Bayesian regression for 3D model fitting

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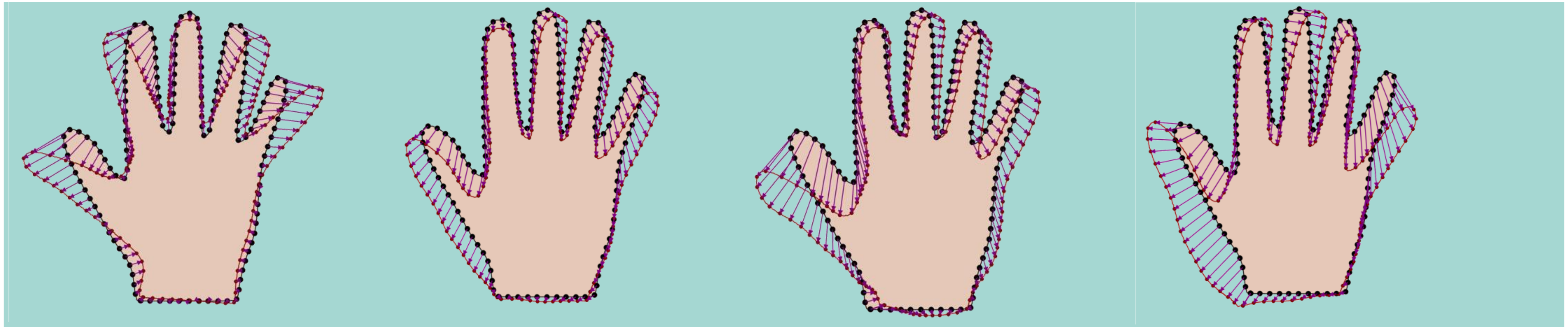
Statistical shape models

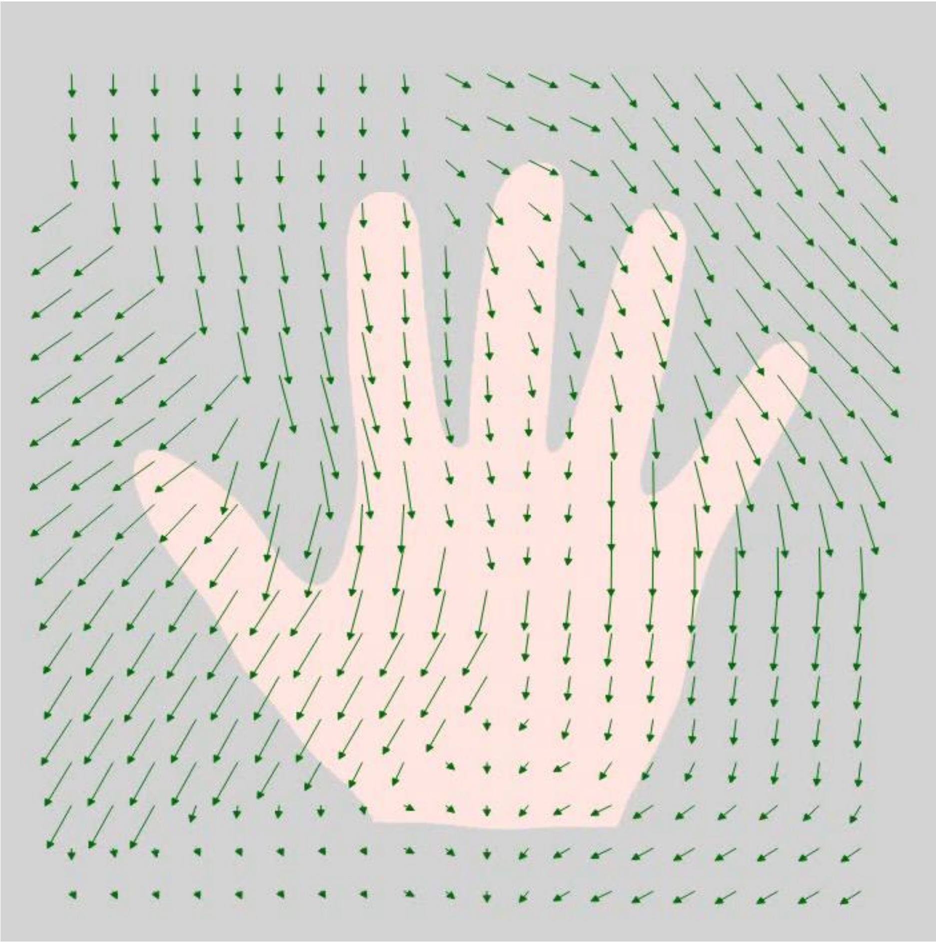
Idea: Models are learned from example deformation fields



$u^1 : \Gamma_R \rightarrow \mathbb{R}^2$ $\quad$ $u^2 : \Gamma_R \rightarrow \mathbb{R}^2$ $\quad$ … $\quad$ $u^n : \Gamma_R \rightarrow \mathbb{R}^2$

$$\mu(x) = \overline{u}(x) = \frac{1}{n}\sum_{i-1}^{n} u^i\,(x) \qquad k(x, x') = \frac{1}{n-1}\sum_{i}^{n}\left(u^i(x) - \overline{u}(x)\right)\left(u^i(x') - \overline{u}(x')\right)^T$$

Statistical shape models

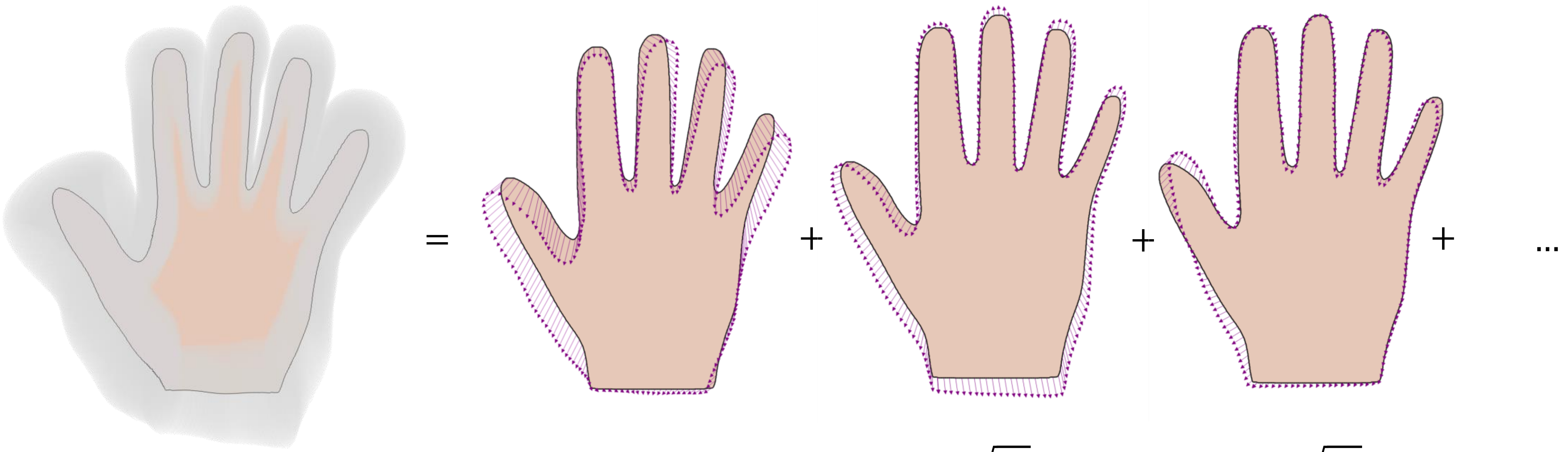


The Karhunen-Loève expansion

We can write $$u \sim GP(\mu, k)$$

as $$u \sim \mu + \sum_{i=1}^{\infty} \alpha_i \sqrt{\lambda_i}\, \phi_i, \;\; \alpha_i \sim N(0,1)$$

ϕ_i is the Karhunen-Loève basis and λ_ia scaling factor



u $\quad\quad \mu \quad\quad \alpha_1 \sqrt{\lambda_1}\phi_1 \quad\quad \alpha_2 \sqrt{\lambda_2}\phi_2$

Low-rank approximation

Approximation of rank r

$$u \sim \mu + \sum_{i=1}^{r} \alpha_i \sqrt{\lambda_i}\, \phi_i, \qquad \alpha_i \sim N(0,1)$$

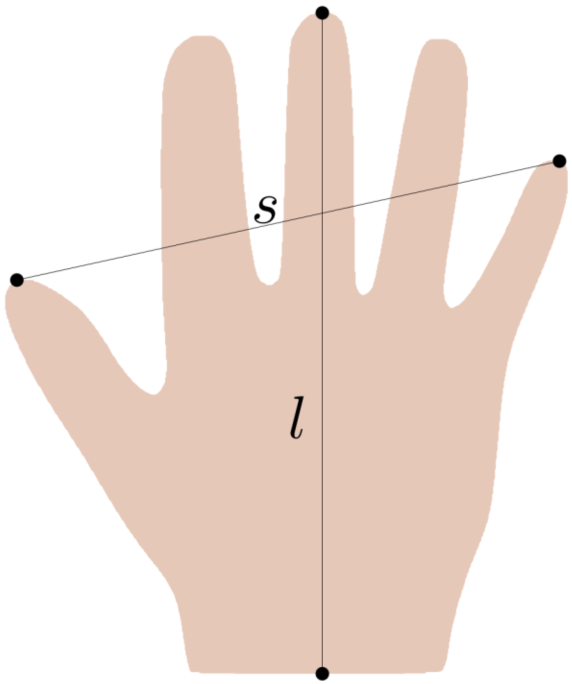
Any deformation u is determined by the coefficients
$\alpha = (\alpha_1, \dots, \alpha_r)$

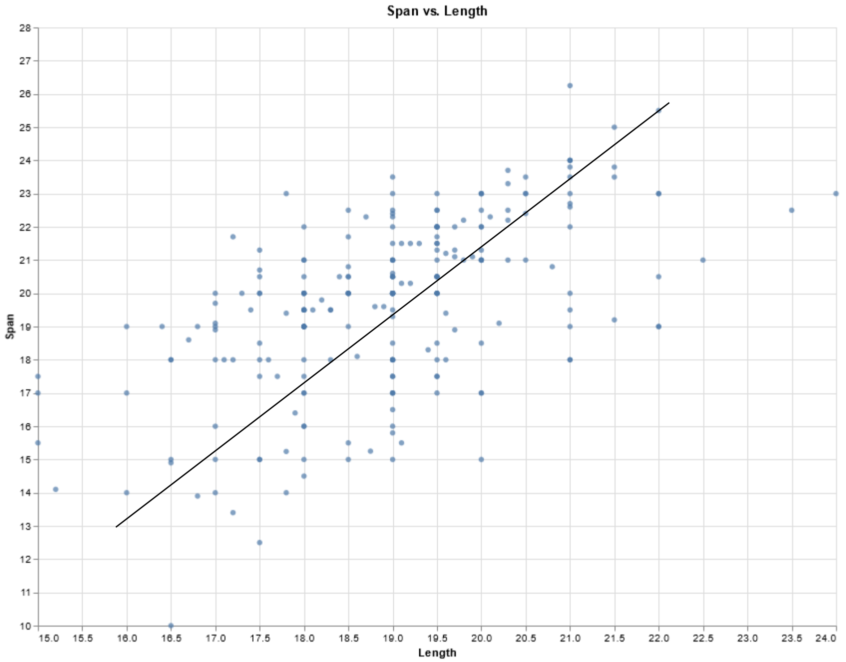
$$p(u) = p(\alpha) = \prod_{i=1}^{r} \frac{1}{\sqrt{2\pi}} \exp(-\alpha_i^2/2)$$

Parametric nonparametrics

- *We use GPs as a modelling tool, and not because of infinite basis functions.*

Revisiting Bayesian linear regression



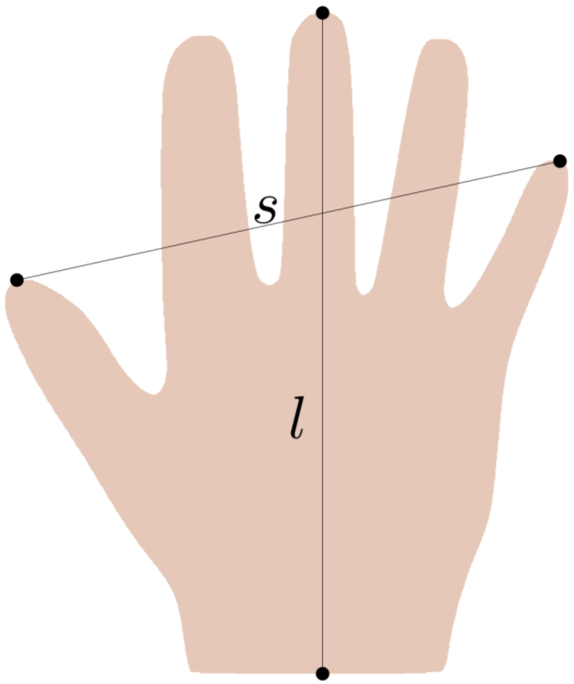


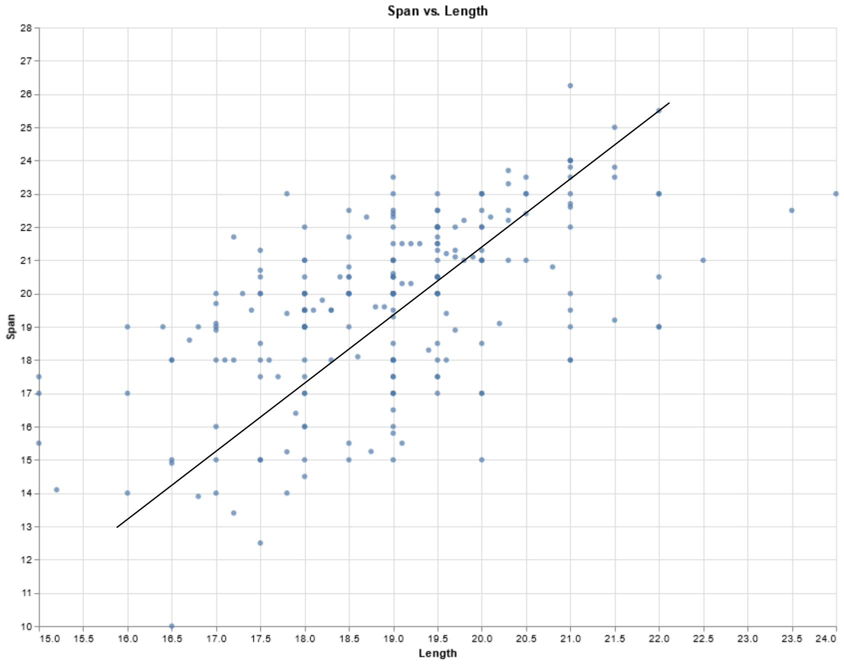
Linear relationship:

$s = a\left(l - \bar{l}\right) + b + \varepsilon,\ \ \varepsilon \sim N(0, \sigma^2)$

Probabilistic Model: $p(s \mid a, b, \sigma^2, l) = N\left(a \cdot \left(l - \bar{l}\right) + b, \sigma^2\right)$ with priors $p(a), p(b), p(\sigma^2)$

Revisiting Bayesian linear regression - Fitting





Linear relationship:

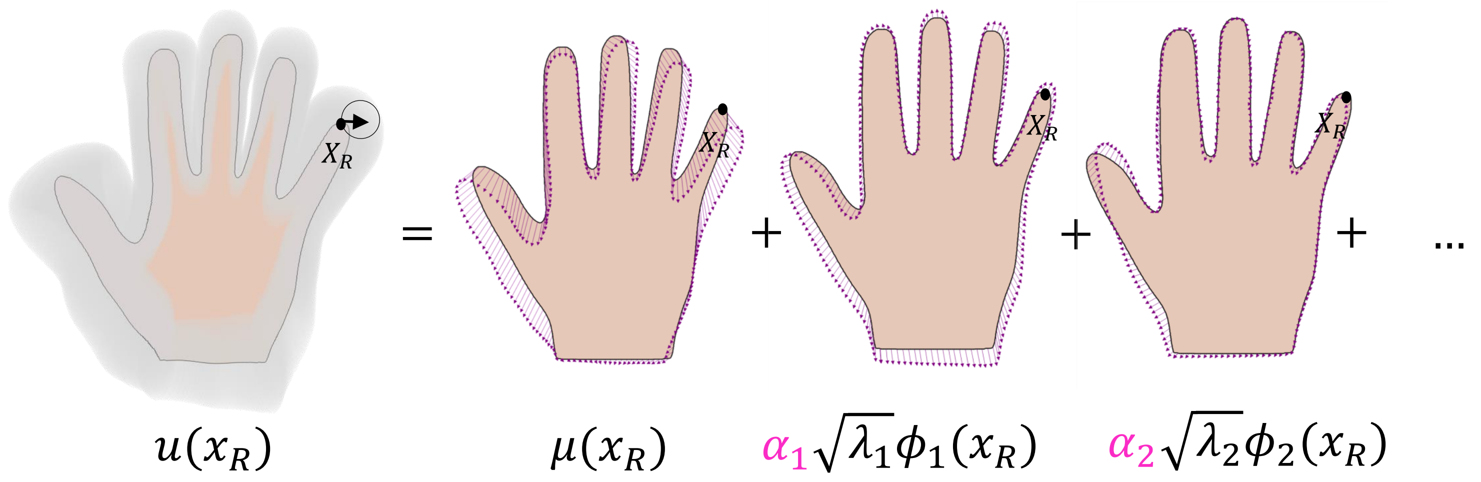
$$s = a\,(l - \bar{l}) + b + \varepsilon,\ \ \varepsilon \sim N(0, \sigma^2)$$

Model is fitted to observations $\hat{s}_1, \hat{l}_1, \dots, \hat{s}_n, \hat{l}_n$ using posterior $\prod_i^n \mathrm{p}(a, b, \sigma^2 \mid \hat{s}_i, \hat{l}_i)$

Bayesian Linear for shape model fitting

Linear relationship: Point $x_T$is given as

$$x_T \sim x_R + \mu(x_R) + \sum_{i=1}^{r} \alpha_i \lambda_i \phi_i(x_R) + \varepsilon,\ \varepsilon = N(\vec{0}, I_{3\times 3}\sigma)$$



Probabilistic Model: $p(x_T \mid \alpha_1, \dots, \alpha_n, \sigma^2) = N(x_R + \mu(x_R) + \sum_{i=1}^{r} \alpha_i \lambda_i \phi_i(x_R) + \varepsilon,\ , \sigma^2)$
with suitable priors $p(\alpha), p(\sigma^2)$

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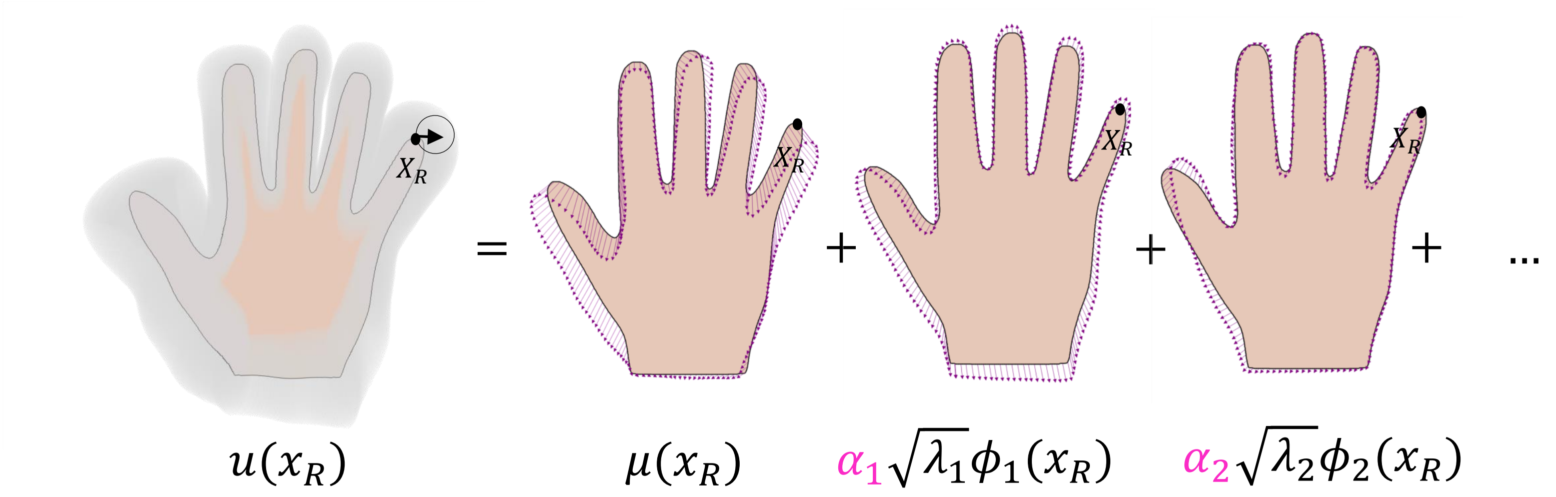

Probabilistic Model: $p(\, x_T \mid \alpha_1, \dots, \alpha_n, \sigma^2 \,) = N(x_R + \mu(x_R) + \sum_{i=1}^{r} \alpha_i \lambda_i \phi_i(x_R) + \varepsilon,\ , \sigma^2)$
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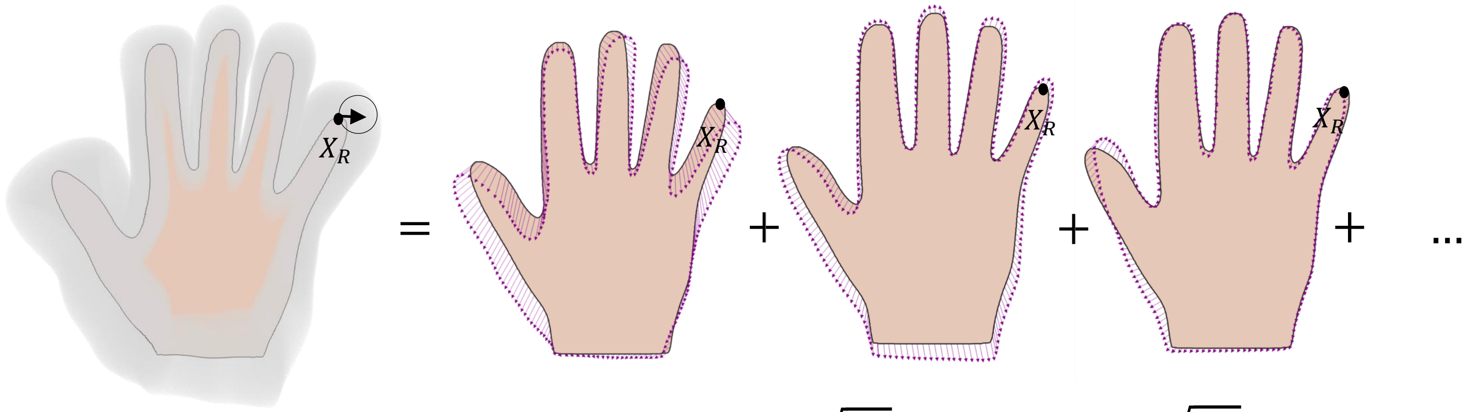
Bayesian Linear for shape model fitting – A complication

Linear relationship: Point x_T is given as

$$x_T \sim x_R + \mu(x_R) + \sum_{i=1}^{r} \alpha_i \lambda_i \phi_i(x_R) + \varepsilon,\ \varepsilon = N(\vec{0}, I_{3\times3}\sigma)$$



$u(x_R)$ $\mu(x_R)$ $\alpha_1\sqrt{\lambda_1}\phi_1(x_R)$ $\alpha_2\sqrt{\lambda_2}\phi_2(x_R)$

Model is fitted to observations $\hat{x}_{T,1}, \hat{x}_{R,1}, \dots, \hat{x}_{T,n}, \hat{x}_{R,n}$ using posterior $\prod_i p(\alpha_1, \dots, \alpha_n, \sigma^2 | \hat{x}_{T,i}\hat{x}_{R,i})$

But what is the observation $\hat{x}_{T,i}$ to $\hat{x}_{R,i}$?
Approximation: Used Closest Point on Target surface to $x_R + \mu(x_R) + \sum_{i=1}^{r} \alpha_i \lambda_i \phi_i(x_R) + \varepsilon,$