Chapter 8

Graph sketching

Definition 8.1 (Streaming connected components problem)**.** *Consider a graph of n vertices and a stream S of edge updates $\{\langle e_t, \pm \rangle\}_{t \in \mathbb{N}^+}$, where edge e_t is either added (+) or removed (-). Assume that S is "well-behaved", that is existing edges are not added and an edge is deleted only if it's already present in the graph.*

At time t, the edge set E_t of the graph $G_t = (V, E_t)$ is the set of edges present after accounting for all stream updates up to time t. How much memory do we need if we want to be able to query the connected components for G_t for any $t \in \mathbb{N}^+$?

Let m be the total number of distinct edges in the stream. There are two ways to represent connected components on a graph:

1. Every vertex stores a label such that vertices in the same connected component have the same label
2. Explicitly build a tree for each connected component — This yields a *maximal forest*

For now, we are interested in building a maximal forest for G_t. This can be done with memory size of $\mathcal{O}(m)$ words[1], or — in the special case of *only edge additions* — $\mathcal{O}(n)$ words[2]. However, these are unsatisfactory as $m \in \mathcal{O}(n^2)$ on a complete graph, and we may have edge deletions. We show how one can maintain a data structure with $\mathcal{O}(n \log^4 n)$ memory, with a randomized algorithm that succeeds in building the maximal forest with success probability $\geq 1 - \frac{1}{n^{10}}$.

[1] Toggle edge additions/deletion per update. Compute connected components on demand.

[2] Use the Union-Find data structure. See https://en.wikipedia.org/wiki/Disjoint-set_data_structure

Coordinator model For a change in perspective[3], consider the following computation model where each vertex acts independently from each other. Then, upon request of connected components, each vertex sends some information to a centralized coordinator to perform computation and outputs the maximal forest.

The coordinator model will be helpful in our analysis of the algorithm later as each vertex will send $\mathcal{O}(\log^4 n)$ amount of data (a local sketch of the graph) to the coordinator, totalling $\mathcal{O}(n \log^4 n)$ memory as required.

8.1 Warm up: Finding the single cut

Definition 8.2 (The single cut problem)**.** *Fix an arbitrary subset $A \subseteq V$. Suppose there is exactly 1 cut edge $\{u, v\}$ between A and $V \setminus A$. How do we output the cut edge $\{u, v\}$ using $\mathcal{O}(\log n)$ bits of memory?*

Without loss of generality, assume $u \in A$ and $v \in V \setminus A$. Note that this is not a trivial problem at first glance since it already takes $\mathcal{O}(n)$ bits for any vertex to enumerate all its adjacent edges. To solve the problem, we use a *bit trick* which exploits the fact that any edge $\{a, b\} \in A$ will be considered twice by vertices in A. Since one can uniquely identify each vertex with $\mathcal{O}(\log n)$ bits, consider the following:

- Identify an edge $e = \{u, v\}$ by the concatenation of the identifiers of its endpoints: $id(e) = id(u) \circ id(v)$ if $id(u) < id(v)$
- Locally, every vertex u maintains

  $$XOR_u = \oplus\{id(e) : e \in S \wedge u \text{ is an endpoint of } e\}$$

  Thus XOR_u represents the bit-wise XOR of the identifiers of all edges that are adjacent to u.
- All vertices send the coordinator their value XOR_u and the coordinator computes

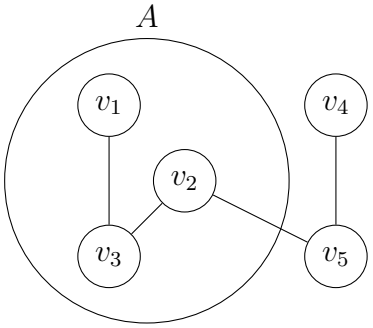
  $$XOR_A = \oplus\{XOR_u : u \in A\}$$

[3] In reality, the algorithm simulates all the vertices' actions so it is not a real multi-party computation setup.

Example Suppose $V = \{v_1, v_2, v_3, v_4, v_5\}$ where $id(v_1) = 000$, $id(v_2) = 001$, $id(v_3) = 010$, $id(v_4) = 011$, and $id(v_5) = 100$. Then, $id(\{v_1, v_3\}) = id(v_1) \circ id(v_3) = 000010$, and so on. Suppose

$$S = \{\langle\{v_1, v_2\}, +\rangle, \langle\{v_2, v_3\}, +\rangle, \langle\{v_1, v_3\}, +\rangle, \langle\{v_4, v_5\}, +\rangle, \langle\{v_2, v_5\}, +\rangle, \langle\{v_1, v_2\}, -\rangle\}$$

and we query for the cut edge $\{v_2, v_5\}$ with $A = \{v_1, v_2, v_3\}$ at $t = |S|$. The figure below shows the graph G_6 when $t = 6$:



Vertex v_1 sees $\{\langle\{v_1, v_2\}, +\rangle, \langle\{v_1, v_3\}, +\rangle$, and $\langle\{v_1, v_2\}, -\rangle\}$. So,

$$\begin{aligned}
XOR_1 &\Rightarrow 000000 && \text{Initialize}\\
&\Rightarrow 000000 \oplus id((v_1, v_2)) = 000000 \oplus 000001 = 000001 && \text{Due to } \langle\{v_1, v_2\}, +\rangle\\
&\Rightarrow 000001 \oplus id((v_1, v_3)) = 000001 \oplus 000010 = 000011 && \text{Due to } \langle\{v_1, v_3\}, +\rangle\\
&\Rightarrow 000011 \oplus id((v_1, v_2)) = 000011 \oplus 000001 = 000010 && \text{Due to } \langle\{v_1, v_2\}, -\rangle
\end{aligned}$$

Repeating the simulation for all vertices,

$$\begin{aligned}
XOR_1 = 000010 &= id(\{v_1, v_2\}) \oplus id(\{v_1, v_3\}) \oplus id(\{v_1, v_2\})\\
&= 000001 \oplus 000010 \oplus 000001\\
XOR_2 = 000110 &= id(\{v_1, v_2\}) \oplus id(\{v_2, v_3\}) \oplus id(\{v_2, v_5\}) \oplus id(\{v_1, v_2\})\\
&= 000001 \oplus 001010 \oplus 001100 \oplus 000001\\
XOR_3 = 001000 &= id(\{v_2, v_3\}) \oplus id(\{v_1, v_3\})\\
&= 001010 \oplus 000010\\
XOR_4 = 011100 &= id(\{v_4, v_5\})\\
&= 011100\\
XOR_5 = 010000 &= id(\{v_4, v_5\}) \oplus id(\{v_2, v_5\})\\
&= 011100 \oplus 001100
\end{aligned}$$

Thus, $XOR_A = XOR_1 \oplus XOR_2 \oplus XOR_3 = 000010 \oplus 000110 \oplus 001000 = 001100 = id(\{v_2, v_5\})$ as expected. Notice that after adding or deleting an edge $e = (u, v)$, updating XOR_u and XOR_v can be done by doing a bit-wise XOR of each of these values together with $id(e)$. Also, the identifier of every edge with both endpoints in A contributes two times to XOR_A.

Claim 8.3. *$XOR_A = \oplus\{XOR_u : u \in A\}$ is the identifier of the cut edge.*

Proof. For any edge $e = (a, b)$ such that $a, b \in A$, $id(e)$ contributes to both XOR_a and XOR_b. So, $XOR_a \oplus XOR_b$ will cancel out the contribution of $id(e)$ because $id(e) \oplus id(e) = 0$. Hence, the only remaining value in $XOR_A = \oplus\{XOR_u : u \in A\}$ will be the identifier of the cut edge since only one of its endpoints lies in A. □

Remark Bit tricks are often used in the random linear network coding literature (e.g. [HMK+06]).

8.2 Warm up 2: Finding one out of $k > 1$ cut edges

Definition 8.4 (The k cut problem)**.** *Fix an arbitrary subset $A \subseteq V$. Suppose there are exactly k cut edges (u, v) between A and $V \setminus A$, and we are given an estimate $\widehat{k}$ such that $\frac{\widehat{k}}{2} \leq k \leq \widehat{k}$. How do we output a cut edge (u, v) using $\mathcal{O}(\log n)$ bits of memory, with high probability?*

A straight-forward idea is to independently mark each edge, each with probability $1/\widehat{k}$. In expectation, we expect one edge to be marked. Denote the set of marked cut edges by E'.

$$
\begin{aligned}
&\Pr[|E'| = 1] \\
&= k \cdot \Pr[\text{Cut edge } \{u, v\} \text{ is marked; others are not}] \\
&= k \cdot (1/\widehat{k})(1 - (1/\widehat{k}))^{k-1} && \text{Edges marked ind. w.p. } 1/\widehat{k} \\
&\geq (\widehat{k}/2)(1/\widehat{k})(1 - (1/\widehat{k}))^{\widehat{k}} && \text{Since } \frac{\widehat{k}}{2} \leq k \leq \widehat{k} \\
&\geq \frac{1}{2} \cdot 4^{-1} && \text{Since } 1 - x \geq 4^{-x} \text{ for } x \leq 1/2 \\
&\geq \frac{1}{10}
\end{aligned}
$$

Remark The above analysis assumes that vertices can locally mark the edges in a consistent manner (i.e. both endpoints of any edge make the same decision whether to mark the edge or not). This can be done with a sufficiently large string of *shared randomness*. We discuss this in Section 8.3.

From above, we know that $\Pr[|E'| = 1] \geq 1/10$. If $|E'| = 1$, we can re-use the idea from Section 8.1. However, if $|E'| \neq 1$, then XOR_A may correspond erroneously to another edge in the graph. In the above example, $id(\{v_1, v_2\}) \oplus id(\{v_2, v_4\}) = 000001 \oplus 001011 = 001010 = id(\{v_2, v_3\})$.

To fix this, we use random bits as edge IDs instead of simply concatenating vertex IDs: Randomly assign (in a consistent manner) to each edge a random ID of ~~$k =$~~ $20 \log n$ bits. Since the XOR of random bits is random, for any edge e, $\Pr[XOR_A = id(e) \mid |E'| \neq 1] = (\frac{1}{2})^{k} = (\frac{1}{2})^{20 \log n}$. Hence,

$$
\begin{aligned}
&\Pr[XOR_A = id(e) \text{ for some edge } e \mid |E'| \neq 1] \\
&\leq \sum_{e \in \binom{V}{2}} \Pr[XOR_A = id(e) \mid |E'| \neq 1] && \text{Union bound over all possible edges} \\
&= \binom{n}{2} (\frac{1}{2})^{20 \log n} && \text{There are } \binom{n}{2} \text{ possible edges} \\
&= 2^{-18 \log n} && \text{Since } \binom{n}{2} \leq n^2 = 2^{2 \log n} \\
&= \frac{1}{n^{18}} && \text{Rewriting}
\end{aligned}
$$

Now, we can correctly distinguish $|E'| = 1$ from $|E'| \neq 1$ and $\Pr[|E'| = 1] \geq \frac{1}{10}$. For any given $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that if we repeat $t = C(\epsilon) \log n$ times, the probability that *all* t tries fail to extract a single cut is $(1 - \frac{1}{10})^t \leq \frac{1}{n^{1+\epsilon}}$.

8.3 Maximal forest with $\mathcal{O}(n \log^4 n)$ memory

Recall that Borůvka's algorithm[4] builds a minimum spanning tree by iteratively finding the cheapest edge leaving connected components and adding them into the MST. The number of connected components decreases by at least half per iteration, so it converges in $\mathcal{O}(\log n)$ iterations.

For any arbitrary cut, the number of edge cuts is $k \in [0, n]$. Guessing through $\widehat{k} = 2^0, 2^1, \ldots, 2^{\lceil \log n \rceil}$, one can use Section 8.2 to find a cut edge:

- If $\widehat{k} \gg k$, the marking probability will select nothing (in expectation).

[4] See https://en.wikipedia.org/wiki/Bor%C5%AFvka%27s_algorithm

- If $\widehat{k} \ll k$, more than one edge will get marked, which we will then detect (and ignore) since XOR_A will likely not be a valid edge ID.

Algorithm 25 ComputeSketches($S = \{\langle e, \pm \rangle, \dots\}, \epsilon, \mathcal{R}$)

```
for i = 1, ..., n do
    XOR_i ← 0^{(20 log n) * log^3 n}                ▷ Initialize log^3 n copies
end for
for Edge update {⟨e = (u, v), ±⟩} ∈ S do            ▷ Streaming edge updates
    for b = log n times do                           ▷ Simulate Borůvka
        for i ∈ {1, 2, ..., log n} do                ▷ log n guesses of k
            for t = C(ε) log n times do              ▷ Amplify success probability
                R_{b,i,t} ← Randomness for this specific instance based on R
                if Edge e is marked w.p. 1/k̂ = 2^{-i}, according to R_{b,i,t} then
                    Compute id(e) using R
                    XOR_u[b, i, t] ← XOR_u[b, i, t] ⊕ id(e)
                    XOR_v[b, i, t] ← XOR_v[b, i, t] ⊕ id(e)
                end if
            end for
        end for
    end for
end for
return XOR_1, ..., XOR_n
```

Using a source of randomness $\mathcal{R}$, every vertex in ComputeSketches maintains $\mathcal{O}(\log^3 n)$ copies of edge XORs using random (but consistent) edge IDs and marking probabilities:

- $\lceil \log n \rceil$ times for Borůvka simulation later
- $\lceil \log n \rceil$ times for guesses of cut size k
- $C(\epsilon) \cdot \log n$ times to amplify success probability of Section 8.2

Then, StreamingMaximalForest simulates Borůvka using the output of ComputeSketches:

- Find an out-going edge from each connected component via Section 8.2
- Join connected components by adding edges to graph

Since each edge ID uses $\mathcal{O}(\log n)$ memory and $\mathcal{O}(\log^3 n)$ copies were maintained per vertex, a total of $\mathcal{O}(n \log^4 n)$ memory suffices. At each step, we

Algorithm 26 StreamingMaximalForest($S = \{\langle e, \pm \rangle, \dots\}, \epsilon$)

$\mathcal{R} \leftarrow$ Generate $\mathcal{O}(\log^2 n)$ bits of shared randomness
$XOR_1, \dots, XOR_n \leftarrow$ ComputeSketches($S, \epsilon, \mathcal{R}$)
$F \leftarrow (V_F = V, E_F = \emptyset)$ ▷ Initialize empty forest
for $b = \log n$ times **do** ▷ Simulate Borůvka
  $C \leftarrow \emptyset$ ▷ Initialize candidate edges
  for Every connected component A in F **do**
    for $i \in \{1, 2, \dots, \lceil \log n \rceil\}$ **do** ▷ Guess A has $[2^{i-1}, 2^i]$ cut edges
      for $t = C(\epsilon) \log n$ times **do** ▷ Amplify success probability
        $R_{b,i,t} \leftarrow$ Randomness for this specific instance
        $XOR_A \leftarrow \oplus\{XOR_u[b, i, t] : u \in A\}$
        if $XOR_A = id(e)$ for some edge $e = (u, v)$ **then**
          $C \leftarrow C \cup \{(u, v)\}$ ▷ Add cut edge (u, v) to candidates
          Go to next connected component in F
        end if
      end for
    end for
  end for
  $E_F \leftarrow E_F \cup C$, removing cycles in $\mathcal{O}(1)$ if necessary ▷ Add candidates
end for
return F

fail to find one cut edge leaving a connected component with probability $\leq (1 - \frac{1}{10})^t$, which can be ~~be~~ made to be in $\mathcal{O}(\frac{1}{n^{10}})$. Applying union bound over all $\mathcal{O}(\log^3 n)$ computations of XOR_A, we see that

$$\Pr[\textit{Any } XOR_A \text{ corresponds wrongly some edge ID}] \leq \mathcal{O}(\frac{\log^3 n}{n^{18}}) \subseteq \mathcal{O}(\frac{1}{n^{10}})$$

So, StreamingMaximalForest succeeds with high probability.

Remark One can drop the memory constraint per vertex from $\mathcal{O}(\log^4 n)$ to $\mathcal{O}(\log^3 n)$ by using a constant t instead of $t \in \mathcal{O}(\log n)$ such that the success probability is a constant larger than $1/2$. Then, simulate Borůvka for $\lceil 2 \log n \rceil$ steps. See [AGM12] (Note that they use a slightly different sketch).

Theorem 8.5. *Any randomized distributed sketching protocol for computing spanning forest with success probability $\epsilon > 0$ must have expected average sketch size $\Omega(log^3 n)$, for any constant $\epsilon > 0$.*

Proof. See [NY18]. □

Claim 8.6. *Polynomial number of bits provide sufficient independence for the procedure described above.*

Remark One can generate polynomial number of bits of randomness with $\mathcal{O}(\log^2 n)$ bits. Interested readers can check out small-bias sample spaces[5]. The construction is out of the scope of the course, but this implies that the shared randomness $\mathcal{R}$ can be obtained within our memory constraints.

[5] See https://en.wikipedia.org/wiki/Small-bias_sample_space