Math 1B Section 101
14 Oct 2009 Quiz #7 (15 min)

Name: ______________________________ Score: /10

Are the series absolutely convergent, conditionally convergent, or divergent. Explain.

1. (6 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2-2009}$

Ans:
Let $b_n = n/(n^2-2009)$. We want to do an Alternating Series Test (AST) on the series $S=\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ but the first few terms are negative. Let N be the smallest integer satisfying $N^2-2009>0$. We write the series as

$$S = A+B = \sum_{n=1}^{N-1}(-1)^{n+1}b_n + \sum_{n=N}^{\infty}(-1)^{n+1}b_n.$$

Since A is finite, S converges if and only if B converges. The terms b_n of B are positive, so we may perform the AST on it. We need to check that

(i) $\lim_{n\to\infty} b_n = 0$,
(ii) $b_{n+1} \le b_n$ for all $n \ge N$.

For (i), we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{n^2-2009} = \lim_{n\to\infty}\frac{1/n}{1-2009/n^2} = 0.$$

For (ii), let $f(x) = \dfrac{x}{x^2-2009}$. By the quotient rule,

$$f'(x) = \frac{1(x^2-2009)-x(2x)}{(x^2-2009)^2} = -\frac{x^2+2009}{(x^2-2009)^2} < 0.$$

This follows because the numerator and denominator are all squares or sums of squares. Thus, $f(x)$ is a decreasing function, so $b_{n+1}\le b_n$ for all $n \ge N$. By the AST, the series B converges, and therefore S also converges.

Now, to check if S converges absolutely, we do a Limit Comparison Test between

$$\sum_{n=1}^{\infty}\left|\frac{n}{n^2-2009}\right| \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{1}{n}.$$

Indeed, we have

$$\lim_{n\to\infty}\frac{|n/(n^2-2009)|}{1/n} = \lim_{n\to\infty}\left|\frac{1}{1-2009/n^2}\right| = 1.$$

The harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. By the LCT, $\sum_{n=1}^{\infty}|n/(n^2-2009)|$ diverges. This means that S is conditionally divergent.

[Please turn over.]

2. (4 points) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{2009^n\,n^{2009}}{n!}$

Ans:
Let $b_n = (-1)^{n+1}\frac{2009^n\,n^{2009}}{n!}$. We do a Ratio Test on the series. Indeed,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| &= \lim_{n\to\infty}\frac{2009^{n+1}\,(n+1)^{2009}}{(n+1)!}\,\frac{n!}{2009^n\,n^{2009}}\\
&= \lim_{n\to\infty}\frac{2009}{n+1}(1+\frac{1}{n})^{2009}\\
&= 0
\end{aligned}$$

Thus, the series $\sum_{n=1}^{\infty} b_n$ converges absolutely.

3. (bonus, 0 points) $\sum_{n=1}^{\infty}(-1)^n\dfrac{(2n)!}{2^{2n}(n!)^2}$

Ans:
Let $b_n = (2n)!/[2^{2n}(n!)^2]$. Then $b_{n+1}/b_n = (2n+1)/(2n+2)$. This proves that b_n is a decreasing sequence. To see that $\lim_{n\to\infty} b_n = 0$, write

$$b_n = \frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n}$$

$$b_n^2 = \Big(\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}\Big)\Big(\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}\Big) < \Big(\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}\Big)\Big(\frac{2}{3}\frac{4}{5}\cdots\frac{2n}{2n+1}\Big) = \frac{1}{2n+1}.$$

By the AST, $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges. To check if the series converges absolutely,

$$b_n = \frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n} = \frac{3}{2}\frac{5}{4}\cdots\frac{2n-1}{2n-2}\frac{1}{2n} > \frac{1}{2n}$$

so by the Standard Comparison Test, $\sum_{n=1}^{\infty} b_n$ diverges. Therefore, the original series is conditionally convergent.

[End of Quiz]

Quiz Statistics

Scores	0	1	2	3	4	5	6	7	8	9	10
	1	0	2	0	2	6	5	9	4	1	0

Average 5.87

Grading Scheme

Q1)
(1 pt) idea of using Alternating Series Test.
(1 pt) noting b_n positive only for $n > \sqrt{2009}$.
(1 pt) showing b_n decreasing.
(1 pt) showing limit of b_n is zero.
(1 pt) idea of using Limit Comparison Test to check absolute convergence.
(1 pt) comparing with $1/n$, showing limit is 1.
Alternatively,
(1 pt) idea of using Integral Test to check absolute convergence.
(1 pt) noting function is decreasing, showing limit is divergent.

Q2)
(1 pt) idea of using Ratio Test.
(1 pt) showing limit of ratio is 0.
(1 pt) stating limit is less than 1.
(1 pt) stating absolute convergence.

Common Mistakes

Q1 Used LCT on $\sum(-1)^{n+1}1/n$ but LCT requires terms to be positive.
Q1 Fail to note that first few terms of b_n is negative so we cannot apply AST there.
Q1 Forgetting to check absolute convergence after checking convergence.
Q2 When we have a limit of the ratio of two polynomials $\lim_{x\to\infty} p(x)/q(x)$, we need to show some proof of how we got the limit. For instance, you can say,
1. degree of $p(x)$ is greater than degree of $q(x)$ so $\lim_{x\to\infty} p(x)/q(x) = 1$
2. degree of $p(x)$ is smaller than degree of $q(x)$ so $\lim_{x\to\infty} p(x)/q(x) = 0$
3. degree of $p(x)$ is equal to degree of $q(x)$ so $\lim_{x\to\infty} p(x)/q(x)$ is the ratio of the leading coefficients of $p(x)$ and $q(x)$.

Alternatively, you can divide the numerator and denominator throughout by x^d where d is the degree of the denominator $q(x)$, and take limits as $x \to \infty$.
Q2 Ratio Test involves checking if the limit L satisfies $L > 1$ or $L < 1$, not $L > 0$ or $L < 0$.
Q1, Q2 Wrong application of Test For Divergence: $\lim |a_n| = 0$ does not imply $\sum |a_n|$ converges.