A Deterministic Approach to Wireless Information Flow in Gaussian Networks

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***Abstract*—We survey papers of the topic of wireless information flow and mainly discuss the deterministic approach proposed by Avestimehr et al. [1]. This deterministic model captures the features of a Gaussian network. We illustrate its usage by some examples and obtain exact capacity of deterministic networks, which is a natural generalization of the max-flow min-cut theorem for wired networks. The insights from the deterministic analysis introduces a new quantize-map-and-forward scheme for Gaussian networks. This scheme can achieve the cut-set upper bound to within a gap which is independent of the channel parameters. The deterministic model is a good method to help infer and analyze the capacity region of Gaussian networks.**

***Index Terms*—Network information theory, relay networks, Gaussian networks.**

I. Introduction

As wireless technology evolves and prevails, how information flows in the wireless networks has become an important topic. In wireless networks, signals from different transmitters may interact with each other at the receiver. On one hand, this facilitates the spread of information among users in a network while on the other hand, it can do harm by potentially creating signal interference among users. This is in striking contrast to wired networks where transmitter-receiver pairs can be regarded as isolated point-to-point links.

In spite of much progress in wired networks, we still know less about wireless networks. We would like to ask in wireless networks:

- How to model the interaction of signals?
- How much information can we transmit?
- How to achieve the capacity?

Over the past decade, some researchers examined scaling laws for multiple independent flows over wireless networks, for example Xie et al. [6]. The goal is to characterize the order of the wireless network capacity as the network size grows. Nevertheless, we still want to characterize the capacity region of a wireless network.

The linear additive Gaussian channel model is a commonly used model to capture signal interactions in wireless channels. However, because of the complexity of the Gaussian model, the capacity of most Gaussian networks is still unknown. In the class, we learn the capacity of point-to-point channel, multiple access channel (MAC) and broadcast channel (BC). However, even the simple two-user unicast interference channel and single-relay network as shown in Figure 1 and 2, we have not yet characterized the capacity region.

Now that deriving the exact capacity region of a Gaussian network is hard, we try to solve it in an approximate manner.

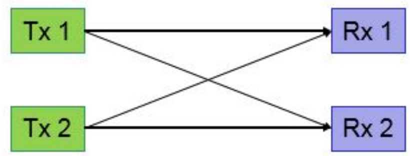

Figure 1. Two-user unicast interference channel. Best known achievable region: Han & Kobayashi 81

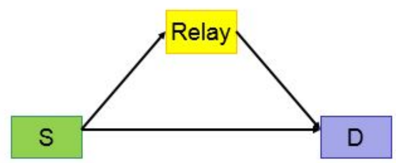

Figure 2. Single-relay network. Best known achievable region: El Gamal & Cover 79

In this paper survey, we mainly study the deterministic method proposed by A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse in [1]. Unlike most literature that investigated the effect of noise, their work focus more on exploiting information from interference since, with coding scheme, we can greatly reduce noise effect .

The report is organized as follows. Section II shows how to construct the so-called "deterministic model" based on the capacity of point-to-point Gaussian channel and illustrate its application to the known MAC and BC Gaussian channels. In section III, we begin to use the cut-set upper bound in T. M. Cover and J. A. Thomas [5] as a baseline to evaluate how good the achievability is. Then we analyze a single-relay deterministic network. We state the main results of the paper [1] in section IV, and explain the main ideas of the proofs. In section V, we survey some works that have adopted the deterministic models to investigate special Gaussian networks. We present a method to efficiently find the maximum flow in the deterministic network. Section VI summarizes the insight and results and how this deterministic model can be helpful as we analyze a Gaussian network.

II. Deterministic Model

Avestimehr et al. [1] build the deterministic model from a point-to-point real scalar Gaussian model, which is described as $y = hx + z$ where $z \sim N(0, 1)$. Setting the transmitter power constraint to 1, and normalizing the transmit and noise powers to be equal to 1, the model can be represented as $y = \sqrt{\mathsf{SNR}}x + z$ and its capacity is well-known as $C_{\mathrm{AWGN}} =$

$\frac{1}{2}\log(1+\mathsf{SNR})$. The next step is to limit the peak power of both transmit and noise signals to 1; thus, the received signal y can be written in binary expansions

$$2^{\frac{1}{2}\log \mathsf{SNR}}\sum_{i=1}^{\infty}x(i)2^{-i}+\sum_{i=-\infty}^{\infty}z(i)2^{-i} \tag{1}$$

which can be simplified as

$$y\approx 2^{n}\sum_{i=1}^{n}x(i)2^{-i}+\sum_{i=1}^{\infty}(x(i+n)+z(i))2^{-i} \tag{2}$$

where $n=\lceil\frac{1}{2}\log\mathsf{SNR}\rceil^{+}$. Ignoring the 1-bit carry-over from the second summation to the first, we can approximate this channel as a pipe, which truncates the transmitted signal, only passing the more significant bits. If we think the transmitted signal as a sequence of bits on different signal level, it is like that the receiver sees the n most significant bits of x without any noise, while the lower bits are corrupted by the noise. This corresponds to the deterministic model shown in Figure 3. Each circle represents a signal level holding a bit. The capacity of such model is $n=\lceil\frac{1}{2}\log\mathsf{SNR}\rceil^{+}$, within 1-bit approximation of the AWGN channel capacity.

Remark. In [1], authors state that the difference is within $\frac{1}{2}$ bit. However, after calculations we think it can actually be as large as 1 bit.

Avestimehr et al. mention that these signals can be created using multilevel lattice codes, which coincides with another group's study.

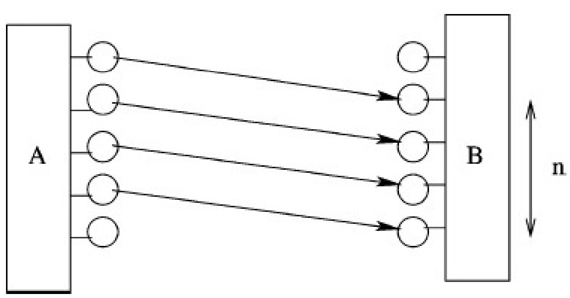

Figure 3. Pictorial representation of the deterministic model for point-to-point channel.

Based on the intuition obtained from above, the model can be directly put on use to MAC scenario. The first task here is to model the superposition effect that does not appear above. In this deterministic model, the signal is operated in the binary field, thus when the receiver receives two bits from different channel on the same signal level, it hears the modulo 2 sum of them. Carry-overs are ignored here, and instead of the wide-used collision model, this setting lets the receiver exploit some information from the superposition. In the Gaussian channel the received signal of a 2-user MAC is $y=h_1x_1+h_2x_2+z$. Assume that $\mathsf{SNR}_2\le\mathsf{SNR}_1$, then analogous to above, using binary expansions we can obtain

$$\begin{aligned}y\approx&2^{n_1}\sum_{i=1}^{n_1-n_2}x_1(i)2^{-i}+2^{n_2}\sum_{i=1}^{n_2}(x_1(i+n_1-n_2)+x_2(i))2^{-i}\\&+\sum_{i=1}^{\infty}(x_1(i+n_1)+x_2(i+n_2)+z(i))2^{-i}\end{aligned} \tag{3}$$

We can model the signal as follows:

- The stronger part of x_1 that is above SNR_2 is received without any contribution from x_2 and z.
- The remaining part of x_1 that is above noise level, and that of x_2 are superposed on each other and received without noise.
- The rest signals are superposed on noise, corrupted and dropped.

The capacity region of this deterministic scenario is the set of nonnegative pairs (R_1,R_2) satisfying

$$\begin{aligned}R_2&\le n_2\\R_1+R_2&\le n_1\end{aligned}$$

while the corresponding capacity region of the Gaussian MAC with average power set to 1 at both transmitters is the set

$$\begin{aligned}R_i&\le\log(1+\mathsf{SNR}_i),\quad i=1,2\\R_1+R_2&\le\log(1+\mathsf{SNR}_1+\mathsf{SNR}_2)\end{aligned}$$

In Figure 4, (a) part demonstrates the MAC deterministic model. In the (b) part the capacity region of the deterministic model is plotted with solid line, while that of the Gaussian MAC is plotted with dashed line. We can notice that even if we ignore all the carry-overs of the addition, the capacity region of both cases are still very close to each other. In fact they are within one bit of each other, and an intuitive explanation is that, in real addition once two bounded signals are added the magnitude can only become as large as twice the larger.

As of the broadcast scenario, the steps of modeling signal strength are analogous. Again assuming $\mathsf{SNR}_2\le\mathsf{SNR}_1$, since MAC and BC are dual networks in deterministic scenario, the capacity region of the deterministic simplified BC model coincides with that of the deterministic MAC. It can be observed in Figure 4 and 5 The encoding and decoding utilizes superposition coding and successive interference cancellation. Again we examine the capacity regions of the deterministic and Gaussian BC and their capacity region are still within one bit of each other. The corresponding figure is shown below. In fact, in both MAC and BC the gap is usually much smaller than one bit given the fact that $\mathsf{SNR}_1\ne\mathsf{SNR}_2$ normally.

Hence, from above we can set our linear finite-field deterministic model as this:

- The relay network is defined using its vertice set $\mathcal{V}$, and the link capacity n_{ij} corresponding to the link from node i to node j.
- The link capacity corresponds to the SNR in the corresponding Gaussian channel.

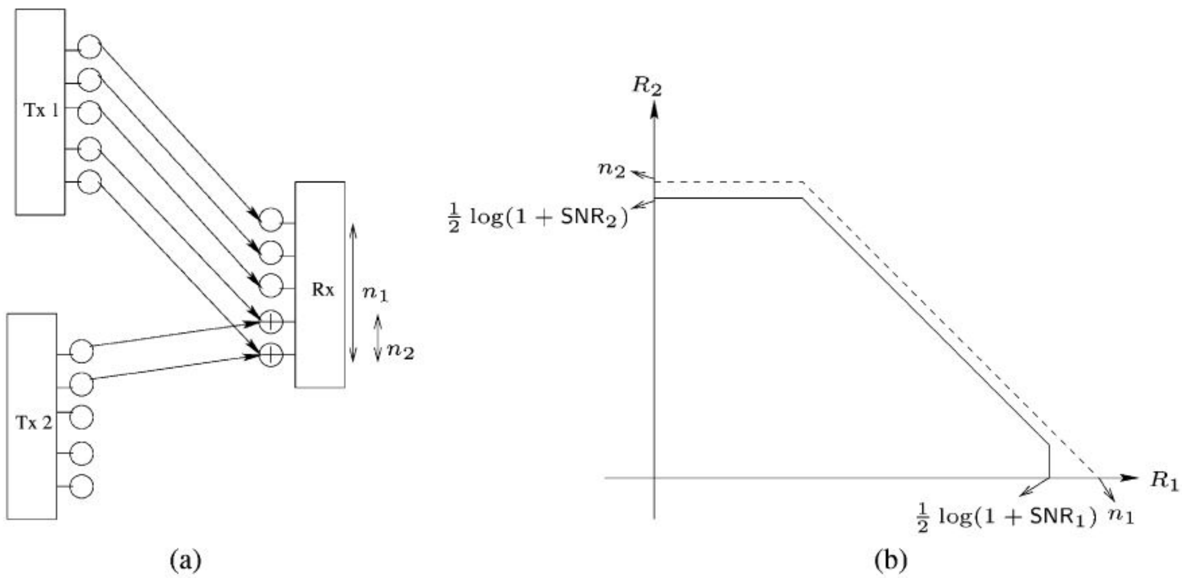

Figure 4. (a) Pictorial representation of the deterministic MAC. (b) Capacity region of Gaussian and deterministic MACs.

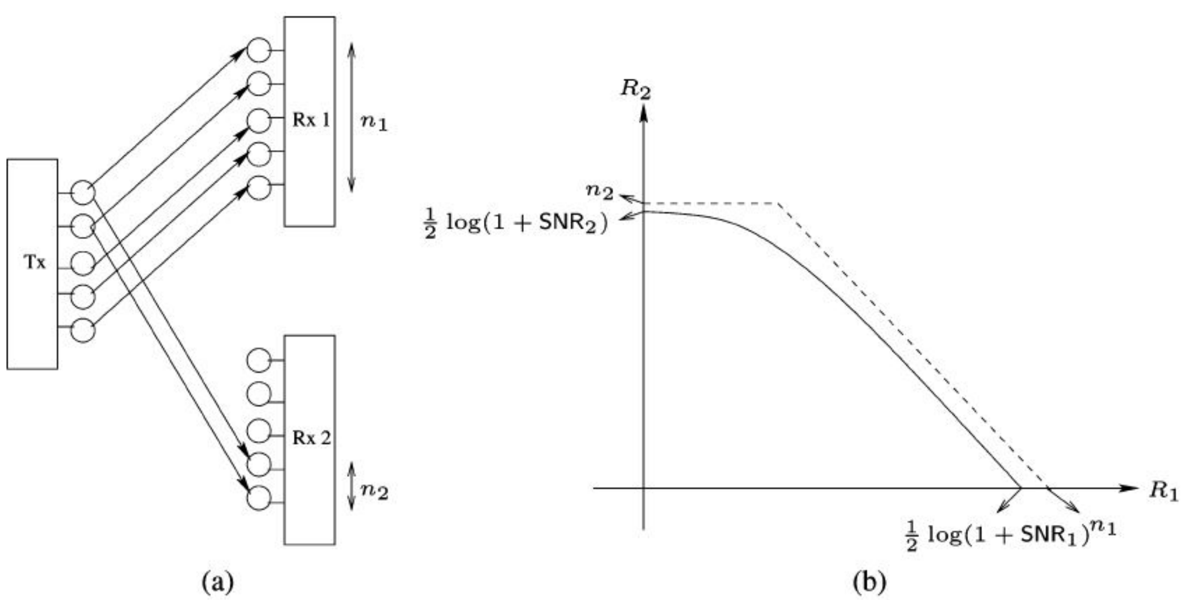

Figure 5. (a) Pictorial representation of the deterministic BC. (b) Capacity region of Gaussian and deterministic BCs.

- At each time t, node i transmits a vector $\mathbf{x}_i[t] \in \mathbb{F}_p^q$ and receives a vector $\mathbf{y}_i[t] \in \mathbb{F}_p^q$ where $q = \max_{i,j}(n(ij))$ and p is the field size.
- The received signal at each node is a deterministic function of the transmitted signals at the other nodes, with each signal truncated and superposed on each other.

III. Deterministic Relay Networks

In the previous section the deterministic model is introduced and used on MAC and BC networks, which we happen to know the capacity region. The next step is to put the model into use on the networks whose capacity region is still unknown. For those networks, we will need a standard to measure our approximation, which is the cut-set upper bound in [5].

$$\bar{C} = \max_{p(\{\mathbf{x}_j\}_{j\in V})} \min_{\Omega\in\Lambda_D} I(\mathbf{y}_{\Omega^c};\mathbf{x}_\Omega|\mathbf{x}_{\Omega^c}) \tag{4}$$

Equation (4) dictates the cut-set upper bound for single-unicast synchronized and causal relay network. Though the relay nodes are allowed to do causal processing, i.e. coding, we do not take coding into use in our model. An intuition for this bound is that given that a cut separates the network into two sides, the information sent by the relay nodes in the destination side, Ω^c, will be superposed on the information flowing from the source side. All the relay nodes will need to

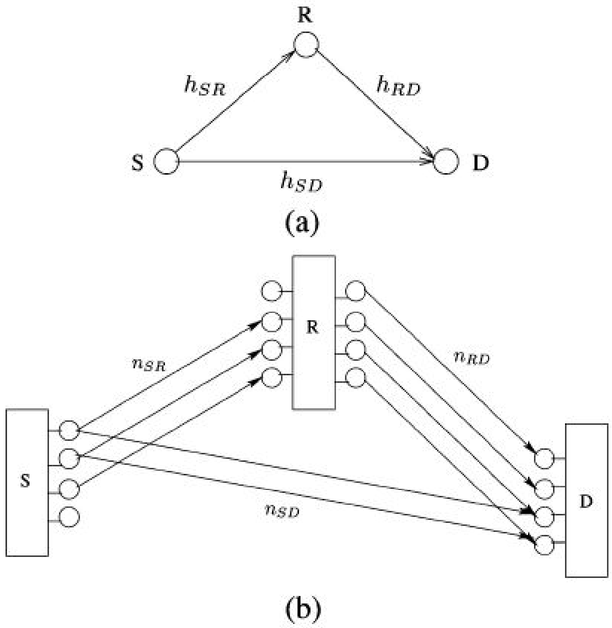

Figure 6. Relay channel: (a) Gaussian model; (b) linear finite-field deterministic model.

be able to distinguish them, and thus the mutual information term is conditioned on $\mathbf{x}_{\Omega^c}$. This cut-set bound is used by the authors to assess how good the approximation is.

The first network topology we look here is the Gaussian relay network. It contains only one relay node R, and using our deterministic model from the previous section we can model the channel strength by n_{SR}, n_{RD} and n_{SD}, respectly. The corresponding network model is shown in the figure below. The capacity of such scenario is simple to compute. It is smaller than both the maximum number of bits that source can broadcast and the maximum number of bits that the destination can achieve. Hence the capacity upper bound is given as

$$\begin{aligned} C^d_{relay} &\leq \min\{\max(n_{SR}, n_{SD}), \max(n_{RD}, n_{SD})\} \\ &= \begin{cases} n_{SD} & \text{, if } n_{SD} > \min(n_{SR}, n_{RD}) \\ \min(n_{SR}, n_{RD}) & \text{, otherwise.} \end{cases} \end{aligned} \tag{5}$$

The achivability is directed from the intuition. If the direct link is better than any of the other link, the relay will be silent. Otherwise it helps the source by decoding the message and sending innovations. The capacity will be increased if the relay can forward more bits onto a non-overlapping signal level to the destination, which is shown in the Figure 6. This is a decode-and-forward scheme, and using such protocol the capacity is within 1 bit/s/Hz of the cut-set upper bound of the single-relay Gaussian network. The proof details can be found in Appendix A of [1].

Avestimehr et al. also point out that the 1-bit gap is at most times conservative. Only if two of the channel gains are exactly the same will the gap be at 1 bit. Given the feature of wireless networks it rarely happens and the gap is at most times close to zero. The Figure 7 plots the gap between deterministic and Gaussian version of single-relay network for different ratio of channel gains. Also, for this particular scenario many other strategy, like compress-and-forward and network coding also achieves to within 1-bit gap.

Avestimehr et al. also investigate other cases such as the diamond networks and the four-relay network. We omit the details here, and illustrate the four-relay in the section V. There is an important result that all the common forwarding strategies fail to achieve within a constant gap to the cut-set bound in more complicated networks. Instead, in the paper [1], a new strategy namely the quantize-map-and-forward is developed. The key point is to quantize the received signal at the distortion of noise power, map them by a randomly-generated Gaussian codebook and forward the codeword. The strategy is universally approximate for arbitrary noisy Gaussian relay networks, which is shown in the next section.

IV. Main Result

We present the main results in [1]. Given a network, the outer bound is evaluated by the cut-set bound as described in (4). Then the question is how to prove the achievability part.

First, we characterize the capacity of a linear finte-field deterministic rely network of single-unicast. After deriving the cut-set bound of such case, we illustrate the major ideas behind the achievability which turn out to match this cut-set bound. Second, we show approximation on the capacity of single-unicast Gaussian relay networks by introducing a new transmission strategy resulting from the insight of deterministic model. The proofs of single-unicast can be extended to multicast case. The last but not the least, we investigate the connection between deterministic models and general Gaussian models. The difference of their individual capacities is at most comparable to network size $|\mathcal{V}|$.

To make matters simple, we only illustrate the ideas here. Please respectively see section V, VI, VII in [1] in order to explore the rigorous proofs.

A. Deterministic Networks

The cut-set upper bound evaluated in single-unicast linear finite-field deterministic relay network is

$$\bar{C} \overset{(a)}{=} \max_{p(\{\mathbf{x}_i\}_{i\in\mathcal{V}})} \min_{\Omega\in\Lambda_D} H(\mathbf{y}_{\Omega^c}|\mathbf{x}_{\Omega^c}) \overset{(b)}{=} \min_{\Omega\in\Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}) \tag{6}$$

(a) follows from we address the deterministic model and (b) comes from features of linear finite-field. We can also derive cut-set bound for multicast case. By showing achievability part, we have the following theorems,

Theorem 4.1. The single-unicast capacity C of a linear finite-field deterministic relay network is given by

$$C = \min_{\Omega\in\Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}) \tag{7}$$

Theorem 4.2. The single-multicast capacity C of a linear finite-field deterministic relay network is given by

$$C = \min_{D\in\mathcal{D}} \min_{\Omega\in\Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}) \tag{8}$$

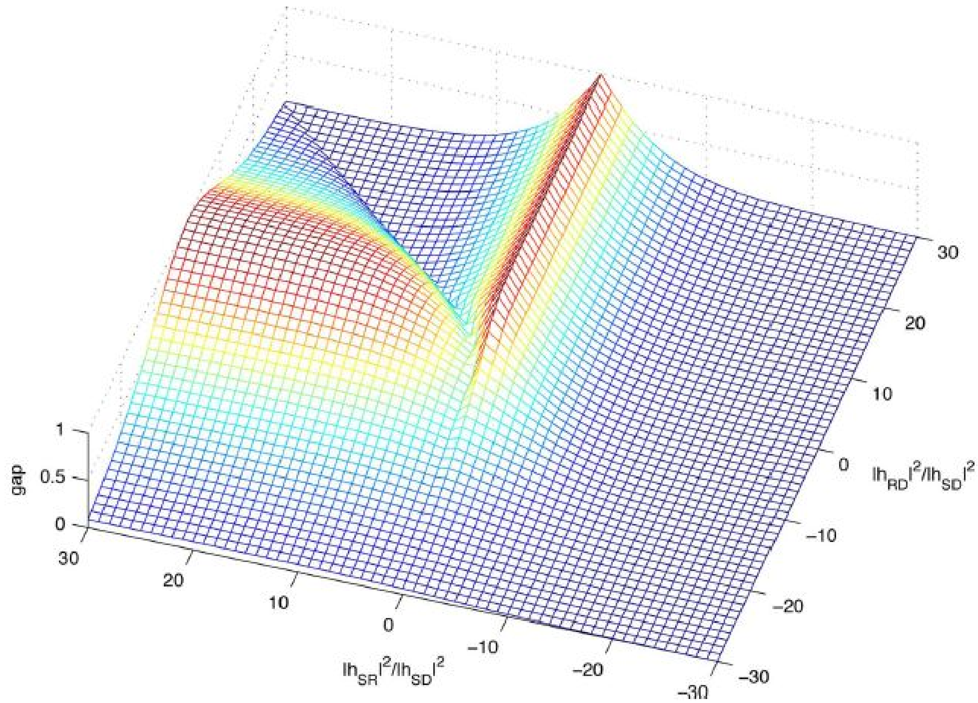

Figure 7. The x and y axis respectively represent the channel gains from relay to destination and source to relay normalized by the gain of the direct link (source to destination) in dB scale. The z axis shows the value of the gap between the cut-set upper bound and the achievable rate of decode-forward scheme in bits/sec/Hz.

where $\mathcal{D}$ is the set of destinations.

Remark. Theorem 4.1 generalizes the classical max-flow min-cut theorem for wired networks and Theorem 4.1 generalizes the network coding result.

The achievability proof is as follows: we focus on networks that have a layered structure, i.e., all path from the source to the destination are equally long. In this way, source messages can each be encoded into one block of symbols and the blocks in turn do not interact with each other as they flow through the deterministic relay network. We use random coding argument in the proof of this case. Then the result of layered networks can be extended to an arbitrary network by taking advantage of time-expansion. The rationale here is to reduce an arbitrary network to the one layered network by expanding the considered network over time, and then we can apply the result in the first step to complete the proof.

1) Layer Network Proof Illustration:

- Encoding at source: source S has messages $W \in [0 : 2^{TR} - 1]$. It generates a codebook with each codeword is a block of T symbols i.i.d. generated over $\mathbb{F}_2$.
- Relay nodes operation: relay j uses a mapping $f_j : \mathcal{Y}_j^T - \mathcal{X}_j^T$ on its received block to next transmitted block. We have linear vector mapping in the deterministic model. The corresponding linear matrix mapping function $\mathbf{F}_j$ is generated uniformly randomly over $\mathbb{F}_2^{qT \times qT}$ (remember q means the length of one symbol).
- Decoding at destination: destination D uses a deterministic mapping from received block to the messages. Decodability equals to invertibility of the mapping function.

To analyze the error probability, we WLOG can assume message $W = 1$ is sent and then apply the concept—distinguishability as taught in the class. The error probability is bounded by

$$P_e \leq 2^{TR}|\Lambda_D|2^{-T\min_{\Omega\in\Lambda_D}\operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})} \tag{9}$$

Thus R is achievable if $R < \min_{\Omega\in\Lambda_D}\operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})$.

For multicast case, we note that for each destination $D \in \mathcal{D}$, the error probability of (9) must hold,

$$\begin{aligned} P_e &\leq 2^{TR}\sum_{D\in\mathcal{D}}|\Lambda_D|2^{-T\min_{\Omega\in\Lambda_D}} \\ &\leq 2^{TR}2^{|\mathcal{V}|}2^{-T\min_{D\in\mathcal{D}}\min_{\Omega\in\Lambda_D}\operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})} \end{aligned} \tag{10}$$

so $R < \min_{D\in\mathcal{D}}\min_{\Omega\in\Lambda_D}\operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})$.

2) Arbitrary Network Proof Illustration: Given the proof for layered networks with equal path lengths, we can tackle general relay networks. The ingredients are developed below.

We first unfold the network $\mathcal{G}$ over time to create a layered network. The idea is to unfold the network to K stages such that ith stage represents what happens in the network during $(i-1)T$ to $iT - 1$ symbol times. More specifically, the time-steps unfolded network $\mathcal{G}_{\text{unf}}^{(K)} = (\mathcal{V}_{\text{unf}}^{(K)}, \mathcal{E}_{\text{unf}}^{(K)})$ is constructed from $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

For $\mathcal{V}_{\text{unf}}^{(K)}$:

- The network $\mathcal{G}_{\text{unf}}^{(K)}$ has $K+2$ stages, labeled as 0 to $K+1$.
- Stage 0 has only node $S[0]$ and stage $K+1$ has only node $D[K+1]$. $S[0]$ and $D[K+1]$ represent respectively the source and the destination in $\mathcal{G}_{\text{unf}}^{(K)}$.
- Each node $v \in \mathcal{V}$ appears at stage i as relay denoted by $v[i], i = 1 \ldots K$.

For $\mathcal{E}_{\text{unf}}^{(K)}$:

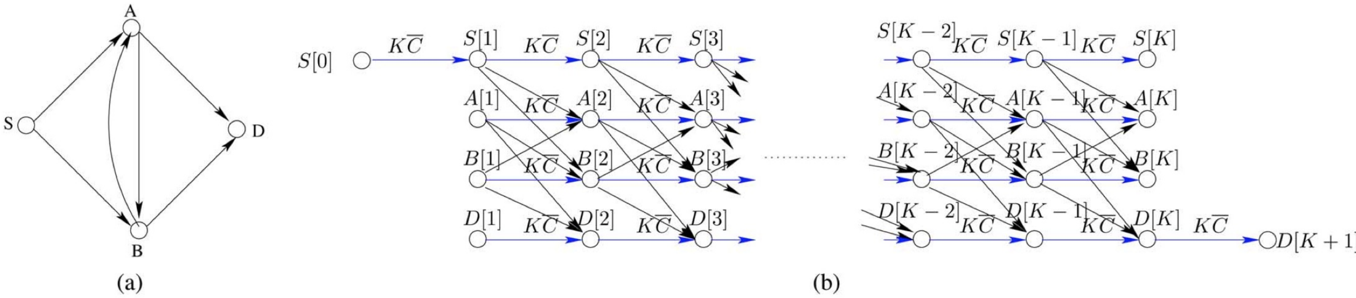

Figure 8. (a) An example of a general deterministic network with unequal paths from S to D. (b) The corresponding unfolded network.

- There are links orthogonal to all other transmissions in the network as shown blue in Figure 8. Each has capacity $K\bar{C}$, where $\bar{C} = \min_{\Omega\in\Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})$ is the min-cut value of $\mathcal{G}$. These links are created if
  - $(S[0], S[1])$ and $(D[K], D[K+1])$.
  - $(v[i], v[i+1]), \forall v \in \mathcal{V}$ and $1 \le i \le K$.
- Node $v[i]$ is connected to node $w[i+1]$ with the linear finite-field deterministic channel of the original network $\mathcal{G}, \forall (v,w) \in \mathcal{E}, v \neq w$.

I illustrate the scheme in an intuitive manner. Since $\mathcal{G}_{\mathrm{unf}}^{(K)}$ is a layered network, we can achieve $R_{\mathrm{unf}}^{(K)} < \bar{C}_{\mathrm{unf}}^{(K)} = \min_{\Omega_{\mathrm{unf}}\in\Lambda_{\mathrm{unf}}} \operatorname{rank}(\mathbf{G}_{\Omega_{\mathrm{unf}},\Omega_{\mathrm{unf}}^c})$. To achieve R in the original network, source $S[i], i = 1\ldots K$ sends one block of T symbols into the network at ith stage. Blocks here can be regarded as symbols in the layered networks. Eventually, destination D receives KT symbols and it can decode the messages because T symbols at each stage ensure the reliable transmission. We require a multiple K of symbols for achieving rate R and therefore $R < \frac{1}{K} R_{\mathrm{unf}}^{(K)}$. The remaining is to prove $\lim_{K\to\infty} \frac{1}{K} R_{\mathrm{unf}}^{(K)} = \bar{C}$ by Squeeze Theorem, which I do not explain here.

B. Gaussian Networks

In Gaussian models, node $j \in \mathcal{V}$ has M_j transmit, and N_j receive antennas. The received signal $\mathbf{y}_j[t]$ at node j at time t is

$$\mathbf{y}_j[t] = \sum_{i\in\mathcal{V}} \mathbf{H}_{ji}\mathbf{x}_i[t] + \mathbf{z}_j[t] \tag{11}$$

where $\mathbf{H}_{ji}$ is a $N_j \times M_i$ complex matrix whose (n,m)-element represent the gain from mth transmit antenna in node i to nth receive antenna in node j. Furthermore, we assume average power constraint equal to 1at each transmit antenna. The noise vector $\mathbf{z}_j$ is modeled as complex Gaussian random vector, and noises at different receivers are assumed mutually independent.

Theorem 4.3 (Approximation Theorem)**.** The capacity Formula of the Gaussian relay network satisfies

$$\bar{C} - \kappa \le C \le \bar{C} \tag{12}$$

where $\bar{C}$ is the cut-set upper bound, and κ is a constant and upper bounded by $12\sum_{i=1}^{|\mathcal{V}|} N_i + 3\sum_{i=1}^{|\mathcal{V}|} M_i$

Theorem 4.4. The capacity $\bar{C}_{\mathrm{mult}}$ of the Gaussian relay network satisfies

$$\bar{C}_{\mathrm{mult}} - \kappa \le C_{\mathrm{mult}} \le \bar{C}_{\mathrm{mult}} \tag{13}$$

where $\bar{C}_{\mathrm{mult}}$ is the multicast cut-set upper bound on the capacity of Formula given by

$$\bar{C}_{\mathrm{mult}} = \max_{p(\{\mathbf{x}_j\}_{j\in\mathcal{V}})} \min_{D\in\mathcal{D}} \min_{\Omega\in\Lambda_D} I(\mathbf{y}_{\Omega^c}; \mathbf{x}_\Omega \,|\, \mathbf{x}_{\Omega^c}) \tag{14}$$

κ is a constant and upper bounded by $12\sum_{i=1}^{|\mathcal{V}|} N_i + 3\sum_{i=1}^{|\mathcal{V}|} M_i$.

Remark. The gap κ holds for all values of the channel gains.

The achievability proof steps are similar to those in linear finite-field deterministic relay networks. We still start from layered networks to arbitrary ones by unfolding the network $\mathcal{G}$ over K stages. Therefore, we only illustrate the major ideas in the proof of layered networks.

The relaying strategy is however different. Inspired by quantized signal levels in deterministic model, we develop a new strategy, quantize-map-and forward, and show that it can achieve the rate to cut-set upper bound within a constant κ.

To prove layered network case, we have two steps: inner code and outer code. Using the relaying strategy and operating over a large block, we can achieve an end-to-end mutual information which is however within a constant gap to the cut-set upper bound. We implement this as inner code. Outer code is to map the message to multiple inner code symbols and send them to the destination. By coding over many such symbols, it is possible to achieve a reliable communication rate arbitrarily close to the mutual information of the inner code, and hence, we finish the proof. The system diagram of our coding strategy is illustrated in Figure 9.

We first define the quantization operation.

Definition 4.1. The quantization operation $[\cdot] : \mathbb{C} \to \mathbb{Z}\times\mathbb{Z}$ maps a complex number $c = x + iy$ to $[c] = ([x],[y])$, where $[x]$ and $[y]$ are the closest integers to x and y, respectively. Since the Gaussian noise at all receive antennas has variance 1, this operation is basically scalar quantization at noise-level.

- Inner code: We first construct the codebooks for node $j \in \mathcal{V}\setminus\{D\}$. The codebook $\mathcal{T}_{x_j}$ is a set of independent complex Gaussian codewords of block length T with components distributed as i.i.d. $\mathcal{CN}(0,1)$.

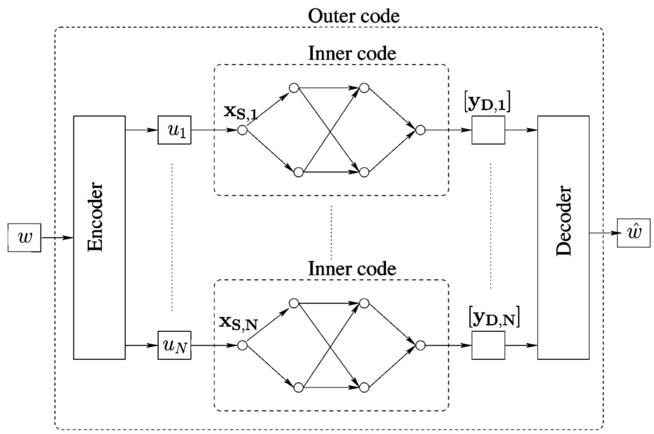

Figure 9. System diagram for achievability of Gaussian relay networks

- Encoding at source: source S has inner code symbols $U \in [0 : 2^{TR_{\text{in}}} - 1]$ and maps it to $\mathcal{T}_{x_S}$.
- Relay nodes operation: relay j receives signal vector $\mathbf{y}_j$, quantizes it to $[\mathbf{y}_j]$ and maps it to $\mathcal{T}_{x_j}$.

- Outer code: Source S encodes messages into $U_1 \ldots U_N$ inner code symbols, Formula. Each inner code symbol is then sent via the inner code over T transmission times, giving an overall transmission rate of R.
- Decoding at destination: Given the knowledge of all the encoding functions at the relays and quantized received signals $[\mathbf{y}_{D,1}] \ldots [\mathbf{y}_{D,N}]$, the destination attempts to decode the message.

Again, we apply the distinguishability to analyze the error probability. The detailed proof is in section VI in [1].

C. Connection between Deterministic and Gaussian Networks

The linear finite-field channel model captures certain high SNR behaviors of the Gaussian channel, but Avestimehr et al. in [1] showed that its capacity is not within a constant gap to the Gaussian capacity for all MIMO channels. The question of interest is: can we extend deterministic model to approximate the Gaussian relay network capacity to within a constant gap?

We find that whatever the noise level, the approximation theorem for the Gaussian network capacity shows a constant gap between outer and inner bounds. Then we refine the deterministic models by ignoring the noise and introducing the MIMO gain matrix $\mathbf{H}_{ji}$.

$$\mathbf{y}_j[t] = \left[\sum_{i \in \mathcal{V}} \mathbf{H}_{ij} x_i[t] \right], \quad j = 1, \ldots, |\mathcal{V}| \tag{15}$$

The improved deterministic model must be least within a constant gap to the capacity of the Gaussian network. We call it truncated deterministic model.

Theorem 4.5. The capacity of any Gaussian relay network, C_{Gaussian}, and the capacity of the corresponding truncated deterministic model, $C_{\text{Truncated}}$, satisfy the following relationship:

$$|C_{\text{Gaussian}} - C_{\text{Truncated}}| \le 33|\mathcal{V}| \tag{16}$$

The proof behind is to derive the difference of their cut-set bounds, apply the achievabilty results and use these inequalities to approximate the interval.

Theorem 4.5 suggests that the gap of approximation in the use of the truncated deterministic model is at most comparable to network size $|\mathcal{V}|$. It is interesting to think whether there exists scheme able to reduce the gap to constant, or it is a inherent feature.

V. Related Works

In this section, we survey literature that investigates the capacity of wireless information flow using deterministic models. In subsection V-D, we point out a method to design algorithms for finding max-flow in deterministic networks. In the following, we briefly discuss what they explore and find in their researches.

A. A Four-Relay Network

Avestimehr et al. explore a certain four-relay network in [1]. As shown in Figure 10, we find an optimal strategy to transmit bits from source to destination. How the bits are sent at source and each relay is illustrated on the corresponding links.

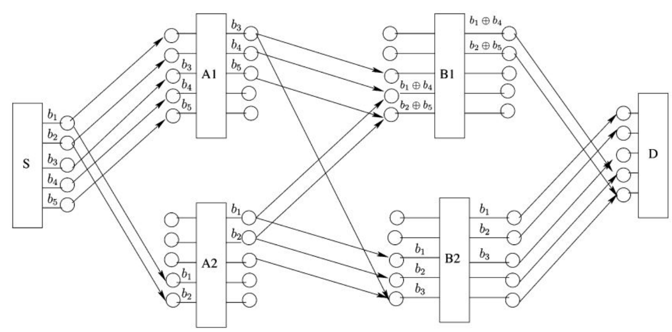

Figure 10. Demonstration of an optimal strategy in a four-relay network.

In this strategy, relays do not decode or partially decode the original flows of bits from received signal vectors. They cooperate, select relevant bits and then forward. Also, in the Gaussian network, we cannot implement this strategy using a standard compress-forward scheme pretending that the two received signals at B_1 and B_2 are jointly Gaussian. In fact, the statistical correlation between the real-valued received signals at B_1 and B_2 is quite weak since their MSBs are totally independent. This insight is apparent when we use deterministic model ans see the received signals as vectors of bits. It can be shown that the compress-forward strategy cannot achieve a constant gap to the cut-set bound.

B. One-Stage N-Relay Diamond Network

A N-relay diamond network consists of a source node connected through a broadcast channel to N relays; the N relays, in turn, are connected to the destination node through a multiple-access channel.

In [1], Avestimehr et al. shortly discuss that (bursty) amplify-and-forward does not achieve a constant when $N = 2$.

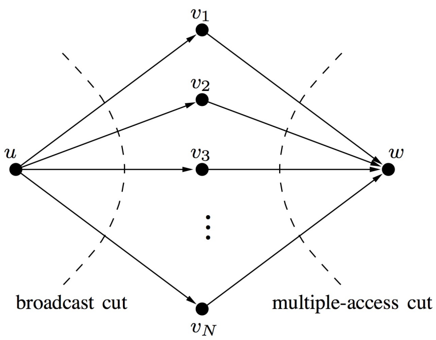

Figure 11. The N-relay diamond network. The source node u transmits a message to the destination node w via the N relays $\{v_n\}_{n=1}^N$.

Niesen et al. in [2] investigate more deeply. They approximate the capacity of the symmetric Gaussian N-relay diamond network first. " Symmetric" means gains of BC and MAC are respectively uniform. Gaps are independent of the channel gains and, unlike the best previously known result, are also independent of the number of relays N in the network. They achieve rates based on bursty amplify-and-forward. The upper bound on capacity is a careful evaluation of the cut-set bound. They also try to approximate the capacity of asymmetric Gaussian N-relay diamond network. At least in the low-rate regime, bursty amplify-and-forward is a good communication scheme for asymmetric diamond networks.

C. ZZ, ZS Two-Stage Interference Network

Mohajer et al. in [3] study a Gaussian relay-interference network. They focus on two-stage relay-interference networks where weak cross-links cause the networks to behave like a chain of Z Gaussian channels. Aim is to approximate the capacity region for such ZZ and ZS networks. The analysis of these Gaussian networks is based on insights gained from an exact characterization of the corresponding linear deterministic model. A new interference management scheme, termed interference neutralization, is proposed to achieve the capacity. It is implemented using structured lattice codes.

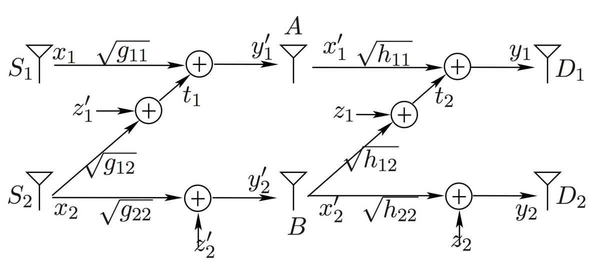

Figure 12. The Gaussian ZZ network

D. Polylinking Networks

This literature in [4] is not to investigate the capacity of Gaussian networks but to design efficient algorithms for finding maximum flow in the deterministic model.

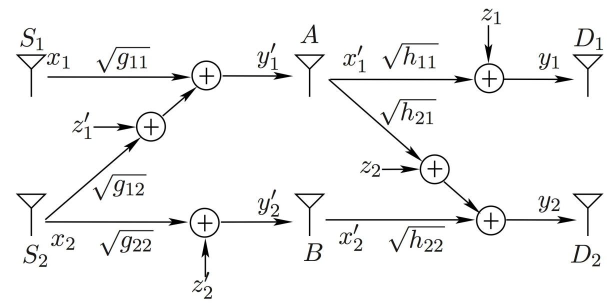

Figure 13. The Gaussian ZS network

Goemans et al. introduce polylinking networks—a flow model based on polylinking systems that generalizes the classical flow model of Ford-Fulkersonon acyclic networks and has applications in the context of wireless networks. They demonstrate that the deterministic model is essentially a linking network. Therefore, results derived for general polylinking systems do not only generalize the known results but also lead to new properties and algorithms. By presenting a compact representation of the deterministic model as a polylinking network, we are able to design a faster algorithm for finding a maximum flow in the deterministic model.

VI. Conclusions

We aim to investigate how to address the wireless information flow and start from popular Gaussian models. Instead of exactly computing Gaussian networks, Avestimehr *et al.* present deterministic models to build insights and base on them to analyze Gaussian models in an engineering sense.

The deterministic models bridge the wired networks and Gaussian networks. In wired networks, nodes can receive information from channel without interference from other channel. Deterministic networks take interference into account by such XOR operations. We can thus think how to extract information from interference, not to treat it as noise. To downplay Gaussian noise, deterministic models use quantized signals and thus we can focus on bits above noise power level. Figure 14 shows the relationship.

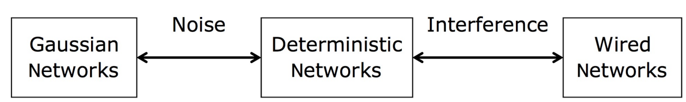

Figure 14. Role of deterministic models between Gaussian networks and wired networks

Deterministic models introduces a new scheme for general Gaussian relay networks called quantize-map-and-forward, and Avestimehr *et al.* prove that it can achieve to within a constant gap to the cut-set bound. The gap does not depend on the SNRs or the channel values of the network, but grows with the network size due to the noise accumulation property of the quantize-map-and-forward scheme. We have not answered the question: whether another scheme exists such that a universal constant gap to the cut-set bound can be attained, independent of network size, or if this is an inherent feature of any scheme for arbitrary networks. We may need to improve cut-set bound or to show the gap is at least comparable to network size.

Deterministic models serve as tools to study capacity region of a Gaussian interference network as we see some examples in section V. The derived capacity of a deterministic network can help us to infer the possible capacity region of the corresponding Gaussian network, and then try to prove it by applying the similar strategy that is used in the underlying deterministic network. Despite approximated results, the deterministic model is a stride in characterizing wireless information flows.

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