Folk Theorem and, in particular, at the thresholds where cooperation is more beneficial than defection in the game of a Prisoners' Dilemma.

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SUMMARY

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Chapter 1

Introduction

Game Theory is the study of interactive decision making and developing strategies through mathematics [4]. It analyses and gives methods for predicting the choices made by players (those making a decision), whilst also suggesting ways to improve their 'outcome' [8]. Here, the abstract notion of utility is what the players wish to maximise (see Chapter 2 in [8] for a detailed discussion on the topic of utility theory or Section 1.3 [13] for a more introductory explanation). One of the earliest pioneers of game theory is mathematician, John von Neumann who, along with economist Oskar Morgenstern, published *The Theory of Games and Economic Behaviour* in 1944 [8]. This book discusses the theory, developed in 1928 and 1940-41, by von Neumann, regarding "games of strategy" and its applications within the subject of economics [12]. Following this, several advancements have been made in the area, including, most notably, John Nash's papers on the consequently named Nash Equilibria in 1950/51 [10, 9]. Due to the "context-free mathematical toolbox" [8] nature of this subject, it has been applied to many areas, from networks ([7] and [11]) to biology ([3] and [1]). In this project, the main focus is on a particular class of theorems, within game theory, known as "Folk Theorems" with application to the game of A Prisoner's Dilemma. These will be defined and discussed in the subsequent sections.

1.1 An Introduction to Games

Consider the following scenario:

Two convicts have been accused of an illegal act. Each of these prisoners, separately, have to decide whether to reveal information (defect) or stay silent (cooperate). If they both cooperate then the convicts are given a short sentence whereas if they both defect then a medium sentence awaits. However, in the situation of one cooperation and one defection, the prisoner who cooperated has the consequence of a long term sentence, whilst the other is given a deal. [5]

This is one of the standard games in game theory known as A Prisoner's Dilemma which has the following normal form representation:

$$\begin{array}{c c} & \begin{array}{cc} \scriptstyle coop & \scriptstyle defect \end{array} \\ \begin{array}{c} \scriptstyle coop \\ \scriptstyle defect \end{array} & \begin{pmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{pmatrix} \end{array}$$

where each cell contains the ordered pair (row player's payoff, column player's payoff). The payoff values are as given in [2] and are used throughout this project.

In general a *normal form* or *strategic form* game is defined by an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where:

- $N = \{1, 2, ..., n\}$ is a finite set of players;
- $S = S_1 \times S_2, \times ... \times S_n$ is the set of strategies for all players in which each vector $(S_i)_{i \in N}$ is the set of strategies for player i [1]; and
- $u_i : S \to \mathbb{R}$ is a payoff function which associates each strategy vector, $\mathbf{s} = (s_i)_{i \in N}$, with a utility [2] $u_i (i \in N)$.

[8]

Before continuing the discussion into the key notions of game theory, it needs to be highlighted that there is an important assumption, which is central to most studies of game theory, entitled *Common Knowledge of Rationality.* This, more formally, is an infinite list of statements which claim:

- The players are rational;
- All players know that the other players are rational;
- All players know that the other players know that they are rational; etc.

Assuming Common Knowledge of Rationality allows for the prediction of rational behaviour through a processes entitled *rationalisation* [6] (see section 4.5 in [8] for an alternative explanation of this assumption).

A strategy for player i, s_i, is *strictly dominated* if there exists another strategy for player i, say $\bar{s}_i$, such that for all strategy vectors $s_{-i} \in S_{-i}$ of the other

[1] Since the game of A Prisoner's Dilemma has a finite strategy set for each player $S_i = \{\text{cooperate}, \text{defect}\}(i \in N)$, in this project only finite strategy spaces are considered.

[2] 'Utility' is referred to as a player's 'payoff' throughout this report.

players,

$$u_i(s_i, s_{-i}) < u_i(\bar{s}_i, s_i).$$

In this case we say that s_i is *strictly dominated* by $\bar{s}_i$. Here, $s_{-i} = \{s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n\}$, i.e. the ith player's strategy has been omitted. The set, S_{-i}, is defined similarly. Looking at the row player's matrix of a Prisoner's Dilemma 1.1 (the first entries in the ordered tuples), it is clear that cooperation is a strictly dominated strategy. Due to the symmetricity of the game, this is also true for the column player. [8]

So far, only the pure strategies, $S_i = \{\text{coop}, \text{defect}\}$, have been discussed, thus the notion of a probability distribution over S_i is now introduced, giving the so-called *mixed strategies*: Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game (with each S_i finite), then a *mixed strategy* for player i is a probability distribution over their strategy set S_i. Define:

$$\Sigma_i := \{\sigma_i : S_i \to [0,1] : \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

to be the set of mixed strategies for player i. Hence, observe that the pure strategies are specific cases of mixed strategies, with $\sigma_i = (1, 0)$ for cooperation and $\sigma_i = (0, 1)$ for defection, in the example of a Prisoner's Dilemma. [8]

1.2 Nash Equilibrium for Normal Form Games

As mentioned above, mathematician, John Nash, introduced the concept of an equilibrium point and proved the existence of mixed strategy Nash Equilibria in all finite games. These notions are central to the study of game theory [8] and hence, in this section, Nash's concepts will be defined and proved in detail.

Firstly, before the definition of a Nash equilibrium, the idea of a *best response* is introduced: For a game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, the strategy, s_i, of the ith player is considered a *best response* to the strategy vector s_{-i} if $u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i})$. [8]

This leads onto the main definition of the section:

Definition 1.2.1. Given a game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, the vector of strategies $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is a *Nash equilibrium* if, for all players $i \in N$, s_i^* is a best response to $s_i^* \in N$. [8]

In other words, s^* is a Nash equilibrium if and only if no player has any reason to deviate from their current strategy s_i^*.

Recall that, in Section 1.1, for any player in A Prisoner's Dilemma, defection dominated cooperation. This leads to the following observation:

The strategy pair (Defect, Defect), is the unique Nash equilibrium for A Prisoner's Dilemma, with a payoff value of 1 for each player. [8]

This can be visualised as followed: Assume the row player uses the following mixed strategy, $\sigma_r = (x, 1-x)$ and, similarly, assume the column player has the strategy, $\sigma_c = (y, 1-y)$. The payoff obtained for the row and column player, respectively, is then:

$$A\sigma_c^T = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 3y \\ 4y+1 \end{pmatrix},$$

$$\sigma_r B = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 1-x \end{pmatrix} = \begin{pmatrix} 3x & 4x+1 \end{pmatrix}$$

Plotting these gives the following:

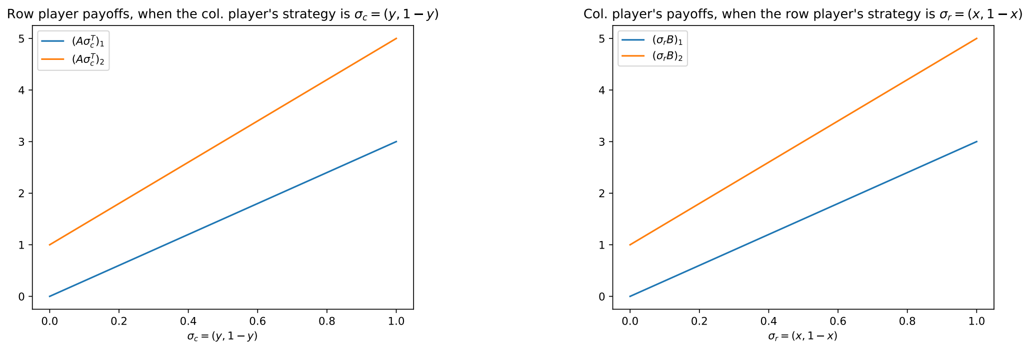

Figure 1.1: Graphs to show the row and column players' payoffs against a mixed strategy.

Chapter 2

Initial Investigation

There exist three algorithms for computing the Nash Equilibria of a game: Support Enumeration, Vertex Enumeration and Lemke Howson algorithm. Thus, firstly an exploration into the properties and execution times was needed to verify which algorithm would be the most appropriate. This investigation, along with a description of each algorithm (in the two-player case), is detailed in the next few sections.

2.1 Support Enumeration Algorithm

For a non-degenerate two player game

$$G = (2, (S_i)_{i=1,2}, (u_i)_{i=1,2})$$

, the following algorithm yields all Nash equilibria:

- For all $1 \leq k \leq min(|S_1|, |S_2|)$,
- and each $I, J \subset S_1, S_2$, respectively, with $|I| = |J| = k$,
- solve $\sum_{i \in I} \sigma_i b_{i,j} = v$, $\sum_{j \in J} a_{i,j} \sigma_j = v$ for all $j \in J$, $i \in I$ respectively
- such that $\sum_{i \in I} \sigma_i = 1$ and $\sum_{j \in J} \sigma_j = 1$ with $\sigma_i, \sigma_j \geq 0$
- and the best response condition in section **??** is satisfied.

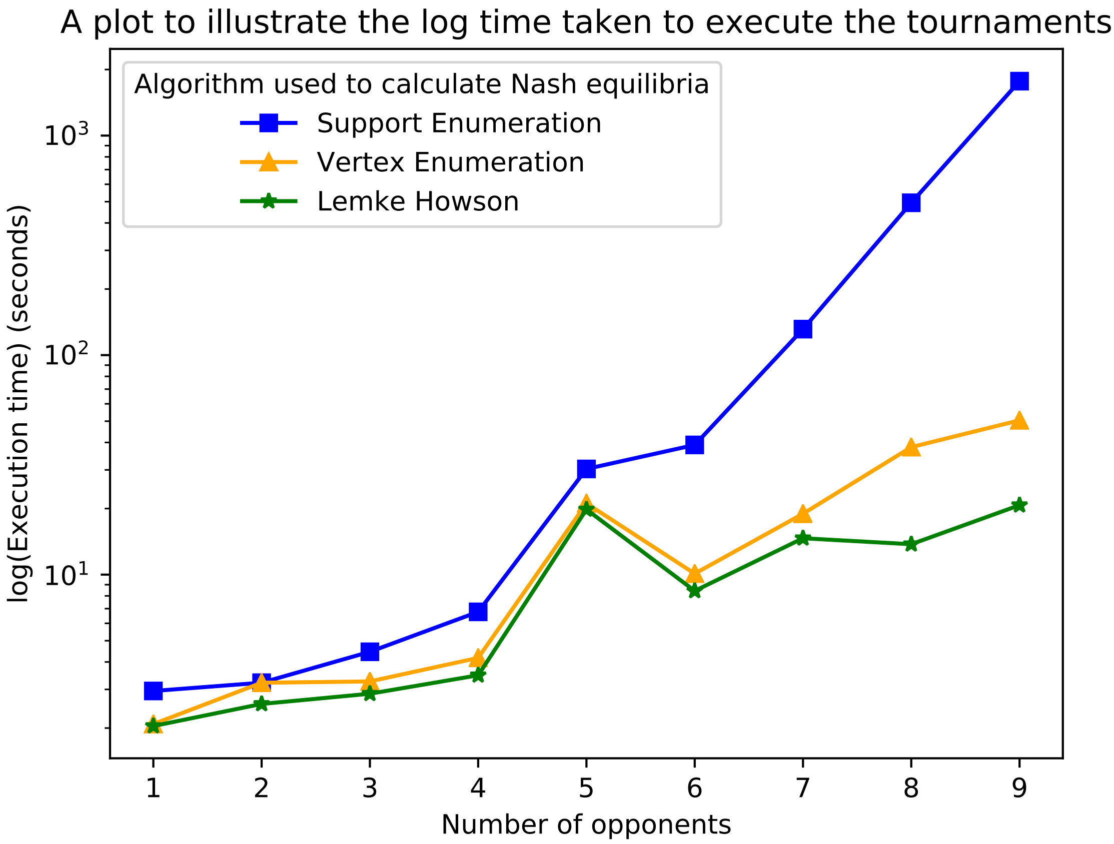

Figure 2.1: A graph to show the results of the timed experiments for each of the three stated algorithms.

2.2 Vertex Enumeration Algorithm

2.3 Lemke Howson Algorithm

2.4 Execution Times

The code used to obtain the results in this section can be found in Appendix ...

From figure **??**, it can be seen that until the Defector has five opponents, each algorithm has approximately the same running time.

After this, as the number of opponents increases it is clear that the support enumeration algorithm blows up significantly quicker exponentially in comparison to boththe Vertex Enumeration and Lemke Howson algorithm. The latter two algorithms seem to have a similar execution time until the number of opponents reaches eight and nine. This is when the Vertex Enumeration algorithm's execution time starts to grow faster than the Lemke Howson algorithm.

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Appendix A

APPENDIX A TITLE

Appendix B

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