Folk Theorem and, in particular, at the thresholds where cooperation is more beneficial than defection in the game of a Prisoners' Dilemma.

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SUMMARY

ACKNOWLEDGEMENTS

Contents

List of Figures

Chapter 1

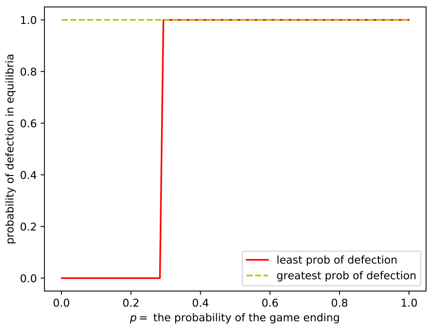
Analyses

In this chapter, an analysis of the data collected via the methods described in the previous chapter (Chapter ??) is given. Firstly, a brief initial overview is provided, where descriptive statistics of the equilibria obtained and the overall characteristics of the strategies used is discussed. Following this, a critical analysis of the p-thresholds obtained is carried out. Here, the environmental effects, on the outcome of the game, discussed include: number of opponents and level of added noise. Note, as of writing, the database currently has 825700 entries (rows) and a total number of 159 tournament sets.

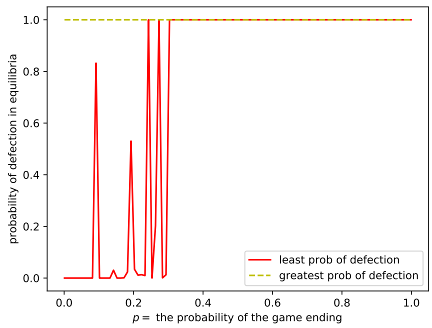
1.1 Initial Analysis

In this section, all the data (including those games which could be degenerate) are considered. Taking a brief look at the graphs produced for each set of tournaments, it can be seen that the main 'shapes' obtained are as seen in Figure 1.1.

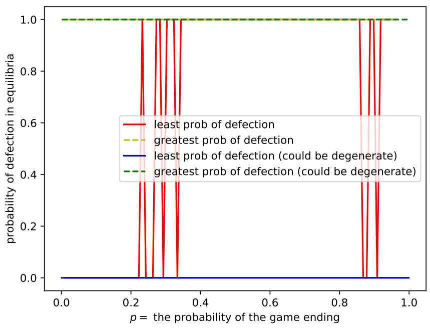
In Figure 1.1a, a clear p-threshold of approximately 0.28 is apparent, clearly visualising the Folk Theorem. In this tournament set the opponent was *Inverse*, a stochastic player, indicating that perhaps the stochasticity of a player does not affect the accuracy of the threshold as first thought. In Figure 1.1b, the precise value of the p-threshold is less clear, lying approximately in the range [0.1, 0.3]. This could be due to the potential degeneracy identified or just the element of randomness that appears within each tournament, possibly magnified by the number of stochastic players. The opponents within this set were: *Feld: 1.0, 0.5, 200*; *Cooperator*; *EvolvedLookerUp2_2_2*; *Tullock: 11*; and *ZD-GEN-2: 0.125, 0.5, 3.* Figure 1.1c shows a potential problem with the visualisation of the Folk Theorem when degenerate games are involved. It becomes unclear as



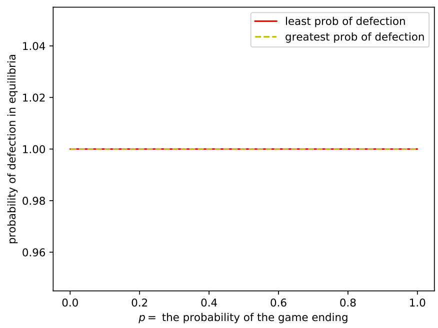
(a) 2-player tournament set with no added noise, one stochastic player and no degeneracy identified.



(b) 6-player tournament set with no added noise, three stochastic players and potential degeneracy was yielded from 10 of the tournaments.



(c) 5-player tournament set with no added noise, two stochastic players and five tournaments played yielded potentially degenerate games.



(d) 4-player game with no added noise, one stochastic player and no degeneracy identified.

Figure 1.1: Example graphs obtained from the experiment.

to what is happening in the graph, especially where the p-threshold lies. The opponents, in this case, were: *Random: 0.5*; *Grumpy: Nice, 10, -10*; *Fortress3*; and *Negation*. Finally, Figure 1.1d gives an example of a tournament set for which there was always a non-zero probability of defection, regardless of the game ending probability. In this case, the precision of game-ending probabilities chosen was not accurate enough to identify the p-threshold. It is implied that the tournament has to have an ending probability of almost 0 (within the interval (0, 0.001) in order for a zero probability of defection to be rational. Here, the opponents of the *Defector* were: *AntiCycler*; *e*; and *Stalker: (D,)*. Similarly, it is observed that some graphs obtained were constant at 0, again indicating that the precision of game-ending probabilities were not fine enough to highlight the p-threshold. This indicates that the ending probability of the tournament has to lie within the interval (0.999, 1). That is, almost immediately, the decision to defect is no longer rational.

The summary statistics gained from running the *describe* method of a pandas database is given in Table **??**. From this, it can be seen that the number of opponents the *Defector* played against ranged from one to seven, with an average of four opponents. Also, as expected, the mean probability of the game ending encountered was 0.5. Observe that, overall, there were $175,399$ distinct tournaments played with a total of 159 distinct sets of strategies. Looking now at the statistics for Nash equilibria, it can be seen that a total of $823,823$ equilibrium points were calculated in this experiment, with an average of $1.914 \approx 2$ equilibria per game. However, observe, at least one game obtained 39 equilibria which will be explored into later on in this section. Considering the probabilities of defection within these equilibria, notice that both the greatest and the least probabilities of defection ranged from zero to one inclusive with a 50th percentile of zero. But, looking at the average values, the least probability has a mean of 0.342 and only just above this, the greatest probability has a mean of 0.460.

Next, further descriptive statistics are calculated for the strategies. This is to obtain a more in-depth view on the types of strategies randomly chosen to play and their characteristics. Executing *value_counts* method on the column of strategy names, it is observed that the player which appeared the most times (9 times) is *ZD-GEN-2: 0.125, 0.5, 3*; followed closely by *Tideman and Chieruzzi* with 7 sets of tournaments. On the other hand 38 out of the 200 strategies playing in this experiment appeared only once. Running the *value_counts* method again, but this time on the memory depths of the strategies found the majority of strate-

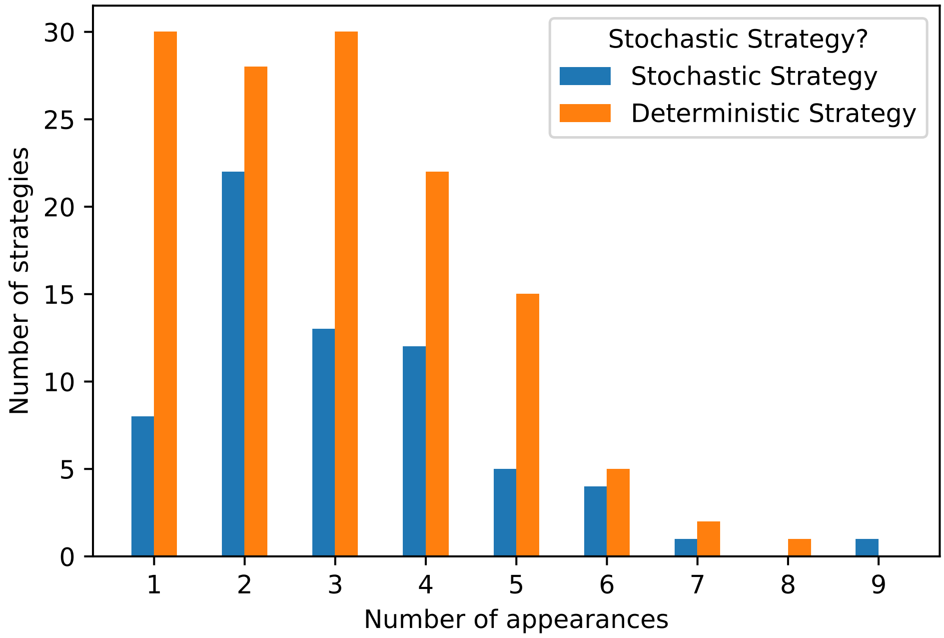


Figure 1.2: A plot to show the ratio of stochastic to deterministic strategies randomly chosen throughout the experiment.

gies to have an infinite memory depth. On the other hand, strategies having no memory or a depth equal to one were also significant. Considering the stochasticity of players alongside how many appearances each strategy made yielded the following chart in Figure 1.2.

It is clear that there is a clear bias towards deterministic strategies in this experiment. However, this is to be expected as running the following code:

```
len(axl.filtered_strategies(filterset={"stochastic": True})),
len(axl.filtered_strategies(filterset={"stochastic": False}))

(86, 156)
```

it can be seen that over half the strategies coded into the Axelrod library are classed as deterministic. Looking at Figure 1.2 again, observe, the majority of deterministic strategies were executed either once or three times. On the other hand, a large proportion of the stochastic strategies were played twice.

Further, the number of Nash equilibria obtained for each game was analysed and their distributions with respect to the number of opponents of the *Defector* plotted. Executing the *value_counts* method on the 'num_of_equilibria' column gave the conclusion that the majority of games (131773) yielded one Nash equilibria with 28793 games obtaining 3 equilibria. Also, the maximum number of equilibria yielded, 39, was for one game, which, doing a search on the database, was found to be a six player game with noise=0.1. The opponents of the *Defector* were: *Inverse Punisher*; *Prober*; *PSO Gambler 2_2_2 Noise 05*; *Handshake*; and *More Tideman and Chieruzzi*. This game was expected to be degenerate however, when taking a closer look at this entry in the database, no degeneracy was identified. This could be worth looking at in further investigation of the work.

Figure 1.3 shows the distributions of the number of Nash equilibria as per the number of players. This visualisation turns out to not be extremely revealing - possibly as an effect of degeneracy - however some insights can be found here. Observe, all of the distributions observed on this plot have a clear modal value of one, that is, irrespective of the number of players, the majority of games yielded only one equilibria. Moreover, there also seems to be an increase in density around 3 equilibria which becomes more prominent as the number of players increases. As can be seen from the plot, the variance in the number of equilibria increases with the number of players, apart from when there were 6 players (5 opponents of the defector), where the spread is maximum. This could be due to the 39 equilibria gained for one game as discussed in the paragraph above. Looking now at the mean of the distributions, observe that these are also slightly increasing as the number of players increase.

1.2 Analysis of the p-Threshold

Firstly, for clarity, here is a restatement of the definition of a *p-threshold*: The probability of the tournament ending for which the least probability of defection in Nash equilibria is non-zero.

In order to analyse the p-thresholds of the tournaments, a csv file was created [1] containing the minimum, mean, median and maximum probabilities for each set of tournaments. This was in order to gain as much information as possible from tournaments which gave graphs such as Figure 1.1b as described above. Within this file, other than the varying thresholds, the information about the number of players, tournament strategy and level of noise was retained. Moreover, it contained a column which identified whether any of the strategy sets lead to possible degenerate games.

Now, an exploration into the overall p-thresholds is given.

[1] Please see Appendix **??** for the code used to obtain these files.

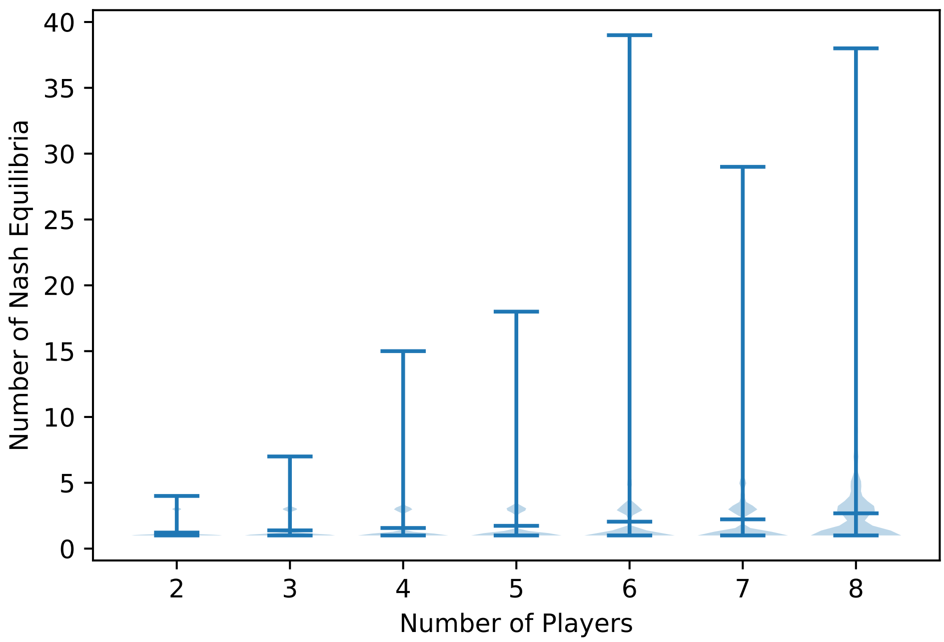
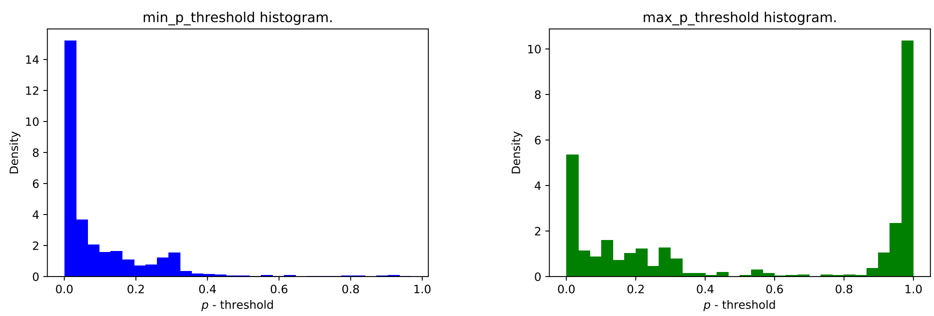


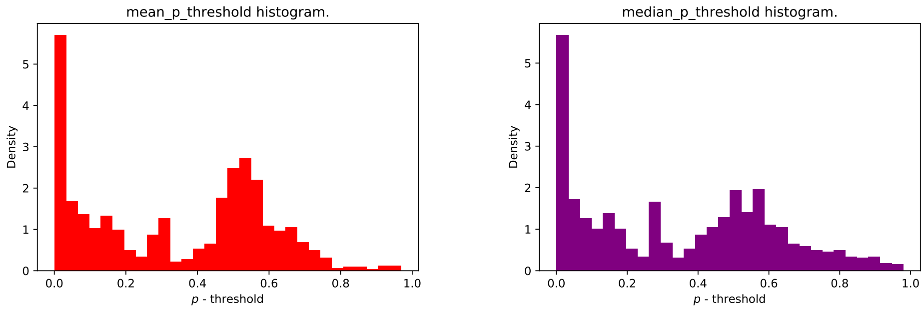
Figure 1.3: A violinplot showing the distribution of the number of equilibria obtained for varying number of players.



(a) A plot to show the minimum p-thresholds.

(b) A plot to show the maximum p-thresholds.

Figure 1.4: Plots to show the p-thresholds for all 1001 sets of tournaments.



(a) A plot to show the mean p-thresholds.

(b) A plot to show the median p-thresholds.

Figure 1.5: Plots to show the p-thresholds for all 1001 sets of tournaments.

Observe, in Figure 1.4a, the majority of minimum thresholds were less than 0.4, with a clear modal value of 0. That is, in a significant proportion of the tournaments, there was no probability of the game ending for which the probability of defection was zero. Now considering Figure 1.4b, it can be seen that the modal value for the maximum threshold is 1. Also, in comparison with the minimum threshold, there is a larger spread in the density. Yet, there is still a peak around zero as with the minimum threshold data.

Looking at the mean and median threshold data, Figure 1.5, observe that the distributions obtained are similar with, again, a modal value of zero, indicating there is always a chance of defection irrespective of the game ending probability. However, in these histograms there is also a clear peak around a threshold of 0.5. This suggests that some of the tournaments may have several p-thresholds, perhaps due to stochasticity, degeneracy or just inevitable randomness that appears in the tournaments. Indeed, obtaining the minimum and maximum p-thresholds for those tournaments where the mean threshold was within the range $[0.5, 0.6]$, it can be seen that, from Figure 1.6 for a significant proportion, their minimum threshold was around zero and their maximum around one. Moreover, looking closer at three of the tournaments which satisfied the above, there is a clear randomness within the corresponding graphs, Figure **??**.

The plots contained in Figure 1.7 were sampled randomly using the random library in Python. What is interesting here is that they all contain varying amounts of additional noise and further analysis into this would be beneficial in seeing if this is one of the main causes for the inaccuracy of the thresholds. For example, looking at how many of the tournaments with this property had an added noise level.

Before continuing onto the general overview of the thresholds for those tourna-

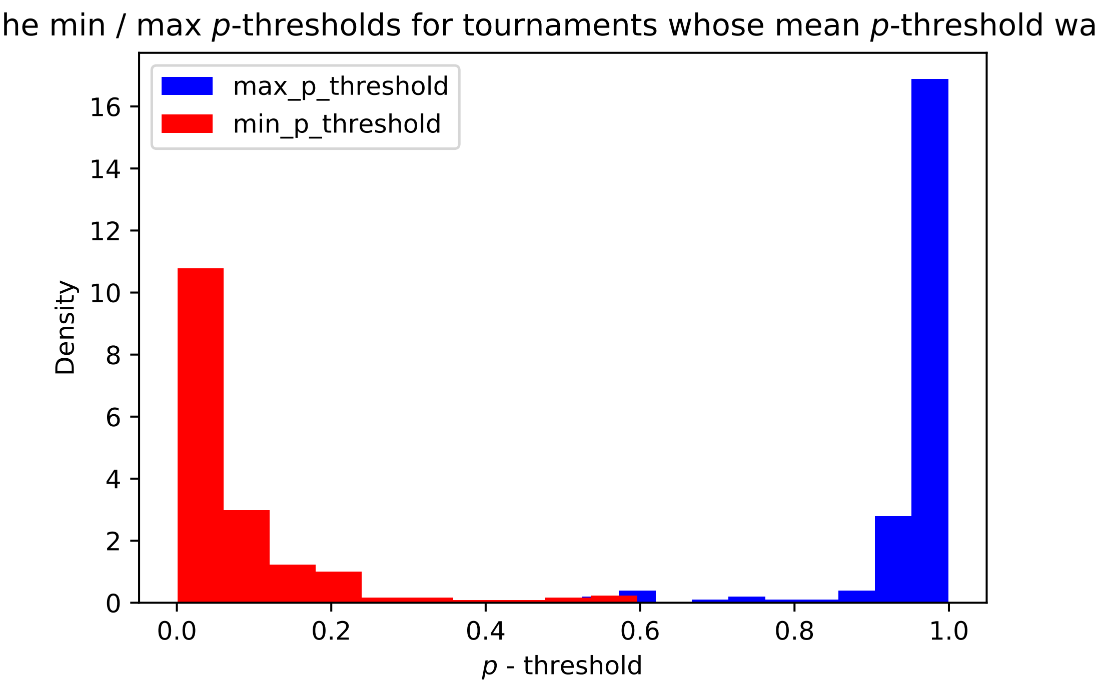
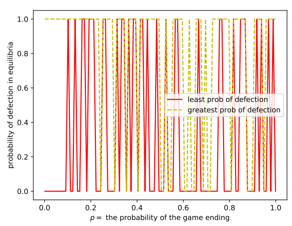
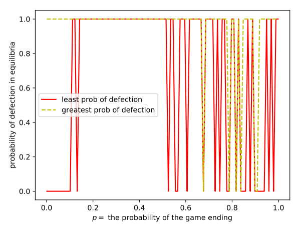


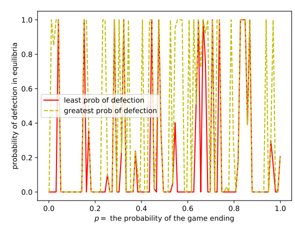
Figure 1.6: A plot to show the minimum and maximum p-thresholds for those tournaments which had an mean threshold within the range $[0.5, 0.6]$.



(a) 6-player tournament set with an additional noise level of 0.2, two stochastic players and no degeneracy was identified.

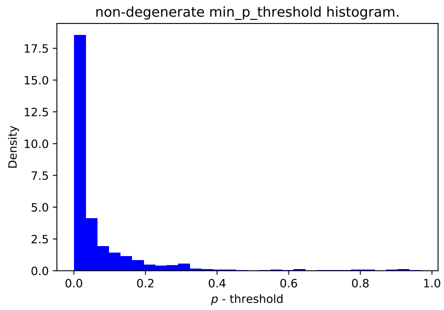


(b) 3-player tournament set with an additional 0.1 of noise, one stochastic player and no degeneracy was identified.

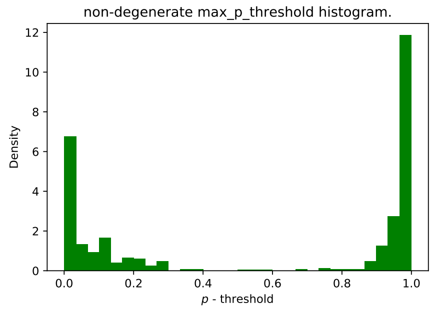


(c) 3-player tournament set with an additional noise level of 0.5, one stochastic player and only one tournament yielded potential degeneracy, with a game-ending prob of 0.9788383838384.

Figure 1.7: Example plots of the tournaments where the mean p-threshold was within the range $[0.5, 0.6]$.

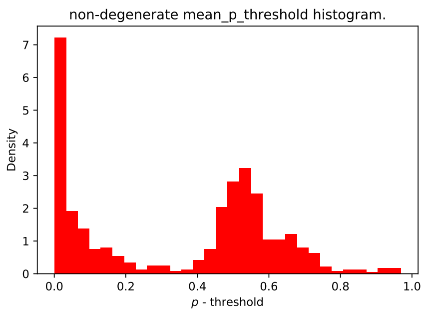


(a) A plot to show the minimum p-thresholds.

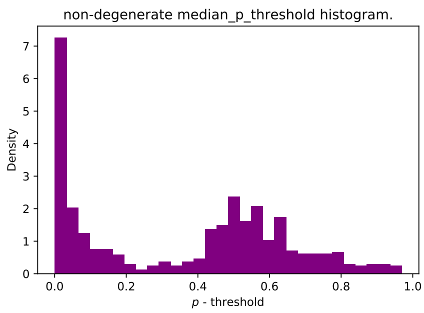


(b) A plot to show the maximum p-thresholds.

Figure 1.8: Plots to show the p-thresholds for all tournaments which led to non-degenerate games.



(a) A plot to show the mean p-thresholds.



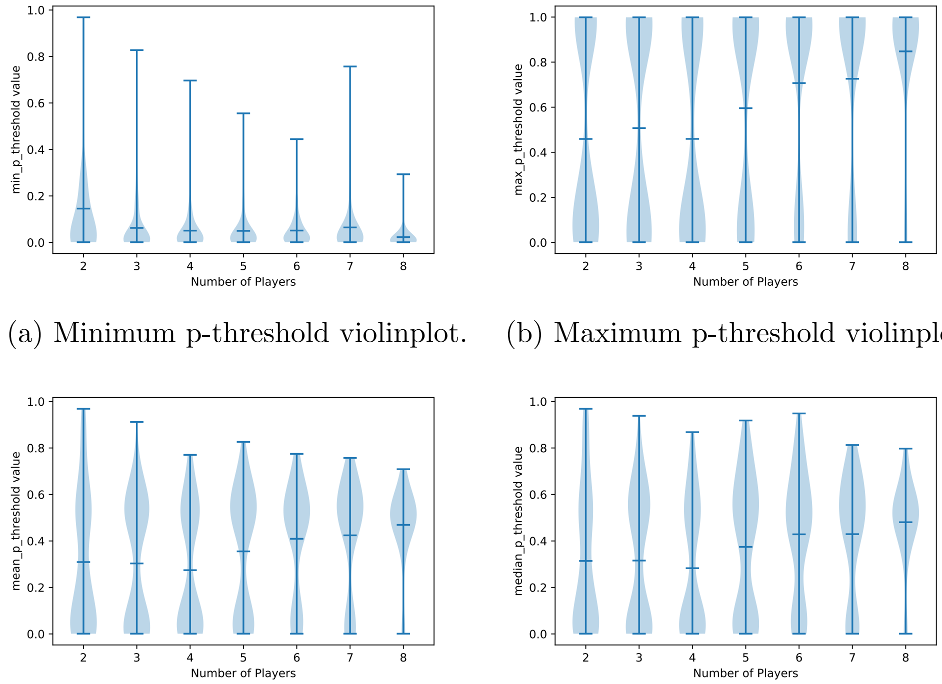
(b) A plot to show the median p-thresholds.

Figure 1.9: Plots to show the p-thresholds for all tournaments which led to non-degenerate games.

ments which led to definite non-degenerate games, a brief point is made regarding those tournaments which, for all probabilities of the game ending, had no positive probability of defection. Indeed, out of the 1754 total tournaments 753 of them had the above property of no apparent threshold (that is, lie within the unobserved interval of (0.999, 1)). That is, the plots obtained for these tournaments were (approximately). Further investigation into these tournaments, as well as those in which the probability of defection was always positive, is highly recommended.

Regarding degeneracy, out of all 1754 tournaments, 372 were highlighted as potentially leading to degenerate games. These are omitted from the following plots in order to focus solely on non-degenerate games.

Comparing Figures 1.8a, 1.8b, 1.9a and 1.9b with Figures 1.4a, 1.4b, 1.5a and 1.5b, respectively, it can be seen that, in general, there is no significant change in the



(a) Minimum p-threshold violinplot. (b) Maximum p-threshold violinplot.

(c) Mean p-threshold violinplot. (d) Median p-threshold violinplot.

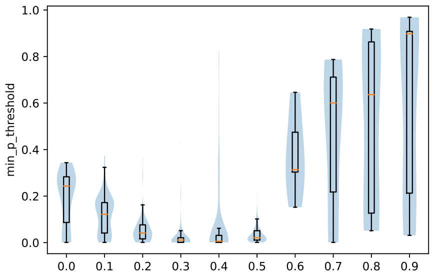
Figure 1.10: Violinplots of the thresholds for each number of opponents.

distributions of the thresholds. But, there is a more prominent peak in Figure 1.5a around 0.3 than in the corresponding non-degenerate plot of Figure 1.9a. Further work regarding the effects of degeneracy is advised as the above discussion seems to indicate that degeneracy does not seem to have as much of an affect on the thresholds as initially expected.

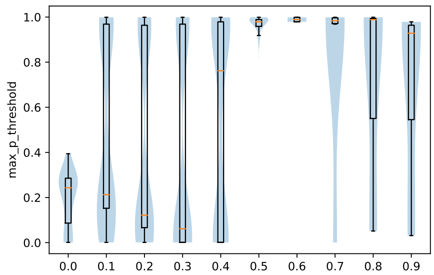
1.2.1 Effects of the Number of Players

In this section, the p-thresholds will be analysed with respect to the number of opponents the *Defector* played against. Note, in this section, only non-degenerate tournaments will be considered.

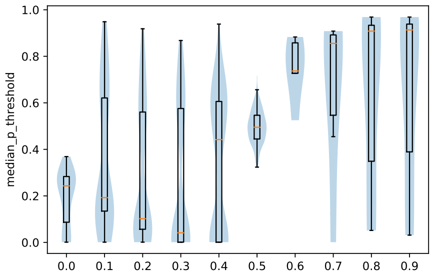
From Figure 1.10c, it can be seen that the distributions of the minimum p-thresholds with respect to the number of players all have a modal value around 0. However, apart from the 7-player tournaments, the spread of the distributions decrease, along with the mean values. Considering the maximum thresholds, Figure **??**, the distributions become bimodal with mode values of around 0 and 1. But, as the number of players increases the modal value at zero becomes less prominent with the 8-player tournament distribution not having a mode around 0. The variance of the distributions are similar and, apart from 4-player tournaments, the means increase with the number of players. Looking now to the



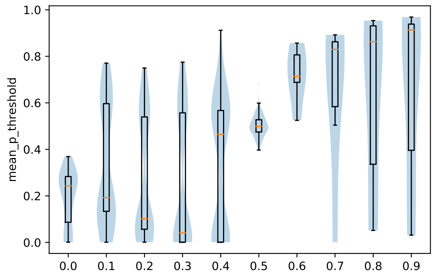
(a) Minimum p-threshold violinplot.



(b) Maximum p-threshold violinplot.



(c) Median p-threshold violinplot.



(d) Mean p-threshold violinplot.

Figure 1.11: Violinplots of the thresholds for each level of noise.

mean and median violinplots, Figures 1.10c and 1.10d respectively, there is no significant difference between the two plots other than the spread is a little larger in the median thresholds. Within the mean plot, Figure 1.10c, it can be seen that the distributions also start off bimodal, at 0 and approximately 0.5. But again, the modal value at zero becomes less distinct with the 8-player tournament distribution being unimodal. The modal value at around 0.5 is a consequence of the reason stated in the previous section. Moreover, observe that, apart from 4-player tournaments, the variance of the distributions seem to decrease with the size of the player set whilst the means increase from a value of approximately 0.3 for 2-player tournaments to around 0.5 for 8-players. Looking at Figure 1.10 as a whole, it is implied that the number of players has no significant effect on the value of the p-threshold.

1.2.2 Effects of Noise

Here, an analysis on the effects of noise on the p-threshold is provided. The addition of noise to a tournament indicates that, with a certain probability, the action of a particular strategy is altered [1]. That is, an action of *coop* changes to *defect* and vice versa.

Figure 1.11a, shows the distribution of the minimum p-thresholds for each level of additional noise added to the tournaments. Here, it can be noted that the distributions of noise levels at least 0.6 have a large variance, indicating that adding a too large noise probability is highly random and thus no conclusions can be drawn. On the other hand, considering the noise levels less than 0.6, observe that the mean p-thresholds decrease from around 0.25 to approximately 0 as the noise increases. These distributions are also clearly unimodal. Looking now at Figure 1.11b, it can be seen that the majority of distributions have a much larger spread here varying over the full spectrum of game-ending probabilities and similar observations can be made from Figures 1.11c and 1.11d.

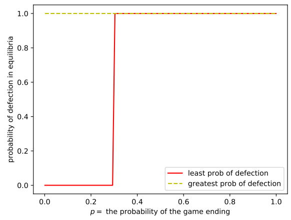
Figure 1.12 shows an example of the same tournament set through a variety of differing noise levels. Indeed, here, it can clearly be seen that the amount of noise does effect the p-threshold. However, this is to be expected since, by definition, as the noise level increases, the *Defector* will be observed as similar to the *Cooperator* when there is no additional noise.

Therefore, on the whole, there are not many significant conclusions that can be made here. This implies that the addition of noise to an already random tournament obscures any possible visions of the threshold, especially as the magnitude increases.

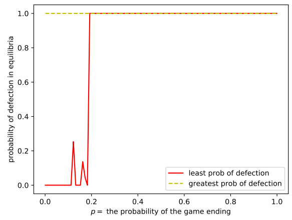
1.3 Conclusions and Further Work

In this section, the beginnings of an analysis into the p-thresholds was discussed. The effects of the number of players and level of additional noise on these thresholds were the main focus with a brief discussion regarding the degeneracy of games also given. This turned out to be a non-trivial task due to the proportion of tournament sets which yielded potential degeneracy at certain game ending probabilities and the inevitability of randomness within the tournaments. However a few points of interest were highlighted and these are summarised here.

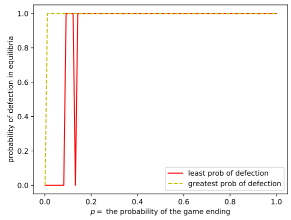
Firstly, the potential degeneracy of games yielded by the tournaments, which was highlighted as a potential factor, at first glance appears to not create as much of an issue as originally thought. The histograms of the p-thresholds, when the tournaments including potential degeneracy were omitted, did not have any significant changes when compared to the original histograms of all tournaments. However, further exploration is advised here. Regarding the number of players in a tournament, it was initially hypothesised that this would be a key factor in the variability of the p-threshold. However, on obtaining the distributions of the thresholds for tournament sets of size one to eight, it was implied that



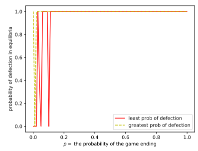
(a) 6-player game with no additional noise.



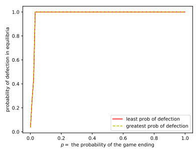
(b) 6-player game with an additional noise level of 0.1.



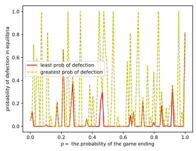
(c) 6-player game with noise level 0.2.



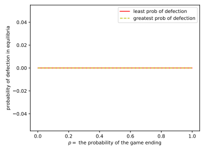
(d) 6-player game with noise level 0.3.



(e) 6-player game with an additional noise level of 0.4.



(f) 6-player game with noise level 0.5.



(g) 6-player game with an additional noise level of 0.6. Note, the remaining noise levels of 0.7, 0.8, 0.9 and 1.0 all yielded the same constant 0 graph as represented here.

Figure 1.12: Observation of one 6-player tournament set through the varying levels of additional noise. There was one stochastic player and 13 out of the 1100 tournaments played yielded potential degenerate games. The opponents here were: *Getzler*; *Punisher*; *Forgiver*; *GrudgerAlternator*; and *GraaskampKatzen*.

the number of players does not have a significant impact on the value of the threshold. Finally, the effect of additional noise on the p-threshold was analysed. Here, it was observed that, as expected, the level of additional noise did affect the p-threshold however there was no significant trends appearing out of the randomness.

As stated above, this is only the very start of an analysis into the p-thresholds described by the 'original' Folk Theorem and the effects of the varying environmental factors which appear in tournaments of the IPD. Thus many questions regarding this are still to be researched and a few recommendations regarding further work are now given. Firstly, it was observed that there were a significant proportion of tournaments for which the graph remained constant at zero or one. That is, the p-threshold was not identified using the precision of game-ending probabilities chosen and therefore must lie within the intervals (0, 0.001) or (0.999, 1), respectively. Hence, these tournament sets could be rerun with a much finer precision within the appropriate intervals to highlight exactly what is happening here. Also, an analysis into the characteristics of the strategies involved and the additional noise levels included in the tournaments could provide a clearer insight into potential reasons for this. Moreover, with regards to analysing the characteristics of players, it is suggested that those tournament sets in which stochastic players were included could be removed and the tournaments rerun. This could help in revealing whether the stochasticity of the player has any effect on the threshold, as the author has hypothesised.

To check the reliability of the data collected, a second experiment is recommended using a different algorithm for calculating the Nash equilibria, for example vertex enumeration. This could be used in comparison with the data already collected to identify whether the algorithms are producing the same Nash equilibria or, more importantly, whether they identified the same games as being degenerate. Furthermore, it is suggested that this experiment be executed with a larger number of tournaments repeats (greater than 500) to observe whether this 'smooths' the payoff matrices with greater success to enable for a clearer visualisation of the p-thresholds. Finally, some multivariate data analysis of the results, for example regression, could provide some more insights into this topic.

Overall, this chapter has been successful in visualising the Folk Theorem using the data collection setup as explained in Chapter **??** and using the plots obtained as in Figure 1.1a.

References

[1] Nikoleta E. Glynatsi and Vincent A. Knight. *A meta analysis of tournaments and an evaluation of performance in the Iterated Prisoner's Dilemma*. 2020. arXiv: `2001.05911 [cs.GT]`.