Skolkovo Institute of Science and Technology

Computational Science and Engineering II: Discretization

Model Order Reduction

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Teaching Assistants: *Ilias Giannakopoulos, Evgenii Tsymbalov*

Thanks to Thanos Polimeridis Skoltech

&

Luca Daniel MIT

OUTLINE

¬⅄ Model Order Reduction of Linear Problems

- **Problem setup**
- **Reduction via "modal analysis"**
  - time domain
  - frequency domain
- **Reduction via transfer function fitting**
  - point matching
  - least square
- **Importance of preserving physical properties**
  - stability
  - passivity/dissipativity

MOTIVATION

$$\frac{dx}{dt} = Ax(t) + bu(t)$$

$$y(t) = c^T x(t)$$

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Suppose

We are just interested in terminal (i.e. input/output) behavior

And we need to compute the output y(t) for many many different input signals u(t)

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We are just interested in terminal (i.e. input/output) behavior

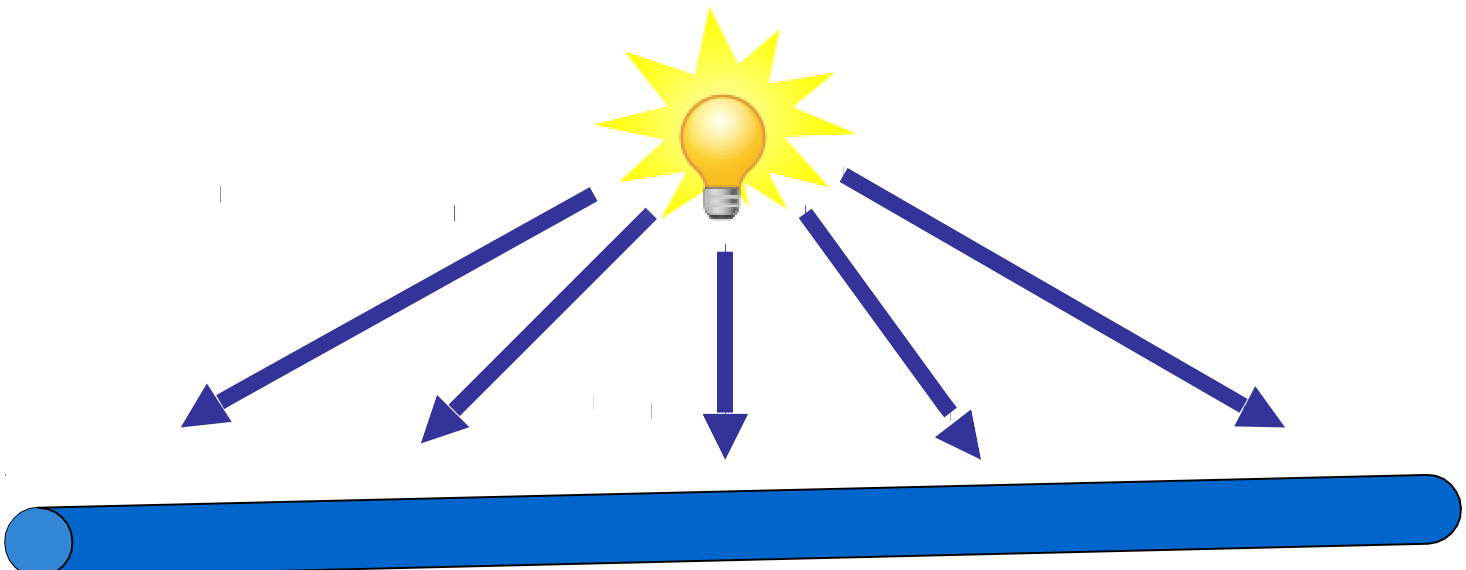
And we need to compute the output y(t) for many many different input signals u(t)

It might be more convenient to

1. do some pre-computation on the original system
2. generate a compact dynamical model
3. re-use the model over and over

$\text{Power of lamp} = u(t)$

Input of Interest

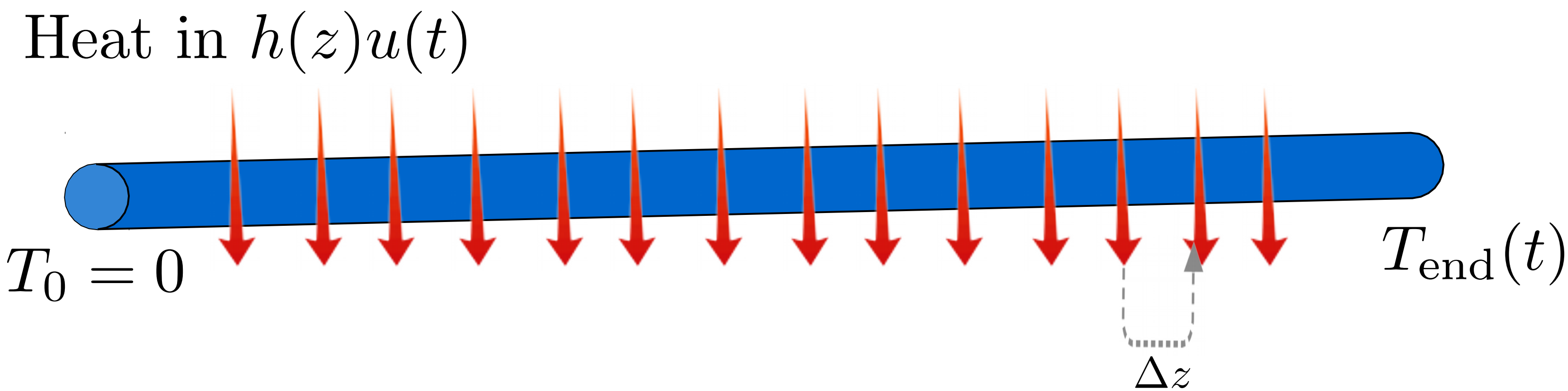


$T_0 = 0$

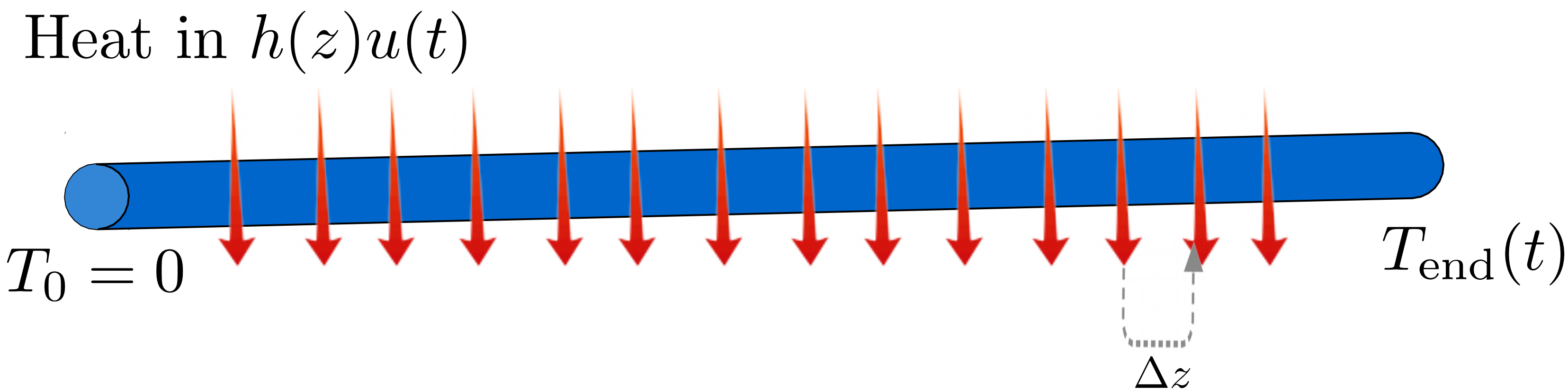
$T_{\text{end}}(t)$

Output of Interest

PROBLEM SETUP



PROBLEM SETUP



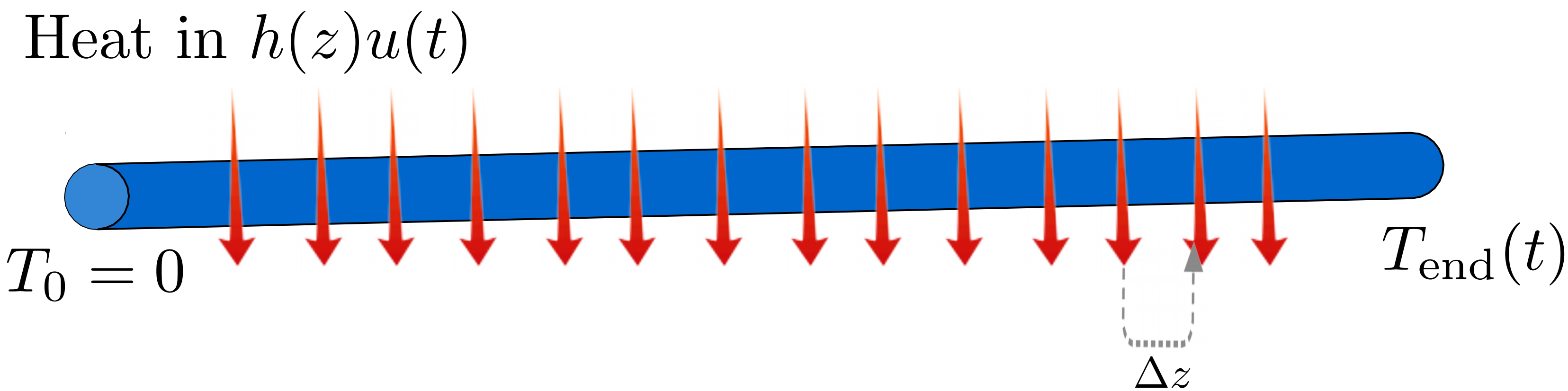
$$\gamma \frac{\partial T(z,t)}{\partial t} = k \frac{\partial^2 T(z,t)}{\partial z^2} + h(z)u(t)$$

specific heat

thermal conductivity

scalar input

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE PROBLEM SETUP



$$\gamma \frac{\partial T(z,t)}{\partial t} = k \frac{\partial^2 T(z,t)}{\partial z^2} + h(z)u(t)$$

specific heat

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scalar input

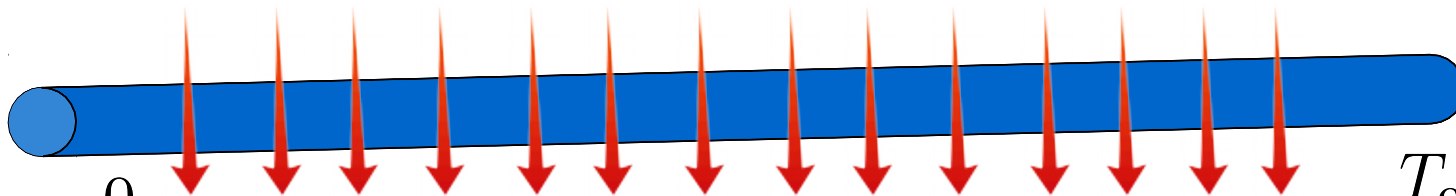
- Spatial Discretization

$$\gamma \frac{\partial \tilde{T}_i}{\partial t} = \frac{k}{(\Delta z)^2}(\tilde{T}_{i+1} - 2\tilde{T}_i + \tilde{T}_{i-1}) + h(z_i)u(t)$$

MODEL ORDER REDUCTION <u>HEAT CONDUCTING BAR EXAMPLE</u>

PROBLEM SETUP

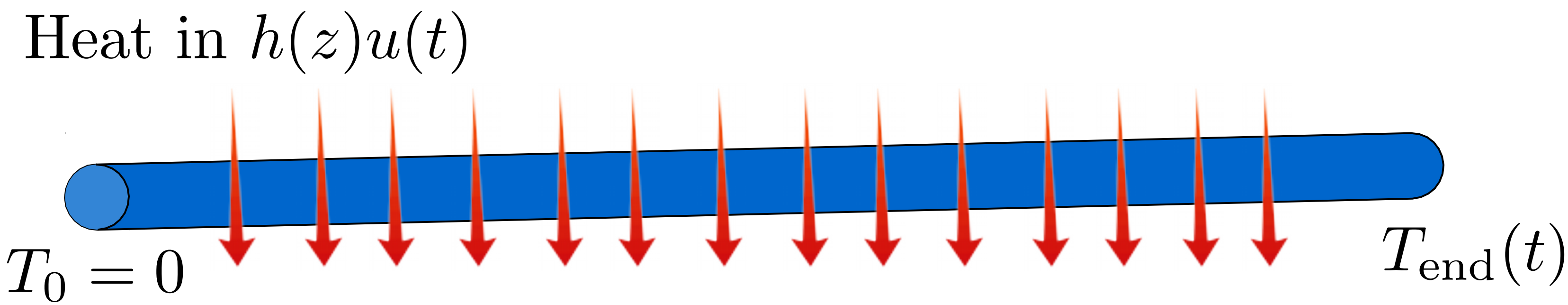
Heat in $h(z)u(t)$



$T_0 = 0$ $\qquad$ $T_{\text{end}}(t)$

$$\frac{dx(t)}{dt} = \underbrace{A}_{N \times N} x(t) + \underbrace{b}_{N \times 1}\underbrace{u(t)}_{\text{Scalar input}} \qquad \underbrace{y(t)}_{\text{Scalar output}} = \underbrace{c^T}_{N \times 1} x(t)$$

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE PROBLEM SETUP



$$\frac{dx(t)}{dt} = Ax(t) + bu(t) \qquad y(t) = c^T x(t)$$

$N \times N$ $\quad$ $N \times 1$ $\quad$ Scalar input $\quad$ Scalar output $\quad$ $N \times 1$

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \qquad b = \frac{1}{\gamma} \begin{bmatrix} h(z_1) \\ h(z_2) \\ \vdots \\ \vdots \\ h(z_N) \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Original Dynamical System - Single Input/Output

$$\frac{dx(t)}{dt} = Ax(t) + bu(t) \qquad y(t) = c^T x(t)$$

A: $N \times N$

b: $N \times 1$

$u(t)$: Scalar input

$y(t)$: Scalar output

c: $N \times 1$

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE PROBLEM SETUP

Original Dynamical System - Single Input/Output

$$\frac{dx(t)}{dt} = Ax(t) + bu(t) \qquad y(t) = c^T x(t)$$

$N \times N$ $N \times 1$ Scalar input Scalar output $N \times 1$

Reduced Dynamical System

$$\frac{dx(t)}{dt} = \tilde{A}x(t) + \tilde{b}u(t) \qquad y(t) = \tilde{c}^T x(t)$$

$q \times q$ $q \times 1$ Scalar input Scalar output $q \times 1$

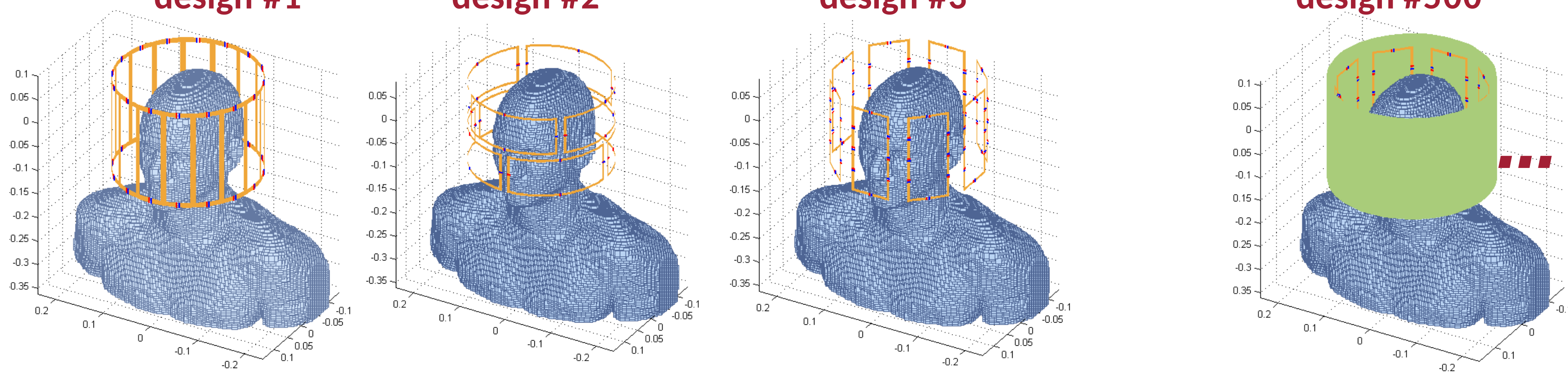
$q << N$

but input/output behavior is well preserved

RF-COIL DESIGN IN MAGNETIC RESONANCE IMAGING

- **Best out of 500 coil designs? Need full EM analysis of each design**

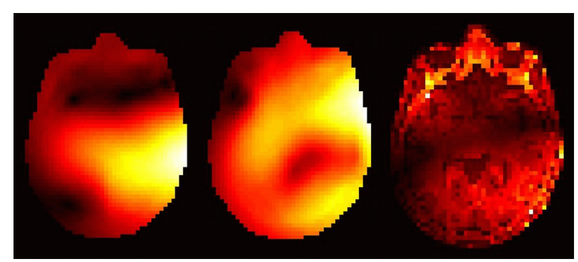
design #1 design #2 design #3 design #500



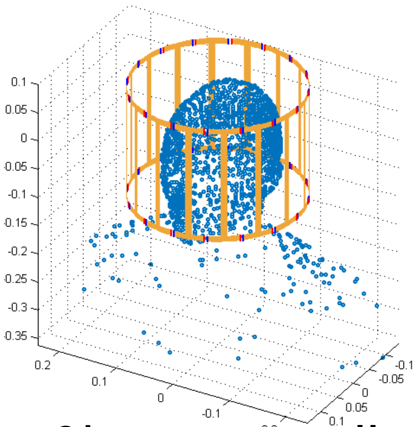
Standard Model
> 1 month
(hours / design)

Reduced Model
over the weekend
(~4min / design)

Full EM analysis



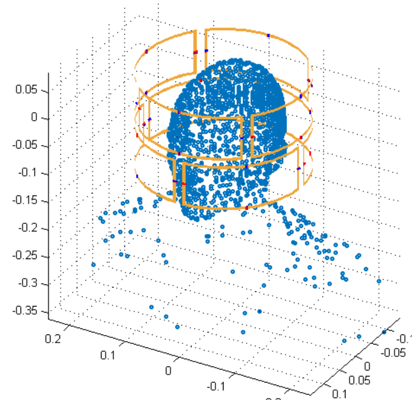
RF-COIL DESIGN IN MAGNETIC RESONANCE IMAGING



~8h29m overall
(~18 min/port)

~462X

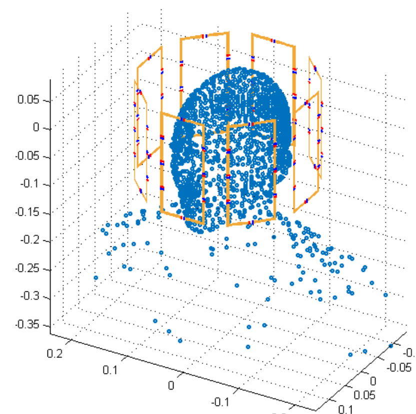
~1min overall



~7h30m overall
(~17 min/port)

~230X

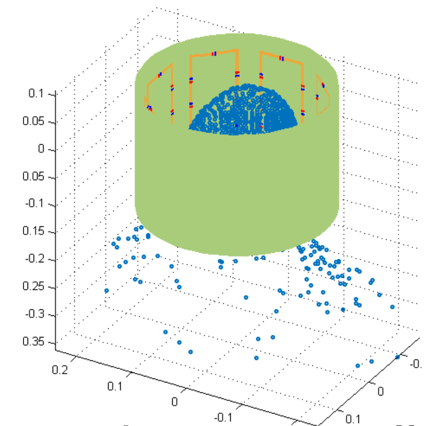
~2min overall



~9h32m overall
(~15 min/port)

~298X

~2min overall



~17h13m overall
(~30 min/port)

~154X

~7min overall

RF-COIL DESIGN IN MAGNETIC RESONANCE IMAGING

IEEE TRANSACTIONS ON

BIOMEDICAL ENGINEERING

A PUBLICATION OF THE IEEE ENGINEERING IN MEDICINE AND BIOLOGY SOCIETY

EMB

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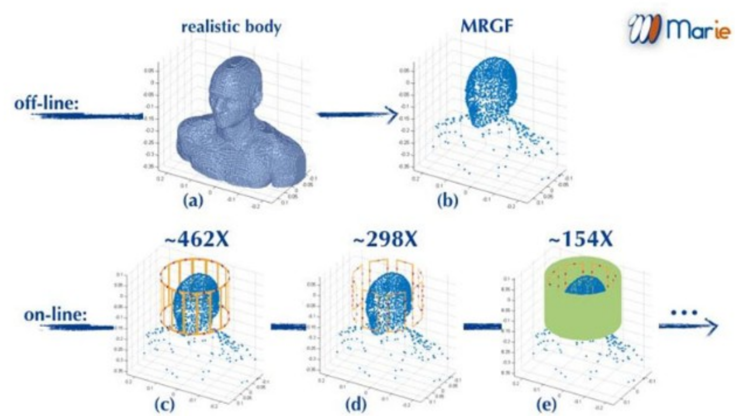

Magnetic resonance-specific framework based on the combination of integral equation methods and domain decomposition approaches for accelerating the numerical modeling of interactions between electromagnetic waves and biological tissue. (a) For any given realistic human body model, (b) pre-compute the magnetic resonance Green function (MRGF) via an ultrafast volume integral equations method and principal component analysis. (c)–(e) Combine the MRGF with standard surface integral equations method to model the performance of multiple transmit/receive coil configurations with more than two orders of magnitude speed-ups and negligible errors. See "Fast Electromagnetic Analysis of MRI Transmit RF Coils Based on Accelerated Integral Equation Methods," by Villena *et al.*, p. 2250.

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EIGENVALUE INTERLUDE

$$\boxed{\frac{dx}{dt} = Ax(t) + bu(t), \quad x(0) = x_0}$$

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Eigendecomposition:

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ V_1 & V_2 & \vdots & V_N \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ V_1 & V_2 & \vdots & V_N \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}^{-1}$$

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Change of variables:

$$V\tilde{x}(t) = x(t), \text{ or } \tilde{x}(t) = V^{-1}x(t)$$

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Change of variables:

$$V\tilde{x}(t) = x(t), \text{ or } \tilde{x}(t) = V^{-1}x(t)$$

Substituting:

$$\frac{dV\tilde{x}(t)}{dt} = AV\tilde{x}(t) + bu(t), \quad V\tilde{x}(0) = x_0$$

EIGENVALUE INTERLUDE

Decoupled equations:

$$\frac{d\tilde{x}(t)}{dt} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \tilde{x}(t) + V^{-1} b u(t)$$

EIGENVALUE INTERLUDE

Decoupled equations:

$$\frac{d\tilde{x}(t)}{dt} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \tilde{x}(t) + V^{-1} b u(t)$$

Output equations:

$$y(t) = c^T x(t) = c^T V \tilde{x}(t) = (V^T c)^T \tilde{x}(t)$$

$$\tilde{b} = V^{-1} b$$

$$\tilde{c} = V^T c$$

MODEL ORDER REDUCTION VIA MODEL ANALYSIS
(i.e. dominant eigenvalues/poles method)

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_1\\ \vdots\\ \vdots\\ \tilde{x}_N\end{bmatrix} = \begin{bmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_N\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \vdots\\ \vdots\\ \tilde{x}_N\end{bmatrix} + \begin{bmatrix}\tilde{b}_1\\ \vdots\\ \vdots\\ \tilde{b}_N\end{bmatrix} u(t)$$

$$y(t) = [\tilde{c}_1 \cdots\cdots \tilde{c}_N]\begin{bmatrix}\tilde{x}_1\\ \vdots\\ \vdots\\ \tilde{x}_N\end{bmatrix}$$

MODEL ORDER REDUCTION VIA MODEL ANALYSIS
(i.e. dominant eigenvalues/poles method)

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_1\\ \vdots\\ \tilde{x}_q\end{bmatrix} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_q\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \vdots\\ \tilde{x}_q\end{bmatrix} + \begin{bmatrix}\tilde{b}_1\\ \vdots\\ \tilde{b}_q\end{bmatrix} u(t)$$

$$y(t) = [\tilde{c}_1 \cdots \tilde{c}_q]\begin{bmatrix}\tilde{x}_1\\ \vdots\\ \tilde{x}_q\end{bmatrix}$$

MODEL ORDER REDUCTION VIA MODEL ANALYSIS
(i.e. dominant eigenvalues/poles method)

Certain modes are not affected by the input:

$$\tilde{b}_{q+1}, \cdots, \tilde{b}_N, \quad \text{are all small}$$

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Certain modes do not affect the output:

$$\tilde{c}_{q+1}, \cdots, \tilde{c}_N, \quad \text{are all small}$$

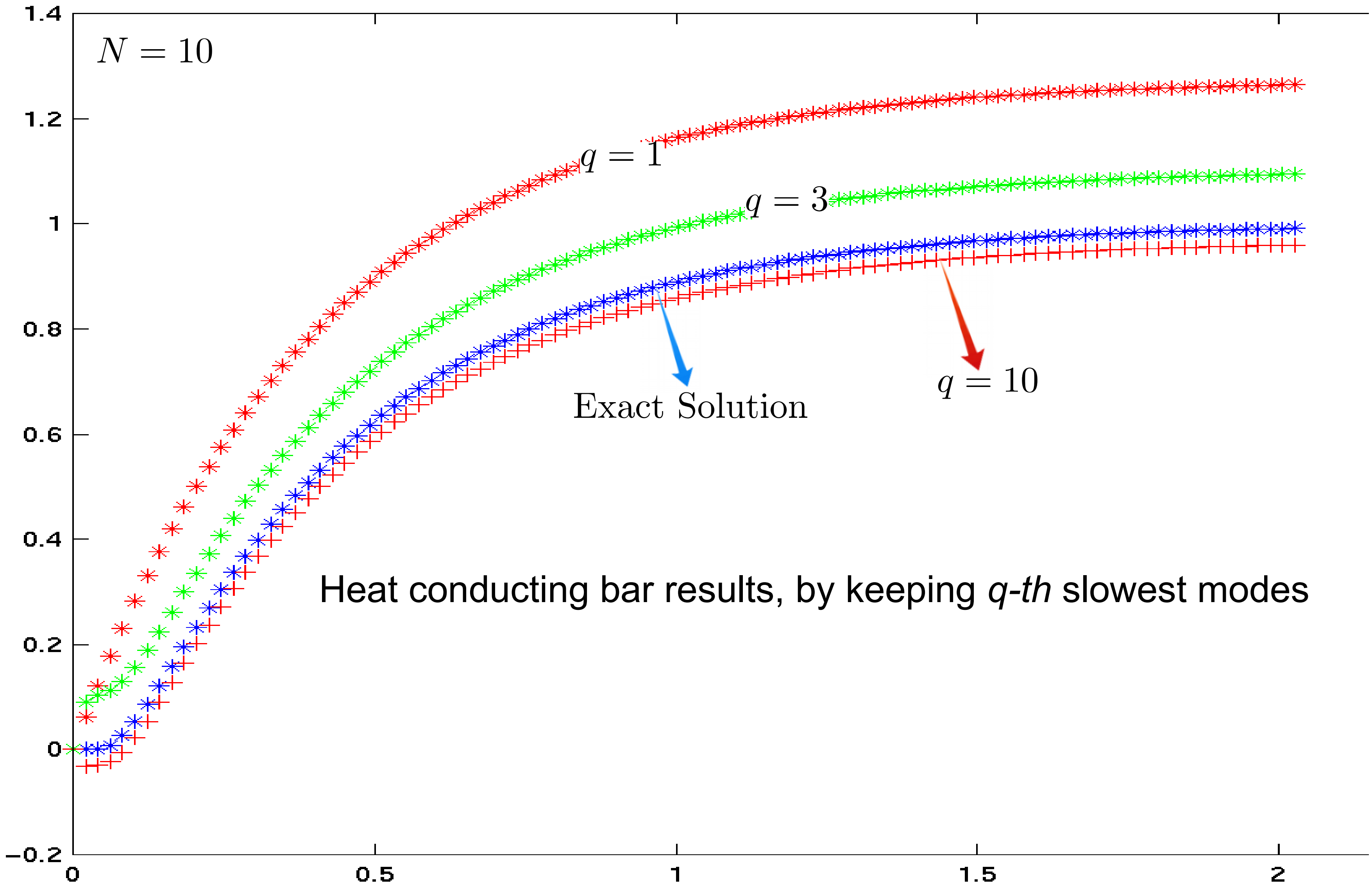
Keep least negative eigenvalues (slowest modes)

Look at response to a constant input:

Small if $|\lambda_i|$ is large

$$\tilde{x}_i(t) = \int_0^t e^{\lambda_i(t-\tau)}\tilde{b}_i u(t) d\tau = -\frac{1}{\lambda_i}(1 - e^{\lambda_i t})\tilde{b}_i u(t)$$

MODEL ORDER REDUCTION VIA MODEL ANALYSIS
(i.e. dominant eigenvalues/poles method)



Heat conducting bar results, by keeping *q-th* slowest modes

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ON TRANSFER FUNCTIONS – LAPLACE TRANSFORM

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Bilateral Laplace transform

$$X(s) = \int_{\infty}^{\infty} x(t)e^{-st}dt$$

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Key transform property

$$sX(s) = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

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$$sX(s) = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

Rewrite the ODE in "Laplace domain"

$$sX(s) = AX(s) + bU(s), \quad Y(s) = c^T X(s)$$

$$Y(s) = c^T (sI - A)^{-1} bU(s)$$

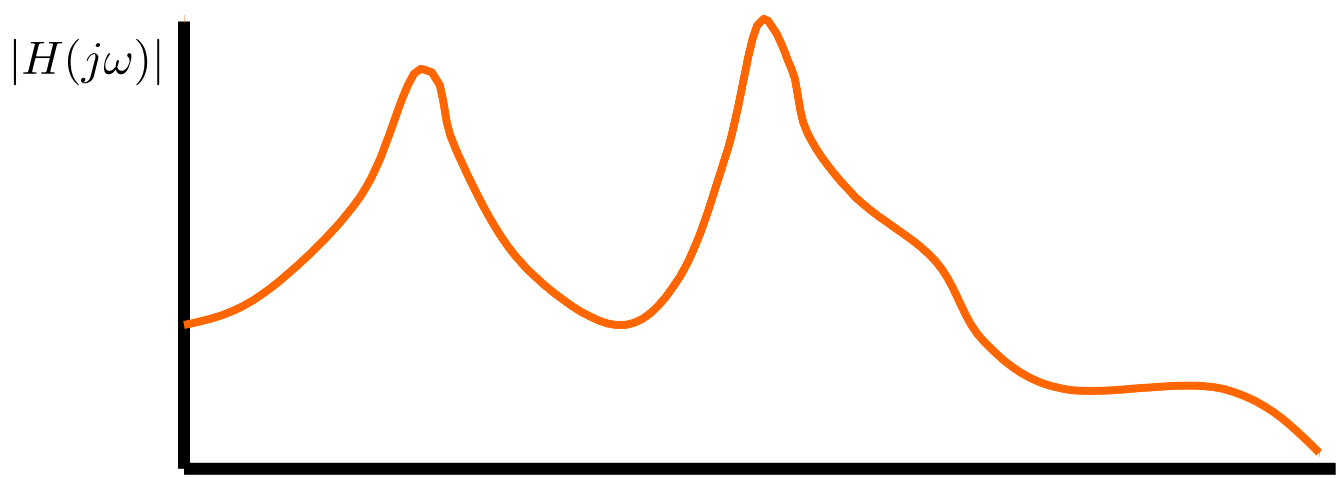
$$H(s) = c^T (sI - A)^{-1} b, \quad \text{Transfer function}$$

ON TRANSFER FUNCTIONS – MEANING OF H(s)

For stable systems $H(j\omega)$ is the frequency response

If $u(t) = e^{j\omega t} \rightarrow$ Sinusoid

Then $y(t) = H(j\omega)e^{j\omega t} \rightarrow$ Sinusoid with shifted phase and amplitude



MODEL (EIGENVALUE) ANALYSIS IN THE FREQUENCY DOMAIN

Transfer function:

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Apply Eigendecomposition $\longrightarrow$ $A = V \Lambda V^{-1}$

$$H(s) = c^T V (sI - \Lambda)^{-1} V^{-1} b$$

MODEL (EIGENVALUE) ANALYSIS IN THE FREQUENCY DOMAIN

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$$= \tilde{c}^T \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_N} \end{bmatrix} \tilde{b} \quad \Longrightarrow \quad H(s) = \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Eliminate each mode for which this term is small

MODEL ORDER REDUCTION via MODEL ANALYSIS (i.e. dominant eigenvalues/poles method)

$$H(s) = \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Pole-Residue Form

MODEL ORDER REDUCTION via MODEL ANALYSIS (i.e. dominant eigenvalues/poles method)

$$H(s) = \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Pole-Residue Form

$$H(s) = \frac{\prod_{i=1}^{N} s - \zeta_i}{\prod_{i=1}^{N} s - \lambda_i}$$

Pole-Zero Form

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Impulse Response

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Impulse Response

Ideas for Reducing Order:

1) Drop terms with small residues $\tilde{c}_i \tilde{b}_i$
2) Drop terms with large negative $Re\{\lambda_i\}$
3) Remove pole/zero near-cancellation stability?

MODEL ORDER REDUCTION via MODEL ANALYSIS (i.e. dominant eigenvalues/poles method)

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Impulse Response

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2) Drop terms with large negative
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stability?

ㄱㅅ**Advantages**

ㄱㅅ 1) Conceptually Familiar

ㄱㅅ 2) Simple physical interpretation: retains dominant system modes/poles

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Pole-Zero Form

$$h(t) = \sum_{i=1}^{N} \tilde{c}_i \tilde{b}_i e^{\lambda_i t}$$

Impulse Response

Rational Function Form

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{N-1} s^{N-1}}{1 + a_1 s + \cdots + a_N s^N}$$

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

Original System Transfer Function:

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{N-1} s^{N-1}}{1 + a_1 s + \cdots + a_N s^N}$$

Rational function

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Rational function

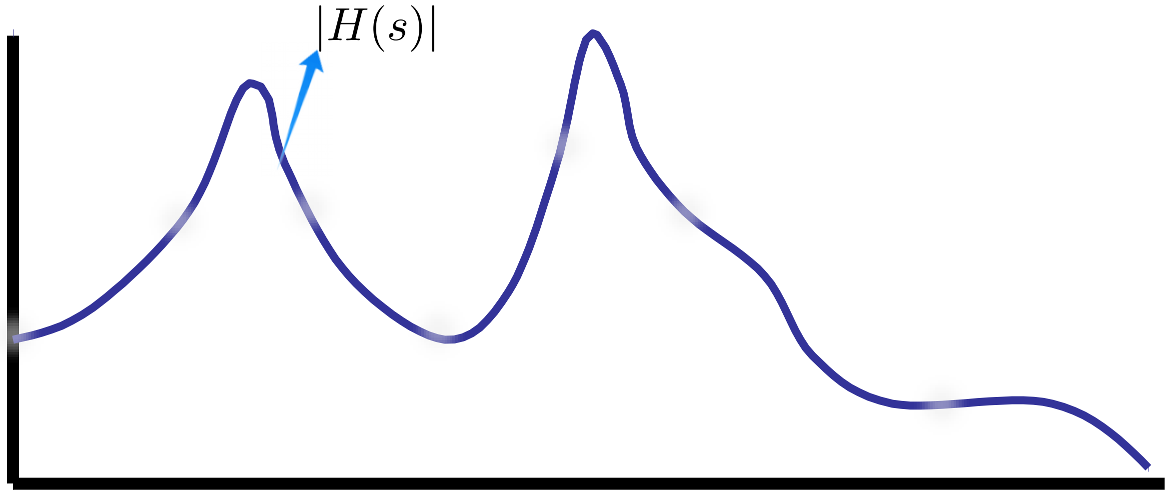
Model Reduction: Find a low order (q<<N) rational function matching

$$\tilde{H}(s) = \frac{\tilde{b}_0 + \tilde{b}_1 s + \cdots + \tilde{b}_{q-1} s^{q-1}}{1 + \tilde{a}_1 s + \cdots + \tilde{a}_q s^q}$$

Reduced Order
Rational function

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

Point matching



MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

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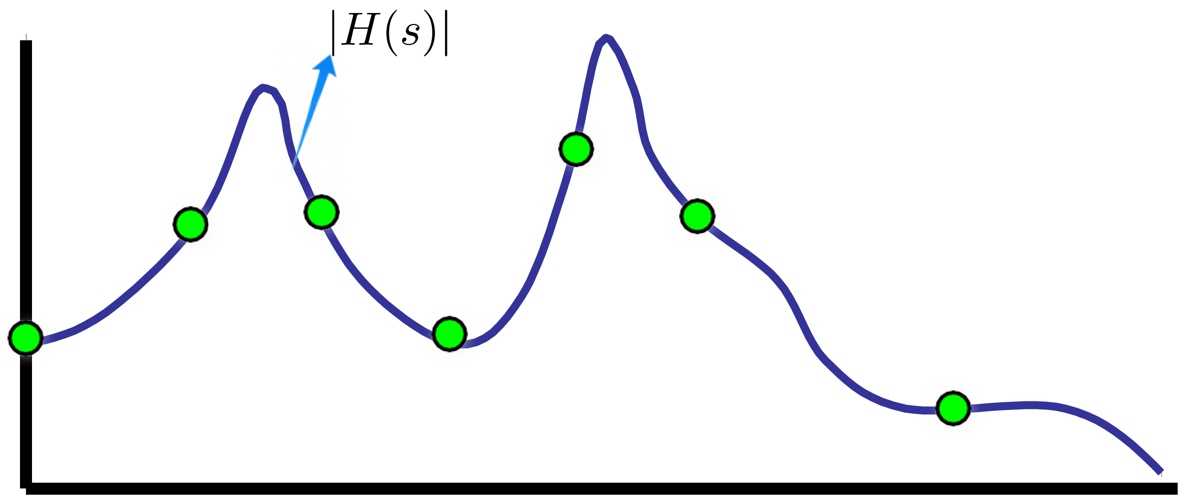
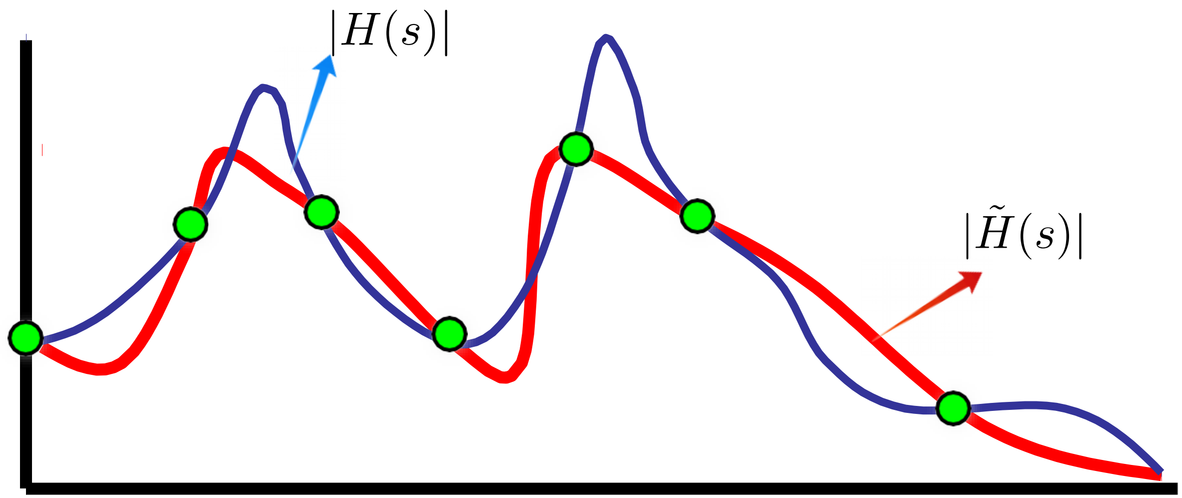


Can Match 2q points

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{q-1} s^{q-1}}{1 + a_1 s + \cdots + a_q s^q}$$

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING
Point matching



Can Match 2q points

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{q-1} s^{q-1}}{1 + a_1 s + \cdots + a_q s^q}$$

Cross multiplying generates a linear system

For i=1 to 2q

$$(1 + \tilde{a}_1 s_i + \cdots + \tilde{a}_q s_i^q) H(s_i) - (\tilde{b}_0 + \tilde{b}_1 s_i + \cdots + \tilde{b}_{q-1} s_i^{q-1}) = 0$$

Point matching

$$\begin{bmatrix} s_1 H(s_1) & s_1^2 H(s_1) & \cdots & -s_1^{q-1} \\ \vdots & \vdots & \cdots & -s_2^{q-1} \\ \vdots & \vdots & \cdots & \vdots \\ s_{2q}^2 H(s_{2q}) & s_{2q}^2 H(s_{2q}) & \cdots & -s_{2q}^{q-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{b}_{q-1} \end{bmatrix} = \begin{bmatrix} -H(s_1) \\ -H(s_2) \\ \vdots \\ -H(s_{2q}) \end{bmatrix}$$

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1) The Columns contain progressively higher powers of the test frequencies: problem is numerically ill-conditioned

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1) The Columns contain progressively higher powers of the test frequencies: problem is numerically ill-conditioned

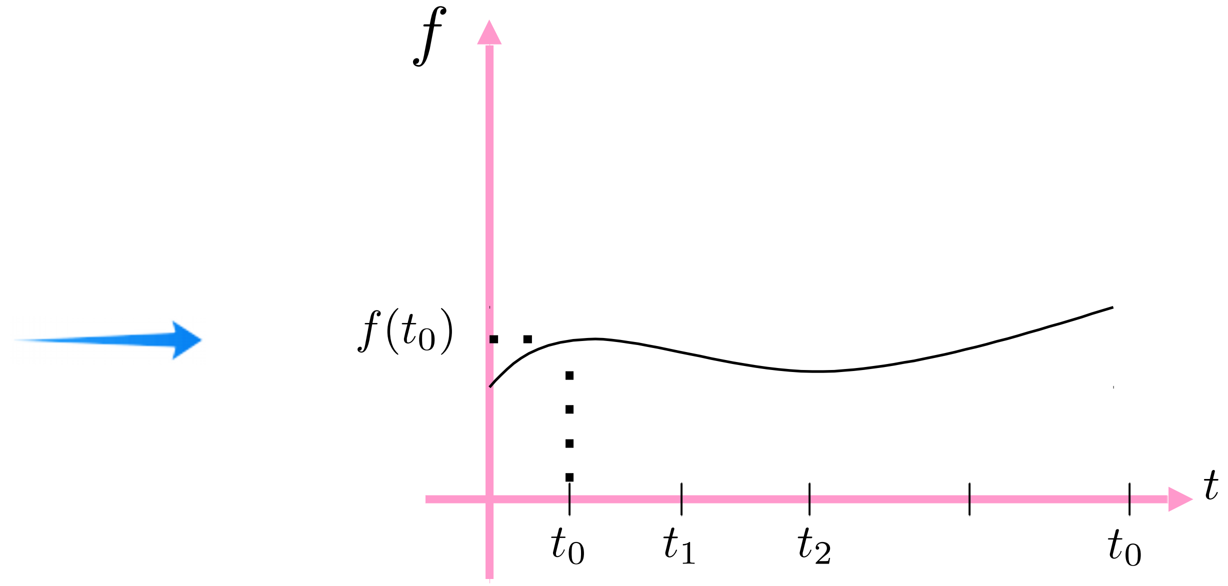
2) Missing data can cause severe accuracy problems

HARD TO SOLVE PROBLEMS

“Polynomial interpolation” example

Table of Data

$$
\begin{array}{ll}
t_0 & f(t_0) \\
t_1 & f(t_1) \\
\vdots & \vdots \\
\vdots & \vdots \\
t_N & f(t_N)
\end{array}
$$



HARD TO SOLVE PROBLEMS

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Table of Data

t_0	$f(t_0)$
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$\vdots$	$\vdots$
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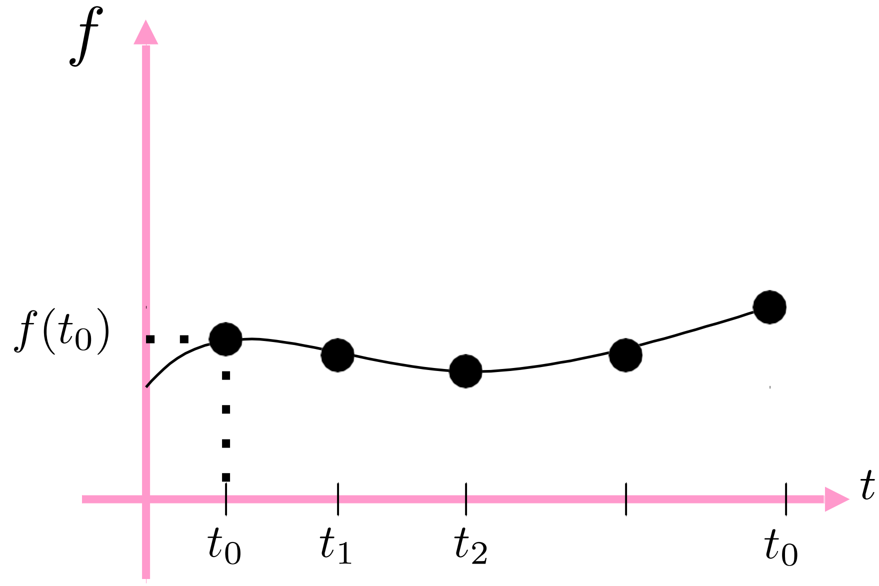


Problem fit data with an Nth order polynomial

$$\mathrm{f(t)} = \mathrm{a}_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N$$

Matrix Form

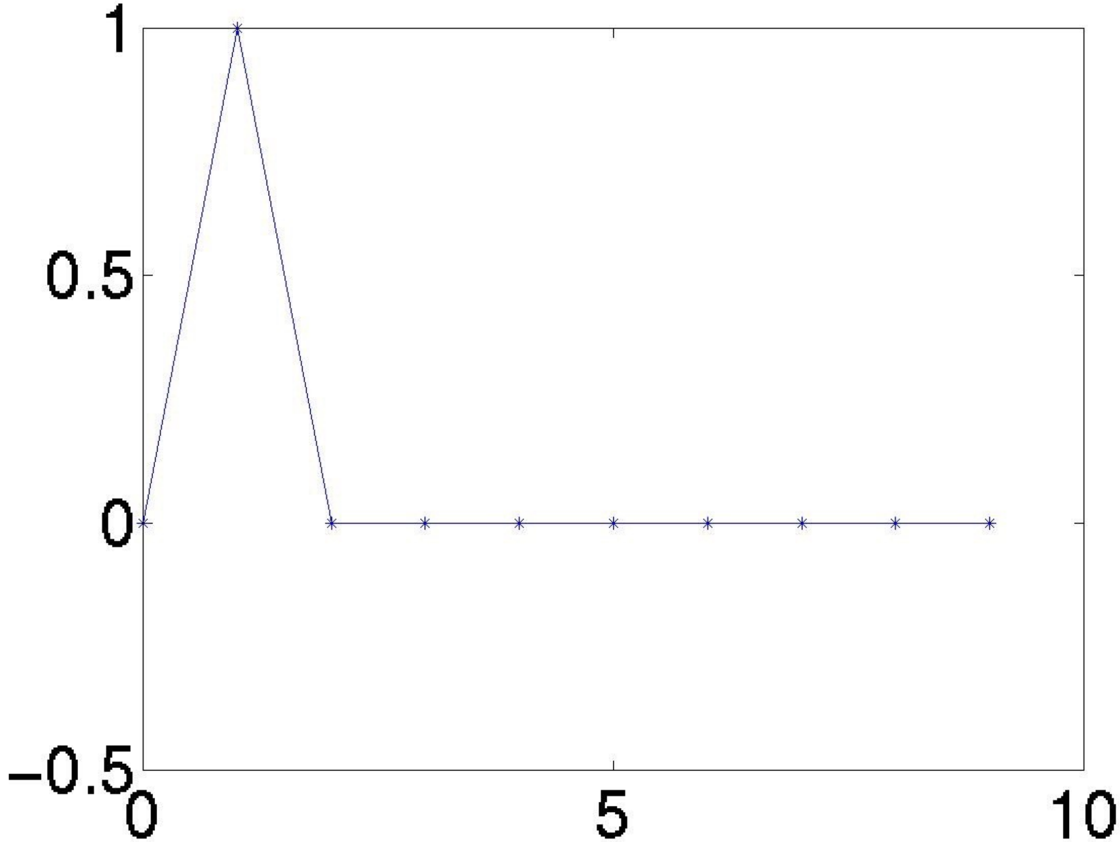
$$\begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^N \\ 1 & t_1 & t_1^2 & \cdots & t_1^N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_N) \end{bmatrix}$$

$A_{\text{interpolation}}$

HARD TO SOLVE PROBLEMS

"Polynomial interpolation" example

Fitting $f(t) = t$



Coefficient Value

Coefficient number

HARD TO SOLVE PROBLEMS

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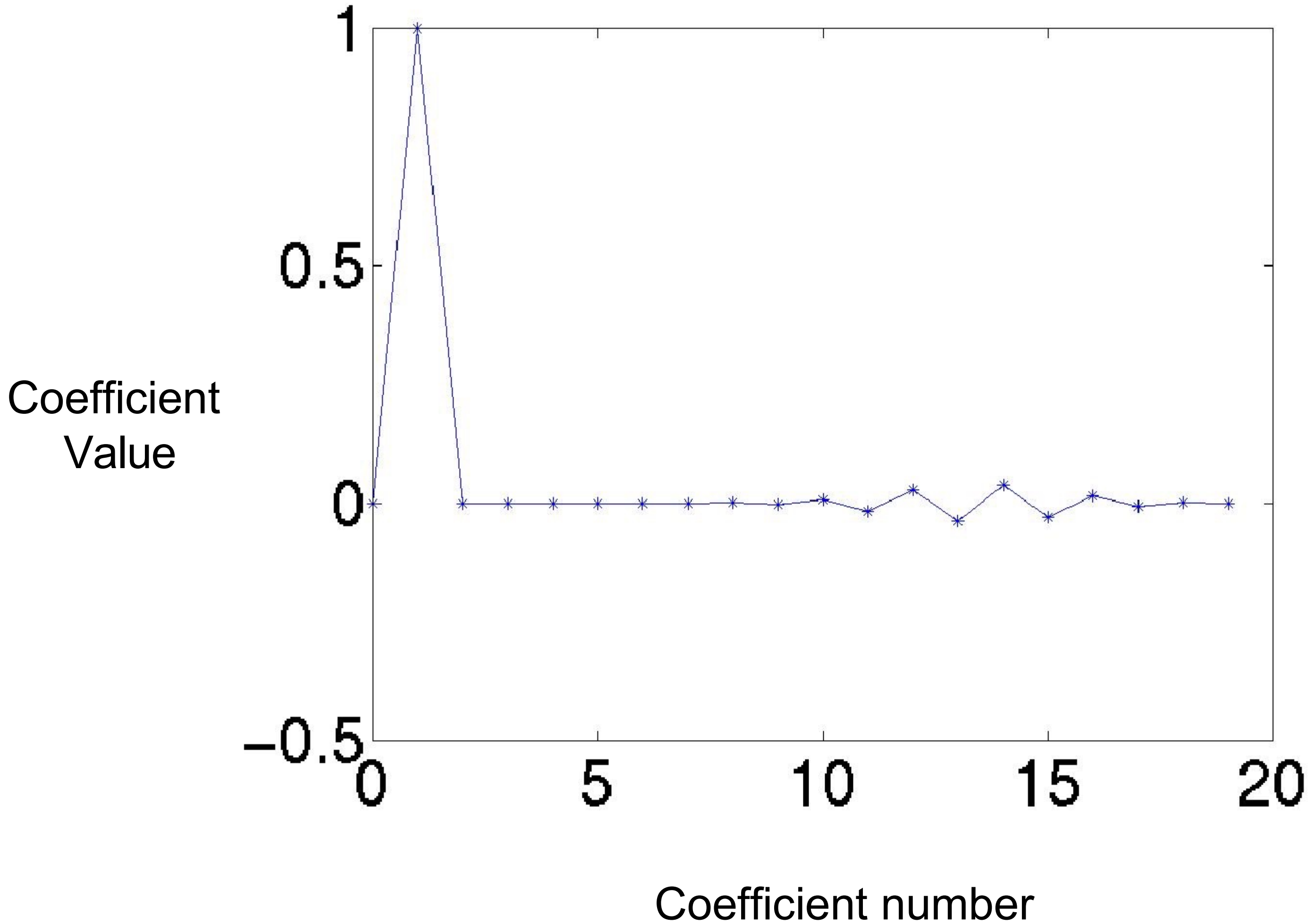

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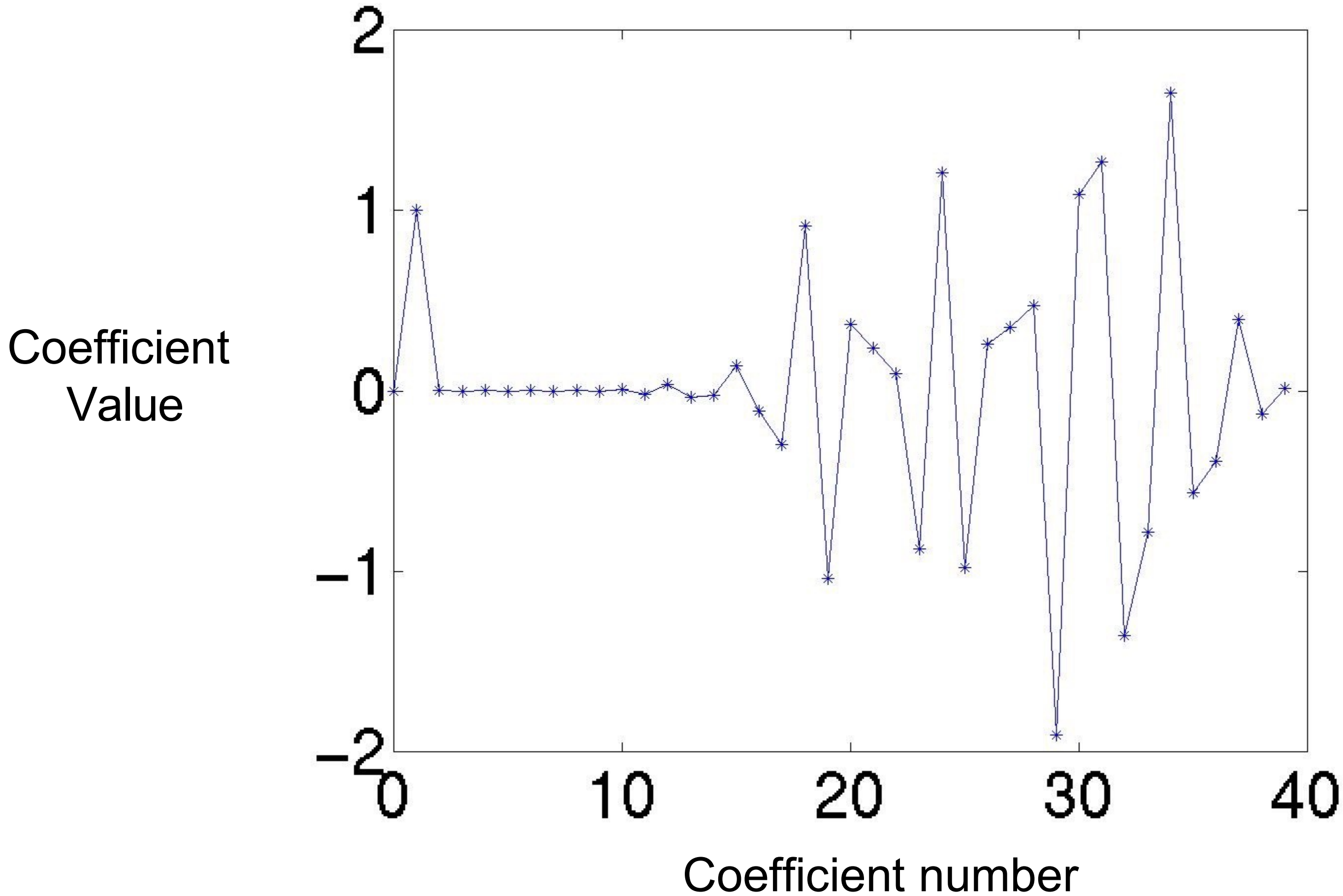

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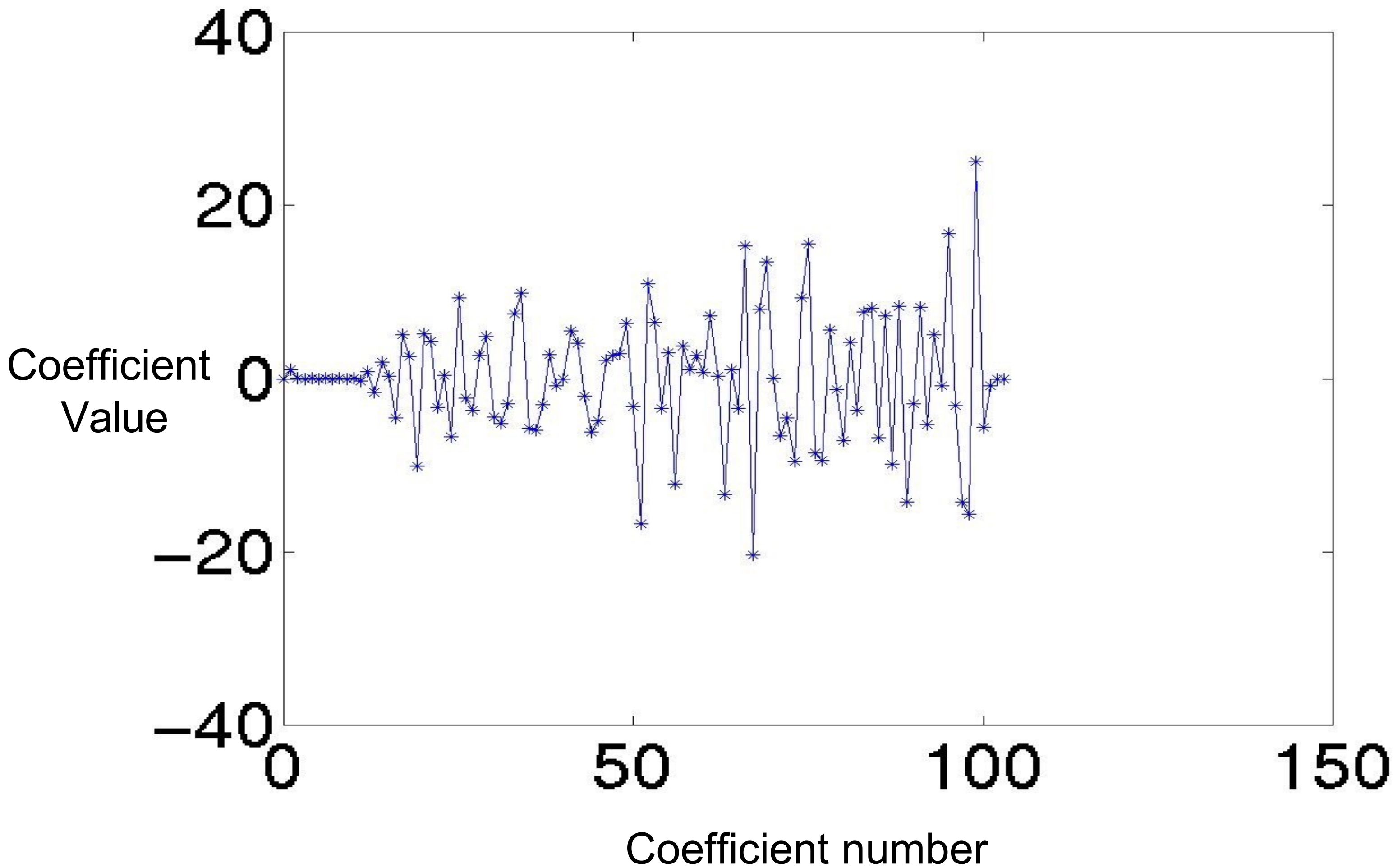
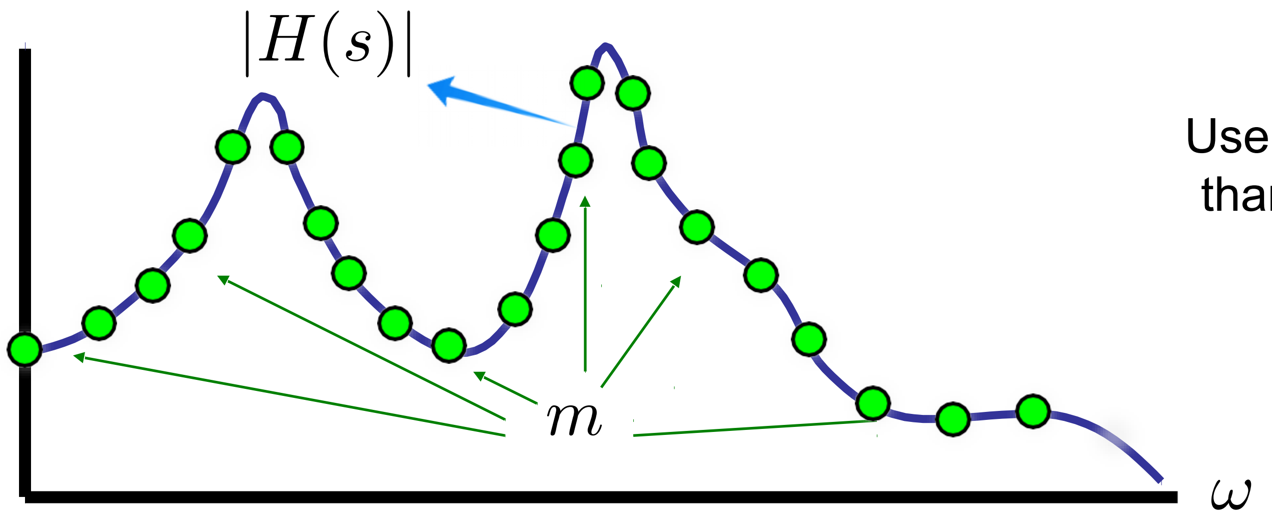


MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING POINT MATCHING USING LEAST SQUARES



Use much lower order than available points
2q<<m

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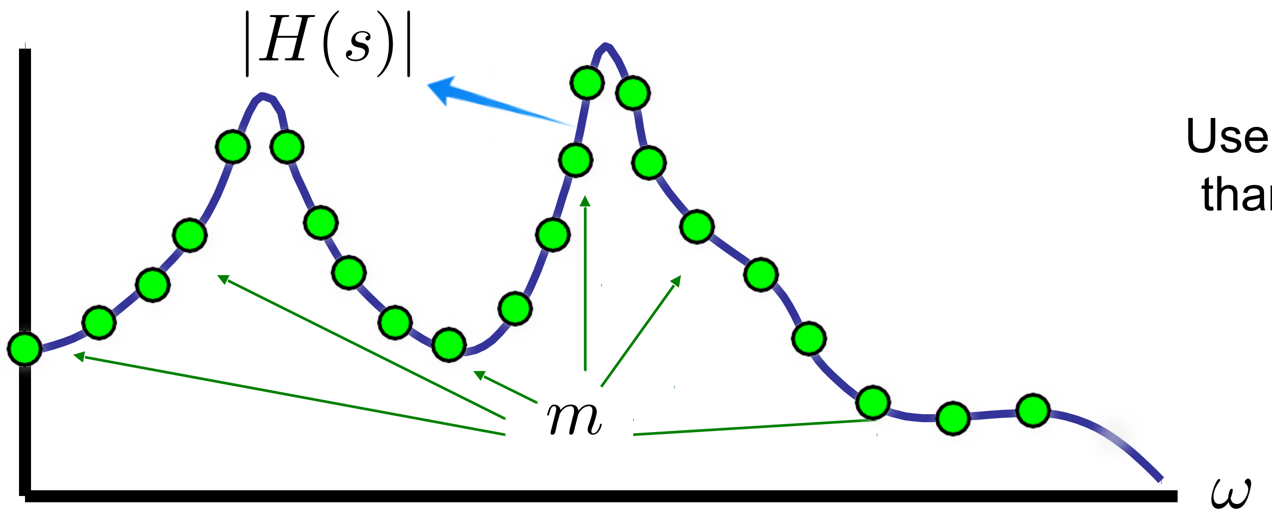
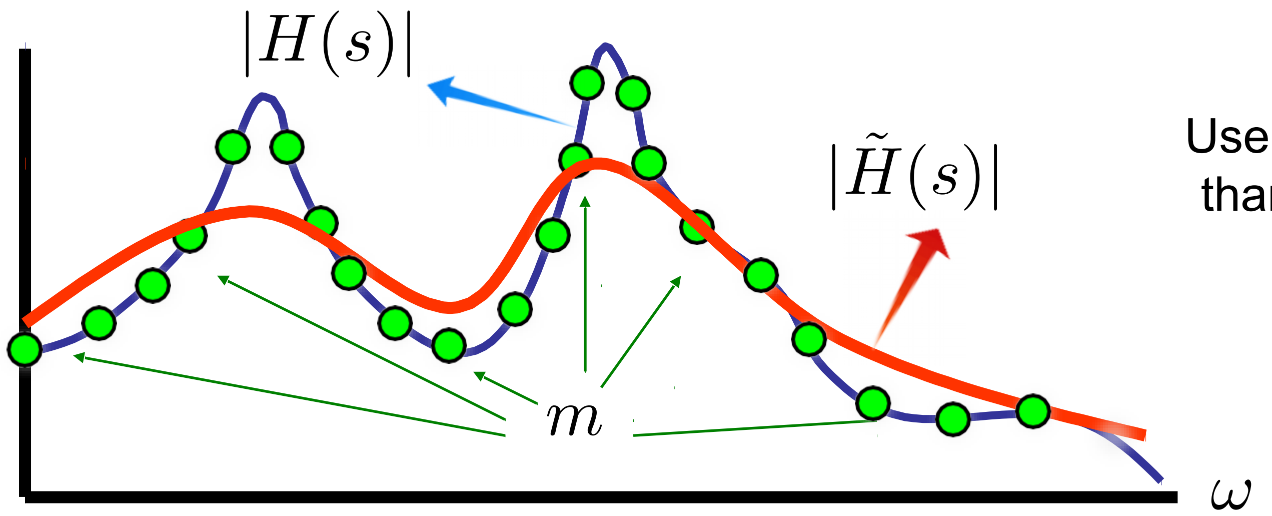


Use much lower order than available points
2q<<m

$$
\begin{bmatrix}
s_1 H(s_1) & s_1^2 H(s_1) & \cdots & -s_1^{q-1} \\
\vdots & \vdots & \cdots & -s_2^{q-1} \\
\vdots & \vdots & \cdots & \vdots \\
s_m^2 H(s_m) & s_m^2 H(s_m) & \cdots & -s_m^{q-1}
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\vdots \\
\tilde{b}_{q-1}
\end{bmatrix}
=
\begin{bmatrix}
-H(s_1) \\
-H(s_2) \\
\vdots \\
-H(s_m)
\end{bmatrix}
$$

Cross multiplying generates a linear TALL SKINNY system

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING POINT MATCHING USING LEAST SQUARES



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Cross multiplying generates a linear TALL SKINNY system
Least Squares problem: Use QR to solve the system

OUTLINE

Model Order Reduction of Linear Problems

- **Problem setup**
- **Reduction via “modal analysis”**
  - time domain
  - frequency domain
- **Reduction via transfer function fitting**
  - point matching
  - least square
- **Importance of preserving physical properties**
  - stability
  - passivity/dissipativity

STABILITY of STATE-SPACE MODELS

Consider a state-space model in isolation (i.e. not connected to other models)

$$\boxed{\frac{dx}{dt} = Ax(t) + bu(t), \quad y(t) = c^T x(t)}$$

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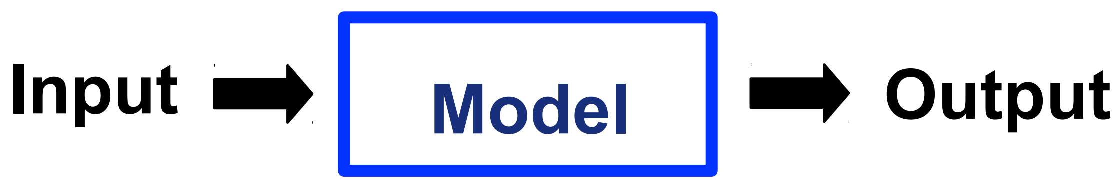

For well-behaved (e.g. bounded) inputs, when will the outputs be well-behaved (e.g. bounded) as well?

STABILITY of STATE-SPACE MODELS

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$$H(s) = \sum_{i=1}^{N} \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i} \qquad h(t) = \sum_{i=1}^{N} \tilde{c}_i \tilde{b}_i e^{\lambda_i t}$$

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- 1) In reality, dynamical models are only useful when connected together with models of other components in a composite simulation

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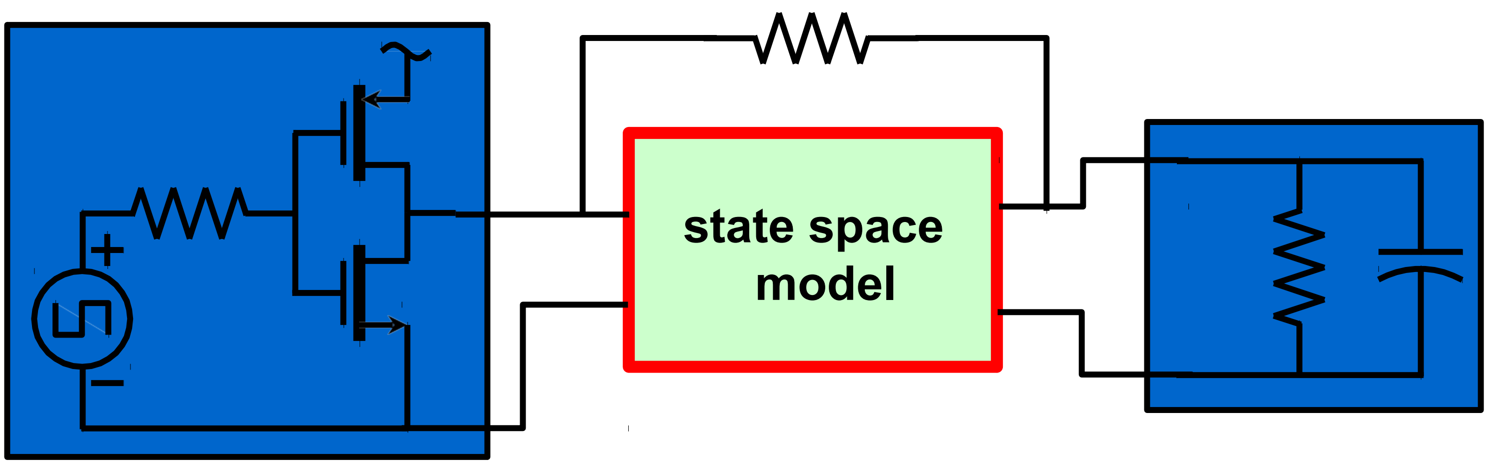

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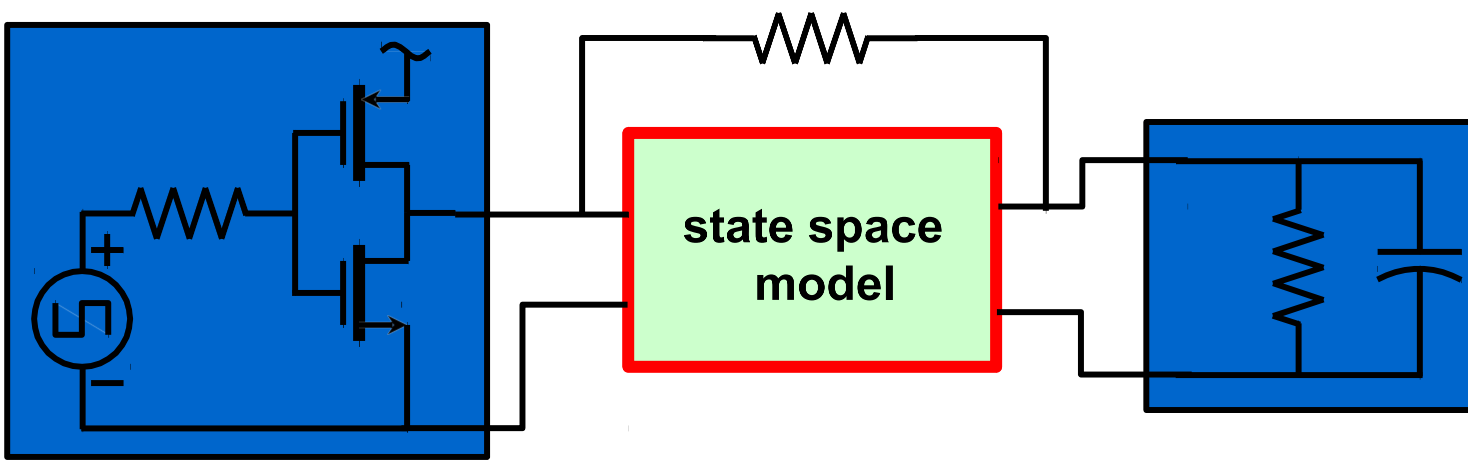

- 3) Can we assure that the simulation of the composite system will be well-behaved?

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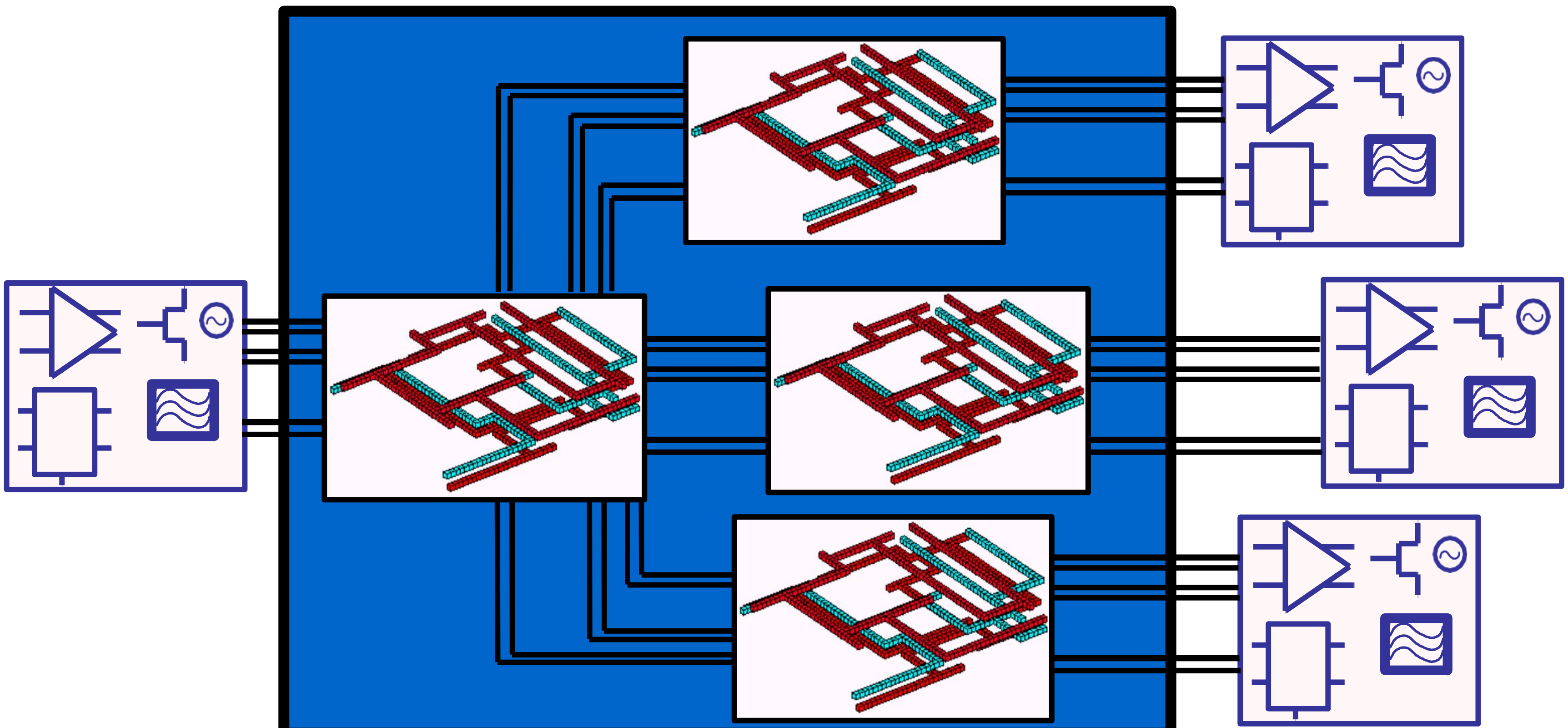
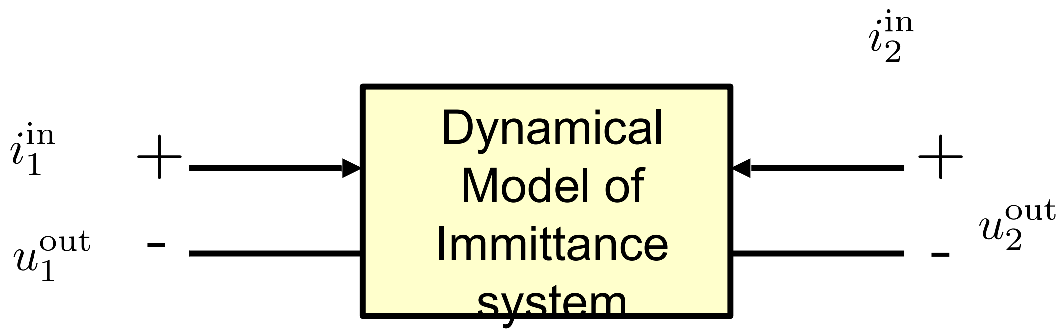
  - stability
  - passivity/dissipativity

INTERCONNECTING PASSIVE/DISSIPATIVE SYSTEMS

- 1) The interconnection of stable models is not necessarily stable
- 2) BUT the interconnection of passive/dissipative models is a passive/dissipative model (and hence also stable)



PASSIVITY/DISSIPATIVITY for IMMITTANCE SYSTEMS



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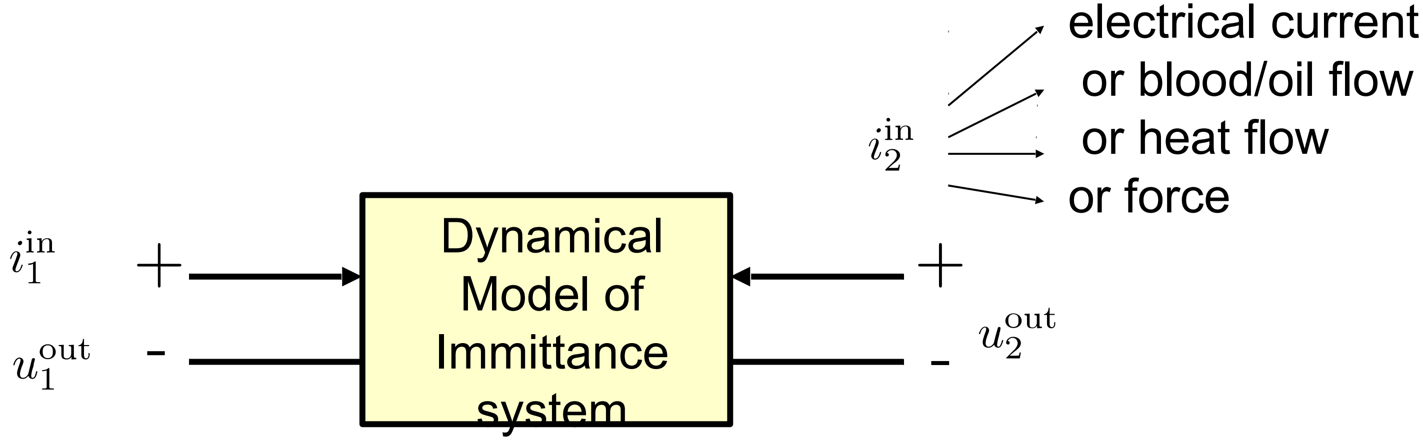

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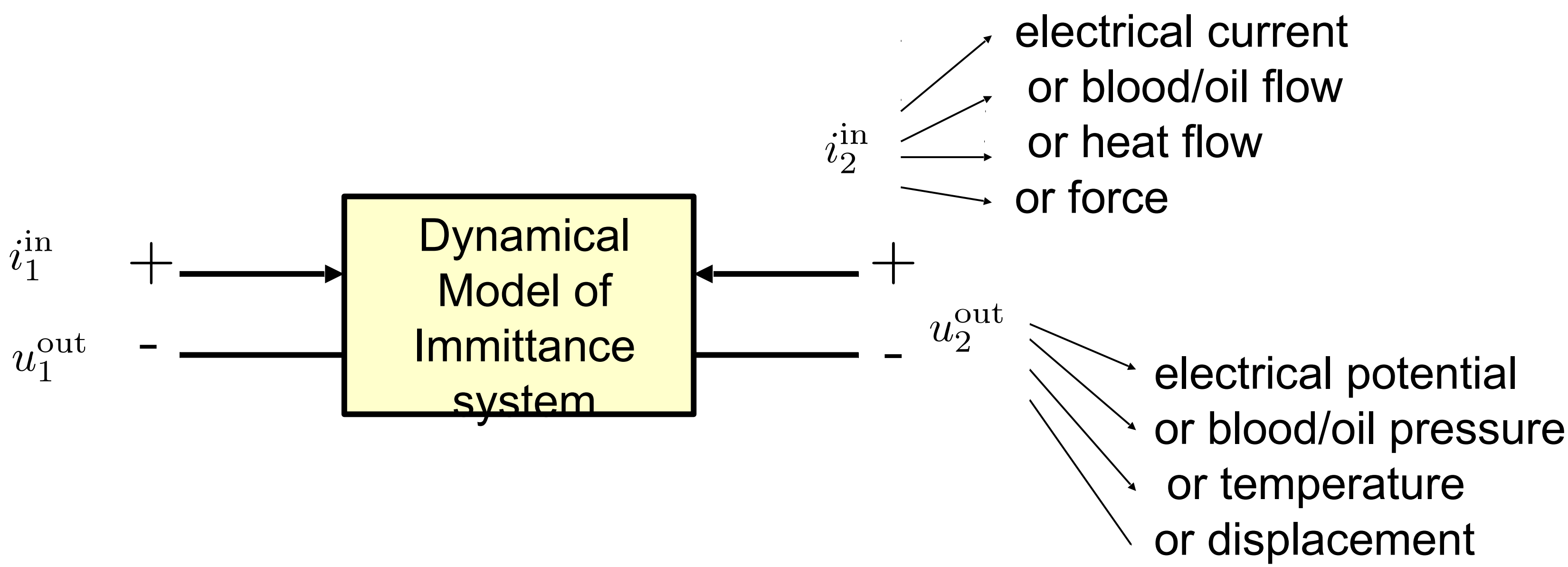

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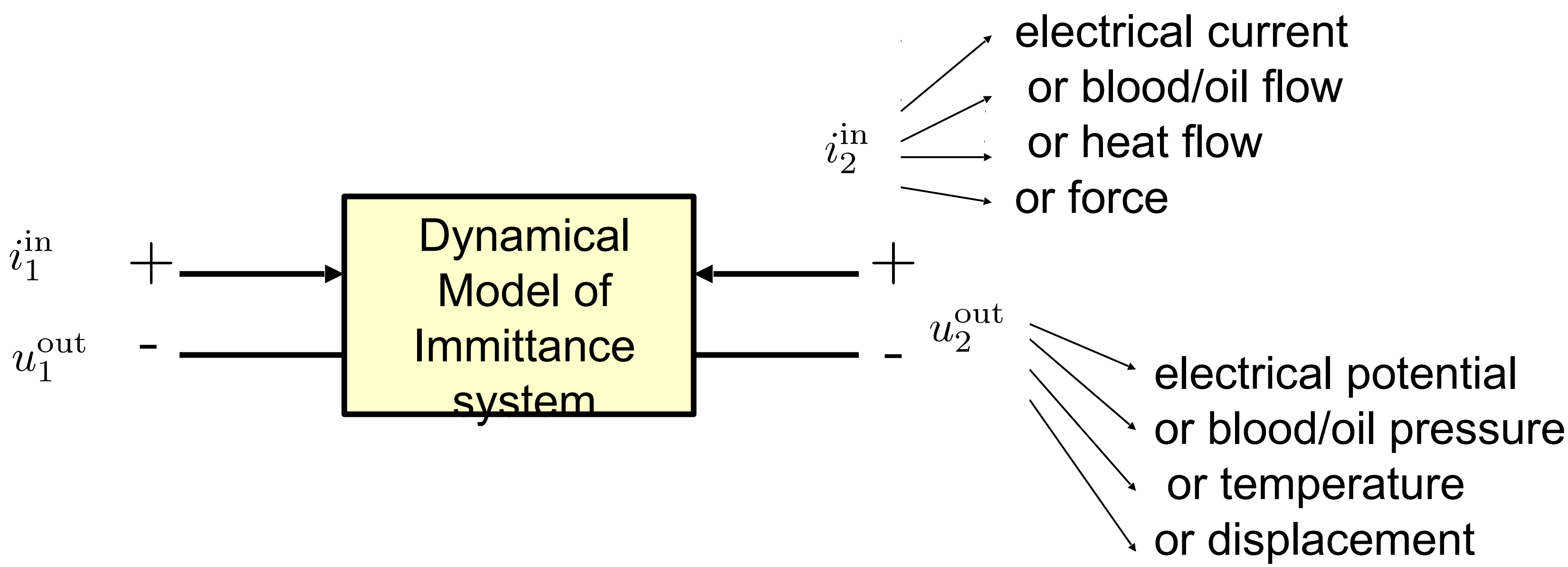

$$\begin{bmatrix} u_1^{\text{out}} \\ u_2^{\text{out}} \end{bmatrix} = H(s) \begin{bmatrix} i_1^{\text{in}} \\ i_2^{\text{in}} \end{bmatrix}$$

In “immittance systems” the quantity $\sum i_k u_k$

has typical the meaning of physical power

PASSIVITY/DISSIPATIVITY

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$$\text{Energy} = \int_{-\infty}^{t} i(\tau)u(\tau)d\tau > 0$$

PASSIVITY/DISSIPATIVITY CONDITION ON TRANSFER SYSTEMS

- For “immitance systems”, passivity/dissipativity is equivalent to positive-realness of the transfer function

$$y(s) = H(s)u(s)$$

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$Re\{H(j\omega)\} \geq 0, \quad \forall \omega$ **It means its real part is positive FOR ANY FREQUENCY**

Hermitian Decomposition

$$A = \underbrace{\frac{A + A^*}{2}}_{\text{Hermitian part of } A} + \underbrace{\frac{A - A^*}{2}}_{\text{skew-Hermitian part of } A}$$

POSITIVITY of QUADRATIC FORMS – HERMITIAN DECOMPOSITION

Hermitian Decomposition

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2}$$

Hermitian part of A

skew-Hermitian part of A

Quadratic Forms

$$Re\{x^* A x\} = x^* \left(\tfrac{A+A^*}{2}\right) x \geq 0 \text{ iff } \left(\tfrac{A+A^*}{2}\right) \geq 0$$

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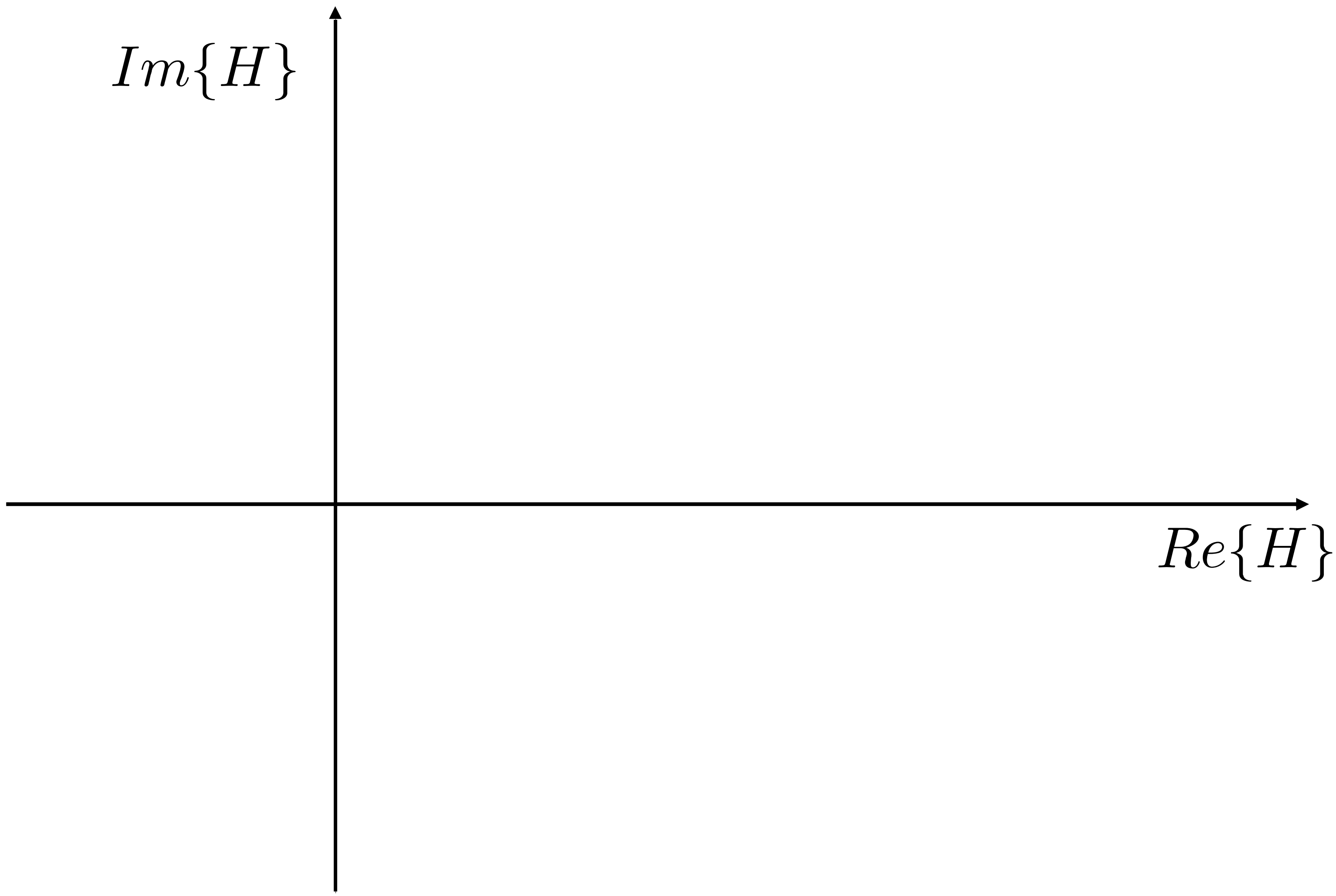
$$Im\{x^* A x\} = x^* \left(\frac{A-A^*}{2}\right) x \geq 0 \text{ iff } \left(\frac{A-A^*}{2}\right) \geq 0$$

Positive real transfer function in the complex plane for different frequencies



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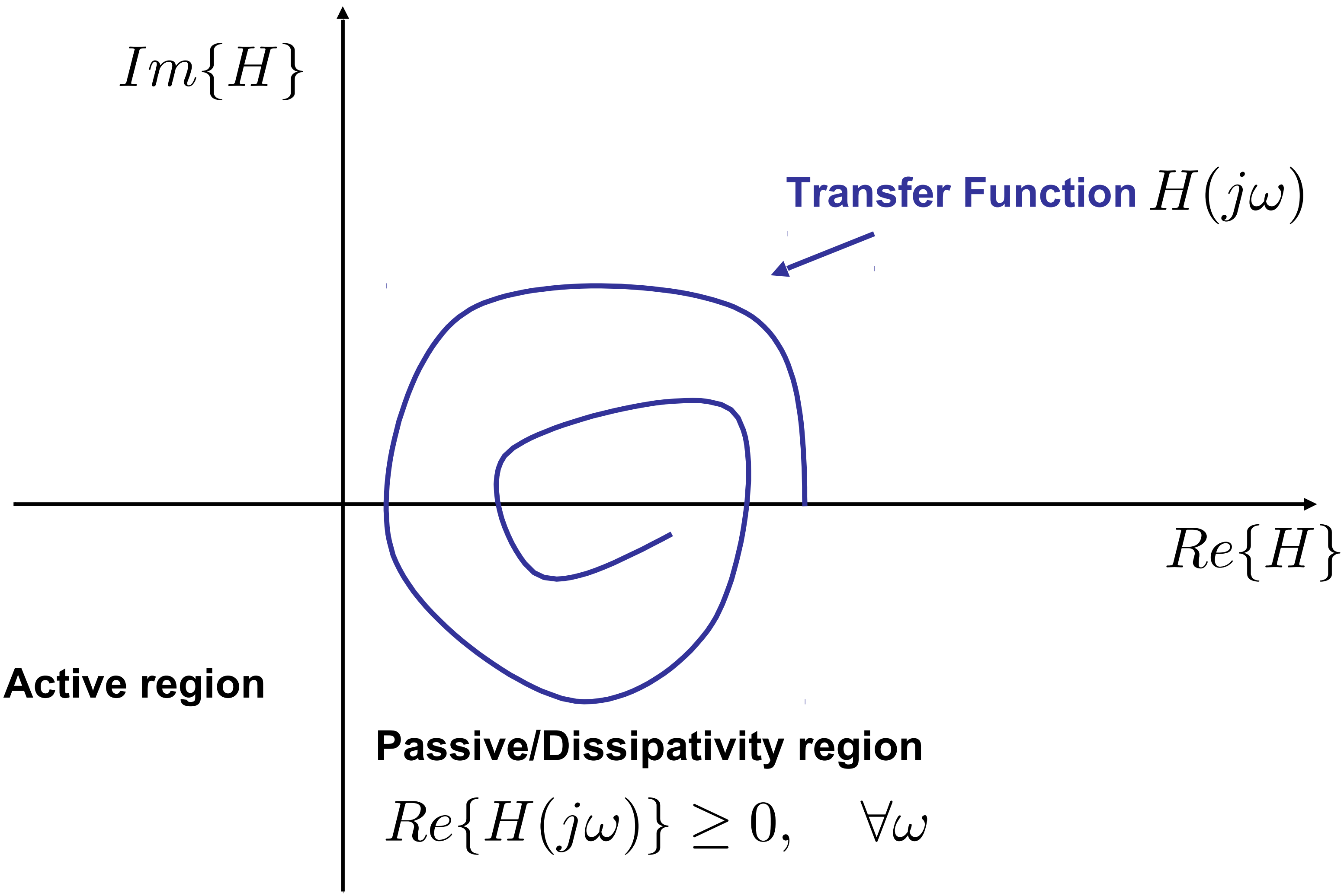

Active region

Passive/Dissipativity region

$$Re\{H(j\omega)\} \geq 0, \quad \forall \omega$$

Sufficient conditions for passivity/dissipativity of immitance dynamical system models

$$sEx = Ax + Bu$$

$$y = Cx$$

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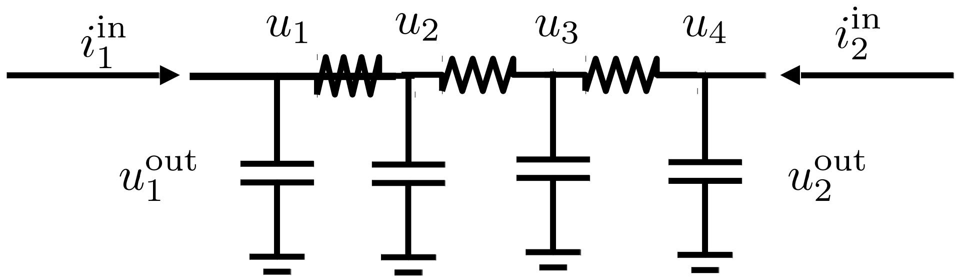
Sufficient conditions for passivity/dissipativity for immitance dynamical models:

$$C = B^T$$

$$x^T E x \geq 0, \quad \forall x$$ **Positive Semidefinite**

$$x^T A x \leq 0, \quad \forall x$$ **Negative Semidefinite**

State-Space Model from Conserv./Constit. Laws



State-Space Model from Conserv./Constit. Laws

When using Nodal Analysis

E is Positive Semidefinite

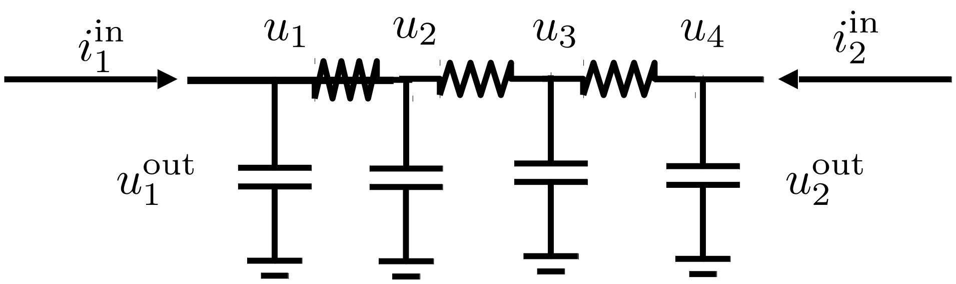
A is Negative Semidefinite

For immitance systems in Nodal Analysis form

$$B = C^T$$

$$\begin{bmatrix} C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{bmatrix} \frac{d}{dt} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = -\frac{1}{R} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_1^{\text{in}}(t) \\ i_2^{\text{in}}(t) \end{bmatrix}$$

$$\begin{bmatrix} u_1^{\text{out}}(t) \\ u_2^{\text{out}}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}$$



Necessary and Sufficient Condition for Passivity

Positive Real (Kalman-Yakubovich-Popov) Lemma

$$\frac{dx}{dt} = Ax + Bu, \quad y = C^T x$$

Necessary and Sufficient Condition for Passivity

Positive Real (Kalman-Yakubovich-Popov) Lemma

$$\frac{dx}{dt} = Ax + Bu, \quad y = C^T x$$

A stable system (A,B,C) is positive real if and only if there exists $P = P^* \geq 0$
Such that the linear matrix inequality is satisfied

$$-A^T P - PA \geq 0$$

$$B^T P = C$$