Math 1B Section 101
09 Sep 2009 Quiz #2

Name: ______________________________ Score: /10

SID#: ______________________________ Time: 15 mins

1. (4 points) Find $\int \frac{x^2+x+1}{(x+1)(x^2+1)}\ dx$.

Ans:

$$\frac{x^2+x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+D}{x^2+1}$$

$$x^2+x+1 = (A+B)x^2 + (B+C)x + (A+D)$$

Thus, $A = B = D = \frac{1}{2}$. Hence,

$$\begin{aligned}\int \frac{x^2+x+1}{(x+1)(x^2+1)}\ dx &= \frac{1}{2}\int \frac{1}{x+1}\ dx + \frac{1}{4}\int \frac{2x}{x^2+1}\ dx + \frac{1}{2}\int \frac{1}{1+x^2}\ dx \\ &= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + C\end{aligned}$$

2. (2 points) Show that $\int \sec^3 x\ dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$.

Ans:

$$\begin{array}{ll} u = \sec x & dv = \sec^2 x dx \\ du = \sec x \tan x dx & v = \tan x \end{array}$$

$$\begin{aligned}\int \sec^3 x\ dx &= \sec x\tan x - \int \sec x \tan^2 x dx \\ &= \sec x\tan x - \int \sec x(\sec^2 x - 1)dx \\ 2\int \sec^3 x\ dx &= \sec x\tan x + \ln|\sec x + \tan x| + C\end{aligned}$$

From the last line, the result follows.

[Please turn over.]

3. (4 points) Find $\int \sqrt{x^2-6x}\ dx$.

Ans:

$$\begin{aligned}
\int \sqrt{x^2-6x}\ dx &= \int \sqrt{(x-3)^2-9}\ dx \\
&= \int \sqrt{u^2-9}\ du \quad \text{after substituting } u = x-3 \\
&= \int \sqrt{9\sec^2\theta - 9}\ \ 3\sec\theta\tan\theta\ d\theta \quad \text{after substituting } u = 3\sec\theta \\
&= 9\int \sec\theta\tan^2\theta\ d\theta \\
&= 9\int \sec\theta(\sec^2\theta - 1)\ d\theta \\
&= 9(\int \sec^3\theta\ d\theta - \int \sec\theta\ d\theta) \\
&= 9(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|) + C \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta| + C
\end{aligned}$$

We now substitute $\sec\theta = \frac{u}{3} = \frac{x-3}{3}$ and $\tan\theta = \frac{\sqrt{x^2-6x}}{3}$ (by drawing a right-triangle).

$$\begin{aligned}
\int \sqrt{x^2-6x}\ dx &= \frac{1}{2}(x-3)\sqrt{x^2-6x} - \frac{9}{2}\ln|\frac{x-3+\sqrt{x^2-6x}}{3}| + C \\
&= \frac{1}{2}(x-3)\sqrt{x^2-6x} - \frac{9}{2}\ln|x-3+\sqrt{x^2-6x}| + C
\end{aligned}$$

(We discarded a constant $\frac{9}{2}\ln 3$ for the last step.)

4. (bonus, 0 points) Find $\int \frac{x\tan^{-1}x}{(1+x^2)^2}\ dx$.

Ans: First, integrate by parts using $u = \tan^{-1}x, dv = \dfrac{x}{(1+x^2)^2}$.

One of the terms will be $\int \frac{1}{(1+x^2)^2}\ dx$.

We integrate this using the trigonometric substitution $x = \tan\theta$.

The integral becomes $\int \cos^2 x\ dx$.

Use a half-angle identity to complete the integration. Replace the θ's with x's.

The answer is

$$\frac{x + (x^2-1)\tan^{-1}x}{4(1+x^2)} + C$$

Quiz Statistics

Score	Count
0	3
1	0
2	1
3	2
4	3
5	1
6	6
7	5
8	1
9	5
10	3

Ave = 5.97

Common Mistakes

Q1. Writing $\frac{B}{x^2+1}$ instead of $\frac{Bx+C}{x^2+1}$ in partial fraction decomposition.
Q3. Substituting $(x-3) = 3\sec\theta$ but not changing the θ's back to x's at the end.
Q3. Substituting $(x-3) = 3\sec\theta$ but forgetting to change $dx = 3\sec\theta\tan\theta d\theta$.

[End of Quiz]