Forecasting, Time Series data

Introduction

- forecast package

forecast package

- automatic forecasting algorithms

forecast package

automatic forecasting algorithms

of two types

- Exponential Smoothing methods
- ARIMA models

forecast package

automatic forecasting algorithms

to

- select the model
- estimate the parameters
- model accuracy measures
- predictions

Time Series methods and models

Time Series components

Any time series can be split into

- Trend
- Seasonal
- random (error)

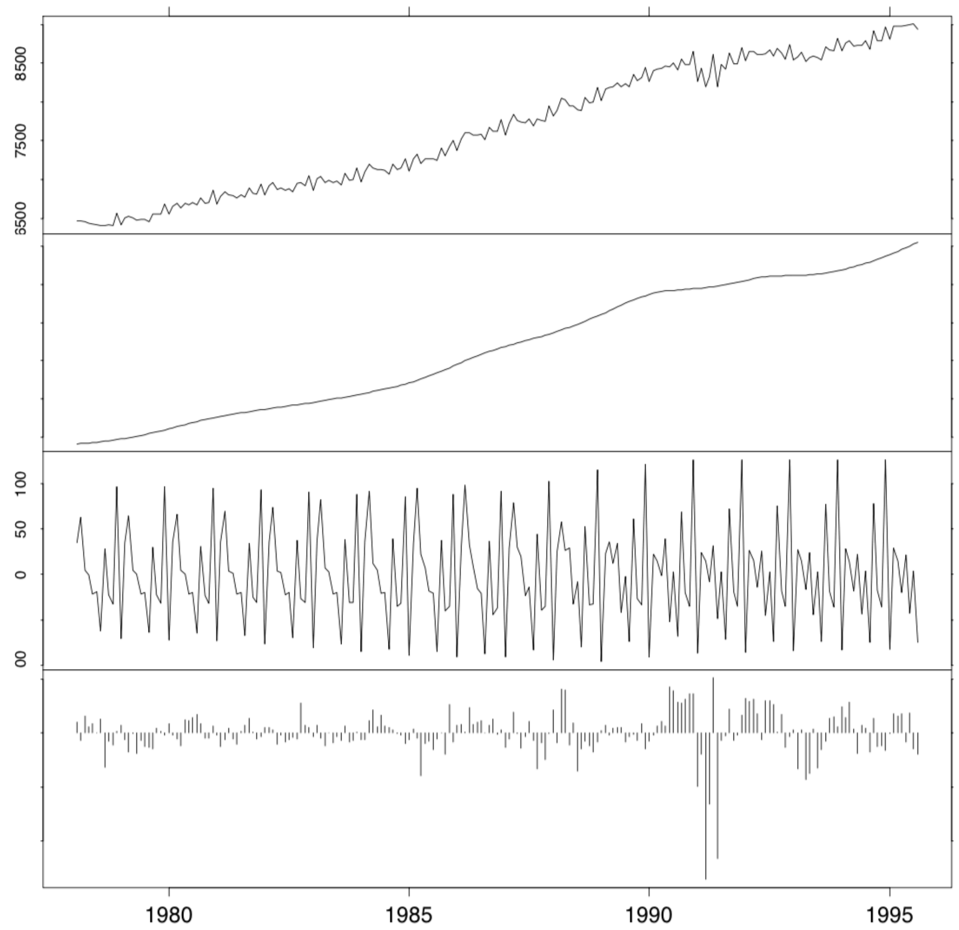
components

Time Series components

Any time series can be split into

- Trend
- Seasonal
- random (error)

components



Time Series components

Any time series can be split into

- Trend The long-term direction of the series
- Seasonal Pattern that repeats with known periodicity
- Cycle Pattern that repeats with regularity but unknown, changing periodicity
- random unpredictable component of the series

Time Series components

Trend and seasonal components can be

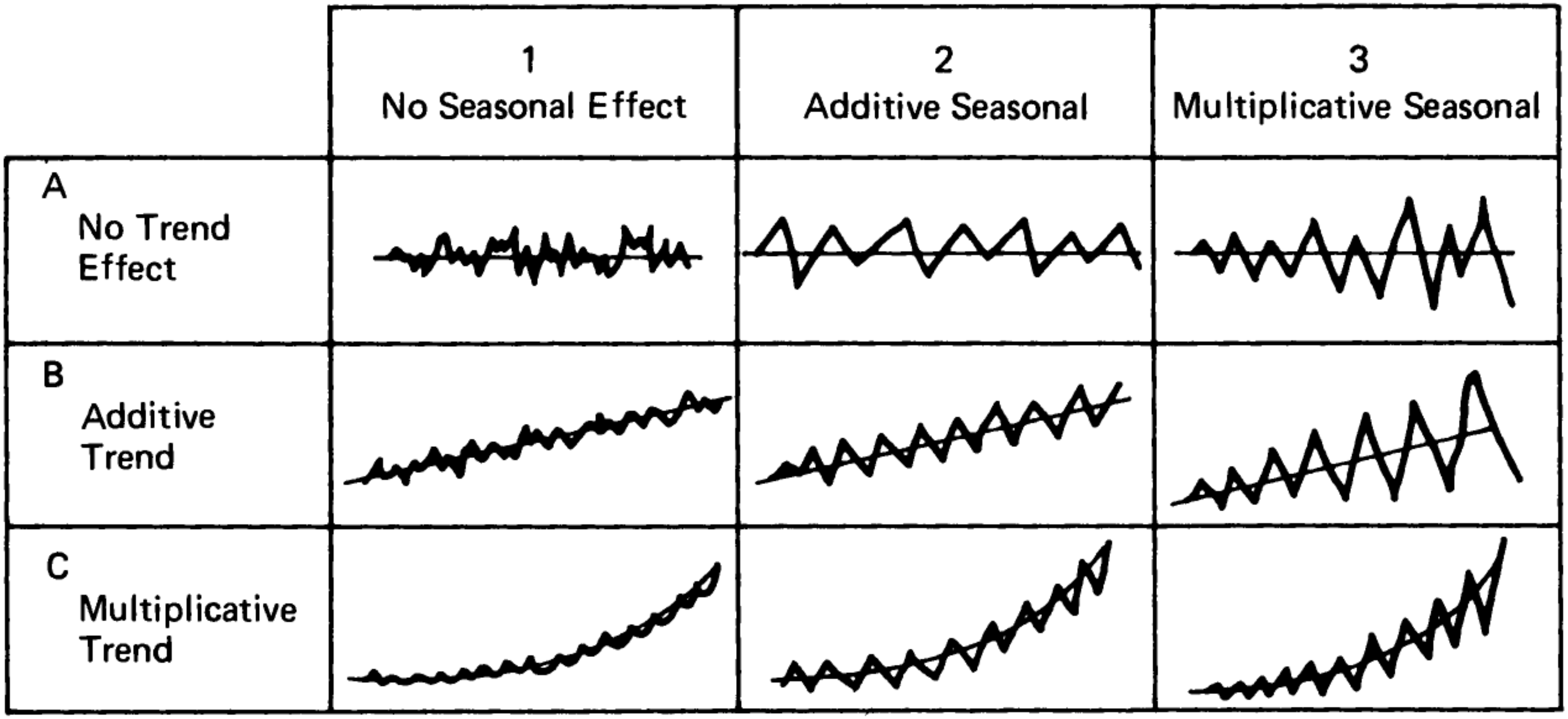
- Additive

  (consecutive differences are constant)

- Multiplicative

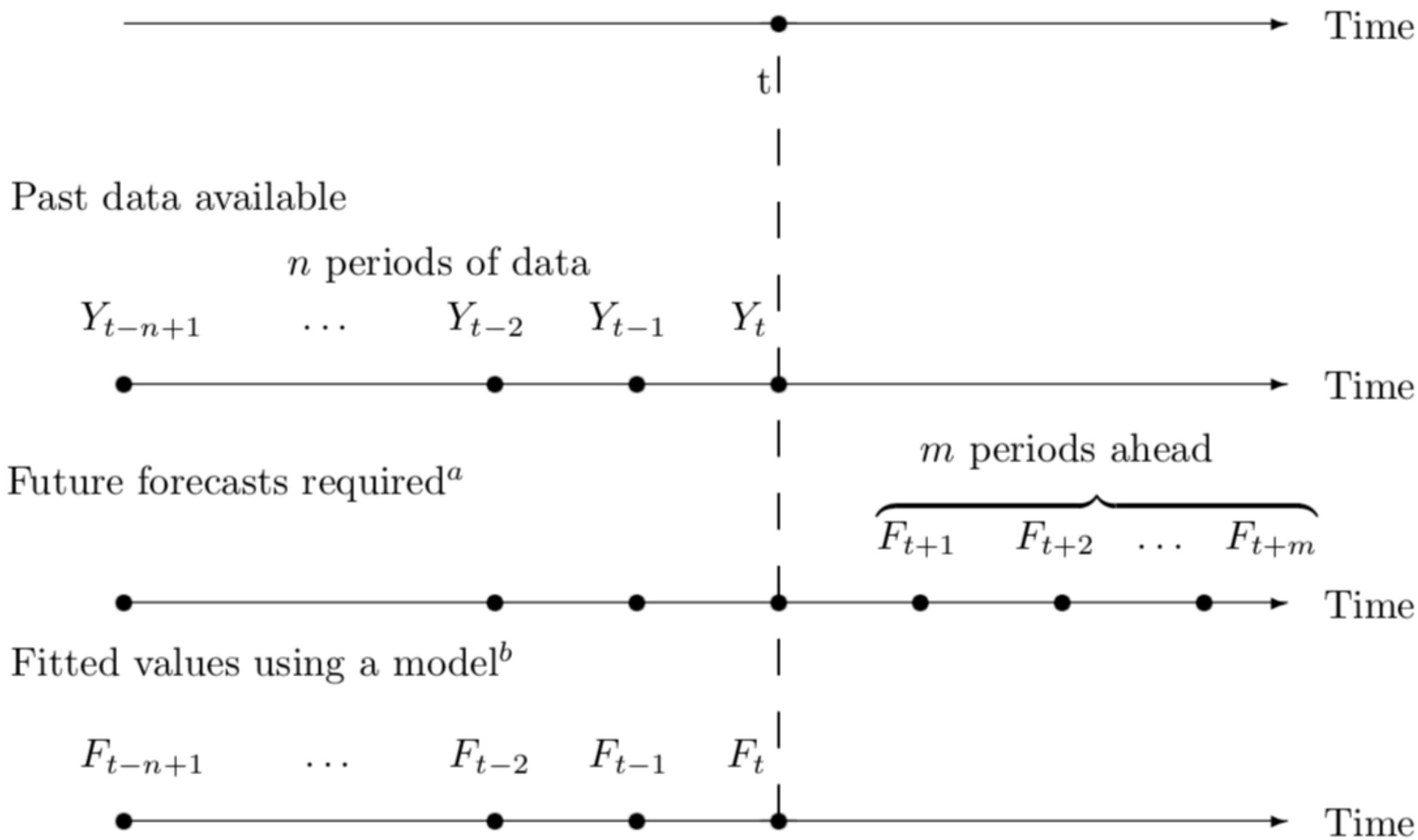
  (consecutive ratios are constant)

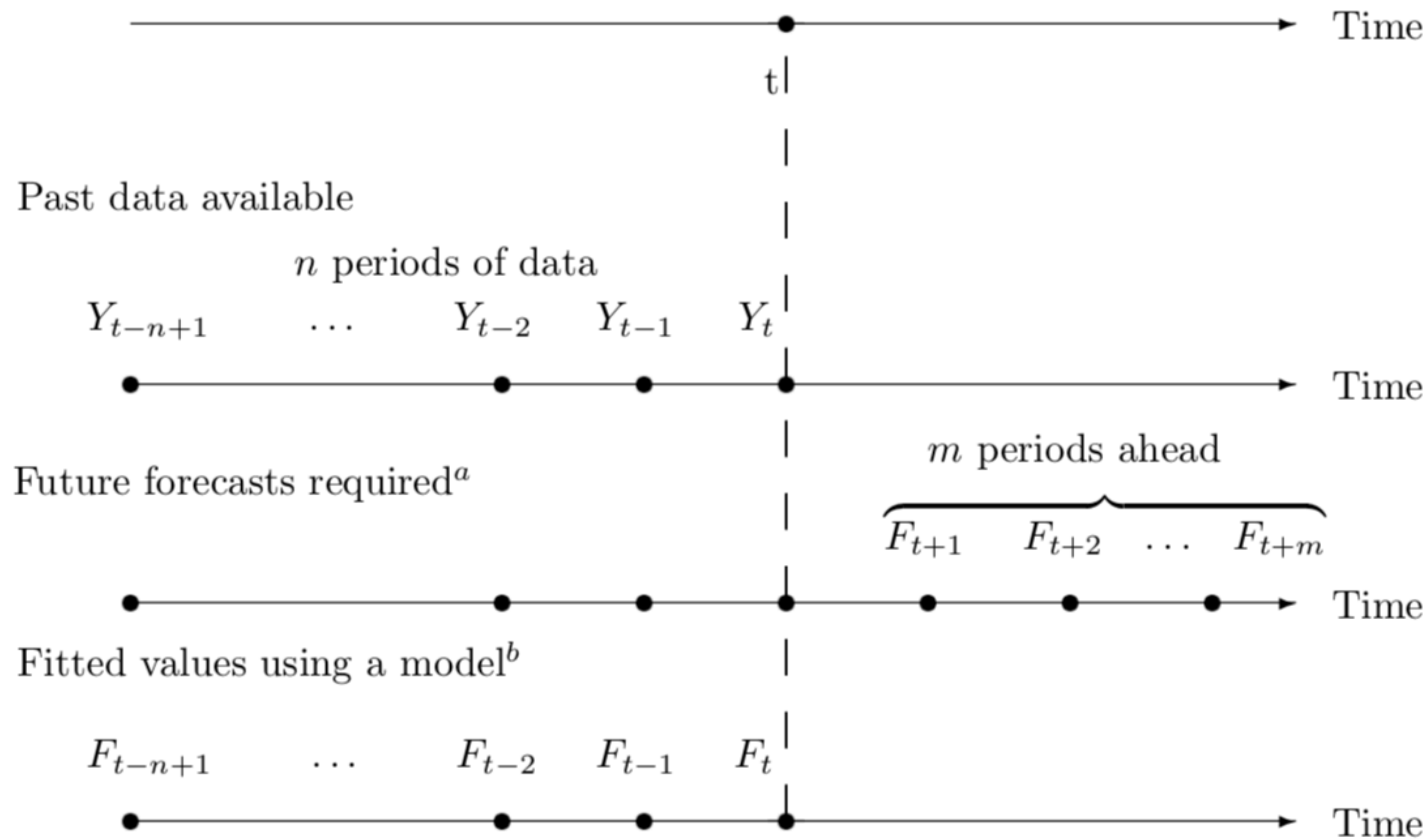
Time Series components



Time Series -notation

. x





Fitting errors

$$(Y_{t-n+1} - F_{t-n+1}), \dots, (Y_{t-1} - F_{t-1}), (Y_t - F_t)$$

Forecasting Errors (when Y_{t+1}, Y_{t+2}, etc., become available)

$$(Y_{t+1} - F_{t+1}), (Y_{t+2} - F_{t+2}), \dots$$

Time Series -errors

$$e_t = Y_t - F_t.$$

. x

$$\text{ME} = \frac{1}{n}\sum_{t=1}^{n} e_t$$ mean error

$$\text{MAE} = \frac{1}{n}\sum_{t=1}^{n} |e_t|$$ mean absolute error

$$\text{MSE} = \frac{1}{n}\sum_{t=1}^{n} e_t^2.$$ mean squared error

$$\text{PE}_t = \left(\frac{Y_t - F_t}{Y_t}\right) \times 100.$$

$$\text{MPE} = \frac{1}{n}\sum_{t=1}^{n} \text{PE}_t$$

$$\text{MAPE} = \frac{1}{n}\sum_{t=1}^{n} |\text{PE}_t|$$

Time Series -errors $e_t = Y_t - F_t.$

Period t	Observation Y_t	Forecast F_t	Error $Y_t - F_t$	Absolute Error $\lvert Y_t - F_t\rvert$	Squared Error $(Y_t - F_t)^2$
1	138	150.25	−12.25	12.25	150.06
2	136	139.50	−3.50	3.50	12.25
3	152	157.25	−5.25	5.25	27.56
4	127	143.50	−16.50	16.50	272.25
5	151	138.00	13.00	13.00	169.00
6	130	127.50	2.50	2.50	6.25
7	119	138.25	−19.25	19.25	370.56
8	153	141.50	11.50	11.50	132.25
Total			−29.75	83.75	1140.20

$$\begin{aligned}
\text{ME} &= -29.75/8 = -3.72\\
\text{MAE} &= 83.75/8 = 10.47\\
\text{MSE} &= 1140.20/8 = 142.52
\end{aligned}$$

Time Series -errors

$$e_t = Y_t - F_t.$$

Period t	Observation Y_t	Forecast F_t	Error $Y_t - F_t$	Percent Error $\left(\frac{Y_t - F_t}{Y_t}\right) 100$	Absolute Percent Error $\left|\frac{Y_t - F_t}{Y_t}\right| 100$
1	138	150.25	−12.25	−8.9	8.9
2	136	139.50	−3.50	−2.6	2.6
3	152	157.25	−5.25	−3.5	3.5
4	127	143.50	−16.50	−13.0	13.0
5	151	138.00	13.00	8.6	8.6
6	130	127.50	2.50	1.9	1.9
7	119	138.25	−19.25	−16.2	16.2
8	153	141.50	11.50	7.5	7.5
Total				−26.0	62.1

$$\text{MPE} = -26.0/8 = -3.3\%$$

$$\text{MAPE} = 62.1/8 = 7.8\%$$

Time Series -methods

Averaging	Exponential smoothing
Simple average	Single exponential smoothing
Moving average	Holt's linear method
	Holt-Winter's method

Time Series –moving average

x

Month	Time period	Observed values (shipments)	Three-month moving average	Five-month moving average
Jan	1	200.0	—	—
Feb	2	135.0	—	—
Mar	3	195.0	—	—
Apr	4	197.5	176.7	—
May	5	310.0	175.8	—
Jun	6	175.0	234.2	207.5
Jul	7	155.0	227.5	202.5
Aug	8	130.0	213.3	206.5
Sep	9	220.0	153.3	193.5
Oct	10	277.5	168.3	198.0
Nov	11	235.0	209.2	191.4
Dec	12	—	244.2	203.5

Analysis of errors

Test periods:	4–11	6–11
Mean Error (ME)	17.71	–1.17
Mean Absolute Error (MAE)	71.46	51.00
Mean Absolute Percentage Error (MAPE)	34.89	27.88
Mean Square Error (MSE)	6395.66	3013.25
Theil's U-statistic	1.15	0.81

Time Series –exponential smoothing

Adjusts the new forecast using the previous error

$$e_t = Y_t - F_t.$$

$$F_{t+1} = F_t + \alpha(Y_t - F_t)$$

equivalent to

$$F_{t+1} = \alpha Y_t + (1 - \alpha)F_t$$

a weighted average of
most recent observation and forecast

Time Series –exponential smoothing

Replacing terms

$$F_{t+1} = \alpha Y_t + (1-\alpha)F_t$$

$$\begin{aligned} &= \alpha Y_t + (1-\alpha)[\alpha Y_{t-1} + (1-\alpha)F_{t-1}] \\ &= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + (1-\alpha)^2 F_{t-1}. \end{aligned}$$

$$\begin{aligned} = \quad & \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \alpha(1-\alpha)^3 Y_{t-3} \\ & + \alpha(1-\alpha)^4 Y_{t-4} + \alpha(1-\alpha)^5 Y_{t-5} + \cdots + \alpha(1-\alpha)^{t-1} Y_1 \\ & + (1-\alpha)^t F_1. \end{aligned}$$

a weighted moving average with decreasing weights

Time Series – Holt linear method

$$F_{t+m} = L_t + b_t m$$

$$\begin{aligned} L_t &= \alpha Y_t + (1-\alpha)(L_{t-1} + b_{t-1}) \\ b_t &= \beta(L_t - L_{t-1}) + (1-\beta) b_{t-1} \end{aligned}$$

Second expression shows slope estimate b(t). It is an average of last slope and current slope b(t-1)

First expression is the updated level L(t) of the time series. Last level L(t-1) is adjusted by adding last trend b(t-1) estimate.

Parameters needed are α, β, and initial values L0, b0

Time Series

- x

Year	Quarter	Occupancy Rate
2009	1	.561
	2	.702
	3	.800
	4	.568
2010	1	.575
	2	.738
	3	.868
	4	.605
2011	1	.594
	2	.738
	3	.729
	4	.600
2012	1	.622
	2	.708
	3	.806
	4	.632
2013	1	.665
	2	.835
	3	.873
	4	.670

Time Series – Seasonal indices

- x

Year	Quarter	t	y_t	$\hat{y} = .639368 + .005246t$	Ratio $\dfrac{y_t}{\hat{y}_t}$
2009	1	1	.561	.645	.870
	2	2	.702	.650	1.080
	3	3	.800	.655	1.221
	4	4	.568	.660	.860
2010	1	5	.575	.666	.864
	2	6	.738	.671	1.100
	3	7	.868	.676	1.284
	4	8	.605	.681	.888
2011	1	9	.594	.687	.865
	2	10	.738	.692	1.067
	3	11	.729	.697	1.046
	4	12	.600	.702	.854
2012	1	13	.622	.708	.879
	2	14	.708	.713	.993
	3	15	.806	.718	1.122
	4	16	.632	.723	.874
2013	1	17	.665	.729	.913
	2	18	.835	.734	1.138
	3	19	.873	.739	1.181
	4	20	.670	.744	.900

Time Series – Seasonal indices

- x

Year	Quarter	t	y_t	$\hat{y} = .639368 + .005246t$	Ratio $\frac{y_t}{\hat{y}_t}$
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	3	19	.873	.739	1.181
	4	20	.670	.744	.900

Time Series – Seasonal indices

- x

	Quarter			
Year	**1**	**2**	**3**	**4**
2009	.870	1.080	1.221	.860
2010	.864	1.100	1.284	.888
2011	.865	1.067	1.046	.854
2012	.879	.993	1.122	.874
2013	.913	1.138	1.181	.900
Seasonal Index	.878	1.076	1.171	.875

Time Series – Seasonal indices

- x

Year	Quarter	t	y_t	$\hat{y} = .639368 + .005246t$	Ratio $\frac{y_t}{\hat{y}_t}$
2009	1	1	.561	.645	.870
	2	2	.702	.650	1.080
	3	3	.800	.655	1.221
	4	4	.568	.660	.860
2010	1	5	.575	.666	.864
	2	6	.738	.671	1.100
	3	7	.868	.676	1.284
	4	8	.605	.681	.888
2011	1	9	.594	.687	.865
	2	10	.738	.692	1.067
	3	11	.729	.697	1.046
	4	12	.600	.702	.854
2012	1	13	.622	.708	.879
	2	14	.708	.713	.993
	3	15	.806	.718	1.122
	4	16	.632	.723	.874
2013	1	17	.665	.729	.913
	2	18	.835	.734	1.138
	3	19	.873	.739	1.181
	4	20	.670	.744	.900

Time Series – Seasonal indices, prediction

forecast trend values

Trend Values

Quarter	t	$\hat{y} = .639 + .00525t$
1	21	$.639 + .00525(21) = .749$
2	22	$.639 + .00525(22) = .755$
3	23	$.639 + .00525(23) = .760$
4	24	$.639 + .00525(24) = .765$

seasonalized forecasts

Quarter	t	Trend Value $\hat{y}_t$	Seasonal Index	Forecast $F_t = \hat{y}_t \times SI_t$
1	21	.749	.878	$.749 \times .878 = .658$
2	22	.755	1.076	$.755 \times 1.076 = .812$
3	23	.760	1.171	$.760 \times 1.171 = .890$
4	24	.765	.875	$.765 \times .875 = .670$

Time Series – Deseasonalizing

- To remove seasonal variation from a time series
- Series with no seasonal component is called *seasonally adjusted*
- It is found by removing the seasonal component, leaving only the trend and the error components

Time Series – Deseasonalizing

- Divide the time series by seasonal indexes

Year	Quarter	Occupancy Rate y_t	Seasonal Index	Seasonally adjusted time series
2009	1	.561	.878	.639
	2	.702	1.076	.652
	3	.800	1.171	.683
	4	.568	.875	.649
2010	1	.575	.878	.655
	2	.738	1.076	.686
	3	.868	1.171	.741
	4	.605	.875	.691
2011	1	.594	.878	.677
	2	.738	1.076	.686
	3	.729	1.171	.623
	4	.600	.875	.686
2012	1	.622	.878	.708
	2	.708	1.076	.658
	3	.806	1.171	.688
	4	.632	.875	.722
2013	1	.665	.878	.757
	2	.835	1.076	.776
	3	.873	1.171	.746
	4	.670	.875	.766

Time Series – Holt Winter's multiplicative

- For series with seasonal component S(t) with period *s*

$$\text{Forecast:} \qquad F_{t+m} = (L_t + b_t m) S_{t-s+m}$$

$$\begin{aligned}
\text{Level:} \qquad L_t &= \alpha \frac{Y_t}{S_{t-s}} + (1-\alpha)(L_{t-1} + b_{t-1}) \\
\text{Trend:} \qquad b_t &= \beta(L_t - L_{t-1}) + (1-\beta) b_{t-1} \\
\text{Seasonal:} \qquad S_t &= \gamma \frac{Y_t}{L_t} + (1-\gamma) S_{t-s}
\end{aligned}$$

Time Series – Holt Winter's multiplicative

- For series with seasonal component S(t) with period *s*

$$\text{Forecast:} \quad F_{t+m} = (L_t + b_t m) S_{t-s+m}$$

$$\text{Level:} \quad L_t = \alpha \frac{Y_t}{S_{t-s}} + (1-\alpha)(L_{t-1} + b_{t-1})$$

$$\text{Trend:} \quad b_t = \beta(L_t - L_{t-1}) + (1-\beta) b_{t-1}$$

$$\text{Seasonal:} \quad S_t = \gamma \frac{Y_t}{L_t} + (1-\gamma) S_{t-s}$$

Seasonal index
at time t

Time Series – Holt Winter's multiplicative

- For series with seasonal component S(t) with period *s*

$$\text{Forecast:} \qquad F_{t+m} = (L_t + b_t m) S_{t-s+m}$$

Deseasonalize series

$$\text{Level:} \qquad L_t = \alpha \frac{Y_t}{S_{t-s}} + (1-\alpha)(L_{t-1} + b_{t-1})$$

$$\text{Trend:} \qquad b_t = \beta(L_t - L_{t-1}) + (1-\beta) b_{t-1}$$

$$\text{Seasonal:} \qquad S_t = \gamma \frac{Y_t}{L_t} + (1-\gamma) S_{t-s}$$

Time Series – Holt Winter's multiplicative

For series with seasonal component S(t) with period *s*

$$\text{Forecast:} \qquad F_{t+m} = (L_t + b_t m) S_{t-s+m}$$

$$\begin{aligned}
\text{Level:} \qquad L_t &= \alpha \frac{Y_t}{S_{t-s}} + (1-\alpha)(L_{t-1} + b_{t-1}) \\
\text{Trend:} \qquad b_t &= \beta(L_t - L_{t-1}) + (1-\beta) b_{t-1} \\
\text{Seasonal:} \qquad S_t &= \gamma \frac{Y_t}{L_t} + (1-\gamma) S_{t-s}
\end{aligned}$$

Parameters are α, β, γ, and initial values, L_s, b_s, $S_1, \ldots, S_s$

Time Series – Holt Winter's additive

$$\text{Forecast:} \quad F_{t+m} = L_t + b_t m + S_{t-s+m}$$

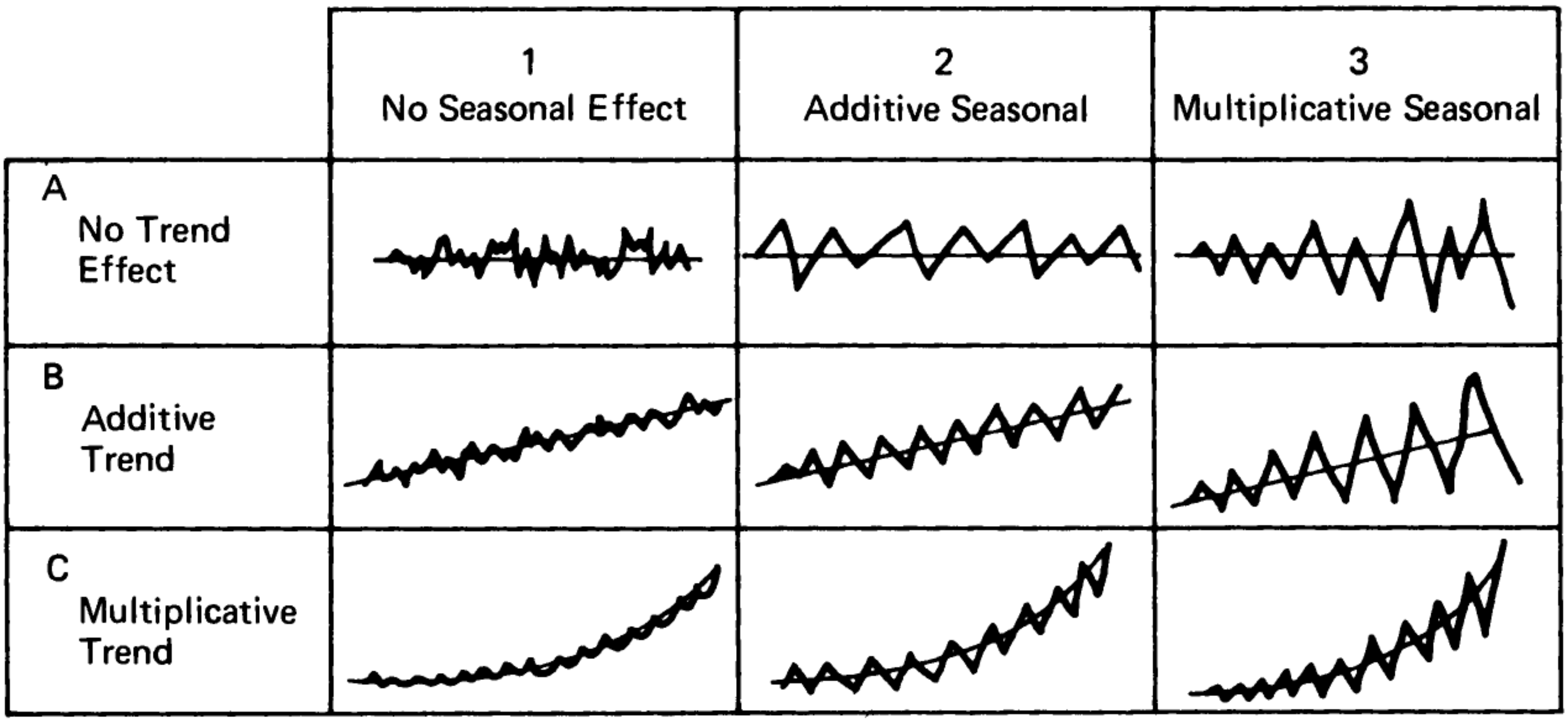
$$\begin{aligned}
\text{Level:} \quad L_t &= \alpha(Y_t - S_{t-s}) + (1-\alpha)(L_{t-1} + b_{t-1}) \\
\text{Trend:} \quad b_t &= \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1} \\
\text{Seasonal:} \quad S_t &= \gamma(Y_t - L_t) + (1-\gamma)S_{t-s}
\end{aligned}$$

Exponential Smoothing methods

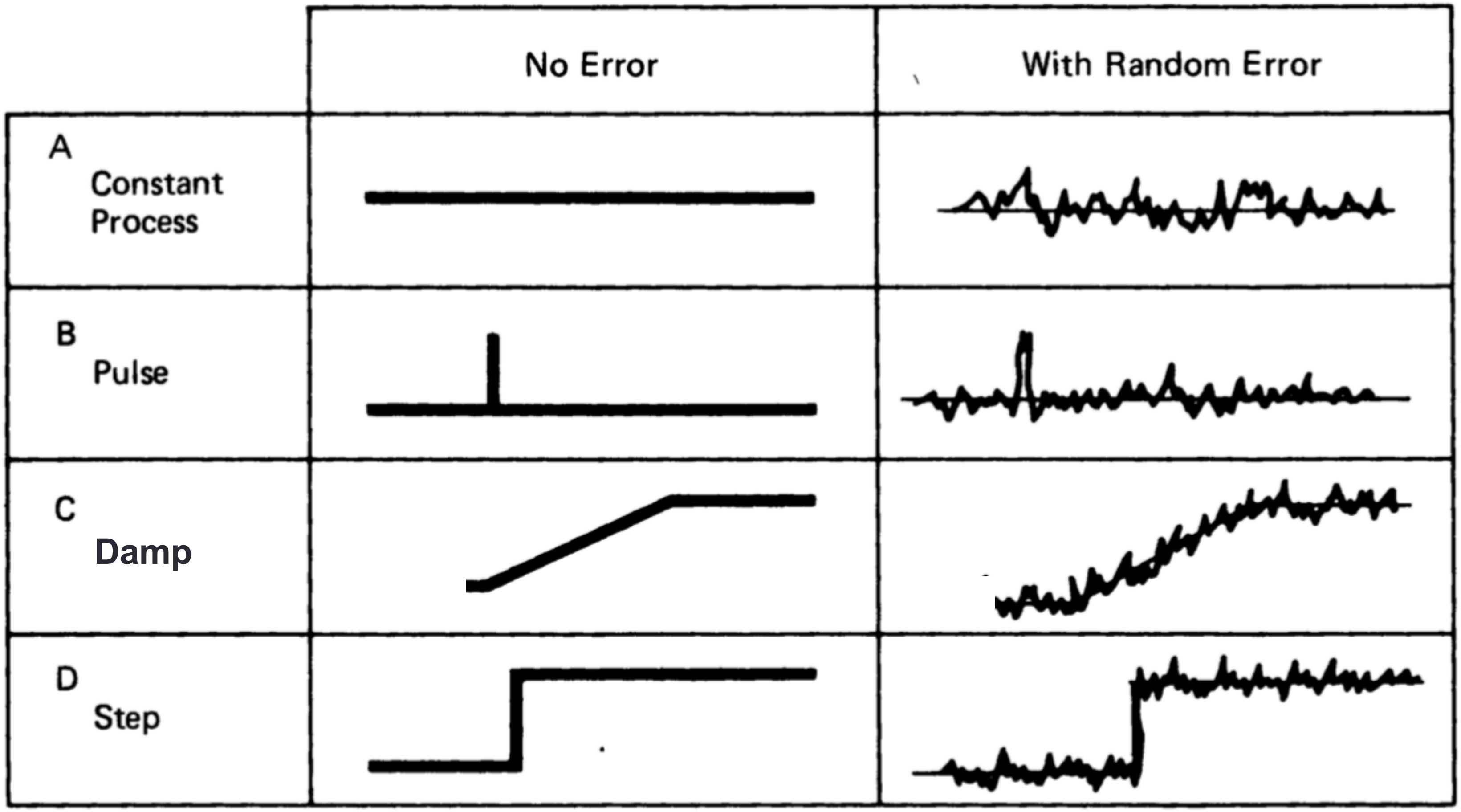
There are 15 methods

Exponential Smoothing methods

There are 15 methods

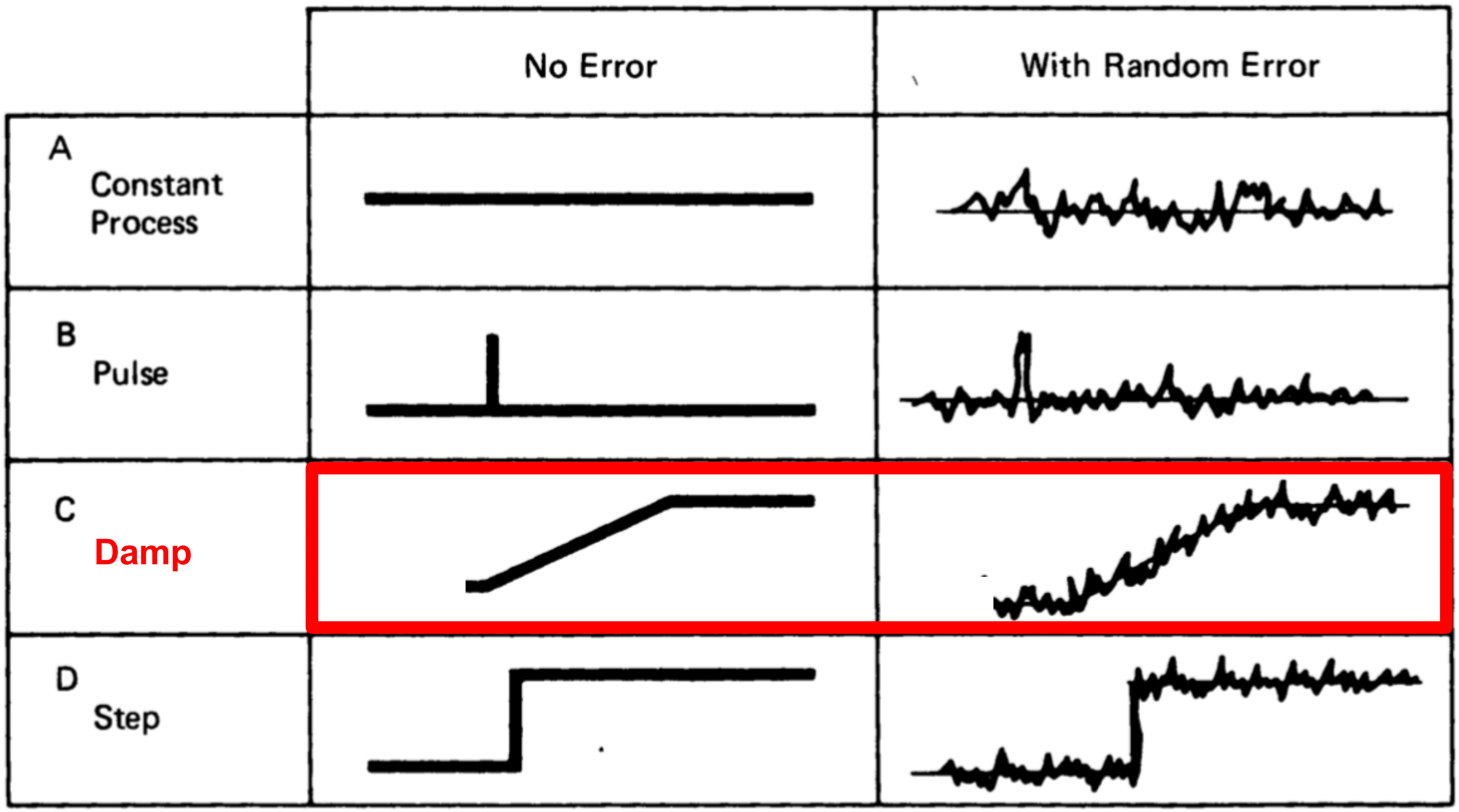


Exponential Smoothing methods



Basic patterns

Exponential Smoothing methods



Damped trend is a trend that does not continue beyond a short time in the future

Exponential Smoothing methods

Trend Component		Seasonal Component N (None)	A (Additive)	M (Multiplicative)
N	(None)	N,N	N,A	N,M
A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
M	(Multiplicative)	M,N	M,A	M,M
M_d	(Multiplicative damped)	M_d,N	M_d,A	M_d,M

Table 1: The fifteen exponential smoothing methods.

Exponential Smoothing methods

Trend Component		Seasonal Component: N (None)	A (Additive)	M (Multiplicative)
N	(None)	SES	N,A	N,M
A	(Additive)	A,N	Holt-Winters	Holt-Winters
A_d	(Additive damped)	Damped Trend	A_d,A	A_d,M
M	(Multiplicative)	M,N	M,A	M,M
M_d	(Multiplicative damped)	M_d,N	M_d,A	M_d,M

Table 1: The fifteen exponential smoothing methods.

Exponential Smoothing methods

Trend Component		Seasonal Component: N (None)	A (Additive)	M (Multiplicative)
N	(None)	N,N	N,A	N,M
A	(Additive)	A,N	A,A	A,M
A_d	(Additive damped)	A_d,N	A_d,A	A_d,M
M	(Multiplicative)	M,N	M,A	M,M
M_d	(Multiplicative damped)	M_d,N	M_d,A	M_d,M

State Space models – Additive errors

Trend Component		Seasonal Component N (None)	Seasonal Component A (Additive)	Seasonal Component M (Multiplicative)
N	(None)	A N,N	A N,A	A N,M
A	(Additive)	A A,N	A A,A	A A,M
A_d	(Additive damped)	A A_d,N	A A_d,A	A A_d,M
M	(Multiplicative)	A M,N	A M,A	A M,M
M_d	(Multiplicative damped)	A M_d,N	A M_d,A	A M_d,M

State Space models – Multiplicative errors

.

Trend Component		Seasonal Component: N (None)	Seasonal Component: A (Additive)	Seasonal Component: M (Multiplicative)
N	(None)	MN,N	MN,A	MN,M
A	(Additive)	MA,N	MA,A	MA,M
A_d	(Additive damped)	MA_d,N	MA_d,A	MA_d,M
M	(Multiplicative)	MM,N	MM,A	MM,M
M_d	(Multiplicative damped)	MM_d,N	MM_d,A	MM_d,M

State Space model – Holt linear method

.

$$
\begin{aligned}
F_{t+m} &= L_t + b_t m \\
\\
L_t &= \alpha Y_t + (1-\alpha)(L_{t-1} + b_{t-1}) \\
b_t &= \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1}
\end{aligned}
$$

$$\hat{y}_{t+m} = \ell_t + b_t\, m$$

$$\ell_t = \alpha\, y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*\,(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$$

State Space model – Holt linear method

m-period forecast $\hat{y}_{t+m} = \ell_t + b_t\, m$

one-period forecast $\hat{y}_{t+1} = \ell_t + b_t$

at time *t* $\hat{y}_t = \ell_{t-1} + b_{t-1}$

one-period forecast error $\varepsilon_t = y_t - \hat{y}_t$

$$\varepsilon_t \sim \mathrm{NID}\left(0, \sigma^2\right)$$

State Space model – Holt linear method

▪

$$
\begin{aligned}
\hat{y}_t &= \ell_{t-1} + b_{t-1} \\
y_t &= \hat{y}_t + \varepsilon_t \\
\ell_t &= \alpha\, y_t + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
&= \ell_{t-1} + b_{t-1} + \alpha\, \varepsilon_t
\end{aligned}
$$

State Space model – Holt linear method

$$
\begin{aligned}
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha\,\varepsilon_t \\
b_t &= \beta^*\,(\ell_t - \ell_{t-1}) + (1-\beta^*)\,b_{t-1} \\
&= \beta^*\,(b_{t-1} + \alpha\,\varepsilon_t) + (1-\beta^*)\,b_{t-1} \\
&= b_{t-1} + \beta^*\alpha\,\varepsilon_t \\
&= b_{t-1} + \beta\,\varepsilon_t
\end{aligned}
$$

State Space models

forecast library

automatic forecasting algorithms

- State Space models for Exponential Smoothing methods
- ARIMA models

State Space models

forecast library functions

- ets(dataframe)

  fits an *Exponential smoothing state space model*

- accuracy(model)

  computes ME, MAE, MPE, MAPE, MASE

- forecast(model)

  computes prediction intervals

State Space models - usnetelec

expsmooth library

data(usnetelec)

125 monthly US government bond annual yields

Jan 1994 – May 2004

fit the model

etsfit = ets(usnetelec)

State Space models - usnetelec

- Multiplicative error, Additive trend,

  No seasonal component

- Initial params are l_o and b_o

```
R> etsfit

ETS(M,A,N)

Call:
 ets(y = usnetelec)

  Smoothing parameters:
    alpha = 0.9999
    beta  = 0.2191

  Initial states:
    l = 254.9338
    b = 38.3125

  sigma:  0.0259

     AIC     AICc      BIC
634.0437 635.2682 644.0803
```

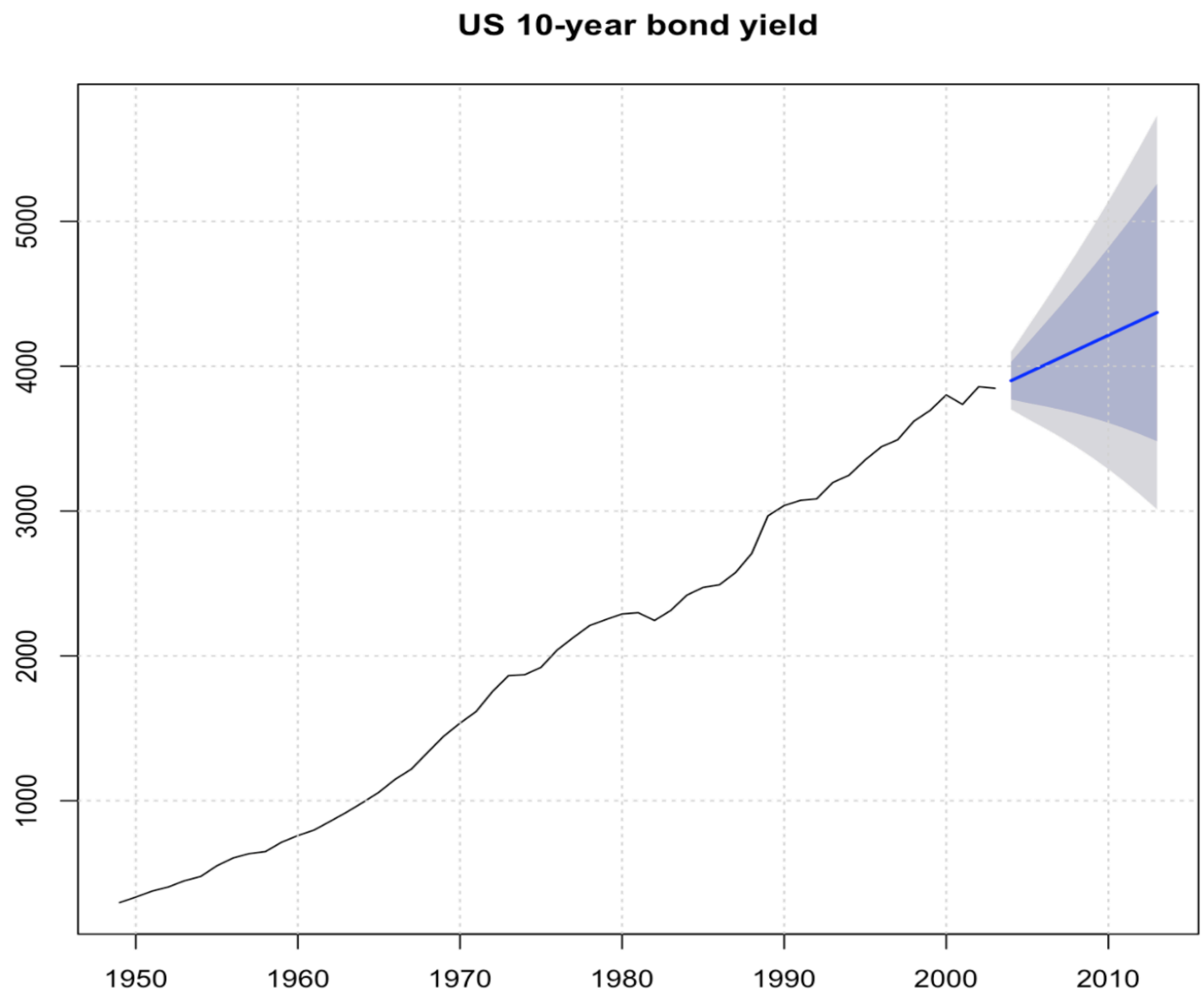
State Space models - usnetelec

```
> forecast(etsfit1)
     Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
2004       3900.329 3770.801 4029.857 3702.233 4098.425
2005       3952.650 3747.279 4158.022 3638.562 4266.738
2006       4004.972 3725.589 4284.355 3577.692 4432.251
2007       4057.293 3701.885 4412.701 3513.743 4600.842
2008       4109.614 3674.968 4544.259 3444.881 4774.347
2009       4161.935 3644.367 4679.503 3370.383 4953.487
2010       4214.256 3609.881 4818.632 3289.944 5138.569
2011       4266.577 3571.428 4961.726 3203.439 5329.716
2012       4318.898 3528.985 5108.812 3110.830 5526.967
2013       4371.220 3482.552 5259.888 3012.119 5730.320
```

```
> accuracy(etsfit1)
                   ME     RMSE      MAE       MPE     MAPE      MASE        ACF1
Training set 1.162583 52.00363 36.77721 0.2629582 1.942062 0.5211014 0.006113498
```

```
> plot(forecast(etsfit1),main='US 10-year bond yield')
> grid()
```

State Space models - usnetelec



State Space models - usnetelec

Save 10 periods for test evaluation

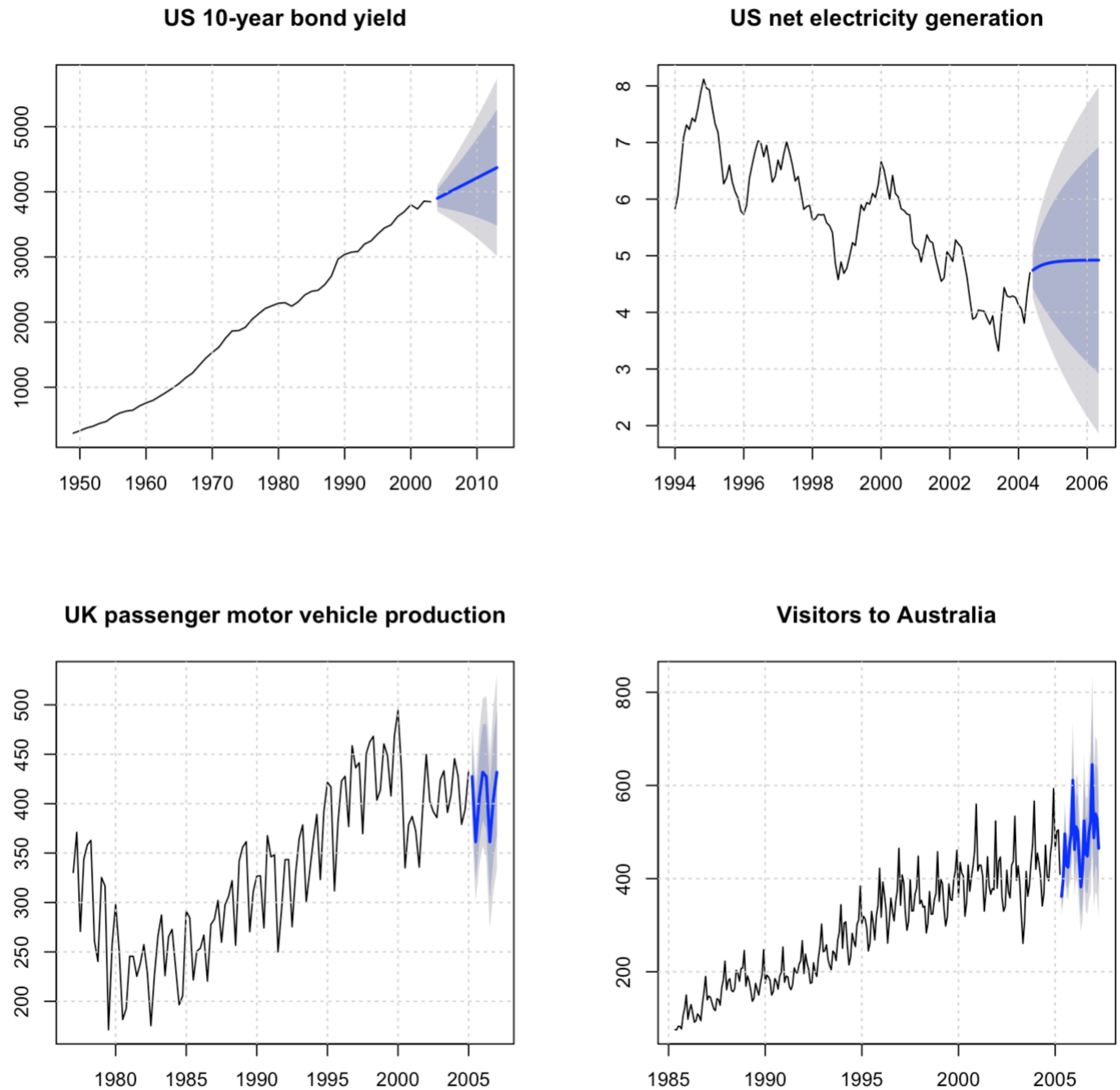
```
> # validation
> str(usnetelec)
 Time-Series [1:55] from 1949 to 2003: 296 334 375 404 447 ...
> length(usnetelec)
[1] 55
> # train performance
> model1 = ets(usnetelec[1:45])
> accuracy(model1)
                   ME    RMSE      MAE       MPE     MAPE      MASE     ACF1
Training set 3.952422 48.8502 33.89778 0.3530148 2.051756 0.4957958 0.225129
> # test performance
> test = ets(usnetelec[46:55],model = model1)
Model is being refit with current smoothing parameters but initial states are being re-estimated.
Set 'use.initial.values=TRUE' if you want to re-use existing initial values.
> accuracy(test)
                   ME     RMSE      MAE        MPE     MAPE      MASE       ACF1
Training set -13.0306 64.02833 45.85866 -0.3446059 1.234965 0.5484756 -0.5879419
```

datasets

from

library

expsmooth



State Space models – 4 datasets

```
> par(mfrow=c(2,2))
> plot(forecast(etsfit1),main='US 10-year bond yield'); grid()
> plot(forecast(etsfit2),main='US net electricity generation'); grid()
> plot(forecast(etsfit3),main='UK passenger motor vehicle production'); grid()
> plot(forecast(etsfit4),main='Visitors to Australia'); grid()
```

State Space models – 4 datasets

- ETS(A,A_d,N) for monthly US 10-year bonds yield ($\alpha = 0.9999$, $\beta = 0.09545$, $\phi = 0.8026$, $\ell_0 = 5.3252$, $b_0 = 0.5934$);
- ETS(M,A,N) for annual US net electricity generation ($\alpha = 0.9999$, $\beta = 0.2191$, $\ell_0 = 254.9338$, $b_0 = 38.3125$);
- ETS(A,N,A) for quarterly UK motor vehicle production ($\alpha = 0.6199$, $\gamma = 1e{-}04$, $\ell_0 = 314.2568$, $s_{-3} = 25.5223$, $s_{-2} = 21.1956$, $s_{-1} = -44.9601$, $s_0 = -1.7579$);
- ETS(M,A,M) for monthly Australian overseas visitors ($\alpha = 0.6146$, $\beta = 0.00019$, $\gamma = 0.1920$, $\ell_0 = 92.9631$, $b_0 = 2.2221$, $s_{-11} = 0.8413$, $s_{-10} = 0.8755$, $s_{-9} = 1.0046$, $s_{-8} = 0.9317$, $s_{-7} = 0.8219$, $s_{-6} = 1.0012$, $s_{-5} = 1.1130$, $s_{-4} = 1.3768$, $s_{-3} = 0.9625$, $s_{-2} = 1.0669$, $s_{-1} = 1.0666$, $s_0 = 0.9378$).