Project Non-linear Optimization (2DME20)

Questions

Department of Electrical Engineering and Department of Mathematics and Computer Science

Preprocessing

Please read the project description first. Select a digital image `file.jpg` and import the image in Matlab by the commands

```
f = imread('file.jpg');
f = double(f);
image(f); colormap grey
```

This results in a mapping $f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ where $f(x, y)$ represents the grey level of the image at position (x, y). Let $x \in \mathbb{X} = [-\frac{X}{2}, \frac{X}{2}]$ and $y \in \mathbb{Y} = [-\frac{Y}{2}, \frac{Y}{2}]$ where $X \times Y$ is the dimension of the frame of the image.

Parametrization of ellipses

Recall that an ellipse is a set of the form

$$\mathcal{E} = \{ \xi \in \mathbb{R}^2 \mid (\xi - c)^\top P(\xi - c) = r^2 \} \tag{1}$$

where $c \in \mathbb{R}^2$ is the center, $P \succ 0$ and $r > 0$.

1. Show that any ellipse (1) can be written as the roots of a polynomial expression in the spatial variables. That is, show that

$$\mathcal{E} = \left\{ \xi \in \mathbb{R}^2 \mid \begin{pmatrix} \xi \\ 1 \end{pmatrix}^\top \begin{pmatrix} Q & d \\ d^\top & e \end{pmatrix} \begin{pmatrix} \xi \\ 1 \end{pmatrix} = 0 \right\} \tag{2}$$

   for suitable $Q = Q^\top \in \mathbb{R}^{2\times 2}$, $d \in \mathbb{R}^2$ and $e \in \mathbb{R}$. Conversely, determine appropriate conditions on (Q, d, e) so that the right hand side of (2) defines an ellipse. In addition, determine the minimal number of variables to uniquely define an ellipse of this form.

2. Show that the ellipse (1) can be written as the image of an operator according to

$$\mathcal{E} = \{ \xi \in \mathbb{R}^2 \mid \xi = e_0 + Ez, \quad \|z\| = 1 \} \tag{3}$$

   where $E : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map and $e_0 \in \mathbb{R}^2$ is a vector. Determine E and e_0 as function of (P, c, r) in (1). Conversely, determine appropriate conditions on (E, e_0) so that the right hand side of (3) defines an ellipse.

Feature extraction

3. Select a favorite and suitable image f for the classification of elliptical features and implement the algorithm in the project description for the detection of edges in the image. Create a feature image f_{feature} from your input image f together with a feature set

$$\mathcal{F} = \{\xi_1, \ldots, \xi_N\} \tag{4}$$

   of detected edges $\xi_i \in \mathbb{X} \times \mathbb{Y}$, $i = 1, \ldots, N$ in your image. Depending on the image, experiment with the variance $\sigma > 0$ and the thresholds $d_1 > 0$ and $d_2 > 0$ to create a suitable feature image. If you don't manage, please create a hand-made feature image by defining a feature set $\mathcal{F}$.

Classification of ellipses

The distance from an arbitrary point $\xi_0 = \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)$ in $\mathbb{R}^2$ to an ellipse $\mathcal{E}$ can be defined in various ways:

- The *residual distance* $d_{\text{res}}(\xi_0, \mathcal{E})$ is the absolute value of

$$r_0 := \begin{pmatrix} \xi_0 \\ 1 \end{pmatrix}^\top \begin{pmatrix} Q & d \\ d^\top & e \end{pmatrix} \begin{pmatrix} \xi_0 \\ 1 \end{pmatrix}. \tag{5}$$

  as associated with the representation (2).

- The *nearest distance* $d_{\text{near}}(\xi_0, \mathcal{E})$ is the number

$$d_{\text{near}} := \min_{\xi \in \mathcal{E}} \|\xi - \xi_0\|. \tag{6}$$

With respect to these distance measures, consider the following questions.

4. Show that r_0 in (5) is negative, zero or positive if ξ_0 lies in, on, or outside $\mathcal{E}$, respectively.
5. Find an expression for the nearest distance d_{near} in (6) in terms of the parameters (E, e_0) in the parametrization (3).

Consider the feature image f_{feature} of item 3 together with the feature set $\mathcal{F}$ as defined in (4).

6. Formulate the classification problem C1 as the optimization problem to minimize the total distance

$$P_{\text{tot}}(\mathcal{E}) := \sum_{i=1}^{N} d_{\text{res}}^2(\xi_i, \mathcal{E})$$

   over (a minimal set of) the parameters in (2). Show the resulting ellipse $\mathcal{E}^*_{\text{tot}}$, give the total distance $P_{\text{tot}}(\mathcal{E}^*_{\text{tot}})$ and verify whether this is a convex optimization problem.

7. Formulate the classification problem C2 as the optimization problem to minimize the maximal distance

$$P_{\max}(\mathcal{E}) := \max_{\xi_i \in \mathcal{F}} d_{\text{res}}(\xi_i, \mathcal{E})$$

   over (a minimal set of) the parameters in (2). Show the resulting ellipse $\mathcal{E}^*_{\max}$, give the maximal distance $P_{\max}(\mathcal{E}^*_{\max})$ and verify whether this is a convex optimization problem.

8. Formulate the classification problem C3 as the optimization problem to minimize the nearest distance

$$P_{\text{near}}(\mathcal{E}) := \sum_{i=1}^{N} d_{\text{near}}^2(\xi_i, \mathcal{E})$$

   over (a minimal set of) the parameters E and e_0 in (3). Show the resulting ellipse $\mathcal{E}^*_{\text{near}}$, give the nearest distance $P_{\text{near}}(\mathcal{E}^*_{\text{near}})$ and verify whether this is a convex optimization problem.

Clustering and classification

A non-zero vector $a \in \mathbb{R}^2$ and a constant $b \in \mathbb{R}$ define two *half-spaces* (or half-planes) $\mathcal{H}_-$ and $\mathcal{H}_+$ by setting

$$\begin{aligned} \mathcal{H}_- &:= \{\xi \in \mathbb{R}^2 \mid a^\top \xi \le b\} \\ \mathcal{H}_+ &:= \{\xi \in \mathbb{R}^2 \mid a^\top \xi > b\}. \end{aligned}$$

These are convex sets and provide a simple clustering of the feature set $\mathcal{F}$ into the two disjoint (and possibly empty) sets

$$\mathcal{F}_- := \mathcal{F} \cap \mathcal{H}_-, \qquad \mathcal{F}_+ := \mathcal{F} \cap \mathcal{H}_+.$$

The classification problems C1, C2, C3 can independently be applied to each of these clusters. The selection of the cluster sets $\mathcal{F}_-$. and $\mathcal{F}_+$ can be performed in a *supervised* or *unsupervised* manner. Supervised selection involves visual inspection for subdividing the image into two half-spaces. In unsupervised selection this process is automated.

9. Formulate an optimization problem that makes the clustering in an unsupervised manner such that the total error

$$\min_{\mathcal{E}_-} \sum_{i=1}^{N_-} d_{\text{res}}^2(\xi_i^-, \mathcal{E}_-) \; + \; \min_{\mathcal{E}_+} \sum_{i=1}^{N_+} d_{\text{res}}^2(\xi_i^+, \mathcal{E}_+)$$

is minimized over all (a, b). Here, N_- and N_+ are the number of elements in the feature sets $\mathcal{F}_- = \{\xi_1^-, \cdots, \xi_{N_-}^-\}$ and $\mathcal{F}_+ = \{\xi_1^+, \cdots, \xi_{N_+}^+\}$, respectively. Verify whether this is a convex optimization problem in the *joint* parameters of (2) and (a, b).

10. Demonstrate the clustering and classification procedures on a suitable image and clearly indicate whether your procedure requires supervised or unsupervised clustering.

Evidently, a further clustering can be made by introducing more half-planes in the configuration space of the image. This leads to a further refined decomposition of $\mathcal{F}$ in clusters in which objects can be classified. The conceptual algorithm for this is similar, but we will not further elaborate on this issues here.

Siep Weiland
Department of Electrical Engineering
Eindhoven University of Technology
October 2022.