Braess's Paradox in Random Networks

Oct 23, 2015

Outline

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Selfish Routing Game

- Source - sink pair (s, t)
- Traffic rate r
- Inifinite number of players (Non-atomic)
- Non-decreasing latency functions
- Nash flow exists
  - Same latency for all players
  - Same common latency in all Nash flows
  - In general inefficient (PoA, PoS...) [2]

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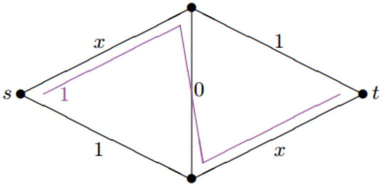

(a) Original Network with equilibrium latency 2

(b) Modified network with equilibrium latency 3/2

Fig. 1: Braess's Paradox

- Counterintuitive
- Removing edges can decrease latency
  (Adding edges can increase latency)
- *PoA* $= 4/3$, the worst PoA in affine latency functions
- Braess ratio = largest factor of latency improvement by removing edges

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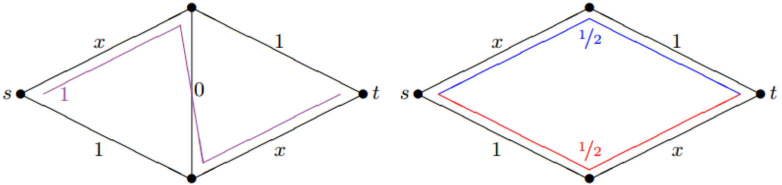

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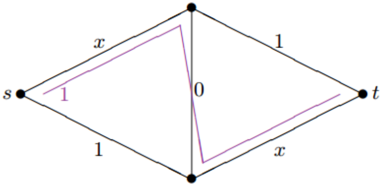

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Braess's Paradox Properties

- History
  - Inspired in 1968
  - Not much knowledge about its mechanism until 2000
- Happens in
  - Many real-world networks, by transportation scientist
  - ER random network (discussed today) [1]
- Doesn't happen in
  - Constant latency (reduced to minimum-cost flow problem)
  - Series-Parallel Network
- *NP-hard* to find the BP and decide which edges to remove [3]

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ER Network Definition

- Graph $G(n, p)$
- Undirected graph on n vertices
- Edge (v, w) exists with probability p, independently, for all pairs of vertices (v, w).

ER Network Properties

- Very simple, good statistical properties
- Sharp threshold for phase changes: giant component, connectivity, etc
- Not a good model for real-world simulation
- Modified version: Small-World networks

Outline

Model Definition

- Underlying graph $G \sim G(n, p)$ with $p = \Omega(n^{(-1/2)+\zeta})$ for some $\zeta > 0$
- Affine latency functions: $l(x) = ax + b, a, b \geq 0$
  - (Simplest functions for Braess's Paradox to occur)
- Independent coefficients model: $a \sim \mathcal{A}, b \sim \mathcal{B}$
  - Reasonable assumptions: $\mathcal{A}$ bounded from 0, $\mathcal{B}$ dense around 0
- The 1/x model: $l(x) = x$ (not independent) with probability q and $l(x) = 1$ with probability $1 - q$

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Braess's Paradox is dense

Theorem

Let $p = \Omega(n^{(-1/2)+\zeta})$ be an edge sampling probability with for some $\zeta > 0$, $\mathcal{A}, \mathcal{B}$ reasonable distributions, there is a constant $\rho = \rho(\zeta, \mathcal{A}, \mathcal{B}) > 1$ such that, with high probability, a random network (G, l) admits a choice of traffic rate r such that the Braess ratio of the instance (G, r, l) is at least ρ.

Theorem

Let p, q, ϵ be constants. With high probability, a sufficiently large random network (G, l) from $\mathcal{G}(n, p, q)$ admits a choice of traffic rate r such that the Braess ratio of the instance (G, r, l) is at least $\frac{4-3pq}{3-2pq} - \epsilon$.

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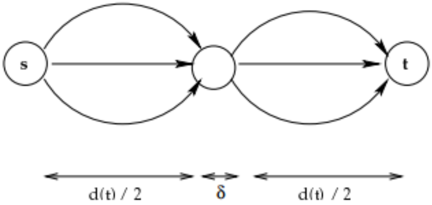
Proof Approach

- Want to show a random network has a "global" structure similar to the 4-node Braess network
- Argue that all "internal vertices" in the Nash flow have almost equal distance from source
- Regard G as two sets of parallel links, latency balanced



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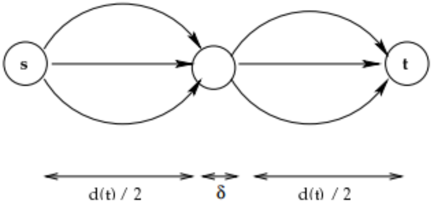

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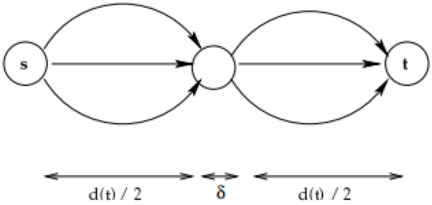
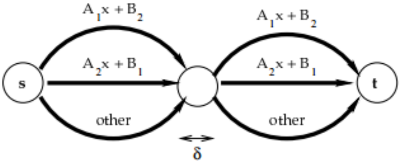


Proof Approach cont'd

- Partition all links into 3 groups: "1-type", "x-type", others
- Delete links inside G to pair up 1-type and x-type edges (Analogous to the removal of the intermediate link in Braess network)
- Argue that latency improved after removal of links



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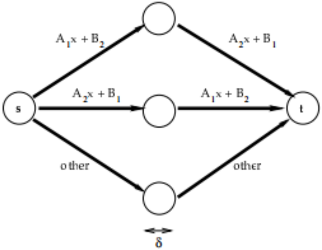

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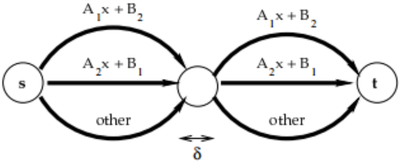

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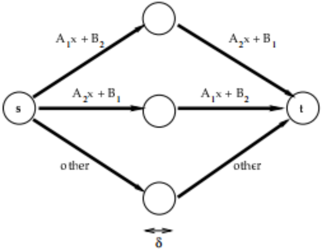

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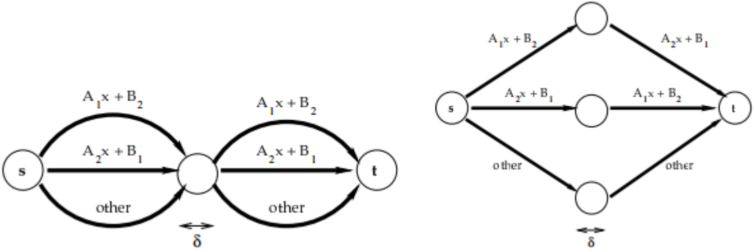


(a) Original Network G (b) Subnetwork G'

Braess's Paradox in Other Random Networks

- In 2010, the main theorem is expanded to sparser, more general networks. [4]
- The proof idea is all the same.

Theorem 3. *Let G be an Erdős-Rényi random graph on n vertices with edge probability p. Let $\mathcal{A}$ and $\mathcal{B}$ be reasonable distributions and let all latency functions have the form $\ell_e(f_e) = a_e f_e + b_e$ where (a_e, b_e) is distributed according to $\mathcal{A} \times \mathcal{B}$. There are constants $\delta > 0$, $c > 1$, and $\rho > 1$ such that, if $\mathbb{P}\left(\mathcal{B} \leq \frac{\delta}{\log(n)}\right) pn \geq c \log(n)$, then there is a flow rate R such that the instance (G, ℓ, R) has Braess's ratio at least ρ with high probability.*

Braess's Paradox in Other Random Networks

- In 2012, analogous result was proved in expander networks with continuous convex latency functions. [5]
- More, more and more x-type, 1-type analysis...

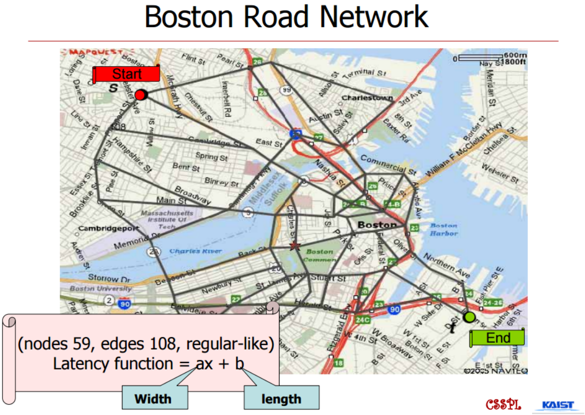
Theorem 1 (nontechnical). *If G is a sufficiently good expander with source-sink pair (s, t) with $\deg(s) \approx \deg(t)$, and the latency functions are randomly chosen from a reasonable class of continuous, convex, latency functions, then with high probability there is a subgraph G' and a traffic rate R such that the selfish routing on G' incurs less latency than the selfish routing on G.*

Finding Best Subnetwork

- In 2013, the same type of analysis leads to an approximation algorithm [6], that runs in
  - polynomial time if the graph has average degree $O(poly(\ln n))$ and the traffic rate is $O(poly(\ln\ln n))$
  - quasipolynomial time if average degree $o(n)$ and traffic rate $O(poly(\ln n))$
- The same property that causes Braess's Paradox to be everywhere makes it easy to find.

Braess's Paradox in Reality

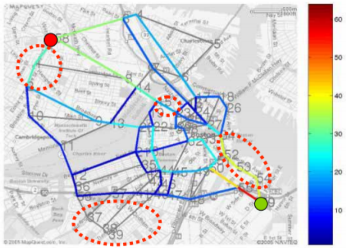
- An 2006 conference talk data [7]: Braess's Paradox in Boston road network



Braess's Paradox in Reality

- Find optimal solution and the best Nash flow for all numbers of agents

Congestion distribution on the edges



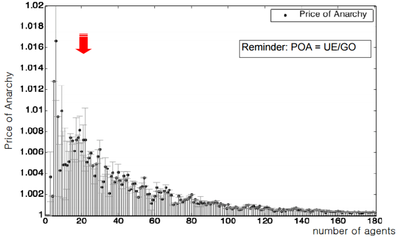
User Optimum **Global Optimum**

Number of Agents: **20**

Braess's Paradox in Reality

- PoA is small, due to affine costs
- They identified which roads are Braess-like (can be removed)
- They also simulated Regular-Like, Barabasi-Albert, ER and Small-World random graphs with various graph sizes

Variation of POA with Agent #



Summary

- Braess's Paradox is dense, easy to find and to resolve in random networks.
- Magic number 4/3:
  - PoA and Braess ratio for the original four-node network
  - The largest PoA for affine-cost network
  - Limit of Braess's ratio for 1/x model
  - NP-hard for any $\frac{4}{3}$-approximation algorithm in deterministic network
- Need extension to more real-world like network models

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