

Shape-modelling

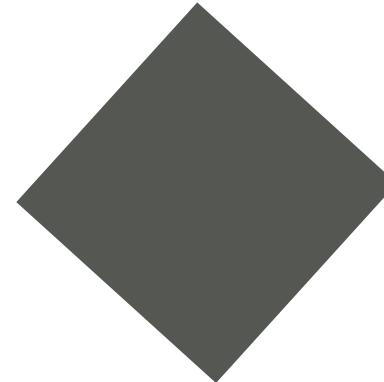
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Statistical shape models



Animations: Jasenko Zivanov

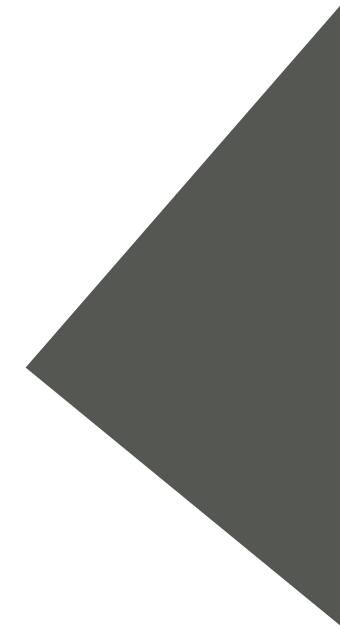
What is a shape?



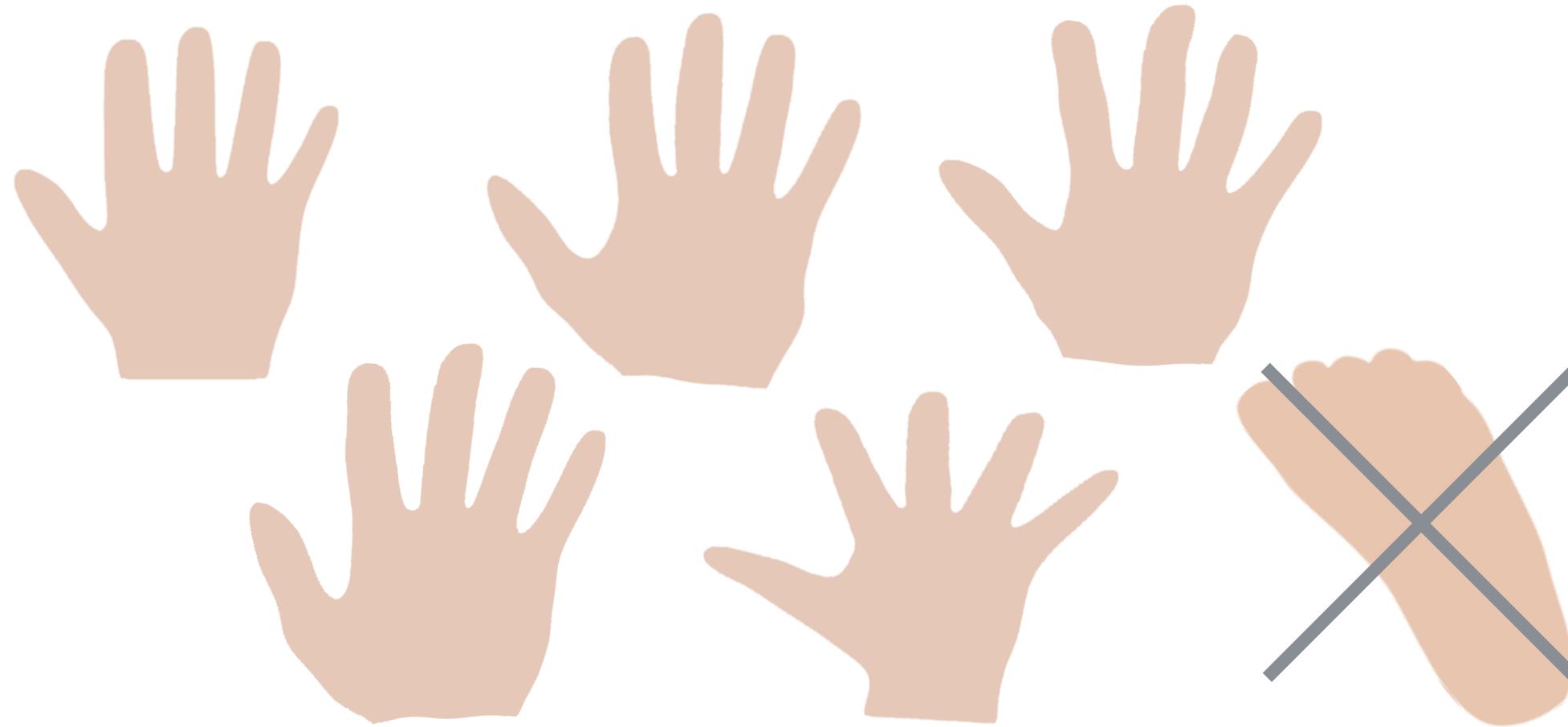
Classical definition

All geometrical information that remains when **location**, **scale** and **rotational effects** are filtered out from an object.

Shape families

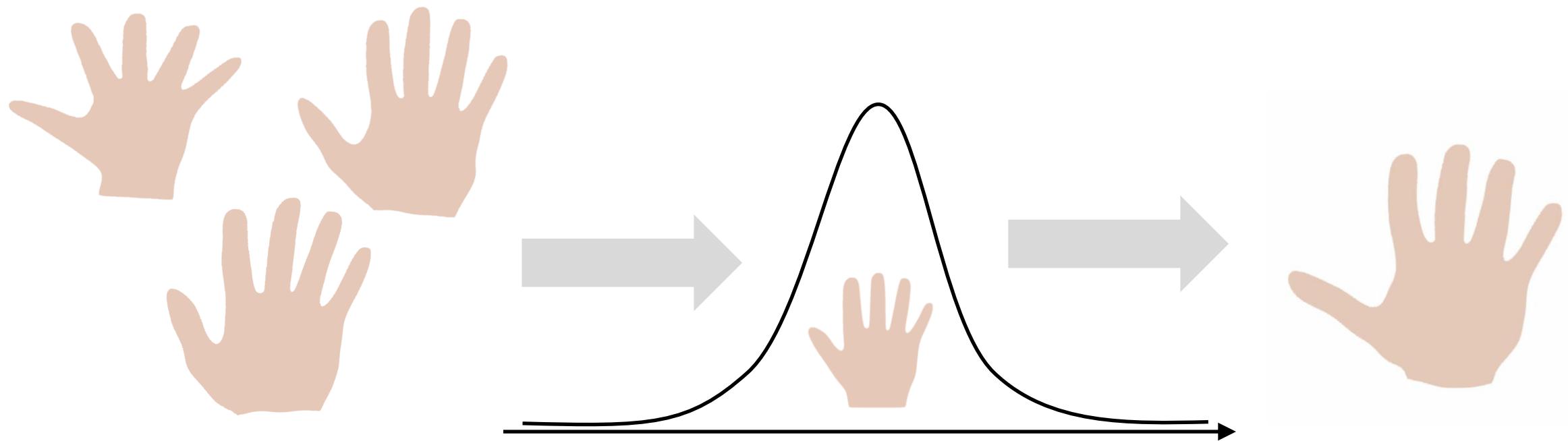


Shape families

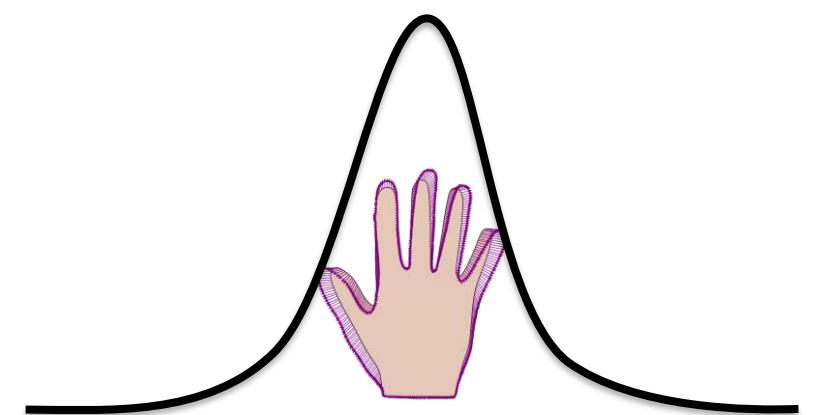


Shape families: Statistical shape models

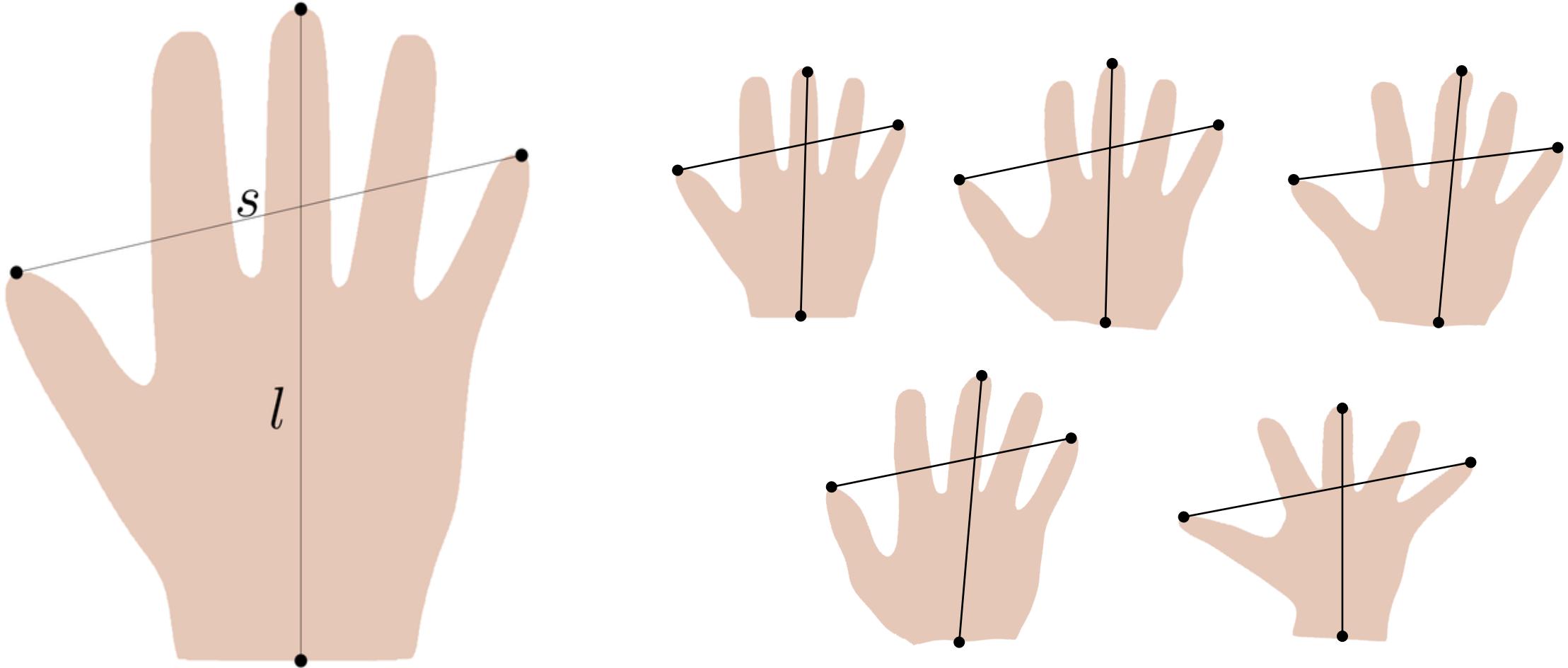
Example shapes



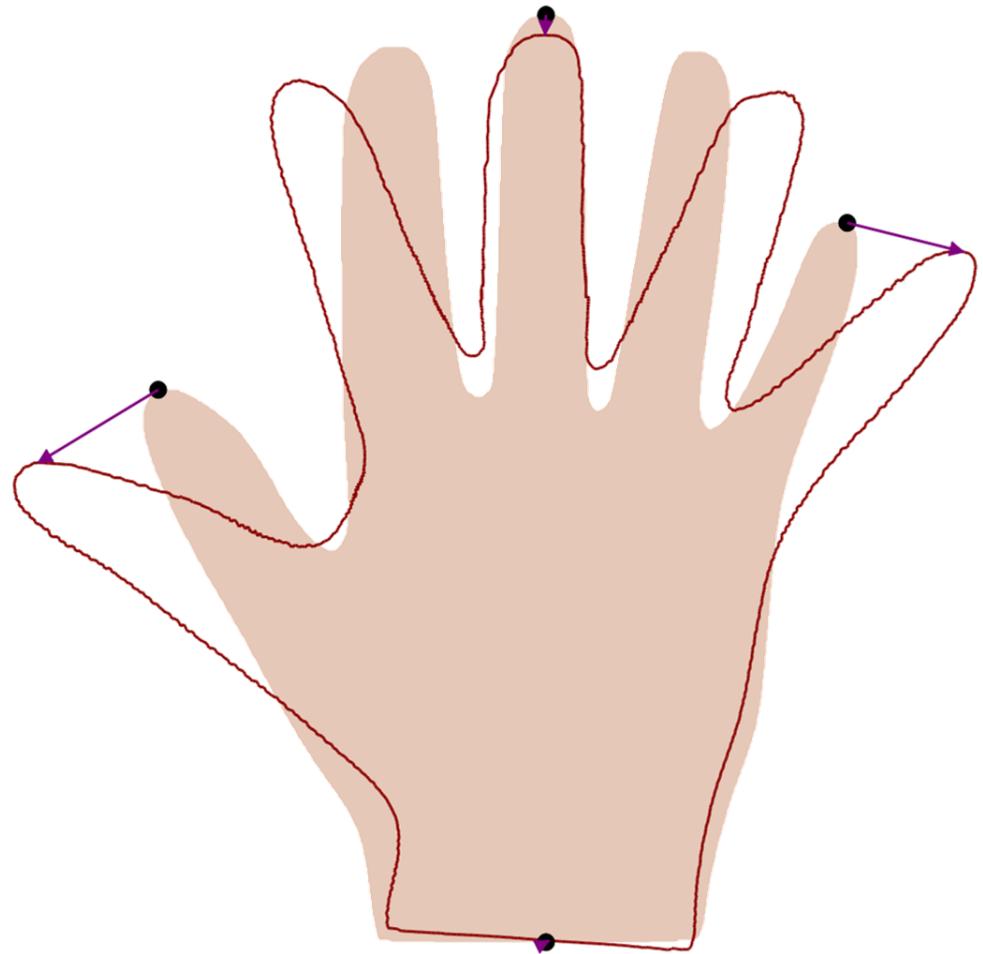
Modelling shape variations using Gaussian processes



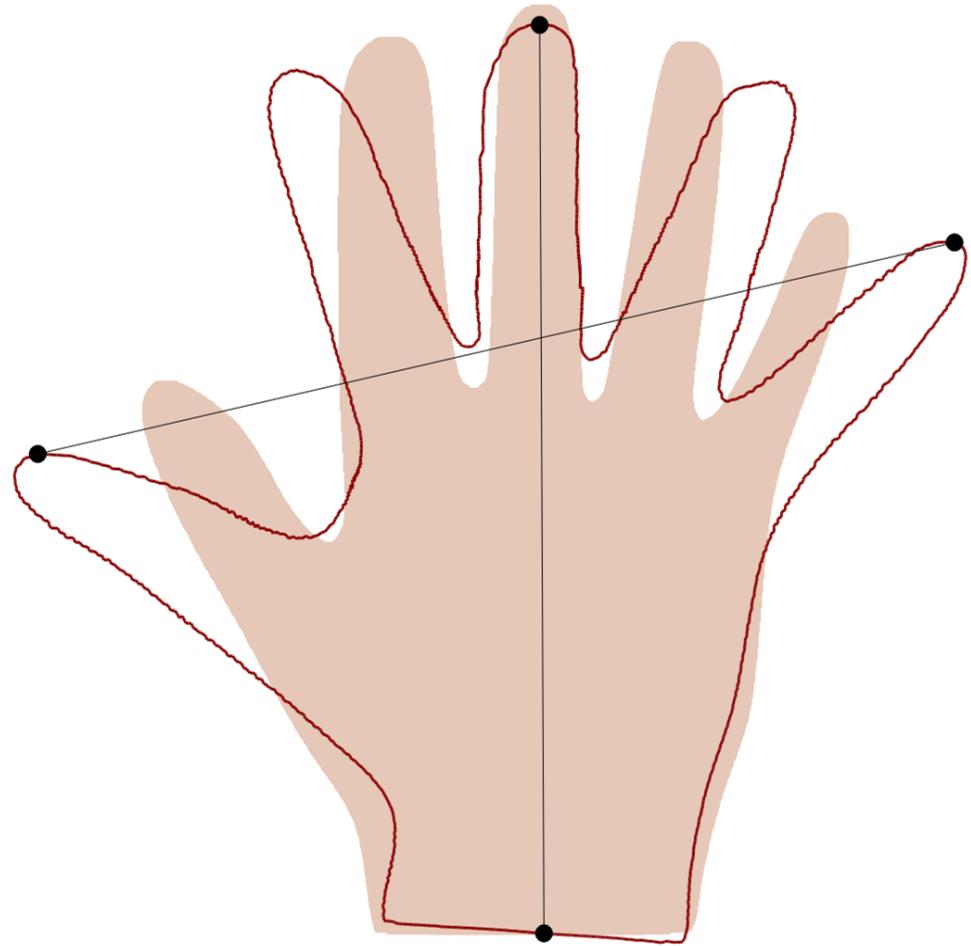
Measurements



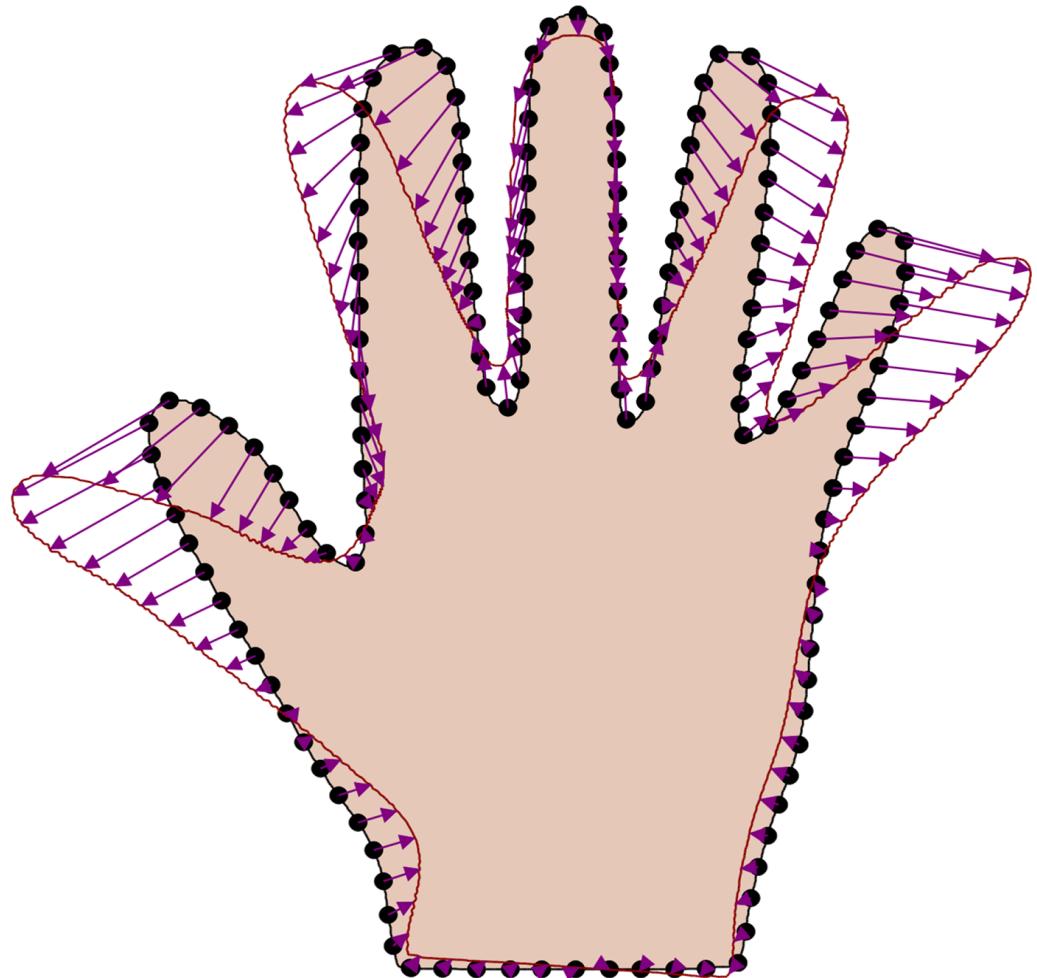
Shape changes as deformations



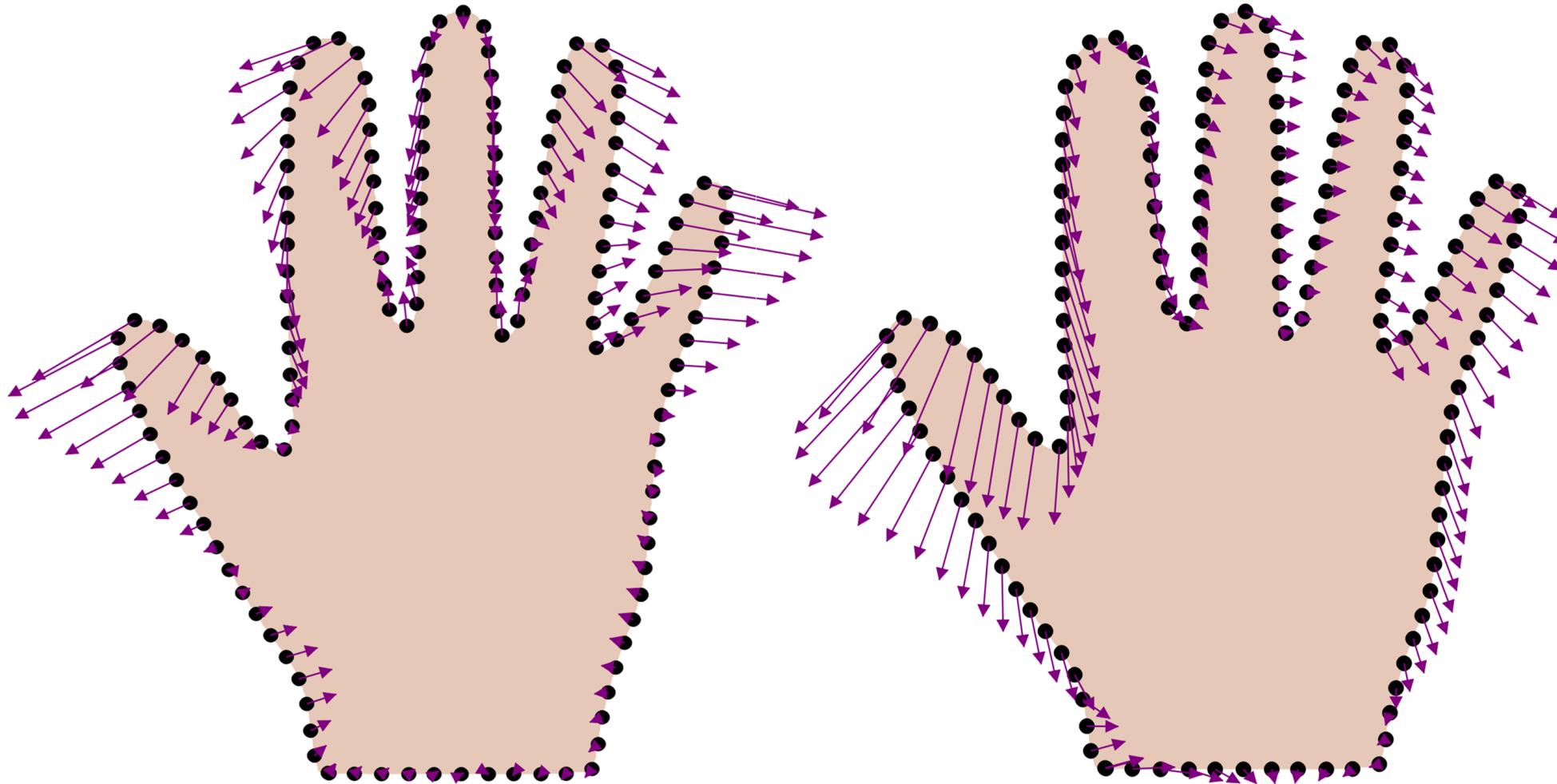
Shape changes as deformations



Shape changes as deformations



Shape changes as deformations



Shape changes as deformations

Set of points defining a reference shape:

$$\Gamma_R = \{x \mid x \in \mathbb{R}^2\}$$

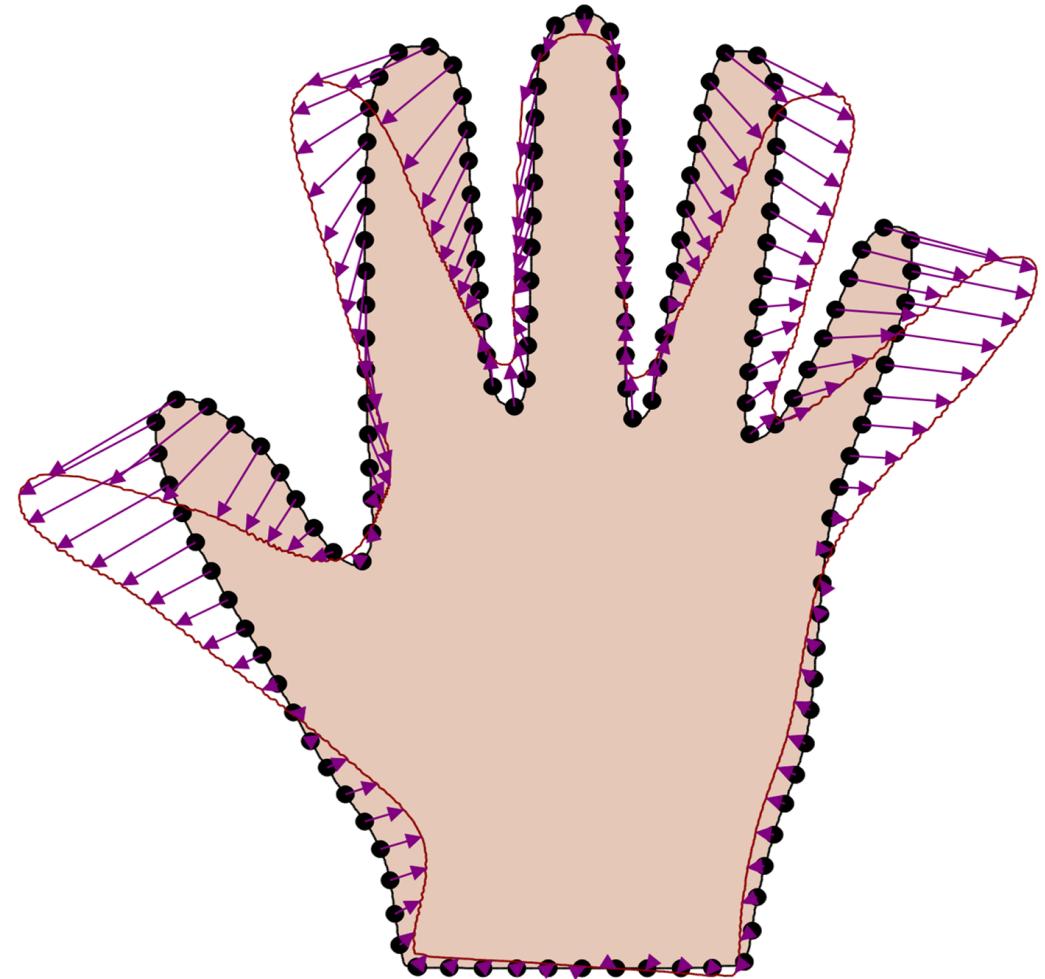
Vector field modelling the deformations

$$u : \Gamma_R \rightarrow \mathbb{R}^2$$

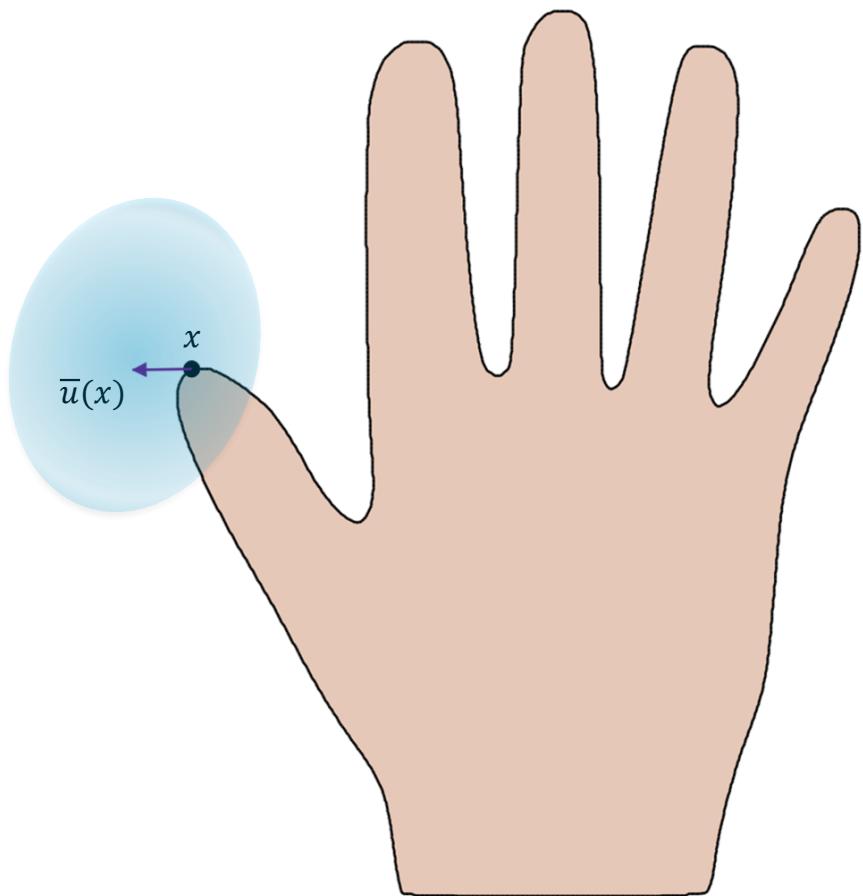
Shape defined by deformation field u

$$\Gamma = \{x + u(x) \mid x \in \Gamma_R\}$$

Our task: Model plausible deformation fields u !

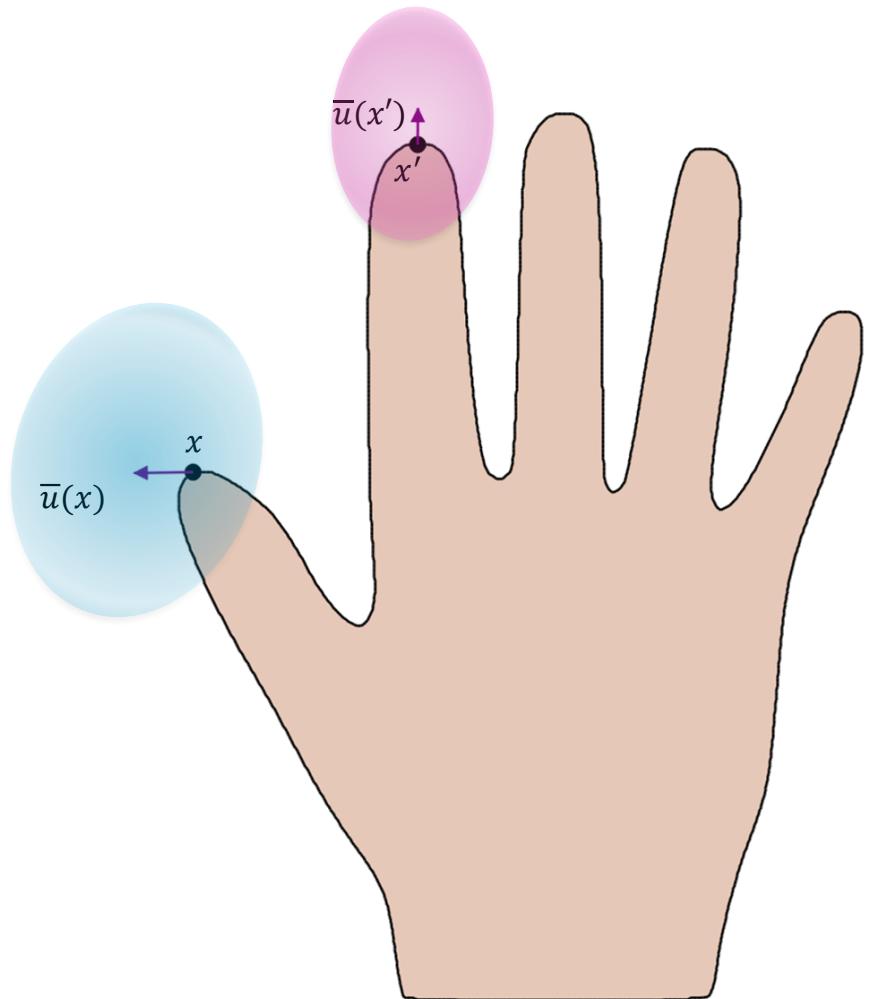


Modelling possible deformations



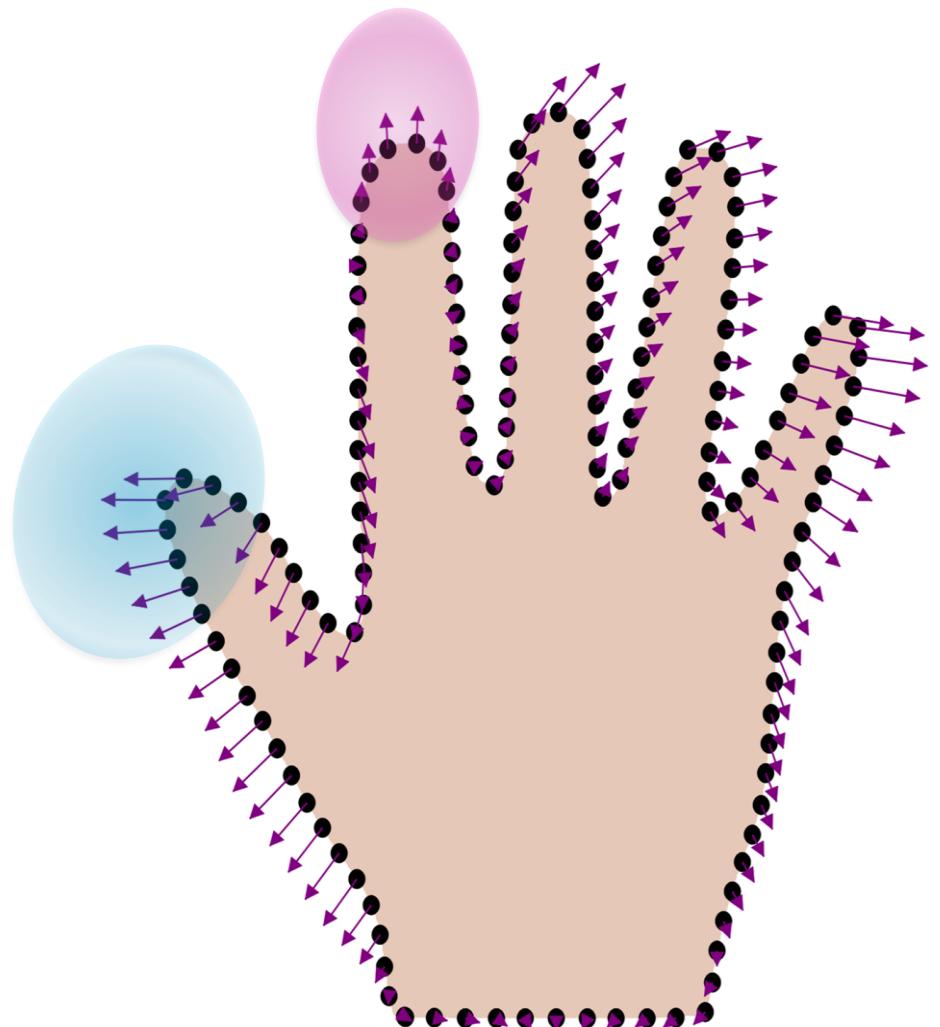
$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$
$$\sim N \left(\begin{pmatrix} \bar{u}_1(x) \\ \bar{u}_2(x) \end{pmatrix}, \begin{pmatrix} \Sigma_{11}(x) & \Sigma_{12}(x) \\ \Sigma_{21}(x) & \Sigma_{22}(x) \end{pmatrix} \right)$$

Modelling possible deformations



$$\begin{pmatrix} u(x) \\ u(x') \end{pmatrix} = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_1(x') \\ u_2(x') \end{pmatrix} \sim N \left(\begin{pmatrix} \bar{u}_1(x) \\ \bar{u}_2(x) \\ \bar{u}_1(x') \\ \bar{u}_2(x') \end{pmatrix}, \begin{pmatrix} \Sigma_{11}(x, x) & \Sigma_{12}(x, x) & \Sigma_{11}(x, x') & \Sigma_{12}(x, x') \\ \Sigma_{21}(x, x) & \Sigma_{22}(x, x) & \Sigma_{21}(x, x') & \Sigma_{22}(x, x') \\ \Sigma_{11}(x', x) & \Sigma_{12}(x', x) & \Sigma_{11}(x', x') & \Sigma_{12}(x', x') \\ \Sigma_{21}(x', x) & \Sigma_{22}(x', x) & \Sigma_{21}(x', x') & \Sigma_{22}(x', x') \end{pmatrix} \right)$$

Modelling possible deformations



Idea

Define

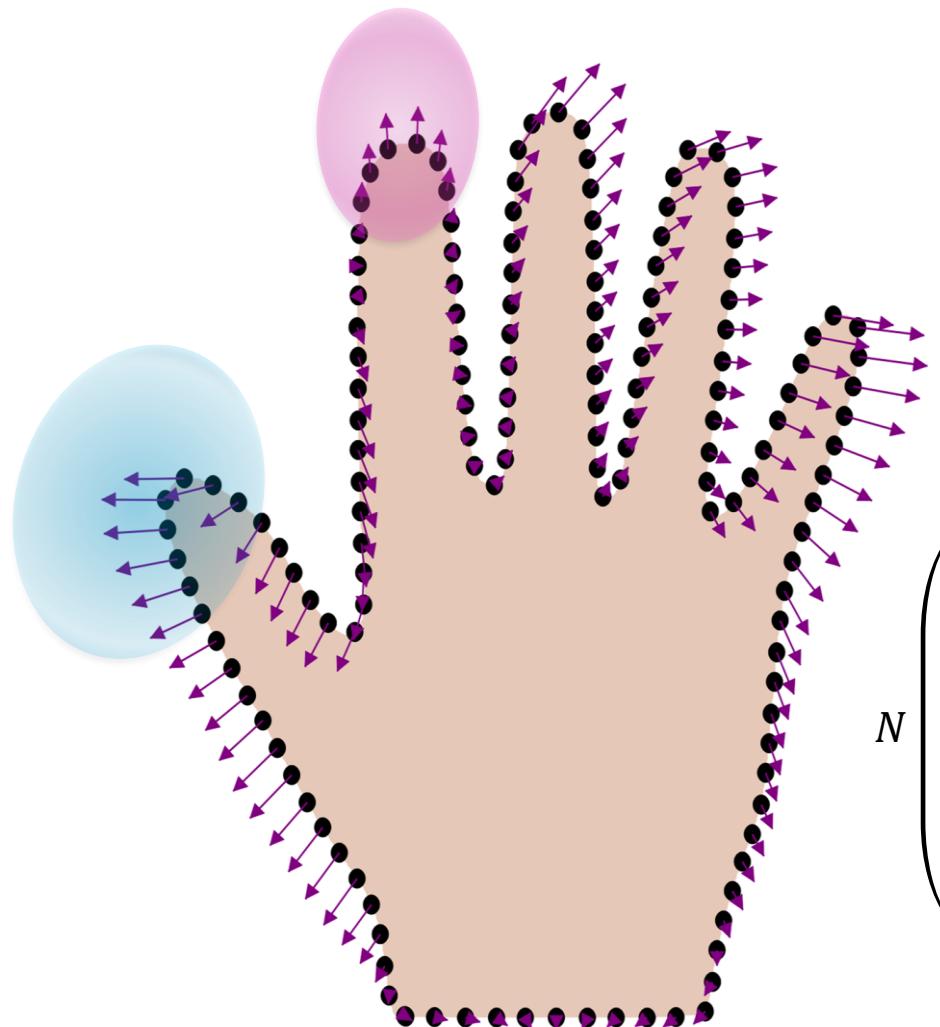
- Mean function: $\mu: \Gamma_R \rightarrow \mathbb{R}^2$
- Covariance function: $k: \Gamma_R \times \Gamma_R \rightarrow \mathbb{R}^{2 \times 2}$

For any finite set Γ_R we can define $u \sim N(\vec{\mu}, K)$, with

$$\vec{\mu} = (\mu(x))_{x \in \Gamma_R}$$

$$K = (k(x, x'))_{x, x' \in \Gamma_R}$$

Modelling possible deformations



$$N \left(\begin{pmatrix} \vdots \\ u(x) \\ \vdots \\ u(x') \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \mu_1(x) \\ \mu_2(x) \\ \vdots \\ \mu_1(x') \\ \mu_2(x') \\ \vdots \end{pmatrix} \right) = \begin{pmatrix} \vdots \\ u_1(x) \\ u_2(x) \\ \vdots \\ u_1(x') \\ u_2(x') \\ \vdots \end{pmatrix} \sim \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{11}(x, x) & k_{12}(x, x) & \cdots & k_{11}(x, x') & k_{12}(x, x') & \cdots & \vdots \\ k_{21}(x, x) & k_{22}(x, x) & \cdots & k_{21}(x, x') & k_{22}(x, x') & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ k_{11}(x', x) & k_{12}(x', x) & \cdots & k_{11}(x', x') & k_{12}(x', x') & \cdots & \vdots \\ k_{21}(x', x) & k_{22}(x', x) & \cdots & k_{21}(x', x') & k_{22}(x', x') & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Formal definition

A Gaussian process

$$p(u) = GP(\mu, k)$$

is a probability distribution over functions

$$u : \mathcal{X} \rightarrow \mathbb{R}^d$$

such that every finite restriction to function values

$$u_X = (u(x_1), \dots, u(x_n))$$

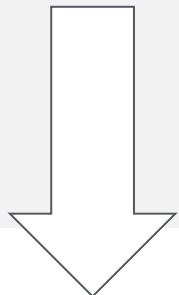
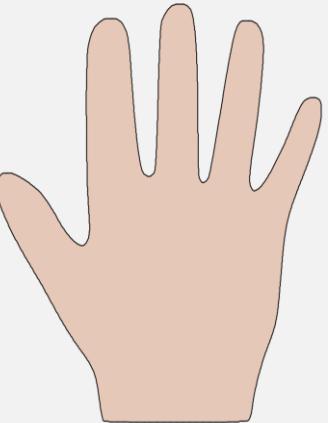
is a multivariate normal distribution

$$p(u_X) = N(\mu_X, k_{XX}).$$

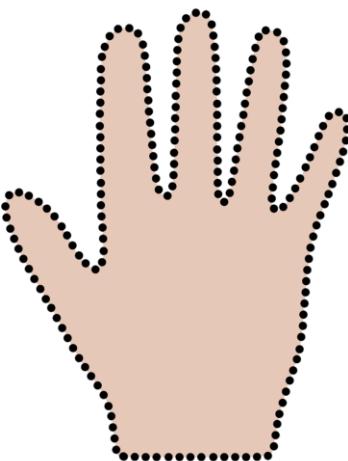
It is completely specified by a mean function μ and covariance function (or kernel) k .

Model continuously – compute discretely

Continuous representation:
 $GP(\mu, k)$

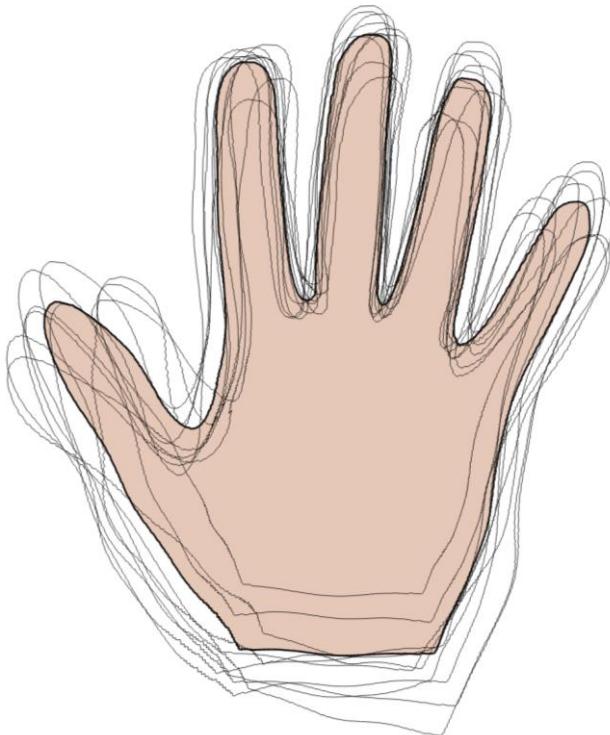


Discrete representation:
 $N(\mu, K)$



The mean function

$\mu: \Gamma_R \rightarrow \mathbb{R}^d$ defines how the average deformation looks like



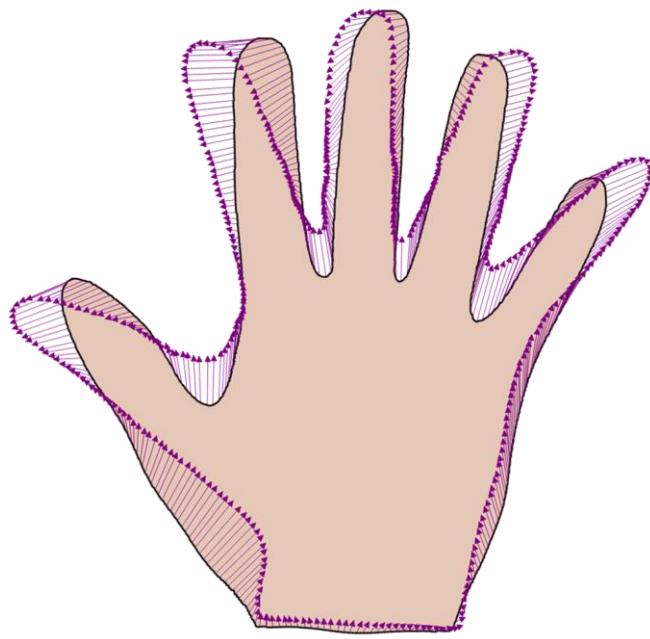
Usual assumption:

- The reference shape is an average shape.

$$\mu(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The covariance function

$k: \Gamma_R \times \Gamma_R \rightarrow \mathbb{R}^{d \times d}$ Defines characteristics of the deformations fields



Usual assumption

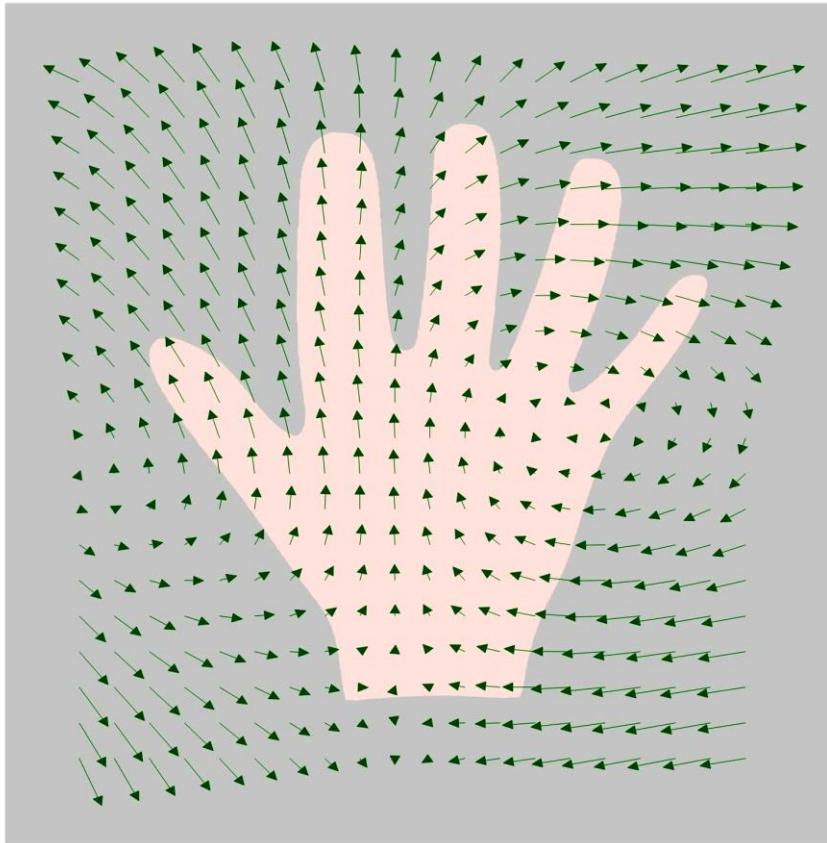
- deformation fields are smooth

Example covariance function

Squared exponential covariance function (Gaussian kernel)

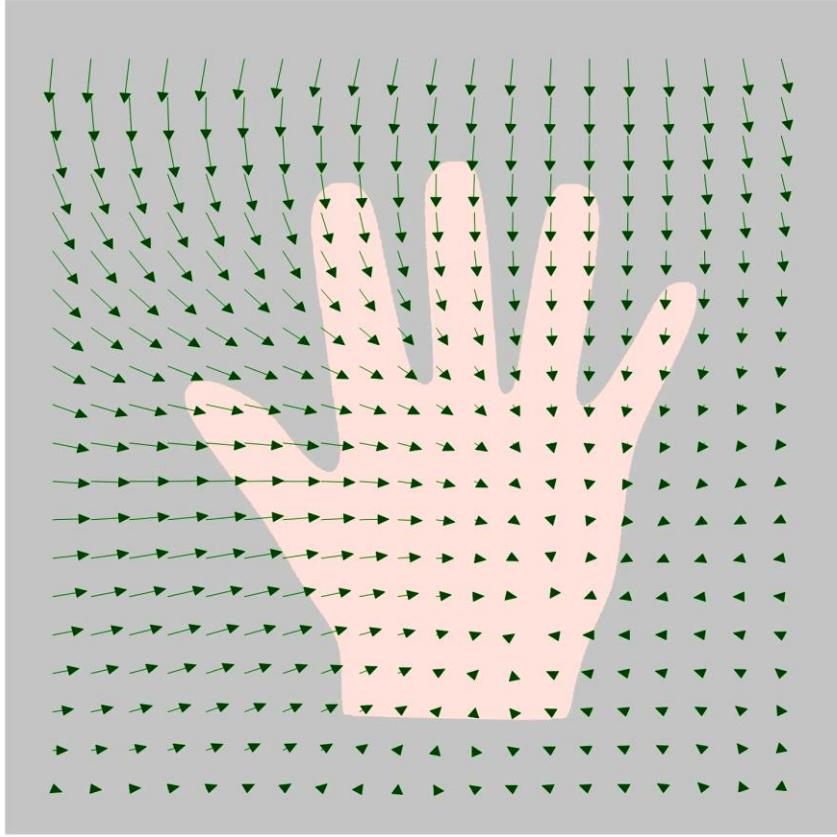
$$k(x, x') = \begin{pmatrix} s_1 \exp\left(-\frac{\|x - x'\|^2}{\sigma_1^2}\right) & 0 \\ 0 & s_2 \exp\left(-\frac{\|x - x'\|^2}{\sigma_2^2}\right) \end{pmatrix}$$

A model for smooth deformations



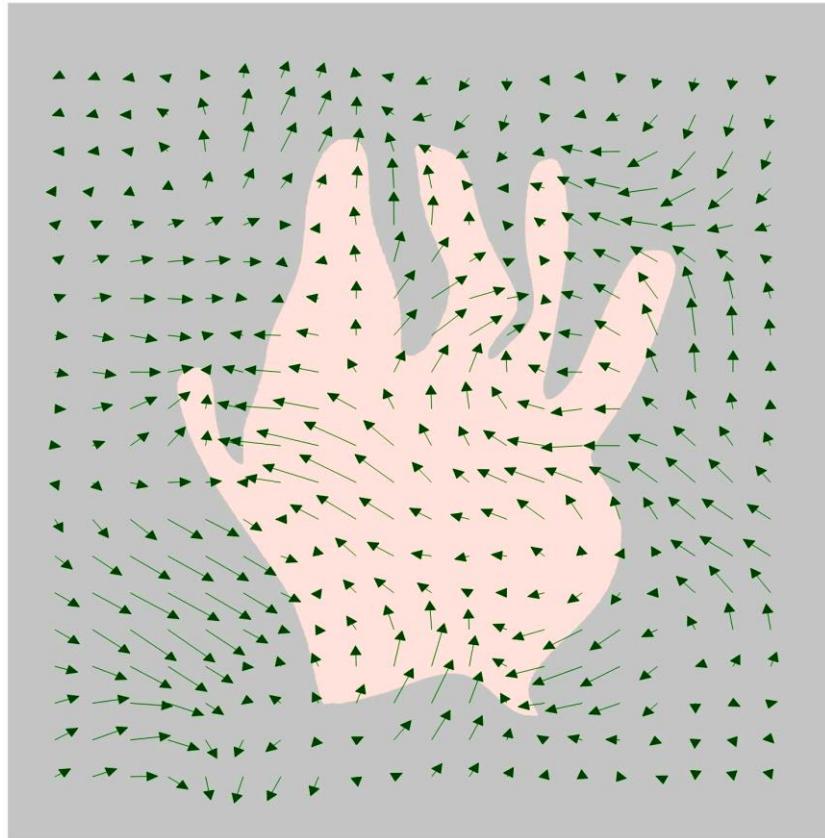
$$s_1 = s_2 \text{ large}, \quad \sigma_1 = \sigma_2 \text{ large}$$

A model for smooth deformations



$$s_1 = s_2 \text{ small}, \quad \sigma_1 = \sigma_2 \text{ large}$$

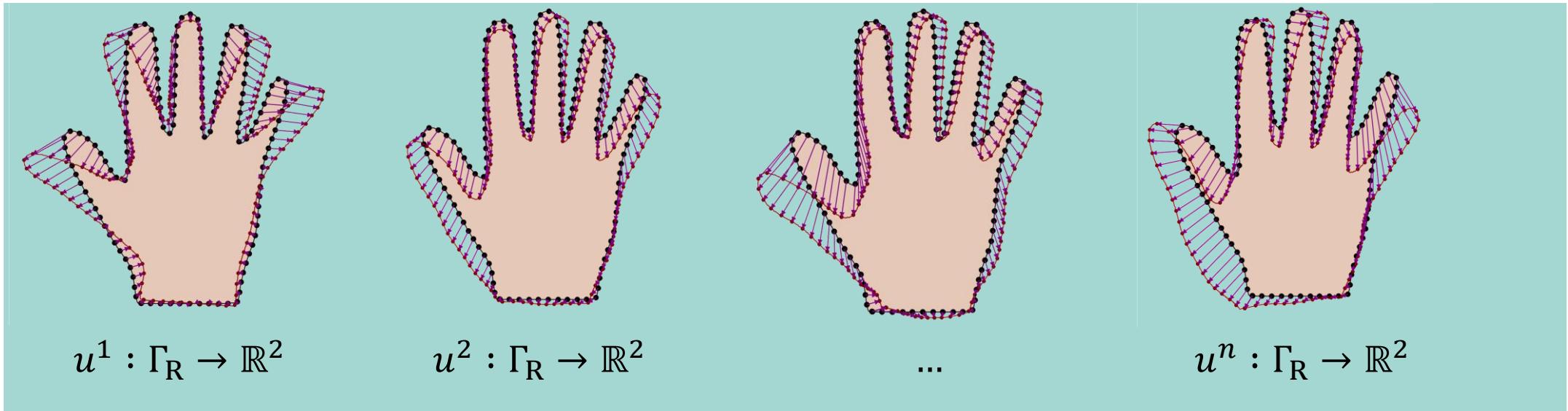
A model for smooth deformations



$$s_1 = s_2 \text{ large}, \quad \sigma_1 = \sigma_2 \text{ small}$$

Statistical shape models

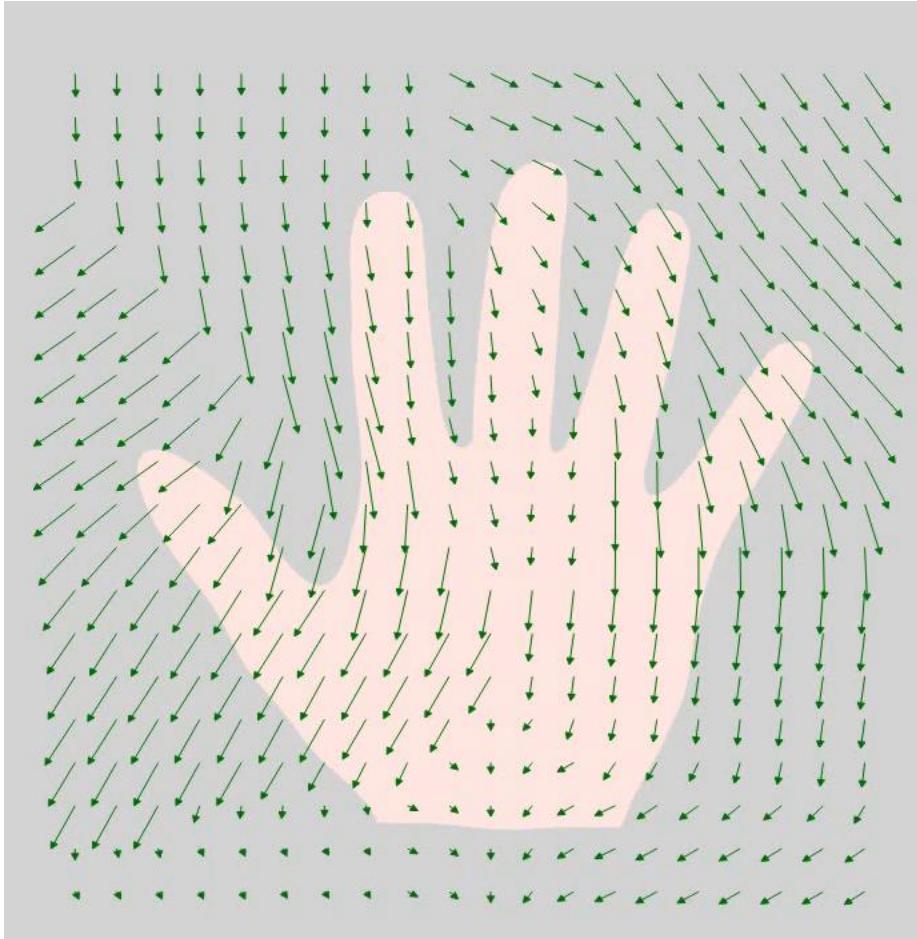
Idea: Models are learned from example deformation fields



$$\mu(x) = \bar{u}(x) = \frac{1}{n} \sum_{i=1}^n u^i(x)$$

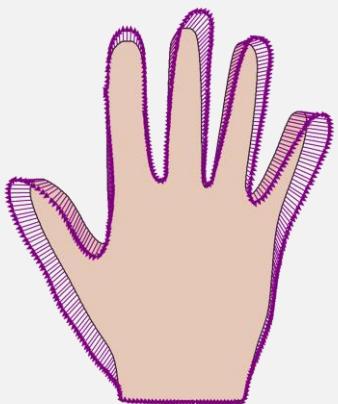
$$k(x, x') = \frac{1}{n-1} \sum_i^n (u^i(x) - \bar{u}(x))(u^i(x') - \bar{u}(x'))^T$$

Statistical shape models



Finite observations revisited

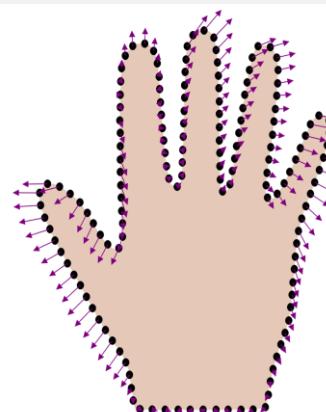
Continuous
representation



$GP(\mu, k)$

Too Large for real
3D shapes

Discrete
representation



$$N \left(\begin{pmatrix} \mu_1(x_1) \\ \mu_2(x_1) \\ \vdots \\ \mu_1(x_n) \\ \mu_2(x_n) \end{pmatrix}, \begin{pmatrix} k_{11}(x_1, x_1) & k_{12}(x_1, x_1) & \dots & k_{11}(x_1, x_n) & k_{12}(x_1, x_n) \\ k_{21}(x_1, x_1) & k_{22}(x_1, x_1) & \dots & k_{21}(x_1, x_n) & k_{22}(x_1, x_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{11}(x_n, x_1) & k_{12}(x_n, x_1) & \dots & k_{11}(x_n, x_n) & k_{12}(x_n, x_n) \\ k_{21}(x_n, x_1) & k_{22}(x_n, x_1) & \dots & k_{21}(x_n, x_n) & k_{22}(x_n, x_n) \end{pmatrix} \right)$$

The Karhunen-Loève expansion

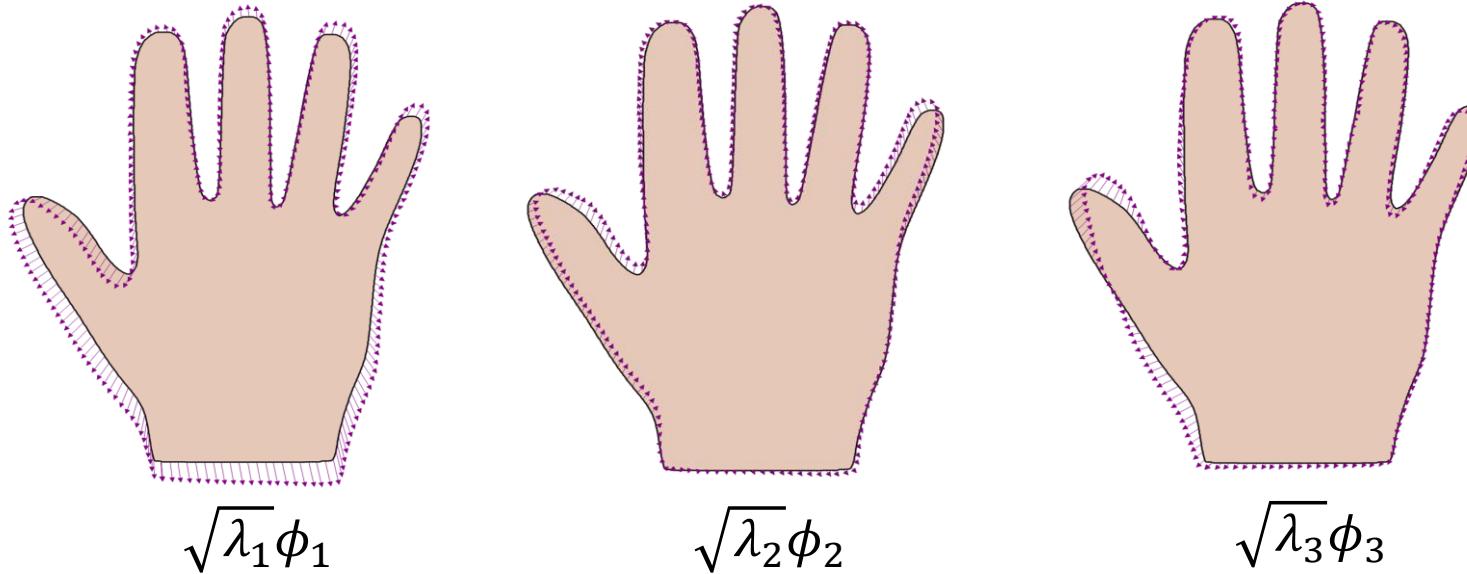
We can write

$$u \sim GP(\mu, k)$$

as

$$u \sim \mu + \sum_{i=1}^{\infty} \alpha_i \sqrt{\lambda_i} \phi_i, \quad \alpha_i \sim N(0, 1)$$

ϕ_i is the Karhunen-Loève basis and λ_i a scaling factor



Low-rank approximation

Approximation of rank r

$$u \sim \mu + \sum_{i=1}^{\textcolor{teal}{r}} \alpha_i \sqrt{\lambda_i} \phi_i, \quad \alpha_i \sim N(0, 1)$$

Any deformation u is determined by the coefficients

$$\alpha = (\alpha_1, \dots, \alpha_r)$$

$$p(u) = p(\alpha) = \prod_{i=1}^r \frac{1}{\sqrt{2\pi}} \exp(-\alpha_i^2/2)$$

Parametric nonparametrics

- *We use GPs as a modelling tool, and not because of infinite basis functions.*