Quantum Information Processing (2020) 19:25
https://doi.org/10.1007/s11128-019-2519-8

Foundation of quantum optimal transport and applications

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Received: 4 August 2019 / Accepted: 22 November 2019 / Published online: 29 November 2019

Abstract

Quantum optimal transportation seeks an operator which minimizes the total cost of transporting a quantum state to another state, under some constraints that should be satisfied during transportation. We formulate this issue by extending the Monge–Kantorovich problem, which is a classical optimal transportation theory, and present some applications. As an example, we address infinitely repeated quantum games and establish the folk theorem of the quantum prisoners' dilemma, which claims mutual cooperation can be an equilibrium of the infinitely repeated quantum game. We also exhibit a series of examples which show generic and practical advantages of the abstract quantum optimal transportation theory.

Keywords Quantum optimal transportation · Monge–Kantorovich problem · Quantum game theory · Folk theorem · Repeated quantum game · Quantum automata

Contents

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1 Introduction

Optimization is ubiquitous in many areas and widely studied in various settings. In the modern physics literature, a preferred physical quantum quantity can be obtained by optimizing (minimizing or maximizing) a certain functional. In this article, we address optimal transportation, which is a problem to find an optimized way of transporting objects. A rigorous mathematical definition of optimal transportation was given by Gaspard Monge in 1781 [1], and since then a number of authors worked on the problem. From a viewpoint of physics, the conventional optimal transportation theory is basically established in a classical manner. Therefore, recent advances in quantum technologies endow us with motivation to address a quantum version of Monge's problem, that is "What is an operator which minimizes the total cost of transporting a quantum state to another state?". As shown later, this problem can be formulated in various ways, depending on constraints imposed on a transportation process. Optimal transportation seeks an optimized map or operator of transportation by a non-perturbative way. This is in contrast to the modern physics, where perturbation is a major method. However, this conventional method, in general, does not always become the best way to investigate non-local physics. Hence, our work presented here has a potential advantage when one explores global quantum physics. In this note, we present some formulations of quantum optimal transportation and apply them to some practical examples.

In the latter part of this work, we consider a repeated quantum game as an example of optimization problems. Particularly, we investigate quantum games, which is a game theory based on quantum mechanics. Game theory looks at optimization problems in terms of economics, where agents aim at maximizing their individual payoff as well as minimizing their own loss. Game theory plays a fundamental role in a fairly large part of the modern economics. From this point of view, one can say that optimal transportation seeks an economically best way of transportation; hence, it will be natural to ask how it can be useful to game theory. There are various definitions of quantum game [2,3], and it is as open to establish the foundation. What we mean by a quantum game in this work is a quantum theory which addresses economical optimization problem. Pursuing quantum games gets more and more important as quantum technology develops. Indeed, many authors work on quantum games from various motivations (see [4–6], for example). Most of the conventional works on quantum games focus on some single-stage quantum games, where each agent plays a quantum strategy only one time. However, in a practical situation, a game is more likely played repeatedly; hence, it is more meaningful to address repeated quantum games. Even for the classical theory of games, it is known that repeated games are very different from single-stage games [7,8]. For example, a Nash equilibrium of a single-stage game is not always an equilibrium of a repeated game and a strategy which is not an equilibrium of a single-stage game has a chance to be an equilibrium of a repeated game. One of the most important aspects of repeated games is the folk theorem, which addresses how to achieve and maintain a socially optimal equilibrium in repeated games. Several conventional works on repeated games have been done for finite stage games [9,10], but nothing has been known for infinitely repeated quantum games. In fact, the folk theorem of quantum games has not been explored yet. To

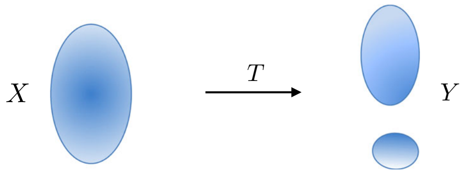

Fig. 1 Image of optimal transportation

open up a new direction of the quantum game theory, in this work, we investigate an infinitely repeated quantum in terms of quantum optimal transportation and show the folk theorem of the repeated quantum game (Theorem 3.1 and Proposition 3.2). This is the first work that investigates an infinitely repeated quantum game and the folk theorem.

Attempt of solving not necessarily physical problems by a physical method is increasingly gaining a lot of interest, due to recent progress in quantum computers and quantum information technology. In fact, solving optimization problems by quantum physics is known as adiabatic quantum computing [8] or quantum annealing [11], which is partly implemented with superconducting qubits [12] and applied to some combinatorial optimization problems [13]. The adiabatic quantum computation is as powerful as universal computation and useful not only to solve combinatorial optimization problems but also to simulate physics and chemistry [14]. Indeed, we later show that some optimal transportation problems can be implemented by a formalism of quantum annealing. While a quantum annealing can solve only discrete problems, in this article, we address problems with uncountable number of degrees of freedom. Though such a computer that solves non-discrete problems has not existed so far, our formulation of continuous quantum Monge problems can be useful when a machine with an ultimate computational capability is realized. Indeed, quantum field theory is quantum mechanics with uncountable number of degrees of freedom and nature implements quantum field theoretical algorithm by a yet-unknown way.

This piece is orchestrated as follows. In Sect. 2, we first give a brief review on the Monge–Kantorovich optimal transportation theory and give various quantum extensions. Some of them have clear physical interpretations, and some require further investigation. In Sect. 3, we address some examples based on our formalism. Particularly, we describe a generic relation among quantum walk, quantum automata, and quantum games in a context of optimal transportation (Fig. 1). Finally, this work is concluded in Sect. 4.

2 Quantum optimal transport

2.1 The Monge–Kantorovich problem

We consider the Monge–Kantorovich problem [1,15]. Let (X, μ), (Y, ν) be two probability spaces. The original Monge problem is to find a bijective map $T : X \to Y$ which minimizes the total cost:

$$C(T)=\int_X c(x,T(x))\mu(\mathrm{d}x), \tag{2.1}$$

where $c(x, T(x))$ is a certain function on X. We write the transported distribution as $\nu = T_\#\mu$. Monge's problem is reformulated by Kantorovich in such a way that one finds an optimal plan $\pi = (1\times T)_\#\mu$, which satisfies $\pi(A, Y) = \mu(A), \pi(X, B) = \nu(B)$ for all measurable sets $A \subset X$, $B \subset Y$ and

$$\int_{X\times Y} c(x,y)\pi(\mathrm{d}x\mathrm{d}y)=\int_X c(x,T(x))\mu(\mathrm{d}x). \tag{2.2}$$

Hence, the optimized transportation plan is $\pi(\mathrm{d}x\mathrm{d}y) = \mu(\mathrm{d}x)\delta_{T(x)}(\mathrm{d}y)$. Let $(T_t(x))_{t\in[0,1]}$ be the associated optimal flow and $(\mu_t)_{t\in[0,1}$ be a family of curves $\mu_t = T_{t\#}\mu$. Then, they naturally obey the equation of continuity

$$\partial_t\mu+\nabla(v\mu)=0, \tag{2.3}$$

where $v(t, T_t(x)) = \frac{\mathrm{d}}{\mathrm{d}t}T_t(x)$.

Now, let us formulate the Monge–Kantorovich problem with a Hamiltonian formalism. Before moving to quantum cases, we first consider the classical problem. Let $q : X \times Y \to [0, 1]$ be a function by which $\mu_0(x)q(x, y)$ indicates the amount transported to y from x. Then, a solution of the classical Monge–Kantorovich problem is a ground state of the following Hamiltonian:

$$\begin{aligned}H=&\int \mathrm{d}x\mathrm{d}y c(x,y)\mu(x)q(x,y)+\int \mathrm{d}x\left(\int \mathrm{d}y\mu(x)q(x,y)-\mu(x)\right)^2\\&+\int \mathrm{d}y\left(\int \mathrm{d}x\mu(x)q(x,y)-\nu(y)\right)^2.\end{aligned} \tag{2.4}$$

The first term is the cost of transportation, the second term is the penalty term which requires for any x the sum of the transported amount from x to y is equal to the sum of the amount $\mu(x)$ at x, and the third term implies that the required amount $\nu(y)$ should be delivered to all y without loss. The problem is solved by finding $\{q(x, y)\}_{(x,y)\in X\times Y}$ that minimizes the Hamiltonian (2.4).

In practice, we can numerically simulate the problem in a discrete situation, by finding the ground state of the classical Hamiltonian

$$\begin{aligned}H=&\sum_{x,y} c(x,y)\mu(x)q(x,y)+\sum_x\left(\sum_y \mu(x)q(x,y)-\mu(x)\right)^2\\&+\sum_y\left(\sum_x \mu(x)q(x,y)-\nu(y)\right)^2.\end{aligned} \tag{2.5}$$

If q takes a value in $\{0, 1\}$, this Hamiltonian (2.5) works for quantum annealing. This corresponds to the Hamiltonian of the Hitchcock transportation problem.

2.2 Quantum optimal transport

In what follows, we work on a Euclidean space $X = Y = \mathbb{R}^d$. Occasionally, we use X or Y to emphasize a reference space and a target space. We may define the quantum optimal transport by saying that find an operation $T : \mathcal{H}(X) \to \mathcal{H}(Y)$ which minimizes the total cost (or maximizes the total reward) when a given wave function $\psi_0(x) = \langle x|\psi_0\rangle$ on M is transported to another $\psi_1(y) = \langle y|\psi_1\rangle$ on N. So we formulate the problem by the following functional $I[T]$ of T defined by

$$\begin{aligned} I[T] = \int_Y \mathrm{d}y \left| \int_X \mathrm{d}x \sqrt{c(x,y)} \langle y|T|x\rangle \langle x|\psi_0\rangle \right|^2 \\ + \int_Y \mathrm{d}y \lambda(y) \left(\langle y|T|\psi_0\rangle - \langle y|\psi_1\rangle \right), \end{aligned} \tag{2.6}$$

where $\lambda(y)$ is a Lagrange multiplier. We expand

$$T|x\rangle = \int_Y \mathrm{d}y T(x,y)|y\rangle, \tag{2.7}$$

where $T(x,y) \in \mathbb{C}$ should satisfy the unitarity condition

$$\int_Y \mathrm{d}y |T(x,y)|^2 = 1, \quad \forall x \in X. \tag{2.8}$$

Alternatively, we can do the same business by introducing an operator $\widehat{CT}$ which acts on $|x\rangle$ as

$$\widehat{CT}|x\rangle = \int_Y \mathrm{d}y' \sqrt{c(x,y')} T(x,y')|y'\rangle \tag{2.9}$$

and redefine the cost term with $\langle y|\widehat{CT}|x\rangle$, which is equivalent to $\sqrt{c(x,y)}\langle y|T|x\rangle$. In fact, the following formula holds:

$$\left\| \int_X \mathrm{d}x \widehat{CT}|x\rangle\langle x|\psi_0\rangle \right\|^2 = \int_Y \mathrm{d}y \left| \int_X \mathrm{d}x \langle y|\widehat{CT}|x\rangle\langle x|\psi_0\rangle \right|^2. \tag{2.10}$$

Proof is simple. The L.H.S. is

$$\int_X \mathrm{d}x\mathrm{d}x' \left(\langle\psi_0|x'\rangle\langle x'|\widehat{CT}^\dagger \right) \left(\widehat{CT}|x\rangle\langle x|\psi_0\rangle \right) \tag{2.11}$$

$$= \int_X \mathrm{d}x\mathrm{d}x' \int_Y \mathrm{d}y\mathrm{d}y' \langle\psi_0|x'\rangle\langle x'|\widehat{CT}^\dagger|y'\rangle\langle y'|y\rangle\langle y|\widehat{CT}|x\rangle\langle x|\psi_0\rangle \tag{2.12}$$

$$= \int_Y \mathrm{d}y \left| \int_X \mathrm{d}x \langle y|\widehat{CT}|x\rangle\langle x|\psi_0\rangle \right|^2. \tag{2.13}$$

So we can interpret the state $\int_X \mathrm{d}x\widehat{CT}|x\rangle\langle x|\psi_0\rangle$ as the quantum version of Monge's integral (2.1). $\left\|\int_X \mathrm{d}x\widehat{CT}|x\rangle\psi_0\rangle\right\|^2$ gives amplitude of states after transportation. In general, the cost $\sqrt{c(x,y)}$ makes a transition process non-unitary, as evolution of a particle interacting with a heat bath. We may restrict to the case $0 \leq |c(x,y)| \leq 1$ for all $x \in X, y \in Y$ and consider the problem $\sup_{\widehat{T}} I[\widehat{T}]$, instead of $\inf_{\widehat{T}} I[\widehat{T}]$. If $c(x,y) = 1$ everywhere, any $\widehat{T}$ which realizes $|\psi_1\rangle = \widehat{T}|\psi_0\rangle$ can be a solution of the problem.

Our framework can address the classical Monge's problem as well. Using the formula

$$\begin{aligned}\left\|\widehat{CT}|x\rangle\right\|^2 &= \int_Y \mathrm{d}y\mathrm{d}y'\sqrt{c(x,y)c(x,y')^*}T(x,y)T(x,y')^*\langle y|y'\rangle \\ &= \int_Y \mathrm{d}yc(x,y)|T(x,y)|^2,\end{aligned} \tag{2.14}$$

we find that the functional of $\widehat{T}$

$$I_1[\widehat{T}] = \int_X \mathrm{d}x\left\|\widehat{CT}|x\rangle\langle x|\psi_0\rangle\right\|^2 \tag{2.15}$$

$$= \int_X \mathrm{d}x\int_Y \mathrm{d}yc(x,y)|T(x,y)|^2\mu(x) \tag{2.16}$$

describes the classical Monge's optimal transportation. Here, $|\langle y|\widehat{T}|x\rangle|^2 = |T(x,y)|^2$ plays the role of $q(x,y)$. Therefore, the classical formulation of the problem with a strict constraint on quantum states is

$$I[\widehat{T}] = \int_Y \mathrm{d}y\int_X dx\left|\sqrt{c(x,y)}\langle y|\widehat{T}|x\rangle\langle x|\psi_0\rangle\right|^2 \tag{2.17}$$

$$+ \int_Y \mathrm{d}y\lambda_1(y)\left(\langle y|\widehat{T}|\psi_0\rangle - \langle y|\psi_1\rangle\right). \tag{2.18}$$

This functional can be also obtained when quantum interactions between two different positions are lost, namely $\langle x|\psi_0\rangle\langle\psi_0|x'\rangle = |\langle x|\psi_0\rangle|^2\delta(x-x')$ holds in the equation (2.12).

Dynamical Approach Now, let us consider a dynamical approach. Let $(\widehat{T_t})_{t\in[0,1]}$ be a family of operators $\widehat{T_t} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$\widehat{T_t}|x\rangle = \int_X \mathrm{d}x'T_t(x,x')|x'\rangle, \tag{2.19}$$

where $T_t(x,x') \in \mathbb{C}$ satisfies $\int_X \mathrm{d}x'|T_t(x,x')|^2 = 1$ for any x and t. With respect to this $\widehat{T}_t$, we consider a family $(\widehat{CT_t})_{t\in[0,1]}$ of operators with cost

$$\widehat{CT_t}|x\rangle = \int_X \mathrm{d}x'c(x,x')T_t(x,x')|x'\rangle. \tag{2.20}$$

Suppose $T_t(x, x')$ and $c(x, x')$ are smooth and finite with respect to any choice of parameters. The problem is to find $\widehat{T} = (\widehat{T}_t)_{t\in[0,1]}$ which satisfies $|\psi_1\rangle = \widehat{T}_1|\psi\rangle$ and minimizes the total cost (or maximizes the total reward) for the quantum case

$$\int_0^1 \mathrm{d}t \left\| \int_X \mathrm{d}x \widehat{CT_t}|x\rangle\langle x|\psi_0\rangle \right\|^2 \tag{2.21}$$

and for the classical case

$$\int_0^1 \mathrm{d}t \int_X \mathrm{d}x \left\| \widehat{CT_t}|x\rangle\langle x|\psi_0\rangle \right\|^2. \tag{2.22}$$

The classical formulation is obtained by another way. The information entropy of a quantum system is expressed with the density operator ρ in such a way that

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho). \tag{2.23}$$

We define $\rho_t(x) = T_t|x\rangle\langle x|T_t^\dagger$ and the trace operation by

$$\mathrm{Tr}\rho_t^T(x) = \int_Y \mathrm{d}y\langle y|\rho_t(x)|y\rangle. \tag{2.24}$$

By definition, $\mathrm{Tr}\rho_t^T(x) = \int_X \mathrm{d}y|T_t(x, y)|^2$, which is equal to 1 due to the unitarity and respects the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\rho_t^T(x) = 0, \quad \forall x \in X. \tag{2.25}$$

Similarly, we define

$$\rho_t^{CT}(x) = \widehat{CT_t}|x\rangle\langle x|\widehat{CT_t}^\dagger, \tag{2.26}$$

whose trace is $\mathrm{Tr}\rho_t^{CT}(x) = \int_Y \mathrm{d}y|\sqrt{c(x, y)}T_t(x, y)|^2$. Using this, we can write the functional $I_t[T]$ in a simple form

$$I_t[T] = \int_X \rho_t^{CT}(x)\mu(x)\mathrm{d}x. \tag{2.27}$$

Moreover,

$$\rho_t^{\psi_0}(x) = \widehat{T}_t|x\rangle\langle x|\psi_0\rangle\langle\psi_0|x\rangle\langle x|\widehat{T}_t^\dagger \tag{2.28}$$

gives the total amount

$$\mathrm{Tr}\rho_t^{\psi_0}(x) = \int_Y \mathrm{d}y|T_t(x, y)|^2\mu(x) \tag{2.29}$$

transported to y from x at t. The unitarity requires

$$\mathrm{Tr}\rho_t^{\psi}(x) = \mu(x). \tag{2.30}$$

We may write

$$\mu_t(y) = \int_X \mathrm{d}x |T_t(x, y)|^2 \mu(x). \tag{2.31}$$

Then, at the end of transportation $t = 1$, the density precisely obeys $\mu_1(y) = |\langle y|\psi_1\rangle|^2 = \nu(y)$, which agrees with the constraint on $\langle y|T_1|\psi_0\rangle = \langle y|\psi\rangle$. In this way, we can recover the classical picture of optimal transportation.

2.2.1 Variant 1

So far, we have discussed the case where initial state is transformed into a target state. In practice, a quantum state is not a physical observable and this constraint is too hard; thereby, it would be better to work with relaxed constraints. One of the most practical requirements is that quantum states are efficiently transported so that the total cost is as small as possible and the transported quantum state forms an expected probability distribution. The corresponding functional is defined with a Lagrange multiplier $\lambda(y)$ in such a way that

$$\begin{aligned} I[\widehat{T}] = &\int \mathrm{d}y \left| \int \mathrm{d}x \sqrt{c(x, y)} \langle y|\widehat{T}|x\rangle\langle x|\psi_0\rangle \right|^2 \\ &+ \int \mathrm{d}y \lambda(y)(\mu(y) - |\langle y|\widehat{T}|\psi_0\rangle|^2). \end{aligned} \tag{2.32}$$

So the problem is to find a unitary operation which minimizes the cost of transporting wave function, when the two probability distributions $\mu(x) = |\psi_0(x)|^2$ and $\nu(y) = |\psi_1(y)|^2$ are given. With respect to a wave function $\psi : \mathbb{R} \to \mathbb{C}$, we define its support $\mathrm{supp}(\psi)$ by

$$\mathrm{supp}(\psi) = \{x \in \mathbb{R} : \psi(x) \neq 0\}. \tag{2.33}$$

Let $\psi_a(x), \psi_b(x)$ be normalized wave functions on X. We write $\psi_{a\pm b}(x) = \frac{1}{\sqrt{2}}(\psi_a(x) \pm \psi_b(x))$. Density distributions $|\psi_{a+b}(x)|^2$ and $|\psi_{a-b}(x)|^2$ become equal to each other if they are not correlated $\mathrm{supp}(\psi_{a+b}) \cap \mathrm{supp}(\psi_{a-b}) = \emptyset$. While the original functional (2.6) requires a coincidence between a mapped state and a target state, the functional (2.32) only demands a coincidence between a mapped distribution and a target distribution. In this sense, the functional (2.32) looks practical.

Moreover, it is also possible to work with the classical formulation of the problem, by optimizing the functional

$$
\begin{aligned}
I[\widehat{T}] = & \int \mathrm{d}y \int \mathrm{d}x \left| \sqrt{c(x, y)} \langle y | \widehat{T} | x \rangle \langle x | \psi_0 \rangle \right|^2 \\
& + \int \mathrm{d}y \lambda(y) (\nu(y) - |\langle y | \widehat{T} | \psi_0 \rangle|^2).
\end{aligned} \tag{2.34}
$$

This functional corresponds to the classical problem by a quantum method. Generally, it would be hard to find the optimized $\widehat{T}$.

2.2.2 Variant 2

We consider quantum optimal transport $\widehat{T}$ so that the transported states become as close as possible to the desired state in along with minimizing the total transportation cost. Instead of using a Lagrange multiplier, we consider fidelity $F(\psi_1, \widehat{T}\psi_0) = \frac{|\langle \psi_1 | \widehat{T} | \psi_0 \rangle|^2}{\| \widehat{T} | \psi_0 \rangle \|^2}$ to measure the quantum distance $D(\psi_1, \widehat{T}\psi_0) = 1 - F(\psi_1, \widehat{T}\psi_0)$ between two states:

$$
I[\widehat{T}] = \int_Y \mathrm{d}y \left| \int_X \mathrm{d}x c(x, y) \langle y | \widehat{T} | x \rangle \langle x | \psi_0 \rangle \right|^2 + D(\psi_1, \widehat{T}\psi_0). \tag{2.35}
$$

Fidelity is a measure of the closeness of two quantum states and useful to classify quantum states. Low-cost quantum teleportation with high fidelity is an example of this case.

2.2.3 Variant 3

Let $|\psi_0\rangle \in \mathcal{H}(X)$ and $|\psi_1\rangle \in \mathcal{H}(Y)$ be given states. Let $\widehat{C} : \mathcal{H}(X) \to \mathcal{H}(Y)$ be a given cost operator, and $\widehat{T} : \mathcal{H}(X) \to \mathcal{H}(X)$ be a unitary operator. We define the problem to find an optimal $\widehat{T}$ which maximizes the following functional:

$$
I[\widehat{T}] = |\langle \psi_1 | \widehat{C}\widehat{T} | \psi_0 \rangle|^2. \tag{2.36}
$$

This should be understood as the problem to find a operator which maximizes the probability amplitude of sending an initial state $|\psi_0\rangle$ to a finial state $|\psi_1\rangle$ with the cost $\widehat{C}$. By inserting $\int_X |x\rangle\langle x| = 1$ and $\int_Y |y\rangle\langle y| = 1$, the problem is equivalent to

$$
I[\widehat{T}] = \left| \int_Y \mathrm{d}y \int_X \mathrm{d}x \langle \psi_1 | y \rangle \langle y | \widehat{C} | x \rangle \langle x | \widehat{T} | \psi_0 \rangle \right|^2. \tag{2.37}
$$

The cost $\langle y | \widehat{C} | x \rangle$ would correspond to $\sqrt{c(x, y)}$ in the previous cases.

It is also possible to consider the cost to obtain a given final state $|\psi_1\rangle \in \mathcal{H}(Y)$ by integrating all possible initial states $|\psi_\lambda\rangle \in \mathcal{H}(X)$ ($\lambda \in [0, 1]$), and satisfies

$$\int_0^1 \mathrm{d}\lambda |\psi_\lambda\rangle\langle\psi_\lambda| = 1. \tag{2.38}$$

Here, we assume that for any $|\psi\rangle \in \mathcal{H}(X)$, there is some λ and $|\psi\rangle = |\psi_\lambda\rangle$. The cost to obtain $|\psi_1\rangle \in \mathcal{H}(Y)$ is

$$I[\widehat{T}] = \int_0^1 \mathrm{d}\lambda |\langle\psi_1|\widehat{C}\widehat{T}|\psi_\lambda\rangle|^2, \tag{2.39}$$

and the problem is to find T which minimizes it. Similarly, we can evaluate the cost to start with a given initial state $|\psi_0\rangle \in \mathcal{H}(X)$, by integrating out all final states

$$I[\widehat{T}] = \int_0^1 \mathrm{d}\lambda |\langle\psi_\lambda|\widehat{C}\widehat{T}|\psi_0\rangle|^2. \tag{2.40}$$

2.2.4 Variant 4

Initiated by the functional (2.37), we formulate a functional by

$$I[\widehat{T}] = \int_Y \mathrm{d}y \left| \int_X \mathrm{d}x \langle\psi_1|y\rangle\langle y|\widehat{C}|x\rangle\langle x|\widehat{T}|\psi_0\rangle \right|^2 \tag{2.41}$$

and the classical case by

$$I[\widehat{T}] = \int_Y \mathrm{d}y \int_X \mathrm{d}x \left| \langle\psi_1|y\rangle\langle y|\widehat{C}|x\rangle\langle x|\widehat{U}|\psi_0\rangle \right|^2. \tag{2.42}$$

It requires a further investigation to unveil how they could be useful to optimal transportation or to any physical system.

2.2.5 Variant 5

Another variation of this game is given by operator $\widehat{CT}$ and maximizing the reward

$$I[\widehat{T}] = \left|\langle\psi_1|\widehat{CT}|\psi_0\rangle\right|^2. \tag{2.43}$$

By inserting $\int_Y \mathrm{d}y|y\rangle\langle y| = 1$ and $\int_X \mathrm{d}x|x\rangle\langle x| = 1$, the functional is

$$\begin{aligned} I[\widehat{T}] &= \left| \int_Y \mathrm{d}y \int_X \mathrm{d}x \langle\psi_1|y\rangle\langle y|\widehat{CT}|x\rangle\langle x|\psi_0\rangle \right|^2 \\ &= \left| \int_Y \mathrm{d}y \int_X \mathrm{d}x \sqrt{c(x, y)}\langle\psi_1|y\rangle\langle y|\widehat{T}|x\rangle\langle x|\psi_0\rangle \right|^2. \end{aligned} \tag{2.44}$$

The functional (2.43) is the transition amplitude of non-unitary scattering process. By integrating out the degree of freedom about initial or final states, we obtain the cost of ending with $|\psi_1\rangle$ or starting with $|\psi_0\rangle$, respectively.

A way to introduce dynamics into our model is to consider the following functional of a family: $\widehat{T} = \{\widehat{T}_t\}_{t\in[0,1]}$ of transportation operators:

$$I[\widehat{T}] = \int_0^1 \mathrm{d}t \left|\langle\psi_1|\widehat{CT_t}|\psi_0\rangle\right|^2. \tag{2.45}$$

This describes the total cost of transporting $|\psi_0\rangle$ to $|\psi_1\rangle$.

3 Applications of quantum optimal transportation

3.1 Costly quantum walk

We consider discrete, one-dimensional, and two-state quantum walk

$$|\psi_t(x)\rangle = \psi_t^L(x)|L\rangle + \psi_t^R(x)|R\rangle \in \mathbb{C}^2, \tag{3.1}$$

where $|L\rangle = \binom{1}{0}$, $|R\rangle = \binom{0}{1}$ and $\psi_t(x)$ satisfy $\sum_{x=-t}^{t} \|\psi_t(x)\|^2 = 1$ for all t. Time evolution $\psi_{t+1}(x) = U_t\psi_t(x)$ of a quantum walker is defined by a unitary matrix U_t in such a way that

$$|\psi_{t+1}(x)\rangle = U_{L,t}\psi_t(x+1) + U_{R,t}\psi_t(x-1), \tag{3.2}$$

where $U_{R,t} + U_{L,t} = U_t$ is a two-by-two unitary matrix and $\psi_t(x)$ is a state on x at t. More explicitly, a state can be written as

$$\psi_{t+1}^R(x) = a_t\psi_t^L(x+1) + b_t\psi_t^R(x+1) \tag{3.3}$$
$$\psi_{t+1}^L(x) = c_t\psi_t^L(x-1) + d_t\psi_t^R(x-1), \tag{3.4}$$

where a_t, b_t, c_t, d_t are components of a unitary matrix

$$\begin{gathered} U_t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}, \quad U_tU_t^\dagger = 1 \\ U_{L,t} = \begin{pmatrix} a_t & 0 \\ c_t & 0 \end{pmatrix}, U_{R,t} = \begin{pmatrix} 0 & b_t \\ 0 & d_t \end{pmatrix}. \end{gathered} \tag{3.5}$$

Suppose the cost of transporting from x to y is given by

$$c(x, y) = y^2 - x^2. \tag{3.6}$$

Then, the cost operator $\widehat{CU_t}$ acts on $\psi_t(x)$ as

$$\widehat{CU_t}\psi_t(x) = \begin{pmatrix} (2x+1)\left(a_t\psi_t^L(x+1) + b_t\psi_t^R(x+1)\right) \\ (-2x+1)\left(c_t\psi_t^L(x-1) + d_t\psi_t^R(x-1)\right) \end{pmatrix}. \tag{3.7}$$

So the problem is to find best choice of a family $\{U_t\}$ that minimizes the total cost $\sum_t \|\widehat{CU_t}\psi_t\|^2$ of transporting an initial state, say, $\psi_0(x) = \delta(x)|R\rangle$ to a given target state or a target distribution.

3.2 Quantum cellular automata

A two-way quantum finite automaton (2QFA) [16] is defined by a 6-tuple

$$M = (Q, \Sigma, \delta, q_0, Q_{\mathrm{acc}}, Q_{\mathrm{rej}}), \tag{3.8}$$

where $Q, q_0, Q_{\mathrm{acc}}, Q_{\mathrm{rej}}$ are a set of states, an initial state, a set of accepted states, and a set of rejected states. $\delta : Q \times \Sigma \times Q \times \mathbb{Z} \to \mathbb{C}$ gives a transition amplitude, namely $\delta(q, a, q', D) = \alpha$ means that, when the machine in a state q reads an input letter a, the transition amplitude of the state into another q' with a head moving to $D \in \mathbb{Z}$ is α. We denote by $|q, x\rangle$ a state of the machine with the head at $x \in \mathbb{Z}$. So for a given input a, a unitary time evolution U^a of a given state $|q, x\rangle$ is expressed as

$$U^a|q, x\rangle = \sum_{q' \in Q, D \in E(x)} \delta(q, a(x), q', D)|q', x + D\rangle, \tag{3.9}$$

where $E(x) \subset \mathbb{Z}$ is a set of lattice vectors defining local neighborhoods for the automaton and $a(x)$ is a's xth input letter. We define some sets by $\mathcal{H}_{\mathrm{acc}} = \mathrm{span}\{|q, x\rangle : |q\rangle \in Q_{\mathrm{acc}}\}$, $\mathcal{H}_{\mathrm{rej}} = \mathrm{span}\{|q, x\rangle : |q\rangle \in Q_{\mathrm{rej}}\}$ and $\mathcal{H}_{\mathrm{non}} = \mathrm{span}\{|q, x\rangle : |q\rangle \in Q \setminus (Q_{\mathrm{acc}} \cup Q_{rej}\}$. Then, the whole Hilbert space is

$$\mathcal{H} = \mathcal{H}_{\mathrm{acc}} \oplus \mathcal{H}_{\mathrm{rej}} \oplus \mathcal{H}_{\mathrm{non}}. \tag{3.10}$$

Let $\pi_\star : \mathcal{H} \to \mathcal{H}_\star$, ($\star$ = acc,rej,non) be the natural projections. We denote by $|\Psi_0\rangle = |q_0, 0\rangle$ an initial state. 2QFA works as follows. (1) Pick up $U_t = U^x$ and operate it to $|\Psi_t\rangle$. We write $|\Psi_{t+1}\rangle = U^x|\Psi_t\rangle$. (2) Measure $|\Psi_{t+1}\rangle$ using projection operators, which make the state shrink to $\frac{1}{\|\pi_\star|\Psi_{t+1}\rangle\|}\pi_\star|\Psi_{t+1}\rangle$. Further processing halts when either accept or reject is output. The dynamics of a quantum cellular automaton can be described by quantum walk. A way of introducing the cost into this study is to define it as the total steps needed for a state to be accepted or rejected. Suppose dimension of $\mathcal{H}_{\mathrm{acc}} \oplus \mathcal{H}_{\mathrm{rej}}$ is finite n, and let $\{\psi_i\}_{i=1,\ldots,n}$ be an orthonormal basis of $\mathcal{H}_{\mathrm{acc}} \oplus \mathcal{H}_{\mathrm{rej}}$. Cost arises while states in $\mathcal{H}_{\mathrm{non}}$ are observed. Therefore, a way of defining the total cost that arises until processing halts at certain $t = \tau \in \mathbb{Z}$ is

$$\tau[U] = \sum_{t=0}^{\tau}\sum_{i=1}^{n} \Delta\left(\langle\psi_i|M|\Psi_t\rangle\right), \tag{3.11}$$

Table 1 Payoffs for agents with respect to a pair of signals $(\omega_1, \omega_2) \in \{(C, C), (C, D), (D, C), (D, D)\}$. Each component consists of positive X, Y, Z such that $Y < Z$

	C	D
C	(X, X)	$(X - Z, X + Y)$
D	$(X + Y, X - Z)$	$(X - Z + Y, X - Z + Y)$

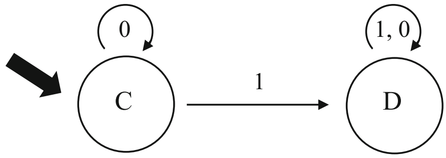

Fig. 2 Strategy profile/state transition diagram of the grim trigger strategy. The bold arrow stands for an initial strategy/state

where $\Delta : \mathbb{C} \ni z \mapsto \Delta(z) \in \{0, 1\}$ is nonzero only at $z = 0$ and M is the measurement operator. The cost (3.11) is actually a functional $\tau[U]$ of $U = \{U_t\}_{t=0,\ldots,\tau}$; therefore, the problem is to find U which minimizes $\tau[U]$. The complexity of this decision problem should be defined by min $\tau[U]$.

3.3 Automata and games

Automata and games are widely studied mostly from a viewpoint of computer games. Conway's life game is a well-known example. But here we like to explore a relation between automata and game theory in economics. Particularly, we are interested in repeated games. Let us first look at classical cases. We prepare a set $\Sigma = \{0, 1\}$ of input letters, a set $Q = \{C, D\}$ of states, and a set $Q_{\mathrm{acc}} \subset \{C, D\}$ of acceptable states. A classical automata consists of those sets, an initial state q_0 and a transition function $\delta : Q \times \Sigma \to Q$, which corresponds to a strategy of a game. Σ corresponds to a set of signals of opponent's strategy. Signals are updated every stage by detecting new ones. As a simple example, we consider a two-person prisoners' dilemma (PD) with monitoring where payoff for each agent is given in Table 1. Let $q_0 = C$ be an initial state of an agent (or an automaton). For instance, the grim trigger strategy is expressed by

$$\begin{aligned} &\delta(C, 0) = C, \quad \delta(D, 0) = D \\ &\delta(C, 1) = D, \quad \delta(D, 1) = D. \end{aligned} \tag{3.12}$$

This means that an agent keeps the cooperative strategy while 0 is observed, but never cooperates once 1 is observed. Interestingly, strategy profiles which game theorists use are almost the same as state transition diagrams which computer scientists use (Fig. 2). Decision making is not a hard task for this automaton, and the complexity of this PD is 1. (We define the complexity by the minimal time step needed for an automaton to make a decision "accept" or "reject.") However, there is yet another way to introduce a cost function, that is, a payoff function. Agents playing a game try to maximize reward (or minimize economic loss) by choosing strategies based on opponents' signals. In a single-stage PD, mutual cooperation (C, C) is not a Nash equilibrium [17]; however, there is a chance that the Pareto optimal strategy (C, C) can be an equilibrium of the repeated game. A study on a repeated game is to find such a non-trivial equilibrium solution that can be established in a long-term relation of agents. The automaton which decides if a mutual cooperation relation is maintained or not has $Q_{\mathrm{acc}} = \{C\}$. From this viewpoint of the games/automata correspondence, mixed strategy games can be seen as stochastic automata models. Quantum games which we consider below is a simple version of quantum automata models (3.8) and (3.9) without degrees of spacial freedom.

3.4 Repeated quantum games

We apply our model to repeated quantum games. Historically, quantum games were proposed by [2,3] and many relevant works have been made. In many cases, however, only a single-stage quantum games are studied and less is known for repeated quantum games. Some finite and small repeated quantum games are proposed by [9,10], whereas infinite cases have not been addressed yet. Particularly, in terms of repeated games, a study on efficient strategy in a long-term relation is critical. In this section, we formulate the problem and solve it affirmatively. For simplicity, we restrict ourselves to two-person prisoners' dilemma where rewards to agents are given in Table 1. A state of each agent is spanned by two vectors $|C\rangle$ and $|D\rangle$, corresponding to Cooperate and Defect:

$$|C\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |D\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.13}$$

We are typically interested in a situation where strategic efficiency is achieved, which we call the folk theorem. So when agents play the quantum prisoners' dilemma (QPD), we consider what gives incentive for agents to keep $|C\rangle \otimes |C\rangle$ in a long-term relation. In every round, each agent chooses a quantum strategy independently. So a strategic state of an agent $i = 1, 2$ is generally defined by

$$|\psi_t^i\rangle = S_t^i|0\rangle, \tag{3.14}$$

where S_t^i should satisfy $\|S_t^i|0\rangle\|^2 = 1$ for any $t = 1, 2, \ldots$, called a quantum strategy. In particular, we define two special quantum strategies by

$$S_C = |C\rangle\langle C| \tag{3.15}$$

$$S_{\overline{C},t} = a_t|C\rangle\langle C| + b_t|D\rangle\langle C|, \tag{3.16}$$

where b_t is some nonzero complex numbers and a_t, b_t satisfy $|a_t|^2 + |b_t|^2 = 1$. The game proceeds as follows. In each stage, each agent decides a quantum strategy and operates it to $|0\rangle$. Neither a quantum strategy nor a quantum state is a physical observable, so agents measure quantum states and reward is paid to agents based on their classical signals $\omega_i \in \{C, D\}$. Information of their outcomes can be either open

Table 2 Monitoring accuracy with respect to each strategy

	C	D
$S_{C,t}$	1	0
$S_{\overline{C},t}$	$\lvert a_t\rvert^2$	$\lvert b_t\rvert^2$

or close to agents. Such games, where a strategy cannot be directly observed, are often referred to as imperfect monitoring games. If open (closed), the game looks similar to a game with public (private) monitoring. Private monitoring is done by observing their opponents' signals and agents cannot know their own signals.

The profit function of agent i is defined by

$$\$_i = (1-\delta)\sum_{t=1}^{\infty}\delta^t \sum_{\omega_1,\omega_2} \$_i(\omega_1,\omega_2)|\langle\omega_1,\omega_2|S_t^1 S_t^2|0,0\rangle|^2, \tag{3.17}$$

where $\delta \in (0,1)$ is a discount factor and $\$_i(\omega_1,\omega_2)$ is i's profit defined in Table 2 when a pair (ω_1,ω_2) of classical outputs is observed. $\$_i(n)$ is a discrete version of (2.45) without degree of freedom about final states. We call $V_i(S_t^1, S_t^2) = \sum_{\omega_1,\omega_2} \$_i(\omega_1,\omega_2)|\langle\omega_1,\omega_2|S_t^1 S_t^2|0,0\rangle|^2$, which is a discrete version of (2.40), the expected payoff of agent i. Regarding the QPD, each player tries to maximize $\$_i$ as much as possible, by choosing $U_t = U_t^1 \otimes U_t^2$.

Theorem 3.1 (Strategic Efficiency) *There is such a quantum strategy for the repeated quantum prisoner's dilemma (RQPD) that is an equilibrium of the repeated game.*

More explicitly, we can show the following statement.

Proposition 3.2 *The (Trigger, Trigger) strategy is an equilibrium of the RQPD.*

Proof The proof is simple. Let $|\psi_0\rangle = |C, C\rangle$ be an initial state of the RQPD. We write $|\overline{C}\rangle = a|C\rangle + b|D\rangle$ with a nonzero $a \in \mathbb{C}$. Suppose agents play the (Trigger, Trigger) strategy, where they repeat cooperation S_C until $|D\rangle$ is observed and, once if $|D\rangle$ is observed, they play a not-cooperate strategy $S_{\overline{C}}$. To complete the proof of the statement, it is sufficed to show that the (Trigger, Trigger) is an equilibrium of the game. Let $r > 0$ be j's probability of playing $S_{\overline{C}}$ when j does observe D. $r = 1$ is called the grim trigger strategy. Suppose both agents play the trigger strategy and their cooperative relation is maintained. Then, the discounted payoff $\$_i^*$ of agent i is $\$_i^* = V_i(S_C, S_C) + \delta\$_i^*$, which implies $\$_i^* = \frac{X}{1-\delta}$. By playing $S_{\overline{C},t}$, agent i can increase the expected payoff by $V_i(S_{\overline{C},t}, S_C) - V_i(S_C, S_C) = |b_t|^2 Y$, but loses the expected reward in the future $\delta r|b_t|^2\$_i^*$. Hence, agent i has no incentive to change the trigger strategy if

$$\$_i(S_{\overline{C},t}, S_C) - \$_i(S_C, S_C) \leq \delta r|b_t|^2\$_i^*. \tag{3.18}$$

Solving this inequality, we obtain the inequality

$$\delta \geq \frac{Y}{rX + Y}. \tag{3.19}$$

Since the R.H.S. is always smaller than 1, there exists $\delta \in (0, 1)$ that satisfies this condition. Therefore, (Trigger,Trigger) is an equilibrium strategy of the RQPD. □

This result agrees with our naive intuition. Since players know that there is no welfare loss while playing S_C, they would not be willing to change their strategies unless $r = 0$ or $X \ll Y$. In addition, our observation in the proof above shows that a quantum game is not quite the same as a classical case. Though quantum games look similar to imperfect monitoring games with mixed strategy, an imperfect monitoring process usually includes measurement errors and hence triggers welfare loss, which describes a loss of economic efficiency that can occur when an equilibrium is not achieved. Therefore, (Trigger,Trigger) unlikely becomes an equilibrium of a repetition of prisoners' dilemma with imperfect monitoring [18]. In contrast to such classical repeated games, our quantum game assures that $|C\rangle$ is observed without any error while S_C is played; therefore, welfare loss never occurs. Indeed, this fact makes the RQPD different from the RCPD (repeated classical prisoner's dilemma). From a viewpoint of the conventional classical game theory, if measurement of signals does not unveil opponents' strategies, which is called conditional independence, "Anti-folk Theorem" claims that only a single-stage Nash equilibrium can be an equilibrium of the RCPD with imperfect private monitoring under conditional independence [19]. In contrast to this, we claim that RQPD always respects the folk theorem, though it also respects conditional independence since agents' quantum states are not entangled at all and measurement does not unveil opponents' strategies.

4 Conclusion and research directions

In this article, we formulated the quantum version of the Monge–Kantorovich problem and addressed several applications, including quantum walk, quantum automata, and repeated quantum games. One can extend our formalism to various ways. More recently, some advantages of quantum optimal transportation were proposed [20]. We leave it to a future work to do a similar observation for our proposed cases. In Sect. 3, we mainly discussed infinitely repeated games and established the folk theorem for the case of quantum prisoners' dilemma. From a perspective of game theory, it is quite natural to discuss repeated games and investigate a possibility of mutual cooperation. As far as the author knows, there were no prior works which addressed infinitely repeated games and the associated folk theorem, whereas single-stage games have been studied by a number of authors. This fact is a bit of surprising, but the author hopes the results in this work endow successors with further motivation. For example, it will be interesting to explore repeated games in terms of quantum error correction. Intuitively speaking, differences from the initial cooperative strategy should be understood as an error, so whether the folk theorem holds or not would correspond to whether such an error can be fixed or not. This interpretation may give more useful and general meanings to repeated quantum games.

Acknowledgements I thank Travis Humble for useful discussion about quantum computation and game theory. I came up with an idea of showing the folk theorem while I discussed with him. Also, I was benefited by discussing with Katsuya Hashino, Kin-ya Oda, and Satoshi Yamaguchi. The author was partly supported by Grant-in-Aid for JSPS Research Fellow, No. 19J11073.

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