

Computational Science and Engineering II: Discretization

Model Order Reduction

Professor: Alexander Shapeev

*Teaching Assistants: Ilias Giannakopoulos, Evgenii
Tsymbalov*

Thanks to Thanos Polimeridis **Skoltech**

&

Luca Daniel 

Model Order Reduction of Linear Problems

Problem setup

Reduction via “modal analysis”

- time domain
- frequency domain

Reduction via transfer function fitting

- point matching
- least square

Importance of preserving physical properties

- stability
- passivity/dissipativity

MOTIVATION

$$\frac{dx}{dt} = Ax(t) + bu(t)$$

$$y(t) = c^T x(t)$$

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Suppose

- 1) We are just interested in terminal (i.e. input/output) behavior
- 2) And we need to compute the **output $y(t)$** for many many different **input signals $u(t)$**

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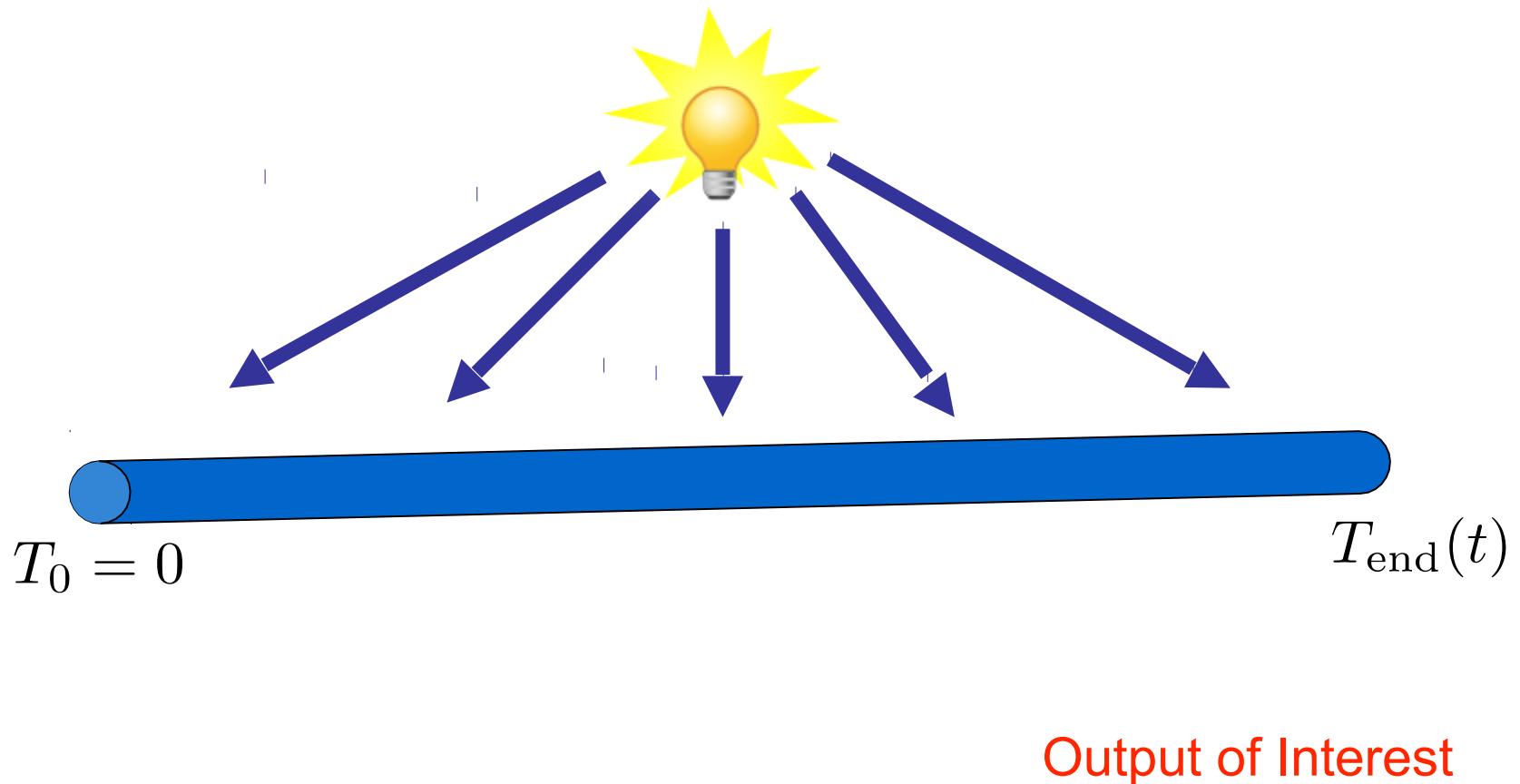
It might be more convenient to

- 1) do some **pre-computation** on the original system
- 2) generate a compact dynamical model
- 3) **re-use** the model over and over

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

Power of lamp = $u(t)$

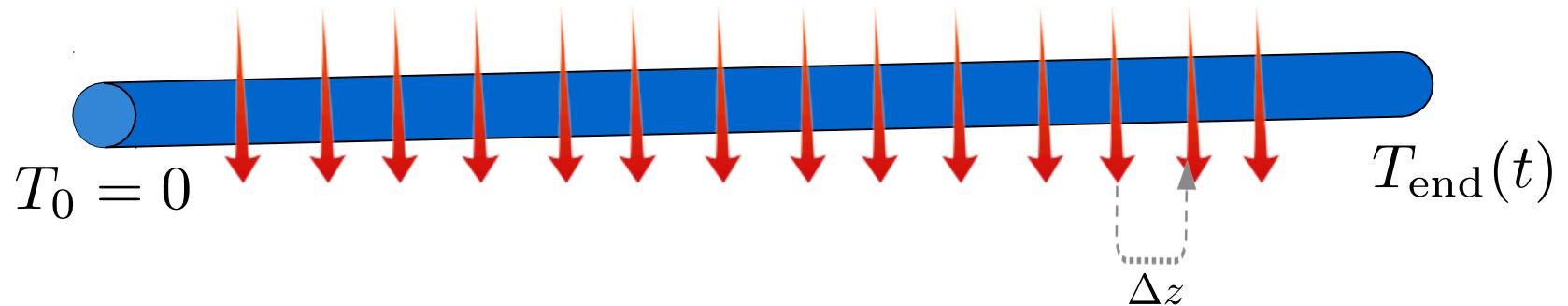
Input of Interest



MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

PROBLEM SETUP

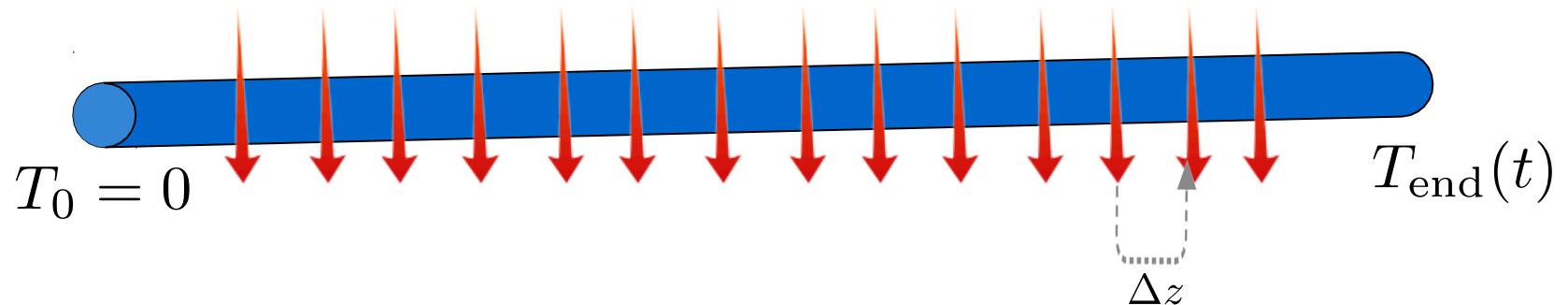
Heat in $h(z)u(t)$



MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

PROBLEM SETUP

Heat in $h(z)u(t)$



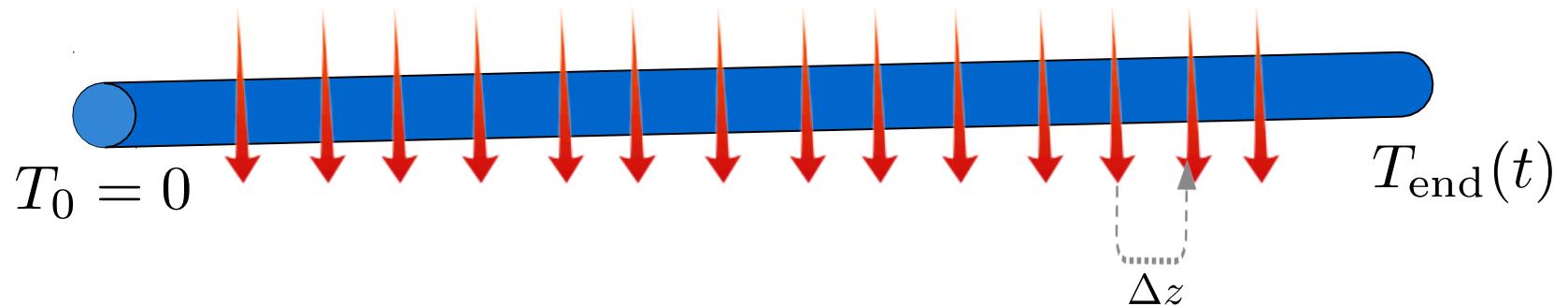
$$\gamma \frac{\partial T(z, t)}{\partial t} = k \frac{\partial^2 T(z, t)}{\partial z^2} + h(z)u(t)$$

specific heat scalar input
 thermal conductivity

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

PROBLEM SETUP

Heat in $h(z)u(t)$



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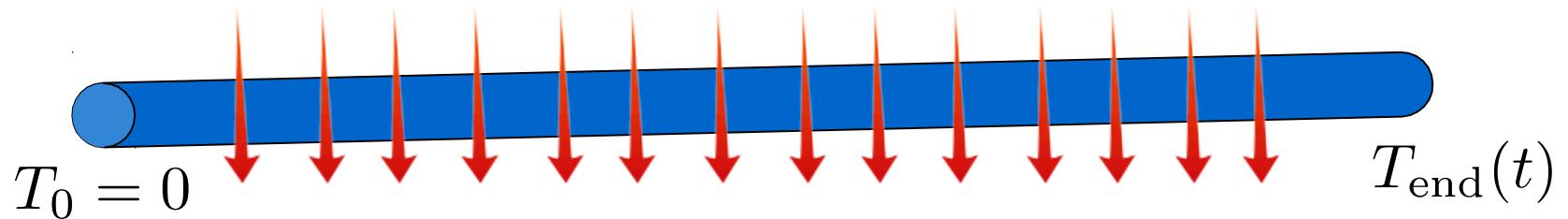
Spatial Discretization

$$\gamma \frac{\partial \tilde{T}_i}{\partial t} = \frac{k}{(\Delta z)^2} (\tilde{T}_{i+1} - 2\tilde{T}_i + \tilde{T}_{i-1}) + h(z_i)u(t)$$

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

PROBLEM SETUP

Heat in $h(z)u(t)$



$$\frac{dx(t)}{dt} = Ax(t) + bu(t)$$

$\overset{\textcolor{blue}{\uparrow}}{N \times N}$ $\overset{\textcolor{blue}{\uparrow}}{N \times 1}$ $\overset{\textcolor{blue}{\uparrow}}{\text{Scalar input}}$

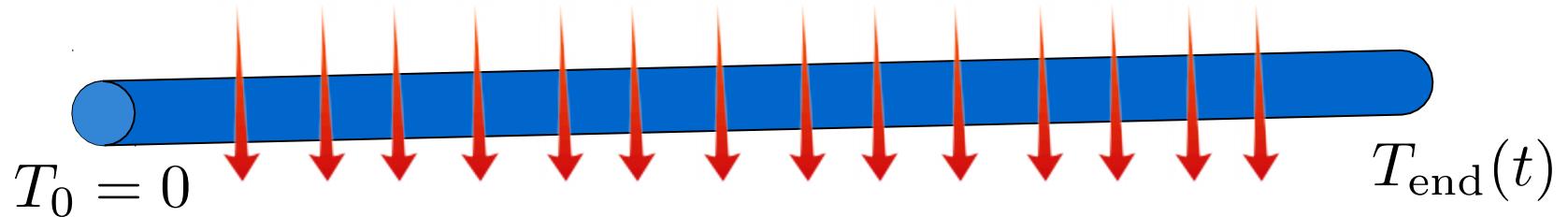
$$y(t) = c^T x(t)$$

$\overset{\textcolor{blue}{\uparrow}}{\text{Scalar output}}$ $\overset{\textcolor{blue}{\uparrow}}{N \times 1}$

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

PROBLEM SETUP

Heat in $h(z)u(t)$



$$\frac{dx(t)}{dt} = Ax(t) + bu(t)$$

↑ ↑ ↑
 $N \times N$ $N \times 1$ Scalar
 input

$$y(t) = c^T x(t)$$

↑ ↑
 Scalar $N \times 1$
 output

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \quad b = \frac{1}{\gamma} \begin{bmatrix} h(z_1) \\ h(z_2) \\ \vdots \\ h(z_N) \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

PROBLEM SETUP

Original Dynamical System - Single Input/Output

$$\frac{dx(t)}{dt} = Ax(t) + bu(t) \quad y(t) = c^T x(t)$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ N \times N & N \times 1 & \text{Scalar input} \end{matrix}$ $\begin{matrix} \uparrow & \uparrow \\ \text{Scalar output} & N \times 1 \end{matrix}$

MODEL ORDER REDUCTION HEAT CONDUCTING BAR EXAMPLE

PROBLEM SETUP

Original Dynamical System - Single Input/Output

$$\frac{dx(t)}{dt} = Ax(t) + bu(t)$$

$\overset{\uparrow}{N \times N}$ $\overset{\uparrow}{N \times 1}$ $\overset{\uparrow}{\text{Scalar input}}$

$y(t) = c^T x(t)$
 $\overset{\uparrow}{\text{Scalar output}}$ $\overset{\uparrow}{N \times 1}$

Reduced Dynamical System

$$\frac{dx(t)}{dt} = \tilde{A}x(t) + \tilde{b}u(t)$$

$\overset{\uparrow}{q \times q}$ $\overset{\uparrow}{q \times 1}$ $\overset{\uparrow}{\text{Scalar input}}$

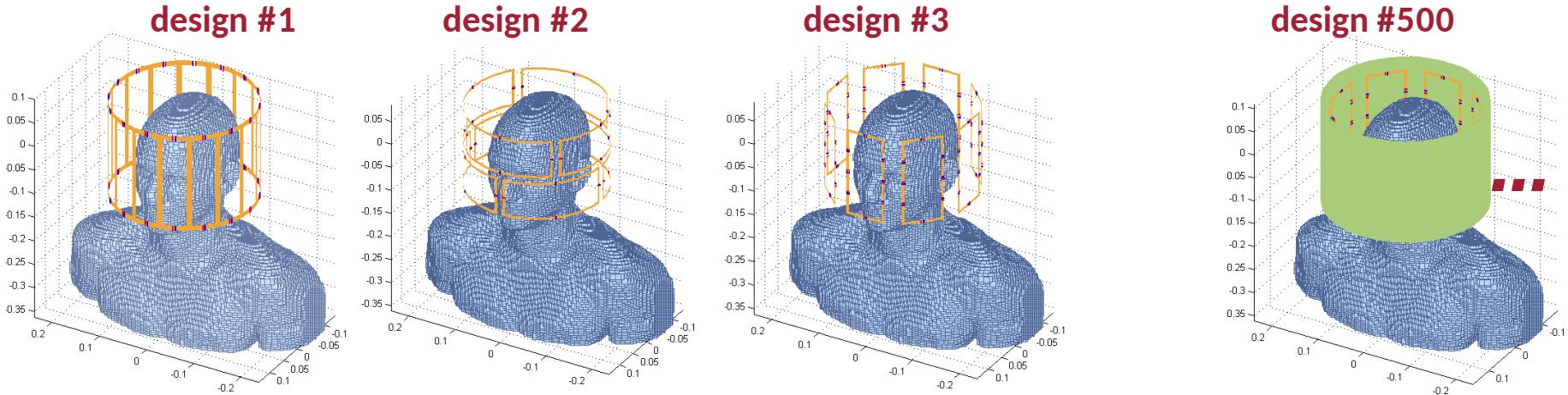
$y(t) = \tilde{c}^T x(t)$
 $\overset{\uparrow}{\text{Scalar output}}$ $\overset{\uparrow}{q \times 1}$

$$q \ll N$$

but input/output behavior is well preserved

RF-COIL DESIGN IN MAGNETIC RESONANCE IMAGING

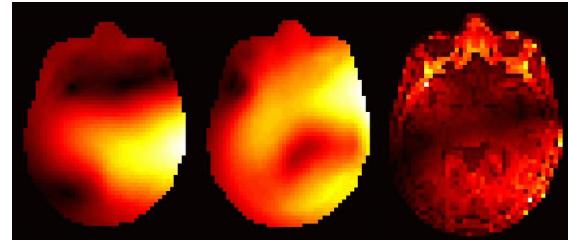
- Best out of 500 coil designs? Need full EM analysis of each design



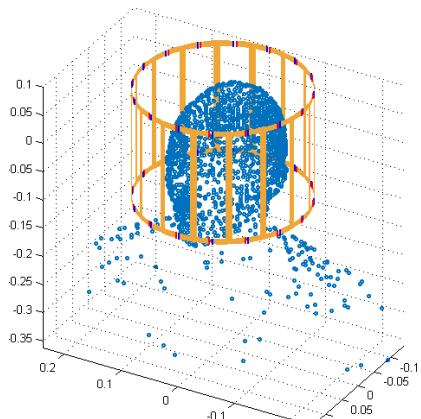
Standard Model
>> 1 month
(hours / design)

Reduced Model
over the weekend
(~4min / design)

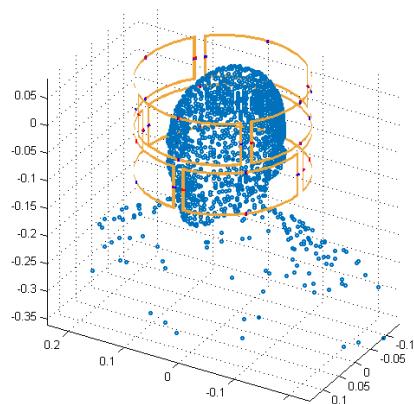
Full EM analysis



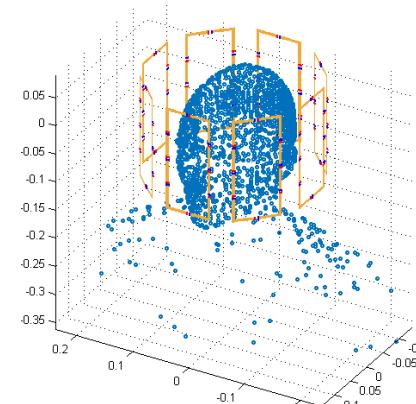
RF-COIL DESIGN IN MAGNETIC RESONANCE IMAGING



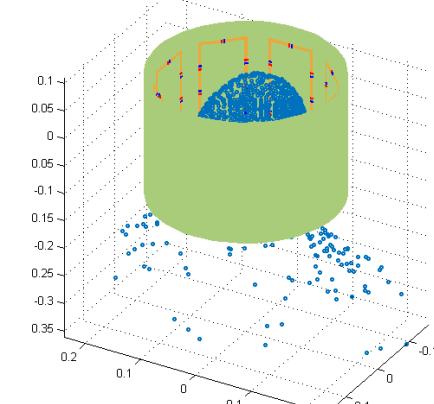
~8h29m overall
(~18 min/port)



~7h30m overall
(~17 min/port)



~9h32m overall
(~15 min/port)



~17h13m overall
(~30 min/port)



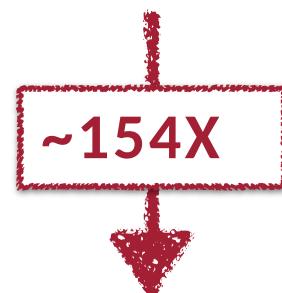
~1min overall



~2min overall



~2min overall



~7min overall

RF-COIL DESIGN IN MAGNETIC RESONANCE IMAGING

IEEE TRANSACTIONS ON BIOMEDICAL ENGINEERING

A PUBLICATION OF THE IEEE ENGINEERING IN MEDICINE AND BIOLOGY SOCIETY



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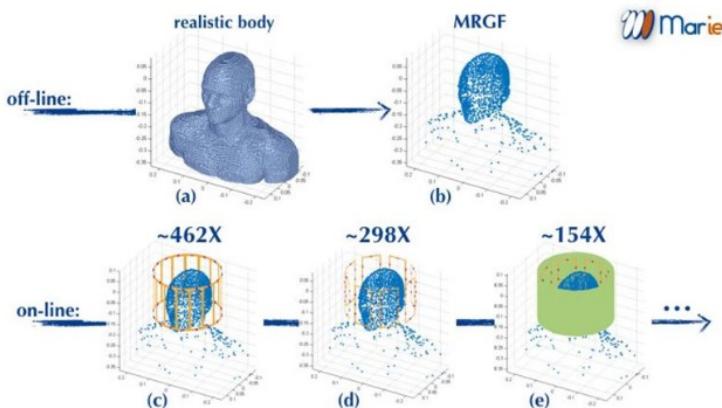
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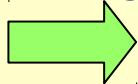


Magnetic resonance-specific framework based on the combination of integral equation methods and domain decomposition approaches for accelerating the numerical modeling of interactions between electromagnetic waves and biological tissue. (a) For any given realistic human body model, (b) pre-compute the magnetic resonance Green function (MRGF) via an ultrafast volume integral equations method and principal component analysis. (c)-(e) Combine the MRGF with standard surface integral equations method to model the performance of multiple transmit/receive coil configurations with more than two orders of magnitude speed-ups and negligible errors. See "Fast Electromagnetic Analysis of MRI Transmit RF Coils Based on Accelerated Integral Equation Methods," by Villena *et al.*, p. 2250.

Model Order Reduction of Linear Problems

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time domain

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EIGENVALUE INTERLUDE

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Eigendecomposition:

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ V_1 & V_2 & \vdots & V_N \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ V_1 & V_2 & \vdots & V_N \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}^{-1}$$

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Change of variables:

$$V\tilde{x}(t) = x(t), \text{ or } \tilde{x}(t) = V^{-1}x(t)$$

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Change of variables:

$$V\tilde{x}(t) = x(t), \text{ or } \tilde{x}(t) = V^{-1}x(t)$$

Substituting:

$$\frac{dV\tilde{x}(t)}{dt} = AV\tilde{x}(t) + bu(t), \quad V\tilde{x}(0) = x_0$$

EIGENVALUE INTERLUDE

Decoupled equations:

$$\frac{d\tilde{x}(t)}{dt} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \tilde{x}(t) + V^{-1}bu(t)$$

EIGENVALUE INTERLUDE

Decoupled equations:

$$\frac{d\tilde{x}(t)}{dt} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \tilde{x}(t) + V^{-1}bu(t)$$

Output equations:

$$y(t) = c^T x(t) = c^T V \tilde{x}(t) = (V^T c)^T \tilde{x}(t)$$

$$\tilde{b} = V^{-1}b$$

$$\tilde{c} = V^T c$$

MODEL ORDER REDUCTION VIA MODEL ANALYSIS

(i.e. dominant eigenvalues/poles method)

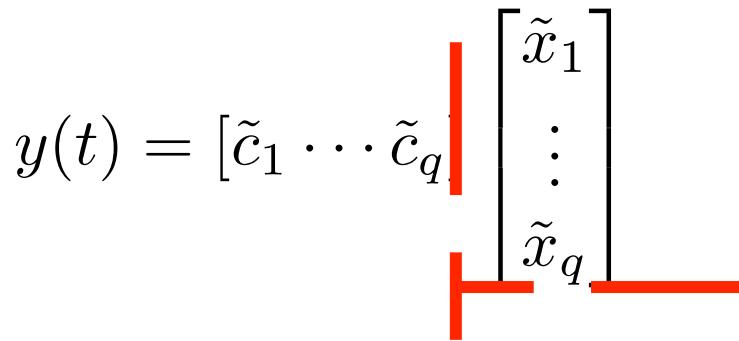
$$\frac{d}{dt} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \vdots \\ \tilde{x}_N \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \vdots \\ \tilde{x}_N \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \vdots \\ \vdots \\ \tilde{b}_N \end{bmatrix} u(t)$$

$$y(t) = [\tilde{c}_1 \dots \dots \tilde{c}_N] \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ \tilde{x}_N \end{bmatrix}$$

MODEL ORDER REDUCTION VIA MODEL ANALYSIS

(i.e. dominant eigenvalues/poles method)

$$\frac{d}{dt} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_q \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_q & \\ & & & \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_q \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_q \end{bmatrix} u(t)$$


$$y(t) = [\tilde{c}_1 \cdots \tilde{c}_q] \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_q \end{bmatrix}$$


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(i.e. dominant eigenvalues/poles method)

Certain modes are not affected by the input:

$$\tilde{b}_{q+1}, \dots, \tilde{b}_N, \quad \text{are all small}$$

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$$\tilde{c}_{q+1}, \dots, \tilde{c}_N, \quad \text{are all small}$$

Keep least negative eigenvalues (slowest modes)

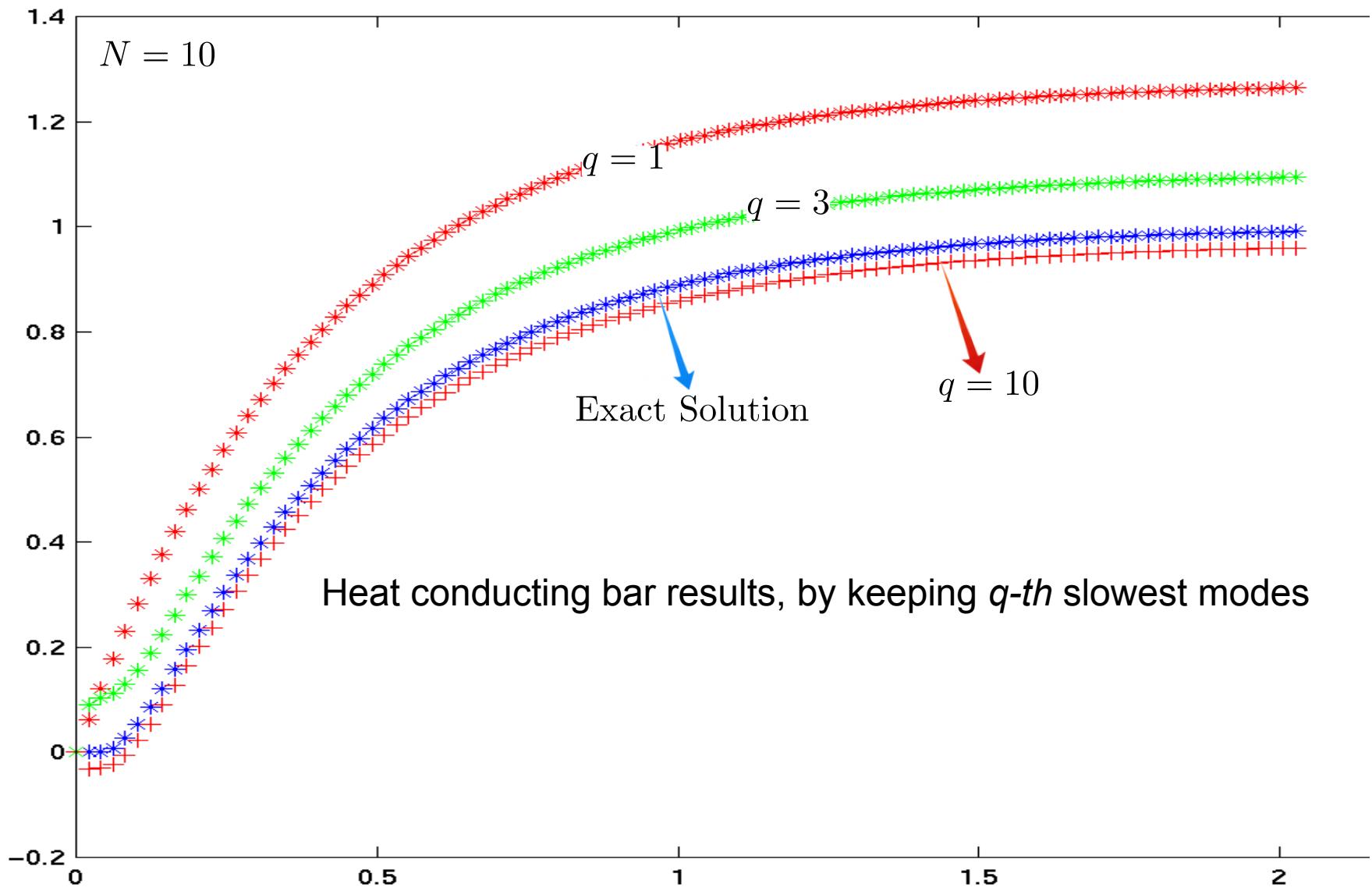
Look at response to a constant input:

$$\tilde{x}_i(t) = \int_0^t e^{\lambda_i(t-\tau)} \tilde{b}_i u(t) d\tau = -\frac{1}{\lambda_i} (1 - e^{\lambda_i t}) \tilde{b}_i u(t)$$

Small if $|\lambda_i|$ is large

MODEL ORDER REDUCTION VIA MODEL ANALYSIS

(i.e. dominant eigenvalues/poles method)



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ON TRANSFER FUNCTIONS – LAPLACE TRANSFORM

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Bilateral Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

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Key transform property

$$sX(s) = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

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Key transform property

$$sX(s) = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

Rewrite the ODE in “Laplace domain”

$$sX(s) = AX(s) + bU(s), \quad Y(s) = c^T X(s)$$

$$Y(s) = c^T (sI - A)^{-1} b U(s)$$

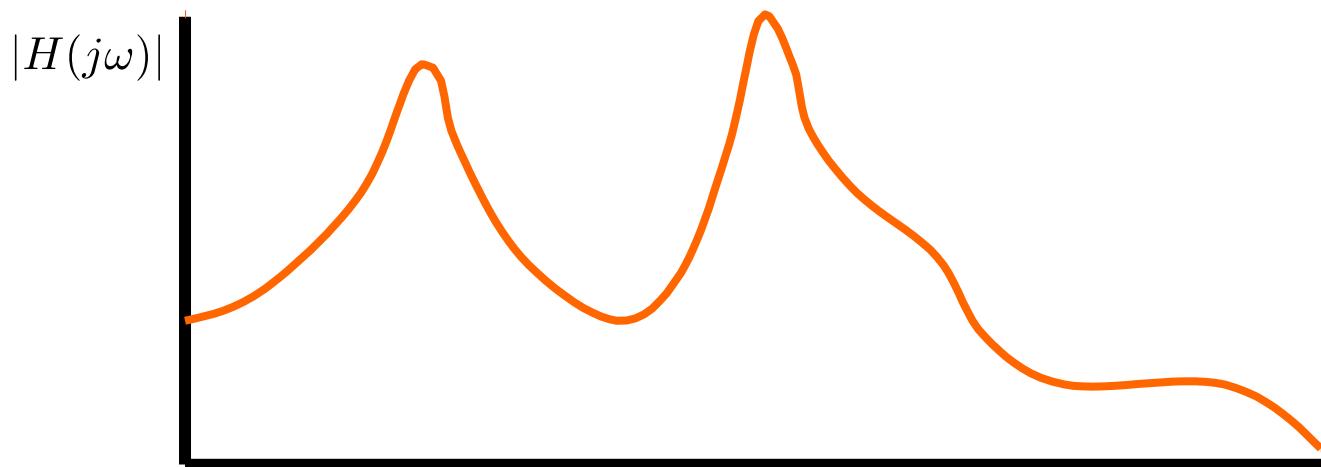
$$H(s) = c^T (sI - A)^{-1} b, \quad \text{Transfer function}$$

ON TRANSFER FUNCTIONS – MEANING OF H(s)

For stable systems $H(j\omega)$ is the frequency response

If $u(t) = e^{j\omega t} \rightarrow$ Sinusoid

Then $y(t) = H(j\omega)e^{j\omega t} \rightarrow$ Sinusoid with shifted phase and amplitude



MODEL (EIGENVALUE) ANALYSIS IN THE FREQUENCY DOMAIN

Transfer
function:

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Apply Eigendecomposition  $A = V \Lambda V^{-1}$

$$H(s) = c^T V (sI - \Lambda)^{-1} V^{-1} b$$

MODEL (EIGENVALUE) ANALYSIS IN THE FREQUENCY DOMAIN

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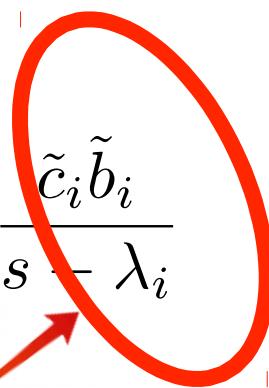
$$H(s) = c^T V (sI - \Lambda)^{-1} V^{-1} b$$

$$= \tilde{c}^T \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_N} \end{bmatrix} \tilde{b}$$

\longrightarrow

$$H(s) = \sum_{i=1}^N \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Eliminate each mode
for which this term is
small



MODEL ORDER REDUCTION via MODEL ANALYSIS (i.e. dominant eigenvalues/poles method)

$$H(s) = \sum_{i=1}^N \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Pole-Residue Form

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$$H(s) = \sum_{i=1}^N \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i}$$

Pole-Residue Form

$$H(s) = \frac{\prod_{i=1}^N s - \zeta_i}{\prod_{i=1}^N s - \lambda_i}$$

Pole-Zero Form

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Pole-Zero Form

$$h(t) = \sum_{i=1}^N \tilde{c}_i \tilde{b}_i e^{\lambda_i t}$$

Impulse Response

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Pole-Zero Form

Ideas for Reducing Order:

- 1) Drop terms with small residues
 $\tilde{c}_i \tilde{b}_i$
- 2) Drop terms with large negative
 $Re\{\lambda_i\}$
- 3) Remove pole/zero near-cancellation
stability?

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Impulse Response

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Advantages

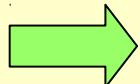
- 1) Conceptually Familiar
- 2) Simple physical interpretation: retains dominant system modes/poles

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Pole-Zero Form

$$h(t) = \sum_{i=1}^N \tilde{c}_i \tilde{b}_i e^{\lambda_i t}$$

Impulse Response

Rational Function Form

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{N-1} s^{N-1}}{1 + a_1 s + \cdots + a_N s^N}$$

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

Original System Transfer Function:

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{N-1} s^{N-1}}{1 + a_1 s + \cdots + a_N s^N}$$



Rational function

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

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Rational function

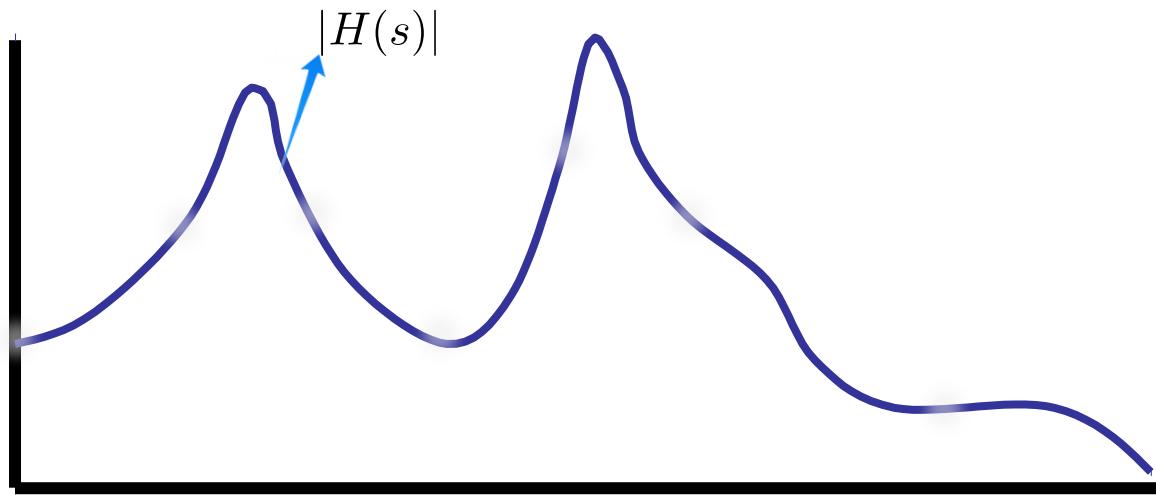
Model Reduction: Find a low order ($q \ll N$) rational function matching

$$\tilde{H}(s) = \frac{\tilde{b}_0 + \tilde{b}_1 s + \cdots + \tilde{b}_{q-1} s^{q-1}}{1 + \tilde{a}_1 s + \cdots + \tilde{a}_q s^q}$$

Reduced Order Rational function

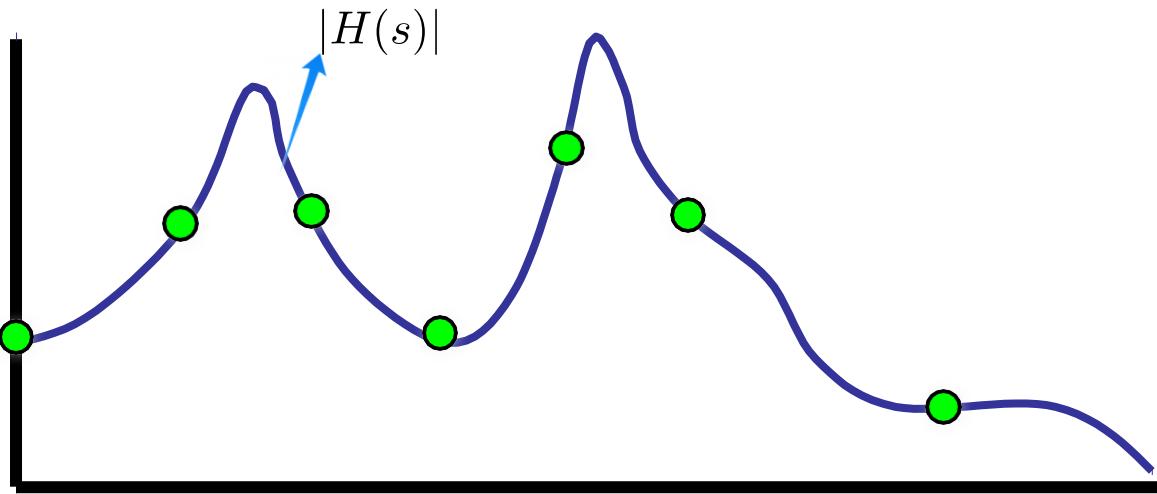
MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

Point matching



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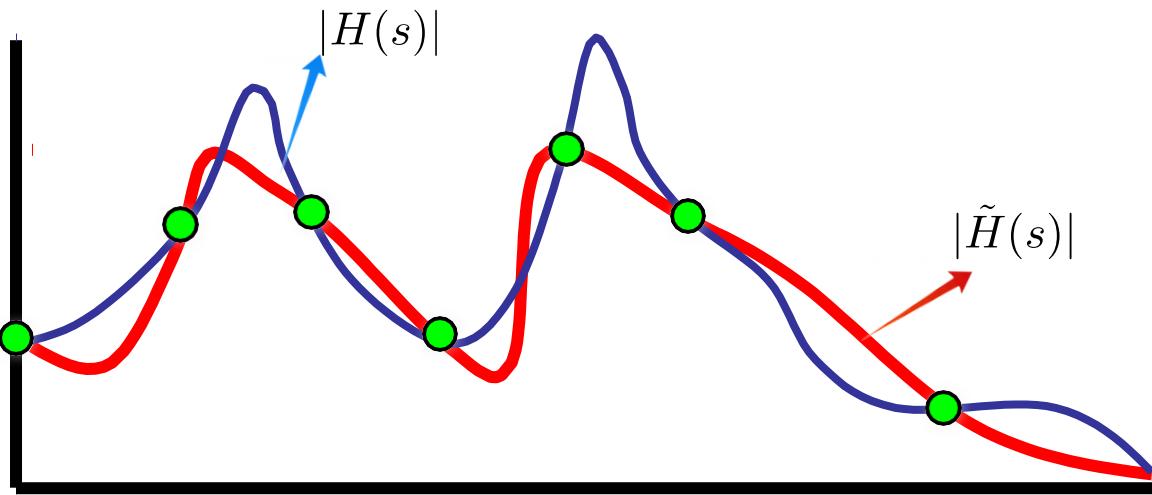


Can Match $2q$ points

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{q-1} s^{q-1}}{1 + a_1 s + \cdots + a_q s^q}$$

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

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$$H(s) = \frac{b_0 + b_1 s + \cdots + b_{q-1} s^{q-1}}{1 + a_1 s + \cdots + a_q s^q}$$

Cross multiplying generates a linear system

For $i=1$ to $2q$

$$(1 + \tilde{a}_1 s_i + \cdots + \tilde{a}_q s_i^q) H(s_i) - (\tilde{b}_0 + \tilde{b}_1 s_i + \cdots + \tilde{b}_{q-1} s_i^{q-1}) = 0$$

MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING

Point matching

$$\begin{bmatrix} s_1 H(s_1) & s_1^2 H(s_1) & \cdots & -s_1^{q-1} \\ \vdots & \vdots & \cdots & -s_2^{q-1} \\ \vdots & \vdots & \cdots & \vdots \\ s_{2q}^2 H(s_{2q}) & s_{2q}^2 H(s_{2q}) & \cdots & -s_{2q}^{q-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{b}_{q-1} \end{bmatrix} = \begin{bmatrix} -H(s_1) \\ -H(s_2) \\ \vdots \\ -H(s_{2q}) \end{bmatrix}$$

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- 1) The Columns contain progressively higher powers of the test frequencies: **problem is numerically ill-conditioned**

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- 1) The Columns contain progressively higher powers of the test frequencies: **problem is numerically ill-conditioned**
- 2) Missing data can cause severe accuracy problems

HARD TO SOLVE PROBLEMS

“Polynomial interpolation” example

Table of Data

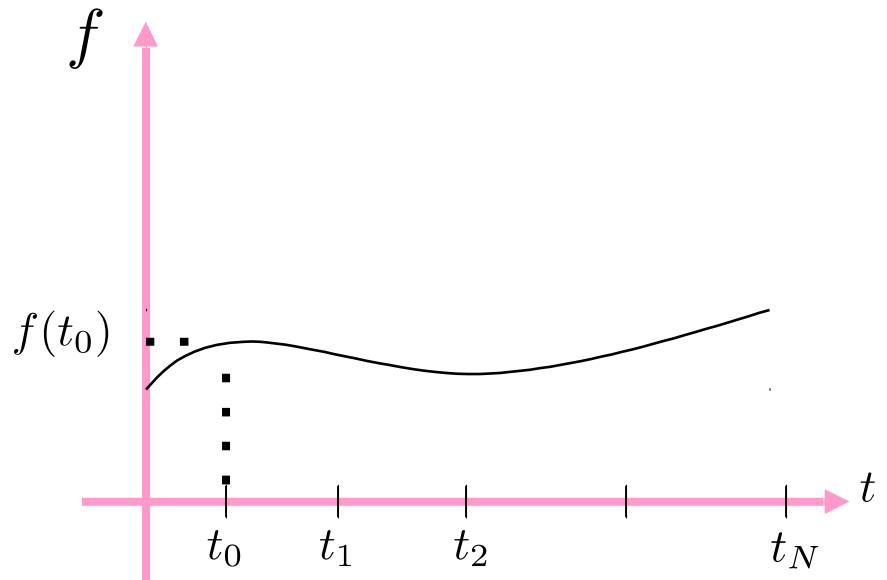
$$t_0 \quad f(t_0)$$

$$t_1 \quad f(t_1)$$

\vdots \vdots

\vdots \vdots

$$t_N \quad f(t_N)$$



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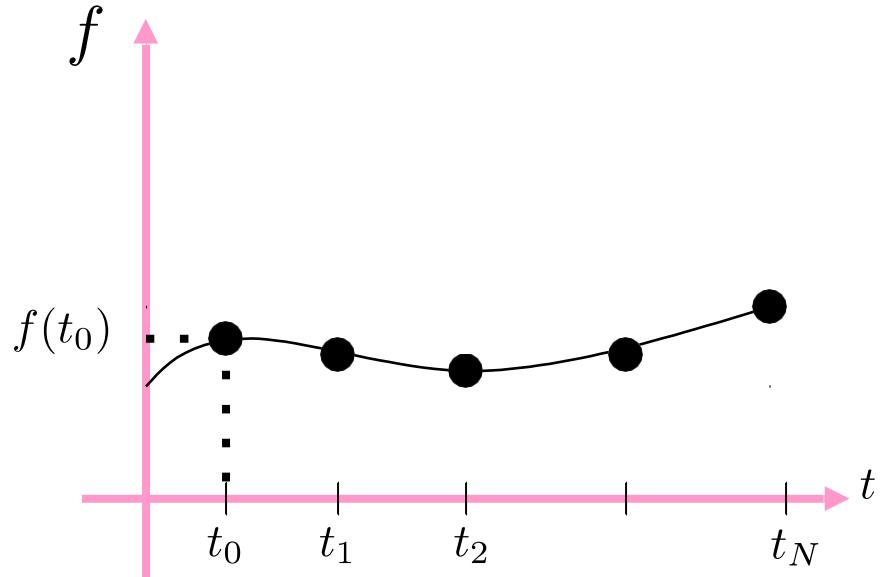
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Problem fit data with an Nth order polynomial

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N$$

HARD TO SOLVE PROBLEMS

“Polynomial interpolation” example

Matrix Form

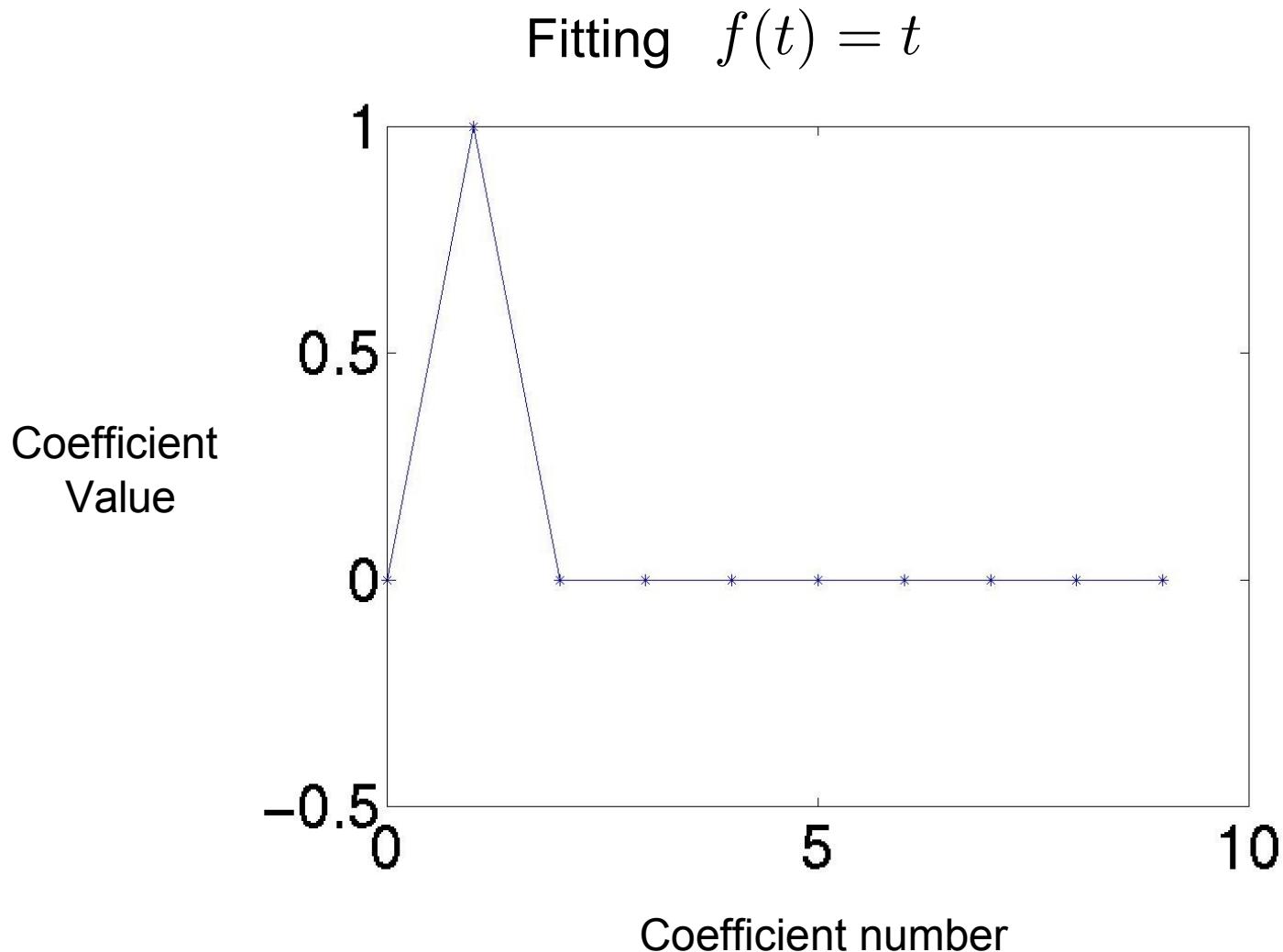
$$\begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^N \\ 1 & t_1 & t_1^2 & \cdots & t_1^N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_N) \end{bmatrix}$$

$A_{\text{interpolation}}$



HARD TO SOLVE PROBLEMS

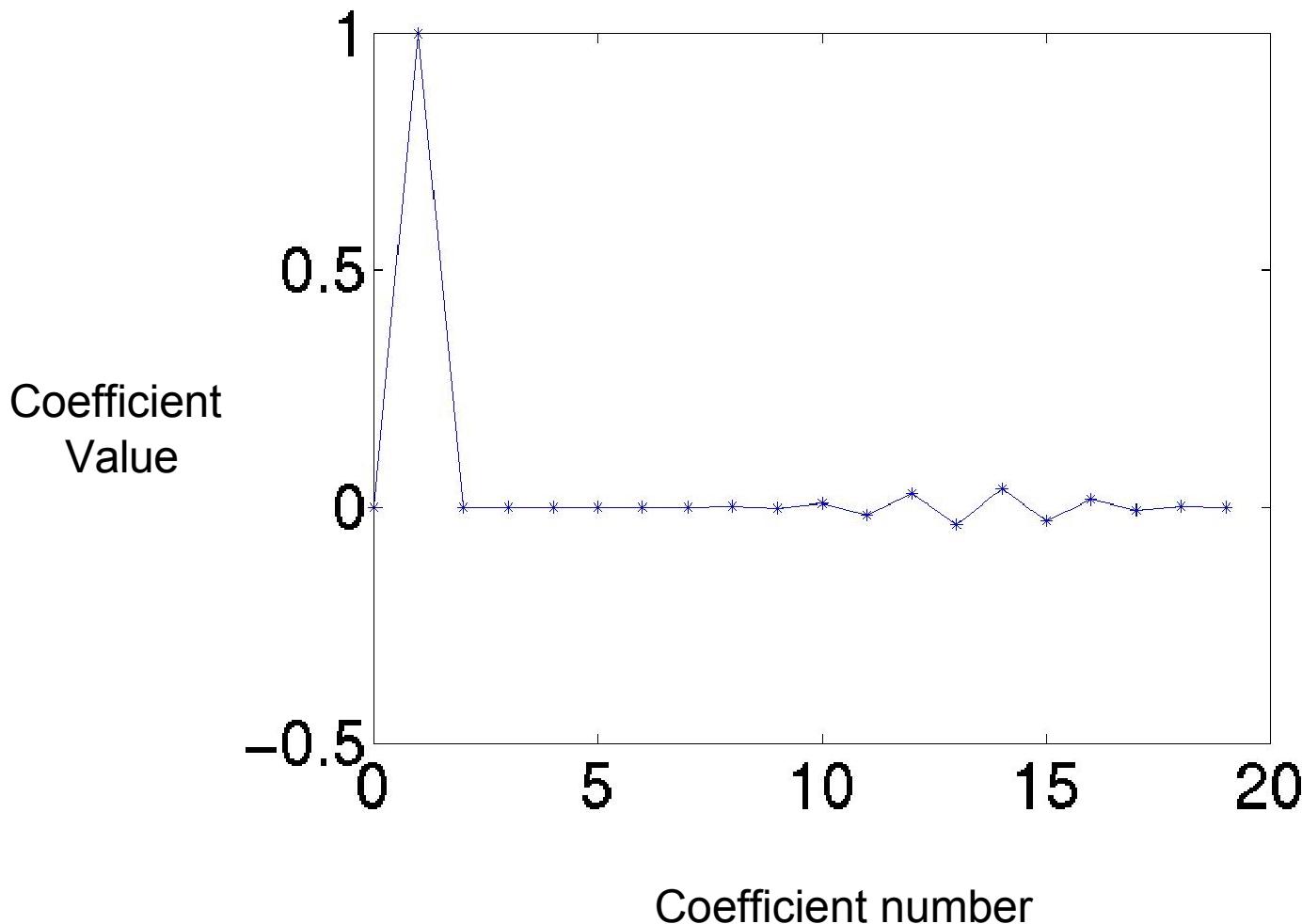
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HARD TO SOLVE PROBLEMS

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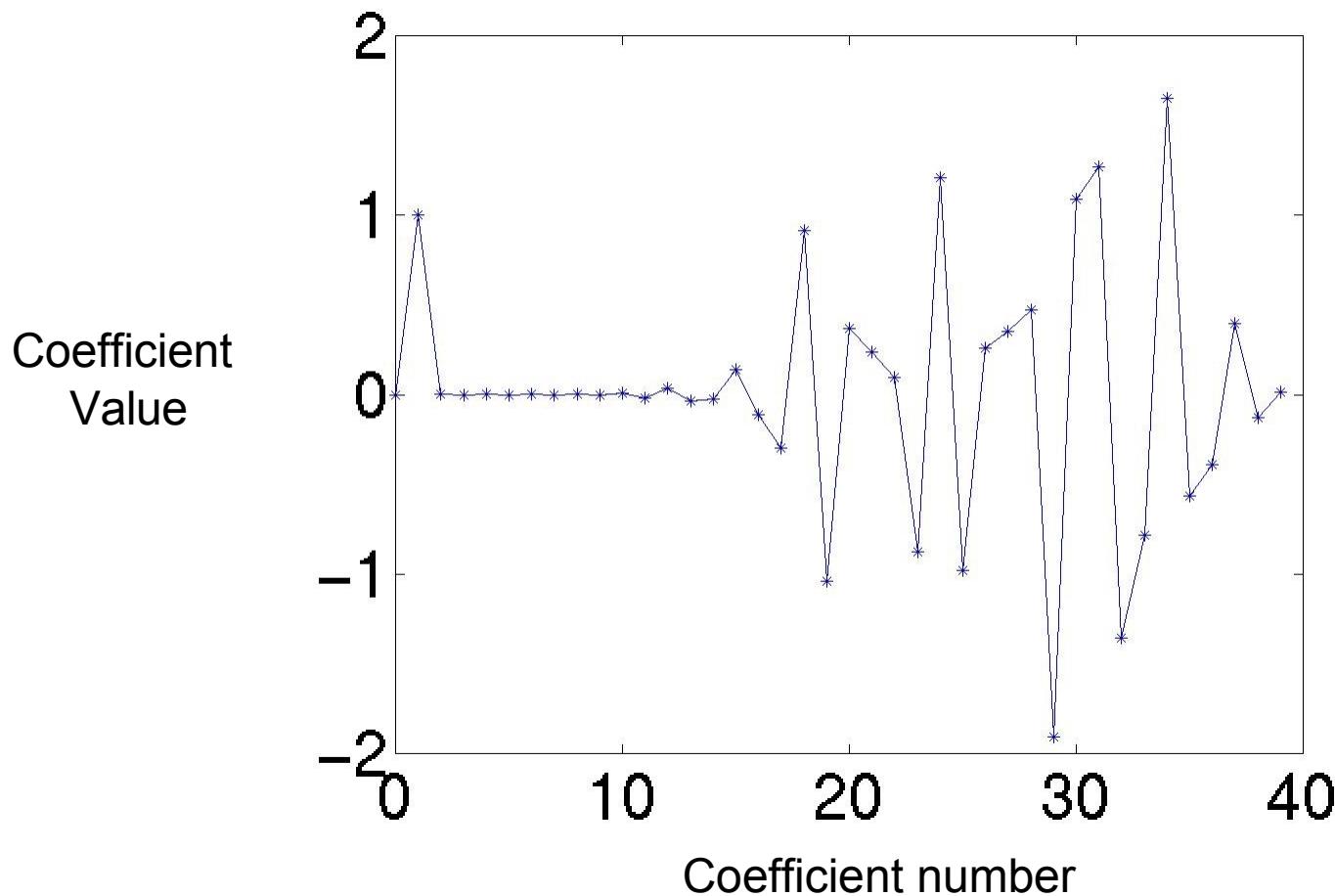
Fitting $f(t) = t$



HARD TO SOLVE PROBLEMS

“Polynomial interpolation” example

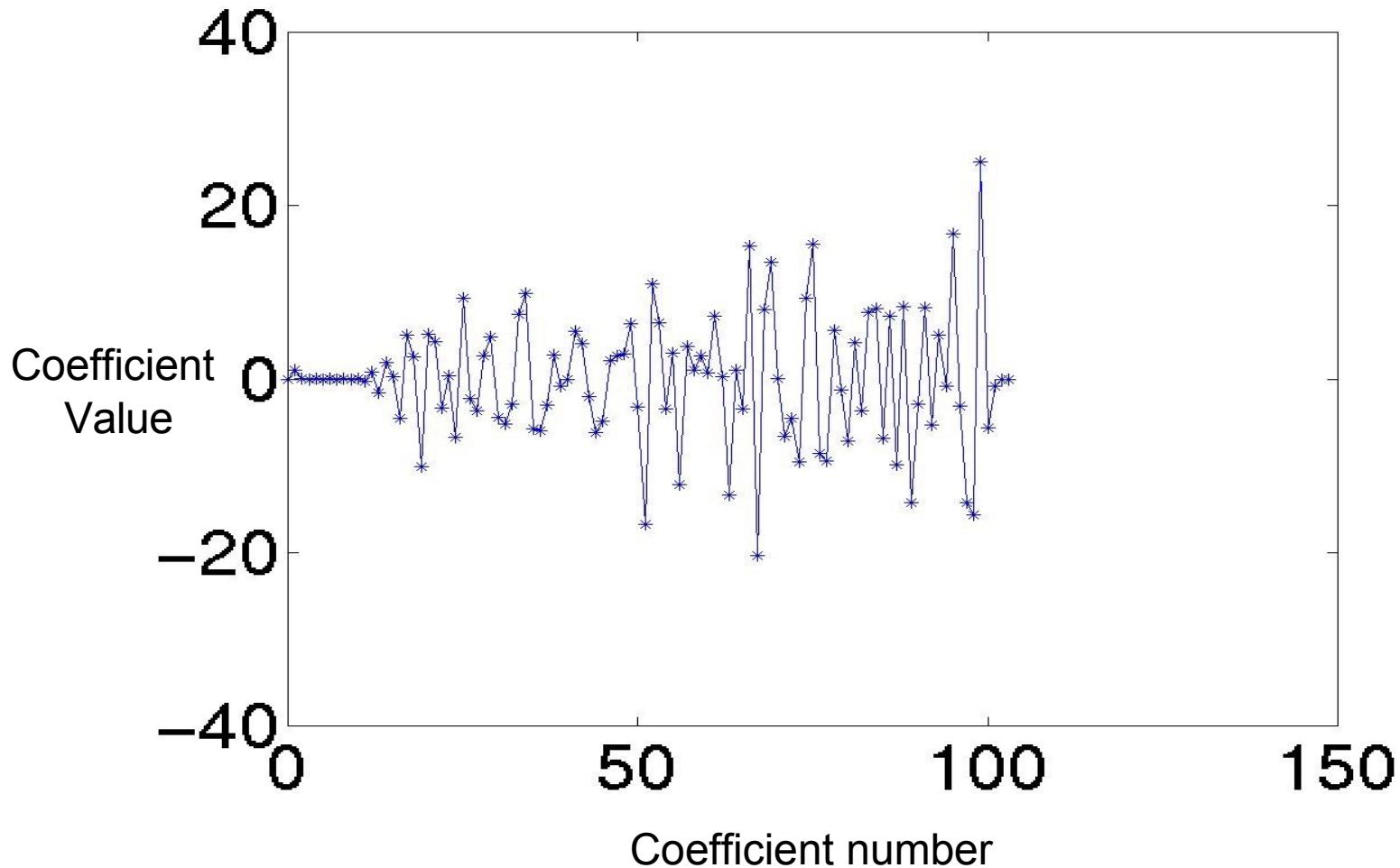
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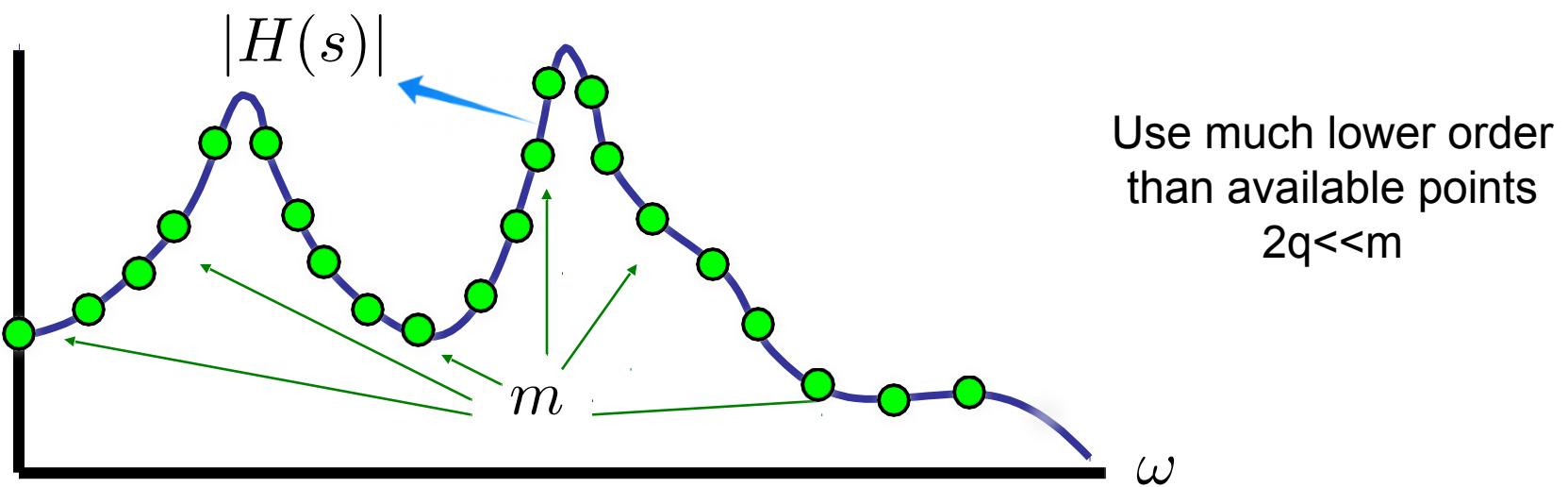
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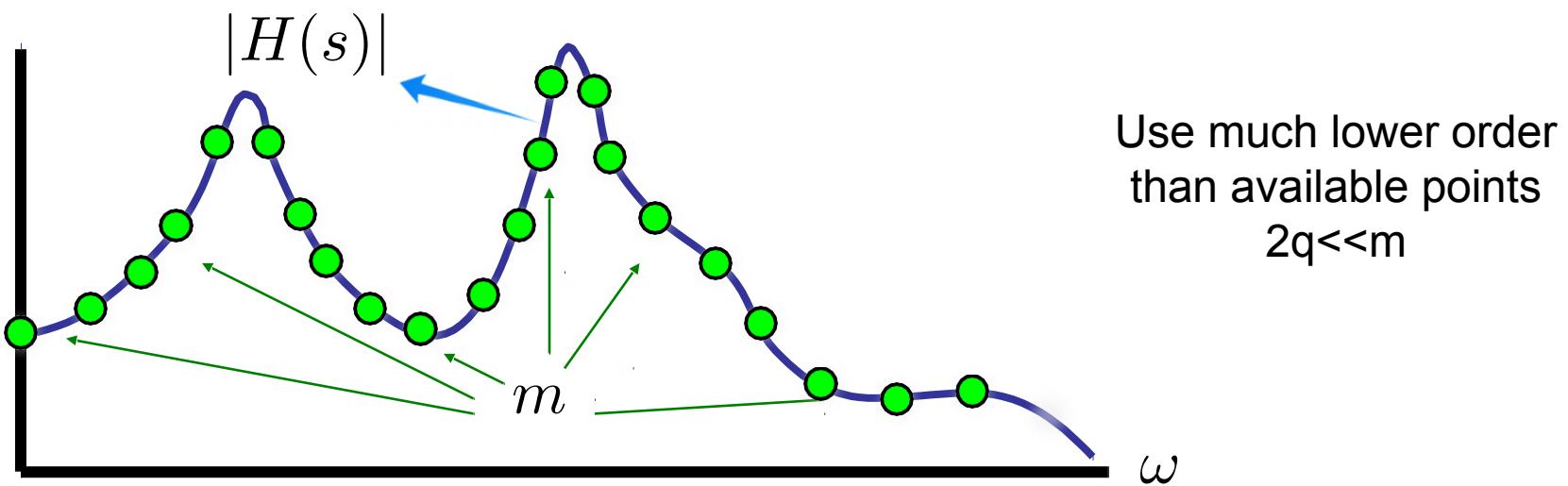
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MODEL ORDER REDUCTION via TRANSFER FUNCTION FITTING POINT MATCHING USING LEAST SQUARES



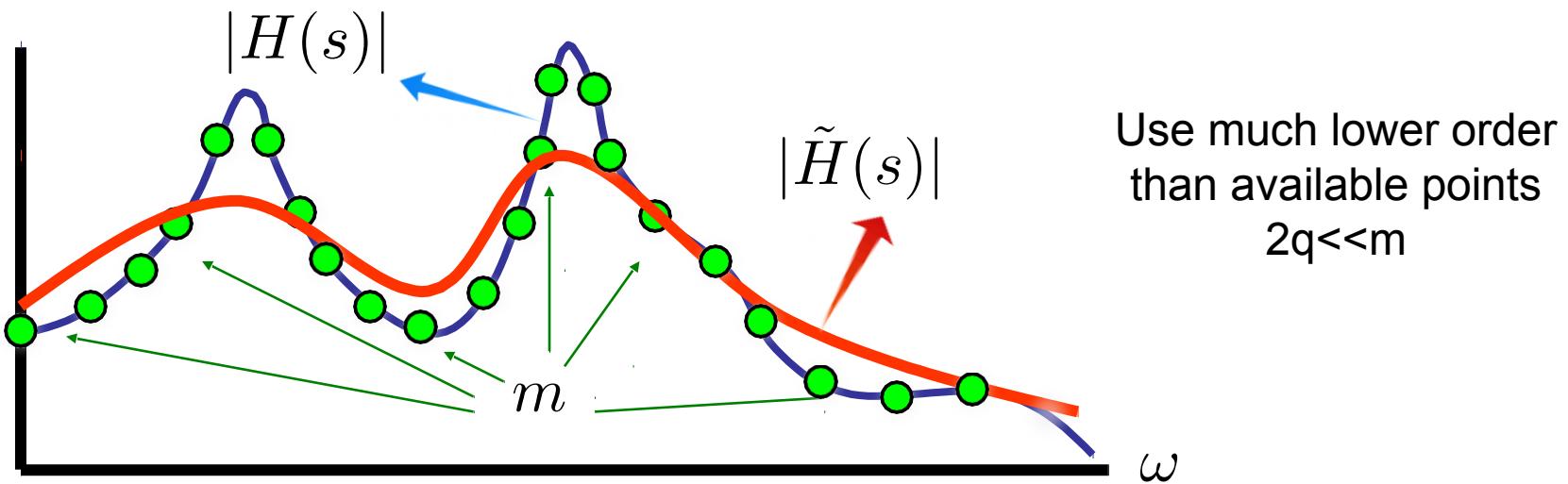
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Cross multiplying generates a linear TALL SKINNY system

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Cross multiplying generates a linear TALL SKINNY system
Least Squares problem: Use QR to solve the system

Model Order Reduction of Linear Problems

Problem setup

Reduction via “modal analysis”

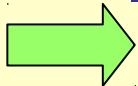
- time domain
- frequency domain

Reduction via transfer function fitting

- point matching
- least square

Importance of preserving physical properties

- stability
- passivity/dissipativity



STABILITY of STATE-SPACE MODELS

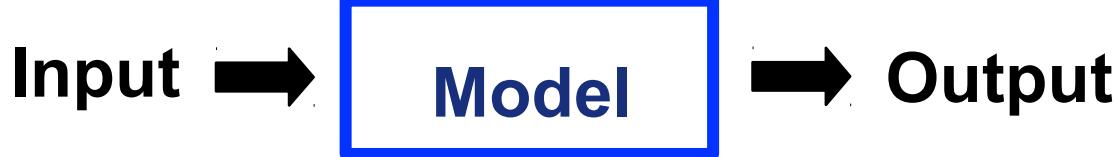
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$$\frac{dx}{dt} = Ax(t) + bu(t), \quad y(t) = c^T x(t)$$

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For well-behaved (e.g. bounded) inputs, when will the outputs be well-behaved (e.g. bounded) as well?

STABILITY of STATE-SPACE MODELS

- 1) From systems theory, the model will be bounded-input / bounded-output (BIBO) stable if the transfer function has no poles in the right half-plane.

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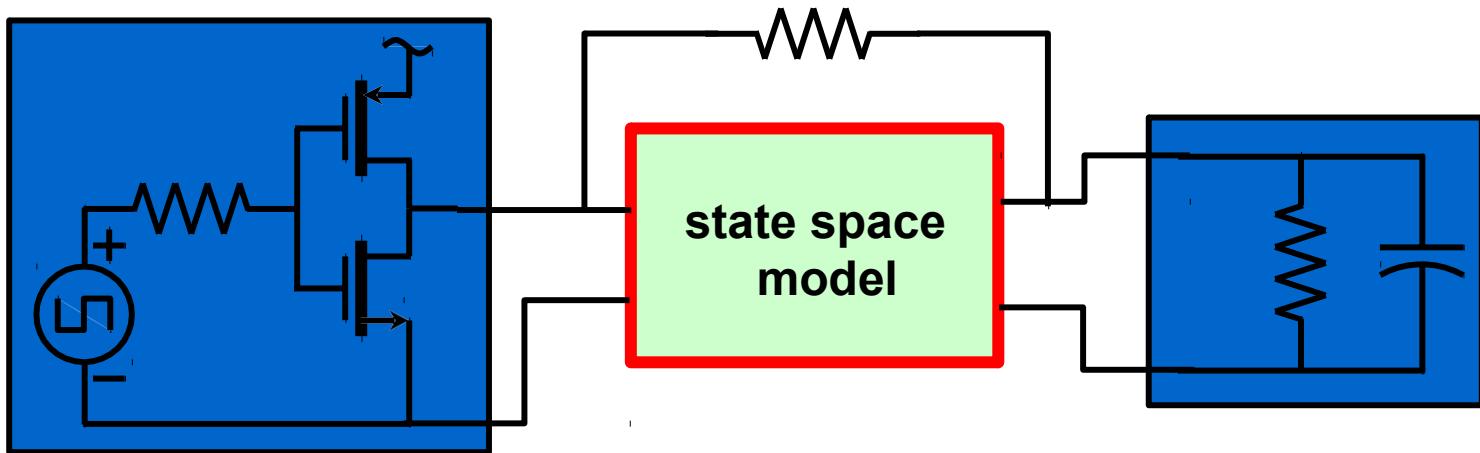
$$H(s) = \sum_{i=1}^N \frac{\tilde{c}_i \tilde{b}_i}{s - \lambda_i} \quad h(t) = \sum_{i=1}^N \tilde{c}_i \tilde{b}_i e^{\lambda_i t}$$

INTERCONNECTED SYSTEMS

- 1) In reality, dynamical models are only useful when connected together with models of other components in a composite simulation

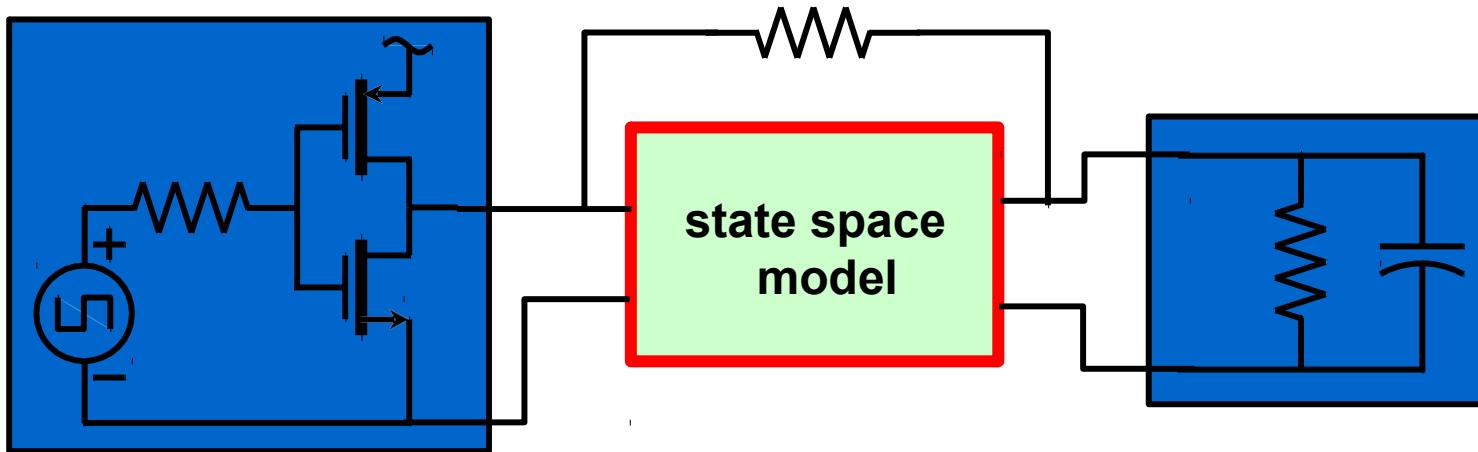
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- 3) Can we assure that the simulation of the composite system will be well-behaved?

Model Order Reduction of Linear Problems

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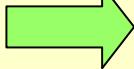
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Reduction via transfer function fitting

- point matching
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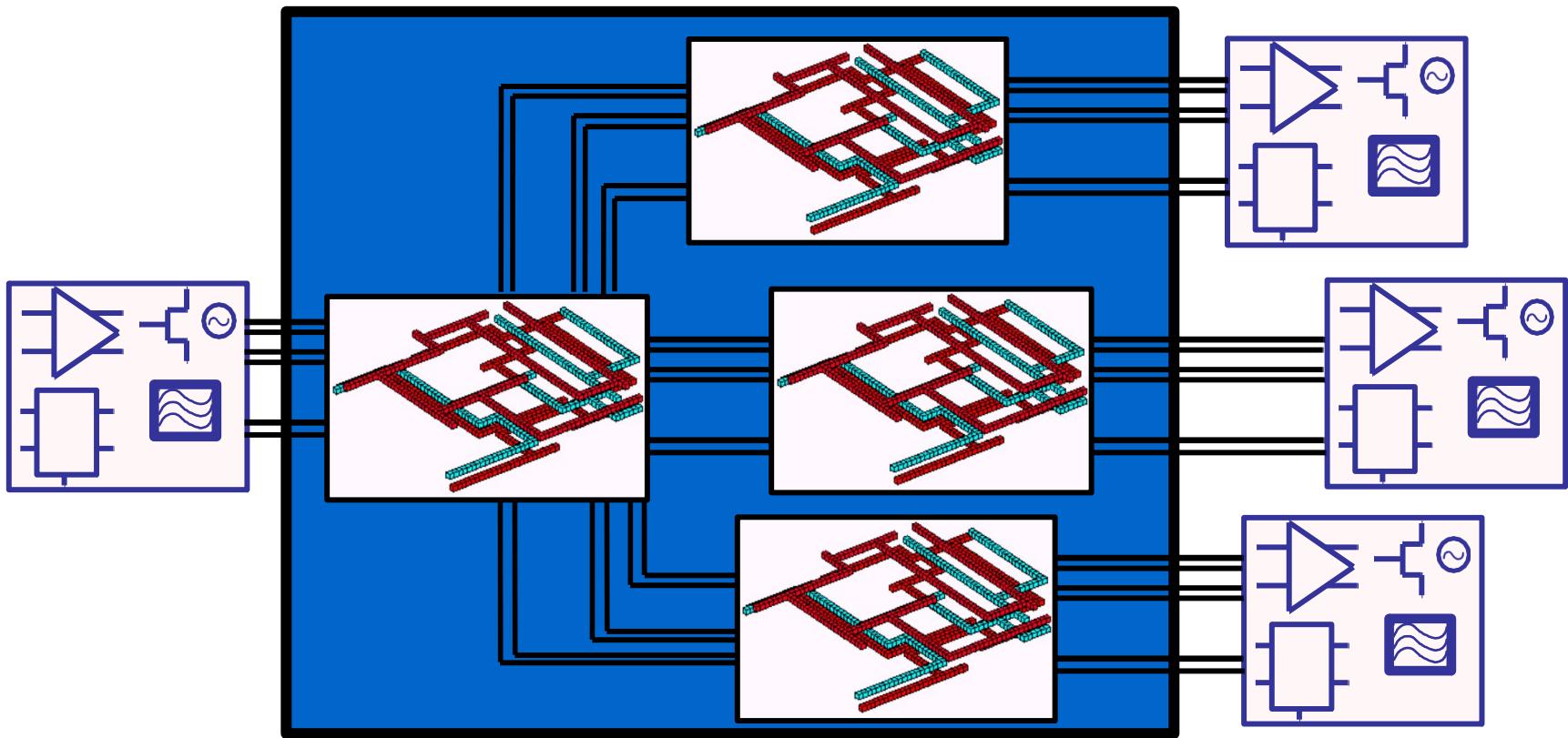
Importance of preserving physical properties

- stability
- passivity/dissipativity

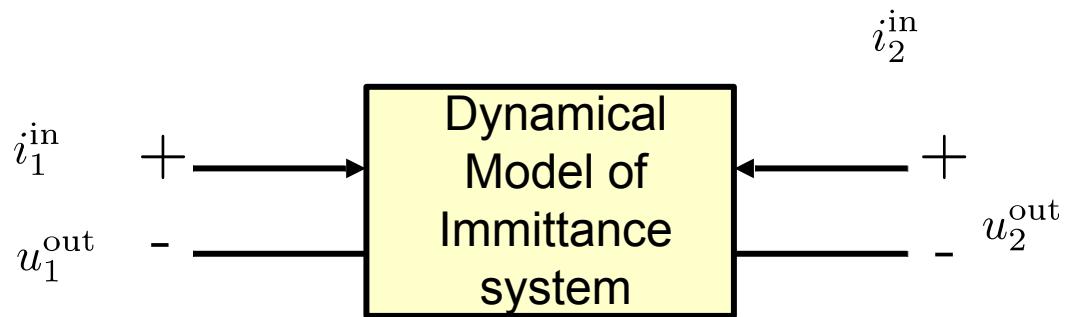


INTERCONNECTING PASSIVE/DISSIPATIVE SYSTEMS

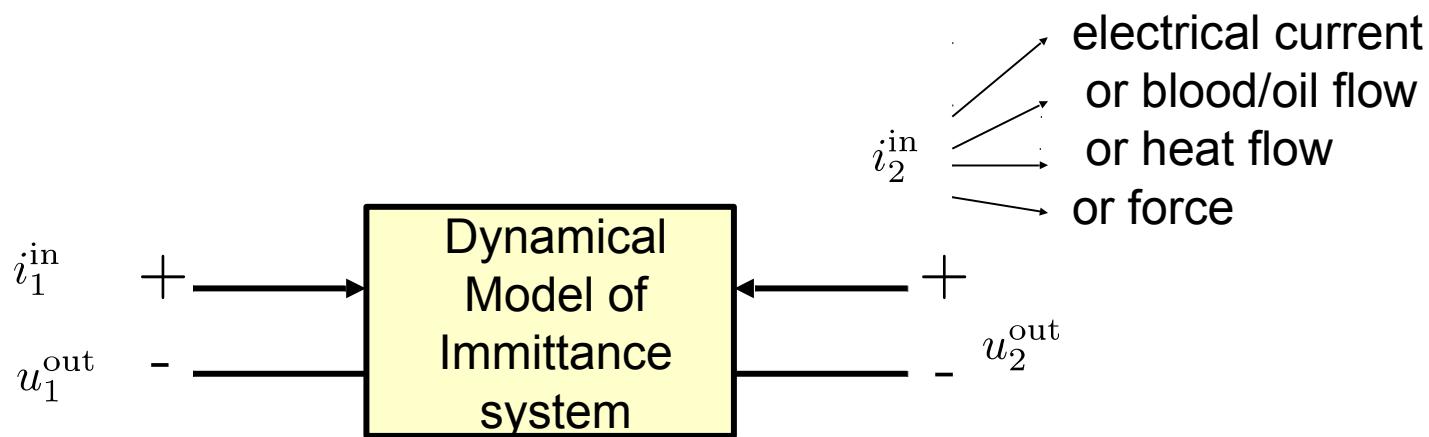
- 1) The interconnection of stable models is not necessarily stable
- 2) BUT the interconnection of passive/dissipative models is a passive/dissipative model (and hence also stable)



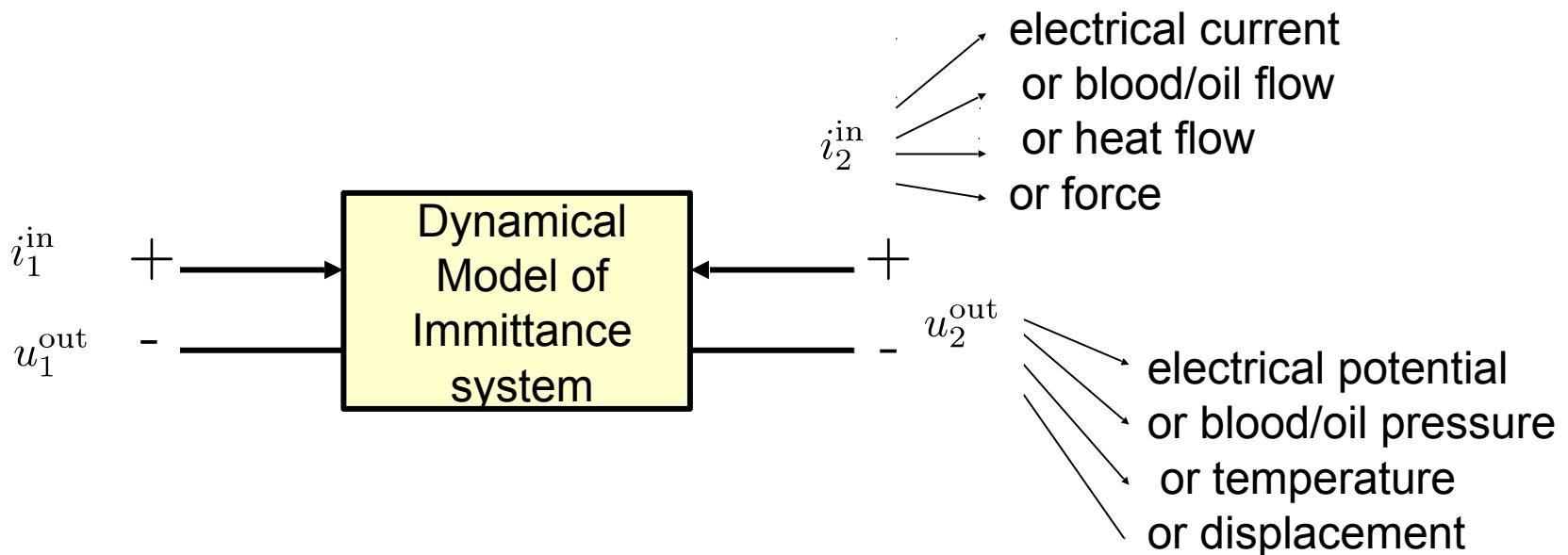
PASSIVITY/DISSIPATIVITY for IMMITTANCE SYSTEMS



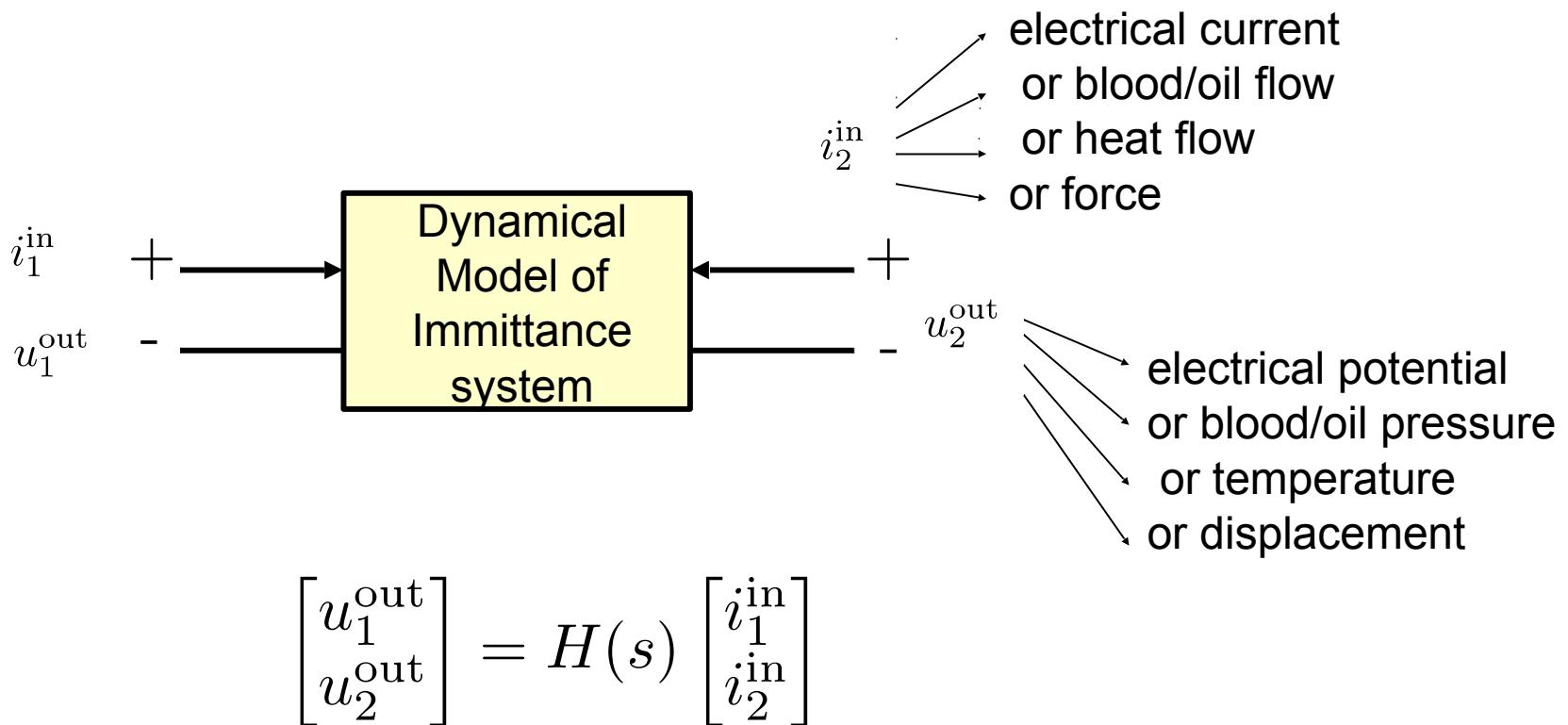
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PASSIVITY/DISSIPATIVITY for IMMITTANCE SYSTEMS



In “**immittance systems**” the quantity $\sum i_k u_k$
has typical the meaning of **physical power**

PASSIVITY/DISSIPATIVITY

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$$\text{Energy} = \int_{-\infty}^t i(\tau)u(\tau)d\tau > 0$$

PASSIVITY/DISSIPATIVITY CONDITION ON TRANSFER SYSTEMS

For “immitance systems”, passivity/dissipativity is equivalent to **positive-realness** of the transfer function

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$$\operatorname{Re}\{H(j\omega)\} \geq 0, \quad \forall \omega$$

It means its real part is positive
FOR ANY FREQUENCY

POSITIVITY of QUADRATIC FORMS – HERMITIAN DECOMPOSITION

Hermitian Decomposition

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Hermitian Decomposition

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2}$$

Hermitian part of A skew-Hermitian part of A

Quadratic Forms

$$\operatorname{Re}\{x^* A x\} = x^* \left(\frac{A + A^*}{2} \right) x \geq 0 \text{ iff } \left(\frac{A + A^*}{2} \right) \geq 0$$

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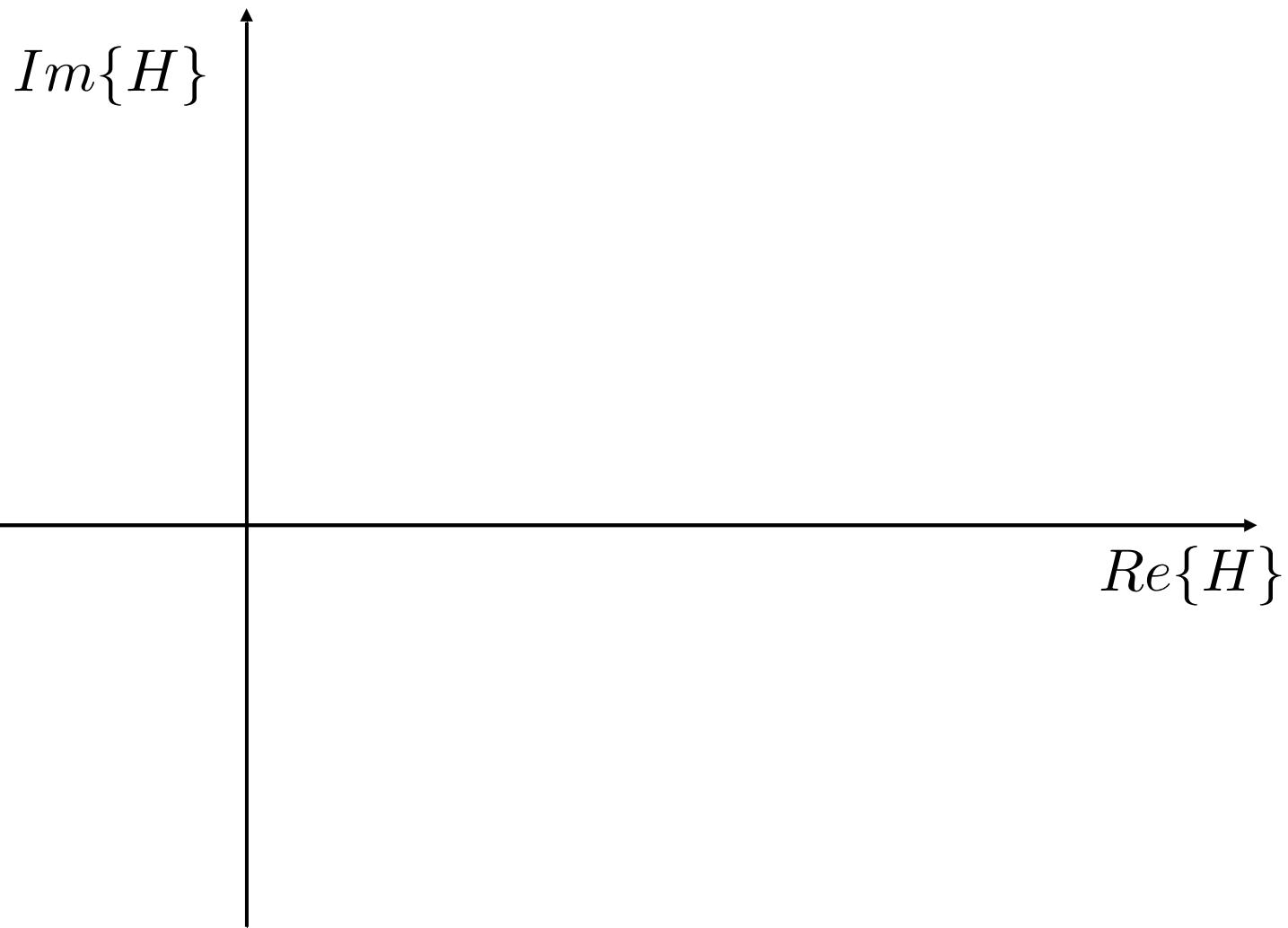
↑ ↑
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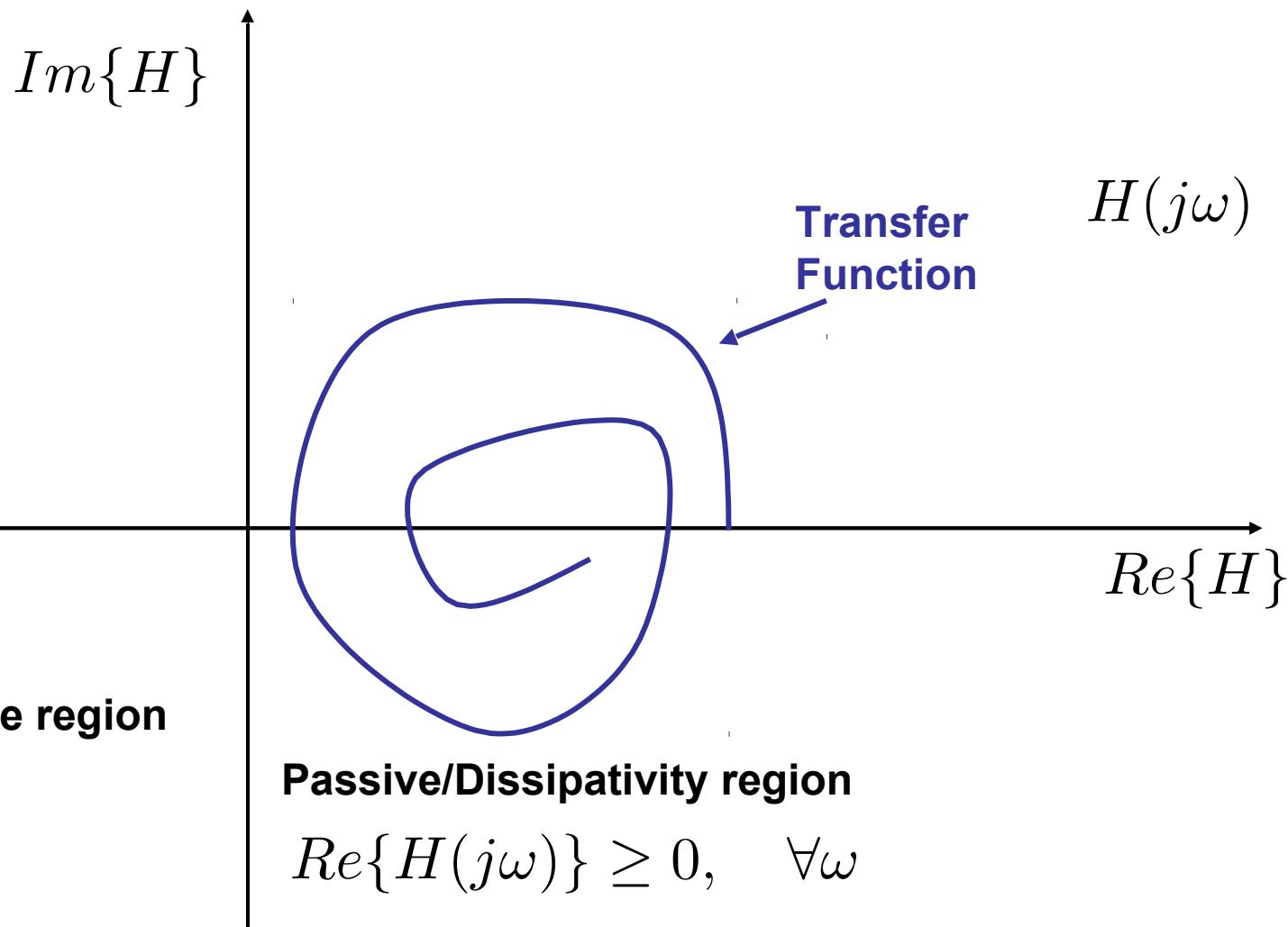
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$$\operatorname{Im}\{x^* A x\} = x^* \left(\frac{A-A^*}{2} \right) x \geq 0 \text{ iff } \left(\frac{A-A^*}{2} \right) \geq 0$$

Positive real transfer function in the complex plane for different frequencies



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Sufficient conditions for passivity/dissipativity of immitance dynamical system models

$$sEx = Ax + Bu$$

$$y = Cx$$

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Sufficient conditions for passivity/dissipativity for immitance dynamical models:

$$C = B^T$$

$$x^T Ex \geq 0, \quad \forall x$$

Positive Semidefinite

$$x^T Ax \leq 0, \quad \forall x$$

Negative Semidefinite

Necessary and Sufficient Condition for Passivity

Positive Real (Kalman-Yakubovich-Popov) Lemma

$$\frac{dx}{dt} = Ax + Bu, \quad y = C^T x$$

Necessary and Sufficient Condition for Passivity Positive Real (Kalman-Yakubovich-Popov) Lemma

$$\frac{dx}{dt} = Ax + Bu, \quad y = C^T x$$

A stable system (A, B, C) is positive real if and only if there exists $P = P^* \geq 0$
Such that the linear matrix inequality is satisfied

$$-A^T P - PA \geq 0$$

$$B^T P = C$$