The Folk Theorem in Repeated Games with Endogenous Termination

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March 5, 2019

Abstract

In the current mobile world, repeated relationships must be self-sustained by the members. We extend the framework of infinitely repeated games to incorporate the possibility that the game is strategically terminated by players. Specifically, we add a voting stage at the beginning of each period where some or all players vote on whether to continue or end the interaction, and if the game ends, players receive predetermined payoffs. We study general majority rules and show that the appropriately modified folk theorem holds except for the unanimity rule. We also derive sufficient conditions for the folk theorem under the unanimity rule.

JEL classification: C73.

Key words: repeated game, folk theorem, termination.

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1 Introduction

Repeated interactions are essential to provide incentives to act non-myopically for self-interested members in an ongoing relationship. However, the possibility of dissolving a relationship becomes more and more relevant in the current economies. For instance, thanks to the improved transportation and information of the global markets, business partners can easily find outside opportunities. Competitive policies, such as the leniency system, are installed to make colluding firms to consider exiting. There are also numerous examples of ongoing relationships with possible voluntary dissolutions such as married couples, employer-worker, firm-customer relationships and international organizations[1]. We can no longer take it for granted that the continuation of a repeated interaction is exogenous. Instead, ongoing relationships must be self-sustained by the members themselves.

In this paper, we construct a general framework of dynamic interactions which extends the infinitely repeated games and incorporates endogenous termination. To formalize voluntary continuation/termination, we add a voting stage at the beginning of each period. Specifically, we consider various majority rules: the game terminates if and only if certain number of players vote to end. This threshold number depends on the situation. For example, in two-player settings, it may be natural to assume that the threshold number is one, i.e., each player can independently walk away and terminate the interaction. By contrast, when more than two members exist, the threshold can vary, and there is also a possibility that some members cannot vote.[2] We call the members who can vote *mobile players* and those who cannot vote *immobile players*.

We study general majority rules in N-player games and identify the environments under which (the appropriately modified version of) the folk theorem holds. The cere-

[1] Fuchs and Lippi (2006) compare monetary union with independent policy making, having a motivation close to this paper. While their focus is on the optimal plan of a two-country stochastic model, we study the entire set of equilibrium payoff vectors of general N-player games without stochasticity.

[2] For example, hierarchical groups may have members who can vote and those who are not allowed to vote. Via actions within the stage game, the latter players can indirectly influence the voters' termination decisions.

brated "folk theorem" by Fudenberg and Maskin (1986) and others[3] of repeated games shows that any mutually beneficial outcome can be sustained in equilibrium when players interact repeatedly and frequently.[4] The key logic is that history-dependent continuation strategies serve as the reward and penalty to discipline the players' current behavior. However, such dynamic incentive schemes work only when players expect future interactions. When the future interaction is endogenous, there arise at least three fundamental issues in designing players' incentives. First, if players plan to punish someone in the future interaction, then game continuation must be incentive compatible. Second, if players want to punish someone by terminating the game, then game termination must be incentive compatible. Third, we have to determine, for each player, whether a dynamic scheme through game continuation or game termination serves as the maximal punishment.

Our contribution in this paper is twofold. The first one is to identify the maximally possible punishment for each player and for each voting structure. We show that the maximal punishment is fully characterized by the static elements of the game.[5] Hence, even though our dynamic game fails to possess a recursive structure (since the game termination is an absorbing state), a dynamic path achieving the severest punishment can be explicitly constructed in a similar fashion to that of Fudenberg and Maskin (1986).

[3]The folk theorem shown by Fudenberg and Maskin (1986) stated that any feasible and individually rational payoff vector of a one-shot game can be sustained as the average payoff vector of a subgame perfect equilibrium of its infinitely repeated game if players are sufficiently patient. Aumann and Shapley (1976) and Rubinstein (1979) established the folk theorem for infinitely repeated games without discounting. Friedman (1971) showed that any outcome that Pareto dominates a Nash equilibrium of the stage game can be supported for infinite repeated games with discounting. Benoit and Krishna (1985) considered finitely repeated games and established a similar result as the folk theorem when the stage game has multiple Nash equilibria. Abreu, Dutta and Smith (1994) and Wen (1994) generalized Fudenberg and Maskin (1986) by deriving a weaker sufficient condition for three or more player cases. See Mailath and Samuelson (2006) for an overview of the literature.

[4]The repeated game literature assumes that either (i) the same fixed set of players interact with one another every period, or (ii) the set of players exogenously vary according to a predetermined process. The models in Fudenberg and Maskin (1986) and the related literature in footnote 1 belong to the category (i). For some specific models in the category of (ii), results similar to the folk theorem have been also obtained. This includes repeated games with long-run and short-run players by Fudenberg, Kreps and Maskin (1990), with overlapping generations of players by Kandori (1992a) and Smith (1992), and with randomly matched opponents by Kandori (1992b), Ellison (1994) and Okuno-Fujiwara and Postlewaite (1995).

[5]As Lemma 1 establishes, the maximally possible punishment for each player is either her stage game minmax value or the (fixed) average after-game payoff, depending on the voting structure.

The second contribution is to establish the folk theorem by clarifying when the above maximal punishment is sustained in equilibrium. That is, we find that the folk theorem such that any payoff vector which dominates our extended minmax point (based on the derived maximal punishment) holds for a wide range of voting rules. For example, even if the dynamic game can be terminated by a single mobile player, i.e., it is easy to end the game, the folk theorem holds. This result suggests that increased mobility does not necessarily hinder cooperation. It is consistent with our casual observation that various long-term cooperative relationships among people still exist despite the increased mobility due to urbanization and technological innovation.

We also find that the folk theorem can fail when it is *difficult* to end the game. Indeed, we show that the folk theorem fails only if the ending rule is unanimous, where the unanimous vote among the mobile players is necessary to terminate the game. We then provide a few sufficient conditions to recover the folk theorem under this unanimity rule.[6]

Finally, let us place our paper in the recent literature of repeated games with endogenous termination. There are two types of model specifications on what happens if an interaction ends: the "recurrent" models and the "termination" models. In the recurrent type models, each player is (randomly) matched with new opponents every time after a repeated interaction ends.[7] Our paper belongs to the termination type models in which the entire game immediately ends once the current relationship is endogenously terminated. This literature goes back at least to the ruin game by Rosenthal and Rubinstein (1984) who assume no discounting. The recent papers with discounting include financial constraint model by Beviá, Corchón and Yasuda (2011) and Wiseman (2017), R&D model by Furusawa and Kawakami (2008), and firing option model by Casas-Arce (2010).[8] Watson (2002) considers an incomplete information two-player model with ter-

[6]The sufficient condition becomes most involved when there is a single mobile player, a special case of the unanimity rule. As we explain in Section 3.3, our condition in this case can be regarded as an extension of the "effective minmax payoff" proposed by Wen (1994).

[7]The research of recurrent models includes Ghosh and Ray (1996), Kranton (1996), Fujiwara-Greve and Okuno-Fujiwara (2009), and Immorlica, Lucier and Rogers (2014).

[8]A recent experimental study by Wilson and Wu (2017) examines the repeated Prisoner's Dilemma with walk away options under the imperfect monitoring situation in which each player cannot directly

mination options. Most of those related papers assume a narrow class of stage games, Prisoner's Dilemma type games with only two players, as well as very specific ending rules.[9] Moreover, few of them investigate the characterization of equilibrium payoffs. Our paper is the first to derive the folk theorem for general N-player stage games with the possibility of endogenous termination and a variety of ending rules.

This paper is organized as follows. Section 2 presents the model and preliminary results. Section 3 gives the main theorems for repeated games with endogenous termination, and Section 4 concludes.

2 Model and Preliminaries

2.1 Repeated Game with Endogenous Termination

The time horizon is discrete and written as period $t = 0, 1, \ldots$. Let $\{1, \ldots, N\} = \mathcal{N}$ be the set of players with a generic element n (sometimes j is also used). At the beginning of each period, as long as the game continues, a subset $\{1, 2, \ldots, M\} =: \mathcal{M} \subset \mathcal{N}$ of players (where $M \geqq 1$) simultaneously vote either "continue" (action 0) or "end" (action 1).[10] We refer to players in $\mathcal{M}$ as **mobile players** and the rest as **immobile players**. The set of mobile/immobile players is exogenous and fixed throughout the game.

Whether the game continues or ends is determined by the **θ-majority rule** such that the game ends if and only if at least $\theta \in \{1, \ldots, M\}$ of mobile players choose "end", or

$$\sum_{m=1}^{M} b_{m,t} \geqq \theta,$$

where $b_{m,t} \in \{0, 1\}$ is the continue/end choice by player m at the beginning of period t. In particular, when $\theta = M$, we call it the **unanimous ending rule**, i.e., all mobile players

observe the other player's actions, but receive public signals which imperfectly reveal the action profiles.

[9]Wiseman (2017) is an exception which considers an N-player oligopoly game.

[10]The case with $M = 0$ corresponds to an ordinary repeated game without endogenous termination, and hence we focus on other cases in this paper.

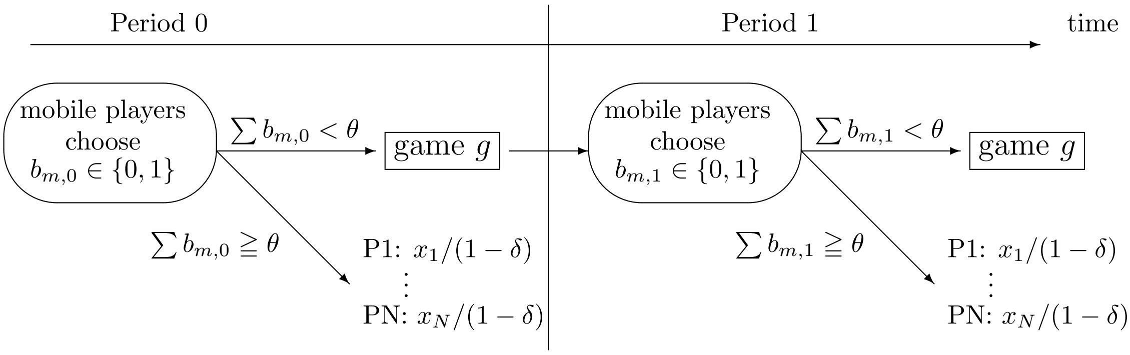

Figure 1: The repeated game with endogenous termination

(not necessarily all players) need to choose "end" in order to terminate the game.[11]

If the game continues, the players engage in a simultaneous-move stage game

$$g : A_1 \times \cdots \times A_N \to \mathbb{R}^N,$$

where A_n is the set of pure actions available to player $n \in \mathcal{N}$ with a generic element a_n. We assume that (i) for each $n \in \mathcal{N}$, A_n is a compact and convex subset of $\mathbb{R}^{\ell_n}$ for some $\ell_n < \infty$, and (ii) g is a continuous function.[12]

We assume that, once the game ends, each player $n \in \mathcal{N}$ receives (an average payoff of) x_n forever, which is exogenously given and does not depend on any history in the game.[13] In other words, termination is the absorbing state that comes with a fixed outside option for each player. Figure 1 illustrates the timeline of the dynamic game, which we refer as the **repeated game with endogenous termination**.

Economic examples of repeated games with endogenous termination are abundant.

[11] The name "mobile players" is easily understood when there are only two players and one of them can unilaterally end the repeated game. Then this player can be called mobile and the other immobile. For general N-player games, the mobile players can be interpreted as senior members of an organization/team who can jointly (but non-cooperatively) choose whether to terminate the organization/team or not.

[12] We can alternatively start with a finite action game $h : S_1 \times \cdots \times S_N \to \mathbb{R}^N$ (where $|S_n| < \infty$ for each $n \in \mathcal{N}$) and let $A_n = \Delta S_n$ and $g = E[h]$. Then assumptions (i) and (ii) are satisfied. However, correlations over $\times_{n=1}^{N} \Delta S_n$ involves a subtle issue of measurability. Hence with this interpretation we only deal with $\Delta(\times_{n=1}^{N} S_n)$ as the correlated action profiles. There is also an issue of observability of mixed actions, see footnote 10 in Section 2.2.

[13] Receiving a fixed payoff during inactive periods is also standard in the relational contract models. See Levin (2003) for example. Fujiwara-Greve and Yasuda (2011) analyze the repeated Prisoner's Dilemma with endogenous termination in which after-game payoffs are stochastic.

Many partnership-type firms (law firms, accounting firms) can be endogenously terminated by some members. Non-market activities such as co-authorships and marriages are also endogenously terminated (usually by all members).

Players use a common discount factor $\delta \in (0,1)$ to evaluate an infinite sequence of one-shot payoffs. That is, if $(a_{1,t}, \ldots, a_{N,t})$ is the vector of stage game actions played in period t and the game is terminated in period τ, then player n's total payoff is

$$\sum_{t=0}^{\tau-1} \delta^t g_n(a_{1,t}, \ldots, a_{N,t}) + \frac{\delta^\tau}{1-\delta} x_n,$$

and her average payoff is

$$(1-\delta) \sum_{t=0}^{\tau-1} \delta^t g_n(a_{1,t}, \ldots, a_{N,t}) + \delta^\tau x_n.$$

We assume the standard perfect monitoring of g; all players observe realized (pure) actions in g taken by all players at the end of each period. Without loss of generality, we assume the existence of a public randomization device to correlate the stage-game action profiles, since we can approximate the average payoff vector of any correlated action profile by an infinite sequence of pure-action profiles. We do not need to assume that proper subsets of players can use a joint randomization device.

For our folk theorem, it is sufficient that players make their strategies contingent only on the action profile histories.[14] For each $t = 0, 1, \ldots$, let $H^t = A^t$ (where $A := A_1 \times \cdots \times A_N$ and A^0 is a singleton of the "null history") be the set of observed action histories up to period t.

A pure strategy of a mobile player $m \in \mathcal{M}$ is a function (as long as the dynamic game continues)

$$s_m : H^t \to \{0,1\} \times A_m$$

[14]To show our folk theorem, we construct equilibria in the situation where players cannot observe the continue/end choice of individual mobile players. If our equilibria exist, we can also construct the corresponding equilibria with the same outcomes when the continue/end choice is observable.

and a pure strategy of an immobile player $i \in \{M+1, \dots, N\}(= \mathcal{N} \setminus \mathcal{M})$ is

$$s_i : H^t \to A_i.$$

The equilibrium concept is subgame perfect equilibrium.

2.2 The Minmax Point

For the game g, the minmax point is defined as follows. For each $j \in \mathcal{N}$, choose $\mu^j = (\mu_1^j, \dots, \mu_N^j) \in A$ so that

$$(\mu_1^j, \dots, \mu_{j-1}^j, \mu_{j+1}^j, \dots, \mu_N^j) \in \arg \min_{a_{-j} \in A_{-j}} \left[\max_{a_j \in A_j} g_j(a_j, a_{-j}) \right],$$

and define

$$v_j^* := \max_{a_j \in A_j} g_j(a_j, \mu_{-j}^j) = g_j(\mu^j).^{15}$$

The one-shot actions $(\mu_1^j, \dots, \mu_{j-1}^j, \mu_{j+1}^j, \dots, \mu_N^j)$ are called **minmax actions** by other players against player j. We call v_j^* player j's **reservation value**[16] and refer to $(v_1^*, \dots, v_n^*)$ as the **minmax point**.

To derive an appropriate minmax point for the repeated game with endogenous termination, we first extend the function g. For each mobile player $m \in \mathcal{M}$, define the extended action space

$$\overline{A}_m = \{0, 1\} \times A_m,$$

with a generic element $\overline{a}_m := (b_m, a_m)$. To unify the notation, for each immobile player $i \in \mathcal{N} \setminus \mathcal{M}$, denote $\overline{A}_i = A_i$ with a generic element $\overline{a}_i (= a_i \in A_i)$, and let $\overline{A} := \overline{A}_1 \times \cdots \times \overline{A}_N$. For each extended action profile $\overline{a} \in \overline{A}$, let $b_m(\overline{a}) \in \{0, 1\}$ be the mobile player m's continue/end choice at $\overline{a}_m = (b_m, a_m)$ and $a(\overline{a}) \in A$ be the induced stage-game action

[15] The existence of μ^j (and v_j^*), while it is not necessarily unique, is guaranteed by our assumptions of compact and convex action spaces and continuous payoff function g_j.

[16] If we interpret A_n as the set of mixed actions of player n, and only realized pure actions are observable, we need to overcome an issue of unobservable mixed actions to impose the reservation value on j. See Fudenberg and Tirole (1991), Chapter 5.1.

profile by all players.

Given a θ-majority rule, we define the extended payoff function for every player $n \in \mathcal{N}$, $\overline{g}_n : \overline{A} \to \mathbb{R}$, as follows.

$$\overline{g}_n(\overline{a}) = \begin{cases} g_n(a(\overline{a})) & \text{if } \sum_{m=1}^{M} b_m(\overline{a}) < \theta, \\ x_n & \text{if } \sum_{m=1}^{M} b_m(\overline{a}) \geqq \theta. \end{cases}$$

This $\overline{g}$ embeds the possibility that mobile players can choose to impose x_n on every player $n \in \mathcal{N}$.[17] Note that our repeated game with endogenous termination is not the repeated game with the stage game $\overline{g}$. Once $\sum_{m=1}^{M} b_m \geqq \theta$ holds in some period, the game falls into the absorbing state and all players receive x_n forever after, i.e., players cannot return to play $\overline{g}$ afterwards. Our model, in this way, captures the irreversible feature of game termination while the (infinitely) repeated $\overline{g}$ does not.[18]

For each $j \in \mathcal{N}$, choose $\overline{\mu}^j = (\overline{\mu}_1^j, \ldots, \overline{\mu}_N^j) \in \overline{A}$ so that

$$(\overline{\mu}_1^j, \ldots, \overline{\mu}_{j-1}^j, \overline{\mu}_{j+1}^j, \ldots, \overline{\mu}_N^j) \in \arg \min_{\overline{a}_{-j} \in \overline{A}_{-j}} \left[\max_{\overline{a}_j \in \overline{A}_j} \overline{g}_j(\overline{a}_j, \overline{a}_{-j}) \right],$$

and[19]

$$\overline{v}_j^* = \max_{\overline{a}_j \in \overline{A}_j} \overline{g}_j(\overline{a}_j, \overline{\mu}_{-j}^j) = \overline{g}_i(\overline{\mu}^j).$$

The extended actions $(\overline{\mu}_1^j, \ldots, \overline{\mu}_{j-1}^j, \overline{\mu}_{j+1}^j, \ldots, \overline{\mu}_N^j)$ are now called **extended minmax actions** against player j. Similarly, we call $\overline{v}_j^*$ player j's **extended reservation value** and refer to $(\overline{v}_1^*, \ldots, \overline{v}_n^*)$ as the **extended minmax point**.

[17] $\overline{g}$ can be interpreted as a reduced form static game derived from the original two-stage game played in each period, as long as the game continues. Wen (2002) derives the folk theorem for the repeated games (without endogenous termination) whose stage games are extensive-form.

[18] Dutta (1995) derives the folk theorem for the general stochastic games with no absorbing state. His model, like the standard repeated games, lacks the irreversible feature and thus is different from ours.

[19] Since g_j is continuous and x_j is an exogenously fixed value, $\overline{\mu}^j$ exists.

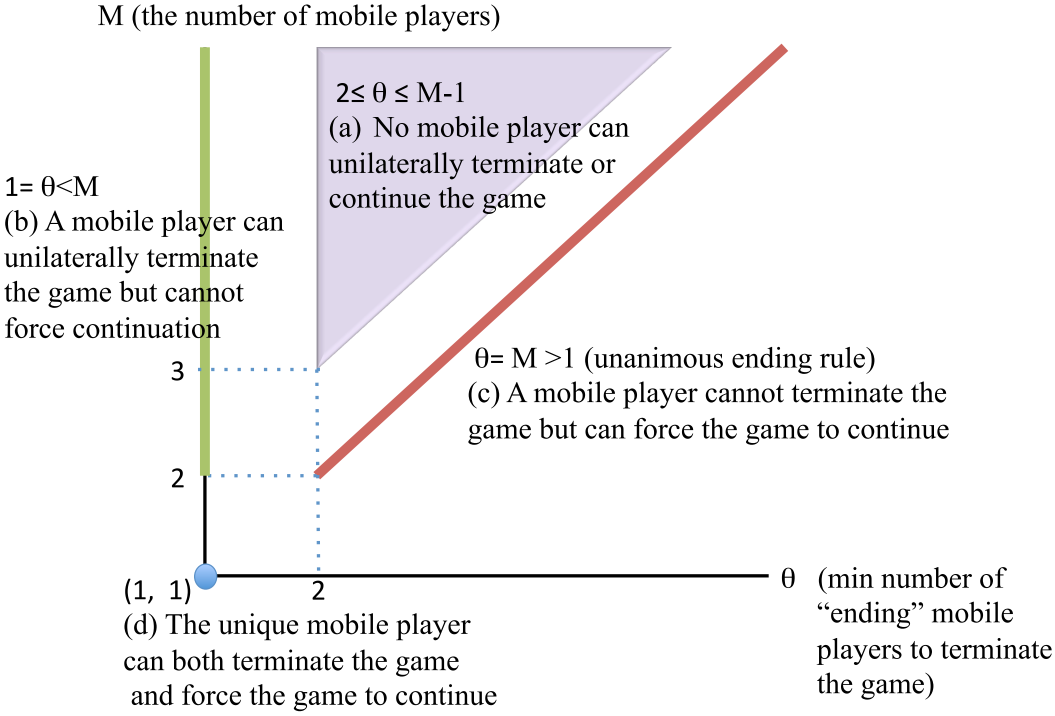

Figure 2: Different "power" of a mobile player

2.3 Preparatory Results

The next lemma provides the connection between the standard reservation value v_n^* and the extended reservation value $\overline{v}_n^*$, which incorporates the possibility of endogenous termination. A mobile player's reservation value varies depending on whether she can unilaterally terminate/continue the game (see Figure 2).

Lemma 1 *For any immobile player $i \in \mathcal{N} \backslash \mathcal{M}$, $\overline{v}_i^* = \min\{x_i, v_i^*\}$ for any $\theta \in \{1, \ldots, M\}$. For any mobile player $m \in \mathcal{M}$, (a) $\overline{v}_m^* = \min\{x_m, v_m^*\}$ if $2 \leqq \theta \leqq M - 1$, (b) $\overline{v}_m^* = x_m$ if $\theta = 1 < M$, (c) $\overline{v}_m^* = v_m^*$ if $\theta = M > 1$, and (d) $\overline{v}_m^* = \max\{x_m, v_m^*\}$ if $\theta = M = 1$.*

Proof. Since immobile player i does not have any influence on whether the game continues or ends, other players can impose her x_i or v_i^* whichever is lower, i.e., $\min\{x_i, v_i^*\}$ is the reservation value. Effectively the same argument applies to a mobile player m if (a) $2 \leqq \theta \leqq M - 1$ holds. In this case, if no other mobile player chooses "end", m cannot unilaterally make the game continue. Similarly, if all other mobile players choose "continue", m cannot unilaterally terminate the game, either.

If (b) $\theta = 1 < M$ holds, a mobile player m can unilaterally terminate the game (independent of other players' actions) but cannot enforce its continuation. Hence she can guarantee herself of x_m.

If (c) $\theta = M > 1$ holds, a mobile player m can unilaterally make the game continue but cannot unilaterally end it. Hence she can guarantee herself of v_m^*.

If (d) $\theta = M = 1$ holds, the unique mobile player m can unilaterally make the game continue and end. Hence she can guarantee $\max\{x_m, v_m^*\}$ as her reservation value. ■

Lemma 1 illustrates that the maximal punishment for each player, depending on the voting structure, is fully characterized by the static elements of the game; our extended reservation value can be explicitly calculated.[20] Henceforth we shall normalize the payoffs so that $(\overline{v}_1^*, \ldots, \overline{v}_N^*) = (0, \ldots, 0)$. Define the set of feasible payoff vectors by[21]

$$V = Conv(\{(v_1, \ldots, v_N) \mid \exists (a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N \text{ with } g(a_1, \ldots, a_N) = (v_1, \ldots, v_N)\}).$$

For the folk theorem, we focus on the set of **modified individually rational payoff vectors**, defined (and assumed to be nonempty[22]) as follows.

$$\overline{V}^* = \{(v_1, \ldots, v_N) \in V \mid v_n > 0 \ \text{ for all } n \in \mathcal{N}; \\ \exists (v_1', \ldots, v_N') \in V;\ v_n > v_n' \ \text{ for all } n \in \mathcal{N}\}.$$

For ordinary repeated games, the minmax point belongs to V and it is sufficient to exclude payoff vectors with some players receiving exactly the reservation value (the north east area of the red dashed lines in Figure 3). By contrast, in the repeated game with endogenous termination, the extended minmax point can be outside of V as Figure

[20]This is in stark contrast to the reservation value defined by Dutta (1995) for the general stochastic games, which depends on the overall structure of the dynamic game. While the existence of the maximal punishment is guaranteed, it is often difficult to solve the reservation value explicitly in his model.

[21]The convex hull of a set X is denoted by $Conv(X)$. For an alternative definition of feasible payoff vectors that include the termination payoff vector, see Appendix B.

[22]By Lemma 1, this assumption must be satisfied for $\theta > 1$. $\overline{V}^*$ becomes empty if and only if $\theta = 1$ and $x_1 > \max_{a \in A} g_1(a)$. This case is not interesting, since the unique mobile player 1 must immediately terminate the game in any equilibrium.

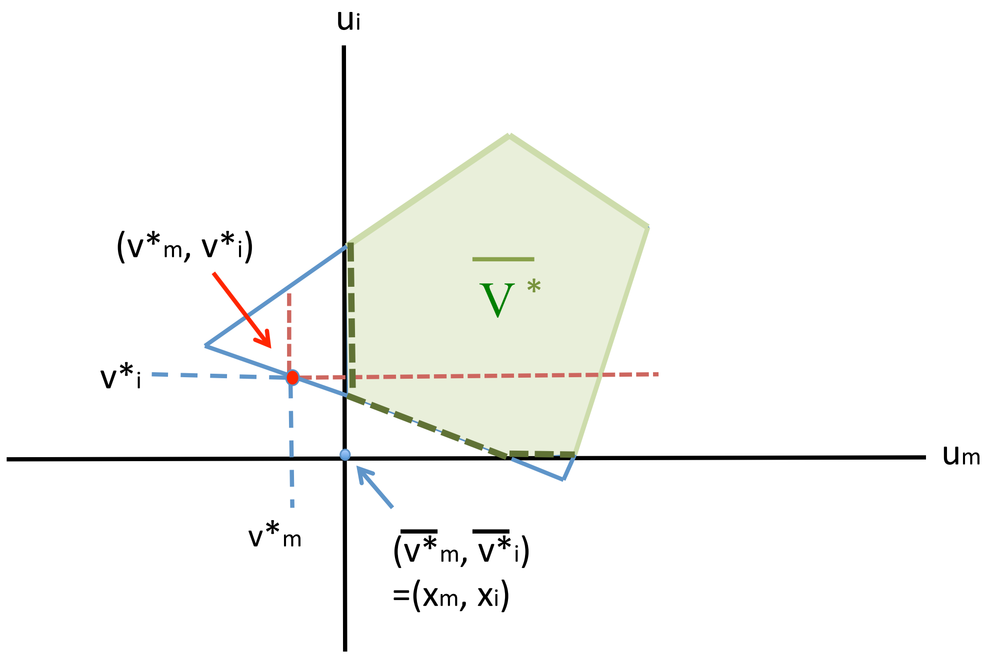

Figure 3: Modified IR payoff vectors for $N = 2$, $M = 1$

3 illustrates. Hence we focus on the payoff vectors which do not belong to the "inefficient boundary" of V.

The next lemma shows that, when $M \geqq 2$, unless the game has the unanimous ending rule ($\theta = M$), a strategy combination in which the game is immediately terminated is a subgame perfect equilibrium in every subgame after each history.

Lemma 2 *Assume* $1 \leqq \theta \leqq M - 1$. *For any period* $t = 0, 1, 2, \ldots,$ *and any history* $h_t \in H^t$, *any (continuation) strategy combination such that all mobile players choose "end" at* t *constitutes a Nash equilibrium of the subgame starting at* h_t.

Proof. Suppose that all M mobile players choose "end". Then, by $\theta \leqq M - 1$, a single mobile player cannot change the outcome by switching to "continue". Since no player can change her continuation payoffs, it is a (weakly) best response for each mobile player to choose "end" (and the game will be terminated immediately). ■

We refer to this equilibrium as the **immediate-ending equilibrium**. Note that the above argument is no longer true if $\theta = M$, since each mobile player can unilaterally change the outcome by switching her action from "end" to "continue", even if all other mobile players choose "end".

The following lemma implies that mobile players can also coordinate to continue the game if two or more mobile players need to agree on terminating the game, i.e., $\theta \geqq 2$.

Lemma 3 *Assume $2 \leqq \theta \leqq M$. Let σ^* be a subgame perfect equilibrium of the ordinary repeated game of g (without termination). Then, the strategy combination such that all mobile players choose "continue" in every period after any history and all players follow σ^* (as long as the game continues) constitutes a subgame perfect equilibrium of the repeated game with endogenous termination.*

Proof. Suppose that all M mobile players choose "continue" after any action history. Then, by $\theta \geqq 2$, a single mobile player cannot change the outcome by switching to "end". Since no player can change her continuation payoffs determined by σ^*, it is a (weakly) best response for each mobile player to choose "continue" and everyone to follow σ^*. ■

We refer to an equilibrium in this class as a **never-ending equilibrium**. For each player n, the worst never-ending equilibrium for n is a candidate of the punishment against n's deviations. Applying the folk theorem for the ordinary repeated game of g, we can characterize this worst equilibrium payoff vector when δ is close to 1.

Let V^* be all feasible payoff vectors that Pareto dominate the (standard) minmax point defined by the stage game g.

$$V^* = \{(v_1, \ldots, v_N) \in V \mid v_n > v_n^* \text{ for all } n \in \mathcal{N}\}.$$

Then, we obtain the following result whose proof is immediate from Lemma 3 and the folk theorem of Fudenberg and Maskin (1986).

Remark 1 *Assume $2 \leqq \theta \leqq M$ and that the dimensionality of V^* equals N, the number of players. Then, for any $(v_1, \ldots, v_N) \in V^*$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

Remark 1 implies that the worst never-ending equilibrium for n converges to v_n^*, n's (standard) reservation value, as δ becomes close to 1. Note, however, that whether the worst never-ending equilibrium exactly achieves v_n^* or not depends on the stage game g.[23]

3 Main Theorems

3.1 Non-Unanimous Ending Rules ($\theta < M$)

The following is the main result of this paper, which establishes that the folk theorem holds for any majority rule except the unanimous one ($\theta = M$).

Theorem 1 *Assume $1 \leqq \theta \leqq M - 1$ and that the dimensionality of $\overline{V}^*$ equals N, the number of players.*[24] *Then, for any payoff vector $(v_1, \ldots, v_N) \in \overline{V}^*$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

Proof. Fix a target payoff vector $(v_1, \ldots, v_N)$ from the interior or on the Pareto frontier of $\overline{V}^*$. Choose a "target" action profile denoted by $\alpha = (\alpha_1, \ldots, \alpha_N) \in \Delta A$ so that $g(\alpha_1, \ldots, \alpha_N) = (v_1, \ldots, v_N)$. There exists $(v_1', \ldots, v_N')$ in the interior of $\overline{V}^*$ such that $v_n > v_n'$ for all $n \in \mathcal{N}$. Since $\overline{V}^*$ has full dimension, there exists $\varepsilon > 0$ such that, for each j,

$$(v_1' + \varepsilon, \ldots, v_{j-1}' + \varepsilon, v_j', v_{j+1}' + \varepsilon, \ldots, v_N' + \varepsilon) \in \overline{V}^*.$$

Let $\beta^j = (\beta_1^j, \ldots, \beta_N^j) \in \Delta A$ be a "reward" action profile which attains this payoff vector (in expectation). Recall that $\overline{\mu}^j = (\overline{\mu}_1^j, \ldots, \overline{\mu}_N^j) \in \overline{A}$ achieves $\overline{g}_j(\overline{\mu}^j) = 0$. Let $w_n^j = g_n(\mu^j)$ be player n's per-period payoff when minmaxing player j and let $\hat{v}_n = \max_{a \in A} g_n(a)$ be

[23]Suppose g is the Prisoner's Dilemma. Then there exists a never-ending equilibrium in which everybody chooses "defection" (as long as the game continues) and each player n simultaneously receives v_n^*.

[24]In other words, the interior of $\overline{V}^*$ relative to n-dimensional space is nonempty. For $N = 2$, we can drop this assumption as in Fudenberg and Maskin (1986). See Appendix A.

player n's greatest one-shot payoff. For each n, choose an integer τ_n such that

$$\frac{\hat{v}_N}{v'_N} < 1 + \tau_n. \tag{1}$$

Now consider the following strategy for a mobile player $m \in M$, starting at phase (A). Let $a^* \in A$ be a (pure-strategy) Nash equilibrium of the stage game g. The existence of a^* is guaranteed by our assumptions (i) and (ii) on g.

(A) Choose "continue" ($b_m = 0$) and α_m each period as long as α was played last period. If player j deviates from (A),[25] then go to (B-j).

(B-j) (i) If $x_j = 0 (= \overline{v}_j^*)$, choose "end" ($b_m = 1$) (and then play a_m^* if the game continues).

(B-j) (ii) If $x_j \neq 0$, choose "continue" ($b_m = 0$) and μ_m^j for τ_j periods, and then go to (C).

(C-j) Choose "continue" ($b_m = 0$) and β_m^j there after.

If player j' deviates in phase (B-j) or (C-j), then begin phase (B-j').

Also, consider the following strategy for each immobile player $i \in \mathcal{N} \setminus \mathcal{M}$ starting at phase (A).

(A) Play α_i each period as long as α was played last period. If player j deviates from (A), then go to (B-j).

(B-j) (i) If $x_j = 0$, play a_i^* (as long as the game continues).

(B-j) (ii) If $x_j \neq 0$, play μ_i^j for τ_j periods, and then go to (C-j).

(C-j) Play β_i^j each period as long as the game continues.

If player j' deviates in phase (B-j) or (C-j), then begin phase (B-j').

We first show that the mobile players do not deviate in the continue/end decision nodes. The assumption $1 \leqq \theta \leqq M - 1$ and Lemma 2 imply that when ending is required, no mobile player deviates. Consider the case when continuation is required. If $x_m \neq 0 (= \overline{v}_m^*)$ holds for some mobile player m, by Lemma 1, $2 \leqq \theta$ $(\leqq M - 1)$ must be satisfied. Then,

[25] If several players simultaneously deviate from (A), then we suppose that everyone ignores the deviation and continues to play α. The same remark applies to other phases than (A).

no single mobile player can unilaterally change the continue/end outcome when all other mobile players jointly choose "continue"/"end". Hence in this case no mobile player deviates. Suppose that $x_m = 0$ holds for all m. Unilateral ending gives 0 to any mobile player m. Except the one-shot payoff w_m^j, all other one-shot payoffs are positive. Hence there exists $\underline{\delta}_M \in (0,1)$ such that for all $\delta > \underline{\delta}_M$ and all $m \in \mathcal{M}$, the continuation payoff of player m is positive after any history. Therefore, no mobile player deviates to end.

It remains to check deviation incentives from stage game actions prescribed in (A) through (C-j)'s. Consider deviation incentives for any player n who has $x_n \neq 0 (= \overline{v}_n^*)$, i.e., the extended reservation value is not the outside option. If player n deviates in phase (A) and then conforms, she receives at most $\hat{v}_n$ in that period, 0 for τ_n periods, and v_n' each period thereafter. Her total payoff is no greater than

$$\hat{v}_n + \frac{\delta^{\tau_n+1}}{1-\delta} v_n'.$$

If she conforms throughout, she obtains $v_n/1-\delta$, so her net gain from a deviation is at most

$$\hat{v}_n + \frac{\delta^{\tau_n+1}}{1-\delta} v_n' - \frac{v_n}{1-\delta}.$$

By $v_n > v_n'$, this is less than

$$\hat{v}_n - \frac{1-\delta^{\tau_n+1}}{1-\delta} v_n'. \tag{2}$$

Since $\frac{1-\delta^{\tau_n+1}}{1-\delta}$ converges to $\tau_n + 1$ as δ tends to 1, condition (1) ensures that (2) is negative for all δ larger than some $\underline{\delta}_n^A < 1$. If player n deviates in (B-n) (ii), i.e., when n is being punished, she obtains at most 0 in that period, and then only lengthens her punishment, postponing the positive payoff v_i'. Since the continuation payoff of such deviation is always less than the one when she conforms, n does not have a deviation incentive. If player n deviates in (B-j) (ii), i.e., when player $j (\neq n)$ is being punished, and then conforms, she receives at most

$$\hat{v}_n + \frac{\delta^{\tau_n+1}}{1-\delta} v_n',$$

which is no greater than $\hat{v}_n + \frac{1}{1-\delta} v_n'$ for any $(1 \leqq)\ \tau \leqq \tau_n$. However, if she does not deviate, she receives

$$\frac{1-\delta^{\tau}}{1-\delta} w_n^j + \frac{\delta^{\tau+1}}{1-\delta}(v_n' + \varepsilon),$$

for some τ between 1 and τ_j, where τ is the number of remaining periods in (B-j) (i). Therefore, her net gain from a one-step deviation is at most

$$\hat{v}_n + \frac{1-\delta^{\tau+1}}{1-\delta}(v_n' - w_n^j) - \delta^{\tau+1} w_n^j - \frac{\delta^{\tau+1}}{1-\delta}\epsilon. \tag{3}$$

As $\delta \to 1$, the second term in (3) remains finite because $\frac{1-\delta^{\tau}}{1-\delta}$ converges to τ, but the third term converges to negative infinity. Thus, there exists $\underline{\delta}_n^{Bji} < 1$ such that for all $\delta > \underline{\delta}_n^{Bji}$, player n will not deviate in phase (B-j) (ii), for any $j \in \mathcal{N}$. Finally the argument for why players do not deviate from (C-j)'s is practically the same as that for phase (A). Therefore, there exists $\underline{\delta}_n < 1$ such that for all $\delta > \underline{\delta}_n$, player n will not deviate in any of phases (A), (B-j) (ii), and (C-j), for any $j \in \mathcal{N}$.

Next, we consider deviation incentives for player n such that $x_n = 0$. If player n deviates in any of the phases (A), (B-j) (ii), and (C-j), she receives at most $\hat{v}_n$ in that period and the game will be immediately terminated. She ends up receiving 0 thereafter, and hence her total payoff is no greater than $\hat{v}_n$. Instead, if she conforms in each phase, she receives $\frac{v_n}{1-\delta}$ in (A), $\frac{1-\delta^{\tau}}{1-\delta} w_n^j + \frac{\delta^{\tau+1}}{1-\delta}(v_n' + \epsilon)$ for some $1 \leqq \tau \leqq \tau_j$ in (B-j) (ii), and $\frac{v_n'+\varepsilon}{1-\delta}$ in (C-j). As $\delta \to 1$, each payoff when she conforms converges to positive infinity, but deviation payoff remains constant. Thus, there exists $\underline{\delta}_n < 1$ such that for all $\delta > \underline{\delta}_n$, player n will not deviate in any of phases (A), (B-j) (ii), and (C-j), for any $j \in \mathcal{N}$. ■

3.2 Unanimous Ending Rules ($\theta = M$)

In the next two sections, we focus on the case that $\theta = M$. The next theorem shows an "anti-folk theorem" result. Namely, under the unanimous ending rule, there is a subset of modified individually rational payoff vectors which cannot be sustained by any subgame perfect equilibrium.

Theorem 2 *Assume $\theta = M$ and that there exist $m \in \mathcal{M}$ and $i \in \mathcal{N} \setminus \mathcal{M}$ such that $x_m < v_m^*$ and $x_i = \overline{v}_i^* < v_i^*$. Then, for any $\delta \in (0,1)$, there is no subgame perfect equilibrium of the repeated game with endogenous termination in which the player i's average payoff v_i is strictly less than v_i^* but greater than the extended reservation value, i.e., $(x_i =)\overline{v}_i^* < v_i < v_i^*$.*

Proof. Suppose that there is a subgame perfect equilibrium in which the player i's average payoff is $v_i \in (\overline{v}_i^*, v_i^*)$. Since the player i can guarantee by herself a reservation payoff of v_i^* when the game continues forever, the game must be terminated, at least with a positive probability, at some period. $\theta = M$ means that all mobile players must choose "end" simultaneously with a positive probability at that period. However, choosing "end" is not incentive compatible for the player m with $x_m < v_m^*$, a contradiction. ■

Intuitively, when $\theta = M$ and there is a mobile player with $x_m < v_m^*$, this m can "block" termination of the game. Hence we cannot sustain a payoff less than the (standard) reservation value v_i^* of the immobile players.

Let us consider whether a folk theorem holds if the conditions of Theorem 2 are not satisfied, i.e., (i) $x_m \geqq v_m^*$ for all $m \in \mathcal{M}$ or (ii) $x_i \geqq v_i^*$ $(= \overline{v}_i^*)$ for all $i \in \mathcal{N} \setminus \mathcal{M}$. The answer is affirmative if there are two or more mobile players and if the worst never-ending equilibrium can impose exactly the reservation value v_m^* on every mobile player simultaneously, such as the Prisoner's Dilemma (see also footnote 15). Then we obtain a characterization of the equilibrium payoff vectors as follows.

Theorem 3 *Assume $\theta = M \geqq 2$ and that there exists $\hat{\delta} \in [0,1)$ such that for any $\delta \in (\hat{\delta}, 1)$, the ordinary repeated game of g admits a subgame perfect equilibrium σ^* such that the equilibrium average payoff is v_m^* for all $m \in \mathcal{M}$. For any payoff vector $(v_1, \ldots, v_N) \in \overline{V}^*$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$, if*

(i) $x_m \geqq v_m^*$, for all $m \in \mathcal{M}$,

(ii) $x_i \geqq v_i^*$*, for all* $i \in \mathcal{N} \setminus \mathcal{M}$*.*

Proof. By Remark 1, any payoff vector which Pareto dominates the (standard) minmax point can be sustained by some subgame perfect equilibrium. Note that, by Lemma 1 (c), the extended reservation value for each mobile player coincides with her reservation value. Hence it remains to prove that each immobile player's modified individually rational payoff is sustainable for sufficiently large δ's.

In case (ii), the extended reservation value for each immobile player coincides with her reservation value of g. Then, Remark 1 directly implies that any modified individually rational payoff for each immobile player is sustained for sufficiently large δ's.

If condition (ii) fails, the extended reservation value for some immobile player $i \in \mathcal{N} \setminus \mathcal{M}$ must be x_i, and it is strictly less than her reservation value v_i^*. To sustain an arbitrary payoff greater than x_i, i must be punished by the immediate game termination once she deviates from the equilibrium path. In so doing, all the mobile players have to choose "end", since we consider the unanimous ending rule ($\theta = M$). As we will show shortly, condition (i) guarantees that such game termination occurs in equilibrium.

In case (i), consider a strategy combination such that if the above immediate game termination does not occur, that is, some mobile player m deviates to choose "continue" when i needs to be punished, then σ^* is played as a never-ending equilibrium. Note that Lemma 3 assures that σ^* can indeed be a never-ending equilibrium. Note also that every mobile player receives v_m^* under σ^* by assumption. Since $x_m \geqq v_m^*$ under (i), each m does not have a strict incentive to deviate, and hence immediate ending can become a subgame perfect equilibrium of the repeated game with endogenous termination. ■

The proof of Theorem 3 relies on the existence of a particular never-ending equilibrium σ^*. Note however that, even if no equilibrium can simultaneously attain the reservation value v_m^* for all $m \in \mathcal{M}$, Lemma 3 (combined with the standard folk theorem) guarantees that there exists a never-ending equilibrium which attains a continuation payoff vector arbitrary close to the (standard) minmax point when a discount factor is sufficiently large.

Now assume that (i)' $x_m > v_m^*$ holds for all $m \in \mathcal{M}$, which is the strict inequality version of condition (i) in Theorem 3. Then, Remark 1 implies that for any payoff vector $\tilde{v}$ such that $\tilde{v}_m \in (v_m^*, x_m)$ for all $m \in \mathcal{M}$, there exists a never-ending equilibrium which attains this payoff vector $\tilde{v}$ when a discount factor is close to 1. Let $\tilde{\sigma}$ denote this equilibrium strategy. In the absence of σ^*, the players can use $\tilde{\sigma}$, instead of σ^*, to achieve an immediate game termination whenever they need to punish an immobile player i by terminating the game.

To sum up the above argument, we establish the following folk theorem.

Theorem 4 *Assume $\theta = M \geqq 2$. For any payoff vector $(v_1, \ldots, v_N) \in \overline{V}^*$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$, if*

(i') $x_m > v_m^*$, for all $m \in \mathcal{M}$,

(ii) $x_i \geqq v_i^*$, for all $i \in \mathcal{N} \setminus \mathcal{M}$.

3.3 Unique Mobile Player ($\theta = M = 1$)

When there is a unique mobile player, i.e., $\mathcal{M} = \{1\}$, it is trivially a unanimous-ending rule. A fundamentally new problem arises that we cannot use the same logic in the previous subsection. Specifically, if $x_1 \geqq v_1^*$, any modified individually rational payoff of (the unique mobile) player 1 under endogenous termination satisfies $v_1 > x_1$, and thus it might be impossible to provide player 1 an incentive to choose "end" to punish an immobile player i with $v_i^* > x_i$. We separate two cases. In the case where the severest punishment for any immobile player can be imposed without terminating the game, the following folk theorem is straightforward.

Remark 2 *Assume $\theta = M = 1$ and $x_i \geqq v_i^*$, for all $i \in \mathcal{N} \setminus \mathcal{M}$. For any payoff vector $(v_1, \ldots, v_N) \in \overline{V}^*$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$,* *there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$.*

Next, suppose that the game termination is needed to punish some immobile player j in the severest way, i.e., $x_j < v_j^*$. In order to induce the unique mobile player to terminate the game, other players need to punish her if the game does not end. However, immobile players who receive the worst possible payoff by game termination never have such an incentive. We divide immobile players into the following two groups.

$$\mathcal{A} = \{i \in \mathcal{N} \setminus \mathcal{M} \mid x_i > v_i^*\} \text{ and } \mathcal{B} = \{j \in \mathcal{N} \setminus \mathcal{M} \mid x_j \leqq v_j^*\}.$$

Note that all immobile players in $\mathcal{B}$ simultaneously receive their severest punishments if the game ends.[26] This means that if the game termination is expected to occur next period, any player $j \in \mathcal{B}$ has no incentive to choose an action which is not one-shot optimal.

Let $br : \times_{i \in \mathcal{A} \cup \{1\}} A_i \to \times_{j \in \mathcal{B}} A_j$ be the $\mathcal{B}$-group's **best reply mapping** such that for any $\mathbf{a}_{-\mathcal{B}} \in \times_{i \in \mathcal{A} \cup \{1\}} A_i$,

$$g_j(br(\mathbf{a}_{-\mathcal{B}}), \mathbf{a}_{-\mathcal{B}}) \geqq g_j(a_j, br(\mathbf{a}_{-\mathcal{B}})_{-j}, \mathbf{a}_{-\mathcal{B}}) \text{ for any } a_j \in A_j$$

simultaneously holds for all $j \in \mathcal{B}$.[27] That is, no immobile player in $\mathcal{B}$ can obtain an immediate deviation gain from the action profile $(br(\mathbf{a}_{-\mathcal{B}}), \mathbf{a}_{-\mathcal{B}})$. To put it differently, for each action profile of non-$\mathcal{B}$ players, $\mathbf{a}_{-\mathcal{B}}$, $br(\mathbf{a}_{-\mathcal{B}})$ essentially solves a Nash equilibrium of the modified game played only by $\mathcal{B}$ players, $g(\cdot, \mathbf{a}_{-\mathcal{B}})$. Since the existence of a Nash equilibrium is guaranteed in our environment, the mapping br must also be well-defined. Because there could be multiple best reply mappings, let

$$v_1^{**}(br) := \min_{\mathbf{a}_{\mathcal{A}} \in \times_{i \in \mathcal{A}} A_i} \max_{a_1 \in A_1} g_1(a_1, \mathbf{a}_{\mathcal{A}}, br(a_1, \mathbf{a}_{\mathcal{A}})), \tag{4}$$

[26] These immobile players are similar to the players who have "equivalent utilities" in the standard repeated games (Abreu, Dutta and Smith (1994) and Wen (1994)).

[27] With a slight abuse of notation, we re-order the action combinations so that j's action is in the first coordinate.

and we define v_1^{**} as follows.

$$v_1^{**} := \min_{br} v_1^{**}(br).$$

Our new concept v_1^{**} extends the effective minmax payoff proposed by Wen (1994), and hence we call it **the extended effective reservation value**.

In the next theorem, we shall provide a sufficient condition for the folk theorem. Our condition requires that x_1 is greater than the one-shot maximal punishment payoff v_1^{**} that other (immobile) players can impose on player 1. Let $\mu^{**} = (\mu_1^{**}, \ldots, \mu_N^{**})$ be the **extended effective minmax actions** against player 1, which achieves v_1^{**}. Note that $v_1^{**} \geqq v_1^*$ is always satisfied.

Theorem 5 *Assume $\theta = M = 1$ and that there exists $i \neq 1$ such that $x_i < v_i^*$. For any payoff vector $(v_1, \ldots, v_N) \in \overline{V}^*$, there exists $\underline{\delta} \in (0,1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player n's average payoff is v_n for all $n \in \mathcal{N}$, if $x_1 \geqq v_1^{**}$ is satisfied.*

Proof. Take any payoff vector $(v_1, \ldots, v_N) \in \overline{V}^*$ and modify the strategy combination in the proof of Theorem 1 at the continuation strategy combination when the mobile player is supposed to choose "end" but deviated to "continue" as follows. Note that $x_1 \geqq v_1^{**}$ implies x_1 $(\geqq v_1^{**})$ $\geqq v_1^*$ and hence $x_1 = \overline{v}_1^* = 0$.

If the unique mobile player 1 deviates and the game continues, the players choose μ^{**}, and the mobile player chooses "end" in the next period. If she further deviates to "continue" in the next period, the players choose μ^{**} and the mobile player chooses "end" in the following period, and so on.

By construction, the mobile player 1 does not have an incentive to deviate from "end", if $x_1 \geqq v_1^{**}$, and hence immediate ending indeed constitutes a subgame perfect equilibrium. When continuation is required, the mobile player 1 does not deviate for sufficiently high δ's, because we can choose $v_1' > 0 = x_1$, when $(v_1, \ldots, v_N) \in \overline{V}^*$. Also by construction, each $i \in \mathcal{B}$ has no incentive to deviate from μ^{**}.

For each $i \in \mathcal{A}$, if he deviates from μ^{**}, go to (B-i) (ii) to minmax player i for τ_i periods, because $v_i^* \ (= 0) < x_i$ for this player. This completes the proof of Theorem 5. ■

Finally, let us show how our v_1^{**} is connected to **the effective minmax payoff** proposed by Wen (1994). In his analysis, the reservation value for player n is modified from the standard reservation value if there exists another player j who has equivalent utilities to n.[28] Suppose that players n and j have equivalent utilities. Then, by definition, it is impossible to punish or reward n alone, separately from j, which implies that we cannot provide any incentive to player j to punish n. Consequently, the standard minmaxing definition must be modified accordingly. Let I_s be the set of players who have the equivalent utilities to n. Wen (1994) defines the effective minmax payoff of player n in game g (which we denote by v_n^W) as follows.

$$v_n^W := \min_a \max_{j \in I_s} \max_{a_j} g_n(a_j, a_{-j}). \tag{5}$$

Let a^W be the corresponding action profile that achieves v_n^W. Since all players in I_s have the equivalent utilities, they simultaneously take their best replies under a^W. Therefore, Wen's effective minmax payoff can be regarded as a special case of our extended effective reservation value. In fact, our definition (4) reduces to the definition (5) when we replace $\mathcal{A}$ with I_s and the unique mobile player with n.

4 Concluding Remarks

We have established various folk theorems when the repeated game can be strategically terminated by players. Theorem 1 is general in the sense that it does not depend on the stage games and holds for all majority rules except for the unanimous ending rule. Under the unanimity rule, the folk theorem may fail (Theorem 2) but we provide sufficient conditions (Theorems 3-5 and Remark 2). Table 1 summarizes our results.

[28] Players n and j have the equivalent utilities if one player's payoff is a positive affine transformation of the other's.

	Environment		folk theorem?
Th. 1	$1 \leqq \theta \leqq M-1$		Yes
Th. 2	$\theta = M$	$\exists m \in \mathcal{M};\ \ x_m < v_m^*$, $\exists i \in \mathcal{N} \setminus \mathcal{M};\ \ x_i = \overline{v}_i^* < v_i^*$	No
Th. 3	$\theta = M \geqq 2$	$\exists \sigma^*$ of g^∞ (i) $\forall m \in \mathcal{M},\ x_m \geqq v_m^*$ (ii) $\forall i \in \mathcal{N} \setminus \mathcal{M},\ x_i \geqq v_i^*$	Yes
Th. 4	$\theta = M \geqq 2$	(i') $\forall m \in \mathcal{M},\ x_m > v_m^*$ (ii) $\forall i \in \mathcal{N} \setminus \mathcal{M},\ x_i \geqq v_i^*$	Yes
Rem. 2	$\theta = M = 1$	$\forall i \in \mathcal{N} \setminus \mathcal{M},\ x_i \geqq v_i^*$	Yes
Th. 5	$\theta = M = 1$	$\exists i \in \mathcal{N} \setminus \mathcal{M};\ x_i < v_i^*$ $x_1 \geqq v_1^{**}$	Yes

Table 1: Summary of the results

An interesting future extension is to allow x_n to depend on the history of the game. On one hand, $x_n(h_t)$ may become large as player n's accumulated payoff up to period t increases, that is, a good performance at the current position brings her a better outside offer. On the other hand, $x_n(h_t)$ may depend on how the game is terminated or the number of mobile players who choose "end". Consider a cartelized oligopoly market where the leniency program is implemented. If a single firm terminates the game (a cartel arrangement) by reporting to the regulatory authority, this firm gets a relatively high after-game payoff through the amnesty rule. If multiple firms report simultaneously, each one gets a low after-game payoff in expectation, since the fine reductions depend on which firm reports first, second and so on.[29]

However, the linkage between play paths and after game payoffs would make it difficult to derive the extended reservation value explicitly. Specifically in the first extension, the recursive structure is lost, which essentially changes our dynamic game. Nonetheless, if the extended reservation value for each player is well-defined, we expect that a similar analysis to this paper is possible.

[29] The details of amnesty rules depend on countries and regions. The immunity of the fine accrues only to the first reporter in the U.S. In EU and Japan, the second and third firms also get reductions of fines.

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Appendix A: When There Are only Two Players

Our Theorem 1 imposes the dimensionality condition, but this can be dropped when there are only two players. The key insight, originally shown by Fudenberg and Maskin (1986), is that two players can take a simple punishment with "mutual minmax actions", which does not require the dimensionality condition, i.e, the dimensionality of $\overline{V}^*$ equals 2. The next theorem establishes the corresponding result in our environment. The additional assumption $M = 2$ is imposed, since the folk theorem fails when $\theta = M = 1$ and the immobile player 2 has $x_2 < v_2^*$, by Theorem 2.

Theorem 6 *Assume $N = M = 2$. Then, for any $(v_1, v_2) \in \overline{V}^*$, there exists $\underline{\delta} \in (0, 1)$ such that, for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player m's average payoff is v_m for $m = 1, 2$.*

Proof. By Lemma 1, (i) $\overline{v}_m^* = x_m$ if $\theta = 1$ $(< M)$, and (ii) $\overline{v}_m^* = v_m^*$ if $\theta = 2$ $(= M)$.

Recall that player m's maximum possible payoff is denoted by $\hat{v}_m = \max_{a_1,a_2} g_m(a_1, a_2)$. Let $\alpha = (\alpha_1, \alpha_2) \in \Delta A$ be a target action profile so that $g(\alpha_1, \alpha_2) = (v_1, v_2) \in \overline{V}^*$.

$\boxed{\text{Case (i): } \theta = 1}$ This is the case in which each mobile player can unilaterally terminate the game. Below we show that the following strategy starting at (A) for each $m \in \{1, 2\}$ attains $(v_1, v_2) \in \overline{V}^*$, for large enough δ.

(A) Choose "continue" and then play α_m each period as long as (α_1, α_2) was played last period. After any unitary deviation from (A), go to (B).

(B) Choose "end".

By Lemma 2, immediate ending prescribed in (B) is incentive compatible, and it is the severest punishment for both players, since $\overline{v}_m^* = x_m$. Therefore, we only need to check deviation incentives from (A). If player m switches from "continue" to "end", she clearly becomes worse off since her lowest possible payoff (x_m) immediately realizes but no deviation gain arises by so doing. If she switches from α_m to any other action and then conforms, she receives at most $\hat{v}_m$ in that period, then 0 due to immediate ending; her total payoff is no greater than $\hat{v}_m$. If she conforms throughout, she obtains $v_m/1-\delta$, which converges to positive infinity as δ tends to 1. Thus, she does not have an deviation incentive for all δ larger than some $\underline{\delta} < 1$.

$\boxed{\text{Case (ii): } \theta = 2}$ This is the case in which no single mobile player can terminate the game. By Lemma 3, any feasible and individually rational payoff vector can be sustained, without the assumption on the dimensionality (by Fudenberg and Maskin (1986)), and the individual rationality coincides with the extended individual rationality.[30] ■

Appendix B: Alternative Set of Feasible Payoff Vectors

In the main text, we defined the feasible payoff vectors as those attainable by playing the stage game, $V = Conv(g(A_1 \times \ldots \times A_N))$. This was because we wanted to investigate sustainability of the "ordinary repeated game outcomes" under the endogenous termination. An alternative definition of feasibility is to include the termination payoff vector, as follows.

$$\hat{V} = Conv(\{(x_1, \ldots, x_N)\} \cup g(A_1 \times \ldots \times A_N)).$$

This is the set considered in the theoretical hypotheses of Wilson and Wu (2017)[31], and by public randomization before playing the game, any payoff vector in $\hat{V}$ is feasible in the repeated game with endogenous termination.

However, as the repeated game with endogenous termination is *not equivalent to* the

[30] Under the conditions of Theorem 6, we do not need to exclude Pareto inefficient vectors from $V^* > 0$. In (i), any feasible payoff vectors in $V^* > 0$ strictly dominates the equilibrium payoff of immediate-ending, i.e., 0, and hence, any Pareto inefficient payoff vectors in $V^* > 0$ can be sustained. In (ii), V^* is defined independently from x_m's, so the downward sloping part in Figure 3 does not exist.

[31] They restrict attention to Prisoner's Dilemma.

repeated game with the extended payoff function $\bar{g}$, the modified individually rational payoff vectors in $\hat{V}$ are not the correct "target" set for the folk theorem. This is because termination is the absorbing state, and if termination is not an equilibrium outcome, it is impossible to sustain a payoff vector which involves $(x_1, \ldots, x_N)$, i.e., the above "extended feasibly" does not expand the set of equilibrium payoff vectors.

Remark 3 *Assume that* $(x_1, \ldots, x_N) \notin V = Conv(g(A_1 \times \ldots \times A_N))$. *For any payoff vector* $(v_1, \ldots, v_N) \in \hat{V}$ *such that* $(v_1, \ldots, v_N) = \alpha(x_1, \ldots, x_N) + (1-\alpha)(w_1, \ldots, w_N)$ *for some* $\alpha \in (0, 1)$ *and* $(w_1, \ldots, w_N) \in V$ *but termination at* $t = 0$ *is not an equilibrium outcome, then* $(v_1, \ldots, v_N)$ *cannot be sustained by a subgame perfect equilibrium, for any* δ.

Proof. Suppose that $(v_1, \ldots, v_N)$ is sustained by a subgame perfect equilibrium σ. Then termination must occur on the equilibrium path of σ with a positive probability after some on-path history h. Since the termination payoffs $(x_1, \ldots, x_N)$ do not depend on the history, the players can reproduce the continuation strategy $\sigma \mid_h$ at $t = 0$ (after the null history). Therefore, if termination after a (public) history h is an equilibrium outcome, then termination in $t = 0$ is an equilibrium outcome, a contradiction. ■

This result holds for any public history model, if the termination payoffs are history independent.

At least, we can show an extension of the folk theorem when termination at $t = 0$ is an equilibrium outcome. Let

$$\hat{V}^* := \left\{ (v_1, \ldots, v_N) \in Conv(\{(x_1, \ldots, x_N)\} \cup \overline{V}^*), \mid v_n > 0 \text{ for all } n \in \mathcal{N} \right\}.$$

Corollary 1 *Assume that the assumptions of Theorem 1 hold (thus immediate termination at* $t = 0$ *is an equilibrium outcome). For any payoff vector* $(v_1, \ldots, v_N) \in \hat{V}^*$*, there exists* $\underline{\delta} \in (0, 1)$ *such that, for any discount factor* $\delta \in (\underline{\delta}, 1)$*, there is a subgame perfect equilibrium of the repeated game with endogenous termination in which player* n*'s average payoff is* v_n *for all* $n \in \mathcal{N}$.

Proof. Take an arbitrary $(v_1, \ldots, v_N) \in \hat{V}^*$. Then there exists a weight $\alpha \in [0, 1]$ and a payoff vector $(w_1, \ldots, w_N) \in \overline{V}^*$ such that $(v_1, \ldots, v_N) = \alpha(x_1, \ldots, x_N) + (1 - \alpha)(w_1, \ldots, w_N)$. Since $(w_1, \ldots, w_N) \in \overline{V}^*$ is sustainable for sufficiently large δ's by Theorem 1, consider a strategy profile such that at $t = 0$, players randomize immediate termination at $t = 0$ with probability α and the equilibrium strategy profile to sustain $(w_1, \ldots, w_N) \in \overline{V}^*$ with probability $1 - \alpha$. Then this publicly randomized strategy profile is a subgame perfect equilibrium for the same δ's as those which make $(w_1, \ldots, w_N)$ an equilibrium payoff vector. ■