Object detection using ellipse fitting on detected edges

Floris Naber 1253514   Shao Hsuan Hung 1723219

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For 2DME20 Non-linear optimization
Group 5

1 Introduction

2 Answers to questions

Parametrization of ellipses

2.1 Question 1

An ellipse is a set of the form:

$$\mathcal{E} = \{\xi \in \mathbb{R}^2 \mid (\xi - c)^\top P(\xi - c) = r^2\}, \tag{1}$$

where $c \in \mathbb{R}^2$ is the center, $P \succ 0$ and $r > 0$. We can write out the right side of this equation in terms of all matrix elements:

$$\begin{aligned} p_{11}\xi_1^2 + p_{22}\xi_2^2 + (p_{12} + p_{21})\xi_1\xi_2 - \Big(2p_{11}c_1 + (p_{12} + p_{21})c_2\Big)\xi_1 - \\ \Big(2p_{22}c_2 + (p_{12} + p_{21})c_1\Big)\xi_2 + p_{11}c_1^2 + p_{22}c_2^2 + (p_{12} + p_{21})c_1c_2 = r^2, \end{aligned} \tag{2}$$

where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ and $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. From this equation it can be seen that the system is over-determined. There are 7 variables in total, but only 5 are needed to make any arbitrary ellipse. As a result we can eliminate 2 variables from the equation.

Firstly, the terms p_{12} and p_{21} exclusively occur together as $p_{12} + p_{21}$. Thus we can replace this term by a single variable and still make any arbitrary ellipse. If we conveniently choose $p_{21} = p_{21}$ such that P is symmetric ($P = P^\top$), we can write $p_{12} + p_{21} = 2p_{12}$ and see that we can choose the single variable p_{12} as needed to make any arbitrary ellipse.

Secondly, in every term of the equation there is at least 1 variable, thus we can normalize over 1 of those variables, such that there are 5 variables left. We can for example normalize over r^2 by defining a matrix $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \frac{1}{r^2}P$ (note: $P = P^\top$, thus $Q = Q^\top$ and $q_{12} = q_{21}$).

By eliminating 2 variables as described above, we can write equation (2) as:

$$q_{11}\xi_1^2 + q_{22}\xi_2^2 + 2q_{12}\xi_1\xi_2 - 2(q_{11}c_1 + q_{12}c_2)\xi_1 - 2(q_{22}c_2 + q_{12}c_1)\xi_2 + q_{11}c_1^2 + q_{22}c_2^2 + 2q_{12}c_1c_2 = 1. \tag{3}$$

Now that we defined $P = P^\top$ and $Q = Q^\top = \frac{1}{r^2}P$, we can rewrite this equation back into matrix form and expand it. In other words, we can write the right side of equation (1) in terms of Q and c and expand it:

$$(\xi - c)^\top Q(\xi - c) = 1 \quad \rightarrow \quad \xi^\top Q\xi - \xi^\top Qc - c^\top Q\xi + c^\top Qc - 1 = 0. \tag{4}$$

Now we can define a scalar constant $e = c^\top Qc - 1$ and a vector $d = -Qc$. We can see that $-c^\top Q = d^\top$, since $-c^\top Q = -c^\top Q^\top = (-Qc)^\top$. Now we can rewrite equation (4) in terms of Q, d and e:

$$\xi^\top Q\xi + \xi^\top d + d^\top \xi + e = 0. \tag{5}$$

We can rewrite this into matrix form:

$$\begin{bmatrix} \xi^\top & 1 \end{bmatrix} \begin{bmatrix} Q\xi + d \\ d^\top \xi + e \end{bmatrix} = \begin{bmatrix} \xi^\top & 1 \end{bmatrix} \begin{bmatrix} Q & d \\ d^\top & e \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} = 0. \tag{6}$$

Thus finally:

$$\mathcal{E} = \left\{ \xi \in \mathbb{R}^2 \mid \begin{bmatrix} \xi \\ 1 \end{bmatrix}^\top \begin{bmatrix} Q & d \\ d^\top & e \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} = 0 \right\}. \tag{7}$$

The conditions for the variables Q, d and e can also be determined:
Since $P \succ 0$ and $r^2 > 0$, and $Q = \frac{1}{r^2}P$, the conditions for Q are: $Q \in \mathbb{R}^{2\times 2}$, $Q = Q^\top$ and $Q \succ 0$. Both P and Q are symmetric positive definite.
Since $d = -Qc$ and $c \in \mathbb{R}^2$, the product $-Qc$ can be any real vector. Thus the condition for d is: $d \in \mathbb{R}^2$.
The constant e is defined as $e = c^\top Qc - 1$. Given that $c = -Q^{-1}d$, we can write e in terms of Q and d: $e = (Q^{-1}d)^\top QQ^{-1}d - 1$. Thus the conditions for e are: $e = d^\top Q^{-1}d - 1$.

To uniquely define an ellipse of this form, we should define 5 variables, 3 variables in the matrix Q (as $Q \in \mathbb{R}^{2\times 2}$ and $Q = Q^\top$), 2 variables in the vector d (since $d \in \mathbb{R}^2$), and we don't need to define e, as e follows from Q and d.

2.2 Question 2

The right side of equation (1) can be written as:

$$\frac{1}{r^2}(\xi - c)^\top P(\xi - c) = 1 = z^\top z, ||z|| = 1, \tag{8}$$

where z is a vector on the unit circle and $P = P^\top$. Since we defined P to be symmetric and we know $P \succ 0$, we can do a Cholesky decomposition: there exists a full rank lower triangular matrix $L^\top \in \mathbb{R}^{2\times 2}$ such that $L^\top L = P$. To find the parameters of L, first we write out P in terms of L:

$$P = L^\top L = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{12} & l_{22} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{12} \\ l_{11}l_{12} & l_{12}^2 + l_{22}^2 \end{bmatrix}. \tag{9}$$

From this it can be seen that all elements of L can be expressed in terms of P, as there are 3 equations and 3 unknowns:

$$\begin{aligned} p_{11} = l_{11}^2 \quad &\rightarrow \quad l_{11} = \sqrt{p_{11}} \\ p_{12} = l_{11}l_{12} \quad &\rightarrow \quad l_{12} = \frac{p_{12}}{\sqrt{p_{11}}} \\ p_{22} = l_{12}^2 + l_{22}^2 \quad &\rightarrow \quad l_{22} = \sqrt{\frac{p_{11}p_{22} - p_{12}^2}{p_{11}}}. \end{aligned} \tag{10}$$

We can now write equation (8) as:

$$\frac{1}{r^2}(\xi - c)^\top L^\top L(\xi - c) = (\frac{1}{r}L(\xi - c))^\top (\frac{1}{r}L(\xi - c)) = z^\top z. \tag{11}$$

From this equation it can be seen that z can be expressed as:

$$z = \frac{1}{r}L(\xi - c). \tag{12}$$

Now we can write ξ in terms of z:

$$\xi = c + rL^{-1}z = e_0 + Ez, \tag{13}$$

where $e_0 = c$ and $E = rL^{-1}$ We can calculate L^{-1} using the general form of the inverse of a matrix:

$$L^{-1} = \begin{bmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{bmatrix}^{-1} = \frac{1}{|L|} \begin{bmatrix} l_{22} & -l_{12} \\ 0 & l_{11} \end{bmatrix} = \frac{1}{l_{11}l_{22}} \begin{bmatrix} l_{22} & -l_{12} \\ 0 & l_{11} \end{bmatrix}. \tag{14}$$

We can now express L^{-1} this in terms of P:

$$L^{-1} = \frac{1}{\sqrt{p_{11}p_{22} - p_{12}^2}} \begin{bmatrix} \sqrt{\frac{p_{11}p_{22}-p_{12}^2}{p_{11}}} & -\frac{p_{12}}{\sqrt{p_{11}}} \\ 0 & \sqrt{p_{11}} \end{bmatrix} = \begin{bmatrix} \sqrt{p_{11}}^{-1} & -\frac{p_{12}}{\sqrt{p_{11}(p_{11}p_{22}-p_{12}^2)}} \\ 0 & \sqrt{\frac{p_{11}}{p_{11}p_{22}-p_{12}^2}} \end{bmatrix}. \tag{15}$$

Thus E is described by:

$$E = rL^{-1} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = r \begin{bmatrix} \sqrt{p_{11}}^{-1} & -\frac{p_{12}}{\sqrt{p_{11}(p_{11}p_{22}-p_{12}^2)}} \\ 0 & \sqrt{\frac{p_{11}}{p_{11}p_{22}-p_{12}^2}} \end{bmatrix} \tag{16}$$

Since $r > 0$ and $P \succ 0$, we know that $p_{11} > 0$ and $|P| = p_{11}p_{22} - p_{12}^2 > 0$. We can immediately see that $e_{11} > 0$, $e_{21} = 0$ and $e_{22} > 0$. For e_{12}, it can be seen that the denominator is in the range $(0, \infty)$. Since p_{12} can be either positive or negative, it can be concluded that e_{12} is in the range $(-\infty, \infty)$ or $e_{12} \in \mathbb{R}$.

To conclude this section. A linear transformation from the unit circle to any ellipse is described by:

$$\mathcal{E} = \{\xi \in \mathbb{R}^2 \mid \xi = e_0 + Ez, ||z|| = 1\}, \tag{17}$$

where $e_0 = c$ and $E = rL^{-1}$, where $L^\top L = P$. The condition for e_0 is $e_0 \in \mathbb{R}^2$. The conditions for E are: $E \in \mathbb{R}^{2\times 2}$ and where the elements of E have the additional conditions: $e_{11} > 0$, $e_{21} = 0$ and $e_{22} > 0$.

Feature extraction

2.3 Question 3

The image chosen as a basis is a clear image of the moon, as it is visually clear how the ellipse should be fitted. In Fig. 1 the image and its transformations can be seen. The feature vector is constructed by simply finding the coordinates in the image where the edge detection algorithm found an edge and normalizing it to the domain $\mathbb{X} = [-X/2,\ X/2]$ and $\mathbb{Y} = [-Y/2,\ Y/2]$. Where X and Y are the dimensions of the image, which in case of the image of the moon is $X \times Y = 220 \times 212$ pixels.

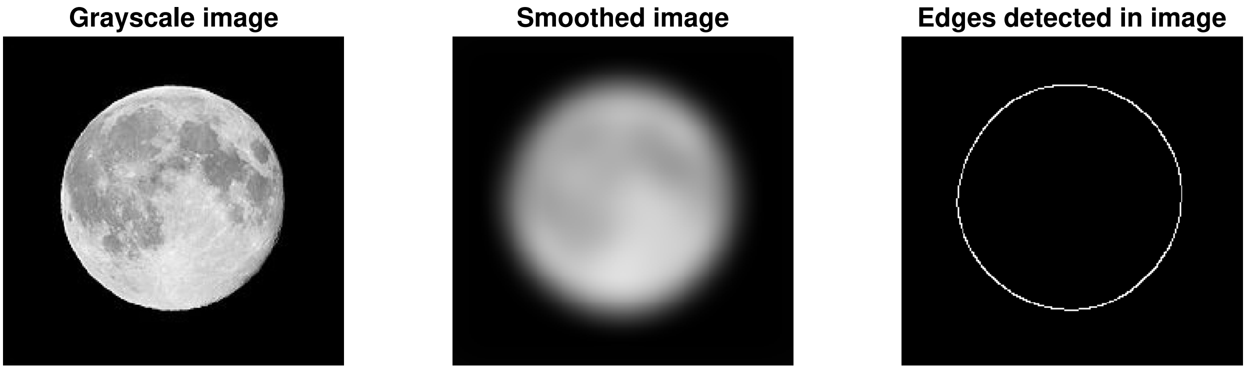

Figure 1: Image of the moon (220×212 pixels) first converted to gray scale (left), then smoothed using a Gaussian filter with $\sigma = 10$ (middle), and finally edge detected using the Canny edge detector with threshold=[0.4, 0.9] and $\sigma = 3$ (right).

Classification of ellipses

2.4 Question 4

Equation (1) describes the set of points on an ellipse. If $(\xi - c)^\top P(\xi - c) < r^2$, then ξ describes an interior point of an ellipse. If $(\xi - c)^\top P(\xi - c) > r^2$ then it describes a point outside of the ellipse. Thus an arbitrary point $\xi_0 \in \mathbb{R}^2$ can be described by:

$$r^2(1 + r_0) := (\xi_0 - c)^\top P(\xi_0 - c), \tag{18}$$

where $r_0 = 0$ if ξ_0 is on the ellipse, $r_0 < 0$ if ξ_0 is inside the ellipse and $r_0 > 0$ if ξ_0 is a point outside of the ellipse. Following the same step as in question 1, this equation can be rewritten as the roots of polynomial expression:

$$1 + r_0 := (\xi_0 - c)^\top Q(\xi_0 - c) = \xi_0^\top Q\xi_0 - \xi_0^\top Qc - c^\top Q\xi_0 + c^\top Qc = \xi_0^\top Q\xi_0 + \xi_0^\top d + d^\top \xi_0 + e + 1, \tag{19}$$

where $Q = Q^\top = \frac{1}{r^2}P$, $d = -Qc$ and $e = c^\top Qc - 1$. The equation can now be written into matrix form:

$$r_0 := \xi_0^\top Q\xi_0 + \xi_0^\top d + d^\top \xi_0 + e = \begin{bmatrix}\xi_0^\top & 1\end{bmatrix}\begin{bmatrix}Q\xi_0 + d \\ d^\top \xi_0 + e\end{bmatrix} = \begin{bmatrix}\xi_0^\top & 1\end{bmatrix}\begin{bmatrix}Q & d \\ d^\top & e\end{bmatrix}\begin{bmatrix}\xi_0 \\ 1\end{bmatrix}. \tag{20}$$

Thus it can be seen that:

$$r_0 := \begin{bmatrix}\xi_0^\top & 1\end{bmatrix}\begin{bmatrix}Q & d \\ d^\top & e\end{bmatrix}\begin{bmatrix}\xi_0 \\ 1\end{bmatrix}, \tag{21}$$

where $r_0 < 0$ describes a point inside of the ellipse, $r_0 > 0$ describes a point outside of the ellipse and $r_0 = 0$ describes a point on the ellipse itself.

2.5 Question 5

The minimum distance from an arbitrary point to an ellipse is described by:

$$d_{\text{near}} := \min_{\xi \in \mathcal{E}} ||\xi - \xi_0||. \tag{22}$$

To get the minimum distance we first need to find the optimal point ξ_{opt} on the ellipse, which is the point where the distance is minimum. It is the point where the gradient of the ellipse ($\nabla\mathcal{E}(\xi_{opt})$) is orthogonal to the vector $\overrightarrow{\xi_{\text{opt}}\xi_0}$. This means the distance is minimum if:

$$\nabla\mathcal{E}(\xi_{\text{opt}}) \cdot (\xi_{\text{opt}} - \xi_0) = 0. \tag{23}$$

The gradient of the ellipse describes the slope of the tangent at ξ_{opt}. So to find the gradient, we first find the tangent to the ellipse in the point ξ_{opt}. Let us consider the polynomial describing an ellipse as in in equation (5) and substitute in the point ξ_{opt}:

$$\xi_{\text{opt}}^\top Q\xi_{\text{opt}} + \xi_{\text{opt}}^\top d + d^\top \xi_{\text{opt}} + e = 0. \tag{24}$$

We can write out this equation in terms of the elements of the matrices:

$$\mathcal{E}(\xi_{\text{opt}}) = q_{11}\xi_x^2 + 2q_{12}\xi_x\xi_y + q_{22}\xi_y^2 + 2d_1\xi_x + 2d_2\xi_y + e = 0, \tag{25}$$

where $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ and $\xi_{\text{opt}} = \begin{bmatrix} \xi_x \\ \xi_y \end{bmatrix}$. Then we take the total derivative:

$$0 = \frac{\partial\mathcal{E}(\xi_{\text{opt}})}{\partial\xi_x} d\xi_x + \frac{\partial\mathcal{E}(\xi_{\text{opt}})}{\partial\xi_y} d\xi_y = \Big(2q_{11}\xi_x + 2q_{12}\xi_y + 2d_1\Big)d\xi_x + \Big(2q_{12}\xi_x + 2q_{22}\xi_y + 2d_2\Big)d\xi_y. \tag{26}$$

Now we can write it as the following which describes the slope of the tangent:

$$\frac{d\xi_y}{d\xi_x} = -\frac{q_{11}\xi_x + q_{12}\xi_y + d_1}{q_{12}\xi_x + q_{22}\xi_y + d_2}. \tag{27}$$

The tangent line is then described by:

$$y - \xi_y = -\frac{q_{11}\xi_x + q_{12}\xi_y + d_1}{q_{12}\xi_x + q_{22}\xi_y + d_2}(x - \xi_x), \tag{28}$$

where x and y are the variables of the tangent line that touches the ellipse in the point ξ_{opt}. We can rewrite the above equation to the following:

$$x(q_{11}\xi_x + q_{12}\xi_y + d_1) + y(q_{12}\xi_x + q_{22}\xi_y + d_2) = q_{11}\xi_x^2 + q_{12}\xi_x\xi_y + d_1\xi_x + q_{12}\xi_x\xi_y + q_{22}\xi_y^2 + d_2\xi_y. \tag{29}$$

The right side of the above equation is almost equal to equation (25). To make it equal we add $(d_1\xi_x+d_2\xi_y+e)$ to both sides of the equation, such that we can set the right side to 0:

$$(q_{11}\xi_x + q_{12}\xi_y)x + (q_{12}\xi_x + q_{22}\xi_y)y + d_1x + d_2y + d_1\xi_x + d_2\xi_y + e = 0. \tag{30}$$

It can now be written back into matrix form:

$$\begin{bmatrix} \xi_x & \xi_y \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \xi_x & \xi_y \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + e = 0. \tag{31}$$

Now we can substitute Q, d and ξ_{opt} back into the equation and get the final form of the equation that describes the tangent line:

$$(\xi_{\text{opt}}^\top Q + d^\top)h + \xi_{\text{opt}}^\top d + e = 0. \tag{32}$$

where $h = \begin{bmatrix} x & y \end{bmatrix}^\top$. The above function describes an affine function. An affine function f has the form $(\nabla f)^\top h + b = 0$, where b is a constant. Applying this definition to the tangent line of the ellipse in the point ξ_{opt}, we can see that the that gradient is described by:

$$\nabla\mathcal{E}(\xi_{\text{opt}}) = Q\xi_{\text{opt}} + d. \tag{33}$$

Thus finally we can get back to equation (23) and substitute in $\nabla\mathcal{E}(\xi_{\text{opt}})$:

$$(Q\xi_{\text{opt}} + d) \cdot (\xi_{\text{opt}} - \xi_0) = (Q\xi_{\text{opt}} + d)^\top(\xi_{\text{opt}} - \xi_0) = 0. \tag{34}$$

We can see that the solution to this polynomial has 2 roots, which is logical since there are 2 points on the ellipse where the gradient is the same. Given that ξ_{opt} is on the ellipse, we can write it as $\xi_{\text{opt}} = e_0 + Ez_{\text{opt}}$, $||z_{\text{opt}}|| = 1$, where z_{opt} is on the unit circle. We can also substitute in $d = -Qc$, where $c = e_0$. The equation simplifies to:

$$(QEz_{\text{opt}})^\top(Ez_{\text{opt}} + e_0 - \xi_0) = 0,\ ||z|| = 1. \tag{35}$$

Furthermore, we can write Q in terms of E as we know $E = rL^{-1}$, where $L^\top L = P$ and $P = r^2 Q$:

$$rI = LE \quad \rightarrow (LE)^\top (LE) = r^2 I \quad \rightarrow E^\top P E = r^2 I \quad \rightarrow E^\top Q E = I \quad \rightarrow Q = (E^\top)^{-1} E^{-1}, \tag{36}$$

where I is the identity matrix. Now we fill Q in terms of E into equation (35):

$$((E^\top)^{-1} z_{\text{opt}})^\top (E z_{\text{opt}} + e_0 - \xi_0) = 0, \; ||z|| = 1 \qquad \rightarrow \qquad z_{\text{opt}}^\top E^{-1} (E z_{\text{opt}} + e_0 - \xi_0) = 0, \; ||z|| = 1. \tag{37}$$

By definition we have that $z_{\text{opt}}^\top z_{\text{opt}} = 1$, thus we get:

$$1 + z_{\text{opt}}^\top E^{-1} (e_0 - \xi_0) = 0, \; ||z|| = 1. \tag{38}$$

We know that z_{opt} lies on the unit circle and can thus be expressed in terms of an angle θ_{opt}, such that $z_{\text{opt}} = \begin{bmatrix} \cos(\theta_{\text{opt}}) & \sin(\theta_{\text{opt}}) \end{bmatrix}^\top$, $\theta \in \mathbb{R}$, $0 \le \theta_{\text{opt}} < 2\pi$. We can solve for θ_{opt} using a root finding algorithm. There will be 2 solutions, since there are 2 points on the ellipse where the gradient is the same. Now that we have 2 optimal points to consider, we can get back to the equation of the minimum distance between an ellipse and an arbitrary point, as described in equation (22), and rewrite it in terms of z_{opt}, E and e_0:

$$d_{\text{near}} := \min_{\xi \in \mathcal{E}} ||\xi - \xi_0|| = \sqrt{(\xi_{\text{opt}} - \xi_0)^\top (\xi_{\text{opt}} - \xi_0)} = \sqrt{(E z_{\text{opt}} + e_0 - \xi_0)^\top (E z_{\text{opt}} + e_0 - \xi_0)}. \tag{39}$$

Plugging in the two possible z_{opt} we get two solutions and the one with the shortest distance is the shortest distance between an ellipse and an arbitrary point. With that we have a method of finding d_{near}.

2.6 Question 6

The residual distance is defined by $d_{res}(\xi_i, \mathcal{E}) = |r_0|$, where $r_0 = \xi_i^\top Q \xi_i + \xi_i^\top d + d^\top \xi_i + e$ as described by equation (20), where ξ_i is an arbitrary point. Now let us write out this matrix equation in terms of all matrix elements:

$$d_{res}(\xi_i, \mathcal{E}) = \left|\xi_i^\top Q \xi_i + \xi_i^\top d + d^\top \xi_i + e\right| = \left| \begin{bmatrix} \xi_{i,x} \\ \xi_{i,y} \end{bmatrix}^\top \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} \xi_{i,x} \\ \xi_{i,y} \end{bmatrix} + \begin{bmatrix} \xi_{i,x} \\ \xi_{i,y} \end{bmatrix}^\top \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}^\top \begin{bmatrix} \xi_{i,x} \\ \xi_{i,y} \end{bmatrix} + e \right|, \tag{40}$$

where $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, $\xi_i = \begin{bmatrix} \xi_{i,x} \\ \xi_{i,y} \end{bmatrix}$ and $\mathcal{E} = \{q_{11}, q_{12}, q_{22}, d_1, d_2\}$. The square of the residual distance to an arbitrary point is described by $d_{res}^2(\xi_i, \mathcal{E}) = |r_0|^2 = r_0^2$. We can write out equations (40) and describe $d_{res}^2(\xi_i, \mathcal{E})$ by:

$$d_{res}^2(\xi_i, \mathcal{E}) = \left(q_{11}\xi_{i,x}^2 + 2q_{12}\xi_{i,x}\xi_{i,y} + q_{22}\xi_{i,y}^2 + 2d_1\xi_{i,x} + 2d_2\xi_{i,y} + e\right)^2. \tag{41}$$

It can be seen that the part inside the square is linear in the parameters, and can be written as an affine matrix equation. Thus the squared residual distance to a single arbitrary point ξ_i is described by:

$$d_{res}^2(\xi_i, \mathcal{E}) = \left(\begin{bmatrix} \xi_{i,x}^2 & 2\xi_{i,x}\xi_{i,y} & \xi_{i,y}^2 & 2\xi_{i,x} & 2\xi_{i,y} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{22} & d_1 & d_2 \end{bmatrix}^\top + e \right)^2. \tag{42}$$

Now we do not consider a single arbitrary point, but the feature set $\mathcal{F} = \{\xi_1, \cdots, \xi_N\}$ that describes a set of edges detected in an image. We define a vector $D_{res}(\mathcal{F}, \mathcal{E})$ that contains all the residual distances to every point $\xi_i \in \mathcal{F}$. We can write $D_{res}(\mathcal{F}, \mathcal{E})$ in terms of $\mathcal{F}$ and the parameters of the ellipse $\mathcal{E}$ as follows:

$$D_{res}(\mathcal{F}, \mathcal{E}) = \left| \begin{bmatrix} \xi_{1,x}^2 & 2\xi_{1,x}\xi_{1,y} & \xi_{1,y}^2 & 2\xi_{1,x} & 2\xi_{1,y} \\ \xi_{2,x}^2 & 2\xi_{2,x}\xi_{2,y} & \xi_{2,y}^2 & 2\xi_{2,x} & 2\xi_{2,y} \\ \xi_{3,x}^2 & 2\xi_{3,x}\xi_{3,y} & \xi_{3,y}^2 & 2\xi_{3,x} & 2\xi_{3,y} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{N,x}^2 & 2\xi_{N,x}\xi_{N,y} & \xi_{N,y}^2 & 2\xi_{N,x} & 2\xi_{N,y} \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{22} \\ d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} e \\ e \\ e \\ \vdots \\ e \end{bmatrix} \right|, \tag{43}$$

where $\mathcal{F} = \{\xi_1, \cdots, \xi_N\} = \{\begin{bmatrix} \xi_{1,x} & \xi_{1,y} \end{bmatrix}^\top, \cdots, \begin{bmatrix} \xi_{N,x} & \xi_{N,y} \end{bmatrix}^\top\}$. We want to minimize the total squared residual distance to all points in the feature set. The total squared distance is described by $\sum_{i=1}^N d_{res}^2(\xi_i, \mathcal{E}) = ||D_{res}(\mathcal{F}, \mathcal{E})||^2$. The total squared distance $P_{tot}(\mathcal{E})$ can then be described as:

$$P_{tot}(\mathcal{E}) = \sum_{i=1}^N d_{res}^2(\xi_i, \mathcal{E}) = \left\| \begin{bmatrix} \xi_{1,x}^2 & 2\xi_{1,x}\xi_{1,y} & \xi_{1,y}^2 & 2\xi_{1,x} & 2\xi_{1,y} \\ \xi_{2,x}^2 & 2\xi_{2,x}\xi_{2,y} & \xi_{2,y}^2 & 2\xi_{2,x} & 2\xi_{2,y} \\ \xi_{3,x}^2 & 2\xi_{3,x}\xi_{3,y} & \xi_{3,y}^2 & 2\xi_{3,x} & 2\xi_{3,y} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{N,x}^2 & 2\xi_{N,x}\xi_{N,y} & \xi_{N,y}^2 & 2\xi_{N,x} & 2\xi_{N,y} \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{22} \\ d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} e \\ e \\ e \\ \vdots \\ e \end{bmatrix} \right\|^2. \tag{44}$$

It can be seen that if we try to minimize $P_{tot}(\mathcal{E})$ by finding the optimal parameter set $\mathcal{E}^*_{tot}$, it is a linear least squares problem, which is also convex optimization problem:

$$P_{tot}(\mathcal{E}^*_{tot}) = \min_{\mathcal{E}_{tot} \in \mathbb{R}^5} ||A\mathcal{E}_{tot} + e_N||^2, \tag{45}$$

where A is the $N \times 5$ matrix as described in equation (44), $\mathcal{E}_{tot} = \begin{bmatrix} q_{11} & q_{12} & q_{22} & d_1 & d_2 \end{bmatrix}^\top$ to be optimized and e_N is the $N \times 1$ matrix containing the elements e as described in equation (44). The constant e is a function of Q and d ($e = d^\top Q^{-1} d - 1$), which we do not know in advance. Thus we normalize over e, such that the constant vector is a vector of 1's. Now we can write out the optimization problem as:

$$-\frac{P_{tot}(\mathcal{E}^*_{tot})}{e} = \min_{x \in \mathbb{R}^5} ||Ax - b||^2, \tag{46}$$

where $x = -\frac{\mathcal{E}_{tot}}{e} = \begin{bmatrix} \frac{q_{11}}{-e} & \frac{q_{12}}{-e} & \frac{q_{22}}{-e} & \frac{d_1}{-e} & \frac{d_2}{-e} \end{bmatrix}^\top$ and $b = \frac{e_N}{e} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top$. Now we can solve for x using linear least squares. Given that the optimization problem is convex, we can find x by finding the point where the gradient $-\frac{P_{tot}(\mathcal{E})}{e}$ is 0. The gradient is described by:

$$-\nabla \frac{P_{tot}(\mathcal{E}_{tot})}{e} = 2A^\top(Ax - b). \tag{47}$$

The point where the gradient is 0 is the optimal point x^*. Thus we can solve for x^*:

$$-\nabla \frac{P_{tot}(\mathcal{E}^*_{tot})}{e} = 0 = 2A^\top(Ax^* - b) \quad \rightarrow \quad x* = (A^\top A)^{-1} A^\top b. \tag{48}$$

After solving for x^* we can solve for $\mathcal{E}^*_{tot}$, as we can express x^* in terms of Q^* and d^* we will have a system of 6 equations and 6 unknowns:

$$\begin{gathered} e^* = d^{*\top} Q^{*-1} d^* - 1 = \frac{q^*_{11} d^{*2}_2 + q^*_{22} d^{*2}_1 - 2q^*_{12} d^*_1 d^*_2}{q^*_{11} q^*_{22} - q^{*2}_{12}} - 1 \\ q^*_{11} = -e^* x^*_1 \qquad q^*_{12} = -e^* x^*_2 \qquad q^*_{22} = -e^* x^*_3 \qquad d^*_1 = -e^* x^*_4 \qquad d^*_2 = -e^* x^*_5, \end{gathered} \tag{49}$$

where $x^* = \begin{bmatrix} x^*_1 & x^*_2 & x^*_3 & x^*_4 & x^*_5 \end{bmatrix}^\top$ and $\mathcal{E}^*_{tot} = \begin{bmatrix} q^*_{11} & q^*_{12} & q^*_{22} & d^*_1 & d^*_2 \end{bmatrix}^\top$, the vector containing the optimal ellipse parameters. First we solve for e^* by substituting all other equations into the equation for e^*:

$$e^* = \frac{-x^*_1 x^{*2}_5 e^* - x^*_3 x^{*2}_4 e^* + 2x^*_2 x^*_4 x^*_5 e^*}{x^*_1 x^*_3 - x^{*2}_2} - 1 \quad \rightarrow \quad e^* = \frac{x^{*2}_2 - x^*_1 x^*_3}{x^*_1 x^*_3 - x^{*2}_2 + x^*_1 x^{*2}_5 + x^*_3 x^{*2}_4 - 2x^*_2 x^*_4 x^*_5}. \tag{50}$$

Now we solve for $\mathcal{E}^*_{tot}$ by substituting e^* in all the other equations:

$$\mathcal{E}^*_{tot} = -e^* \odot x^*, \tag{51}$$

where $\odot$ denotes element-wise multiplication. Now that we solved for the ellipse parameters we can solve for the total distance $P_{tot}(\mathcal{E}^*_{tot})$:

$$P_{tot}(\mathcal{E}^*_{tot}) = ||A\mathcal{E}^*_{tot} + e^*_N||^2, \tag{52}$$

where $e^*_N = \begin{bmatrix} e^* & \cdots & e^* \end{bmatrix}$. The results of the linear least squares method can be seen in Fig. 2. The total squared residual distance $P_{\text{tot}}(\mathcal{E}^*_{\text{tot}}$ from the ellipse to all detected edges is 0.113, 17.31 and 38.28 for the fits of the moon, eggplant and lizard respectively. It can be seen that the fit of the ellipse on the moon is very good. This makes sense since the moon is a sphere (approximately), of which an 2D-image is a circle, which is an ellipse. The fit of the ellipse on the eggplant is also quite good. It can be seen that the fit of the ellipse for the lizard is very different from the other 2. This because the image is small and the there are sparse measurements at the top of the lizard, where the fit is the worst. Thus the best fit happens to be a very large ellipse of which one side approximates all measurements.

2.7 Question 7

The residual distance is defined as before:

$$d_{res}(\xi_i, \mathcal{E}) = \left| \begin{bmatrix} \xi^2_{i,x} & 2\xi_{i,x}\xi_{i,y} & \xi^2_{i,y} & 2\xi_{i,x} & 2\xi_{i,y} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{22} & d_1 & d_2 \end{bmatrix}^\top + e \right|. \tag{53}$$

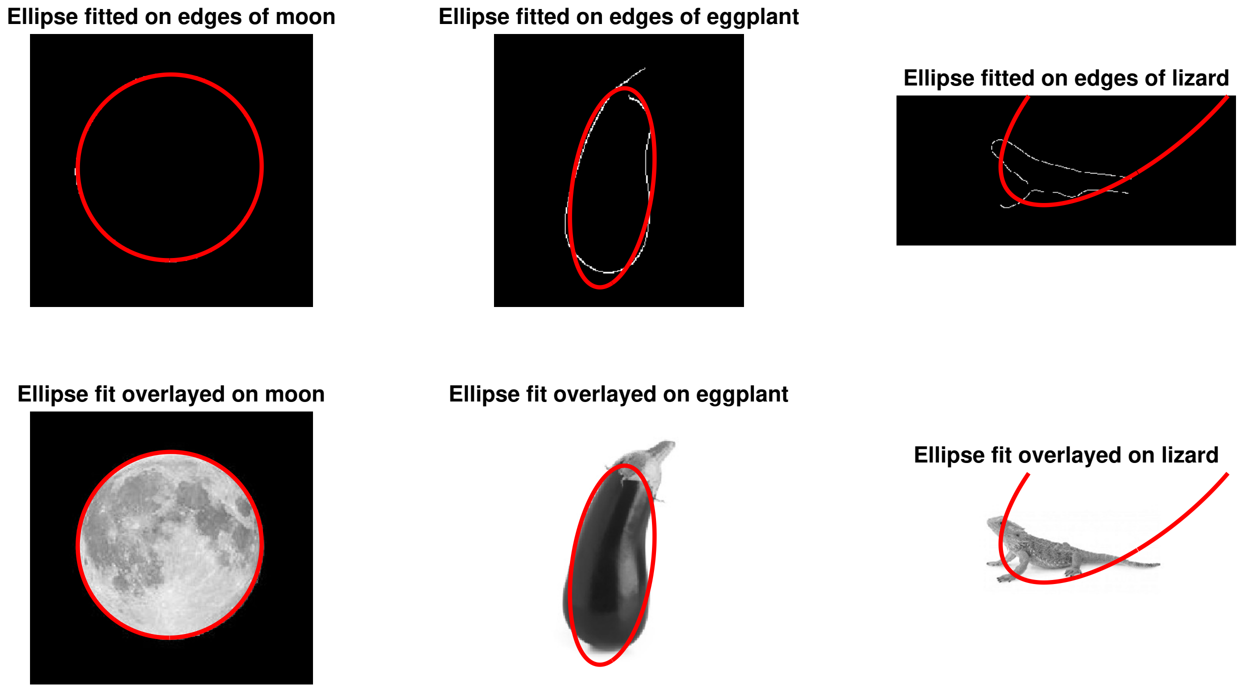

Figure 2: Ellipse fitted using linear least squares of the residual distance of the detected edges of 3 different images (moon (left), eggplant (middle) and lizard (right)). The ellipses are overlayed on the original gray scale images (bottom row).

Now we are interested in minimizing the maximum residual distance to a point ξ_i. The residual distance is an absolute value. Thus it is equal to the L-infinity norm. As a result we have:

$$P_{max}(\mathcal{E}) := \max_{\xi_i \in \mathcal{F}} d_{res}(\xi_i, \mathcal{E}) = \left\| \begin{bmatrix} d_{res}(\xi_1, \mathcal{E}) \\ d_{res}(\xi_2, \mathcal{E}) \\ d_{res}(\xi_3, \mathcal{E}) \\ \vdots \\ d_{res}(\xi_N, \mathcal{E}) \end{bmatrix} \right\|_\infty , \tag{54}$$

where $\xi_i \in \mathcal{F}$ and $\mathcal{F} = \{\xi_0, \xi_1\, \xi_2 \cdots \xi_N\}$ is the feature set. We can formulate this problem as a linear program, where we try to minimize the variable r with the constraint that r should be bigger or equal to $d_{\text{res}}(\xi_i, \mathcal{E})$ for all $\xi_i \in \mathcal{F}$:

$$\begin{array}{lll} \text{minimize} & r & \\ \text{subject to} & d_{res}(\xi_i, \mathcal{E}) \leq r, & i = 1, \dots, N \\ & D_{\text{res}}(\mathcal{F}, \mathcal{E}) = |A\mathcal{E}_{max} + e_N|, & \end{array} \tag{55}$$

where $D_{res}(\mathcal{F}, \mathcal{E})$, A, $\mathcal{E}_{max}$, and e_N are:

$$D_{res}(\mathcal{F}, \mathcal{E}) = \begin{bmatrix} d_{\text{res}}(\xi_1, \mathcal{E}) \\ d_{\text{res}}(\xi_2, \mathcal{E}) \\ d_{\text{res}}(\xi_3, \mathcal{E}) \\ \vdots \\ d_{\text{res}}(\xi_N, \mathcal{E}) \end{bmatrix} = |A\mathcal{E}_{tot} + e_N| = \left| \begin{bmatrix} \xi_{1,x}^2 & 2\xi_{1,x}\xi_{1,y} & \xi_{1,y}^2 & 2\xi_{1,x} & 2\xi_{1,y} \\ \xi_{2,x}^2 & 2\xi_{2,x}\xi_{2,y} & \xi_{2,y}^2 & 2\xi_{2,x} & 2\xi_{2,y} \\ \xi_{3,x}^2 & 2\xi_{3,x}\xi_{3,y} & \xi_{3,y}^2 & 2\xi_{3,x} & 2\xi_{3,y} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{N,x}^2 & 2\xi_{N,x}\xi_{N,y} & \xi_{N,y}^2 & 2\xi_{N,x} & 2\xi_{N,y} \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{22} \\ d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} e \\ e \\ e \\ \vdots \\ e \end{bmatrix} \right| . \tag{56}$$

This linear program cannot be solved as the variable e depends on Q and d, namely $e = d^\top Q^{-1} d - 1$. Thus we normalize over $-e$ to create the following linear program:

$$\begin{array}{lll} \text{minimize} & r & \\ \text{subject to} & \frac{d_{res}(\xi_i, \mathcal{E})}{-e} \leq r, & i = 1, \dots, N \\ & \frac{D_{\text{res}}(\mathcal{F}, \mathcal{E})}{-e} = |Ax - b|, & \end{array} \tag{57}$$

where $x = -\frac{\mathcal{E}_{max}}{e} = \begin{bmatrix} \frac{q_{11}}{-e} & \frac{q_{12}}{-e} & \frac{q_{22}}{-e} & \frac{d_1}{-e} & \frac{d_2}{-e} \end{bmatrix}^\top$ and $b = \frac{e_N}{e} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top$. As d_{res} is an absolute value. It is not linear, as there is a discontinuity in the origin. To make it linear. We define 2 new non-negative variables to make the problem linear:

$$\begin{aligned} D_{\text{res}}(\mathcal{F},\mathcal{E}) &= D^{+}_{\text{res}}(\mathcal{F},\mathcal{E}) + D^{-}_{\text{res}}(\mathcal{F},\mathcal{E}) \\ Ax - b &= D^{+}_{\text{res}}(\mathcal{F},\mathcal{E}) - D^{-}_{\text{res}}(\mathcal{F},\mathcal{E}), \end{aligned} \tag{58}$$

where $D^{+}_{\text{res}}(\mathcal{F},\mathcal{E}) = \begin{bmatrix} d^{+}_{\text{res}}(\xi_1,\mathcal{E}) & \cdots & d^{+}_{\text{res}}(\xi_N,\mathcal{E}) \end{bmatrix}$ and $D^{-}_{\text{res}}(\mathcal{F},\mathcal{E}) = \begin{bmatrix} d^{-}_{\text{res}}(\xi_1,\mathcal{E}) & \cdots & d^{-}_{\text{res}}(\xi_N,\mathcal{E}) \end{bmatrix}$ The linear program now can be written as:

$$\begin{array}{lll} \text{minimize} & r & \\ \text{subject to} & \frac{d^{+}_{\text{res}}(\xi_i,\mathcal{E})+d^{-}_{\text{res}}(\xi_i,\mathcal{E})}{-e} \leq r, & i = 1,\dots,N \\ & \frac{D^{+}_{\text{res}}(\mathcal{F},\mathcal{E})-D^{-}_{\text{res}}(\mathcal{F},\mathcal{E})}{-e} = Ax - b, & \\ & r,\ d^{+}_{\text{res}}(\xi_i,\mathcal{E}),\ d^{-}_{\text{res}}(\xi_i,\mathcal{E}) \quad \geq 0 & \\ & r,\ d^{+}_{\text{res}}(\xi_i,\mathcal{E}),\ d^{-}_{\text{res}}(\xi_i,\mathcal{E}) \quad \in \mathbb{R}. & \end{array} \tag{59}$$

Since the objective function and constraint functions are all linear function, this problem is formulated as convex problem. This linear program can be solved. After solving for the optimal set of values x^*, we can solve for the optimal set of ellipse parameters $\mathcal{E}^*_{max}$ in the same way as was done for the linear least squares method. We express x^* in terms of Q^* and d^* and solve the system of 6 equations and 6 unknowns:

$$\begin{gathered} e^* = d^{*\top} Q^{*-1} d^* - 1 = \frac{q^*_{11} d^{*2}_2 + q^*_{22} d^{*2}_1 - 2 q^*_{12} d^*_1 d^*_2}{q^*_{11} q^*_{22} - q^{*2}_{12}} - 1 \\ q^*_{11} = -e^* x^*_1 \qquad q^*_{12} = -e^* x^*_2 \qquad q^*_{22} = -e^* x^*_3 \qquad d^*_1 = -e^* x^*_4 \qquad d^*_2 = -e^* x^*_5, \end{gathered} \tag{60}$$

where $x^* = \begin{bmatrix} x^*_1 & x^*_2 & x^*_3 & x5^*_4 & x^*_5 \end{bmatrix}^\top$ and $\mathcal{E}^*_{max} = \begin{bmatrix} q^*_{11} & q^*_{12} & q^*_{22} & d^*_1 & d^*_2 \end{bmatrix}^\top$, the vector containing the optimal ellipse parameters. First we solve for e^* by substituting all other equations into the equation for e^*:

$$e^* = \frac{-x^*_1 x^{*2}_5 e^* - x^*_3 x^{*2}_4 e^* + 2 x^*_2 x^*_4 x^*_5 e^*}{x^*_1 x^*_3 - x^{*2}_2} - 1 \quad \rightarrow \quad e^* = \frac{x^{*2}_2 - x^*_1 x^*_3}{x^*_1 x^*_3 - x^{*2}_2 + x^*_1 x^{*2}_5 + x^*_3 x^{*2}_4 - 2 x^*_2 x^*_4 x^*_5}. \tag{61}$$

Now we solve for $\mathcal{E}^*_{max}$ by substituting e^* in all the other equations:

$$\mathcal{E}^*_{max} = -e^* \odot x^*, \tag{62}$$

where $\odot$ denotes element-wise multiplication. The maximum distance can be easily found as the optimal value of r, namely r^* is bounded by the maximum distance, but normalized over $-e$. Thus the maximum distance is described by $P_{\text{max}}(\mathcal{E}^*_{\text{max}}) = -e^* r^*$. Another way the maximum distance can be found is from the optimal set of parameters E^*_{max}. The maximum distance is defined as

$$P_{max}(\mathcal{E}^*_{max}) = \max |A\mathcal{E}^*_{max} + e^*_N| = \max |A\mathcal{E}^*_{max}| + e^*, \tag{63}$$

where $e^*_N = \begin{bmatrix} e^* & \cdots & e^* \end{bmatrix}$. The results of the linear least squares method can be seen in Fig. 3. The maximum residual distance $P_{\text{max}}(\mathcal{E}^*_{\text{max}}$ from the ellipse to a detected edge is 0.036, 0.349 and 0.807 for the fits of the moon, eggplant and lizard respectively. These values are very small compared to pixel distances in the image. This is because the residual distance is not actually the distance from a point to the ellipse, but simply a parameter that we know is 0 if the point is on the ellipse, or close to 0 when it is close to the ellipse. It can be seen that the fit of the ellipse on the moon is very good again. The fit for the eggplant is different from before, slightly taller. The fit to the lizard is also very different from before, the ellipse is much smaller.

2.8 Question 8

The nearest distance d_{near} is defined as:

$$d_{\text{near}} = \min_{\xi \in \mathcal{E}} ||\xi - \xi_0|| = ||E z_{\text{opt}} + e_0 - \xi_0||, \tag{64}$$

where z_{opt} is the point on the unit circle corresponding to the point ξ on the ellipse that is closest to ξ_0. This point z_{opt} is described by an implicit expression:

$$1 + z^\top_{\text{opt}} E^{-1}(e_0 - \xi_0) = 0,\ ||z|| = 1. \tag{65}$$

We know that z_{opt} lies on the unit circle and can thus be expressed in terms of an angle θ_{opt}, such that $z_{\text{opt}} = \begin{bmatrix} \cos(\theta_{\text{opt}}) & \sin(\theta_{\text{opt}}) \end{bmatrix}^\top$, $\theta \in \mathbb{R}$, $0 \leq \theta_{\text{opt}} < 2\pi$. We can solve for θ_{opt} using a root finding algorithm.

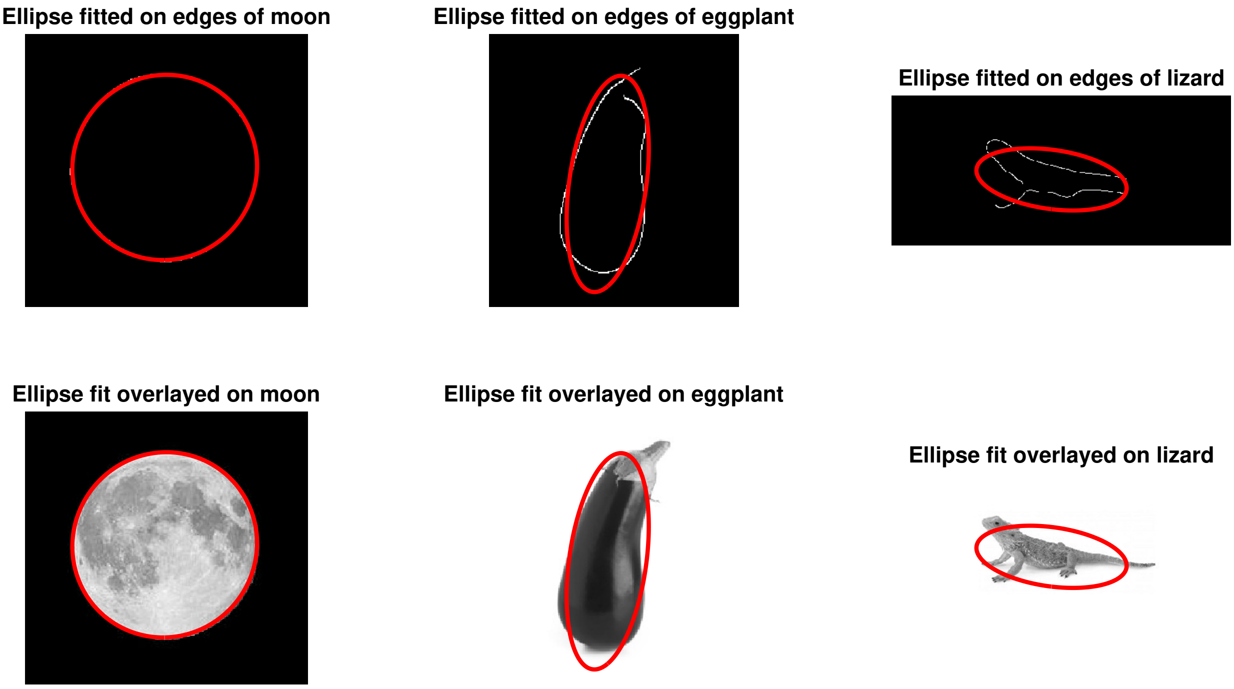

Figure 3: Ellipse fitted using minimizing maximum residual distance of the detected edges of 3 different images (moon (left), eggplant (middle) and lizard (right)) using a linear program. The ellipses are overlayed on the original gray scale images (bottom row).

It can be seen that this function is not convex, since there are 2 solutions. Although it is not convex, there are only 2 solutions and it is easy to check which one is the best one; the one that gives the shortest distance between ξ and ξ_0 Furthermore we have that the nearest distance squared is described by:

$$d^2_{\text{near}}(z_{\text{opt}}, \mathcal{E}) = ||Ez_{\text{opt}} + e_0 - \xi_0||^2 = (Ez_{\text{opt}} + e_0 - \xi_0)^\top (Ez_{\text{opt}} + e_0 - \xi_0). \tag{66}$$

We can expand this equation:

$$d^2_{\text{near}}(z_{\text{opt}}, \mathcal{E}) = z_{\text{opt}}^\top E^\top E z_{\text{opt}} + 2e_0^\top E z_{\text{opt}} + e_0^\top e_0 - 2\xi_0^\top e_0 - 2\xi_0^\top E z_{\text{opt}} + \xi_0^\top \xi_0. \tag{67}$$

It can be seen that this system is a non-linear least squares problem, which may or may not be convex. From looking at the problem of minimizing the sum of squared shortest distance to an ellipse, it appears this problem could be convex. A good way to solve this system would be to use an iterative method with a safety feature, such as Newton's trust region method. On every iteration we solve for z_{opt} for all measurements and subsequently do the Newton step. If the step is larger than a defined radius from the current estimate, then we calculate the lowest value of the objective function on the circle with that radius away from the current estimate. Then after doing a step we again compute z_{opt} for all measurements and again do a Newton step. We iterate until the Newton step is within the radius.

Clustering and classification

2.9 Question 9

As mentioned in Question 6, the summation of the square residual distance is $P_{tot}(\mathcal{E}) = \sum_{i=1}^{N} d_{res}^2(\xi_i, \mathcal{E})$ can be solved by the least square method, and the optimal point is $X^* = (A^T A)^{-1} A^T b$. So the original objective function can be written into:

$$\min_{\mathcal{E}-} \sum_{i=1}^{N-} \|A_{i-}(A_{i-}^T A_{i-}) A_{i-}^T B + B\|^2 + \min_{\mathcal{E}+} \sum_{i=1}^{N+} \|A_{i+}(A_{i+}^T A_{i+}) A_{i+}^T B + B\|^2 \tag{68}$$

We can set up the variable that the linear problem is:

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{N_-} \|A_{i-}(A_{i-}^T A_{i-}) A_{i-}^T B + B\|^2 + \sum_{i=1}^{N_+} \|A_{i+}(A_{i+}^T A_{i+}) A_{i+}^T B + B\|^2 & \\ \text{subject to} & a^T \xi_{i-} - b \leq 0 \qquad i = 1, \ldots, N^- & \\ & -a^T \xi_{i+} + b \leq 0 \qquad i = 1, \ldots, N^+ & \\ & a, \xi_{i+}, \xi_{i-} \qquad \in \mathbb{R}^2 & \\ & b \qquad \in \mathbb{R} & \end{array} \tag{69}$$

2.10 Question 10

For the supervised clustering method, we can set up a binary parameter to that show the feature set is above or below the linear function. We define that the Y is a binary variable. When $Y_i = 1$ meaning that the the feature ξ_i is above the linear function, which is on the H_+ plane. When $Y_i = 0$ meaning that the the feature ξ_i is below the linear function, which is on the H_- plane. For supervised clustering method, it involves visual inspection to label the feature set ξ_i. After we label all the feature set, we can build up the mix integer problem:

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{N_-} \|A_{i-}(A_{i-}^T A_{i-}) A_{i-}^T B + B\|^2 + \sum_{i=1}^{N_+} \|A_{i+}(A_{i+}^T A_{i+}) A_{i+}^T B + B\|^2 & \\ \text{subject to} & a^T \xi_{i-} - b \leq 0 \qquad i = 1, \ldots, N^- & \\ & -a^T \xi_{i+} + b \leq 0 \qquad i = 1, \ldots, N^+ & \\ & a, \xi_{i+}, \xi_{i-} \qquad \in \mathbb{R}^2 & \\ & b \qquad \in \mathbb{R} & \\ & Y_i \leq 1, \qquad i = 1, \ldots, N & \\ & Y_i \geq 0, \qquad i = 1, \ldots, N & \end{array} \tag{70}$$