (r, θ, ϕ) as labelled in the diagram, write down the Lagrangian of the problem. How many degrees of freedom does the problem have?

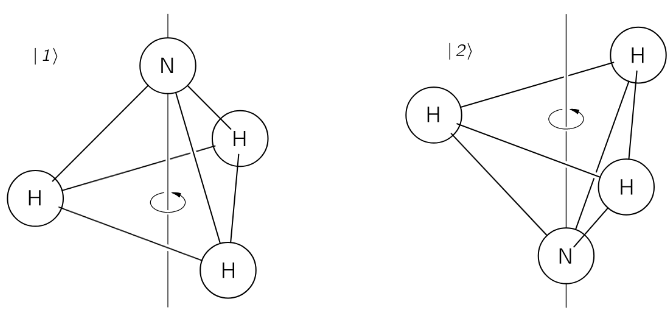
(b) Derive the pendulum's equations of motion.

(c) Show that the magnitude of the angular momentum about the z axis, L_z, is equal to the generalized momentum, p_ϕ. Why is the generalized momentum conserved in time?

(d) Solve the equations of motion in the special case where $\dot{\phi} = 0$, and θ is small. What situation does this correspond to?

2) Ammonia molecule

Consider a free ammonia molecule at rest, which can be described by a vector in a two-dimensional Hilbert space. The orthonormal basis $\{|1\rangle, |2\rangle\}$ formed by basis vectors $|1\rangle$ and $|2\rangle$ correspond to the following geometrical configurations of the molecule:



The Hamiltonian of the molecule in this basis is represented as:

$$\hat{H} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}, \tag{4}$$

where E_0 and A are positive real numbers. In this problem, we study the stability of these geometric configurations of ammonia.

(a) Show that the eigenvalues of $\hat{H}$ are $E_0 + A$ and $E_0 - A$, with corresponding eigenvectors $|E_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ and $|E_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$.

(b) Construct the propagator, $\hat{U}(t)$.

(c) Suppose the initial state $|\psi(0)\rangle$ of the molecule is $|1\rangle$. Determine its time evolution, i.e. find $|\psi(t)\rangle$.

(d) Determine the probability of finding the state to be $|1\rangle$ at time t.

(e) Sketch the probability as a function of time. What can you say about the stability of the initial state $|1\rangle$? Give an example of a state that is stable.

3) Canonical brackets

(a) Show the *canonical commutation relation* between the position and momentum operators, which is:

$$[\hat{x}, \hat{p}] = i\hbar\hat{\mathbb{1}}, \tag{5}$$

where $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator between two arbitrary operators $\hat{A}$ and $\hat{B}$. (Hint: The commutator is an operator. Work in the position basis and apply this operator to an arbitrary state $|\psi\rangle$.)

(b) The Heisenberg-Robertson inequality for arbitrary operators $\hat{A}$ and $\hat{B}$ is given by:

$$(\Delta A)^2_\rho (\Delta B)^2_\rho \geq \frac{1}{4}\langle i[\hat{A}, \hat{B}]\rangle^2_\rho. \tag{6}$$

Recover Heisenberg's uncertainty relation using this inequality.

(c) The quantum mechanical canonical commutation relation has an analogue in the Hamiltonian formalism of classical mechanics. Instead of the commutator $[\hat{A}, \hat{B}]$ between two operators, the *Poisson brackets* between two classical dynamical variables $\omega(x,p)$ and $\lambda(x,p)$ are defined as:

$$\{\omega, \lambda\} \equiv \frac{\partial \omega}{\partial x}\frac{\partial \lambda}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial \lambda}{\partial x}. \tag{7}$$

Show that:

$$\{x, p\} = 1. \tag{8}$$

(d) Show that Hamilton's canonical equations of motion can be written in terms of the Poisson brackets:

$$\dot{x} = \{x, H\} \tag{9}$$

$$\dot{p} = \{p, H\}. \tag{10}$$