Fourier Analysis Outside Dimension One

Michael Christ

October 2007

Fourier Series and Transform

$\mathbb{R}^d$ acts on $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ by translations $x \mapsto x+y = \tau_y(x)$. Each of the functions $e^{in\cdot x}$ (inner product of n, x) is a joint eigenfunction of all τ_y (eigenvalues e^{iny}). The (complex) Hilbert space $L^2(\mathbb{T}^d)$ decomposes into the original direct sum of these one-dimensional eigensubspaces. Thus any function is expressed as

$$f(x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n)e^{inx},$$

and one has Parseval's identity

$$\|f\|_{L^2}^2 = \sum_{n\in\mathbb{Z}^d} |\widehat{f}(n)|^2$$

and convergence in L^2 norm:

$$\|f - \sum_{|n|\leq N} \widehat{f}(n)e^{inx}\|_{L^2} \to 0 \text{ as } N \to \infty.$$

Similarly for $\mathbb{R}^d$

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$$

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi$$

$$(2\pi)^{-d} \int_{|\xi|\le N} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi \to f$$

in L^2 norm as $N \to \infty$.

The utility of Fourier series/integrals derives largely from their connection with *group structure*.

Example: Since the Laplace operator

$$\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$$

is invariant under translations, it is diagonalized by the Fourier "basis", and the solution of the linear Schrödinger equation

$$iu_t(t,x) = \Delta_x u(t,x) \quad \text{with} \quad u(0,x) = u_0(x),$$

which describes a quantum electron in a vacuum, is consequently

$$u(t,x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{-i|\xi|^2 t} e^{ix\cdot\xi} \widehat{u_0}(\xi)\, d\xi,$$

a formula scarcely more complicated than that expressing u_0 in terms of $\widehat{u_0}$.

Fourier invented Fourier series in order to solve the heat equation in the same way.

Convergence

Fourier's claim that any function was expressible as $\sum_n a_n e^{inx}$ was promptly contested. Convergence of Fourier series/integrals became the subject of much study. Many theorems assert convergence, with variations in
(i) Sense in which series converges
(ii) Method of summation of series.

Some theorems about the $\boxed{\text{one-dimensional}}$ case:

• M. Riesz ($\approx$ 1935): For any $p \in (1, \infty)$, the partial sums $S_N(f)(x) = \sum_{|n| \leq N} \widehat{f}(n) e^{in \cdot x}$ converge to f in L^p norm, if $f \in L^p$.

• Fejér: The *averages* of the first N partial sums behave better, converging also in L^1 and C^0 norms.

• Carleson 1966: For $f \in L^p(\mathbb{T})$, $S_N(f)(x) \to f(x)$ for almost every x if $p > 1$.

There were also negative results, e.g. Kolmogorov $\approx$ 1920: There exists $f \in L^1(\mathbb{T})$ whose Fourier series converges *nowhere*.

L^p norms are very natural

$\star$ Many laws in physical science are expressed in powers. If

$$f(x) = \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then $\|f\|_{L^p} = |E|^{1/p}$.

$\star$ Functional analysis (closed graph theorem, uniform boundedness principle, Banach-Alaoglu Theorem) tells us that soft conclusions (e.g. existence, uniqueness) are often equivalent to certain inequalities.

$\star$ Experience has shown that the deepest inequalities are often expressed in terms of power laws, and in particular, L^p norms. (OK, logs and exponentials, too)

$\star$ Consider the *nonlinear Schrödinger equation*

$$iu_t = \Delta_x u \pm |u|^2 u.$$

There is a natural conservation law

$$\|u(t,x)\|_{L^2}^2 \equiv \|u(0,x)\|_{L^2}^2,$$

but the cubic term on the right doesn't make any sense unless u is at least in L^3; one is led to study solutions in stronger norms than L^2.

Convergence questions have repeatedly turned out to be linked to other, seemingly less technical, questions. Fourier's original claim that his solutions of the heat equation satisfied prescribed initial conditions was really a claim about an alternative method of summation of Fourier series.

Riesz's theorem on L^p convergence of Fourier series is equivalent to the L^p boundedness of a single operator, the Hilbert transform H, which has an alternative description as the mediator between the real and imaginary parts of holomorphic functions of one complex variable:

If $u + iv$ is a holomorphic function on the upper half plane which is continuous on the closed half plane, then the boundary values of u, v on $\mathbb{R}$ satisfy

$$v = H(u).$$

Indeed, Riesz's proof relied on this connection with complex analysis. Riesz's theorem wasn't extended to higher dimensions until the Calderón-Zygmund revolution of the early 1950's. They freed the subject from its 1D shackles and gained many applications to partial differential equations — but lost the connection with convergence questions!

Not so fast

What is a partial sum of a *multidimensional* Fourier series? One can sum over all lattice points inside a $\boxed{\text{cube}}$: with $n = (n_1, \cdots, n_d)$,

$$\tilde{S}_N(f)(x) = \sum_{|n_j| \le N \ \forall j} \widehat{f}(n) e^{inx}.$$

Resulting theory is very much parallel to 1D theory.

Alternative (*Spherical summation*): Sum over lattice points in a $\boxed{\text{ball}}$:

$$S_R(f)(x) = \sum_{|n| \le R} \widehat{f}(n) e^{inx}.$$

S_R is intimately associated with the *spectral analysis of* Δ; because

$$\widehat{\Delta f}(\xi) = |\xi|^2 \widehat{f}(\xi),$$

S_R is orthogonal projection onto span of eigen-functions of Δ with eigenvalues $\le R$.

One can do the same for multidimensional Fourier integrals. It turns out that the two theories are essentially identical; very often, a theorem about one setting ($\mathbb{T}^d$ or $\mathbb{R}^d$) directly implies the corresponding theorem about the other.

Of course $S_R f \to f$ in L^2 norm for any $f \in L^2$ as $R \to \infty$.

It was realized early on that an analogue of Fejér's theorem holds: If one averages over different parameters R by inserting a factor of

$$(1 - |\xi/R|^2)^\alpha$$

into the sum or integral, then one has convergence in all L^p norms provided that* $\alpha > \frac{d-1}{2}$.

*Your truly studied the borderline case $\alpha = \frac{d-1}{2}$ in the mid-80s.

Holy Cow

Theorem. [C. Fefferman, 1971] The operator

$$Pf(x) = \int_{|\xi|\le 1} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

is $\boxed{\text{not}}$ bounded on $L^p(\mathbb{R}^d)$ unless $d = 1$ or $p = 2$.

Corollary. For $\mathbb{T}^d$, for any $p \neq 2$ there exists $f \in L^p(\mathbb{T}^d)$ for which $\|S_R(f)\|_{L^p} \to \infty$ as $R \to \infty$.

Fefferman proved this by making a connection with the *Besicovitch set*, a fundamental example from (what had been thought to be) another subject.

Before we get to that, I want to emphasize (next slide) that Fefferman's counterexample didn't kill off this line of inquiry — far from it.

The modified conjecture

With a tiny change, spherical summation of Fourier sums/integrals is possibly salvagable:

Conjecture: The modified partial "sums"

$$\int_{|\xi|\leq R} (1-|\xi/R|)^{\alpha}\widehat{f}(\xi)e^{ix\xi}\,d\xi$$

converge to $f \in L^p$ in $L^p(\mathbb{R}^d)$ norm provided that

$$\alpha > 0 \qquad \text{and} \qquad \frac{2d}{d+1} < p < \frac{2d}{d-1}.$$

(Range of p can't be enlarged.)

This turns out to be a deep problem, with many connections to "other" topics in mathematics.

What is known:

- True in dimension 2 (Carleson and Sjölin, 1972)
- Optimal $\alpha = \alpha(p)$ is known for all $p > \frac{2(d+1)}{d-1}$ in $\mathbb{R}^d$ for all dimensions. (Fefferman dissertation, 1970)
- Optimal $\alpha(p)$ is known for a slightly larger range of p in all dimensions (Bourgain 1991, and others subsequently)

Two Applications

Let $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$.

If u is a solution of the linear Schrödinger equation $iu_t = \Delta_x u$ with initial datum $u|_{t=0} = u_0 \in L^2(\mathbb{R}^d)$ then

$$\int_{\mathbb{R} \times \mathbb{R}^d} |u(t,x)|^p \, dx \, dt < \infty$$

for $p = 2(d+2)/d$.

If instead u is a solution of the Schrödinger equation with driving force $F(t,x)$:

$$iu_t = \Delta_x u + F$$

then

$$\|u\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \le C \|F\|_{L^q(\mathbb{R} \times \mathbb{R}^d)}$$

for $p = 2(d+2)/d$ and $q = 2(d+2)/(d+4)$.

Kakeya and Besicovitch

Kakeya asked: Let $E \subset \mathbb{R}^2$ be a compact set, which contains a line segment of length 1 pointing in **every** direction. How small can the volume of E be?

Besicovitch (1928) constructed such a set having Lebesgue measure $\boxed{\text{zero}}$. (In 1938 Besicovitch published a fundamental theorem concerned with projections of fractals and other irregular sets. He apparently didn't notice until the 1960's that his 1928 example was actually a corollary of his general 1938 theorem!)

Fefferman realized that Besicovitch's set was the key to a counterexample for the Fourier integral problem.

Kahane (1969) gave alternative construction of compact $E \subset \mathbb{R}^2$ containing a unit line segment in every direction:

First construct a Cantor set $\mathcal{C} \subset [0,1]$:

- Discard middle half of $[0,1]$, leaving 2 segments of length $\frac{1}{4}$.
- Discard middle half of each of these 2 segments, leaving 4 segments of length $\frac{1}{16}$.
- Continue; get 2^k segments of length 4^{-k} at stage k.

Now set

$E_0 = \{(0,t) : t \in \mathcal{C}\}, \quad E_1 = \{(1, \frac{1}{2}t) : t \in \mathcal{C}\}.$

Define $E =$ union of all line segments having one endpoint in E_0, and one endpoint in E_1.

The slope of a line segment joining $(1, \frac{1}{2}t)$ to $(0, t')$ is $t' - \frac{1}{2}t$. The set of all such *differences* contains an interval, so one gets lots of directions.

$(x, y) \in E$ if and only if x can be expressed as
$\boxed{\frac{1}{2}yt + (1-t)y' \text{ for some } y, y' \in \mathcal{C}}$.
For almost every y, the set of all such sums has 1D Lebesgue measure zero.

Fefferman realized that because the boundary of the unit disk $\{\xi : |\xi| \leq 1\}$ had a continuum of tangent directions, the Besicovitch set was potentially lurking.

He constructed a function f which was a sum of many "wavelets" living on $\delta \times 1$ rectangles with different orientations, all packed inside a near-Besicovitch set of arbitrarily small measure, so that when $\widehat{f}$ was modified by restricting to $|\xi| \leq 1$, each rectangle moved in such a way that they became pairwise disjoint.

Besicovitch only resolved a "soft" version of a more quantitative question.

For any compact set E, there are various notions of the $\boxed{\text{dimension}}$ of E; Besicovitch himself was one of the founders of this theory.

Dimension* of E: Cover E by balls of radius δ in a nearly optimal way. If it takes on the order of $\delta^{-\gamma}$ δ-balls to cover E as $\delta \to 0$, then E has (Minkowski) dimension γ.

*This definition is suitable for typical fractals; it must be modified for less regular sets.

Question: Suppose that $E \subset \mathbb{R}^d$ contains a unit line segment in every direction. How small can the *dimension* of E be?

Conjecture: No such set in $\mathbb{R}^d$ can have dimension $< d$.

True in $\mathbb{R}^2$ (Cordoba 1977).

All progress on spherical summation of Fourier integrals since approx 1980 (which is to say, since 1990) has gone by
(i) first improving the best dimension bound for Kakeya-Besicovitch sets, then
(ii) combining this with additional orthogonality arguments.

Bourgain 1991: Proved via very simple *combinatorial* reasoning that dimension(E) must be* $\geq \frac{d+1}{2}$. This was on the same level as Fefferman's 1970 work.
Bourgain then improved this to $\frac{d+1}{2} + \varepsilon(d)$ and went on to (very slightly) improve the spherical summation results in dimensions ≥ 3.
Bourgain used *bushes*:

*Christ-Duoandikoetxea-Rubio de Francia had proved a stronger inequality in a more complicated way in 1985, but had lacked the combinatorial insight, and hadn't interpreted it in terms of dimension.

Wolff (1995) improved dimension lower bound to $\frac{d+2}{2}$. This remains one of the last clean results in the subject. Wolff used *hairbrushes*:

In late 1990s, T. Gowers turned his attention (from investigations into the structure of Banach spaces) to a problem in additive combinatorics.

Szemerédi's theorem states: Let $E \subset \mathbb{Z}$ be a set of *positive density* in the sense that:

$$\limsup_{N\to\infty} \frac{\#(E \cap [1, 2, \cdots, N])}{N} > 0.$$

Then for any k, E contains an *arithmetic progression of length* k.
(that is, $a, a+d, a+2d, \cdots, a+(k-1)d$ all belong to E.)

- Case $k = 3$ was done by Roth (1953) using Fourier analysis; $k = 4$ and then $k > 4$ by Szemerédi (1969, 1975) by combinatorial methods.
- Gowers (1998, 2001) pushed through a (very original) extension of the original Fourier approach of Roth, to establish a quantitative version of Szemeredi's theorem.
- Bourgain (1999) found in a certain combinatorial idea in Gowers' work inspiration for a new attack on the Kakeya/Besicovitch problem.

Bourgain's lemma

(as improved by Katz and Tao 1999)

Lemma. Let A, B be finite subsets of any Abelian group with $\#(A) \leq N$ and $\#(B) \leq N$. Let $G \subset A \times B$, and suppose that

$$\#\{a + b : (a, b) \in G\} \leq N.$$

Then

$$\#\{a - b : (a, b) \in G\} \leq N^{2-\eta}$$

for $\eta = \frac{1}{6}$.

Trivial bound is N^2; the exponent η measures the improvement. The optimal η remains unknown; an example shows that one can't beat a certain constant which is approximately 0.37.

Application. [Katz and Tao 1999] Let $E \subset \mathbb{R}^d$ contain a unit line segment in every direction. Then E has dimension $\geq \frac{4d}{7} + \frac{3}{7}$.

(Wolff's lower bound was: $\frac{d}{2} + 1$.)

Relevance

If E is a small set then there exist parallel *and equally spaced* hyperplanes H_1, H_2, H_3 such that $E \cap H_j$ is small (by Roth's theorem).

Let $A = E \cap H_1$ and $B = E \cap H_3$, regarded as subsets of $\mathbb{R}^{d-1}$ by translation.

Let $G =$ set of all $(a, b) \in A \times B$ such that segment $[a, b]$ lies in E. Slope of this line is constant multiple of $a - b$. Thus get lots of differences if E is a Kakeya/Besicovitch set.

$E \cap H_2$ contains $\frac{1}{2}(a + b)$ for all $(a, b) \in G$. Thus sum set is small if E is small.

Bookkeeping produces dimension-dependent bound out of *dimension-independent* lemma.

Christ noticed that this type of lemma is connected with variants of one of the most classical inequalities of Fourier analysis. $\boxed{\text{Convolution}}$ of two functions can be defined by

$$f * g(x) = \int_{\mathbb{R}^d} f(x-t)g(x+t)\,dt.$$

For *indicator functions* $f = \chi_A$ and $g = \chi_B$, there is the power law bound

$$\|(f * g)\|_{L^q} \leq |A|^{1/p}|B|^{1/p}$$

for $q^{-1} + 1 = 2p^{-1}$, provided that $1 \leq p, q \leq \infty$.

Variants (one-dimensional for simplicity):

$$T_\theta(A, B, C) = \int_{-1}^{1} \chi_A(x-t)\chi_B(x+t)\chi_C(x-\theta t)\,dt.$$

Question: Are there any nontrivial estimates of the form $\|(f * g)\|_{L^q} \leq (|A| \cdot |B| \cdot |C|)^{1/p}$ with $\boxed{q < 1}$? It turns out that such an inequality implies a sums-differences type lemma for $\mathbb{R}$, with sums/differences replaced by certain linear combinations with coefficients depending on θ, *provided that* $q < 1$.

Warning: $q = 1$ is like the speed of light; everything goes in reverse for $q < 1$ . . .

Theorem.

• For $\theta \notin \mathbb{Q}$ there are *no such inequalities*.

• For any $\theta \in \mathbb{Q}$ such an inequality holds for all $q_0 < q$, for a certain threshold $q_0(\theta) < 1$.

• The optimal threshold $q_0(\theta)$ depends on Diophantine properties of θ.

• The optimal threshold can be arbitrarily close to 1

• So far as I know, the optimal exponent q_0 is not known for even a single value of θ.

For rational θ, the inequalities hold for arbitrary Abelian groups after a change of variables to clear denominators.

Variants of Kahane's construction of Kakeya/Besicovitch sets are clearly visible in the counterexamples.

This connects with another apparently very challenging open problem. In 1997, M. Lacey and C. Thiele established multilinear L^p inequalities for the *bilinear Hilbert transform*

$$B(f,g)(x) = \int_{\mathbb{R}} f(x-t)g(x+t)\,\frac{dt}{t},$$

resolving a conjecture of Calderón dating to the 1960s. Their work gave new insight into, and a systematized and somewhat simplified proof of, Carleson's 1966 theorem on the almost everywhere convergence of Fourier series.

(Their work has also spawned an impressive yet terrifying string of 60 page papers . . .)

It remains entirely open, and clearly outside the range of existing techniques, to decide whether *trilinear* integrals

$$\int_{-1}^{1} f(x-t)g(x+t)h(x-\theta t)\,\frac{dt}{t}$$

have any meaning at all, even for *bounded* functions f, g, h.

One now knows that there are certain restrictions on the possible combinations of L^p norms for which inequalities are valid for these operators, depending on Diophantine properties of θ.

So near, yet so far

Bennett, Carbery, and Tao (2006) completely solved, in the affirmative, a multilinear version of the problem on dimensions of Kakeya-Besicovitch sets.

Let $0 < \delta \ll 1$. A δ-tube is a rectangle in $\mathbb{R}^d$, whose sides have lengths $(\delta, \delta, \cdots, \delta, 1)$, with arbitrary orientation.

For $1 \leq j \leq d$ let T_j be a finite collection of $\#(T_j)$ δ-tubes τ whose long sides are nearly parallel to the coordinate vector e_j.

Consider the *multiplicity functions*

$\mu_j(x) =$ number of δ-tubes in T_j containing x.

Theorem. [Bennett-Carbery-Tao] For any such families of δ-tubes,

$$\| \prod_{j=1}^{d} \mu_j \|_{L^{1/(d-1)}(\mathbb{R}^d)} \leq C_\varepsilon \delta^{-\varepsilon} \delta^{(d-1)d} \prod_{j=1}^{d} \#(T_j)$$

for all $0 < \delta \leq 1$ and arbitrarily small ε.

This would follow from a dual version of the conjecture on the dimension of Kakeya/Besicovitch sets, which boils down to an $L^{d/(d-1)}$ bound for the multiplicity function associated to a single family of δ-tubes. Thus the corresponding bound for a $\boxed{d\text{-fold product}}$ is in $L^{1/(d-1)}$.

The idea of Bennett, Carbery, and Tao (which had earlier been used by Carlen, Lieb, and Loss to treat a simpler family of problems, but which I would not have expected to work here) was

- To let the family of δ-tubes evolve by a heat equation

- so that in the limit as time $\to +\infty$, after a rescaling, all the tubes pass through the origin

- in such a way that the norm of $\prod_j \mu_j$ is a strictly increasing function of time.

If all tubes pass through a common point, then the inequality is obvious.

(I would like to see a more conventional proof, which does not rely on any type of monotonicity.)

We've come full circle, in more ways than one.

• Fourier set out to solve the heat equation;
• This gave rise to convergence questions;
• These led to geometric questions about dimensions of Kakeya/Besicovitch sets.

In one line of development,
$\star$ These geometric questions impinged on issues in additive number theory and a Fourier-theoretic attack on a combinatorial problem,
$\star$ Which connected back to work on almost everywhere convergence of Fourier series.

♡ In another line of development, the heat equation was used to resolve a version of the geometric question, thus shedding some light on convergence of Fourier series.