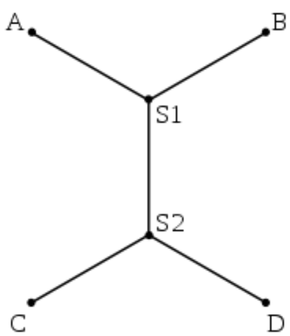
Approximation algorithms, Part II

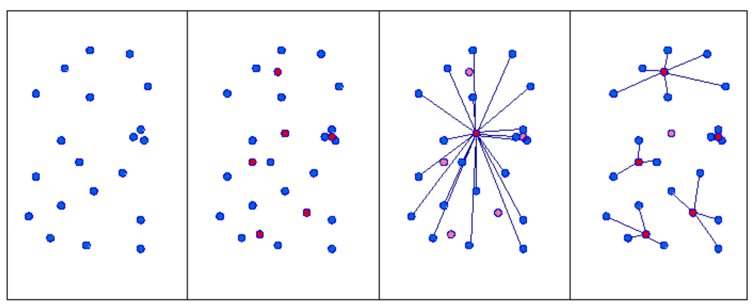
Problems

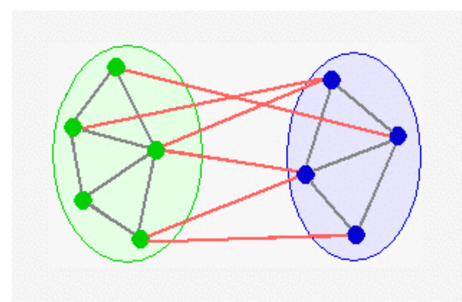
Steiner forest
Facility location
Maximum cut
Sparsest cut...

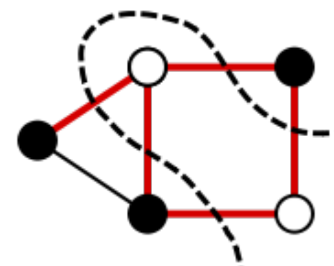
Techniques

Linear programming duality
Semi-definite programming
(More) geometric embeddings









Linear programming duality

Technique 1
Linear programming duality

Bounding the value of an LP

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$

$$\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad \mathbf{(1)}$$

$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad \mathbf{(2)}$$

$$\mathbf{x_1, x_2, x_3 \geq 0} \quad \mathbf{(3, 4, 5)}$$

How do we certify that OPT is at most 54?

How do we certify that OPT is at most 54?

$$\begin{aligned} &\min \mathbf{7x_1 + x_2 + 5x_3} : \\ &\mathbf{x_1 - x_2 + 3x_3 \geq 10} && \mathbf{(1)} \\ &\mathbf{5x_1 + 2x_2 - x_3 \geq 6} && \mathbf{(2)} \\ &\mathbf{x_1, x_2, x_3 \geq 0} && \mathbf{(3, 4, 5)} \end{aligned}$$

Try (7,0,1):

- feasible
- objective value = 54

How do we certify an upper bound on OPT?

For minimization:

exhibit a feasible solution

its value is an upper bound

Bounding the value of an LP

$$\begin{array}{ll} \min \mathbf{7x_1 + x_2 + 5x_3}: & \\ \mathbf{x_1 - x_2 + 3x_3 \geq 10} & \mathbf{(1)} \\ \mathbf{5x_1 + 2x_2 - x_3 \geq 6} & \mathbf{(2)} \\ \mathbf{x_1, x_2, x_3 \geq 0} & \mathbf{(3, 4, 5)} \end{array}$$

How do we certify that OPT is at least 10?

How do we certify that OPT is at least 10?

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$

$$\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad (\mathbf{1})$$

$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad (\mathbf{2})$$

$$\mathbf{x_1, x_2, x_3 \geq 0} \quad (\mathbf{3, 4, 5})$$

$$\mathbf{7x_1 \geq x_1 \text{ and } x_2 \geq -x_2 \text{ and } 5x_3 \geq 3x_3}$$

$$\mathbf{\text{so } 7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10}$$

A better lower bound for OPT

$$\begin{aligned} &\min \mathbf{7x_1 + x_2 + 5x_3} : \\ &\mathbf{x_1 - x_2 + 3x_3 \geq 10} \qquad \mathbf{(1)} \\ &\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \qquad \mathbf{(2)} \\ &\mathbf{x_1, x_2, x_3 \geq 0} \qquad \mathbf{(3, 4, 5)} \end{aligned}$$

$\mathbf{2 \times (1) + (2)}$ **implies:** $\mathbf{7x_1 + 5x_3 \geq 26.}$

so $\mathbf{7x_1 + x_2 + 5x_3 \geq 7x_1 + 5x_3 \geq 26}$

How do we certify a lower bound on OPT?

For minimization:
exhibit a convex combination of constraints
if each coefficient is less than in objective
then RHS is a lower bound.

What is the best upper bound we can obtain?

Among (x1,x2,x3) such that

$$x_1 - x_2 + 3x_3 \geq 10$$
$$5x_1 + 2x_2 - x_3 \geq 6$$
$$x_1, x_2, x_3 \geq 0$$

Choose the one that minimizes

$$7x_1 + x_2 + 5x_3$$

What is the best lower bound we can obtain?

$$\min 7x_1 + x_2 + 5x_3 :$$
$$x_1 - x_2 + 3x_3 \geq 10 \quad (1)$$
$$5x_1 + 2x_2 - x_3 \geq 6 \quad (2)$$
$$x_1, x_2, x_3 \geq 0 \quad (3, 4, 5)$$

Among the convex combinations of constraints

$$y_1 \times (1) + y_2 \times (2)$$

such that

$$7 \geq y_1 + 5y_2 \text{ and } 1 \geq -y_1 + 2y_2 \text{ and } 5 \geq 3y_1 - y_2$$

Choose the one that maximizes

$$10y_1 + 6y_2$$

It's a linear program!

$$
\begin{aligned}
&\min \mathbf{7x_1 + x_2 + 5x_3} : \\
&\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad (\mathbf{1}) \\
&\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad (\mathbf{2}) \\
&\mathbf{x_1, x_2, x_3 \geq 0} \quad (\mathbf{3, 4, 5})
\end{aligned}
$$

Lower bound LP

$$
\begin{aligned}
&\max \mathbf{10y_1 + 6y_2} : \\
&\mathbf{y_1 + 5y_2 \leq 7} \quad (\mathbf{1'}) \\
&\mathbf{-y_1 + 2y_2 \leq 1} \quad (\mathbf{2'}) \\
&\mathbf{-3y_1 - y_2 \leq 5} \quad (\mathbf{3'}) \\
&\mathbf{y_1, y_2 \geq 0} \quad (\mathbf{4', 5'})
\end{aligned}
$$

Primal LP (P)

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$
$$\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad (\mathbf{1})$$
$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad (\mathbf{2})$$
$$\mathbf{x_1, x_2, x_3 \geq 0} \quad (\mathbf{3}, \mathbf{4}, \mathbf{5})$$

Dual LP (D)

$$\max \mathbf{10y_1 + 6y_2} :$$
$$\mathbf{y_1 + 5y_2 \leq 7} \quad (\mathbf{1'})$$
$$\mathbf{-y_1 + 2y_2 \leq 1} \quad (\mathbf{2'})$$
$$\mathbf{-3y_1 - y_2 \leq 5} \quad (\mathbf{3'})$$
$$\mathbf{y_1, y_2 \geq 0} \quad (\mathbf{4'}, \mathbf{5'})$$

Linear programming duality

Linear programming duality

Primal LP (P)

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$
$$\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad (\mathbf{1})$$
$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad (\mathbf{2})$$
$$\mathbf{x_1, x_2, x_3 \geq 0} \quad (\mathbf{3}, \mathbf{4}, \mathbf{5})$$

Dual LP (D)

$$\max \mathbf{10y_1 + 6y_2} :$$
$$\mathbf{y_1 + 5y_2 \leq 7} \quad (\mathbf{1'})$$
$$\mathbf{-y_1 + 2y_2 \leq 1} \quad (\mathbf{2'})$$
$$\mathbf{3y_1 - y_2 \leq 5} \quad (\mathbf{3'})$$
$$\mathbf{y_1, y_2 \geq 0} \quad (\mathbf{4'}, \mathbf{5'})$$

Lemma

$$\min\{\mathbf{7x_1 + x_2 + 5x_3 : x \ feasible}\} \geq \max\{\mathbf{10y_1 + 6y_2 : y \ feasible}\}$$

Applying the same ideas to a maximization problem

$$\max \mathbf{10y_1 + 6y_2} :$$

$$\mathbf{y_1 + 5y_2 \leq 7} \qquad (\mathbf{1'})$$

$$\mathbf{-y_1 + 2y_2 \leq 1} \qquad (\mathbf{2'})$$

$$\mathbf{3y_1 - y_2 \leq 5} \qquad (\mathbf{3'})$$

$$\mathbf{y_1, y_2 \geq 0} \qquad (\mathbf{4', 5'})$$

How do we prove a lower bound?

$$
\begin{aligned}
&\max \mathbf{10y_1 + 6y_2} : \\
&\quad \mathbf{y_1 + 5y_2 \leq 7} && (\mathbf{1'}) \\
&\mathbf{-y_1 + 2y_2 \leq 1} && (\mathbf{2'}) \\
&\mathbf{3y_1 - y_2 \leq 5} && (\mathbf{3'}) \\
&\mathbf{y_1, y_2 \geq 0} && (\mathbf{4', 5'})
\end{aligned}
$$

Exhibit a feasible (y1,y2)
its value is a lower bound

How do we prove an upper bound?

$$\max \mathbf{10y_1 + 6y_2} :$$

$$\mathbf{y_1 + 5y_2 \leq 7} \qquad (\mathbf{1'})$$

$$\mathbf{-\,y_1 + 2y_2 \leq 1} \qquad (\mathbf{2'})$$

$$\mathbf{3y_1 - y_2 \leq 5} \qquad (\mathbf{3'})$$

$$\mathbf{y_1, y_2 \geq 0} \qquad (\mathbf{4', 5'})$$

Convex combination of (1'),(2'),(3')
s.t. coeff of y1 is at least 10
and coeff of y2 is at least 6

Upper bound, formally:

$$\max \mathbf{10y_1 + 6y_2} :$$
$$\mathbf{y_1 + 5y_2 \le 7} \quad (\mathbf{1'})$$
$$\mathbf{-y_1 + 2y_2 \le 1} \quad (\mathbf{2'})$$
$$\mathbf{3y_1 - y_2 \le 5} \quad (\mathbf{3'})$$
$$\mathbf{y_1, y_2 \ge 0} \quad (\mathbf{4', 5'})$$

$$\mathbf{z_1 \times (1') + z_2 \times (2') + z_3 \times (3')}$$

If $\mathbf{z_1 - z_2 + 3z_3 \ge 10}$ and $\mathbf{5z_1 + 2z_2 - z_3 \ge 6}$

Then, upper bound for OPT:

$$\mathbf{7z_1 + z_2 + 5z_3}$$

Best upper bound:

$$\begin{aligned}
&\max \mathbf{10y_1 + 6y_2} : \\
&\quad \mathbf{y_1 + 5y_2 \le 7} \qquad (\mathbf{1'}) \\
&\ -\mathbf{y_1 + 2y_2 \le 1} \quad (\mathbf{2'}) \\
&\mathbf{3y_1 - y_2 \le 5} \quad (\mathbf{3'}) \\
&\mathbf{y_1, y_2 \ge 0} \qquad\qquad (\mathbf{4', 5'})
\end{aligned}$$

$$\begin{aligned}
&\min \mathbf{7z_1 + z_2 + 5z_3} : \\
&\mathbf{z_1 - z_2 + 3z_3 \ge 10} \\
&\mathbf{5z_1 + 2z_2 - z_3 \ge 6} \\
&\mathbf{z_1, z_2, z_3 \ge 0}
\end{aligned}$$

Maximization LP

$$\max \mathbf{10y_1 + 6y_2} :$$
$$\mathbf{y_1 + 5y_2 \leq 7} \qquad (\mathbf{1}')$$
$$\mathbf{-y_1 + 2y_2 \leq 1} \qquad (\mathbf{2}')$$
$$\mathbf{-3y_1 - y_2 \leq 5} \qquad (\mathbf{3}')$$
$$\mathbf{y_1, y_2 \geq 0} \qquad (\mathbf{4}', \mathbf{5}')$$

Minimization LP

$$\min \mathbf{7z_1 + z_2 + 5z_3} :$$
$$\mathbf{z_1 - z_2 + 3z_3 \geq 10}$$
$$\mathbf{5z_1 + 2z_2 - z_3 \geq 6}$$
$$\mathbf{z_1, z_2, z_3 \geq 0}$$

Surprise!

Primal

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$
$$\mathbf{x_1 - x_2 + 3x_3 \geq 10}$$
$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6}$$
$$\mathbf{x_1, x_2, x_3 \geq 0}$$

Dual

$$\max \mathbf{10y_1 + 6y_2} :$$
$$\mathbf{y_1 + 5y_2 \leq 7} \qquad (1')$$
$$\mathbf{-y_1 + 2y_2 \leq 1} \qquad (2')$$
$$\mathbf{-3y_1 - y_2 \leq 5} \qquad (3')$$
$$\mathbf{y_1, y_2 \geq 0} \qquad (4', 5')$$

Dual of dual

$$\min \mathbf{7z_1 + z_2 + 5z_3} :$$
$$\mathbf{z_1 - z_2 + 3z_3 \geq 10}$$
$$\mathbf{5z_1 + 2z_2 - z_3 \geq 6}$$
$$\mathbf{z_1, z_2, z_3 \geq 0}$$

The dual of the dual is the primal!

Lemma

$$\min\{\mathbf{7x_1 + x_2 + 5x_3} : \mathbf{x \ feasible}\} \geq$$
$$\max\{\mathbf{10y_1 + 6y_2} : \mathbf{y \ feasible}\}$$

Linear programming duality Theorem

$$\min\{\mathbf{7x_1 + x_2 + 5x_3} : \mathbf{x \ feasible}\} =$$
$$\max\{\mathbf{10y_1 + 6y_2} : \mathbf{y \ feasible}\}$$

In general

(P)

$$\min \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{A}\mathbf{x} \geq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

$\longleftrightarrow$ duality

(D)

$$\max \mathbf{b} \cdot \mathbf{y} :$$
$$\mathbf{A}^{\mathbf{T}}\mathbf{y} \leq \mathbf{c}$$
$$\mathbf{y} \geq \mathbf{0}$$

Strong duality Theorem in general

(P)

$$\min \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{A}\mathbf{x} \geq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

(D)

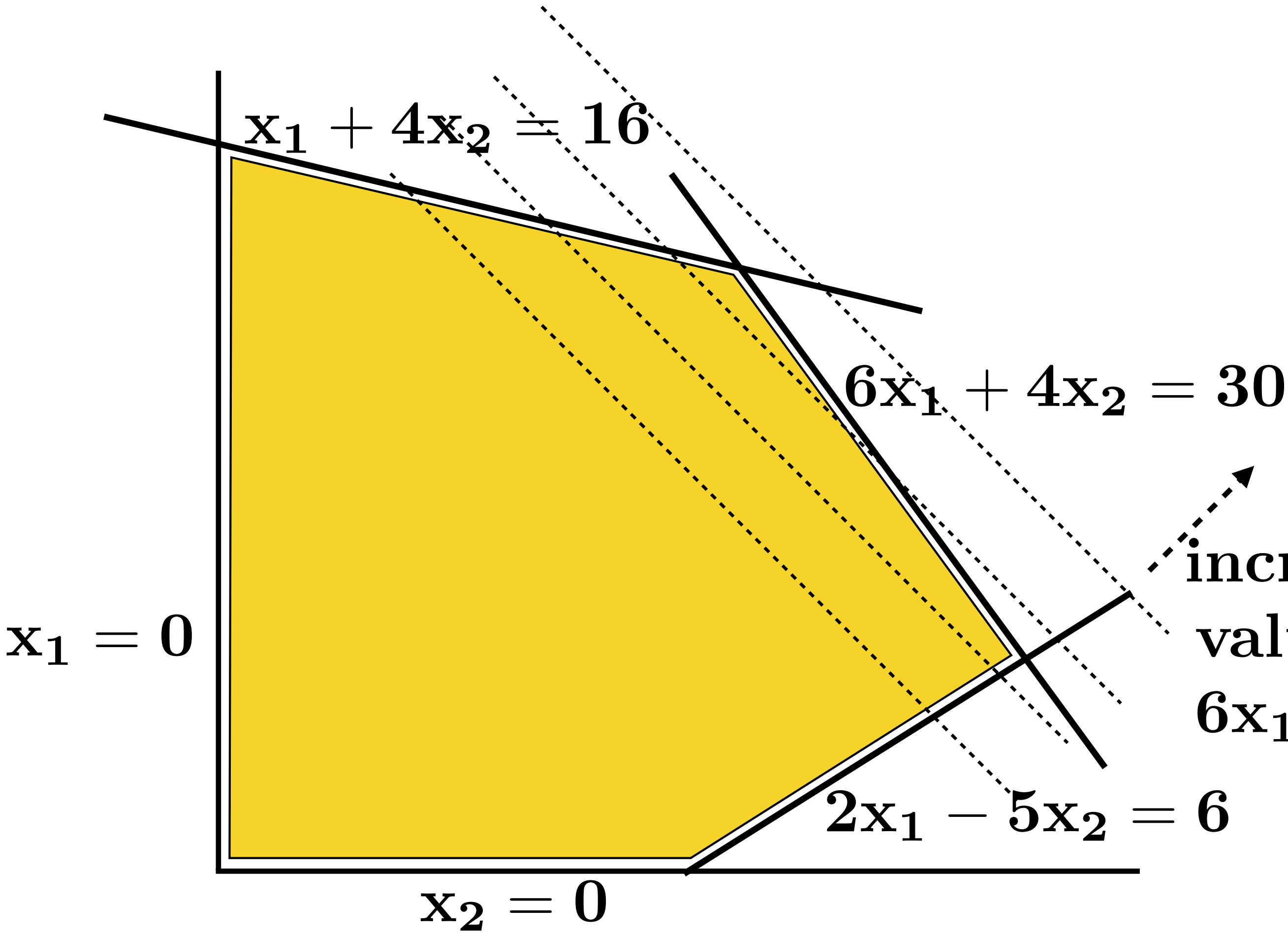
$$\max \mathbf{b} \cdot \mathbf{y} :$$
$$\mathbf{A}^{\mathbf{T}}\mathbf{y} \leq \mathbf{c}$$
$$\mathbf{y} \geq \mathbf{0}$$

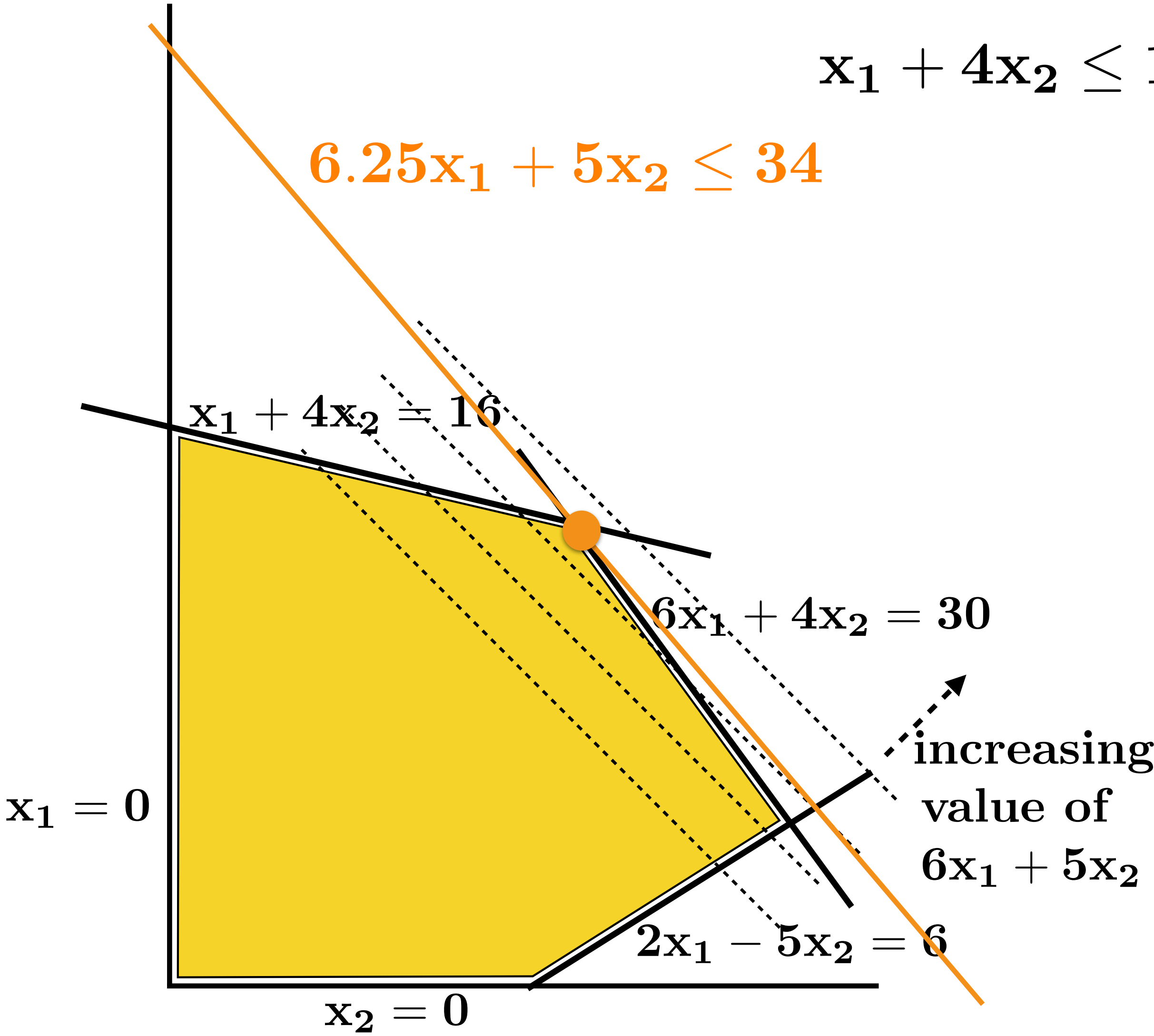
Four possible cases:

- (P) is empty, (D) has value $+\infty$
- (D) is empty, (P) has value $-\infty$
- value(P)=value(D)
- ((P) and (D) both empty)

What about the geometry?

$$\max \mathbf{6x_1 + 5x_2} :$$
$$\mathbf{x_1 + 4x_2 \leq 16}$$
$$\mathbf{6x_1 + 4x_2 \leq 30}$$
$$\mathbf{2x_1 - 5x_2 \leq 6}$$
$$\mathbf{x_1, x_2 \geq 0}$$

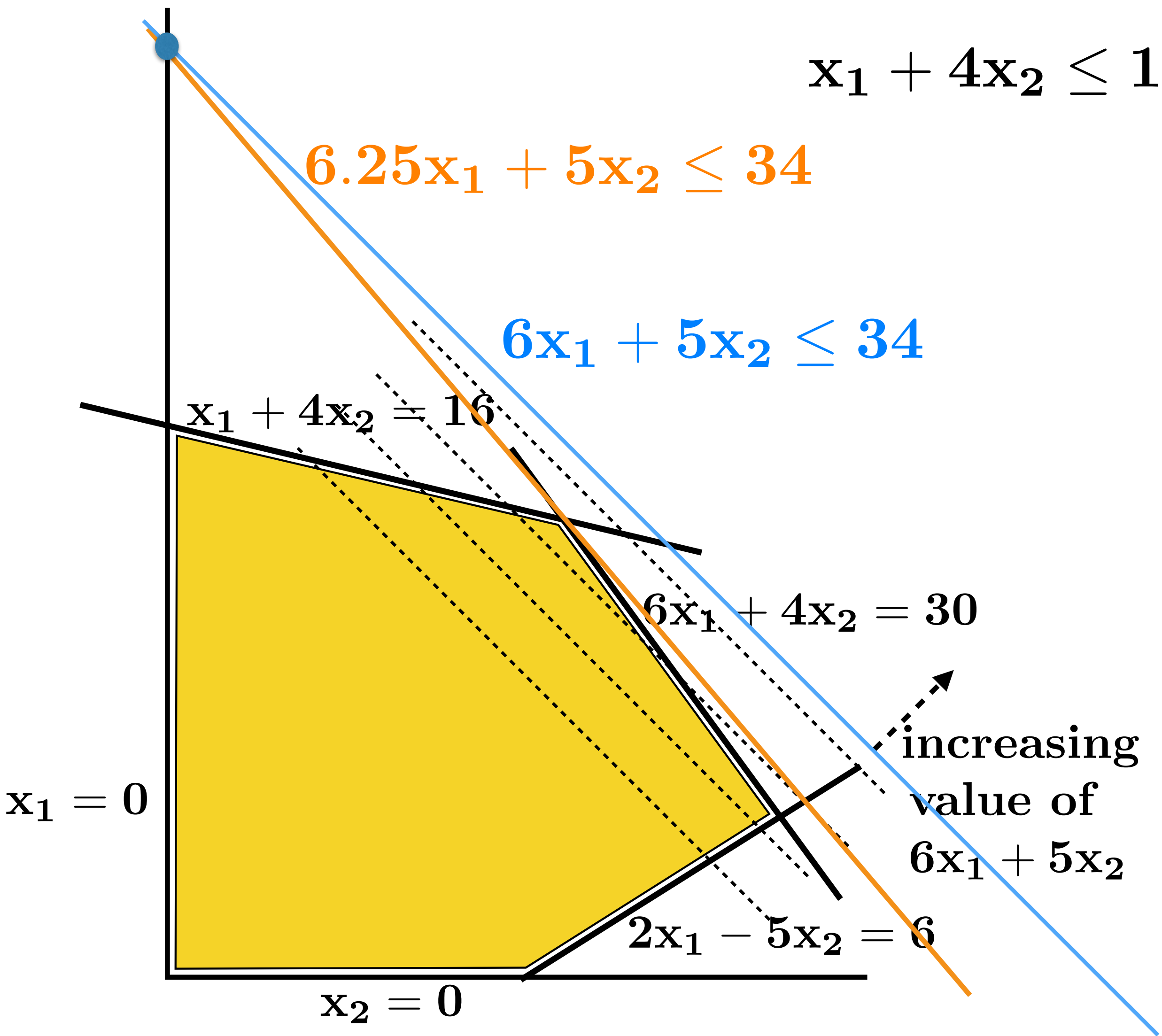




$$\mathbf{x_1 + 4x_2 \le 16 \Rightarrow .25x_1 + x_2 \le 4}$$

$$\frac{\mathbf{6x_1 + 4x_2 \le 30}}{\mathbf{6.25x_1 + 5x_2 \le 34}}$$

$$\mathbf{\Rightarrow 6x_1 + 5x_2 \le 34}$$



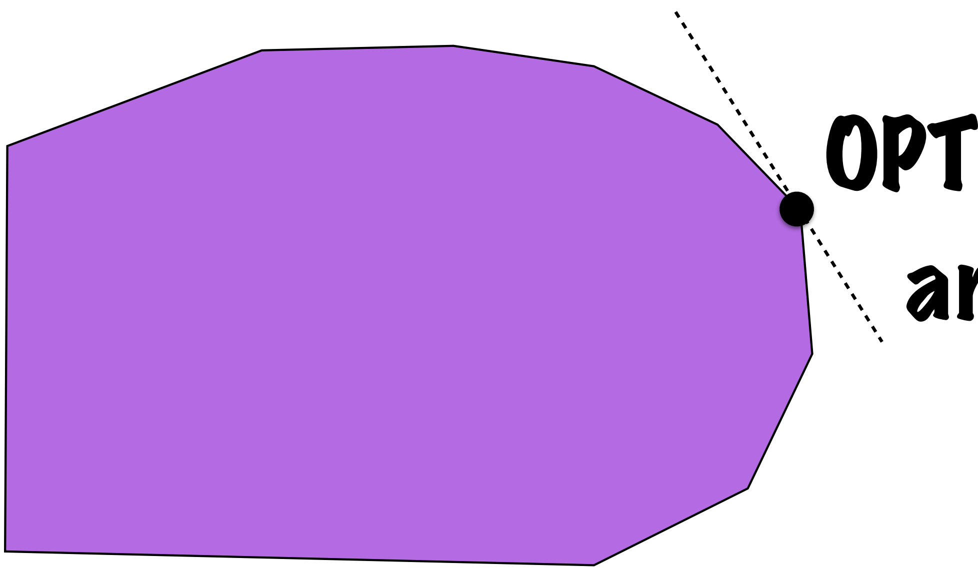
$$
\begin{array}{rl}
x_1 + 4x_2 \leq 16 \Rightarrow & .25x_1 + x_2 \leq 4 \\
& 6x_1 + 4x_2 \leq 30 \\
\hline
& 6.25x_1 + 5x_2 \leq 34 \\
\Rightarrow & 6x_1 + 5x_2 \leq 34
\end{array}
$$

Linear programming duality Theorem

(P) = (D)

$$\max \mathbf{c} \cdot \mathbf{x} : \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \quad \mathbf{x} \ge \mathbf{0}$$

$$\min \mathbf{b} \cdot \mathbf{y} : \quad \mathbf{A}^{\mathrm{T}}\mathbf{y} \ge \mathbf{c} \quad \mathbf{y} \ge \mathbf{0}$$



There exist constraints of (P) and a convex combination that imply exactly the right upper bound

Strong duality Theorem in general

(P)

$$\min \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{A}\mathbf{x} \geq \mathrm{b}$$
$$\mathbf{x} \geq 0$$

(D)

$$\max \mathbf{b} \cdot \mathbf{y} :$$
$$\mathbf{A}^{\mathbf{T}}\mathbf{y} \leq \mathbf{c}$$
$$\mathbf{y} \geq \mathbf{0}$$

Four possible cases:

- (P) is empty, (D) has value $+\infty$
- (D) is empty, (P) has value $-\infty$
- value(P)=value(D)
- [(P) and (D) empty]

Proof of the weak duality theorem

$$\max \mathbf{c}\cdot\mathbf{x} : \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \quad \mathbf{x} \ge \mathbf{0}$$

$$\le$$

$$\min \mathbf{b}\cdot\mathbf{y} : \quad \mathbf{A}^{\mathbf{T}}\mathbf{y} \ge \mathbf{c} \quad \mathbf{y} \ge \mathbf{0}$$

(P)

$$\begin{aligned}
&\max \mathbf{c_1x_1} + \mathbf{c_2x_2} + \cdots + \mathbf{c_nx_n} : \\
&\mathbf{a_{11}x_1} + \mathbf{a_{12}x_2} + \cdots + \mathbf{a_{1n}x_n} \le \mathbf{b_1} \\
&\mathbf{a_{21}x_1} + \mathbf{a_{22}x_2} + \cdots + \mathbf{a_{2n}x_n} \le \mathbf{b_2} \\
&\cdots \\
&\mathbf{a_{m1}x_1} + \mathbf{a_{m2}x_2} + \cdots + \mathbf{a_{mn}x_n} \le \mathbf{b_m} \\
&\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n} \ge \mathbf{0}
\end{aligned}$$

(D)

$$\begin{aligned}
&\min \mathbf{b_1y_1} + \mathbf{b_2y_2} + \cdots + \mathbf{b_my_m} : \\
&\mathbf{a_{11}y_1} + \mathbf{a_{21}y_2} + \cdots + \mathbf{a_{m1}y_m} \ge \mathbf{c_1} \\
&\mathbf{a_{12}y_1} + \mathbf{a_{22}y_2} + \cdots + \mathbf{a_{n2}y_m} \ge \mathbf{c_2} \\
&\cdots \\
&\mathbf{a_{1n}y_1} + \mathbf{a_{2n}y_2} + \cdots + \mathbf{a_{mn}y_m} \ge \mathbf{c_n} \\
&\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m} \ge \mathbf{0}
\end{aligned}$$

(P)

$$\max \mathbf{c_1x_1 + c_2x_2 + \cdots + c_nx_n} :$$
$$\mathbf{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1}$$
$$\mathbf{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2}$$
$$\cdots$$
$$\mathbf{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m}$$
$$\mathbf{x_1, x_2, \ldots, x_n \geq 0}$$

(D)

$$\min \mathbf{b_1y_1 + b_2y_2 + \cdots + b_my_m} :$$
$$\mathbf{a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1}$$
$$\mathbf{a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_m \geq c_2}$$
$$\cdots$$
$$\mathbf{a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n}$$
$$\mathbf{y_1, y_2, \ldots, y_m \geq 0}$$

Take x feasible for (P), y feasible for (D)
Must prove:

$$\mathbf{c_1x_2 + \cdots + c_nx_n \leq b_1y_1 + \cdots + b_my_m}$$

Must prove: $\mathbf{c_1x_2 + \cdots + c_nx_n \le b_1y_1 + \cdots + b_my_m}$

1. Use constraints of (D)

(D)

$$\begin{aligned}
&\min \mathbf{b_1y_1 + b_2y_2 + \cdots + b_my_m} : \\
&\mathbf{a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1} \\
&\mathbf{a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_m \ge c_2} \\
&\cdots \\
&\mathbf{a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n} \\
&\mathbf{y_1, y_2, \ldots, y_m \ge 0}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{c_1x_1 + \cdots + c_nx_n \le} \\
&\mathbf{(a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m)x_1 + \cdots +} \\
&\mathbf{(a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m)x_n}
\end{aligned}$$

2. Invert summations

$$
\begin{aligned}
&(\mathbf{a_{11}y_1} + \mathbf{a_{21}y_2} + \cdots + \mathbf{a_{m1}y_m})\mathbf{x_1} + \cdots + \\
&(\mathbf{a_{1n}y_1} + \mathbf{a_{2n}y_2} + \cdots + \mathbf{a_{mn}y_m})\mathbf{x_n} = \\
&(\mathbf{a_{11}x_1} + \cdots + \mathbf{a_{1n}x_n})\mathbf{y_1} + \cdots + \\
&(\mathbf{a_{m1}x_1} + \cdots + \mathbf{a_{mn}x_n})\mathbf{y_m}
\end{aligned}
$$

3. Use constraints of (P)

(P)

$$\begin{aligned}
&\max \mathbf{c_1x_1 + c_2x_2 + \cdots + c_nx_n} : \\
&\mathbf{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1} \\
&\mathbf{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2} \\
&\cdots \\
&\mathbf{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m} \\
&\mathbf{x_1, x_2, \ldots, x_n \geq 0}
\end{aligned}$$

$$\mathbf{(a_{11}x_1 + \cdots + a_{1n}x_n)y_1 + \cdots + (a_{m1}x_1 + \cdots + a_{mn}x_n)y_m \leq b_1y_1 + \cdots + b_my_m}$$

In summary

$$\max \mathbf{c_1x_1 + c_2x_2 + \cdots + c_nx_n} :$$
$$\mathbf{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1}$$
$$\mathbf{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2}$$
$$\cdots$$
$$\mathbf{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m}$$
$$\mathbf{x_1, x_2, \ldots, x_n \geq 0}$$

$$\min \mathbf{b_1y_1 + b_2y_2 + \cdots + b_my_m} :$$
$$\mathbf{a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1}$$
$$\mathbf{a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_m \geq c_2}$$
$$\cdots$$
$$\mathbf{a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n}$$
$$\mathbf{y_1, y_2, \ldots, y_m \geq 0}$$

Given x feasible for (P), y feasible for (D):

$$\mathbf{c_1x_1 + \cdots + c_nx_n \leq b_1y_1 + \cdots + b_my_m}$$

So:

$$\max\{\mathbf{c_1x_1 + \cdots + c_nx_n : x \in (P)}\} \leq \min\{\mathbf{b_1y_1 + \cdots + b_my_m : y \in (D)}\}$$

QED

Putting a linear program in form appropriate for taking dual

$$\max \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

Proof by example

$$\min \mathbf{2x_1 - 3x_2 + x_3} :$$
$$\mathbf{x_1 + x_2 = 4}$$
$$\mathbf{x_2 - 4x_3 \geq 5}$$
$$\mathbf{x_2 \geq 0}$$

1. Transform min into max

$$\max \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

$$\min \mathbf{2x_1} - \mathbf{3x_2} + \mathbf{x_3} :$$
$$\mathbf{x_1} + \mathbf{x_2} = \mathbf{4}$$
$$\mathbf{x_2} - \mathbf{4x_3} \geq \mathbf{5}$$
$$\mathbf{x_2} \geq \mathbf{0}$$

$$\Downarrow$$

$$\max -\mathbf{2x_1} + \mathbf{3x_2} - \mathbf{x_3} :$$
$$\mathbf{x_1} + \mathbf{x_2} = \mathbf{4}$$
$$\mathbf{x_2} - \mathbf{4x_3} \geq \mathbf{5}$$
$$\mathbf{x_2} \geq \mathbf{0}$$

2. Transform equalities into inequalities

$$\max \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{Ax} \le \mathbf{b}$$
$$\mathbf{x} \ge \mathbf{0}$$

$$\max -\mathbf{2x_1} + \mathbf{3x_2} - \mathbf{x_3} :$$
$$\mathbf{x_1} + \mathbf{x_2} = \mathbf{4}$$
$$\mathbf{x_2} - \mathbf{4x_3} \ge \mathbf{5}$$
$$\mathbf{x_2} \ge \mathbf{0}$$

$$\Downarrow$$

$$\max -\mathbf{2x_1} + \mathbf{3x_2} - \mathbf{x_3} :$$
$$\mathbf{x_1} + \mathbf{x_2} \ge \mathbf{4}$$
$$\mathbf{x_1} + \mathbf{x_2} \le \mathbf{4}$$
$$\mathbf{x_2} - \mathbf{4x_3} \ge \mathbf{5}$$
$$\mathbf{x_2} \ge \mathbf{0}$$

3. Make inequalities in the correct direction

$$\max \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{Ax} \leq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

$$\max -\mathbf{2x_1} + \mathbf{3x_2} - \mathbf{x_3} :$$
$$\mathbf{x_1} + \mathbf{x_2} \geq \mathbf{4}$$
$$\mathbf{x_1} + \mathbf{x_2} \leq \mathbf{4}$$
$$\mathbf{x_2} - \mathbf{4x_3} \geq \mathbf{5}$$
$$\mathbf{x_2} \geq \mathbf{0}$$

$$\downarrow$$

$$\max -\mathbf{2x_1} + \mathbf{3x_2} - \mathbf{x_3} :$$
$$-\mathbf{x_1} - \mathbf{x_2} \leq -\mathbf{4}$$
$$\mathbf{x_1} + \mathbf{x_2} \leq \mathbf{4}$$
$$-\mathbf{x_2} + \mathbf{4x_3} \leq -\mathbf{5}$$
$$\mathbf{x_2} \geq \mathbf{0}$$

4. Reduce to non-negative variables

$$\begin{aligned}&\max \mathbf{c}\cdot\mathbf{x}:\\ &\mathbf{A}\mathbf{x}\leq \mathbf{b}\\ &\mathbf{x}\geq \mathbf{0}\end{aligned}$$

$$\begin{aligned}&\max -\mathbf{2}\mathbf{x_1}+\mathbf{3}\mathbf{x_2}-\mathbf{x_3}:\\ &-\mathbf{x_1}-\mathbf{x_2}\leq -\mathbf{4}\\ &\mathbf{x_1}+\mathbf{x_2}\leq \mathbf{4}\\ &-\mathbf{x_2}+\mathbf{4}\mathbf{x_3}\leq -\mathbf{5}\\ &\mathbf{x_2}\geq \mathbf{0}\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}&\max -\mathbf{2}(\mathbf{x_1^+}-\mathbf{x_1^-})+\mathbf{3}\mathbf{x_2}-(\mathbf{x_3^+}-\mathbf{x_3^-}):\\ &-(\mathbf{x_1^+}-\mathbf{x_1^-})-\mathbf{x_2}\leq -\mathbf{4}\\ &(\mathbf{x_1^+}-\mathbf{x_1^-})+\mathbf{x_2}\leq \mathbf{4}\\ &-\mathbf{x_2}+\mathbf{4}(\mathbf{x_3^+}-\mathbf{x_3^-})\leq -\mathbf{5}\\ &\mathbf{x_2},\mathbf{x_1^+},\mathbf{x_1^-},\mathbf{x_3^+},\mathbf{x_3^-}\geq \mathbf{0}\end{aligned}$$

Done!

Complementary slackness conditions

Weak duality proof in one line

$$\sum_i c_i x_i \le \sum_{i=1}^{n} (A^T y)_i x_i = \sum_{j=1}^{m} (Ax)_j y_j \le \sum_j b_j y_j$$

for optimal solutions: x opt for (P), y opt for (D):

$$\sum_i c_i x_i = \sum_j b_j y_j$$

So in above proof, all inequalities are equalities

$$\forall i : c_i x_i = (A^T y)_i x_i$$
$$\forall j : b_j y_j = (Ax)_j y_j$$

$$c_i x_i = (\sum_j a_{ij} y_j) x_i$$

$$c_i \leq (\sum_j a_{ij} y_j)$$ (constraint of (D))

so: either $c_i = \sum_j a_{ij} y_j$

or $x_i = 0$

$$b_j y_j = (\sum_i a_{ij} xi) y_j$$

$$b_j \leq (\sum_i a_{ij} xi)$$ (constraint of (P))

so: either $b_j = \sum_i a_{ij} xi$

or $y_j = 0$

Complementary slackness conditions

If x is optimal for (P) and y optimal for (D)

then for every i:

$$c_i = \sum_j a_{ij} y_j \text{ or } x_i = 0$$

and for every j:

$$b_j = \sum_i a_{ij} xi \text{ or } y_j = 0$$

LP duality in the design of approximation algorithms

Review: Approximation algorithm by LP-rounding

1. Find a LP relaxation for the problem
2. Find the optimal (fractional) solution x
3. “Round” x to output an integer solution x’

Review: Analysis (maximization problem)

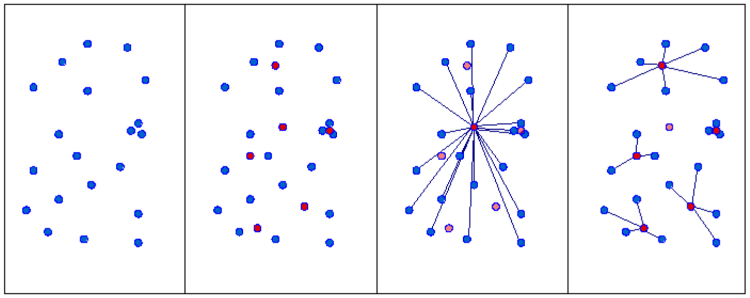
1. LP Relaxation: value(x) > OPT
2. Rounding is s.t. value(x’) > value(x)/c
3. Together: value(output) > OPT/c

Approximation algorithm primal-dual

1. Find a LP relaxation (P) for the problem
2. Let (D) be the dual LP.
3. "Construct" integer solutions x for (P), y for (D)

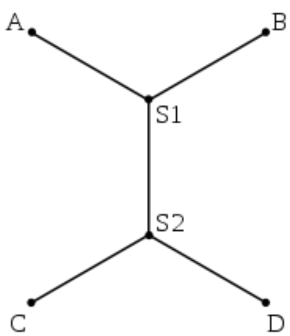
Analysis (maximization problem)

1. Solutions are constructed s.t. value(x) >value(y)/c
2. Weak duality: value(y)>OPT
3. Together: value(output x) >OPT/c.

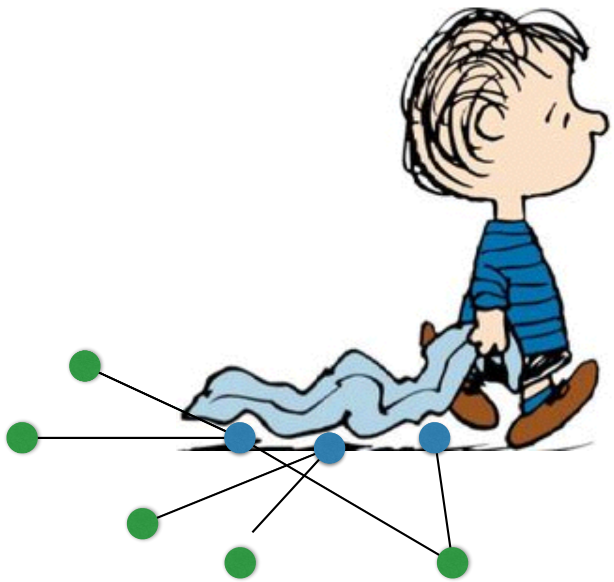


Primal-dual approach for

- Steiner forest
- Facility location



Primal-dual algorithm for Vertex Cover



$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u^*}$$
$$\mathbf{x_u^* + x_v^* \geq 1}$$
$$\mathbf{0 \leq x_u^* \leq 1}$$

Primal $\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$

$$\mathbf{x_u} + \mathbf{x_v} \geq \mathbf{1} \quad \forall \mathbf{uv} \in \mathbf{E}$$

$$\mathbf{x_u} \geq \mathbf{0} \quad \forall \mathbf{u} \in \mathbf{V}$$

$$\begin{array}{cc}
 & \begin{array}{ccccc} (\mathbf{y}) & (\mathbf{u}) & (\mathbf{v}) & (\mathbf{x}) & \end{array} \\
\begin{array}{c} (\mathbf{e} = \mathbf{uv}) \\ \\ (\mathbf{e}' = \mathbf{ux}) \\ \\ (\mathbf{e}'' = \mathbf{uy}) \end{array} &
\begin{pmatrix}
\cdots & 0\;1\;0 & \cdots 0\;1\;0 & \cdots & \cdots 0 \\
\cdots & 0\;0\;0 & \cdots & \cdots & \cdots 0 \\
\cdots & 0\;1\;0 & \cdots & 0\;1\;0 & \cdots 0 \\
\cdots & 0\;0\;0 & \cdots & \cdots & \cdots 0 \\
\cdots 0\;1\;0 \cdots & 0\;1\;0 & \cdots & \cdots & \cdots 0
\end{pmatrix}
\end{array}$$

Dual $\max \sum_{\mathbf{e}} \mathbf{y_e}$

$$\sum_{\mathbf{e:u \in e}} \mathbf{y_e} \leq \mathbf{w_u} \quad \forall \mathbf{u} \in \mathbf{V}$$

$$\mathbf{y_e} \geq \mathbf{0} \quad \forall \mathbf{e} \in \mathbf{E}$$

“Construct” integer solutions x for (P), y for (D)

$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$$
$$\mathbf{x_u} + \mathbf{x_v} \geq \mathbf{1} \quad \forall \mathbf{uv} \in \mathbf{E}$$
$$\mathbf{x_u} \geq \mathbf{0} \quad \forall \mathbf{u} \in \mathbf{V}$$

$$\max \sum_{\mathbf{e}} \mathbf{y_e}$$
$$\sum_{\mathbf{e:u \in e}} \mathbf{y_e} \leq \mathbf{w_u} \quad \forall \mathbf{u} \in \mathbf{V}$$
$$\mathbf{y_e} \geq \mathbf{0} \quad \forall \mathbf{e} \in \mathbf{E}$$

Start with: $\mathbf{x} = (\mathbf{0}, \cdots, \mathbf{0}), \mathbf{y} = (\mathbf{0}, \cdots, \mathbf{0})$

$\mathbf{x}$ has low value but is not feasible
$\mathbf{y}$ is feasible but has low value

“Construct” integer solutions x for (P), y for (D)

$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$$
$$\mathbf{x_u} + \mathbf{x_v} \geq \mathbf{1} \quad \forall \mathbf{uv} \in \mathbf{E}$$
$$\mathbf{x_u} \geq \mathbf{0} \quad \forall \mathbf{u} \in \mathbf{V}$$

$$\max \sum_{\mathbf{e}} \mathbf{y_e}$$
$$\sum_{\mathbf{e:u \in e}} \mathbf{y_e} \leq \mathbf{w_u} \quad \forall \mathbf{u} \in \mathbf{V}$$
$$\mathbf{y_e} \geq \mathbf{0} \quad \forall \mathbf{e} \in \mathbf{E}$$

Start with: $\mathbf{x} = (\mathbf{0}, \cdots, \mathbf{0}), \mathbf{y} = (\mathbf{0}, \cdots, \mathbf{0})$

Repeat:

pick $\mathrm{e} = \mathrm{uv}$ **such that** $\mathrm{x_u} + \mathrm{x_v} < 1$

increase $\mathrm{y_e}$ **until**

$$\sum_{\mathbf{f:u \in f}} \mathrm{y_f} = \mathrm{w_u} \text{ or } \sum_{\mathbf{f:v \in f}} \mathrm{y_f} = \mathrm{w_v}$$

first case: $\mathrm{x_u} \leftarrow 1$ **second case:** $\mathrm{x_v} \leftarrow 1$

$\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$

$\mathbf{x_u + x_v \geq 1} \quad \forall \mathbf{uv \in E}$

$\mathbf{x_u \geq 0} \quad \forall \mathbf{u \in V}$

$\max \sum_{\mathbf{e}} \mathbf{y_e}$

$\sum_{\mathbf{e:u \in e}} \mathbf{y_e \leq w_u} \quad \forall \mathbf{u \in V}$

$\mathbf{y_e \geq 0} \quad \forall \mathbf{e \in E}$

Repeat:

pick $\mathbf{e = uv}$ such that $\mathbf{x_u + x_v < 1}$

increase $\mathbf{y_e}$ until

$\sum_{\mathbf{f:u \in f}} \mathbf{y_f = w_u}$ or $\sum_{\mathbf{f:v \in f}} \mathbf{y_f = w_v}$

first case: $\mathbf{x_u \leftarrow 1}$ second case: $\mathbf{x_v \leftarrow 1}$

Invariants: $\mathbf{y}$ remains feasible throughout

$\mathbf{x}$ has fewer and fewer violated constraints

In the end: both are feasible

Repeat:

pick $e = uv$ **such that** $x_u + x_v < 1$

increase y_e **until**

$\sum_{f:u\in f} y_f = w_u$ **or** $\sum_{f:v\in f} y_f = w_v$

first case: $x_u \leftarrow 1$ **second case:** $x_v \leftarrow 1$

Invariant: **for every** $u \in V$

$\sum_{f:u\in f} y_f = w_u \text{ or } x_u = 0$

and for every $e = uv \in E$

$y_e = 0 \text{ or } x_u + x_v = 1 \text{ or } x_u + x_v = 2$

$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$$
$$\mathbf{x_u + x_v \geq 1} \quad \forall \mathbf{uv \in E}$$
$$\mathbf{x_u \geq 0} \quad \forall \mathbf{u \in V}$$

$$\max \sum_{\mathbf{e}} \mathbf{y_e}$$
$$\sum_{\mathbf{e:u\in e}} \mathbf{y_e \leq w_u} \quad \forall \mathbf{u \in V}$$
$$\mathbf{y_e \geq 0} \quad \forall \mathbf{e \in E}$$

$$\begin{aligned}
\sum_{\mathbf{u}} \mathbf{w_u x_u} &= \sum_{\mathbf{u:x_u \neq 0}} \mathbf{w_u x_u} \\
&= \sum_{\mathbf{u:x_u\neq 0}} \sum_{\mathbf{v:uv\in E}} \mathbf{y_{uv} x_u} \\
&= \sum_{\mathbf{u}} \sum_{\mathbf{e=uv\in E}} \mathbf{y_e x_u} \\
&= \sum_{\mathbf{e}} \Big(\sum_{\mathbf{u\in e}} \mathbf{x_u}\Big) \mathbf{y_e} \\
&\leq \sum_{\mathbf{e}} \mathbf{2y_e} \\
&= \mathbf{2} \sum_{\mathbf{e}} \mathbf{y_e} \leq \mathbf{2 \cdot OPT}
\end{aligned}$$

Primal-dual algorithm for vertex cover

Repeat:

pick $e = uv$ such that $x_u + x_v < 1$

increase y_e until

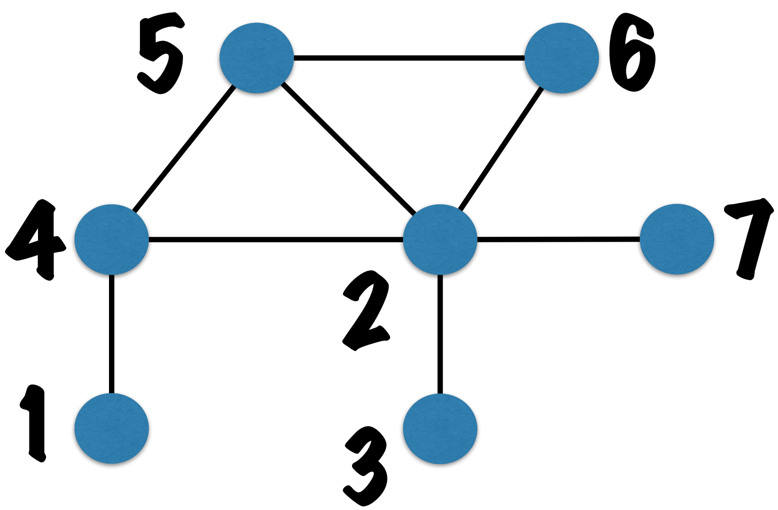
$$\sum_{f:u \in f} y_f = w_u \text{ or } \sum_{f:v \in f} y_f = w_v$$

first case: $x_u \leftarrow 1$ second case: $x_v \leftarrow 1$

Theorem: the primal-dual algorithm for vertex cover is a 2-approximation

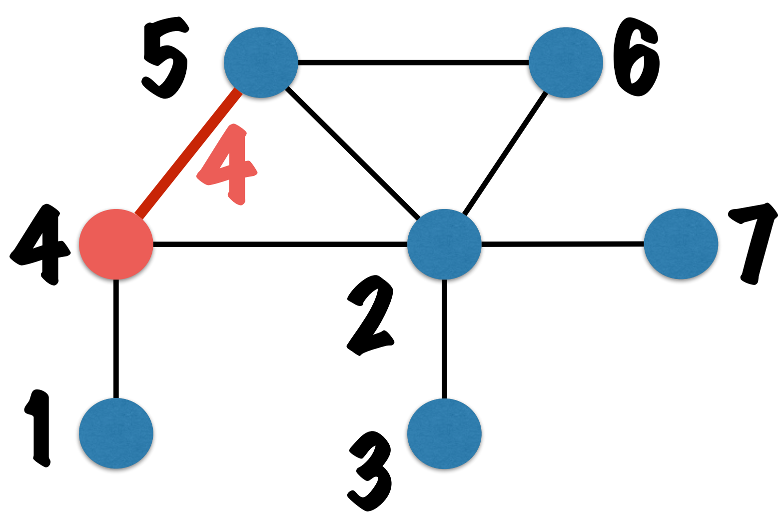
Example

vertex weights



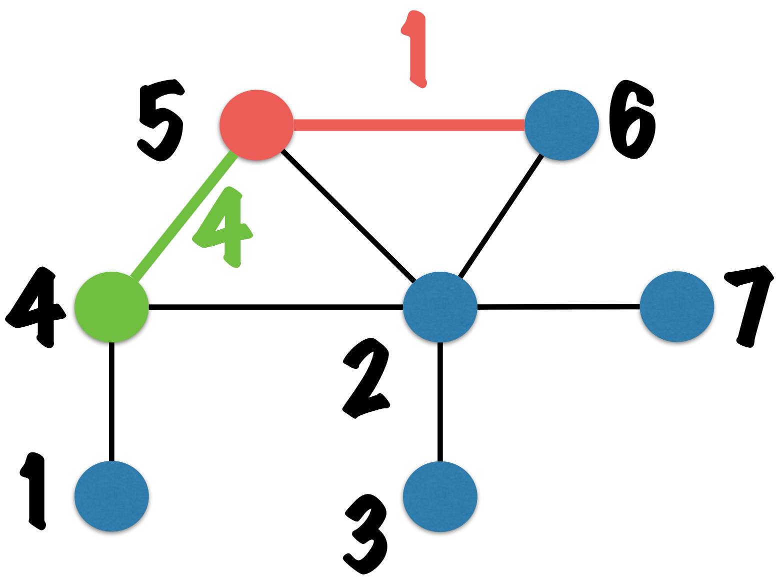
Example

vertex weights



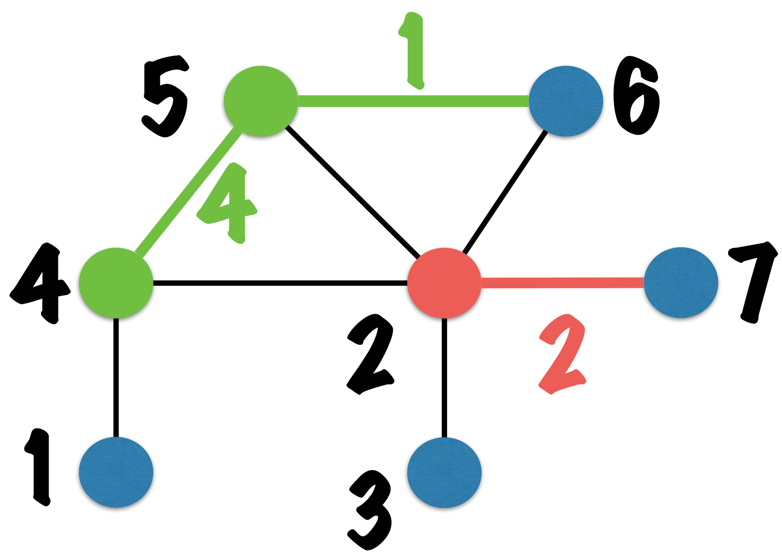
Example

vertex
weights



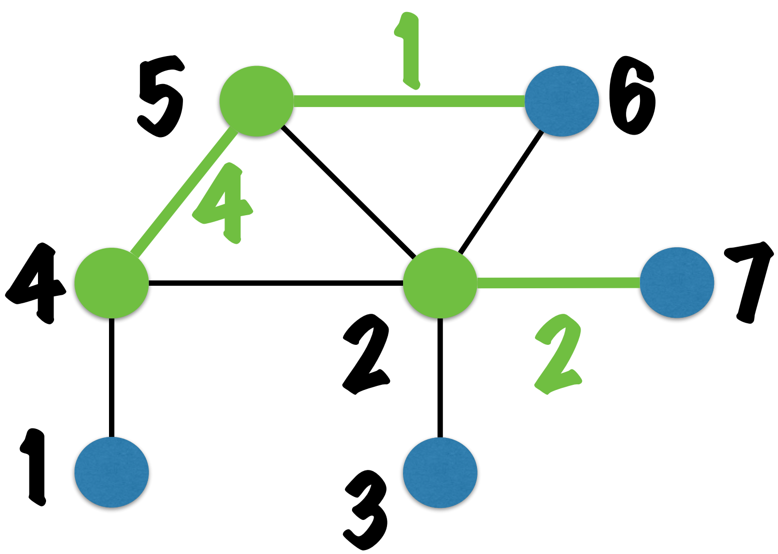
Example

vertex weights



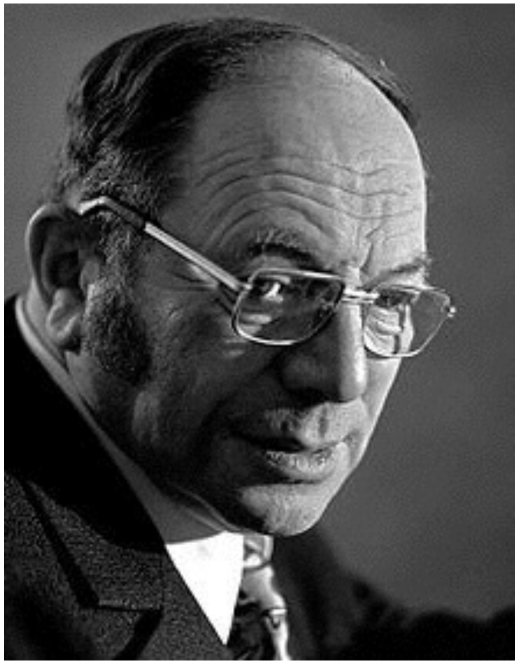
Example

vertex weights

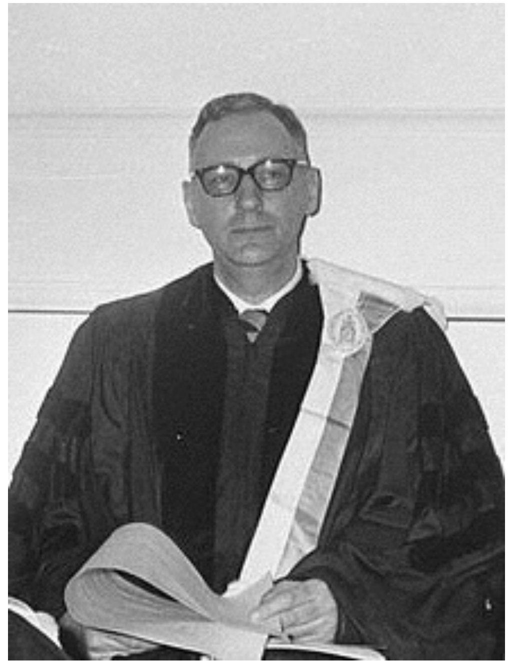


Output {2,4,5}
OPT={1,2,5}

History of LP duality



Kantorovich



Koopsmans

Hitchcock

Economic problems

World War II logistics

Transportation problems

Dantzig
Planning for US air force

von Neumann
Game theory

A Theorem on Linear Inequalities (1948)

For approximation algorithms




Agrawal,
Klein,
Ravi

Goemans,
Williamson