Skoltech

Skolkovo Institute of Science and Technology

Numerical PDEs Guest Lecture

Some Aspects of Integral Equation Methods

Thanos Polimeridis

Skoltech - March 1, 2016

Why IE in NumPDEs course?

PDE solvers (FEM, FDTD, etc)

- Simple
- General purpose
- Sparse linear systems or not at all
- Large number of unknowns
- Absorbing boundary conditions

Integral equations

- Dimensionality reduction
- Automatically satisfy radiation conditions
- High-order approximations
- High complexity

Matrix inverse

$$A^{-1}A = AA^{-1} = I$$

$$A\mathbf{u} = \mathbf{f} \Rightarrow \mathbf{u} = A^{-1}\mathbf{f}$$

$$u_i = \left(A^{-1}\mathbf{f}\right)_i = \sum_j \left(A^{-1}\right)_{i,j} f_j$$

$\left(A^{-1}\right)_{i,j}$ is the **"effect" (solution) at** i **of a source at** j

[and column j of A^{-1} is the effect (solution) **everywhere** from the source at j]

Operator inverse

$$\mathcal{A}u = f \Rightarrow u = \mathcal{A}^{-1} f$$

$\mathcal{A}^{-1}$ is **whatever operator gives the solution** $u = \mathcal{A}^{-1} f$ for any f

$$\boxed{u(\mathbf{x}) = \mathcal{A}^{-1} f = \int_\Omega d^d\mathbf{x}'\, G(\mathbf{x} - \mathbf{x}') f(\mathbf{x}')}$$

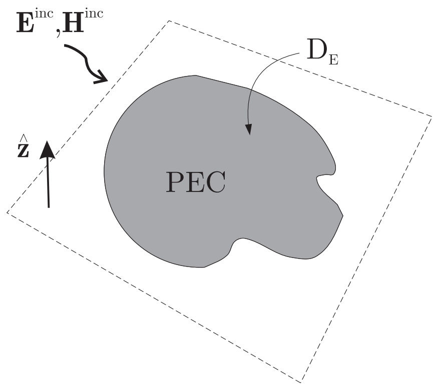
Green function

$$\boxed{u(\mathbf{x}) = \mathcal{A}^{-1} f = \int_\Omega d^d\mathbf{x}'\, G(\mathbf{x} - \mathbf{x}')\, f(\mathbf{x}')}$$

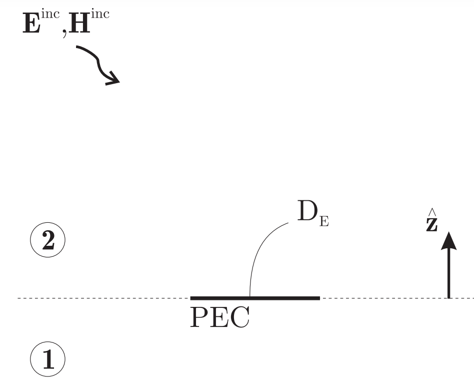
$G(\mathbf{x} - \mathbf{x}')$: **Green function** or **fundamental solution** to the PDE

: **is the effect at** x **from a source “at”** x'

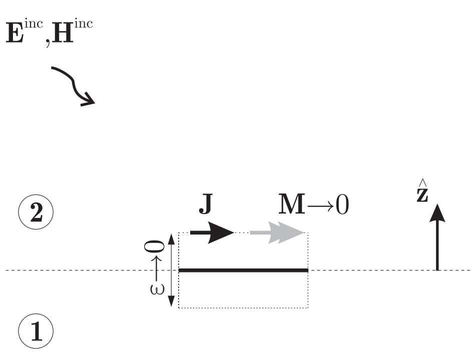
EM scattering example



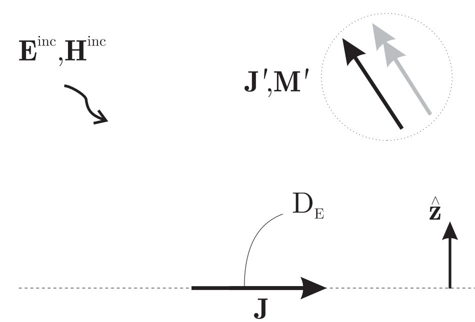
(a) Perfect Electric Conductor



(b) Profile view



(c) Equivalent “mathematical” surface

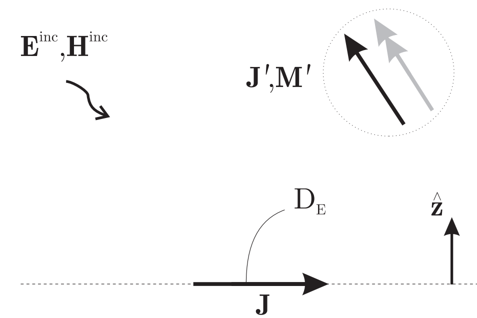


(d) Equivalent scattering problem

Surface equivalence principle

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = i\omega\mu\mathbf{J}'(\mathbf{r})$$

$$+\,\text{Boundary Conditions}$$



$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - k^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}')\overline{\mathbf{I}}$$

$$+\,\text{Boundary Conditions}$$

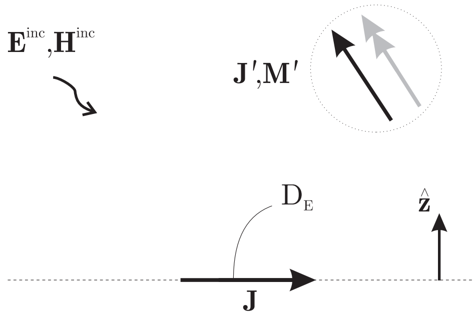
Surface Equivalence Principle

$$\int_{D_E} \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')dD = -\hat{z} \times \mathbf{E}^{\rm inc}(\mathbf{r})$$

IE vs FEM

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 \mathbf{E}(\mathbf{r}) = i\omega\mu \mathbf{J}'(\mathbf{r})$$

$$+ \text{Boundary Conditions}$$



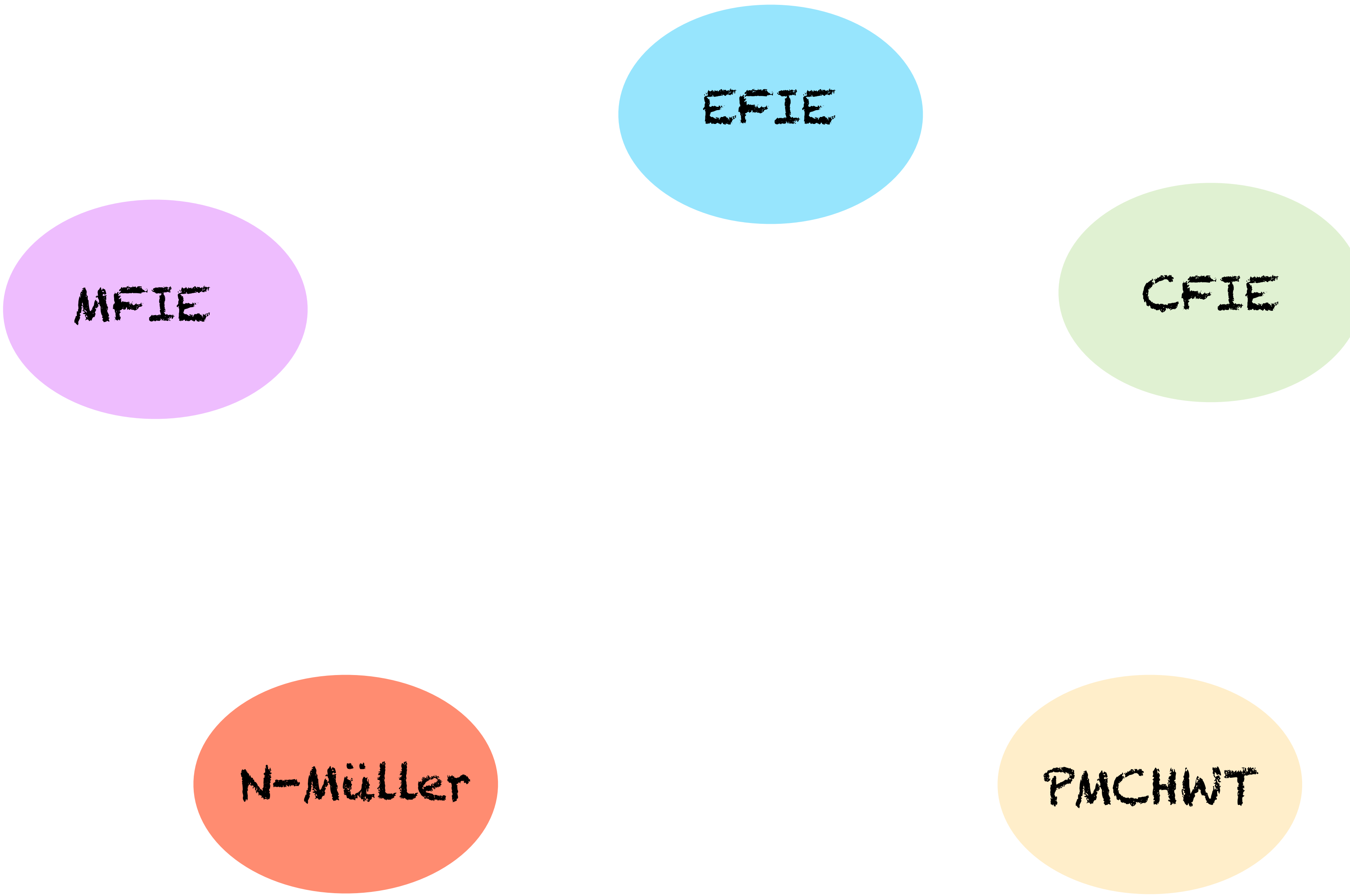
IE

$$\boxed{\int_{D_E} \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dD = -\hat{z} \times \mathbf{E}^{\text{inc}}(\mathbf{r})}$$

FEM

$$\boxed{\int_V [\nabla \times \mathbf{E}' \cdot \nabla \times \mathbf{E} - k^2 \mathbf{E} \cdot \mathbf{E}'] dV + \oint_{\partial V} \mathbf{E}' \cdot \hat{\mathbf{n}} \times \nabla \times \mathbf{E} dS = i\omega\mu \int_V \mathbf{E}' \cdot \mathbf{J}}$$

Variational-Based Galerkin SIE



Variational-Based Galerkin SIE

EFIE

MFIE

CFIE

✓Reduce dimensionality

✓Satisfy by definition the radiation condition

✓Green function is an exact propagator

N-Müller

PMCHWT

Variational-Based Galerkin SIE

EFIE

MFIE

CFIE

✓Symmetric Impedance Matrix

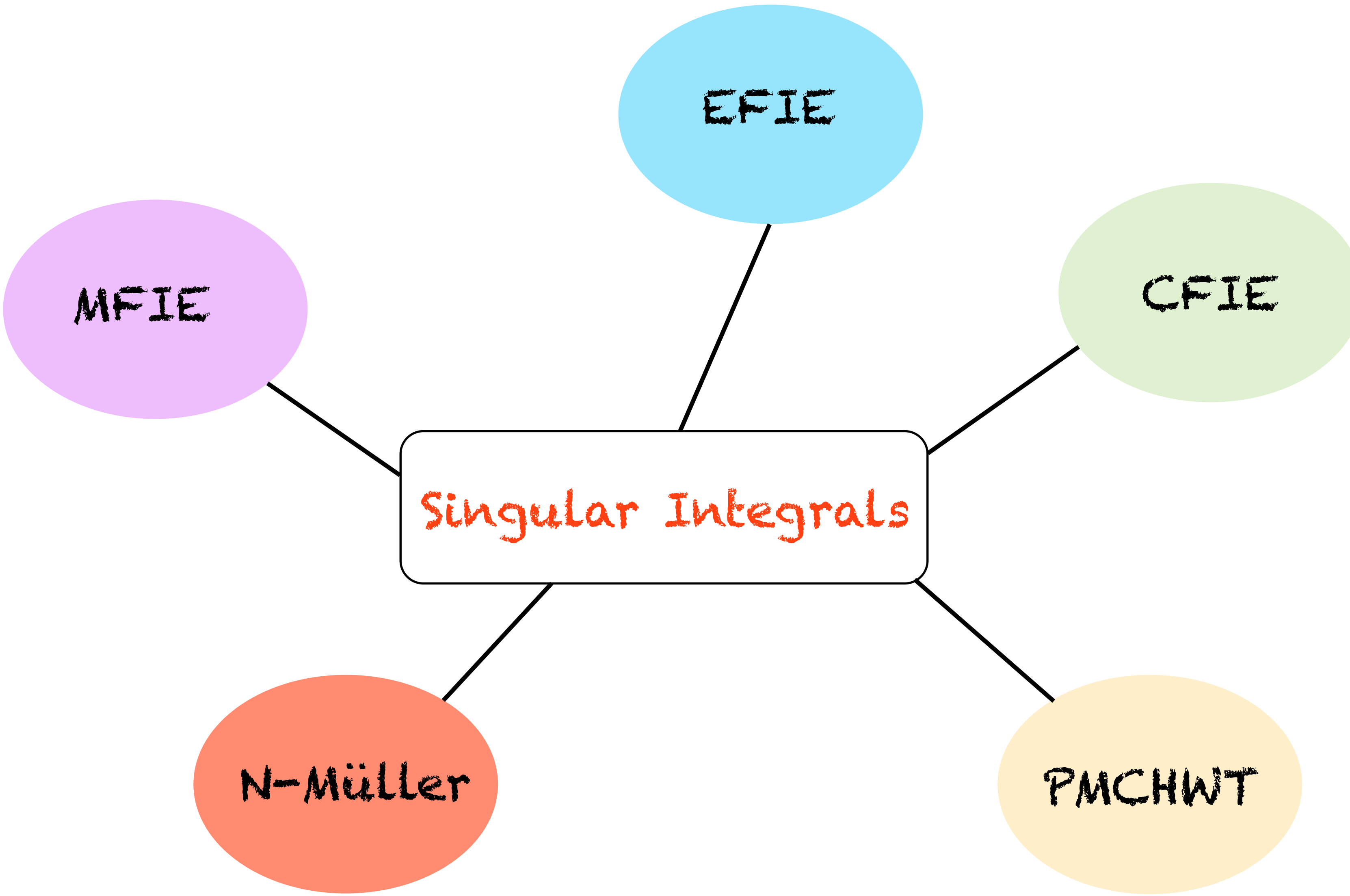
✓Converge quasi-optimally for smooth surfaces

✓Stability, consistency and convergence

N-Müller

PMCHWT

Variational-Based Galerkin SIE



Singular Galerkin Inner Products

Maxwell single layer potential (MSLP):

$$\boldsymbol{\mathcal{L}}(\mathbf{f})(\mathbf{r}) := ik\boldsymbol{\mathcal{S}}(\mathbf{f})(\mathbf{r}) - \frac{1}{ik}\nabla\mathcal{S}(\nabla_s' \cdot \mathbf{f})(\mathbf{r})$$

Maxwell double layer potential (MDLP):

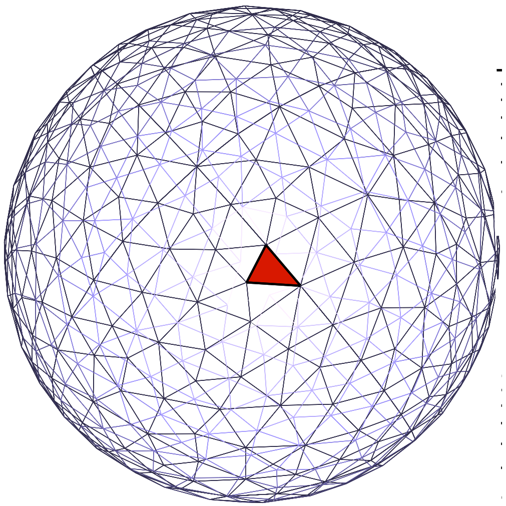
$$\boldsymbol{\mathcal{K}}(\mathbf{f})(\mathbf{r}) := \nabla \times \boldsymbol{\mathcal{S}}(\mathbf{f})(\mathbf{r})$$

$$\boldsymbol{\mathcal{S}}(\mathbf{f})(\mathbf{r}) := \int_\Gamma G(\mathbf{r},\mathbf{r}')\mathbf{f}(\mathbf{r}')dS'$$

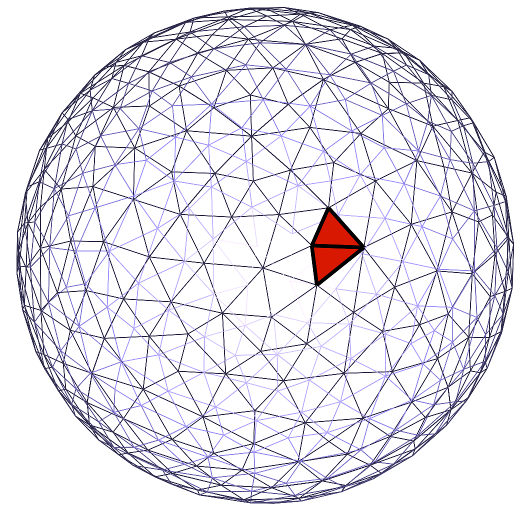
$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Singular Galerkin Inner Products PProductsProducts

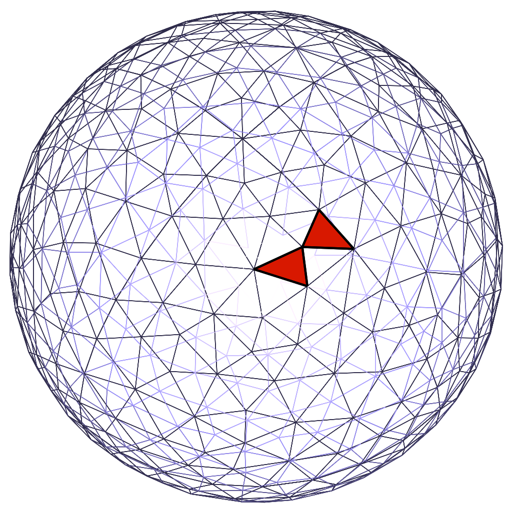
Self-Term (ST)



Edge Adjacent (EA)



Vertex Adjacent (VA)



Singular Galerkin Inner Products

Weakly Singular Integrals (WSI) associated to MSLP:

$$
\begin{aligned}
(I^l_{\boldsymbol{\mathcal{L}}})_{m,n} :=& ik \int_{E_P} \mathbf{f}_m \cdot \int_{E_Q} G\, \mathbf{f}_n \, dS' dS \\
&+ \frac{1}{ik} \int_{E_P} \nabla_s \cdot \mathbf{f}_m \int_{E_Q} G\, \nabla_s^{'} \cdot \mathbf{f}_n \, dS' dS
\end{aligned}
$$

Strongly Singular Integrals (SSI) associated to MDLP:

$$
(I^p_{\boldsymbol{\mathcal{K}}})_{m,n} := \int_{E_P} \mathbf{f}_m \cdot \int_{E_Q} \nabla G \times \mathbf{f}_n \, dS' dS
$$

Singular Galerkin Inner Products

Most popular approaches

- Singularity cancellation
- Singularity Subtraction

Main drawback

- Treat the 4D integral as 2D + 2D

Singular Galerkin Inner Products

Most popular approaches

- Singularity cancellation
- Singularity Subtraction

Main drawback

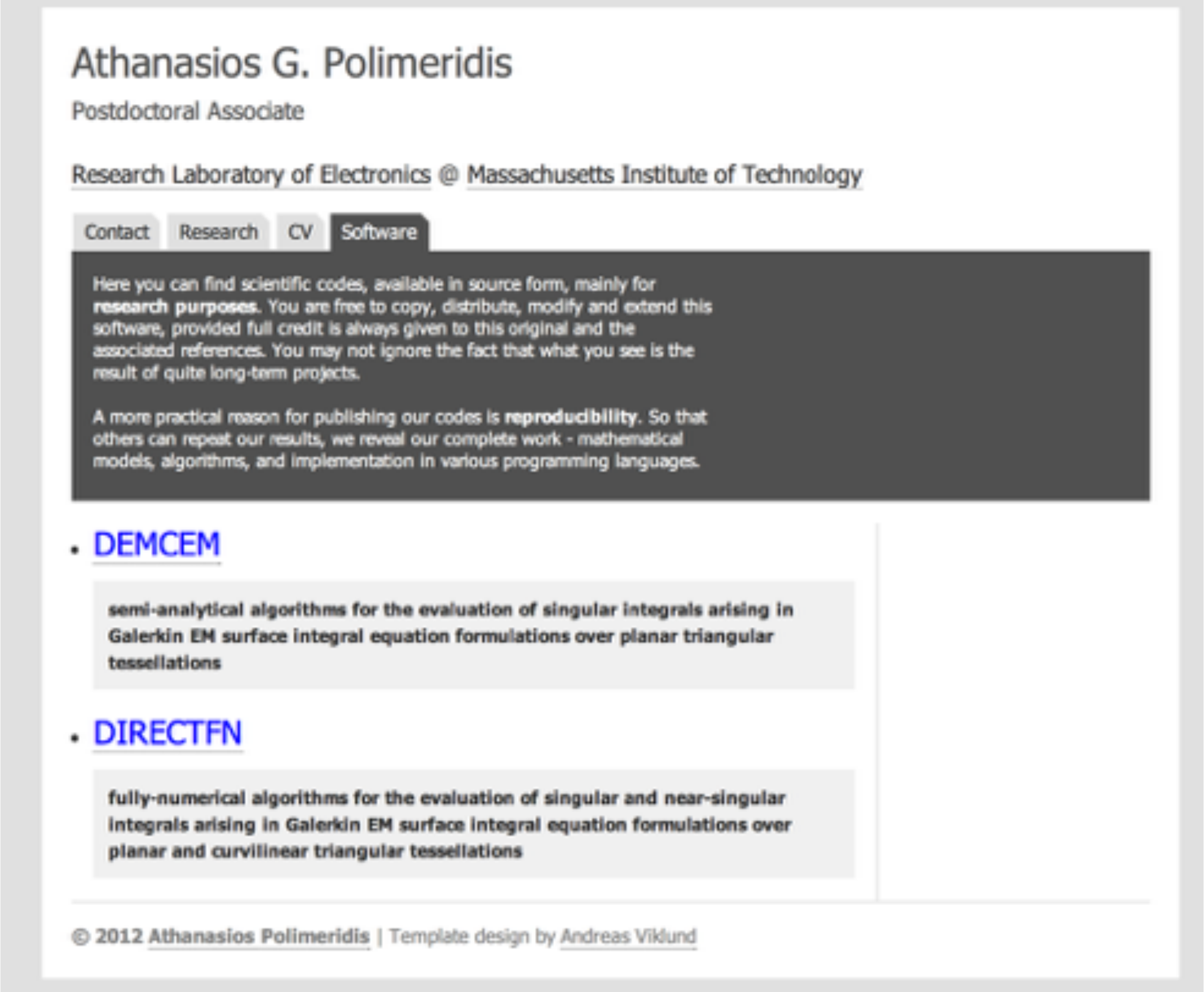
- Treat the 4D integral as 2D + 2D

2D source

2D observation

$$(I^l_{\mathcal{L}})_{m,n} := ik \int_{E_P} \mathbf{f}_m \cdot \int_{E_Q} G\, \mathbf{f}_n \, dS' dS + \frac{1}{ik} \int_{E_P} \nabla_s \cdot \mathbf{f}_m \int_{E_Q} G\, \nabla_s' \cdot \mathbf{f}_n \, dS' dS$$

Singular Galerkin Inner Products



DEMCEM

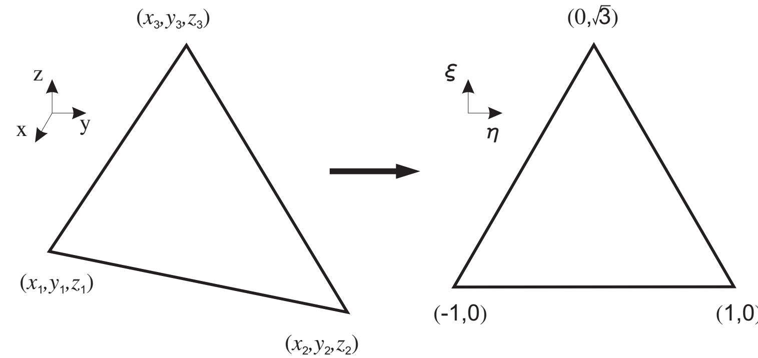
Key features

- ➡4D approach - no outer-inner 2D-2D
- ➡Singularity cancellation via variable transformations
- ➡Re-ordering of the integrations
- ➡Analytical integrations

Equilateral parameter space

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{x_2+x_1}{2} \\ \frac{y_2+y_1}{2} \\ \frac{z_2+z_1}{2} \end{bmatrix} + \begin{bmatrix} \frac{x_2-x_1}{2} & \frac{2x_3-x_1-x_2}{2\sqrt{3}} \\ \frac{y_2-y_1}{2} & \frac{2y_3-y_1-y_2}{2\sqrt{3}} \\ \frac{z_2-z_1}{2} & \frac{2z_3-z_1-z_2}{2\sqrt{3}} \end{bmatrix} \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$

$$= [\mathbf{A}] + [\mathbf{Q}] \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$



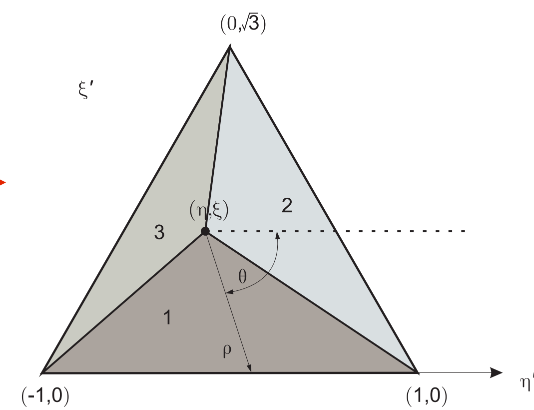
$$I = (J_p J_q) \int_{-1}^{1} d\eta \int_{0}^{\xi(\eta)} d\xi \int_{-1}^{1} d\eta' \int_{0}^{\xi(\eta')} d\xi'$$

DEMCEM - ST

$$
\begin{aligned}
\left(I_{\mathcal{L}}^{\mathrm{ST}}\right)_{m,n}^{\mathrm{sub}_1} &= \int_0^{\pi/3} \left[\Phi_{m,n}^{a}(\Psi) + \Phi_{m,n}^{e}(\Psi) + \Phi_{m,n}^{f}(\Psi)\right] d\Psi \\
&+ \int_{\pi/3}^{2\pi/3} \left[\Phi_{m,n}^{b}(\Psi) + \Phi_{m,n}^{g}(\Psi)\right] d\Psi \\
&+ \int_{2\pi/3}^{\pi} \left[\Phi_{m,n}^{c}(\Psi) + \Phi_{m,n}^{d}(\Psi) + \Phi_{m,n}^{h}(\Psi)\right] d\Psi
\end{aligned}
$$

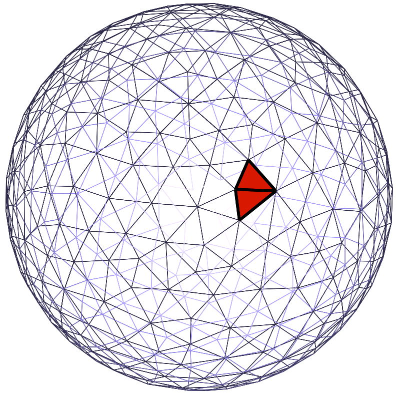
$$
\left(I_{\mathcal{L}}^{\mathrm{ST}}\right)_{m,n}^{\mathrm{sub}_2} = \left(I_{\mathcal{L}}^{\mathrm{ST}}\right)_{m,n}^{\mathrm{sub}_1}\Bigg|_{\substack{\alpha_{cc}\to\alpha_{cc,2}\\ \alpha_{cs}\to\alpha_{cs,2}\\ \alpha_{ss}\to\alpha_{ss,2}}}
$$

$$
\left(I_{\mathcal{L}}^{\mathrm{ST}}\right)_{m,n}^{\mathrm{sub}_3} = \left(I_{\mathcal{L}}^{\mathrm{ST}}\right)_{m,n}^{\mathrm{sub}_1}\Bigg|_{\substack{\alpha_{cc}\to\alpha_{cc,3}\\ \alpha_{cs}\to\alpha_{cs,3}\\ \alpha_{ss}\to\alpha_{ss,3}}}
$$



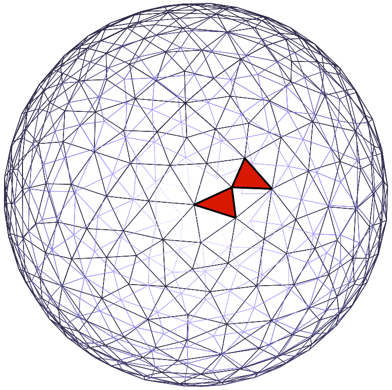
DEMCEM - EA

$$
\begin{aligned}
(I^{\mathrm{EA}}_{\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}})_{m,n} = & \int_0^{\pi/3} d\theta \int_0^{\Psi_B} \chi^a(\theta,\Psi)d\Psi + \int_{\pi/3}^{\pi/2} d\theta \int_{\Psi_A}^{\Psi_B} \chi^a(\theta,\Psi)d\Psi \\
& + \int_{\pi/3}^{\pi/2} d\theta \int_0^{\Psi_A} \chi^b(\theta,\Psi)d\Psi + \int_{\pi/2}^{\pi} d\theta \int_0^{\Psi_A} \chi^c(\theta,\Psi)d\Psi \\
& + \int_0^{\pi/2} d\theta \int_{\Psi_B}^{\pi/2} \chi^d(\theta,\Psi)d\Psi + \int_{\pi/2}^{\pi} d\theta \int_{\Psi_A}^{\pi/2} \chi^d(\theta,\Psi)d\Psi
\end{aligned}
$$



DEMCEM - VA

$$
\begin{aligned}
(I^{\mathrm{VA}}_{\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}})_{m,n} = \int_0^{\pi/3} d\theta_p \int_0^{\pi/3} d\theta_q \Big[\int_0^{\Psi_1} \mathcal{F}^{\alpha}_{m,n}(\theta_p, \theta_q, \Psi) d\Psi \\
+ \int_{\Psi_1}^{\pi/2} \mathcal{F}^{\beta}_{m,n}(\theta_p, \theta_q, \Psi) d\Psi \Big]
\end{aligned}
$$



DEMCEM - Wrap up

Original 4D WSI and SSI arising in Galerkin SIE boil down to

1D smooth integrals for WSI over ST

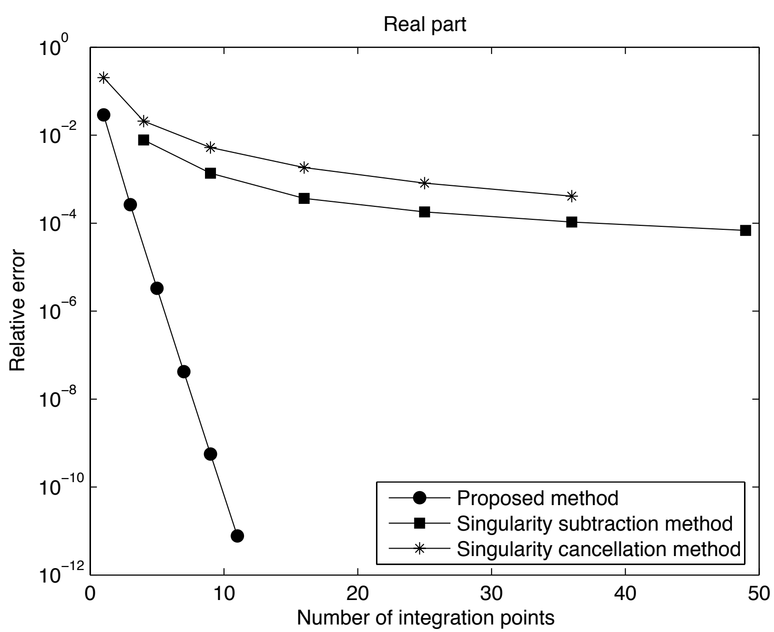
2D smooth integrals for WSI and SSI over EA

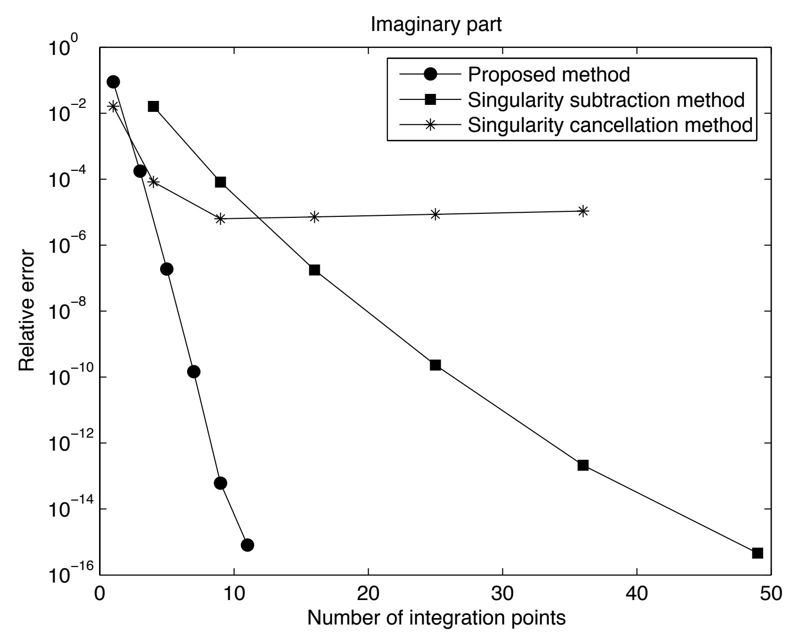
3D smooth integrals for WSI and SSI over VA

All kernels are sufficiently smooth functions!

DEMCEM - Results

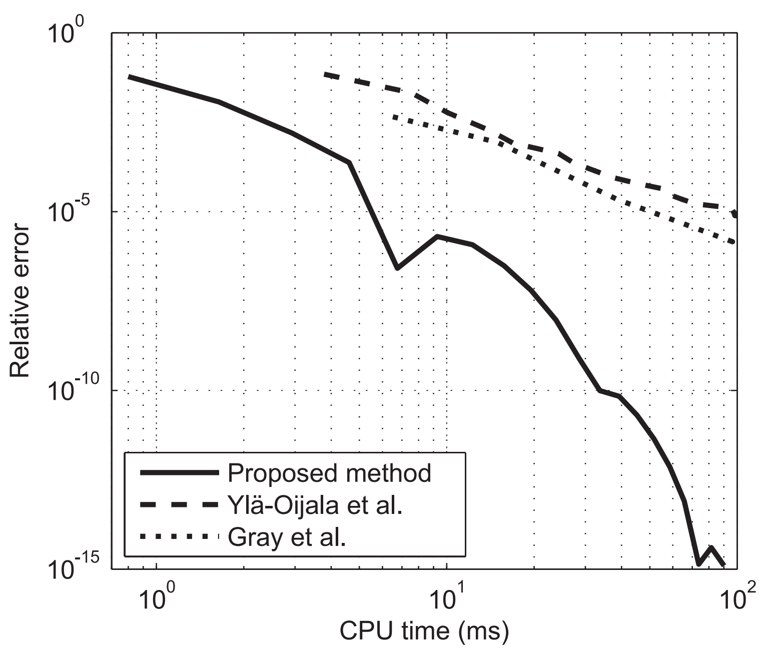
$$\mathcal{I} := \int_{E_P} \zeta_1 \int_{E_Q} G(\mathbf{r}, \mathbf{r}')\, \zeta_1'\, dS dS'$$

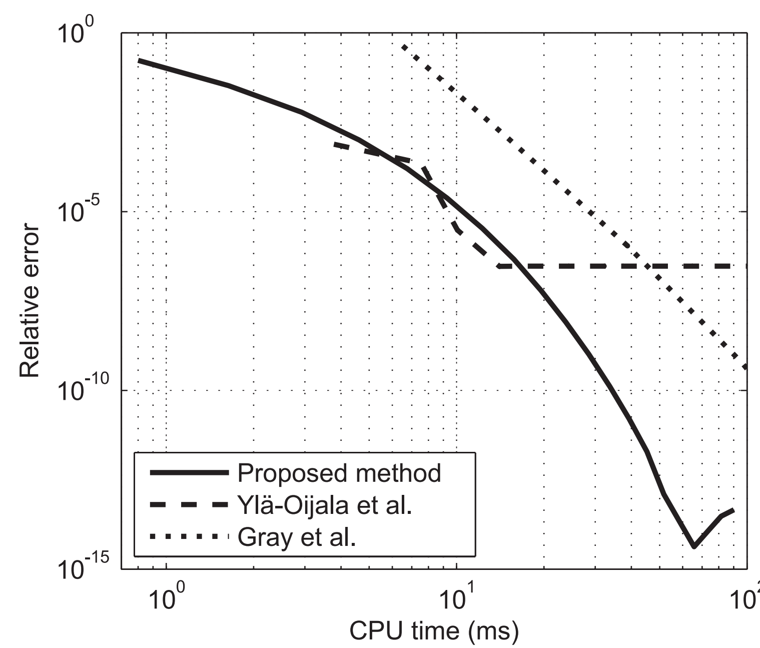




DEMCEM - Results

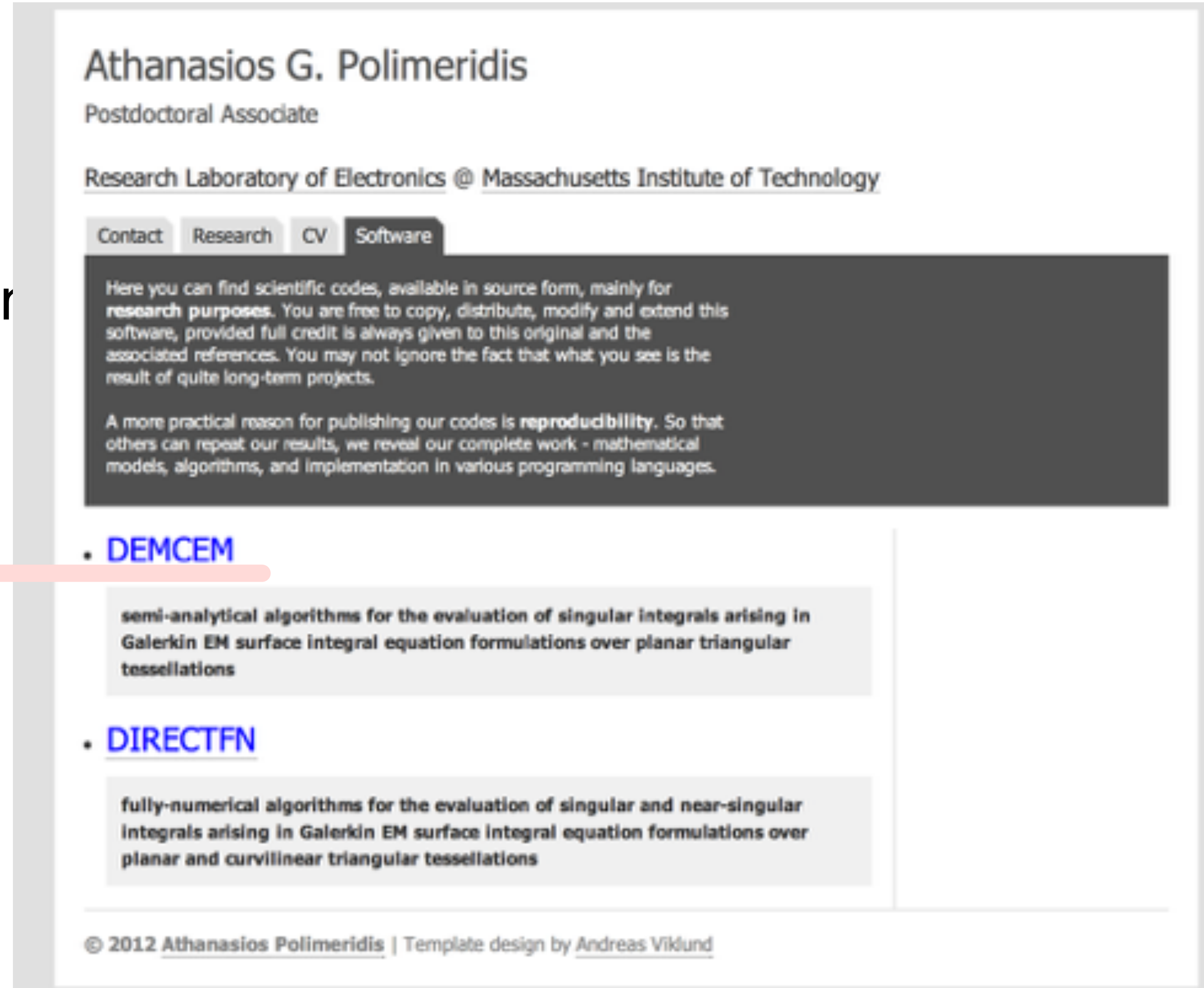
$$(I_{\boldsymbol{\mathcal{K}}}^{\mathrm{EA}})_{3,1} := \int_{E_P} \mathbf{f}_3 \cdot \int_{E_Q} \nabla G \times \mathbf{f}_1' \, dS' dS$$





DEMCEM - Software

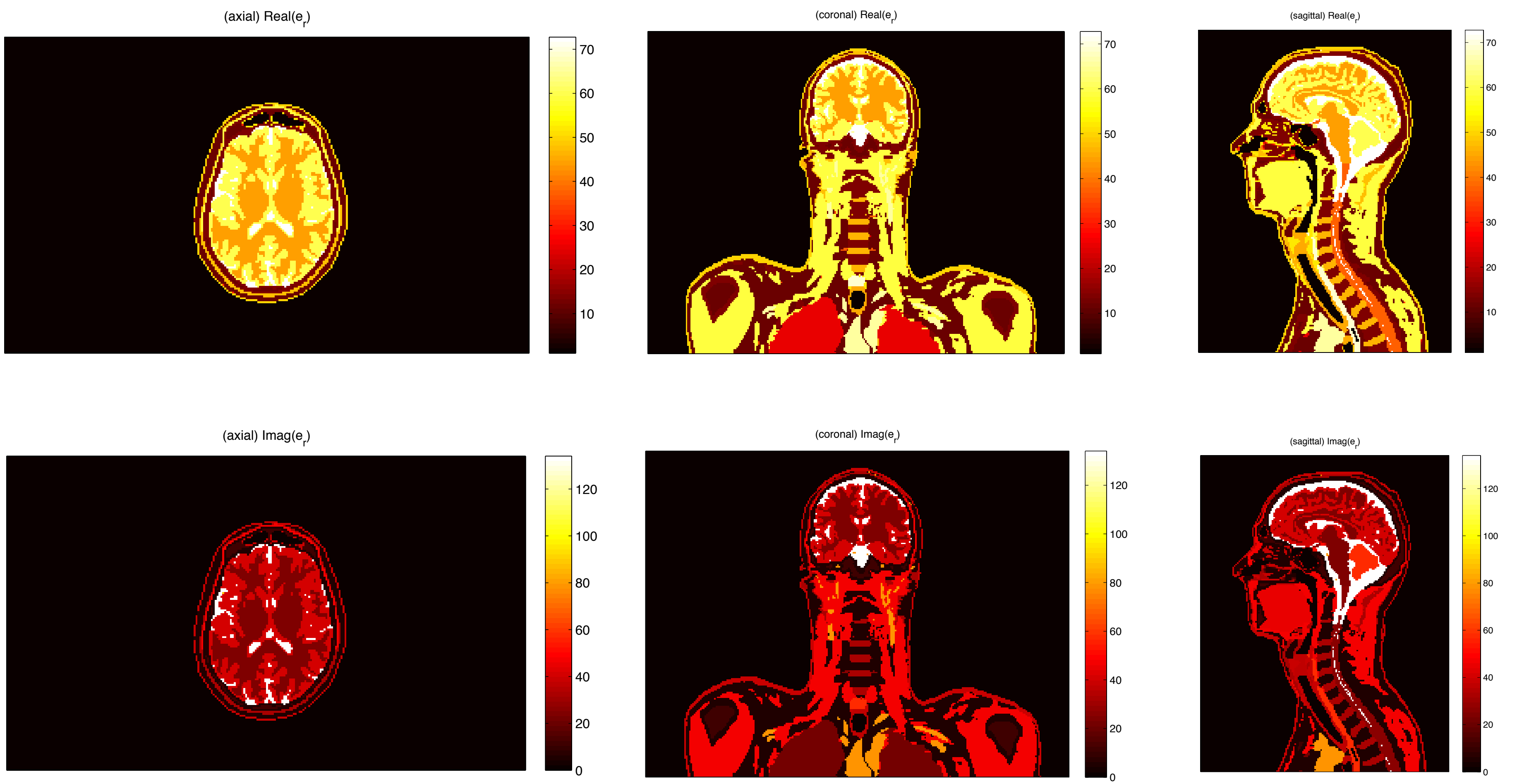
- ✓MATLAB
- ✓C
- ✓MATLAB plugi



> 20,000 code lines

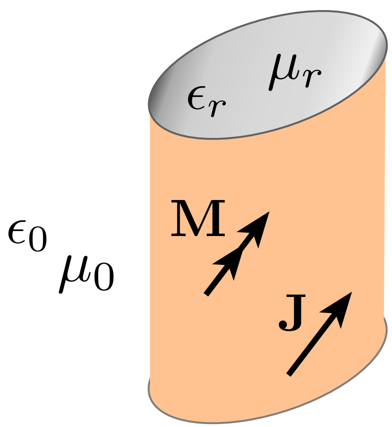
MRI-Specific Integral Equations

Electric Properties @ 7T (298.2 MHz)



MRI-Specific Integral Equations

Representation formulas for total fields

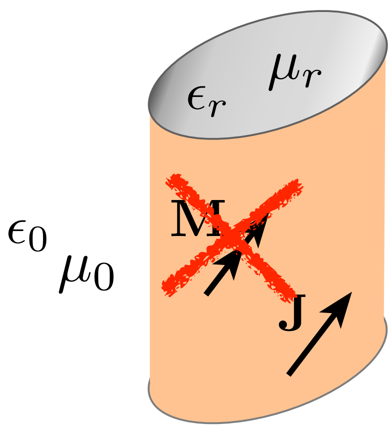


$$\mathbf{e} = \mathbf{e}^{\mathrm{inc}} + \mathbf{e}^{\mathrm{sca}} = \mathbf{e}^{\mathrm{inc}} + \frac{1}{c_\epsilon}\mathcal{L}\mathbf{j} - \mathcal{K}\mathbf{m}$$

$$\mathbf{h} = \mathbf{h}^{\mathrm{inc}} + \mathbf{h}^{\mathrm{sca}} = \mathbf{h}^{\mathrm{inc}} + \frac{1}{c_\mu}\mathcal{L}\mathbf{m} + \mathcal{K}\mathbf{j}$$

MRI-Specific Integral Equations

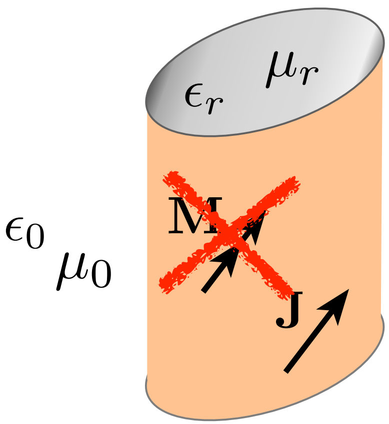
Representation formulas for total fields



$$\mathbf{e} = \mathbf{e}^{\text{inc}} + \mathbf{e}^{\text{sca}} = \mathbf{e}^{\text{inc}} + \frac{1}{c_\epsilon}\mathcal{L}\mathbf{j} - \cancel{\mathcal{K}\mathbf{m}}$$

$$\mathbf{h} = \mathbf{h}^{\text{inc}} + \mathbf{h}^{\text{sca}} = \mathbf{h}^{\text{inc}} + \cancel{\frac{1}{c_\mu}\mathcal{L}\mathbf{m}} + \mathcal{K}\mathbf{j}$$

Representation formulas for total fields



$$\mathbf{e} = \mathbf{e}^{\mathrm{inc}} + \mathbf{e}^{\mathrm{sca}} = \mathbf{e}^{\mathrm{inc}} + \frac{1}{c_\epsilon}\mathcal{L}\mathbf{j} - \cancel{\mathcal{K}\mathbf{m}}$$

$$\mathbf{h} = \mathbf{h}^{\mathrm{inc}} + \mathbf{h}^{\mathrm{sca}} = \mathbf{h}^{\mathrm{inc}} + \cancel{\frac{1}{c_\mu}\mathcal{L}\mathbf{m}} + \mathcal{K}\mathbf{j}$$

$$\mathcal{L}\mathbf{u} \triangleq (k_0^2 + \nabla\nabla\cdot)\mathcal{S}\left(\mathbf{u};\Omega\right)(\mathbf{r})$$

$$\mathcal{K}\mathbf{u} \triangleq \nabla \times \mathcal{S}\left(\mathbf{u};\Omega\right)(\mathbf{r})$$

$$\mathcal{S}\left(\mathbf{u};\Omega\right)(\mathbf{r}) \triangleq \int_\Omega G(\mathbf{R})\mathbf{u}(\mathbf{r}')d\mathbf{r}'$$

$$\mathbf{j}(\mathbf{r}) \triangleq c_\epsilon \chi_\epsilon(\mathbf{r})\,\mathbf{e}(\mathbf{r})$$

$$c_\epsilon = j\omega\epsilon_0$$
$$\chi_\epsilon = \epsilon_r(\mathbf{r}) - 1$$

MRI-Specific Integral Equations

EVIE $$(\mathcal{I} - \mathcal{L}\mathcal{M}_{\chi_\epsilon})\,\mathbf{e} = \mathbf{e}^{\mathrm{inc}}$$

DVIE $$(\mathcal{I} - \mathcal{L}\mathcal{M}_{\chi_\epsilon})\,\mathcal{M}_{\epsilon_r^{-1}}\mathbf{d} = \epsilon_0\mathbf{e}^{\mathrm{inc}}$$

JVIE $$(\mathcal{I} - \mathcal{M}_{\chi_\epsilon}\mathcal{L})\,\mathbf{j} = c_\epsilon\mathcal{M}_{\chi_\epsilon}\,\mathbf{e}^{\mathrm{inc}}$$

MRI-Specific Integral Equations

mapping properties!

EVIE	$(\mathcal{I} - \mathcal{L}\mathcal{M}_{\chi_\epsilon})\,\mathbf{e} = \mathbf{e}^{\text{inc}}$	$\mathcal{H}(curl) \to \mathcal{H}(curl)$
DVIE	$(\mathcal{I} - \mathcal{L}\mathcal{M}_{\chi_\epsilon})\,\mathcal{M}_{\epsilon_r^{-1}}\mathbf{d} = \epsilon_0\mathbf{e}^{\text{inc}}$	$\mathcal{H}(div) \to \mathcal{H}(curl)$
JVIE	$(\mathcal{I} - \mathcal{M}_{\chi_\epsilon}\mathcal{L})\,\mathbf{j} = c_\epsilon\mathcal{M}_{\chi_\epsilon}\,\mathbf{e}^{\text{inc}}$	$\mathcal{L}^2 \to \mathcal{L}^2$

$$\mathcal{H}(curl) := \left\{\mathbf{f} \middle| \mathbf{f} \in \mathcal{L}^2 \wedge \nabla \times \mathbf{f} \in \mathcal{L}^2\right\}$$

$$\mathcal{H}(div) := \left\{\mathbf{f} \middle| \mathbf{f} \in \mathcal{L}^2 \wedge \nabla \cdot \mathbf{f} \in \mathcal{L}^2\right\}$$

MRI-Specific Integral Equations

EVIE $$\left(\mathcal{M}_{\epsilon_r} - \mathcal{N}\mathcal{M}_{\chi_\epsilon}\right)\mathbf{e} = \mathbf{e}^{\mathrm{inc}}$$

DVIE $$\left(\mathcal{I} - \mathcal{L}\mathcal{M}_{\chi_\epsilon}\right)\mathcal{M}_{\epsilon_r^{-1}}\mathbf{d} = \epsilon_0\mathbf{e}^{\mathrm{inc}}$$

JVIE $$\left(\mathcal{M}_{\epsilon_r} - \mathcal{M}_{\chi_\epsilon}\mathcal{N}\right)\mathbf{j} = c_\epsilon\mathcal{M}_{\chi_\epsilon}\mathbf{e}^{\mathrm{inc}}$$

$$\mathcal{L}\mathbf{f} = \mathcal{N}\mathbf{f} - \mathbf{f}$$

$$\mathcal{N}\mathbf{u} \triangleq \nabla \times \nabla \times \mathcal{S}\left(\mathbf{u};\Omega\right)(\mathbf{r})$$

MRI-Specific Integral Equations

$$\lim_{\epsilon_r \to \infty} \text{EVIE}: \qquad (\mathcal{I} - \mathcal{N})\mathbf{e} = 0$$

$$\lim_{\epsilon_r \to \infty} \text{DVIE}: \qquad -\mathcal{L}\mathbf{d} = \epsilon_0 \mathbf{e}^{\text{inc}}$$

$$\lim_{\epsilon_r \to \infty} \text{JVIE}: \qquad (\mathcal{I} - \mathcal{N})\mathbf{j} = c_\epsilon \, \mathbf{e}^{\text{inc}}$$

MRI-Specific Integral Equations

$$\lim_{\epsilon_r \to \infty} \text{EVIE}: \qquad (\mathcal{I} - \mathcal{N})\mathbf{e} = 0$$

$$\lim_{\epsilon_r \to \infty} \text{DVIE}: \qquad -\mathcal{L}\mathbf{d} = \epsilon_0 \mathbf{e}^{\text{inc}}$$

$$\lim_{\epsilon_r \to \infty} \text{JVIE}: \qquad (\mathcal{I} - \mathcal{N})\mathbf{j} = c_\epsilon\, \mathbf{e}^{\text{inc}}$$

$$\mathbf{JVIE_I}: \qquad \left(\mathcal{M}_{\epsilon_r} - \mathcal{M}_{\chi_\epsilon}\mathcal{N}\right)\mathbf{j} = c_\epsilon \mathcal{M}_{\chi_\epsilon}\, \mathbf{e}^{\text{inc}}$$

$$\mathbf{JVIE_{II}}: \qquad \left(\mathcal{I} - \mathcal{M}_{\tau_\epsilon}\mathcal{N}\right)\mathbf{j} = c_\epsilon \mathcal{M}_{\tau_\epsilon}\, \mathbf{e}^{\text{inc}}$$

$$\tau_\epsilon = \chi_\epsilon / \epsilon_r$$

MRI-Specific Integral Equations

$$\lim_{\epsilon_r \to \infty} \text{EVIE}: \qquad (\mathcal{I} - \mathcal{N})\mathbf{e} = 0$$

$$\lim_{\epsilon_r \to \infty} \text{DVIE}: \qquad -\mathcal{L}\mathbf{d} = \epsilon_0 \mathbf{e}^{\text{inc}}$$

$$\lim_{\epsilon_r \to \infty} \text{JVIE}: \qquad (\mathcal{I} - \mathcal{N})\mathbf{j} = c_\epsilon\, \mathbf{e}^{\text{inc}}$$

$$\mathbf{JVIE_I}: \qquad \left(\mathcal{M}_{\epsilon_r} - \mathcal{M}_{\chi_\epsilon}\mathcal{N}\right)\mathbf{j} = c_\epsilon \mathcal{M}_{\chi_\epsilon}\, \mathbf{e}^{\text{inc}}$$

$$\mathbf{JVIE_{II}}: \qquad \left(\mathcal{I} - \mathcal{M}_{\tau_\epsilon}\mathcal{N}\right)\mathbf{j} = c_\epsilon \mathcal{M}_{\tau_\epsilon}\, \mathbf{e}^{\text{inc}}$$

$$\tau_\epsilon = \chi_\epsilon / \epsilon_r$$

MRI-Specific Integral Equations

$\mathcal{N}$

$$\mathrm{IN}_{m,n}^{pq} = \int_{V_m} \mathbf{f}_m^p \cdot \nabla \times \nabla \times \int_{V_n'} G \mathbf{f}_n^q \, dV' dV$$

$$\begin{aligned} \mathrm{IN}_{m,n}^{pq} &= -\oint_{S_m} (\hat{\mathbf{n}} \times \mathbf{f}_m^p) \cdot \int_{V_n'} \nabla G \times \mathbf{f}_n^q \, dV' dS \\ &= -\oint_{S_m} (\hat{\mathbf{n}} \times \hat{\mathbf{p}}) \cdot \int_{V_n'} \nabla G \times \hat{\mathbf{q}} \, dV' dS \end{aligned}$$

$$\mathrm{IN}_{m,n}^{pq} = \oint_{S_m} (\hat{\mathbf{n}} \times \hat{\mathbf{p}}) \cdot \oint_{S_n'} (\hat{\mathbf{n}}' \times \hat{\mathbf{q}}) \, G \, dS' dS$$

MRI-Specific Integral Equations

$\mathcal{N}$

$$\mathrm{IN}^{pq}_{m,n} = \sum_k \sum_l (\hat{\mathbf{n}}_k \times \hat{\mathbf{p}}) \cdot (\hat{\mathbf{n}}'_l \times \hat{\mathbf{q}}) \, J^{kl}_{m,n}$$

$$J^{kl}_{m,n} = \int_{S_k} \int_{S_l} G(\mathbf{R}) dS' dS$$

$1/R$ *

***DEMCEM* comes into play!**

MRI-Specific Integral Equations

$\mathcal{K}$

$$\boxed{\mathrm{IK}^{pq}_{m,n} = \int_{V_m} \mathbf{f}^p_m(\mathbf{r}) \cdot \nabla \times \int_{V'_n} G\, \mathbf{f}^q_n(\mathbf{r}')\, dV' dV}$$

$$\mathrm{IK}^{pq}_{m,n} = -\,(\hat{\mathbf{p}} \times \hat{\mathbf{q}}) \cdot \int_{V_m} \int_{V'_n} \nabla G\, dV' dV$$

$$\begin{aligned}\mathrm{IK}^{pq}_{m,n} &= (\hat{\mathbf{p}} \times \hat{\mathbf{q}}) \cdot \int_{V_m} \oint_{S'_n} G \hat{\mathbf{n}}'\, dS' dV \\ &= (\hat{\mathbf{p}} \times \hat{\mathbf{q}}) \cdot \oint_{S'_n} \hat{\mathbf{n}}' \int_{V_m} G\, dV\, dS'\end{aligned}$$

MRI-Specific Integral Equations

$\mathcal{K}$

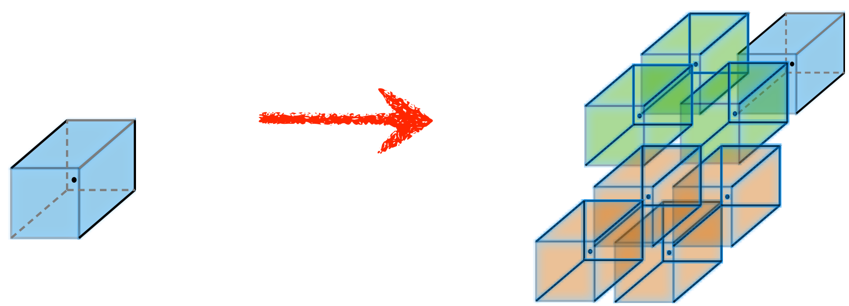
$$\int_{V_m} G(\mathbf{R})\, dV = \int_{V_m} \nabla \cdot \mathbf{F}(\mathbf{R})\, dV = \oint_{S_m} \hat{\mathbf{n}} \cdot \mathbf{F}(\mathbf{R})\, dS$$

$$\mathrm{IK}_{m,n}^{pq} = (\hat{\mathbf{p}} \times \hat{\mathbf{q}}) \cdot \oint_{S_n'} \hat{\mathbf{n}}' \oint_{S_m} \hat{\mathbf{n}} \cdot \mathbf{F}(\mathbf{R}) dS dS'$$

$$\mathbf{F}(\mathbf{R}) = \mathbf{R} \int_0^1 t^2 G(tR)\, dt = \frac{1}{(jk_0)^2} \nabla \left[G(\mathbf{R}) - G_0(\mathbf{R}) \right]$$

MRI-Specific Integral Equations

Toeplitz → *Circulant*



$$Ax = b$$ → *Iterative Solver* → *M-v product* → *FFT* *N* log(*N*)

[Polimeridis et al., Journal of Computational Physics 2013]

MRI-Specific Integral Equations

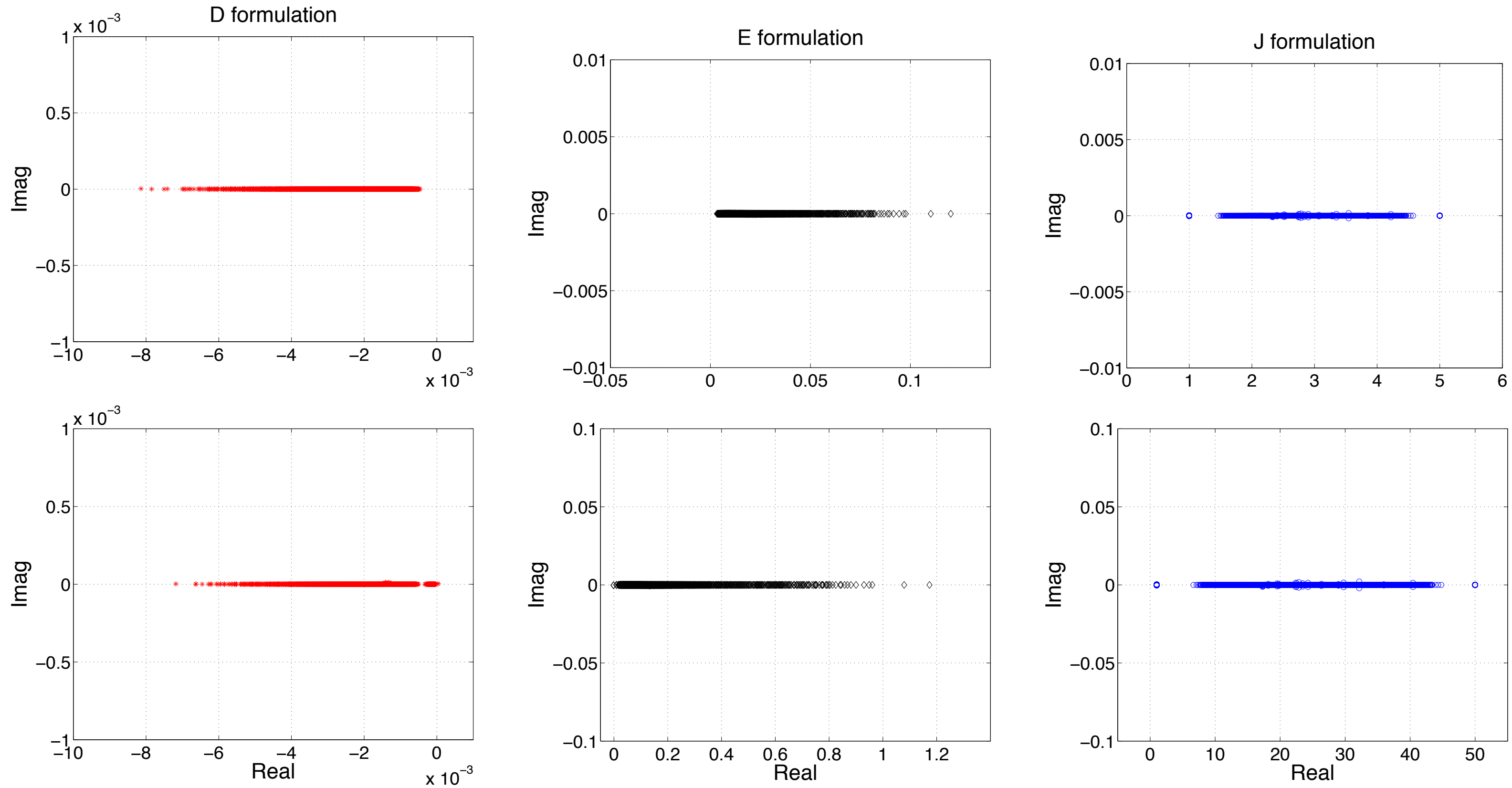
Sphere

$ka = 0.05$

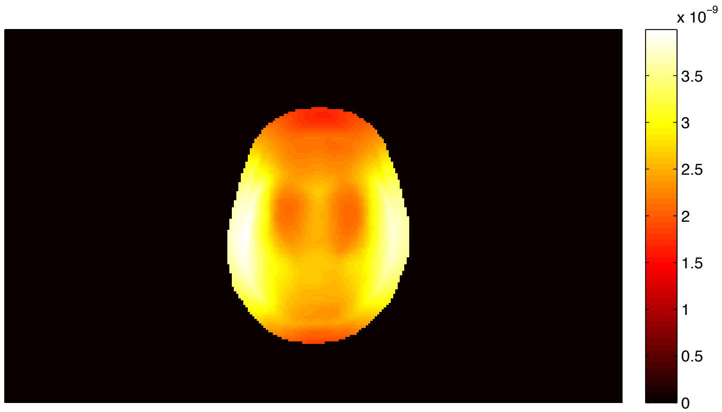
Formulation	Cond $\varepsilon_r = 5$	Cond $\varepsilon_r = 50$
D	18	257
E	37	853
J	5	50

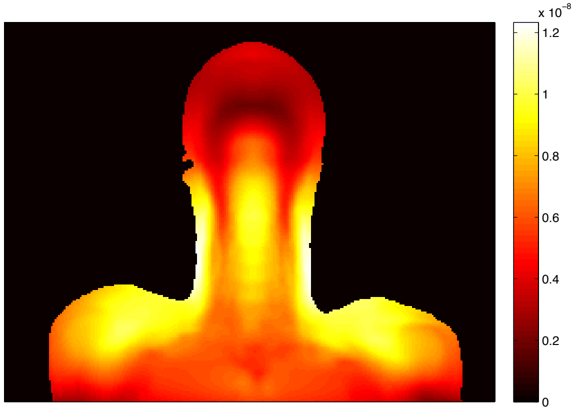
$\epsilon_r = 5$

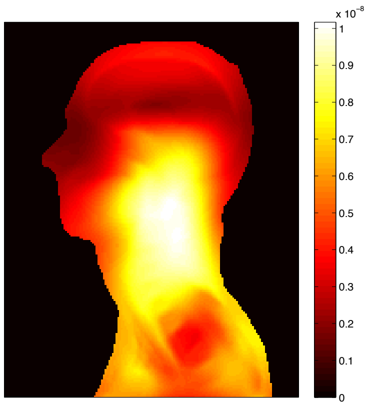
$\epsilon_r = 50$



MRI-Specific Integral Equations



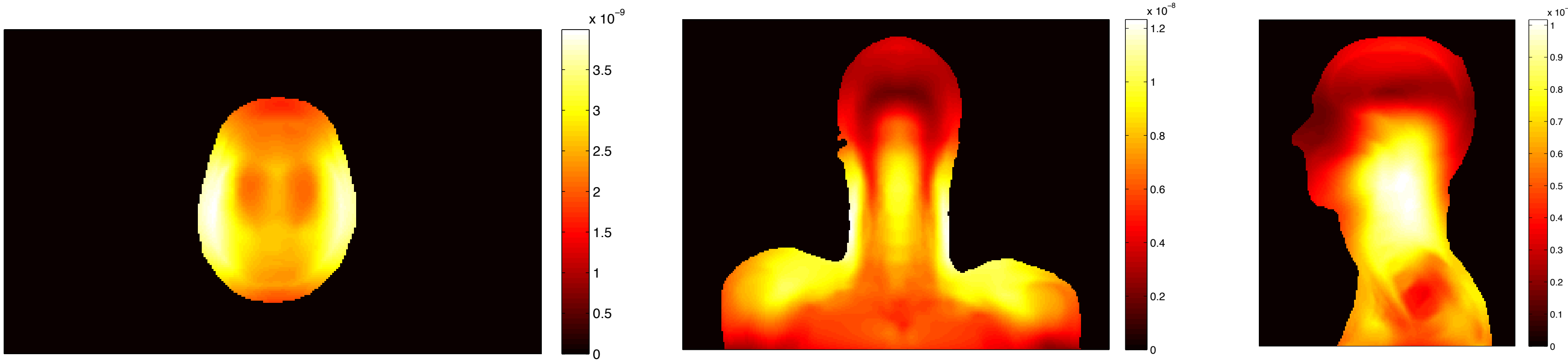




		OFFLINE	GMRES	GMRES (40)	GMRES (40,5)	BICG	BICGSTAB	QMR	TFQMR
5mm	Serial Parallel Speed-Up	20 s 5 s 4×	15 s 7 s 2.1×	15 s 5 s (3 s) 3.0× (5×)	13 s 5 s 2.6×	28 s 4 s 7.0×	16 s 3 s 5.3×	23 s 4 s 5.7×	17s 3 s 5.6×
2.5mm	Serial Parallel Speed-Up	146 s 27 s 5.4×	146 s 65 s 2.2×	142 s 48 s (23 s) 2.9× (6.1×)	125 s 42 s 2.9×	276 s 40 s 6.9×	162 s 25 s 6.4×	266 s 40 s 6.6×	174 s 32 s 5.4×

3,000,000 unknowns!

MRI-Specific Integral Equations



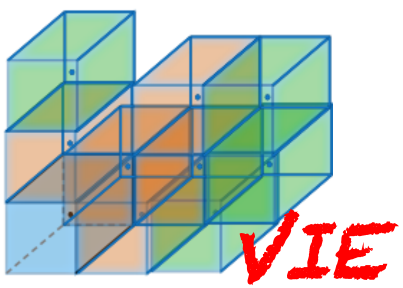
		OFFLINE	GMRES	GMRES (40)	GMRES (40,5)	BICG	BICGSTAB	QMR	TFQMR
5mm	Serial Parallel Speed-Up	20 s 5 s 4×	15 s 7 s 2.1×	15 s 5 s (3 s) 3.0× (5×)	13 s 5 s 2.6×	28 s 4 s 7.0×	16 s 3 s 5.3×	23 s 4 s 5.7×	17s 3 s 5.6×
2.5mm	Serial Parallel Speed-Up	146 s 27 s 5.4×	146 s 65 s 2.2×	142 s 48 s (23 s) 2.9× (6.1×)	125 s 42 s 2.9×	276 s 40 s 6.9×	162 s 25 s 6.4×	266 s 40 s 6.6×	174 s 32 s 5.4×

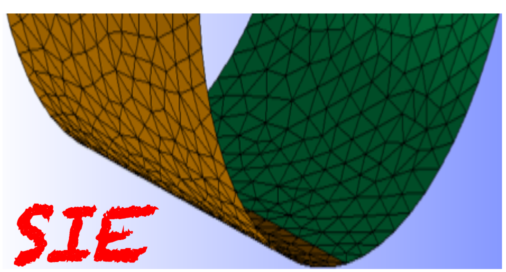
3,000,000 unknowns!

MRI-Specific Integral Equations

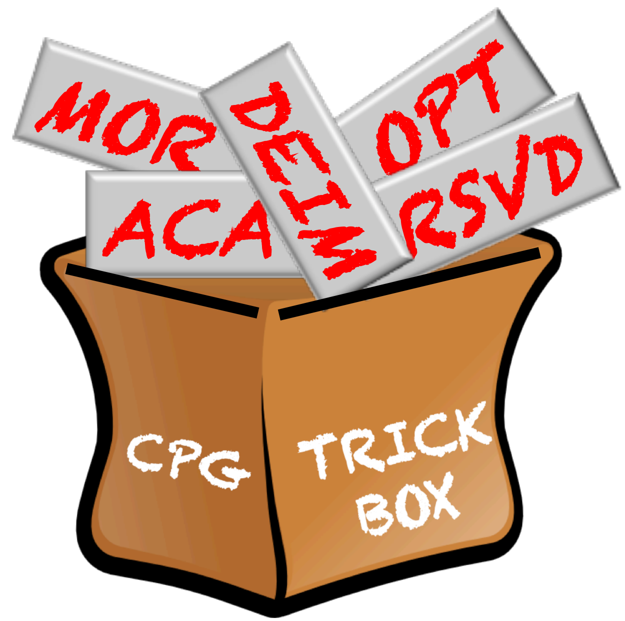
MAgnetic Resonance Integral Equation suite

EM Solvers

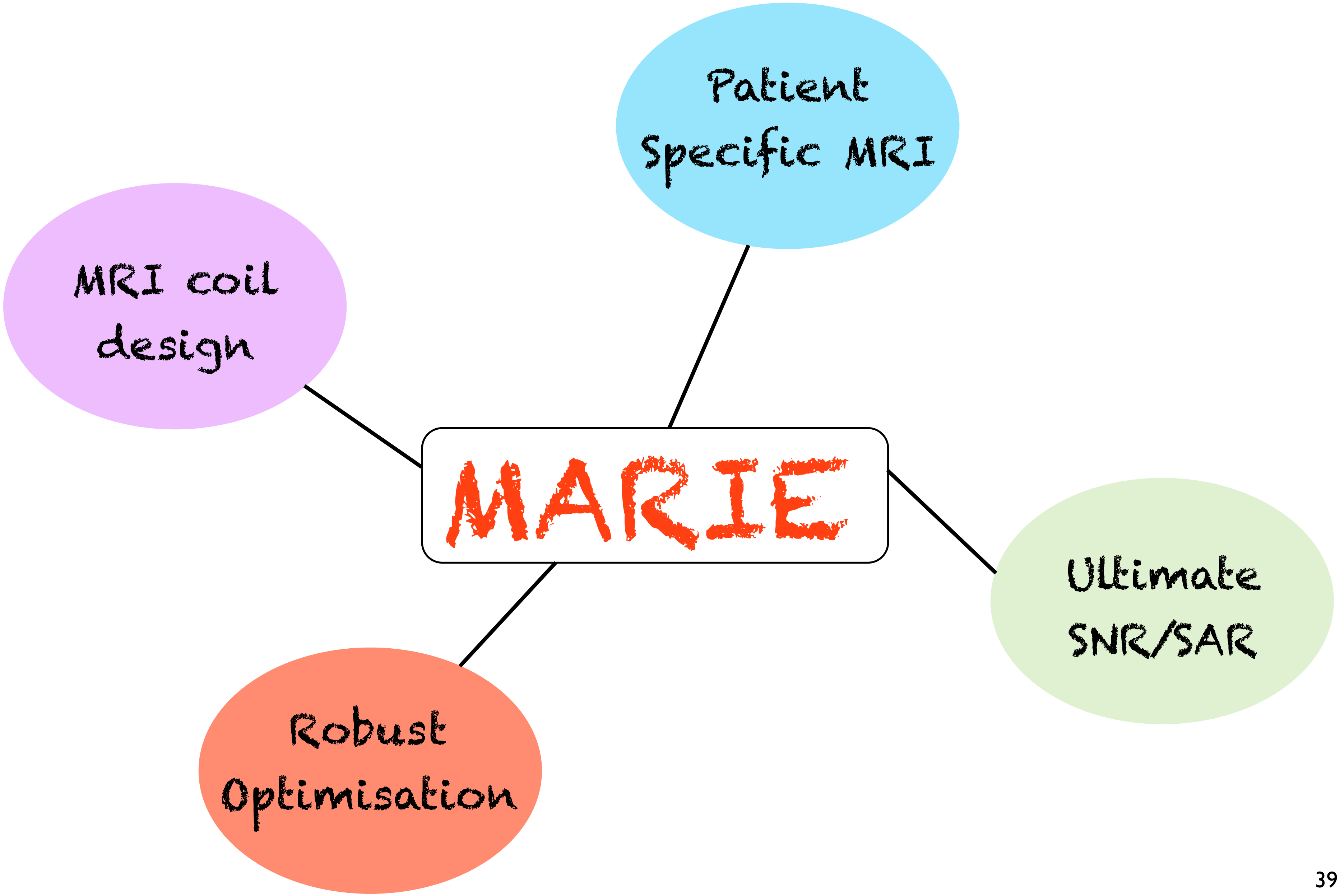




Algorithms



MRI-Specific Integral Equations



magnetic resonance integral equation

Release Notes
ß version

Matlab
Open-source

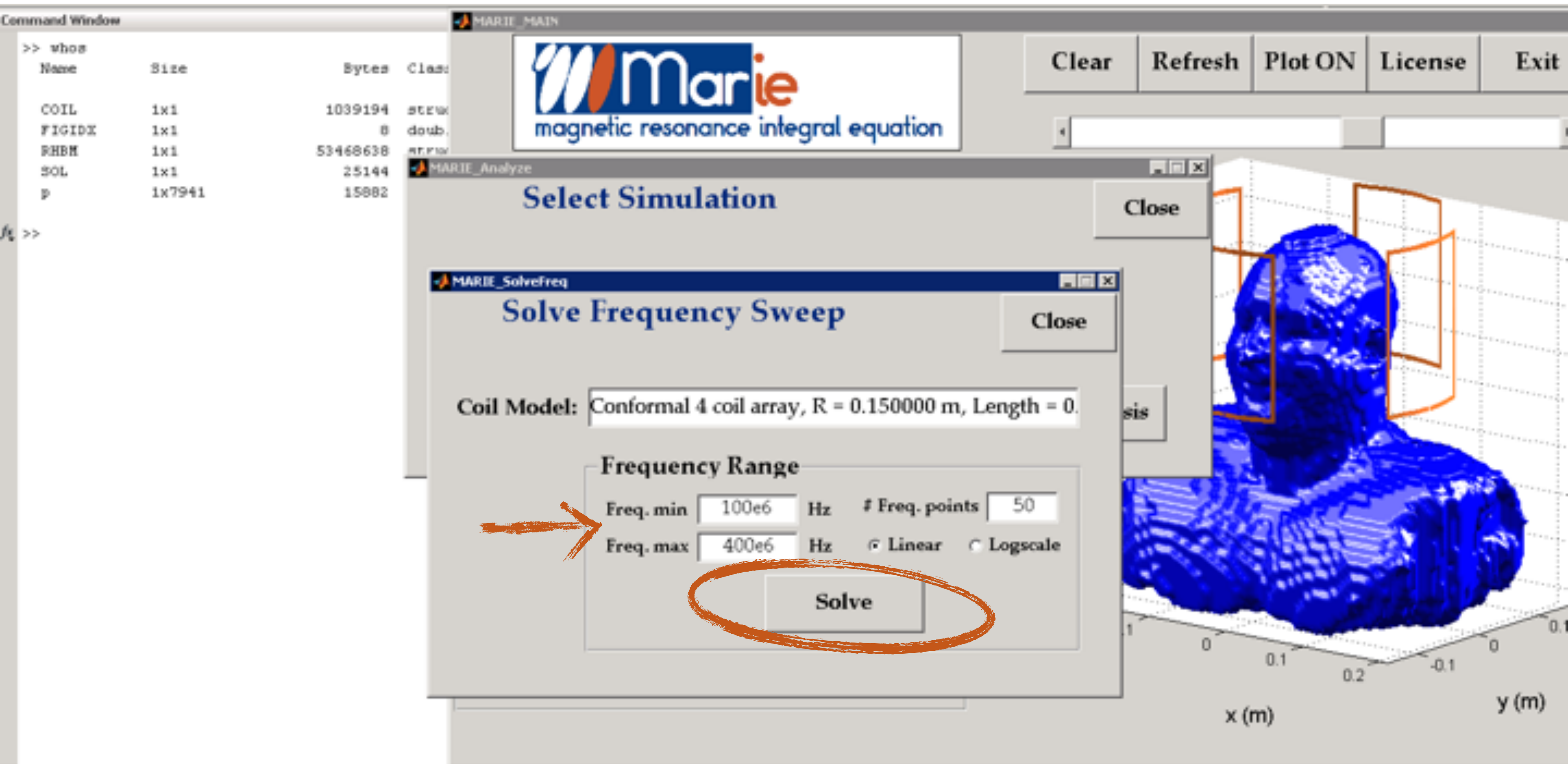
Jorge Fernández Villena, Athanasios G. Polimeridis
Computational Prototyping Group, RLE MIT

Beta Version, Spring 2015

www.rle.mit.edu/cpg

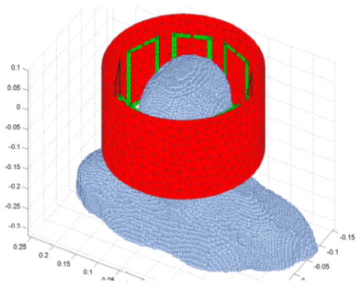
gui - analysis/actions box

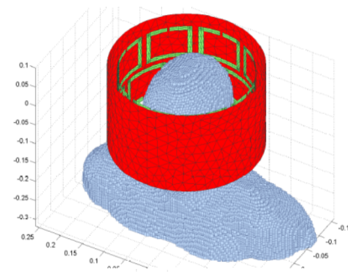
- **Coil analysis**
  - select the frequency range, number of points and distribution
  - push the “solve” button to run the simulation



MRI-Specific Integral Equations

MRI coil design



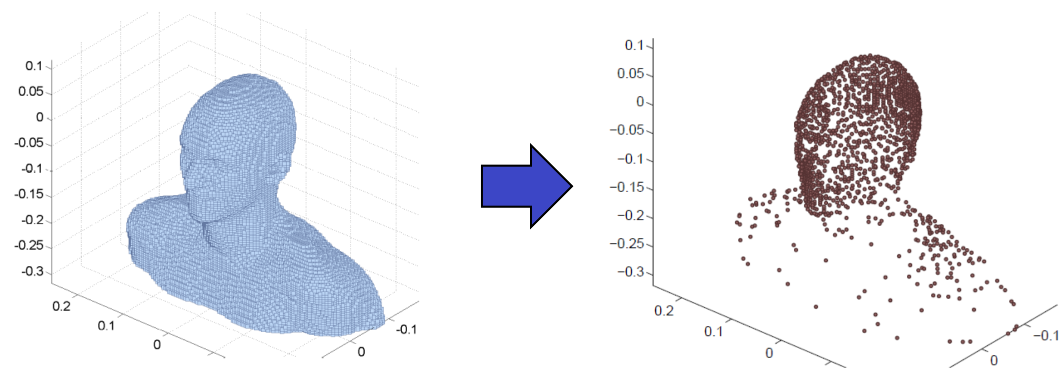


• • •

Realistic DUKE model, 4mm @7T

- 3,512,880 variables total
- 621,426 unknowns in body model
- Use 1,012 vectors in basis
- Reduce to 1,660 interpolation points!

- Pre-computation time: approx. 35h
- Storage required: 9GB



>100x speed-up

Why IE in NumPDEs course?

PDE solvers (FEM, FDTD, etc)

- Simple
- General purpose
- Sparse linear systems or not at all
- Large number of unknowns
- Absorbing boundary conditions

Integral equations

- Dimensionality reduction
- Automatically satisfy radiation conditions
- High-order approximations
- High complexity