BASIC SERIES AND CONVERGENCE TESTS

Let $\sum_{n=1}^{\infty} a_n$ be a series.

Name	Conditions	Converges if	Diverges if
Test For Divergence (TFD)	$\lim \lvert a_n \rvert \neq 0$		conditions satisfied
Standard Comparison Test (SCT)	another series $\sum_{n=1}^{\infty} b_n$ $a_n, b_n \geq 0$ for all n	$a_n \leq b_n$ for all n $\sum b_n$ conv	$b_n \leq a_n$ for all n $\sum b_n$ div
Limit Comparison Test (LCT)	another series $\sum_{n=1}^{\infty} b_n$ $a_n, b_n \geq 0$ for all n $\lim_{n\to\infty} a_n/b_n = L$	$L < \infty$ $\sum b_n$ conv	$0 < L$ $\sum b_n$ div
Integral Test (IT)	$a_n = f(n)$ f continuous f positive f decreasing	$\int_1^{\infty} f(x)\,dx$ conv	$\int_1^{\infty} f(x)\,dx$ div
Alternating Series Test (AST)	$a_n = (-1)^{n+1} b_n$ b_n positive $\lim_{n\to\infty} b_n = 0$ b_n decreasing	conditions satisfied	
Ratio Test (RaT)	$\lim_{n\to\infty} \lvert a_{n+1}/a_n \rvert = L$	$L < 1$ absolutely conv	$L > 1$
Root Test (RoT)	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = L$	$L < 1$ absolutely conv	$L > 1$

Remarks:

1. Remember absolute value signs in ratio and root tests!
   They prove not just convergence but absolute convergence.

2. Check if a function is increasing/decreasing using derivatives,
   or using sum, product, composition, taking powers of increasing/decreasing functions.

3. If the test doesn't work for the whole series, try removing the first few terms
   and test for $\sum_{n=N}^{\infty} a_n$ for some large enough N.

BASIC SERIES

Name	Description	Converges if	Diverges if
Geometric series	$\sum_{n=1}^{\infty} ar^{n-1}$	$-1 < r < 1$, sum is $\frac{a}{1-r}$	otherwise
p-series	$\sum_{n=1}^{\infty} 1/n^p$	$p > 1$	otherwise

ESTIMATES AND BOUNDS

Given a convergent series $\sum_{i=1}^{\infty} a_i$ with *sum* s,
define *partial sum* $s_n = \sum_{i=1}^{n} a_i$ and *remainder* $R_n = s - s_n$.
Given a Taylor series $T(x) = \sum_{i=0}^{\infty} c_i(x-a)^i$ for $f(x)$ around $x = a$,
define *Taylor polynomial* $T_n(x) = \sum_{i=0}^{n} c_i(x-a)^i$ and *remainder* $R_n(x) = T(x) - T_n(x)$.

Name	Estimate for R_n	Bound for s
Remainder Estimate for Integral Test	IF $\sum a_n$ satisfies conditions for the Integral Test, THEN $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$	IF $\sum a_n$ satisfies conditions for the Integral Test, THEN $s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$
Remainder Estimate for Std Comp Test	IF $\sum b_n$ convergent series $0 \le a_n \le b_n$ for all n, THEN $R_n \le \sum_{i=0}^{\infty} b_i - \sum_{i=0}^{n} b_i$	
Remainder Estimate for Alternating Series	IF $\sum a_n$ satisfies conditions for the Alt Series Test, THEN $\lvert R_n\rvert \le b_n$	IF $\sum a_n$ satisfies conditions for the Alt Series Test, THEN $s_{2n} \le s \le s_{2n+1}$
Taylor's Inequality	Even if we do not know whether $f(x)$ equals its Taylor series $T(x)$, For all $\lvert x-a\rvert \le R$, $\lvert R_n(x)\rvert \le \frac{M}{(n+1)!}\lvert x-a\rvert^{n+1}$ $M = \max_{\lvert x-a\rvert \le R} \lvert f^{(n+1)}(x)\rvert$	

Taylor Series of $f(x)$ around $x = a$

$$T(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots$$

n-th degree Taylor Polynomial

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Useful expansions

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && \text{for } -1 < x < 1 \\
e^x &= \sum_{n=0}^{\infty} x^n/n! && \text{for all } x \\
\ln(1-x) &= \sum_{n=1}^{\infty} -\frac{x^n}{n} && \text{for } -1 \le x < 1 \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} && \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} && \text{for all } x \\
\tan^{-1} x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} && \text{for } -1 \le x \le 1 \\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n, && \text{for } |x| < 1 \text{ (endpts?)} \\
\binom{k}{n} &= \frac{k(k-1)\cdots(k-n+1)}{n!} && \text{for all real numbers } k
\end{aligned}$$

KEY ⋆ Definitions • Theorem ∘ Remark

USEFUL FACTS

- Let $p(x), q(x)$ be polynomials, $L = \lim_{x\to\infty} p(x)/q(x)$.
  IF $\deg p(x) > \deg q(x)$ THEN $L = \infty$.
  IF $\deg p(x) < \deg q(x)$ THEN $L = 0$.
  IF $\deg p(x) = \deg q(x)$ THEN $L =$ ratio of leading coefs.

- $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$ (good for Root Tests).
  Generally, write limits of this sort as
  $$\lim_{n\to\infty} p(n)^{q(n)} = \lim_{n\to\infty} e^{q(n)\ln p(n)} = e^{\lim\ q(n)\ln p(n)}$$
  and apply L'Hospital's Rule to $\dfrac{\ln p(n)}{1/q(n)}$

- $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$ for all real numbers x.

- $e = 1 + 1/1! + 1/2! + 1/3! + \cdots$.

11.1 SEQUENCES

⋆ sequence, limit, convergent, divergent, $\lim a_n = \infty$
  increasing, decreasing, monotone.
  bounded above, bounded below, bounded.

- Sum, difference, product, quotient, scaling,
  taking powers of convergent sequences

- Sequence comes from function
  IF $\lim f(x) = L$, $a_n = f(n)$
  THEN $\lim a_n = L$

- Applying a continuous function
  IF $\lim a_n = L$, f continuous at L
  THEN $\lim f(a_n) = f(L)$

- $\lim |a_n| = 0 \;\Leftrightarrow\; \lim a_n = 0$

- Squeeze theorem

- Monotone convergent theorem

- $\lim r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{div} & \text{else} \end{cases}$

∘ Proof by induction: Base case, induction step.

∘ When $a_n = f(n)$, it is possible a_n has a limit even
  though $f(x), x \to \infty$ does not, e.g. $f(n) = \sin(\pi n)$.

11.2 SERIES

⋆ series, partial sum, sum of a series.
  convergent, divergent series.

- IF $\sum a_n$ conv THEN $\lim |a_n| = 0$.
  IF $\lim |a_n|$ is non-zero or non-existent THEN $\sum a_n$ div.

- IF $\sum a_n$ conv THEN $\sum c\, a_n$ conv.

- $\begin{array}{rl} \text{conv} \pm \text{conv} & = \text{conv} \\ \text{conv} \pm \text{div} & = \text{div} \\ \text{div} \pm \text{div} & = \text{unknown} \end{array}$

- Geometric series (see formula sheet).

11.3 The Integral Test and Estimates of Sums

- p-series (see formula sheet).
- Integral Test (see formula sheet).
- Remainder estimate (see formula sheet).
- Bounds for the sum (see formula sheet).

11.4 The Comparison Tests

- Standard Comparison Test (see formula sheet).
- Limit Comparison Test (see formula sheet).

- ◦ Try to use the LCT if possible. If not, it usually means the series contain functions that oscillate, e.g. $e^{\sin n}$. Remove such functions using an SCT and then apply an LCT to finish the problem.

11.5 Alternating Series

- ⋆ alternating series
- • Alternating Series Test (see formula sheet).
- • Alternating Series Estimate Theorem (for formula sheet).

11.6 Absolute Convergence and the Ratio and Root Tests

- ⋆ absolute convergence, conditional convergence.
- • IF a series is absolutely convergence, THEN it is convergent.
- • Ratio Test (see formula sheet).
- • Root Test (see formula sheet).
- • <u>Rearrangements</u>
  IF abs conv THEN any rearrangement has the same sum.
  IF cond conv THEN rearrangements can achieve any sum.

11.7 Strategy for Testing Series (see formula sheet).

- ◦ ALWAYS start by doing the Test For Divergence, in case the series diverges for very simple reasons.
- ◦ Sometimes, you need to apply several tests, e.g.
  $$\sum_{n=1}^{\infty} \frac{e^{\sin n}}{n^3 - 2n^2 + 3n - 4} \text{ (use SCT, followed by LCT)}$$

11.8 Power Series

- ⋆ power series, center, coefficients.
- ⋆ Radius of convergence R: the power series $\sum c_n (x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.
- ⋆ Interval of convergence: set of values of x for which the power series converges. Four possibilities:
  $(a-R, a+R), (a-R, a+R], [a-R, a+R), [a-R, a+R]$.
- ◦ When checking endpoints of interval of convergence, the AST is frequently used for one of them.
- • Given power series $\sum c_n (x-a)^n$, there are 3 possibilities:
  1. series converges only when $x = a$, i.e. $R = 0$.
  2. series has radius of convergence $0 < R < \infty$.
  3. series converges for all x, i.e. $R = \infty$.
- ◦ The prev theorem is useful for problems where you can show easily that a power series converges for all $|x| < d$ (so $R \geq d$) but you still need to show the radius of convergence $R = d$. Do that by finding x with $|x| = R$ where the power series diverges and apply the theorem to get $R \leq d$.

11.9 Representation of Functions as Power Series

- ◦ Expansions of $1/(1-x)$, $1/(1-x)^2$, $\ln(1-x)$, $\tan^{-1} x$.
- ◦ Manipulating other functions to look like one of the above.
- • <u>Term-by-term Differentiation and Integration</u>
  Given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$,
  $\frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$
  $\int \sum_{n=0}^{\infty} c_n (x-a)^n = C + \sum_{n=0}^{\infty} c_n (x-a)^{n+1}/n+1$
  The radius of convergence remains the same.
- ◦ Remember to drop the constant term when differentiating. Remember to add a $+C$ or find C when integrating.
- ◦ When adding two series with radii of convergence R_1, R_2. the radius of the resulting series need *not* be $\min(R_1, R_2)$.

11.10 Taylor and Maclaurin Series

- ⋆ Given $f(x)$, we can associate
  Taylor Series around $x = a$: $T(x)$
  n-degree Taylor polynomial: $T_n(x)$
  Remainder: $R_n(x) = f(x) - T_n(x)$

- ⋆ We say $f(x) = T(x)$ at $x = b$ if $T_n(b) \to f(b)$ as $n \to \infty$.
- • IF $f(x) =$ some power series for $|x - a| < R$
  THEN $f(x) = T(x)$ for $|x - a| < R$.
- • IF $\lim_{n\to\infty} |R_n(x)| = 0$ for $|x - a| < R$
  THEN $f(x) = T(x)$ for $|x - a| < R$.
- • Taylor's Inequality (see formula sheet).
- ∘ e.g. function not equal to its Taylor series: $f(x) = e^{-1/x^2}$
- ∘ Multiplication and division of power series.
- ∘ Using series to do limits.
  $\lim_{x\to b} \sum_{n=0}^{\infty} c_n(x - a)^n = \sum_{n=0}^{\infty} c_n(b - a)^n$ for $|b - a| < R$.
  Reason: Power series are continuous functions in $|x - a| < R$.

11.11 Applications of Taylor Polynomials

- ∘ Say we want to approximate $f(b)$ for some b.

  Expand $f(x)$ around a center $x = a$ where a is close to b and the values of $f(a), f'(a), f''(a)$, etc. are known.
- ∘ Say we are approximating $f(x)$ with $T_n(x)$:

  If we want to know the error for a fixed value of x,
  use Remainder Estimate for Alt Series or Integral Test.

  If we want to know the error as x ranges over $|x - a| < R$,
  use Taylor's Inequality.
- ∘ Application to physics

  The problem will sometimes tell you a bunch of stuff which you translate as $f(0) = a$, $f'(0) = b$, $f''(0) = c$, etc.
  From this, construct $T(x) = a + bx + (c/2)x^2 + \ldots$

9.1 Modeling with Differential Equations

- ⋆ differential equation, order.
  solution, general solution.
- ⋆ equilibrium solution: a constant solution $y = C$.
  find by setting $y' = 0$ in differential eqn and solving it.
- ⋆ initial condition, initial value problem.

9.2 Direction Fields and Euler's Method

- ⋆ direction field, solution curve.
- ⋆ autonomous differential equation $y' = f(y)$.
  if $y = g(x)$ is a solution, so is $y = g(x + C)$.
- ∘ Graphical method:
  1. draw direction field.
  2. draw solution curve.
- ∘ Numerical method: Euler's method, step size h.
  Solving $y' = F(x, y), y(x_0) = y_0$.
  1. Set $x_n = x_0 + nh$ for $n \geq 1$.
  2. Recursively, $y_{n+1} = y_n + hF(x_n, y_n)$ for $n \geq 0$.

9.3 Separable Equations

- ⋆ separable equations $\dfrac{dy}{dx} = g(x)f(y)$

  solution: $\displaystyle\int \frac{1}{f(y)}\, dy = \int g(x)\, dx + C$
- ∘ Dividing by $f(y)$ above, we may lose the solution $f(y) = 0$.
- ∘ If $|y| = f(x)$, usually the general solution is $y = \pm f(x)$.
  (need to check by substituting back into differential eqn.)
- ∘ If $y = \pm e^C f(x)$ and $y = 0$ are solutions,
  rewrite as $y = Af(x)$ where A is any real number.
- ⋆ orthogonal trajectories:
  Given family of curves $y = f(k, x)$,
  1. Differentiate the formula.
  2. Write k in terms of y and x.
  3. If family of curves has diff eqn $y' = F(x, y)$,
  then orth trajectories has diff eqn $y' = -1/F(x, y)$.
- ⋆ mixing problems:
  $\dfrac{dy}{dt} =$ (rate in) − (rate out).