Folk Theorem and, in particular, at the thresholds where cooperation is more beneficial than defection in the game of a Prisoners' Dilemma.

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SUMMARY

ACKNOWLEDGEMENTS

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Chapter 1

Introduction

Game Theory is the study of interactive decision making and developing strategies through mathematics [6]. It analyses and gives methods for predicting the choices made by players (those making a decision), whilst also suggesting ways to improve their 'outcome' [14]. Here, the abstract notion of utility is what the players wish to maximise (see Chapter 2 in [14] for a detailed discussion on the topic of utility theory or Section 1.3 [20] for a more introductory explanation). One of the earliest pioneers of game theory is mathematician, John von Neumann who, along with economist Oskar Morgenstern, published *The Theory of Games and Economic Behaviour* in 1944 [14]. This book discusses the theory, developed in 1928 and 1940-41, by von Neumann, regarding "games of strategy" and its applications within the subject of economics [19]. Following this, several advancements have been made in the area, including, most notably, John Nash's papers on the consequently named Nash Equilibria in 1950/51 [16, 15]. Due to the "context-free mathematical toolbox" [14] nature of this subject, it has been applied to many areas, from networks [13, 17] to biology [5, 1]. In this project, the main focus is on a particular class of theorems, within game theory, known as "Folk Theorems" with application to the game of A Prisoner's Dilemma. These will be defined and discussed in the subsequent sections.

1.1 An Introduction to Games

Consider the following scenario:

> Two convicts have been accused of an illegal act. Each of these prisoners, separately, have to decide whether to reveal information (defect) or stay silent (cooperate). If they both cooperate then the convicts are given a short sentence whereas if they both defect then a medium sentence awaits. However, in the

situation of one cooperation and one defection, the prisoner who cooperated has the consequence of a long term sentence, whilst the other is given a deal [9].

This is one of the standard games in game theory known as A Prisoner's Dilemma. It has four distinct outcomes, for the given two player version, which can be represented as a table (see Table **??**). Each coordinate (a, b) in the table represents the utility values obtained for each player, where a is the utility value obtained by the row player and b is the utility gained by the column player. These utility values are as given in [2] and are used throughout this project. More formally, the game can be represented as the following matrix:

$$\begin{array}{c c} & \begin{array}{cc} \scriptstyle coop & \scriptstyle defect \end{array} \\ \begin{array}{c} \scriptstyle coop \\ \scriptstyle defect \end{array} & \begin{pmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{pmatrix} \end{array}$$

which is known as a *normal form* representation of the game. The following definition is adapted from [14].

In general a *normal form* or *strategic form* game is defined by an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where:

- $N = \{1, 2, \ldots, n\}$ is a finite set of players;
- $S = S_1 \times S_2, \times \cdots \times S_n$ is the set of strategies for all players in which each vector $(S_i)_{i \in N}$ is the set of strategies for player i [1]; and
- $u_i : S \to \mathbb{R}$ is a payoff function which associates each strategy vector, $\mathbf{s} = (s_i)_{i \in N}$, with a utility [2] $u_i(i \in N)$.

Yet another way of representing this game is as a pair of matrices, A, B, defined as follows:

$$A = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \text{ and } B = A^T = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$$

This way of defining games allow for the use of calculating payoffs for each player (see Section 1.2).

Before continuing the discussion into the key notions of game theory, it needs to be highlighted that there is an important assumption, which is central to most studies of game theory, entitled *Common Knowledge of Rationality.* This, more formally, is an infinite list of statements which claim:

- The players are rational;

[1]Since the game of A Prisoner's Dilemma has a finite strategy set for each player $S_i = \{\text{cooperate}, \text{defect}\}(i \in N)$, in this project only finite strategy spaces are considered.

[2]'Utility' is referred to as a player's 'payoff' throughout the remainder of this report.

- All players know that the other players are rational;
- All players know that the other players know that they are rational; etc.

Assuming Common Knowledge of Rationality allows for the prediction of rational behaviour through a process entitled *rationalisation* [10] (see section 4.5 in [14] for an alternative explanation of this assumption).

A strategy for player i, s_i, is *strictly dominated* if there exists another strategy for player i, say $\bar{s}_i$, such that for all strategy vectors $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) < u_i(\bar{s}_i, s_i).$$

In this case we say that s_i is *strictly dominated* by $\bar{s}_i$. Here, $s_{-i} = \{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\}$, i.e. the ith player's strategy has been omitted. The set, S_{-i}, is defined similarly. Looking at the row player's matrix of a Prisoner's Dilemma 1.1 (the first entries in the ordered tuples), it is clear that cooperation is a strictly dominated strategy. Due to the symmetricity of the game, this is also true for the column player. [14]

So far, only the pure strategies, $S_i = \{\text{coop}, \text{defect}\}$, have been discussed, thus the notion of a probability distribution over S_i is now introduced, giving the so-called *mixed strategies* as defined in [14]: Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game (with each S_i finite), then a *mixed strategy* for player i is a probability distribution over their strategy set S_i. Define:

$$\Sigma_i := \{\sigma_i : S_i \to [0,1] : \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

to be the set of mixed strategies for player i. Hence, observe that the pure strategies are specific cases of mixed strategies, with $\sigma_i = (1, 0)$ for cooperation and $\sigma_i = (0, 1)$ for defection, in the example of a Prisoner's Dilemma.

This leads onto the following definition of a *mixed extension* of a game, taken from [14]: Let G be a finite normal form game as above, with $S = S_1 \times S_2 \times \cdots \times S_N$ defining the pure strategy vector set and each pure strategy set, S_i being non-empty and finite. Then the *mixed extension* of G is denoted by

$$\Gamma = (N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N}),$$

and is the game in which, Σ_i is the ith player's strategy set and $U_i : \Sigma \to \mathbb{R}$ is the corresponding payoff function, where each $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in \Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_N$ is mapped to the payoff

$$U_i = \mathbb{E}_\sigma(u_i(\sigma)) = \sum_{(s_1, s_2, \ldots, s_N) \in S} u_i(s_1, s_2, \ldots, s_N)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_N(s_n) \qquad (1.1)$$

for all players $i \in N$.

1.2 Nash Equilibrium for Normal Form Games

As mentioned above, mathematician, John Nash, introduced the concept of an equilibrium point and proved the existence of mixed strategy Nash Equilibria in all finite games. These notions are central to the study of game theory [14] and hence, in this section, Nash's concepts will be defined and proved in detail.

Firstly, before the definition of a Nash equilibrium, the idea, as given in [14] of a *best response* is introduced: For a game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, the strategy, s_i, of the ith player is considered a *best response* to the strategy vector s_{-i} if $u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i})$.

This leads onto the main definition of the section:

Definition 1.2.1. Given a game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, the vector of strategies $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a *Nash equilibrium* if, for all players $i \in N$, s_i^* is a best response to $s_i^* \in N$. [14]

In other words, s^* is a Nash equilibrium if and only if no player has any reason to deviate from their current strategy s_i^*.

Recall that, in Section 1.1, for any player in A Prisoner's Dilemma, defection dominated cooperation. This leads to the following observation:

The strategy pair (Defect, Defect), is the unique Nash equilibrium for A Prisoner's Dilemma, with a payoff value of 1 for each player. [14]

This can be visualised as followed: Assume the row player uses the following mixed strategy, $\sigma_r = (x, 1-x)$, i.e. the probability of cooperating is x and the probability of defecting is $1-x$. Similarly, assume the column player has the strategy, $\sigma_c = (y, 1-y)$. The payoff obtained for the row and column player, respectively, is then:

$$A\sigma_c^T = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 3y \\ 4y+1 \end{pmatrix},$$

$$\sigma_r B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3x & 4x+1 \end{pmatrix}$$

Plotting these gives the following: From Figure 1.1 it is clear that, regardless of the strategy played by the opponent, defection is indeed the only rational move for one to play. Thus, both players have no incentive to deviate if and only if both play the strategy $\sigma = (0, 1)$, i.e. defection for every single game of A Prisoner's Dilemma.

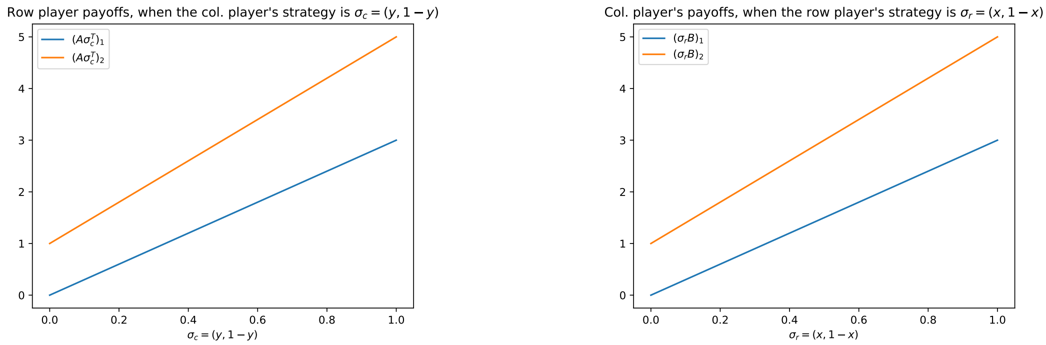

Figure 1.1: Graphs to show the row and column players' payoffs against a mixed strategy.

On the other hand, consider, for example, a game with no dominated strategies [3]:

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Are Nash Equilibria guaranteed to exist? This result is given in the next theorem, taken from [15], Nash's second paper on equilibria in games.

Theorem 1.2.1. Every finite game has an equilibrium point.

The proof of Theorem includes the use of a *fixed point theorem* and thus, a short sub-section regarding one such result is given, for completeness, before providing a formal proof of 1.2.

1.2.1 Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem is a result from the theory of topology. Named after the Dutch mathematician, L.E.J. Brouwer, it was proven in 1912 [4]. However, before stating this notion, a few conditions regarding the properties of sets are recalled.

The following three definitions appear as in [22, 3, 21] respectively:

Definition 1.2.2. A set $X \subseteq \mathbb{R}^d$ is called *convex* if it contains all line segments connecting any two points $\mathbf{x}_1, \mathbf{x}_2 \in X$.

Definition 1.2.3. An *open cover* of a set $S \subset X$, a topological space, is a collection of open sets $A_1, A_2, \ldots \subset X$ such that $A_1 \cup A_2 \cup \ldots \supset S$, that is, the

[3] The game highlighted here is another standard used in game theory entitled *Matching Pennies*. Moreover, it is what is defined as a *zero-sum* game. Interested readers are encouraged to read Example 4.21 of [20] for an introduction to the game and Section 4.12 in [14] for an explanation of zero-sum games.

union of the open sets contain S.

Definition 1.2.4. A subset $S \subseteq X$, a topological space, is called *compact* if, for each open cover of S, there is a finite sub-cover of S.

The presentation of Brouwer's Fixed Point Theorem is now given as in [14]

Theorem 1.2.2. Let $X \subseteq \mathbb{R}^n$ be a non-empty convex and compact set, then each continuous function $f : X \to X$ has a fixed point.

In other words if X and f satisfy the conditions given above then there exists a point $x \in X$ such that $f(x) = x$.

Since this project is regarding game theory, rather than topology, the proof to the above theorem is omitted. However, the interested reader is referred to [] for an in-depth consideration into the theory of topology.

1.2.2 Proof of Nash's Theorem (Theorem 1.2)

The proof provided here is adapted from the original, by Nash, as presented in [15] (with extra notes adapted from [14]). According to [14], the general idea here is to define a function, which satisfies the conditions required for Theorem 1.2.2, using the payoff functions on the set of mixed strategies. Then by identifying each equilibrium point with a fixed point of the function, the required result is obtained.

Proof. Firstly, a brief restatement of the notation needed is provided for clarity. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a finite game with mixed extension $\Gamma = (N, (\Sigma_i)_{i \in N}, (U_i)_{i \in N})$. Here, N denotes the number of players; $S = S_1 \times S_2 \times \ldots \times S_N$ is the set of pure strategies for all players, with $(S_i)_{i \in N}$ the pure strategy set for player i; Σ is defined similarly but relating to mixed strategies; and $U_i : \Sigma \to \mathbb{R}$ are the payoff functions as given in equation 1.1. Recall that σ_{-i} is used to denote the strategy choice of all players *except* the ith player.

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ be a tuple of mixed strategies and $U_{i,t}(\sigma)$ be the ith player's payoff if they changed to their tth pure strategy and all other players continue to use their mixed strategy. Now, define function $f : \Sigma \to [0, \infty)$ such that

$$f_{i,t}(\sigma) = \max\left(0, U_{i,t}(\sigma) - U_i(\sigma)\right) \tag{1.2}$$

and also let

$$\sigma_i' = \frac{\sigma_i + \sum_t f_{i,t}(\sigma) s_i^t}{1 + \sum_t f_{i,t}(\sigma)} \tag{1.3}$$

be a modification of each $\sigma_i \in \sigma$, with $\sigma' = (\sigma'_1, \sigma'_2, \ldots, \sigma'_N)$. In words, this modification increases the proportion of the pure strategy t used in σ_i if the payoff gained by the ith player is larger when they replace their mixed strategy by t. Else, it remains the same if doing this decreases their payoff as $f_{i,t}(\sigma) = 0$ in this case. Note, the denominator ensures that the ending vector is still a probability distribution by standardising.

The aim is to apply Theorem 1.2.2 to the mapping $T : \sigma \to \sigma'$ and show that it's fixed points correspond to Nash equilibria. Thus, firstly compactness and convexity of the set σ is shown along with continuity of the function f.

Observe that each σ_i can be represented by a point in a simplex in a real vector space with the vertices given by the pure strategies, s_i^t. Therefore, it follows that the set Σ_i is convex and compact. Using the result, *If* $A \subseteq \mathbb{R}^n$ *and* $B \subseteq \mathbb{R}^m$ *are convex compact sets then the set* $A \times B$ *is a convex compact subset of* $\mathbb{R}^{n+m}$, as highlighted in [14] gives the convexity and compactness of the set Σ, the cross product of all Σ_is.

The continuity of the function f depends upon the continuity of the payoff functions U_i. As given in [14], this is shown by first proving that the U_i are multilinear functions in the variables $(\sigma_i)_{i \in N}$ and then applying the fact that any multilinear function over Σ is a continuous function. [4] The result then follows.

Hence, by Theorem 1.2.2, the mapping T must have at least one fixed point. The proof is concluded by showing that any fixed points of T are Nash equilibria.

Suppose σ is such that $T(\sigma) = \sigma$. Then, the proportion of s_i^t used in the mixed strategy σ_i must not be altered by T. Therefore, in σ'_i, the sum $\sum_t f_{i,t}(\sigma)$, in the denominator, must equal zero. Otherwise, the total sum of the denominator will be greater than one, decreasing the proportion of s_i^t. This implies that for all pure strategies q, $f_{i,q}(\sigma) = 0$. That is, player i can not improve their payoff by adopting any of the pure strategies. Note, this is true for all i and q by definition of $T(\sigma) = \sigma$ and thus no player is able to improve their payoff. By definition 1.2.1, this is exactly the conditions of a Nash equilibrium.

Now assume σ is a Nash equilibrium. Then, by definition, it must be that $f_{i,q}(\sigma) = 0$ for all pure strategies q for all players, $i \in N$. Note, if $f_{i,q}(\sigma) \neq 0$, then the ith player would benefit from changing their strategy to the pure strategy q, which violates the condition for a Nash equilibrium. From this it follows that $T(\sigma) = \sigma$, that is, σ is a fixed point of T. This concludes the proof.

[4] For a detailed consideration of the continuity of the payoff functions, please see [14], pages 148-149.

□

1.3 Repeated Games

The folk theorems studied in this project are a consequence of games which are repeated several times (not just once). Thus, before discussing the main ideas, the theory of both finitely- and infinitely- repeated games is presented.

Firstly, a couple of alterations to the terminology used in previous sections is redefined for conciseness and to be consistent with the literature []. What was known as a 'game' will become known as a *stage game* to highlight the fact that a one-off game is being considered. Also, what was defined previously as a 'strategy' will now be referred to as an *action* to differentiate it from a strategy of a repeated game (see Section 1.3.1).

1.3.1 Finite Repeated Games

According to [12], a *T-stage repeated game*, $T < \infty$ is when the stage game, G, is played T times, over discrete time intervals. Each ith player has a strategy based on previous 'rounds' of the game and the payoff of a repeated game is calculated as the total sum of the stage game payoffs.

Prior to giving the notion of a strategy in a repeated game, the idea of *history*, within the context of repeated games, is provided. The *history*, $H(t)$ of a repeated game is the knowledge of previous actions of all players up until the tth stage game, assumed to be known by player i for all $i \in 1, \dots, N$. Note that, when $t = 0$, $H(0) = \underbrace{(\emptyset, \emptyset, \dots, \emptyset)}_{N\text{times}}$, since no stage games have yet been played.

As given in [11], a *strategy* of a T-stage repeated game is defined to be a mapping from the complete history so far to an action of the stage game, that is

$$\tau_i : \cup_{t=0}^{T-1} Ht \to a_i.$$

Here, $H(t)$ is the history of play as defined above and a_i is the ith player's action of the stage game.

Consider, for example, the environment in which the stage game of a Prisoner's Dilemma is repeated each time. This is known as the *Iterated Prisoner's Dilemma* (IPD) and has been a popular topic of research for many years see Chapter **??**. Note that the ojective here is to maximise your payoff. The player:

No matter what my opponents play, I will always defect,

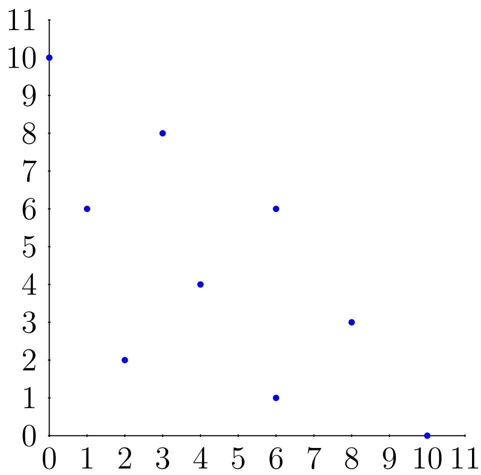

Figure 1.2: A plot to show the possible payoffs of the game between two players in which A Prisoner's Dilemma is repeated twice.

commonly known as the 'Defector' has the following strategy mapping:

$$\tau_i : \cup_{t=0}^{T-1} Ht \to a_i,$$

where $a_i = D$ for all time periods $\tau \geq 0$. Other common IPD strategies include:

Cooperator - *No matter what my opponents play, I will always cooperate*;
Random - *I will either cooperate or defect with a probability of 50%*; and
Tit For Tat - *I will start by cooperating but then will duplicate the most recent decision of my opponent.*

Figure 1.2 shows the possible payoffs obtained in a 2-stage repeated IPD with two players:

Now, what about Theorem 1.2? Are there any Nash equilibria in repeated games? In fact, it can be proven that there exist many equilibria in repeated games [7]. The next result, adapted from [11] guarantees at least one Nash equilibria.

Theorem 1.3.1. Consider a T-stage repeated game with $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ as the stage game, $0 < T < \infty$. Define by $\boldsymbol{\sigma}^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_N^*)$, a stage Nash equilibrium of G. Then the sequence in which $\boldsymbol{\sigma}^*$ is continuously played is a Nash equilibrium of the T-stage repeated game.

Proof. Since $\boldsymbol{\sigma}^*$ is a stage Nash equilibrium, it is, in particular, a Nash equilibrium of the Tth stage game. Thus, no player has any reason to deviate here. But then $\boldsymbol{\sigma}^*$ was also played at the $(T-1)$th stage also, meaning there is still no reason to deviate. Therefore, continuing via backwards induction gives the required result. $\square$

Hence, for the T-stage IPD, every player executing the Defector strategy yields a Nash equilibrium. However, do there exist equilibria for which selecting the action 'cooperate' is more beneficial?

In preparation to answer this, the next Subsection 1.3.2 discusses the case when $T \to \infty$ and results linked to *infinitely repeated games*.

1.3.2 Infinite Repeated Games

The Folk Theorem discussed in Section 1.4 considers a stronger notion of equilibria known as *subgame perfect equilibria*. In order to fully understand this notion, a new representation of games is introduced.

Extensive Form Games

According to [14], an *extensive form game* is given by the ordered vector $\Gamma = (N, V, E, x_0, (V_i)_{i\in N}, O, u)$ where N is a finite set of players; (V, E, x_0) is a *game tree* [5]; $(V_i)_{i\in N}$ is a partition of the set $V \setminus L$, where L is the set of all leaves, or terminal points, of the game tree; O is the set of outcomes for the game; and u is a function which maps each leaf in L to an outcome in O.

This leads on to the following definition, adapted from [20]:
A player's *information set* is a subset of the nodes in a game tree where:

- Only the player concerned is deciding;
- This player is not aware of which node has been reached except that it is definitely one of the elements found in this set.

In Figure **??**, the extensive form representation of the game A Prisoner's Dilemma is provided. Here, only two players are considered and any information sets are represented by a dashed line. Note, any normal form game can be represented as an extensive form game.

According to [20], a *subgame* is a sub-graph of the game tree such that:

- The sub-graph begins at a decision node, say x_i;
- This node, x_i, is the only element contained in its information set;
- The sub-graph contains all of the decision nodes which follow x_i.

This leads to the following definition of *subgame perfect equilibria*, adapted from [20].

[5] The triple (V, E, x_0) is defined as a *tree* if the set of vertices, V, and the set of edges, E, create a *directed graph* (that is, each element in E is an ordered tuple) and x_0 represents the root, or starting node, of the graph.

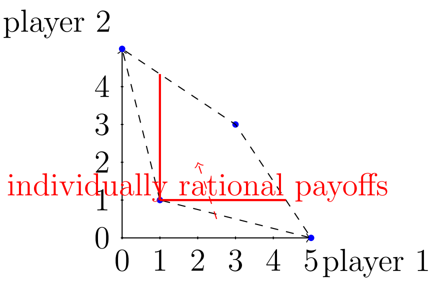

Figure 1.3: A plot highlighting the possible individually rational payoffs for the game of A Prisoner's Dilemma.

Definition 1.3.1. A *subgame perfect equilibrium* is a Nash equilibrium which satisfies the condition that the strategies played define a Nash equilibrium in every subgame.

Hence the strategy defined in Theorem 1.3.1 is a subgame perfect equilibrium. A few final definitions are now highlighted before introducing the Folk Theorem.

Final Definitions Needed

Now, in order to be able to discuss the payoffs of strategies in infinite games; the notion of a *discounted payoff* is introduced. This is defined in [8] as:

$$V_i(\sigma) = \sum_{t=1}^{\infty} \delta^{t-1} U_i(\sigma),$$

where the discount factor, δ, can be thought of as the probability that the game continues. That is, the probability that another stage game will be played. Using this, the *average payoffs*, per stage game, can be defined by:

$$\frac{1}{\bar{T}} V_i(\sigma) = (1-\delta) V_i(\sigma),$$

where $\bar{T} = \frac{1}{1-\delta}$ is the average length of a game [8].

Finally, Figure 1.3 shows those payoffs which are individually rational for a two player version of A Prisoner's Dilemma. In general, an *individually rational payoff* is an average payoff which exceeds the payoffs obtained in the stage Nash equilibria for all players [8]. Note, often the Nash equilibrium payoffs are not the optimal payoff which players could achieve.

In the next section (Section 1.4), the main theorem of this project will be stated and proved.

1.4 Folk Theorem

According to [20], the class of theorems known as Folk Theorems are so-called because the result was well-known before a formal proof was provided. In general, these theorems state that players can achieve a better payoff than the Nash equilibrium (if the Nash equilibrium payoff is not optimal) when the stage game is repeated many times and the probability of the game continuing is large enough.

It is believed that Friedman, 1971 ([7]) was one of the first to publish a formal proof to the widely accepted Folk Theorem []. Thus, the presentation of the statement and proof given here is adapted from [7] as well as from [8].

Before stating the theorem, the list of assumptions which Friedman [7] requires the infinite repeated game to satisfy is given.

1. The mixed action sets, Σ_i are compact and convex for all $i \in N$.
2. The payoff functions, $U_i : \Sigma \to \mathbb{R}$, are continuous and bounded for all $i \in N$.
3. The $U_i(\sigma)$s are quasi-concave [6] functions of $sigma_i$ for all $i \in N$.
4. If $U_i' \leq U_i''$, for all $i \in N$ and $U_i', U_i'' \in H$, then, for all $U_i' \leq U \leq U_i''$, $U \in H$. Here, H is defined to be the set of feasible payoffs, $\{U(\sigma) : \sigma \in \Sigma\}$, where $U(\sigma) = (U_1(\sigma), U_2(\sigma), \ldots, U_N(\sigma))$.
5. H^* is concave, where $H^* \subset H$ denotes the set of all Pareto optimal payoffs. [7].
6. All stage games are identical in the infinitely repeated game.
7. The discount parameter, δ, is equal in all time periods.
8. The stage game has a unique Nash equilibrium.
9. The Nash equilibrium is not Pareto optimal.

Note, Friedman [7] later goes on to prove that assumptions six to nine can be removed with only a small effect on the result. However, since the only game being studied in this project is the IPD (which satisfies all the above assumptions), this

[6] According to [18], a real-valued function f, defined on a convex subset $C \subset \mathbb{R}^n$, is *quasi-concave* if for all $\alpha \in \mathbb{R}$, the set $\{x \in C : f(x) \geq a\}$ is convex.

[7] Friedman defines a *Pareto optimal payoff* as a point in the payoff space $U_(\sigma^*)$ which satisfies the conditions: $\sigma^* \in \Sigma$ and $U_i(\sigma^*) > U_i(\sigma)$ for all $i \in N$

generalisation will be omitted. Thus, only the proof of the original theorem will be provided.

Theorem 1.4.1. If the above assumptions are all satisfied for the given infinite repeated game, then for any individually rational payoff V_i (greater than the payoff yielded by the stage Nash equilibrium σ^{ne}), there exists a discount parameter δ^* such that for all δ_i, $0 < \delta^* < \delta_i < 1$ there is a subgame perfect Nash equilibria with payoffs equal to V_i.

Proof. Consider the set of all actions which yield greater payoffs than the Nash equilibrium, denoted by:

$$B = \{\sigma : \sigma \in \Sigma, U_i(\sigma) > U_i(\sigma^{ne}), i = 1, 2, \ldots, N\}$$

and define the following trigger strategy:

$$\sigma_{i1} = \sigma_i', \sigma_{it} = \begin{cases} \sigma_i', & \text{if } \sigma_{j\tau} = \sigma_j' \; j \neq i, \tau = 1, 2, \ldots, t-1, t = 2, 3, \ldots \\ \sigma_i^{ne}, & \text{otherwise,} \end{cases} \tag{1.4}$$

where $\sigma_i' \in B$. In words, the ith player will choose σ_i' unless any other player does not play σ_j', in which case they continue by playing their Nash equilibrium action, σ_i^{ne}. Now, by definition, the strategy in 1.4 is an equilibrium of the repeated game if

$$\sum_{\tau=0}^{\infty} \delta_i^{\tau} U_i(\sigma_i') > U_i(\sigma_{-i}', t_i) + \sum_{\tau=1}^{\infty} \delta_i^{\tau} U_i(\sigma^{ne}), \quad i = 1, 2, \ldots, N,$$

which can be rearranged to

$$\frac{\delta_i}{1-\delta_i}[U_i(\sigma') - U_i(\sigma^{ne})] > U_i(\sigma_{-i}', t_i) - U_i(\sigma\prime), \quad i = 1, 2, \ldots, N,$$

where $U_i(\sigma_{-i}', t_i) = \max_{\sigma_i \in \Sigma_i} U_i(\sigma_{-i}', \sigma_i)$, $t_i \in \Sigma_i$.

To check if this strategy is indeed a best response to all others players, who are executing the same strategy in 1.4, for the ith player consider their alternatives. They have two options: either the ith player executes the strategy in question, or they play the strategy in which $\sigma_{i1} = t_i$ and then $\sigma_{i\tau} = \sigma^{ne}$ will be the best response as every other player will convert to $\sigma_{j\tau} = \sigma^{ne}$, for all $\tau > 1$. Note that any other strategy is weakly dominated by one of these two, since playing t_i in any other stage $\tau \neq 1$ will yield less gains as these stages have increased discounting.

Now, if the discounted loss from playing the Nash equilibria,

$$\frac{\delta_i}{1-\delta_i}[U_i(\sigma') - U_i(\sigma^{ne})], \tag{1.5}$$

is greater than the gain achieved by playing t_i against σ'_{-i}, then the rational strategy choice for player i, assuming all other players are executing 1.4, is to play 1.4.

Observe, as the discount parameter, δ, tends to one from below, the discounted loss in 1.4 tends to infinity. However, the gain obtained from playing t_i, that is, $U_i(\sigma'_{-i}, t_i) - U_i(\sigma\prime)$ is finite. Thus, for all $\sigma'_i \in B$ there exists a $\delta^* \in (0, 1)$ such that for all $\delta_i > \delta^*$, the strategy 1.4 is optimal against the same strategy for all players $j \neq i$. Therefore, if the conditions are true for all players $i = 1, 2, \ldots, N$, the strategy $(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_N)$, where $\bar{\sigma}_i$ denotes 1.4, yields a Nash equilibrium.

Finally, by construction, the strategy 1.4 is indeed a subgame perfect equilibrium.

□

1.5 Aims of the Project

This project stemmed from an initial idea used in Game Theory Coursework written by the author. In this coursework, regarding Nash equilibria and repeated games, the two graphs, as presented in Figure **??**, were obtained.

These two graphs were obtained using a similar method to this project and so please see Chapter **??** for the details. The plots show the least probability of defection obtained in the Nash equilibria of the game. The game matrices were defined by repeating an IPD tournament 100 times for the Defector strategy, and three other opponents, with 100 distinct probabilities of the game ending.

From Figure **??**, it can be seen that there appears to be a 'clear probability' of the game ending at which the least probability of defection jumps from zero to one. (Throughout the rest of this project, this 'clear probability', p, will be referred to as the *p-threshold* of a game.) However, what was most intriguing was that the two different games obtained had a different p-threshold (approximately 0.25 for Figure **??** but for Figure **??** the threshold appears at around 0.5). This inspired the idea to investigate whether there are any specific characteristics of an IPD tournament that affect the value of this threshold. That is, does the number or the stochasticity of players, for example, cause the probability of a game ending for which the least probability of defection jumps to increase or decrease?

Therefore, the aims of this project are as follows:

1. To look thoroughly into the recent (and past) literature of research already produced in the areas of Folk Theorems and the IPD;

2. To execute a large experiment involving many tournaments of the IPD

with different types / numbers of players to obtain graphs similar to those in Figure **??**;

3. To perform analyses on where the p-thresholds seem to lie and whether it is affected by the different environments of the tournaments (that is, differing number of players, stochasticity within the tournament itself and the players, etc.); and

4. To explore other 'Folk-like' Theorems and perform similar experiments and analyses.

References

[1] IB Adeoye et al. "Application of game theory to horticultural crops in south-west Nigeria." In: *Journal of Agricultural and Biological Science* 7.5 (2012), pp. 372–375.

[2] Robert Axelrod. "Effective choice in the prisoner's dilemma". In: *Journal of conflict resolution* 24.1 (1980), pp. 3–25.

[3] Margherita Barile. *Open Cover*. [Accessed on; 06/01/2020]. Wolfram MathWorld. URL: `http://mathworld.wolfram.com/OpenCover.html`.

[4] Stephan C. Carlson. *Brouwer's fixed point theorem*. [Accessed on: 06/01/2020]. Encyclopaedia Britannica. Sept. 13, 2016. URL: `https://www.britannica.com/science/Brouwers-fixed-point-theorem`.

[5] Bor-Sen Chen, Chia-Hung Chang, and Hsiao-Ching Lee. "Robust synthetic biology design: stochastic game theory approach". In: *Bioinformatics* 25.14 (2009), pp. 1822–1830.

[6] Oxford English Dictionary. *game theory, n.* Online. Accessed: 08.10.2019. Mar. 2013. URL: `https://www.oed.com/view/Entry/319028?redirectedFrom=game+theory&`.

[7] James W Friedman. "A non-cooperative equilibrium for supergames". In: *The Review of Economic Studies* 38.1 (1971), pp. 1–12.

[8] Vincent Knight. *Chapter 10 - Infinitely Repeated Games*. [Accessed on: 15/01/2020]. 2017. URL: `https://vknight.org/Year_3_game_theory_course/Content/Chapter_10_Infinetely_Repeated_Games/`.

[9] Vincent Knight. *Chapter 2 - Normal Form Games*. Online. [Accessed on: 13/10/2019]. 2017. URL: `https://vknight.org/Year_3_game_theory_course/Content/Chapter_02-Normal_Form_Games/`.

[10] Vincent Knight. *Chapter 3 - Rationalisation*. Online. Accessed on: 23/10/2019. 2019. URL: `https://vknight.org/gt/chapters/03/`.

[11] Vincent Knight. *Chapter 8 - Repeated Games*. [Accessed on: 08/01/2020]. 2019. URL: `https://vknight.org/gt/chapters/08/`.

[12] Vincent Knight. *Chapter 9 - Finitely Repeated Games*. Online. Accessed on 06/01/2020. 2017. URL: `https://vknight.org/Year_3_game_theory_course/Content/Chapter_09_Finitely_Repeated_Games/`.

[13] Xiannuan Liang and Yang Xiao. "Game theory for network security". In: *IEEE Communications Surveys & Tutorials* 15.1 (2012), pp. 472–486.

[14] Michael Maschler, Eilon Solan, and Shmuel Zamir. *Game Theory*. Cambridge University Press, 2013. DOI: `10.1017/CBO9780511794216`.

[15] John Nash. "Non-cooperative games". In: *Annals of mathematics* (1951), pp. 286–295.

[16] John F Nash et al. "Equilibrium points in n-person games". In: *Proceedings of the national academy of sciences* 36.1 (1950), pp. 48–49.

[17] V. Srivastava et al. "Using game theory to analyze wireless ad hoc networks". In: *IEEE Communications Surveys Tutorials* 7.4 (Apr. 2005), pp. 46–56. DOI: `10.1109/COMST.2005.1593279`.

[18] Christopher Stover. *Quasi-Concave Function*. [Accessed on: 15/01/2020]. Wolfram Mathworld. URL: `http://mathworld.wolfram.com/Quasi-ConcaveFunction.html`.

[19] John Von Neumann, Oskar Morgenstern, and Harold William Kuhn. *Theory of games and economic behavior (commemorative edition)*. Princeton university press, 2007.

[20] James N. Webb. *Game Theory: Decisions, Interaction and Evolution*. Springer Undergraduate Mathematics Series. Springer, 2007. ISBN: 978-1-84628-423-6.

[21] Eric W. Weisstein. *Compact Space*. [Accessed on: 06/01/2020]. Wolfram Mathworld. URL: `http://mathworld.wolfram.com/CompactSpace.html`.

[22] Eric W. Weisstein. *Convex*. [Accessed on: 06/01/2020]. Wolfram MathWorld. URL: `http://mathworld.wolfram.com/Convex.html`.