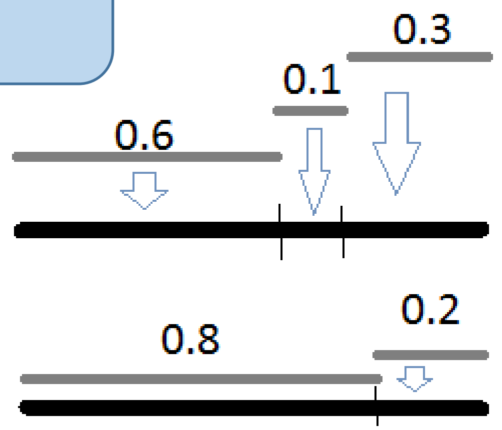
Improved Approximation Algorithm for Two-Dimensional Bin Packing

Arindam Khan (Georgia Tech)

Joint work with Nikhil Bansal (TU Eindhoven)

Bin Packing Problem: (One Dimension)

- Given : n items with sizes $s_1, s_2 \ldots s_n$, s.t. $s_i \in (0,1]$
- Goal: Pack all items into min # of unit bins.
- Example: items {0.8, 0.6, 0.3, 0.2, 0.1} can be packed in 2 unit bins: {0.8, 0.2} and {0.6, 0.3, 0.1}.



- NP Hardness from *Partition*

- Cannot distinguish in poly time if need 2 or 3 bins

- This does not rule out OPT+1 guarantee.

- Insightful to consider Asymptotic approximation ratio.

Asymptotic Approximation Ratio

- (Absolute) Approximation Ratio: $\max_{I} \{Algo(I)/OPT(I)\}$

- Asymptotic Approximation Ratio (AAR):

  $$\lim_{\{n\to\infty\}} \sup\{ \frac{Algo(I)}{OPT(I)} \,|OPT(I) = n \}$$

- AAR $\rho \Rightarrow Algo(I) = \rho.OPT(I) + O(1)$.
- Asymptotic Polynomial Time Approximation Scheme (APTAS):

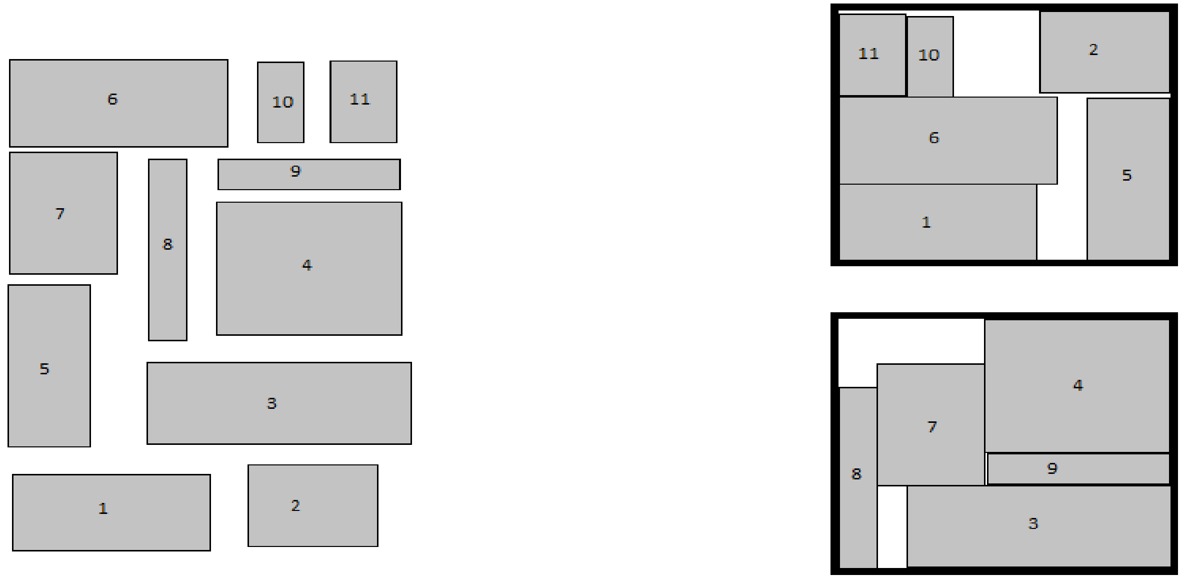
  If $Algo(I) = (1+\epsilon)\, Opt(I) \; + \; f(\epsilon)$

- 1 D Bin Packing: AAR $OPT + logOPT.loglogOPT$ [Rothvoss FOCS'13]

Two-Dimensional Geometric Bin Packing

- Given: Collection of rectangles(by width, height)
- Goal: Pack them into minimum number of unit square bins.

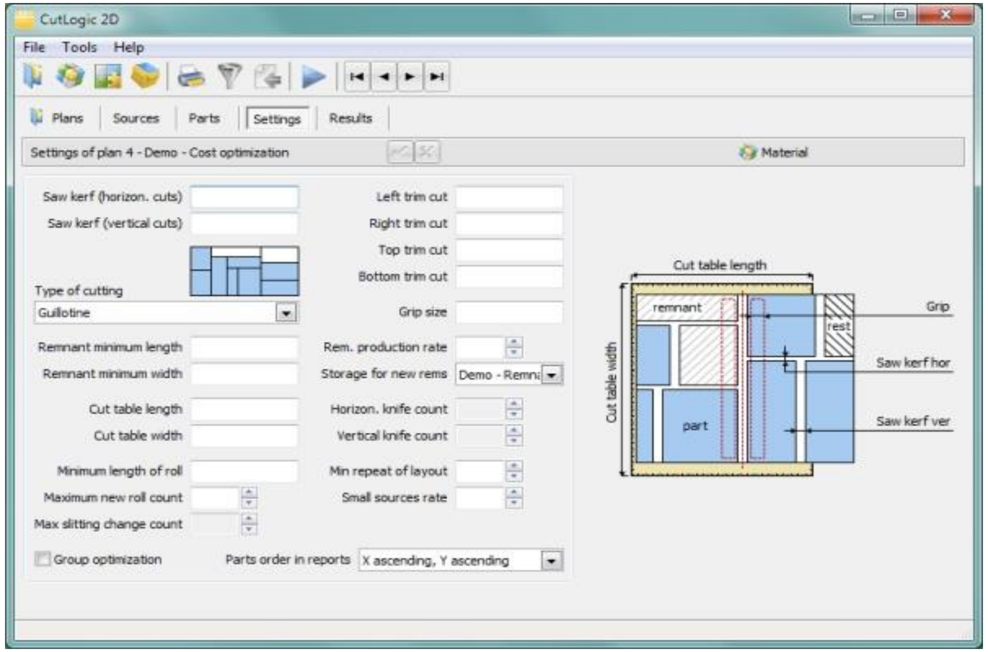
- Orthogonal Packing: rectangles packed parallel to bin edges.
- With 90 degree *Rotations* and *without rotations*.



Applications:

- Cloth cutting, steel cutting, wood cutting
- Placing ads in newspapers
- Memory allocation in paging systems

- Truck Loading
- Palletization by robots



A tale of Approximibility

- Reduction from *Partition:* NP-hard to decide if we need 1 or 2 bins to pack all rectangles.
- Tight 2 (absolute) approximation Algorithm. [Harren VanStee Approx 2009]
- Algorithm: (Asymptotic Approximation)
- 2.125 [Chung Garey Johnson SIAM JADM1982]
- $2+\epsilon$ [Kenyon Remilla FOCS 1996]
- 1.69 [Caprara FOCS 2002]
- 1.52 [Bansal-Caprara-Sviridenko FOCS 2006]
- 1.5 [Jansen-Praedel SODA 2013]
- Hardness:
- No APTAS (from 3D Matching)[Bansal-Sviridenko SODA 2004],
- 3793/3792(with rotation), 2197/2196(w/o rotation) [Chlebik-Chlebikova]

Our Results: [Bansal,K.]

- Algorithm:
- $(1 + \ln(1.5)) = 1.405$ Approximation Algorithm.
  - using Round & Approx framework for rounding based algorithms.
  - Jansen-Praedel 1.5 approximation algorithm as a subroutine.

- Hardness:
- 4/3 for constant number of rounding based algorithm.
- 3/2 for *input agnostic* rounding based algorithms.

Configuration LP

- *Bin packing problem can be viewed as a special case of set cover.*

Set Cover:

Items $i_1, i_2, \ldots\ i_n$; Sets $C_1, C_2, \ldots\ C_m$.

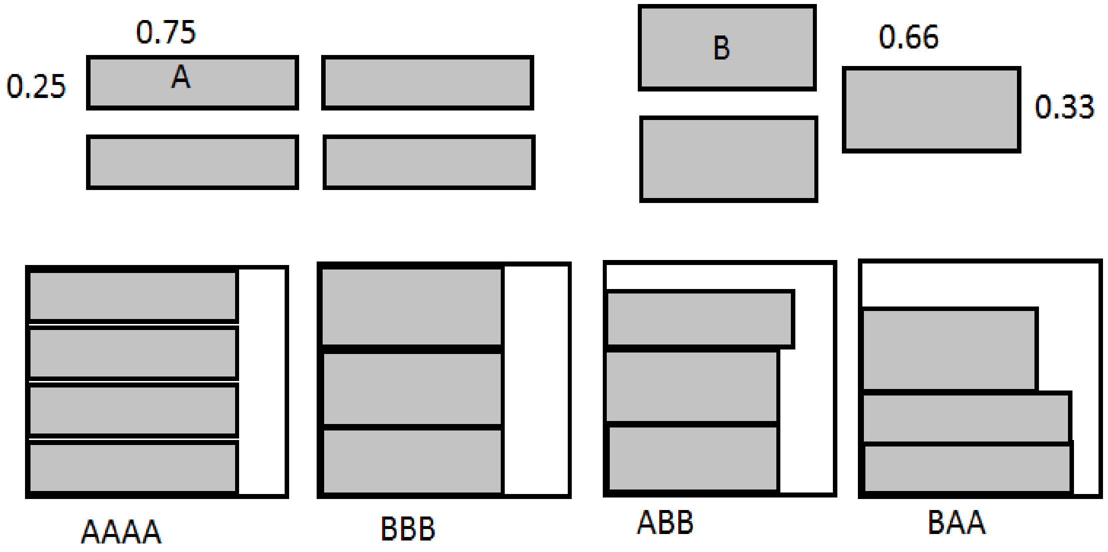
Choose fewest sets s.t. each item is covered.

Bin Packing:

Sets are implicit:

known as configurations.

Any subset of items that fit feasibly in a bin.



Configuration LP

- $\mathbb{C}$*: set of configurations(possible way of feasibly packing a bin).*

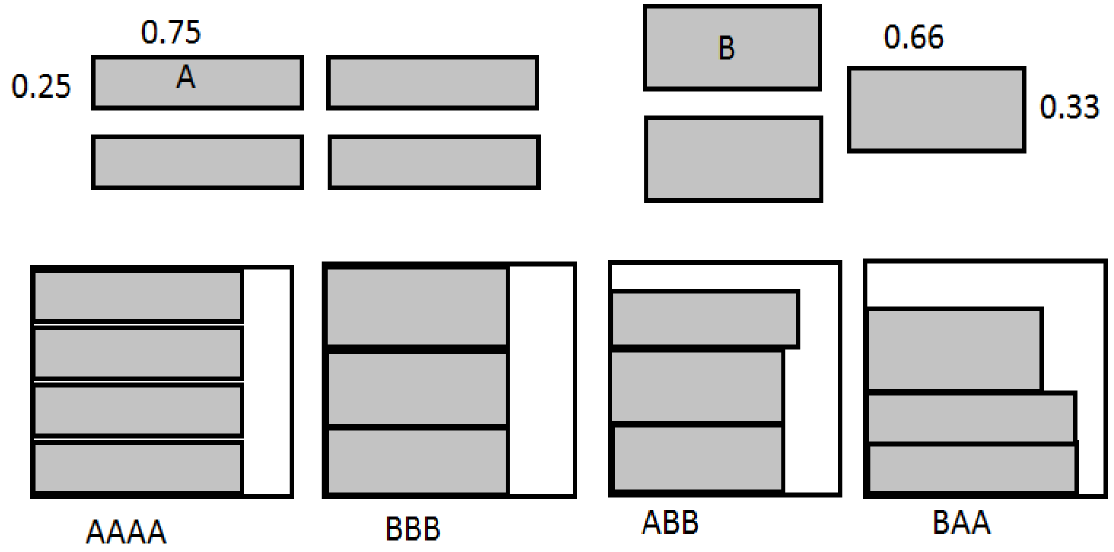
Primal:

$$\min \{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \ (i \in I), x_C \geq 0 \ (C \in \mathbb{C}) \}$$

Objective: min # configurations(bins)

Constraint:

For each item, at least one configuration containing the item should be selected.



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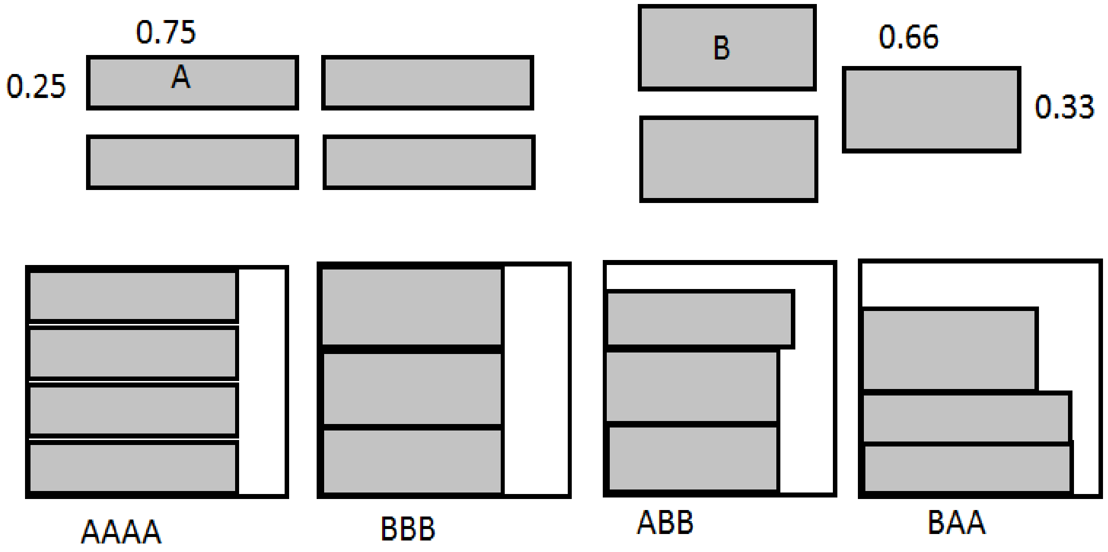
Gilmore Gomory LP for multiple identical items:

$Min\ \{1^T x\colon\ Ax \geq b,\ x_C \geq 0(C \in \mathbb{C})\}$

Columns: Feasible configurations

Rows: Items (or types of items)



Configuration LP

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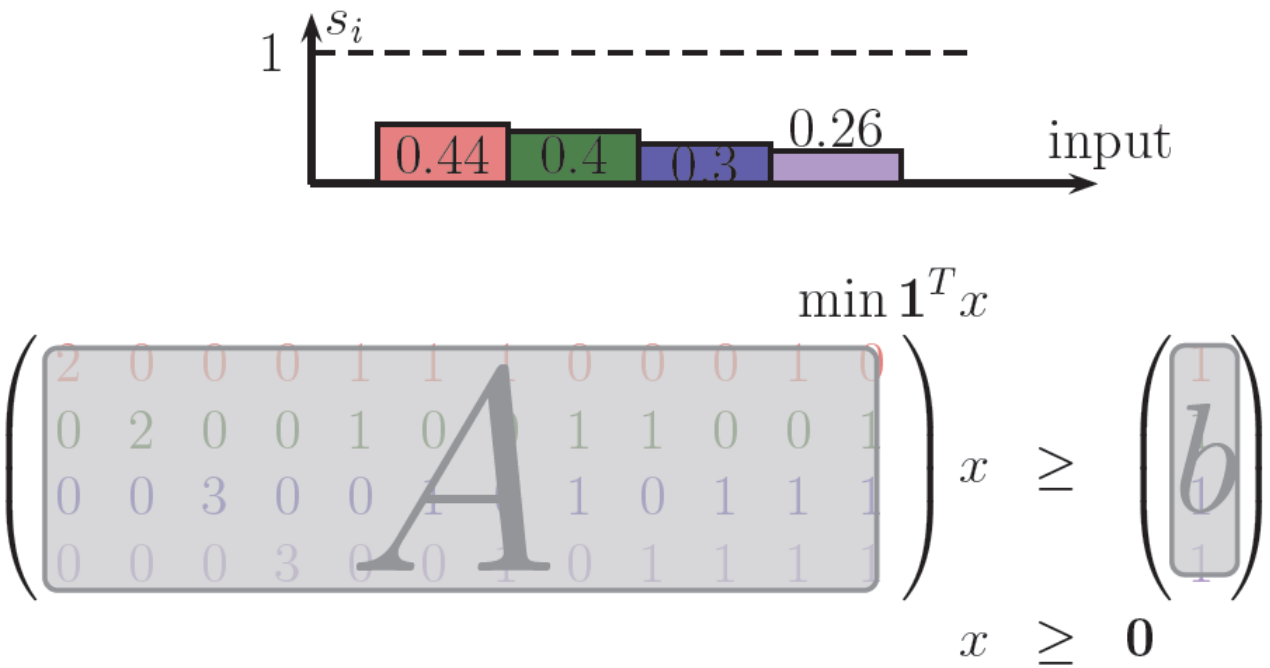

Rothvoss '13

Configuration LP

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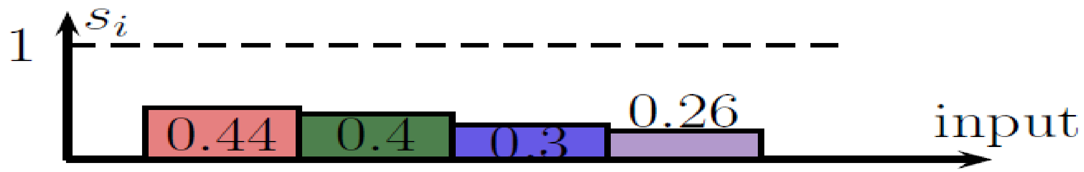
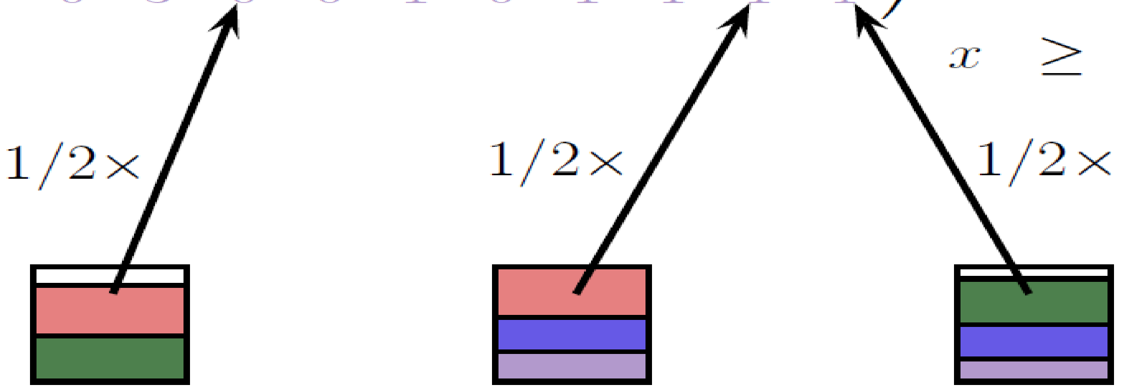


$$\min \mathbf{1}^T x$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x \geq \mathbf{0}$$



Rothvoss ‘13

Configuration LP

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$$\min \{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \; (i \in I), x_C \geq 0 \; (C \in \mathbb{C}) \}$$

Dual:

$$\max \{ \sum_{i \in I} v_i : \sum_{i \in C} v_i \leq 1 \; (C \in \mathbb{C}), v_i \geq 0 (i \in I) \}$$

Dual Separation problem =>
2-D Geometric Knapsack problem:
Given one bin, pack as much area as possible.
[BCJPS ISAAC 2009]

- Problem: Exponential number of configurations!
- Solution: Can be solved within $(1 + \epsilon)$ accuracy using separation problem for the dual.

General Framework [BCS FOCS 06]

- Given a packing problem

1. If can solve the configuration LP

$$\min \left\{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \ (i \in I), x_C \geq 0 \ (C \in \mathbb{C}) \right\}$$

2. There is a ρ approximation subset-oblivious algorithm.

- Then there is (1+ln ρ) approximation.

Subset Oblivious Algorithms

- There exist k weight (n - dim) vectors $w^1, w^2 \ldots w^k$ s.t.

For every subset of items $S \subseteq I$, and ε > 0

1) $OPT\ (I) \geq max_j\ (\sum_{i \in I} w_i^j\ )$

2) $Alg\ (S) \leq \rho\ max_j(\sum_{i \in S} w_i^j\ ) + \varepsilon\ OPT(I) + O(1)$

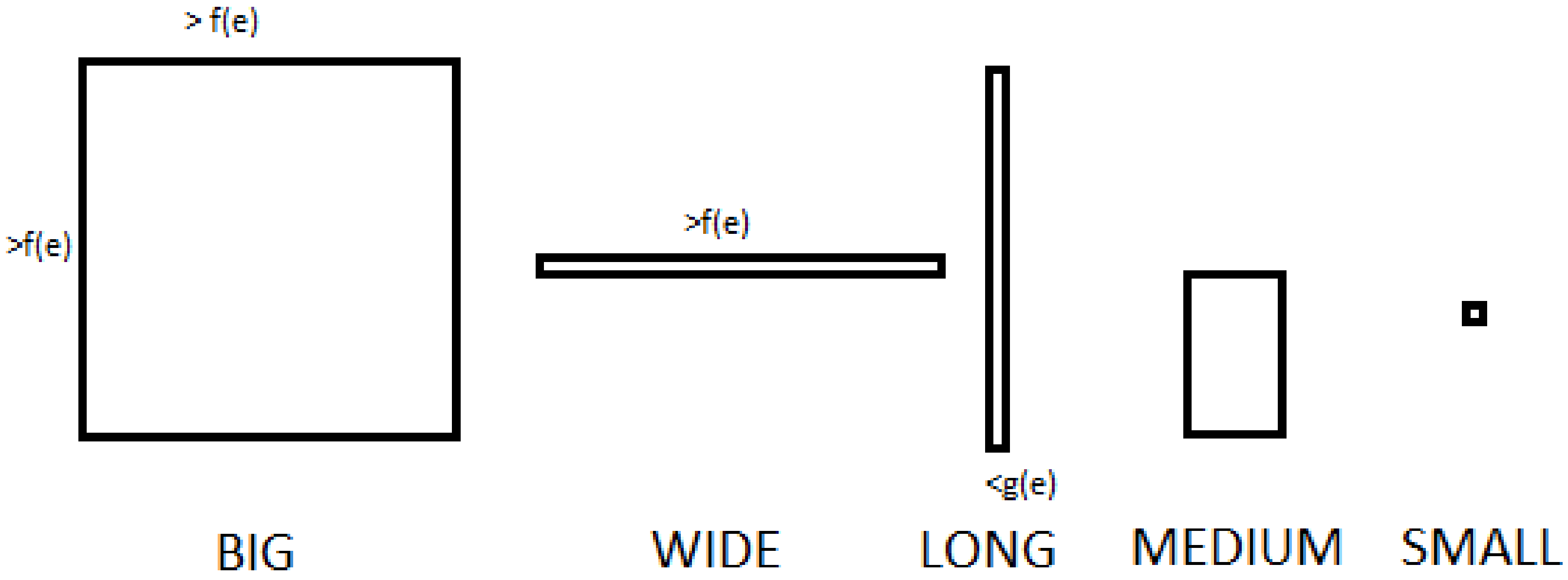
- Based on dual weight function.
- Cumbersome and Complex!

Rounding based Algorithms:

- Our Key Contribution:
  Subset Oblivious techniques work for any rounding based algorithms.
- Rounding based algorithms are ubiquitous in bin packing.
- Items are replaced by slightly larger items from O(1) types to reduce # item types and thus #configurations.
- Example: Linear grouping, Geometric Grouping, Harmonic Rounding.
- Jansen Praedel 1.5 Approximation algorithm is a nontrivial rounding based algorithm.

Rounding based Algorithms

- Classification of items into big, wide, long, medium and small by defining two parameters $f(\epsilon)$ and $g(\epsilon)(\ll f(\epsilon))$ such that total volume of medium rectangles is $\epsilon.Vol(I)$.



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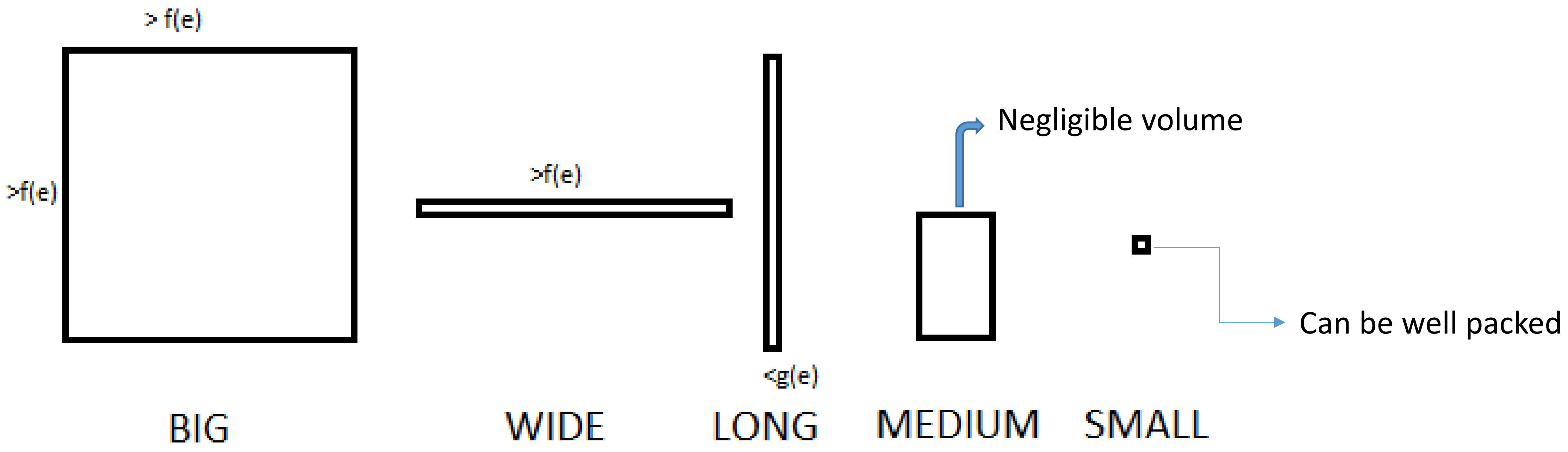

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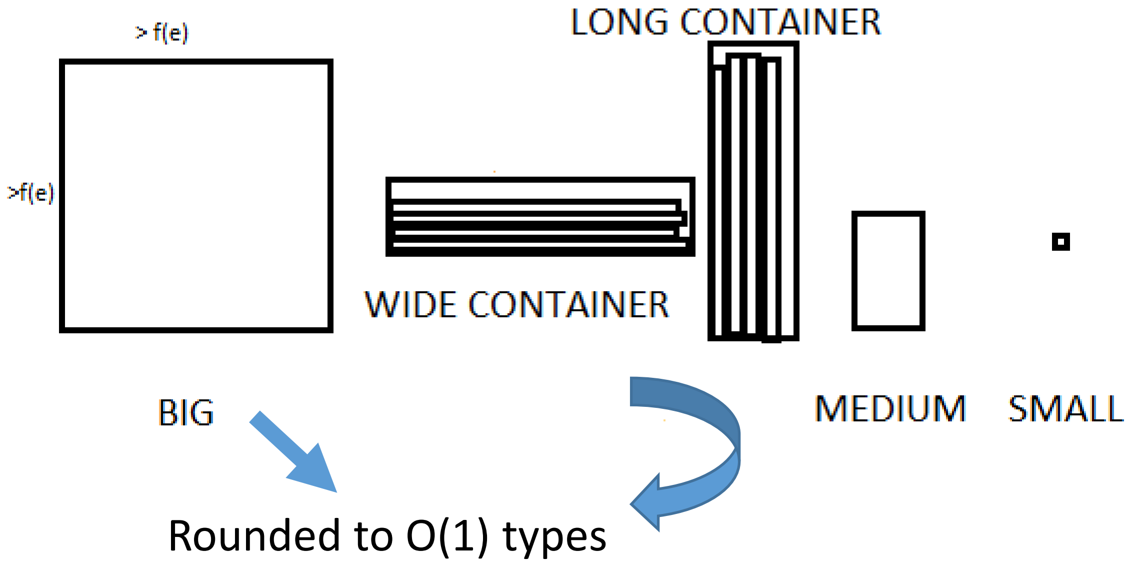


Skewed items are packed into containers where
(i) it has all items of the same type,
(ii) has large size in each dimensions and
(iii) items are packed into containers with a negligible loss of volume.

Round and Approx Framework (R & A)

- 1. Solve configuration LP using APTAS. Let $z^* = \sum_{\{C \in \mathbb{C}\}} x_C^*$.

Primal:

$$\min \left\{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \;(\, i \in I),\, x_C \geq 0 \;(C \in \mathbb{C}) \right\}$$

Round and Approx Framework (R & A)

- 1. Solve configuration LP using APTAS. Let $z^* = \sum_{\{C\in\mathbb{C}\}} x_C^*$.
- 2. Randomized Rounding: For q iterations :

  select a configuration C' at random with probability $\frac{x_{C'}^*}{z^*}$.

Primal:

$$\min \{ \sum_{C} x_C : \sum_{C\ni i} x_C \geq 1 \, (i \in I), x_C \geq 0 \, (C \in \mathbb{C}) \}$$

Round and Approx Framework (R & A)

- 1. Solve configuration LP using APTAS. Let $z^* = \sum_{\{C \in \mathbb{C}\}} x^*_C$.
- 2. Randomized Rounding: For q iterations :

  select a configuration C' at random with probability $\frac{x^*_{C'}}{z^*}$.
- 3. Approx: Apply a ρ approximation rounding based algorithm A on the residual instance S.
- 4. Combine: the solutions from step 2 and 3.

R & A Rounding Based Algorithms

- Probability item i left uncovered after rand. rounding

$= \left(1 - \sum_{\{C \ni i\}} \frac{x_C^*}{z^*}\right)^q \leq \frac{1}{\rho}$ by choosing q= $\lceil (\ln \rho) LP(I) \rceil$.

- Number of items of each type shrinks by a factor ρ

  e.g., $\mathbb{E}\left[\left|B_j \cap S\right|\right] = \frac{|B_j|}{\rho}$.

- Concentration using Chernoff bounds.

Proof Sketch

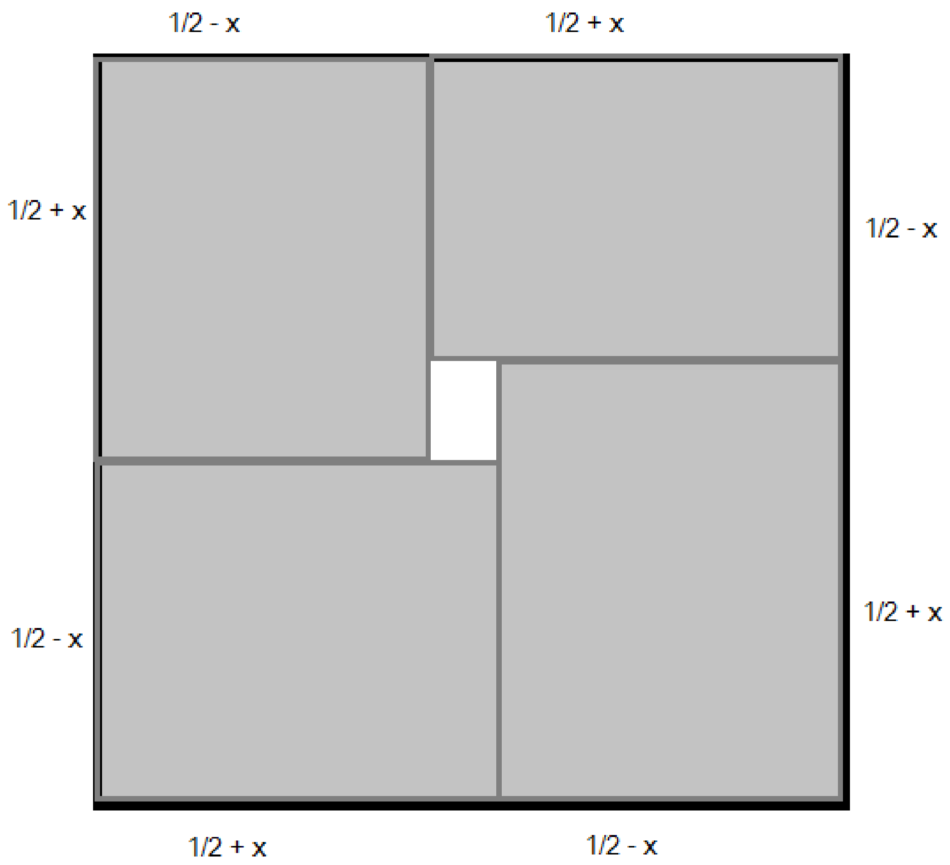
- Rounding based Algo : O(1)types of items $= O(1)$ number of constraints in Configuration LP.
- $ALGO(S) \approx OPT(\tilde{S}) \approx LP(\tilde{S})$.
- As # items for each item type shrinks by ρ, $LP(\tilde{S}) \approx \frac{1+\epsilon}{\rho} LP(\tilde{I})$.
- ρ Approximation: $LP(\tilde{I}) \leq \rho\, OPT(I) + O(1)$.
- $ALGO(S) \approx OPT(\tilde{S}) \approx OPT(I)$.

Proof Sketch

- Thm: R&A gives a $(1 + \ln \rho)$ approximation.
- Proof:
- $\text{Randomized Rounding}$: q= $\ln \rho . LP(I)$
- Residual Instance S $= (1 + \epsilon)OPT(I) + O(1)$.

- Round + Approx => $(\ln \rho + 1 + \epsilon)OPT(I) + O(1)$.

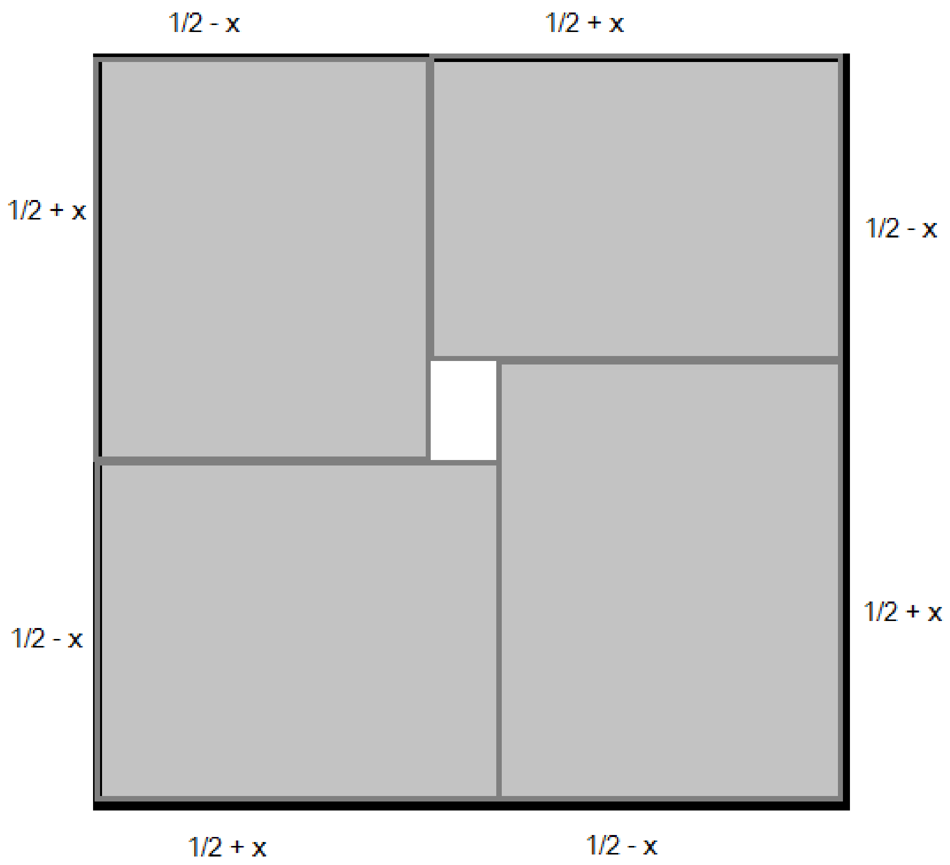
Hardness of 4/3: for Constant Rounding.



Items are rounded to O(1) types.

Only three rounded items
can be packed in a bin.

Hardness of 3/2: for Input-Agnostic Rounding

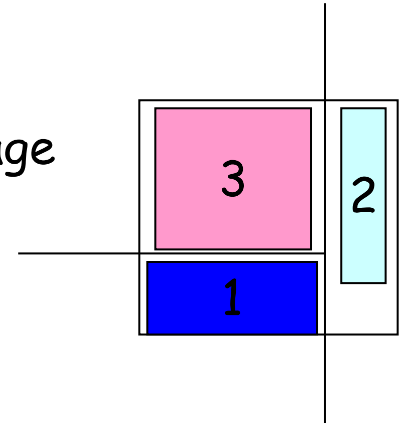


- Input agnostic rounding: Items are rounded to values independent of input values (e.g. Harmonic rounding, JP rounding).
- At least 3m/2 bins are needed to pack 4m such items.

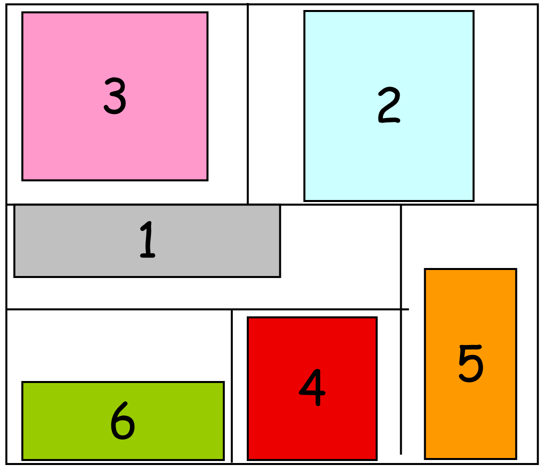
Open Problems:

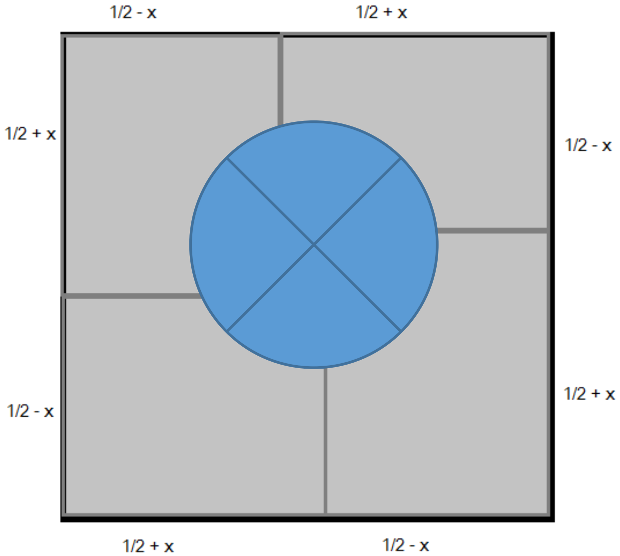
- Settle the gap between upper bound(1.405) and lower bound(1.0003).
- A 4/3 approximation algorithm based on constant rounding. => 1+ln(4/3) using R&A
- Guillotine Cut: Edge to Edge cut across a bin

2-stage



4-stage





- There is an APTAS for Guillotine Packing [BLS FOCS 2005].
- Settle the ratio between best Guillotine Packing and best 2D general packing :

  lower bound 1.33 and upper bound 1.69.

Questions!