Submodular Maximization and Influence Maximization

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Matroid

- A set $(\mathcal{N}, \mathcal{I})$ is matroid if:

1. The empty set is independent.
   - $\emptyset \in \mathcal{I}$
2. Every subset of an independent set is independent.
   - For each $S \subseteq T \subseteq \mathcal{N}$, if $T \in \mathcal{I}$, then $S \in \mathcal{I}$
3. If A and B are two independent set and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is an independent set.

- Ex. No cycles subgraph

Matroid

- $w(e) \geq 0$
- The greedy algorithm can be used to find a maximum-weight basis of the matroid, by starting from the empty set and repeatedly adding one element at a time, at each step choosing a maximum-weight element among the elements whose addition would preserve the independence of the augmented set.

- Ex. Minimum spanning tree problem

Submodular function

- If Ω is a finite set, a submodular function is a set function $f: 2^{\Omega} \rightarrow \mathbb{R}_{+}$, where 2^{Ω} denotes the power set of Ω, which satisfies one of the following equivalent definitions.

1. For every $X, Y \subseteq \Omega$ with $X \subseteq Y$ and every $x \in \Omega \backslash Y$ we have that $f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$.
2. For every, $S, T \subseteq \Omega$ we have that $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$.
3. For every $X \subseteq \Omega$ and $x_1, x_2 \in \Omega \backslash X$ we have that $f(X \cup \{x_1\}) + f(X \cup \{x_2\}) \geq f(X \cup \{x_1, x_2\}) + f(X)$.

Submodular function

- Monotone submodular maximization problem
- Greed algorithm always produces a solution whose value is at least $1 - \left(1 - \frac{1}{k}\right)^k$ times the optimal value. This bound can be achieved for each k and has a limiting value of $1 - \frac{1}{e}$.

- Ex. Maximum coverage problem

f is monotone if

$$A \subset B \Rightarrow f(A) \leq f(B)$$

Supermodular function

- If Ω is a finite set, a supermodular function is a set function $f: 2^{\Omega} \rightarrow \mathbb{R}_{+}$, where 2^{Ω} denotes the power set of Ω, which satisfies one of the following equivalent definitions.

1. For every $X, Y \subseteq \Omega$ with $X \subseteq Y$ and every $x \in \Omega \backslash Y$ we have that $f(X \cup \{x\}) - f(X) \leq f(Y \cup \{x\}) - f(Y)$.
2. For every, $S, T \subseteq \Omega$ we have that $f(S) + f(T) \leq f(S \cup T) + f(S \cap T)$.
3. For every $X \subseteq \Omega$ and $x_1, x_2 \in \Omega \backslash X$ we have that $f(X \cup \{x_1\}) + f(X \cup \{x_2\}) \leq f(X \cup \{x_1, x_2\}) + f(X)$

- Ex. Set cover problem ($\big(H(n)\big)$-approximation)

Maximizing the spread of influence in a social network

- For an arbitrary instance of the Linear Threshold Model, the resulting influence function is submodular.
- For an arbitrary instance of the Independent Cascade Model, the resulting influence function is submodular.

- $\left(1-\frac{1}{e}-\varepsilon\right)$-approximation algorithm by using $(1+\gamma)$-approximate values for the function to be optimized