An Empirical Study on the Folk Theorem

Student: Sophie Shapcott

Supervisor(s): Dr Vince Knight and Henry Wilde

Academic Year: 2019/20

School of Mathematics,
Cardiff University

A final year undergraduate project submitted in partial fulfilment of the requirements for MMORS (Masters in Mathematics, Operational Research and Statistics) taught programme.

Summary

"Under what conditions will cooperation emerge in a world of egoists without central authority?" Robert Axelrod provides justification for this study in the first section of his book [6]. Today, many examples can be given in which cooperation has evolved in situations where, in the short term, it may be preferable to counteract. Due to this, a class of theorems have emerged over the past fifty years, providing explanation for the unintuitive phenomena.

This project consists of an empirical study into these theorems, entitled 'Folk Theorems', which are key in the repeated games theory. The aims of the project include: an in-depth review of academic literature regarding the theorems; the execution of a large experiment based on the 'original' folk theorem of Friedman [27] with the Iterated Prisoner's Dilemma; and an analysis of the effects of different tournament characteristics on the p-threshold [1] described in the folk theorems. These ideas are extended from a third year assignment, completed by the author, in Game Theory.

Firstly, after an introduction to the theory of one-shot and repeated games in chapter 1, a search of the folk theorem literature is provided in chapter 2. This reveals the vast directions of research in the area for the past fifty years. Many generalisations and refinements of the folk theorem have been analysed since the first written papers of the 1970s. The games for which the notion has been applied to range from complete information games to games with imperfect private monitoring. However, in certain cases, the strategies used in the proof of these are unstable. Also, in situations where an individual deviator cannot be identified, a much smaller set of payoffs is achieved, yielding the so-called 'Anti-Folk Theorems'. More recently, focus has been on the application of the folk theorem to different scenarios including: computing and quantum transportation. Finally, it is concluded that, to the best of the author's knowledge, this is the fist study to execute an experiment of this size on the folk theorem.

[1] The p-threshold is defined as the probability of the tournament ending for which the least probability of defection in Nash equilibria becomes zero.

Following this, a detailed description of how the experiment was set-up and executed is given in **??**, with justifications to the specific methods and software chosen. After considering the benefits and drawbacks of varying file formats for storing data, it is stated that a relational database, in particular SQLite, would be the most appropriate. This is due to: the existence of libraries in Python enabling easier access of the data; the general robustness of databases; and their ability to perform out of memory operations. Due to the pre-existing game theoretic libraries, Axelrod and Nashpy, Python was chosen as the language of implementation and ensuring good software development principles were followed is highlighted as a priority. The volume of data which was intended to be collected meant that remote computing was required and this is explained here. The choice of support enumeration for the calculation of Nash equilibria and the potential issues which may be faced due to degeneracy is also discussed.

The main analysis of the data collected is provided in **??**. An initial analysis discussing the characteristics of the strategies used and the overall summary statistics is detailed, before the p-thresholds are explored. The tournament characteristics focused on are: the number of opponents the Defector had, and the level of standard Iterated Prisoner's Dilemma noise included. However, this is concluded as a non-trivial task due to the uncertainty of degeneracy. Also, the inevitability of randomness within the tournaments meant that a lot more data than what was obtained is required. Indeed, there are three sources of noise impacting the tournaments. On the other hand, the graphs that were yielded are successful in visualising the folk theorem.

Finally, **??** details the conclusions of the research prior to giving recommendations for future work. Amongst these are suggestions on how further characteristics of the tournament set could be studied, and the potential to predict the p-threshold via regression analysis.

Acknowledgements

First and foremost, I would like to express my sincere thanks and gratitude to my project supervisors Dr Vince Knight and Henry Wilde for their continual advice, support and invaluable ideas throughout the completion of this project. You have always been willing to lend a hand or provide encouragement when it was needed the most. I do not think I could have been given better supervisors and I apologise for not achieving everything in the plan. You have assisted me through what will probably be the largest project I will ever do!

There are many people who have helped and supported me, both inside and outside university, over the past four years of my undergraduate degree here at Cardiff School of Mathematics. If I was to mention all the names, this would be longer than my whole project! Therefore, this 'thank you' is for all those who I have had the pleasure of being taught by and working with during my degree. You have made this journey an interesting and enjoyable one and I will continue to be inspired by the passion all the lecturers have for what they do.

Also, I would like to give special mention to Nikoleta, who, even though is in her final few months of a PhD, found time to talk with me and make sure I was still operating on this planet. You encouraged me and always had a positive thing to say when I was convinced everything was going to end terribly. So thank you very much for being such a great friend and best of luck with the completion of your thesis!

Last but most certainly not least, I want to say a huge thank you to my Mum, Dad and Zoe for their continual love and support. To say these have been a challenging four years is an understatement but you were always there for me through it all. If it was not for your belief that I could achieve it I do not think I could have completed this project, let alone the past four years! My Mum, thank you for keeping me fuelled with coffee and biscuits through the long days and for being a shoulder to cry on when necessary. I will give you enough hair dye to cover all those grey ones I caused! Dad, thank you for making me laugh with your terrible jokes and I might replace all that coffee I had from ‘your’ jar one day. Finally Zoe, (yes, you are getting a mention because, yes, you did play a key part) thank you for keeping me sane and being an awesome sister.

To all those who I have forgotten to mention: my sincerest apologies and thank you for all your support.

Contents

List of Figures

Chapter 1

Introduction

World War I, a time of harsh conflict and battle, provided an example of how cooperation need not evolve from friendship. Indeed, [6] states that small units of common soldiers on the Western Front were able to execute a "live and let live" system, even against the will of the officers. They knew that "if the British shelled the Germans, the Germans replied; and the damage was equal" (quoted from [17] as given in [6]). Moreover, this was achieved without a direct truce as officers forbade it. The analysis of such circumstances, and other situations involving choice, is covered by an area of mathematics entitled *game theory*.

According to [20], *game theory* is the study of interactive decision making and the development of strategies through mathematics. It analyses and gives methods for predicting the choices made by players (those making a decision), whilst also suggesting ways to improve their 'outcome' [52]. Here, the abstract notion of utility is the outcome players wish to maximise. For further information on the topic of utility theory, readers are referred to Chapter 2 in [52] for a detailed discussion or Section 1.3 in [69] for a more introductory explanation.

One of the earliest pioneers of game theory is mathematician, John von Neumann who, along with economist Oskar Morgenstern, published *The Theory of Games and Economic Behaviour* in 1944 [52]. This book [66] discusses the theory, developed in 1928 and 1940, by von Neumann, regarding "games of strategy" and its applications within the subject of economics. Following this, several advancements have been made in the area including, most notably, John Nash's papers on the consequently named Nash Equilibria in 1950-51 [56, 57]. Due to the "context-free mathematical toolbox" [52] nature of this subject, it has been applied to many areas, from networks [51, 64] to biology [2, 15].

In this project, the main focus is a class of theorems within game theory, known

C, C	C, D
D, C	D, D

Table 1.1: Outcomes for a game of the Prisoner's Dilemma.

as "Folk Theorems". These ideas assist in the analysis of long-term behaviour and evolution of cooperative strategies. In particular, the theory will be applied to the game of a Prisoner's Dilemma which is introduced in subsequent sections. The structure of this report is as follows: chapter 2 provides a literature review on the topic of folk theorems, before **??** discusses the development of a large experiment regarding these notions in the Prisoner's Dilemma. **??** analyses the results obtained whilst **??** provides the conclusions and recommendations for further study. However, first, the remainder of chapter 1 is dedicated to the key definitions and theorems required for a study in game theory.

Unless stated otherwise, the definitions and notation in this chapter have been adapted from [52].

1.1 An Introduction to Games

Consider the following scenario:

> Two convicts have been accused of an illegal act. Each of these prisoners, separately, have to decide whether to reveal information (defect) or stay silent (cooperate). If they both cooperate then the convicts are given a short sentence whereas if they both defect then a medium sentence awaits. However, in the situation of one cooperation and one defection, the prisoner who cooperated has the consequence of a long term sentence, whilst the other is given a deal [44, 52].

This is one of the standard games in game theory known as the Prisoner's Dilemma (PD). It has four distinct outcomes, for the given two player version, which can be viewed in Table 1.1. Note, following the standard literature, cooperation and defection is indicated by C and D, respectively.

More formally, the game can be represented as the following matrix:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3)(R,R) & (0,5)(S,T) \\ (5,0)(T,S) & (1,1)(P,P) \end{pmatrix} \end{array} \tag{1.1}$$

where each coordinate (a, b) in the table represents the utility values obtained for each player, where a is the utility value obtained by the row player and b is the utility gained by the column player. These utility values (payoffs) [1] are as given in [7] and used throughout this project. In general, the PD payoffs are constrained by the two conditions:

$$T > R > P > S \tag{1.2}$$

and

$$2R > T + S \tag{1.3}$$

where (1.1) ensures that D is preferable to C and yet (1.1) ensures that mutual C is best [45, 62]. The matrix given in (1.1) is known as a *normal form* representation of the game.

Definition 1.1.1. In general a *normal form* or *strategic form* game is defined by an ordered triple $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where:

- $N = \{1, 2, \dots, n\}$ is a finite set of players;
- $S = S_1 \times S_2, \times \cdots \times S_n$ is the set of strategies for all players in which each vector $(S_i)_{i\in N}$ is the set of strategies for player i [2]; and
- $u_i : S \to \mathbb{R}$ is a payoff function which associates each strategy vector, $s = (s_i)_{i\in N}$, with a utility $u_i(i \in N)$.

Yet another way of representing this game is as a pair of matrices, A, B, defined as follows:

$$A = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \text{ and } B = A^T = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} \tag{1.4}$$

This way of defining games allows for the use of linear algebraic expressions in the calculation of utilities (see section 1.2).

Before continuing the discussion into the key notions of game theory, it needs to be highlighted that there is an important assumption, which is central to most studies of game theory, entitled *Common Knowledge of Rationality*. This, more formally, is a recurring list of beliefs which claim:

- The players are rational;

[1] 'Utility' is referred to as a player's 'payoff' throughout the remainder of this report.

[2] Since the game of the PD has a finite strategy set for each player $S_i = \{\mathrm{C}, \mathrm{D}\}(i = 1, 2, \dots, n)$, in this project only finite strategy spaces are considered.

- All players know that the other players are rational;
- All players know that the other players know that they are rational;
- etc.

Assuming Common Knowledge of Rationality allows for the prediction of rational behaviour through a process known as *rationalisation* [45]. See Section 4.5 in [52] for an alternative explanation of this assumption.

Thus far, only the pure strategies, $S_i = \{C, D\}$, have been discussed, hence the notion of a probability distribution over S_i is now introduced, giving the so-called *mixed strategies*.

Definition 1.1.2. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game, then a *mixed strategy* for player i is a probability distribution over their strategy set S_i. The set of mixed strategies for player i is defined by

$$\Sigma_i = \left\{ \sigma_i : S_i \to [0, 1] \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\} \tag{1.5}$$

Hence, observe that the pure strategies are specific cases of mixed strategies, with $\sigma_i = (1, 0)$ for cooperation and $\sigma_i = (0, 1)$ for defection, in the PD.

This leads onto the following definition of a *mixed extension* for a game.

Definition 1.1.3. Let G be a finite normal form game as above, with $S = S_1 \times S_2 \times \cdots \times S_N$ defining the pure strategy vector set and each pure strategy set, S_i being non-empty and finite. Then the *mixed extension* of G is denoted by

$$\Gamma = (N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N}), \tag{1.6}$$

and is the game in which, Σ_i is the ith player's strategy set and $U_i : \Sigma \to \mathbb{R}$ is the corresponding payoff function, where each $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in \Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_N$ is mapped to the payoff:

$$U_i = \mathbb{E}_\sigma(u_i(\sigma)) = \sum_{(s_1,s_2,\ldots,s_N)\in S} u_i(s_1, s_2, \ldots, s_N)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_N(s_n) \tag{1.7}$$

for all players $i \in N$.

1.2 Nash Equilibrium for Normal Form Games

As mentioned above, mathematician, John Nash, introduced the concept of an equilibrium point and proved the existence of mixed strategy Nash Equilibria in all finite games. These notions are central to the study of game theory [52] and hence, in this section, Nash's concepts will be defined and proved in detail.

Firstly, the idea of a *best response* is introduced.

Definition 1.2.1. For a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, the strategy, s_i, of the ith player is considered a *best response* to the strategy vector s_{-i} if $u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i})$.

This leads onto the main definition of the section.

Definition 1.2.2. Given a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ and its mixed extension, Γ, the vector of strategies $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* if, for all players $i \in N$, σ_i^* is a best response to $\sigma_{-i}^* \in N$.

In other words, σ^* is a Nash equilibrium if and only if no player has any reason to deviate from their current strategy σ_i^*.

The following observation is highlighted as an example.

The strategy pair (D, D), is the unique Nash equilibrium for the PD, with a payoff value of 1 for each player.

Assume the row player uses the following mixed strategy, $\sigma_r = (x, 1-x)$, that is, the probability of cooperating is x and the probability of defecting is $1-x$. Similarly, assume the column player has the strategy, $\sigma_c = (y, 1-y)$. The payoff obtained for the row and column player, respectively, is then:

$$A\sigma_c^T = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 3y \\ 4y+1 \end{pmatrix},$$

$$\sigma_r B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3x & 4x+1 \end{pmatrix}$$

Plotting these gives the graphs as seen in Figure 1.1.

From Figure 1.1 it is clear that, regardless of the strategy played by the opponent, defection is indeed the only rational move. Thus, the players have no incentive to deviate if and only if both play the strategy $\sigma = (0, 1)$, that is, defection for every one-shot game of the PD.

This next result is taken from [56], Nash's second paper on equilibria in games. The notion obtained here is fundamental to many areas of game theory, including the folk theorems.

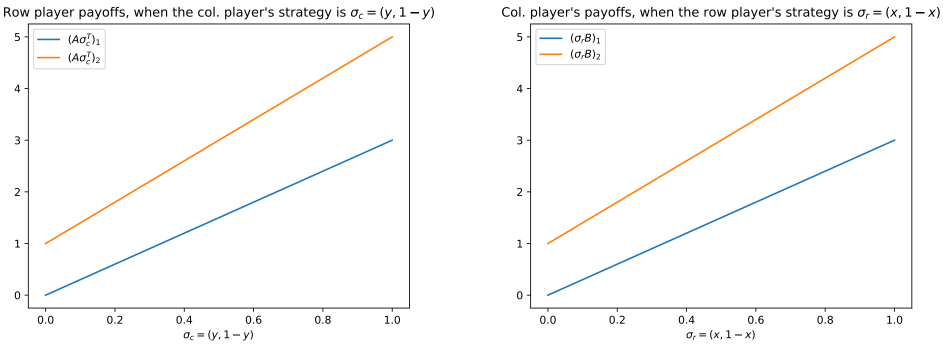


Figure 1.1: Graphs to show the row and column players' payoffs against a mixed strategy.

Theorem 1.2.1. Every finite game has an equilibrium point.

The proof of Theorem 1.2 includes the use of a *fixed point theorem*. Thus, a short sub-section regarding one such result is given for completeness, before providing a formal proof of 1.2.

1.2.1 Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem is a result from the field of topology. Named after the Dutch mathematician, L.E.J. Brouwer, it was proven in 1912 [13]. However, before stating this notion, a few conditions regarding the properties of sets are recalled.

The following three definitions appear as in [8, 70, 71] for 1.2.1, 1.2.1, and 1.2.1, respectively.

Definition 1.2.3. A set $X \subseteq \mathbb{R}^d$ is called *convex* if it contains all line segments connecting any two points $x_1, x_2 \in X$.

Definition 1.2.4. An *open cover* of a set $S \subset X$, a topological space, is a collection of open sets $A_1, A_2, \ldots \subset X$ such that $A_1 \cup A_2 \cup \ldots \supset S$, that is, the union of the open sets contain S.

Definition 1.2.5. A subset $S \subseteq X$, a topological space, is called *compact* if, for each open cover of S, there is a finite sub-cover of S.

The presentation of Brouwer's Fixed Point Theorem is now given as in [52].

Theorem 1.2.2. Let $X \subseteq \mathbb{R}^n$ be a non-empty convex and compact set, then each continuous function $f : X \to X$ has a fixed point.

In other words if X and f satisfy the conditions given above then there exists a point $x \in X$ such that $f(x) = x$.

Since this project is regarding game theory, rather than topology, the proof to the above theorem is omitted. However, the interested reader is referred to [35] for an in-depth consideration into the theory of topology.

1.2.2 Proof of Nash's Theorem

The proof provided is adapted from the original, as presented in [56], with extra notes from [52]. According to [52], the general idea is to define a function, which satisfies the conditions required for Theorem 1.2.1, by using the payoff functions on the set of mixed strategies. Then, through identifying each equilibrium point with a fixed point of the function, the required result is obtained.

Firstly, a brief restatement of the notation needed is provided for clarity. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a finite game with mixed extension $\Gamma = (N, (\Sigma_i)_{i \in N}, (U_i)_{i \in N})$. Here, $N = \{1, \ldots, n\}$ denotes the set of players; $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of pure strategies for all players, with $(S_i)_{i \in N}$ the pure strategy set for player i; Σ is defined similarly but relating to mixed strategies; and $U_i : \Sigma \to \mathbb{R}$ are the payoff functions as given in (1.1.3).

Proof. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a tuple of mixed strategies and $U_{i,t}(\sigma)$ be the ith player's payoff if they changed to their s_i^tth pure strategy and all other players continue to use their mixed strategy. Now, define function $f : \Sigma \to [0, \infty)$ such that

$$f_{i,t}(\sigma) = \max\left(0, U_{i,t}(\sigma) - U_i(\sigma)\right) \tag{1.8}$$

and also let

$$\sigma_i' = \frac{\sigma_i + \sum_t f_{i,t}(\sigma) s_i^t}{1 + \sum_t f_{i,t}(\sigma)} \tag{1.9}$$

be a modification of each $\sigma_i \in \sigma$, with $\sigma' = (\sigma_1', \sigma_2', \ldots, \sigma_n')$. In words, this modification increases the proportion of the pure strategy s_i^t used in σ_i if the payoff gained by the ith player is larger when they replace their mixed strategy by s_i^t. Else, it remains the same if doing this decreases their payoff as $f_{i,t}(\sigma) = 0$ in this case. Note, the denominator ensures that the ending vector is still a probability distribution by standardising.

The aim is to apply Theorem 1.2.1 to the mapping $T : \sigma \to \sigma'$ and show that its fixed points correspond to Nash equilibria. Thus, firstly compactness and convexity of the set Σ is shown along with continuity of the function f.

Claim 1: The set Σ is compact and convex. Observe that each σ_i can be represented by a point in a simplex in a real vector space with the vertices given by the pure strategies, s_i^t. Therefore, it follows that the set Σ_i is convex and compact. Using the result, *If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are convex compact sets then the set $A \times B$ is a convex compact subset of* $\mathbb{R}^{n+m}$ (highlighted in [52]), gives the convexity and compactness of the set Σ, the cross product of all Σ_is.

Claim 2: The function f is continuous. The continuity of the function f depends upon the continuity of the payoff functions U_i. As given in [52], this is shown by first proving that the U_i are multilinear functions in the variables $(\sigma_i)_{i \in N}$ and then applying the fact that any multilinear function over Σ is a continuous function [3]. The result then follows.

Hence, by Theorem 1.2.1, the mapping T must have at least one fixed point. The proof is concluded by showing that any fixed points of T are Nash equilibria and vice versa.

Claim 3: Any fixed point of T is a Nash equilibrium. Suppose σ is such that $T(\sigma) = \sigma$. Then the proportion of s_i^t used in the mixed strategy σ_i must not be altered by T. Therefore, in σ_i', the sum $\sum_t f_{i,t}(\sigma)$ in the denominator must equal zero, otherwise the total sum of the denominator will be greater than one, decreasing the proportion of s_i^t. This implies that for all pure strategies s_i^q, $f_{i,q}(\sigma) = 0$. That is, player i can not improve their payoff by adopting any of the pure strategies. Note, this is true for all i and s_i^q by definition of $T(\sigma) = \sigma$ and thus no player is able to improve their payoff. By 1.2, this is exactly the conditions of a Nash equilibrium.

Claim 4: Any Nash equilibrium is a fixed point of T. Assume σ is a Nash equilibrium. Then, by definition, it must be that $f_{i,q}(\sigma) = 0$ for all pure strategies q for all players, $i \in N$. Note, if $f_{i,q}(\sigma) \neq 0$, then the ith player would benefit from changing their strategy to the pure strategy s_i^q, which violates the condition for a Nash equilibrium. From this it follows that $T(\sigma) = \sigma$, that is, σ is a fixed point of T. This concludes the proof. $\square$

1.3 Repeated Games

The folk theorems studied in this project are a consequence of games which are repeated several times. Indeed, repeated games provide more insight into how and why cooperation can evolve. Moreover, there are cases in which further equilibria become supported when compared to the one-shot equivalent. It could also be argued that repeated games provide more realistic results regarding interactions, since the majority of situations are faced on a regular basis. Thus,

[3]For a detailed consideration of the continuity of the payoff functions see [52], pages 148–149.

before discussing the main statements of the study, the theory of both finitely- and infinitely- repeated games is presented.

Firstly, a couple of alterations to the terminology used in previous sections is redefined, to be consistent with the literature. The notion of a 'game' will become known as a *stage game* to highlight the fact that a one-off game is being considered. Also, what was defined previously as a 'strategy' will now be referred to as an *action* to differentiate it from a strategy of a repeated game, see subsection 1.3.1.

1.3.1 Finite Repeated Games

According to [47], a *T-stage repeated game*, $T < \infty$ is when the stage game, G, is played T times, over discrete time intervals. Each player has a strategy based on previous 'rounds' of the game and the payoff of a repeated game is calculated as the total sum of the stage game payoffs.

Prior to giving a formal description of a strategy in a repeated game, the idea of *history*, within the context of repeated games, is provided.

Definition 1.3.1. The *history*, $H(t)$ of a repeated game is the knowledge of previous actions of all players up until the tth stage game, assumed to be known by all players. Note that, when $t = 0$, $H(0) = \underbrace{(\emptyset, \emptyset, \dots, \emptyset)}_{N\text{times}}$, since no stage games have yet been played.

Definition 1.3.2. As given in [46, 52], a *strategy* of a T-stage repeated game is defined to be a mapping from the complete history so far to an action of the stage game, that is

$$\tau_i : \bigcup_{t=0}^{T-1} H(t) \to a_i. \tag{1.10}$$

Here, $H(t)$ is the history of play as defined in 1.3.1 and a_i is the ith player's action of the stage game.

Consider, for example, the environment in which the stage game PD is repeated each time. This is known as the *Iterated Prisoner's Dilemma* (IPD) and has been a popular topic of research for many years [4]. Note that the objective here is to maximise payoff. The player:

No matter what my opponents play, I will always defect,

[4]The interested reader is referred to the following papers [31, 38, 58] for good reviews regarding the IPD.

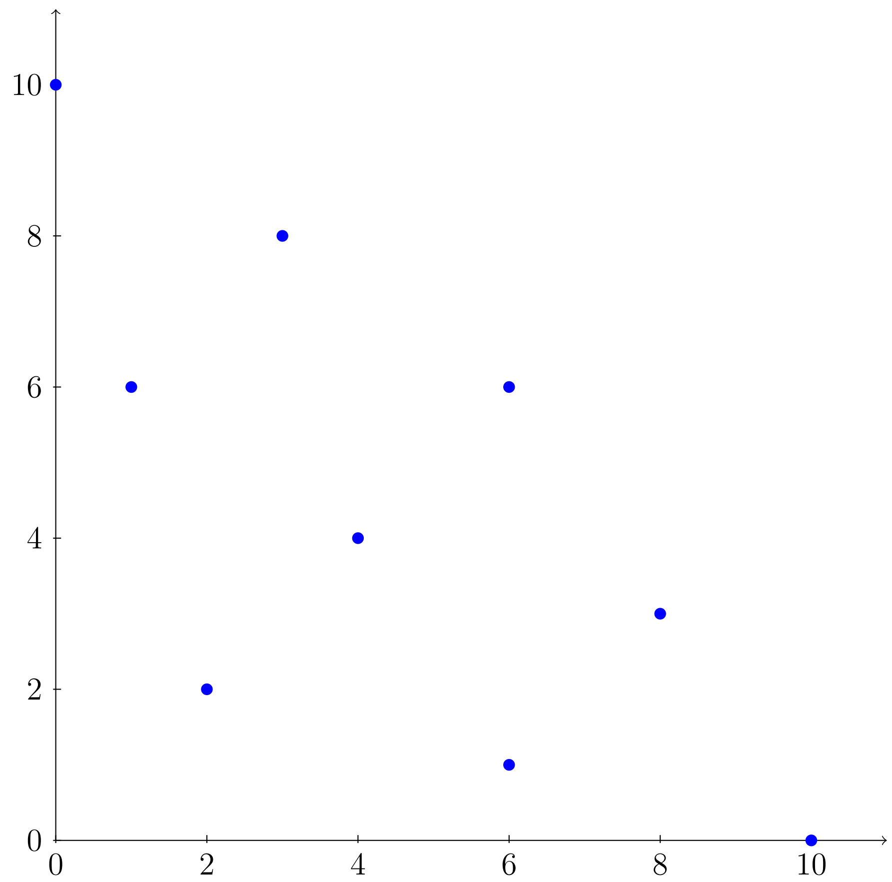


Figure 1.2: A plot to show the possible payoffs of the game between two players in which A Prisoner's Dilemma is repeated twice.

commonly known as the 'Defector' has the following strategy mapping:

$$\tau_i : \bigcup_{t=0}^{T-1} H(t) \to a_i, \tag{1.11}$$

where $a_i = D$ for all time periods $\tau \geq 0$. Other common IPD strategies include:

- Cooperator — *No matter what my opponents play, I will always cooperate*;
- Random — *I will either cooperate or defect with a probability of 50%*; and
- Tit For Tat — *I will start by cooperating but then will duplicate the most recent decision of my opponent.*

Figure 1.2 shows the possible payoffs obtained in a 2-stage repeated IPD with two players.

Now, a discussion on Nash equilibria in repeated games is provided. It can be proven that there exist many equilibria in repeated games [26]. The next result, adapted from [46, 52] guarantees at least one.

Theorem 1.3.1. Consider a T-stage repeated game with $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ as the stage game, $0 < T < \infty$. Define by $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$, a stage Nash equilibrium of G. Then the sequence in which σ^* is continuously played is a Nash equilibrium of the T-stage repeated game.

Proof. Since σ^* is a stage Nash equilibrium, it is, in particular, a Nash equilibrium of the Tth stage game. Thus, no player has any reason to deviate here. But then σ^* was also played at the $(T-1)$th stage, meaning there is still no reason to deviate. Therefore, continuing via backwards induction gives the required result. □

Hence, for the T-stage IPD, all players executing the Defector strategy yields a Nash equilibrium. However, it could be argued that this does not explain why cooperation evolves in many situations.

1.3.2 Infinite Repeated Games

This section discusses the case when $T \to \infty$ and results linked to *infinitely repeated games*. These provide a more realistic framework for analysing behaviours.

Extensive Form Games

The Folk Theorem discussed in section 1.4 considers a stronger refinement of Nash equilibria, for repeated games, known as *subgame perfect equilibria*. In order to fully understand this notion, a new representation of games is introduced.

Definition 1.3.3. An *extensive form game* is given by the ordered vector $\Gamma = (N, V, E, x_0, (V_i)_{i\in N}, O, u)$ where $N = \{1, 2, \ldots, n\}$ is a finite set of players; (V, E, x_0) is a *game tree* [5]; $(V_i)_{i\in N}$ is a partition of the set $V \setminus L$, where L is the set of all leaves, or terminal points, of the game tree; O is the set of outcomes for the game; and u is a function which maps each leaf in L to an outcome in O.

This leads on to the following definition, adapted from [69].

Definition 1.3.4. A player's *information set* is a subset of the nodes in a game tree where:

- Only the player concerned is deciding;

[5] The triple (V, E, x_0) is defined as a *tree* if the set of vertices, V, and the set of edges, E, create a *directed graph*, that is, each element in E is an ordered tuple. The root, or starting node, of the graph is represented by x_0

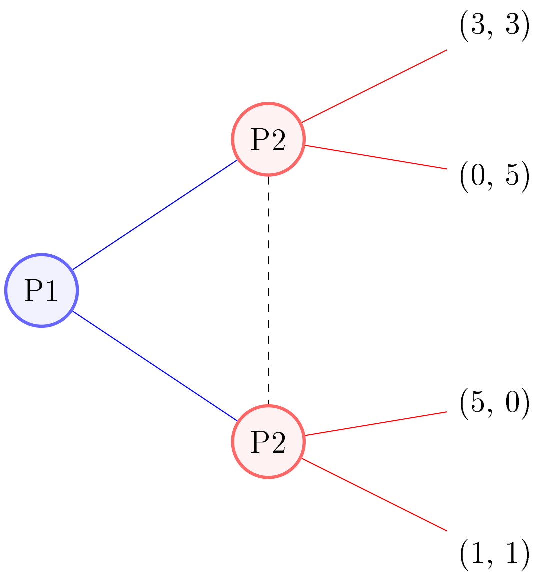


Figure 1.3: The extensive form representation of the PD.

- This player is not aware of which node has been reached, except that it is definitely one of the elements found in this set.

In Figure 1.3, the extensive form representation of the PD is provided. Here, only two players are considered and any information sets are represented by a dashed line. Any normal form game can be represented as an extensive form game.

Definition 1.3.5. According to [69], a *subgame* is a sub-graph of the game tree such that:

- The sub-graph begins at a decision node, say x_i;
- This node, x_i, is the only element contained in its information set; and
- The sub-graph contains all of the decision nodes which follow x_i.

This leads to the following definition of *subgame perfect equilibria*, also adapted from [69].

Definition 1.3.6. A *subgame perfect equilibrium* is a Nash equilibrium which satisfies the condition that the strategies played define a Nash equilibrium in every subgame.

Hence the strategy defined in Theorem 1.3.1 is a subgame perfect equilibrium. A few final definitions are now highlighted before introducing the Folk Theorem.

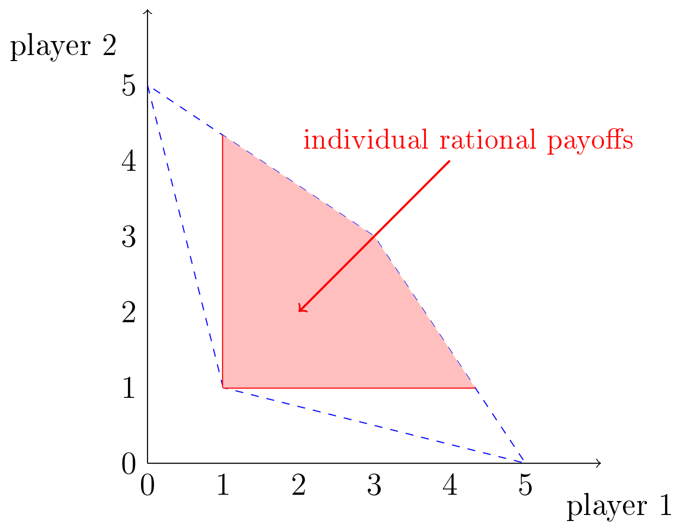


Figure 1.4: A plot highlighting the individually rational payoffs for the PD.

Final Definitions Needed

Now, in order to be able to discuss the payoffs of strategies in infinite games, a few final definitions are required.

Definition 1.3.7. In [43] a *discounted payoff* is defined as:

$$V_i(\sigma) = \sum_{t=1}^{\infty} \delta^{t-1} U_i(\sigma), \tag{1.12}$$

where the discount factor, δ, can be thought of as the probability that the game continues. That is, the probability that another stage game will be played.

Equation 1.3.2 can be used to define *average payoffs*.

Definition 1.3.8. According to [43], the *average payoffs* per stage game, are given by:

$$\frac{1}{\overline{T}} V_i(\sigma) = (1-\delta) V_i(\sigma), \tag{1.13}$$

where $\overline{T} = \frac{1}{1-\delta}$ is the average length of a game.

Finally, Figure 1.4 shows those payoffs which are individually rational for a two player version of PD. In general, an *individually rational payoff* is an average payoff which exceeds those obtained in the stage Nash equilibria for all players [43]. Often the Nash equilibrium payoff is not the optimal payoff players could achieve.

1.4 Folk Theorem

This section contains the statement and proof of the main theorem in this project.

According to [69], the Folk Theorems are so-called because their results were well-known before a formal proof was provided. In general, these theorems state that players can achieve a better payoff than the Nash equilibrium (if the Nash equilibrium payoff is not optimal) when the stage game is repeated many times and the probability of the game continuing is high enough.

It is believed that [26] was one of the first to provide a formal proof to the widely accepted Folk Theorem [1, 69]. Thus, the presentation of the statement and proof given here is adapted from [26] as well as [43].

Theorem 1.4.1. Assume the conditions provided in subsection 1.4.1 are satisfied for the given infinite repeated game. Then, for any individually rational payoff V_i, there exists a discount parameter δ^* such that for all δ_i, $0 < \delta^* < \delta_i < 1$ there is a subgame perfect Nash equilibria with payoffs equal to V_i.

1.4.1 Assumptions

Here, the assumptions which Friedman [26] requires the infinite repeated game to satisfy in order for Theorem 1.4 to hold are listed.

1. The mixed action sets, Σ_i are compact and convex for all $i \in N$.
2. The payoff functions, $U_i : \Sigma \to \mathbb{R}$, are continuous and bounded for all $i \in N$.
3. The $U_i(\sigma)$s are quasi-concave [6] functions of $sigma_i$ for all $i \in N$.
4. If $U_i' \leq U_i''$, for all $i \in N$ and $U_i', U_i'' \in \mathcal{U}$, then, for all $U_i' \leq U \leq U_i''$, $U \in \mathcal{U}$. Here, $\mathcal{U}$ is defined to be the set of feasible payoffs, $\{U(\sigma) : \sigma \in \Sigma\}$, where $U(\sigma) = (U_1(\sigma), U_2(\sigma), \ldots, U_N(\sigma))$.
5. $\mathcal{U}^*$ is concave, where $\mathcal{U}^* \subset \mathcal{U}$ denotes the set of all Pareto optimal payoffs [7].
6. All stage games are identical in the infinitely repeated game.
7. The discount parameter, δ, is equal in all time periods.

[6] According to [65], a real-valued function f, defined on a convex subset $C \subset \mathbb{R}^n$, is *quasi-concave* if for all $\alpha \in \mathbb{R}$, the set $\{x \in C : f(x) \geq a\}$ is convex.

[7] The paper [26] defines a *Pareto optimal payoff* as a point in the payoff space $U_i(\sigma^*)$ which satisfies the conditions: $\sigma^* \in \Sigma$ and $U_i(\sigma^*) > U_i(\sigma)$ for all $i \in N$

8. The stage game has a unique Nash equilibrium.

9. The Nash equilibrium is not Pareto optimal [8].

Note [26] later goes on to prove that assumptions six to nine can be removed with only a small effect on the result. However, since the game being studied in this project is the IPD (which satisfies all the above assumptions), this generalisation will be omitted. Only the proof of the original theorem will be provided.

1.4.2 Proof of the Folk Theorem

Proof. Consider the set of all actions which yield greater payoffs than the Nash equilibrium, denoted by:

$$B = \{\sigma : \sigma \in \Sigma, U_i(\sigma) > U_i(\sigma^*), i \in N\} \tag{1.14}$$

where σ^* is the Nash equilibrium strategy. Define the following trigger strategy:

$$\sigma_{i1} = \sigma_i'; \quad \sigma_{it} = \begin{cases} \sigma_i', & \text{if } \sigma_{j\tau} = \sigma_j' \ j \neq i, \tau = 1, 2, \ldots, t-1, t = 2, 3, \ldots \\ \sigma_i^*, & \text{otherwise,} \end{cases} \tag{1.15}$$

where $\sigma_i' \in B$. In words, the ith player will choose σ_i' unless any other player does not play σ_j', in which case they continue by playing their Nash equilibrium action, σ_i^*.

Now, by definition, the strategy in (1.4.2) is an equilibrium of the repeated game if

$$\sum_{\tau=0}^{\infty} \delta_i^{\tau} U_i(\sigma_i') > U_i(\sigma_{-i}', t_i) + \sum_{\tau=1}^{\infty} \delta_i^{\tau} U_i(\sigma^*), \quad i \in N, \tag{1.16}$$

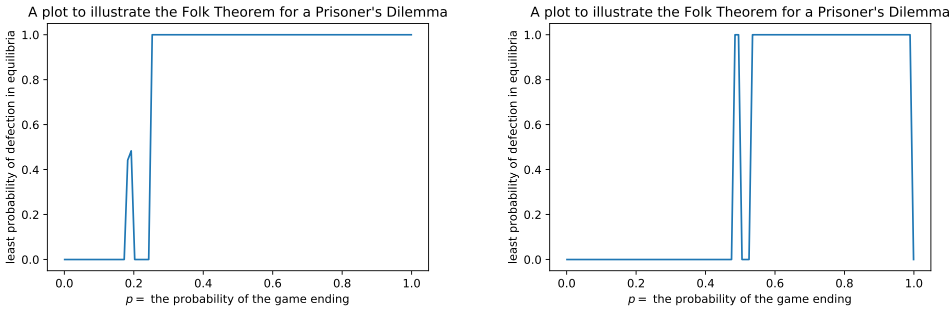
which can be rearranged to

$$\frac{\delta_i}{1-\delta_i}[U_i(\sigma') - U_i(\sigma^*)] > U_i(\sigma_{-i}', t_i) - U_i(\sigma\prime), \quad i \in N, \tag{1.17}$$

where $U_i(\sigma_{-i}', t_i) = \max_{\sigma_i \in \Sigma_i} U_i(\sigma_{-i}', \sigma_i)$, $t_i \in \Sigma_i$. Note, $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$

To check if this strategy is indeed a best response to all others players, who are executing the same strategy in (1.4.2), consider their alternatives. The ith player has two options. Either they execute the strategy (1.4.2), or they play the strategy in which $\sigma_{i1} = t_i$. The latter implies $\sigma_{i\tau} = \sigma^*$ will be the best response as every other player will convert to $\sigma_{j\tau} = \sigma^*$, for all $\tau > 1$. Note that any other strategy is weakly dominated by one of these two, since playing t_i in any other stage $\tau \neq 1$ will yield less gains due to increased discounting.

[8] That is, the payoff yielded from the Nash equilibrium is not a Pareto optimal payoff.



(a) A plot of the least probabilities of defection when playing against the strategies: Cooperator, TitForTat and Random. Here, the p-threshold is approximately 0.25.

(b) A plot of the least probabilities of defection when playing against the strategies: Winner21, AntiTitForTat and OmegaTFT. Here, the p-threshold is around 0.5.

Figure 1.5: Original plots obtained which influenced the subject of this project.

Now if, from playing the Nash equilibria, the discounted loss

$$\frac{\delta_i}{1-\delta_i}[U_i(\sigma') - U_i(\sigma^*)], \tag{1.18}$$

is greater than the gain achieved by playing t_i against σ'_{-i}, then the rational strategy choice for player i, assuming all other players are executing (1.4.2), is to play (1.4.2).

Observe, as the discount parameter, $\delta \to 1$ from below, the discounted loss in (1.4.2) tends to infinity. However, the gain obtained from playing t_i, that is, $U_i(\sigma'_{-i}, t_i) - U_i(\sigma\prime)$ is finite. Thus, for all $\sigma'_i \in B$ there exists a $\delta^* \in (0,1)$ such that for all $\delta_i > \delta^*$, the strategy (1.4.2) is optimal against the same strategy for all players $j \neq i$. Therefore, if the conditions are true for all players $i = 1, 2, \dots, n$, the strategy $(\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_n)$, where $\bar{\sigma}_i$ denotes (1.4.2), yields a Nash equilibrium.

Finally, by construction, the strategy (1.4.2) is indeed a subgame perfect equilibrium. □

1.5 Aims of the Project

This project stemmed from an initial idea presented in a game theory assignment completed by the author. The topic of this coursework was Nash equilibria of repeated games and the two graphs, as presented in Figure 1.5, were obtained.

Figure 1.5 was obtained from repeating IPD tournaments which are implemented in Python via the package Axelrod [19]. In general, these tournaments are a group of strategies, who all compete in a variety of round-robin, two-player IPDs with the aim of achieving the largest payoffs. The implementation in Axelrod allows for the simulation and analysis of IPD tournaments under different environments. For example:

- It is possible to vary the number of strategies making the groups competing in the tournament[9].
- Both finite and infinite IPD can be considered. The infinite IPD is simulated through the scenario of a probabilistic ending, denoted throughout this study as p_e[10]. Note, p_e is related to the discount parameter, δ, introduced in Equation 1.3.2, by $p_e = 1 - \delta$.
- Varying levels of noise can be introduced. This is referred to as *standard PD noise*[11] within this report. According to [32], *standard PD noise* is the probability, p_n, of an action being altered within any particular round. That is, the probability of a C being seen as a D and vice versa.

The two plots seen in **??** were yielded from setting $p_n = 0$, the player set size equal to four, p_e taking 100 distinct values within (0, 1) and each tournament to repeat 100 times. The graphs show the least probability of defection obtained in the Nash equilibria of the corresponding game. Note, the 'corresponding game' here refers to the matrix of mean payoff values which can be calculated from the tournament results of each strategy (see **??** for further explanation of this).

From Figure 1.5, it can be seen that there is a clear game-ending probability p_e for which the least probability of defection goes to zero. In this project, this probability is defined as a *p-threshold*. However, what was most intriguing was that the two different games obtained had a different p-threshold. This is approximately 0.25 for Figure 1.5a but for Figure 1.5b the threshold appears at around 0.5. This initiated the idea to investigate whether there are any specific characteristics of an IPD tournament that affect the value of the p-threshold.

Therefore, the aims of this project are as follows:

---

[9]In this project, the group of strategies will be referred to as a *player set* for clarity. Consider Figure 1.5a, the player set here consists of the strategies: *Cooperator*, *TitForTat*, *Random* and *Defector*

[10]The probability of the game ending, p_e, will also be referred to as a game-ending probability, in this project.

[11]Note, there are three potential sources of noise within an IPD tournament. In order to differentiate between them the following terms are used. *Standard PD noise* to refer to the probability of an action being altered; *stochastic player noise* to refer to the noise induced by a stochastic strategy; and *unexpected noise* to refer to noise which is expected from running numerical experiments.

1. To provide a review of past and present literature already published in the field of folk theorems;
2. To develop a program which executes a large experiment involving tournaments of the IPD with differing environments to obtain graphs similar to those in Figure 1.5; and
3. To perform analyses on where the p-thresholds seem to lie and whether it is affected by the change in the number of players, levels of standard PD noise, etc.

Chapter 2

Literature Review

The folk theorems are a class of results which generally state that for repeated games, any feasible and individually rational payoff vector can be achieved as a subgame perfect equilibrium if the players are patient enough [50]. The origin of these theorems is unknown however written proof and research involving these ideas first appeared in [5, 27, 63] in the 1970s. Since then, many generalisations and refinements of the ideas have been explored, for different games, including: games with private monitoring [36, 55, 61], sequential games [11, 33, 72] and games of complete information [1, 9, 10], to name a few. Due to the identification of further equilibria (as compared with the stage game), which are key in predicting future behaviour, these theorems have been commented on as 'fundamental' in the theory of non-cooperative games by [36, 48]. On the other hand, the majority of the strategies used in the proofs assume the identification of individual deviators [54] which may not be realistic in certain situations. Hence, an area of research on the so-called 'anti-folk theorems' was introduced [54, 61, 75]. Folk theorems appear to still be an active area of research today [37, 60, 67] with many differing applications. Therefore, in this chapter, a review of the literature on this topic is provided with papers ranging from the 'original' ideas in the 1970s to applications of the theorems in 2020.

2.1 First Papers

According to [1], the earliest work on Folk-type theorems is [27]. In [27], infinitely repeated games with discounting are considered. In particular, focus is on a class of strategies, now known as *grim trigger*. These are used to prove that, for any feasible and individually rational payoff vector, there exists a discount parameter such that a subgame perfect equilibrium with payoffs equal to that vector exist.

This is first shown under the constraints of: identical stage games, constant discount parameter and unique Nash equilibrium that is not Pareto optimal. This is in addition to those made on the strategy and payoff spaces. However, a more general result is then given which removes these restrictive conditions. The application of oligopoly is used throughout [27]. Moreover, he introduces the notion of 'temptation'. He motivates this through the explanation that 'threat' is no longer "credible" since players are unable to communicate in non-cooperative games. On the other hand, 'temptation' is said to be analogous to 'threat'.

In contrast to this, [5] presents folk theorems for infinite games without discounting, assuming payoffs take a "limiting average form". This choice is justified through the statement that, as the discount rate approaches zero, the limit of the discounted sum behaves similarly to the limiting average payoff form. However, trigger strategies are still required in the proof and, for simplicity, two player games are assumed. Two folk theorems are proven in this paper with the latter one discussing subgame perfect equilibria similar to [27], whilst the former is a more generic version of the theorem. In [5] it is discussed that, though the generic version of the theorem does exist, the subgame perfect equilibrium points allow for more 'believable' behaviour. They conclude by considering an example in which payoffs are discounted. A similar approach is taken in [63], with the statement of a folk theorem for infinite games with no discounting and the existence of subgame perfect equilibria. The only difference being the use of an "overtaking criterion" instead of a "limit of the mean criterion" as in [5].

2.2 Games with Complete Information

Folk theorems of games assuming complete information are studied in [1, 10]. That is, all players have common knowledge of payoff functions and strategies. Paper [1] focuses on the necessary and sufficient conditions required for a folk theorem proof. They state that feasibility and individual rationality of the payoff vector are necessary, and also approximately sufficient, conditions for a payoff to be in the equilibrium. This is followed by a discussion on the full dimensionality constraint first introduced by [29] and often used in proving folk theorems. It is highlighted in [1] that the equality between the dimension of the convex hull of feasible payoffs and the number of players is a sufficient condition. This provides motivation for the main result of the paper; the introduction of a 'non equivalent utilities' condition, which is proved to be sufficient and almost necessary for a folk theorem. The condition, as indicated here, is weaker than the aforementioned full dimensionality condition. According to [1], it only requires "no pair of players have equivalent utility functions".

Similarly, a folk theorem for complete information games with subgame perfect equilibria is proved in [10]. However, instead of exponential discounting as in [1, 29], he assumes discounting is present-biased. That is, the discount function implies a player is more willing to alter an event in the future than altering a current event. The folk theorem is proved in cases of where the player is time consistent (prefers to maximise their initial preferences) and time inconsistent (prefers to maximise current preferences). In [10], the notion of 'patience' is taken to be the sum of all discount factors. Also considering time consistent and inconsistent players [50], discusses folk theorems with respect to time-dependent discounting. In contrast to [10], the long-term characterisation of 'patience' is taken as the discount factors at all stages uniformly converge to one. Motivation for the study of time-dependent discounting is explained by empirical studies which seem to suggest that a player's time is unstationary, rather than stationary which is assumed in most discounted game models.

2.3 Games with (Imperfect) Private Monitoring

In the late 1990s attention turned to games with imperfect monitoring. That is, a player's actions can no longer be observed accurately, instead public or private signals are detected [22]. According to [55], games with (imperfect) public monitoring were the first to be considered. More recent studies in this area include [14, 41]. Repeated games with public monitoring are stated in [41] to give a suitable model for studying long term relationships. They say intuition suggests that cooperation becomes easier to sustain as action observability improves. However, [41] continues to show that this intuition is false, by considering a repeated public monitored game which satisfies "the limit perfect public equilibrium payoff set can achieve full efficiency asymptotically as public information becomes less sensitive to hidden actions". They give an example which violates the sufficient condition given in [53] yet a folk theorem can still be obtained. This paper mainly focuses on the PD but, similarly to [1], they state an example which satisfies the folk theorem without the full dimensionality condition.

The notion of a robust equilibrium to incomplete information is considered in [14]. That is, if the equilibrium yielded is near the original equilibrium for all perturbed games consisting of a small independent and identically distributed information shock. Conversely to [27], they discuss the implication of grim trigger strategies not being robust when the aim is to sustain cooperation. A folk theorem is proved in robust equilibria for games of public monitoring, however [14] highlights that a much stronger condition is needed in comparison to the full dimensionality requirement in [53].

As of 2004, according to [55], repeated games with private monitoring was a relatively new area of study. In this paper, conditional independence of signals is assumed in order to show a folk theorem for the IPD. Applications to duopolies are then described. Another study to consider a folk theorem for the IPD is [25]. Using a similar definition of robustness to [14], [25] proves that for a discounted PD with private monitoring technologies, a folk theorem with robust equilibrium strategies can be obtained. In particular, they consider almost-perfect private monitoring and a limit folk theorem (for sequential equilibria) follows. In a similar paper, [24] introduce the idea of "belief-free" equilibrium strategies; a property which implies the belief of an opponent's history is not required when obtaining a best response. They use these strategies in proving a folk theorem for the two-player PD however, for general games, the set of belief-free payoffs is not large enough to provide a folk theorem. Moreover, [24] highlight that, for a larger number of players, the calculation of the payoff set becomes significantly harder.

In contrast to this, [36] discusses a folk theorem using strategies that, although are not belief-free, still make beliefs "irrelevant" at the start of each T-block of stage games. Their result is much more generic than [24] since it applies to N-player finite games under the assumption of full dimensionality with private, but almost-perfect, monitoring. The T-block strategies mentioned above are modified in [73, 74] such that they "can support any vector in the belief-free equilibrium payoff set". This modification results in belief-free equilibrium strategies. The results in [73, 74] are generalisations from the two-player PD in [24] to the corresponding N-player game.

2.4 Games with Communication

Academics introduced games with communication to deal with the complications faced with imperfect monitoring. For example, according to [39], games with public monitoring can obtain a folk theorem under weaker assumptions than those given in [53], if communication is introduced. In this paper, communication is a message, taken from the set of possible actions, which the players give simultaneously after choosing an action and observing a signal. He proves a folk theorem for symmetric games with four or more players, without the assumption of the number of signals relevant to the number of actions.

Regarding private monitoring, [59] claims repeated games are "very difficult to analyse without communication". Thus examples of papers, proving folk theorems for these games with communication, include [28, 49, 59]. A 'Nash threats' folk theorem is proved in [28], similar to [27], in the case of almost public information games, without independent signals, for two players. In this paper, communication is defined in the form of announcements where each set of announcements is the same for all players. The decision to only consider two player games is justified in [28] by highlighting that, although the results can be generalised, in certain cases it can be seen as advantageous to have additional players. Similarly, [59] proves a 'Nash threats' folk theorem for private monitoring games with communication. He increases the number of environments where the folk theorem is applicable through developing further the idea of "delayed communication", as given in [18]. In addition, [59] uses the assumptions of correlated private signals; and each player's deviation from strategy is statistically identifiable from the other players' signals. A model of private monitoring and communication within games is also considered in [49]. He has the aim of increasing the number of applicable environments for [40] frequent communication folk theorem. The paper states that [40] assumed only private signals were publicised in this theorem but if other information was useful and communication was free and legal then players would also want to share their actions. This motivates the reasoning behind the paper. However, assumptions of full dimensionality, and the number of actions and signals, are still required.

Another paper which considers communication is [12], regarding self referential games with codes of conduct. These codes are descriptions of how the players and opponents should play and an application to computer algorithms is provided. Two folk theorems are proved: one assuming common knowledge of the codes of conduct, and the other where only certain players observe certain codes. The latter is the main result of the paper and is motivated by the fact that, often, individuals have good knowledge of those closest to them but not the whole community. Moreover, [12] obtain the sufficient condition that, with public communication, if every player is observed by another two opponents, then a folk theorem is yielded.

2.5 Finite Horizon Games

Another area of interest is the existence of folk theorem-type results for games of finite repetition. In [9], the case of finitely repeated games of complete information and the associated subgame perfect equilibria is explored. Despite the existence of games which, when repeated finitely, "produce no non-cooperative equilibrium outcomes"; they state there may be subgame perfect equilibria of finite-repeated games, when the corresponding single game has multiple equilibria. Indeed, using their "three phase punishment", [9] prove that "any rational and feasible payoff vector can be obtained in the limit". This is assuming the feasible payoff region has dimension equal to the number of players and each player has two Nash equilibrium payoff values. In a similar manner, [4] considers an alternative version of the PD in which an additional strategy is included. This gives a second pure-strategy equilibrium and a folk theorem result for the finitely repeated version of this game.

Although not strictly a finite game, [30] study a repeated game in which players may "strategically terminate" it. In particular, this involves the incorporation of a voting-step, at the start of each repetition, where a certain number of players decide whether or not to keep interacting. This is motivated by the increasing possibilities of ending business partnerships due to more technology and knowledge. A general folk theorem for any stage game (with the additional voting), which is satisfied "for all majority rules except the unanimous ending" is proved by [30]. Indeed, for the unanimous ending rule, they show that the theorem may not hold but sufficient conditions are provided for when it is satisfied.

2.6 Stochastic and Sequential Games

Other game types to have associated folk theorem results include stochastic [23] and overlapping generation [11, 33].

It is explained in [23] that, often, the standard assumption of "a completely unchanging environment", within the theory of repeated games, is not reliable in applications. This reasoning is the motivation for studying, the more generic, stochastic games. These games may not have a pre-decided stage game, instead a 'state variable' is used to represent its environment which alters according to"initial conditions, player's actions, and the transition law". The paper [23] discusses equilibrium payoffs in the case of very patient players, without the need for the Markovian property. Specifically, perfect monitoring is assumed, along with asymptotic state independence and either of payoff asymmetry or full dimensionality. Two folk theorems are proved in [23]: one with unobservable mixed strategies (in which case, fully dimensionality is required) and also, similar to [1], one with the slightly weaker condition of payoff asymmetry (here, mixed strategies have to be observed).

Both[11, 33] provide insight into folk theorems associated with overlapping generation games. These are similar to the repeated normal form games except that the players are considered to be finite. That is, each player is involved in a certain number of stage games before they are replaced by another, identical player. A variety of folk theorems are proved in [33] both with and without discounting and / or observable mixed strategies. He shows that the full dimensionality assumption is not required in these games since players are assumed to not end simultaneously. On the other hand, although [11] does provide a folk theorem, his main result is an anti-folk theorem, see section 2.7. He studies games with imperfect public monitoring and states that, in such overlapping generational games, cooperation becomes impossible in contrast to repeated games. Furthermore, [11] constructs a mixed strategy equilibrium folk theorem. However, he goes on to show that these strategies are unstable to perturbations, resulting in an anti-folk theorem.

A similar study by [3] looks at dynastic repeated games. These differ from the overlapping generation games in two aspects: perfect observation of the past is assumed, and the payoffs obtain no dynastic component. Under the assumptions of full dimensionality and the existence of a payoff vector which strictly Pareto dominates the stage game equilibrium, [3] proves a folk theorem for private communication games, with greater than three players, in sequential equilibria.

Considering now sequential games, in which players do not choose their action simultaneously, [72] introduces a concept of effective minimax values before proving a corresponding folk theorem. In his model, players pick actions in groups. Thus the effective minimax value is defined to be the lowest equilibrium payoff a player will receive, even if none of their opponents have equivalent utilities. According to [72], his folk theorem can be applied to other game models as it is a “uniform characterisation”.

2.7 Anti-Folk Theorems

A common theme in the proofs of folk theorems is the use of strategies which “identify and punish” deviators [54]. However, as soon as the game contains incomplete / imperfect information, deviators cannot necessarily be identified. This yields a much smaller equilibrium set and these results are termed “Anti-Folk Theorems”. This description of anti-folk theorems is adapted from [54], who state the original term was given in [21, 42]. An anti-folk theorem is proved in [54] using the “long-run average criterion” instead of the discounted criterion.

Considering similar models to [11, 33], an anti-folk theorem for a limited-observability overlapping generations model is obtained in [75]. They show that cooperation cannot be sustained when new players can only observe recent history. This is in contrast to the folk theorems obtained under the assumption of common knowledge of all past actions. Though the results in [75] are restrictive in certain cases, they justify the work by stating it is suitable for modelling “high turnover” rates.

Another game model where an anti-folk theorem has resulted is a repeated game with private monitoring [61]. In this paper the following three assumptions are made: infinite and connected private monitoring (that is, infinitely many connected signals); finite past; and independent and identically distributed shocks affect the payoffs. Under these assumptions, [61] shows the violation of the folk theorem. That is, the equilibria of the repeated game consist only of the equilibria of the single game.

2.8 Evolutionary Stability

Recently, there has been research into the results of the folk theorem with respect to the evolutionary game theoretic paradigm. Indeed, [48] state that the folk theorem is often used to characterise evolutionary stable strategies. This is since exact solutions using theoretical results from evolutionary game theory are hard to obtain. However, they show the assumption that the folk theorem yields all Nash equilibria is misleading. This is achieved by defining “type-k equilibria” which are a refinement of the Nash equilibria. The set of type-k equilibria is proved in [48] to be contained within the set of repeated-game Nash equilibria using “reactive strategies”.

In contrast to [24, 73, 74], who discuss folk theorems using belief-free strategies, [34] discusses the instability in an evolutionary sense. He shows that the belief-free equilibria are not robust to small perturbations in games with private monitoring and, in certain cases, this is extreme. Similar to [48], he states that Nash equilibria are used to predict evolutionary behaviour since they are thought of as stable. However, [34] goes on to show that only the choice of repeated stage-game Nash equilibria satisfy evolutionary stability.

2.9 Recent Applications

In recent years, studies have been applying results of the folk theorem in various areas, for example, to create algorithms. This section briefly discusses a few of these.

The folk theorem is used in [16] for the creation of a model which aims to suppress the effects of distributed denial of service attacks. They claim that all networks suffer from attacks to infrastructure and services. Thus, [16] use the programmability of software defined network environments to perform a game theoretic analysis. An algorithm is created for reward and punishment based on the Nash folk theorem. Similarly, [67] make use of the cooperative equilibrium solution from the folk theorem in [27] to create an algorithm suggested to optimise a 'multi-period production planning based real-time scheduling method', for a job shop. Also, [68] uses the result of the folk theorem in the IPD to study the cooperation rates of varying agent strategies in a multi-agent system.

Another application of the folk theorem is an algorithm used to obtain equilibria of a discounted repeated game [60]. A new algorithm, entitled "Communicate & Agree", is introduced in [60] to find equilibria in incomplete information, but perfect monitoring, games. Using the folk theorem in the algorithm enables the payoffs obtained to be potentially higher than those achieved by repeating the Nash equilibria of the single game. However, [60] go on to highlight that the algorithm is not always guaranteed to find equilibria. They say it is dependent on: the discount factor, sampling density, and whether it is a zero-sum game or not. Finally, in a different area, [37] discuss the potential of using game theoretic ideas in quantum optimal transport. In particular, he defines the Quantum PD and explores the possibility of a quantum folk theorem in relation to the corresponding repeated game.

2.10 Conclusion

In this section, an overview into research regarding the folk theorem has been provided. The research history of the folk theorem spans from the 1970s until now, with many different models being considered. Examples include: games with complete information, games with imperfect private monitoring and finite-horizon games. However, there have also been studies into situations where the folk theorem does not hold, or the equilibrium strategies used in proving the theorems are unstable and / or not robust.

References

[1] Dilip Abreu, Prajit K. Dutta, and Lones Smith. "The Folk Theorem for Repeated Games: A NEU Condition". In: *Econometrica* 62.4 (1994), pp. 939–948. URL: https://www.jstor.org/stable/2951739.

[2] IB Adeoye et al. "Application of game theory to horticultural crops in south-west Nigeria." In: *Journal of Agricultural and Biological Science* 7.5 (2012), pp. 372–375.

[3] L. Anderlini, D. Gerardi, and R. Lagunoff. "A "super" folk theorem for dynastic repeated games". In: *Economic Theory* 37.3 (2008). Cited By :6, pp. 357–394. URL: www.scopus.com.

[4] VERA ANGELOVA et al. "CAN SUBGAME PERFECT EQUILIBRIUM THREATS FOSTER COOPERATION? AN EXPERIMENTAL TEST OF FINITE-HORIZON FOLK THEOREMS". In: *Economic Inquiry* 51.2 (Sept. 2011). cited By 2, pp. 1345–1356. DOI: 10.1111/j.1465-7295.2011.00421.x. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-84874353398&doi=10.1111%2fj.1465-7295.2011.00421.x&partnerID=40&md5=83e392d6d27e1febb1639f619a306357.

[5] RJ Aumann and L Shapley. "Long term competition". In: *A Game Theoretic Analysis* (1976).

[6] R. Axelrod. *The Evolution Of Cooperation*. Basic Books, 1984. ISBN: 9780465021215. URL: https://books.google.co.uk/books?id=029p7dmFicsC.

[7] Robert Axelrod. "Effective choice in the prisoner's dilemma". In: *Journal of conflict resolution* 24.1 (1980), pp. 3–25.

[8] Margherita Barile. *Open Cover*. [Accessed on; 06/01/2020]. Wolfram MathWorld. URL: http://mathworld.wolfram.com/OpenCover.html.

[9] Jean-Pierre Benoit and Vijay Krishna. "Finitely Repeated Games". In: *Econometrica* 53.4 (July 1985), p. 905. DOI: 10.2307/1912660.

[10] Axel Bernergård. "Self-control problems and the folk theorem". In: *Journal of Economic Behavior & Organization* 163 (July 2019), pp. 332–347. DOI: 10.1016/j.jebo.2019.05.004.

[11] V. Bhaskar. "Informational Constraints and the Overlapping Generations Model: Folk and Anti-Folk Theorems". In: *Review of Economic Studies* 65.1 (1998). Cited By :42, pp. 135–149. URL: www.scopus.com.

[12] Juan I. Block and David K. Levine. "A folk theorem with codes of conduct and communication". In: *Economic Theory Bulletin* 5.1 (Sept. 2016), pp. 9–19. DOI: 10.1007/s40505-016-0107-y.

[13] Stephan C. Carlson. *Brouwer's fixed point theorem*. [Accessed on: 06/01/2020]. Encyclopaedia Britannica. Sept. 13, 2016. URL: https://www.britannica.com/science/Brouwers-fixed-point-theorem.

[14] Sylvain Chassang and Satoru Takahashi. "Robustness to incomplete information in repeated games". In: *Theoretical Economics* 6.1 (Jan. 2011). cited By 12, pp. 49–93. DOI: 10.3982/te795. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-78651454708&doi=10.3982%2fTE795&partnerID=40&md5=53dbe8255f303468a1e5d95095c08326.

[15] Bor-Sen Chen, Chia-Hung Chang, and Hsiao-Ching Lee. "Robust synthetic biology design: stochastic game theory approach". In: *Bioinformatics* 25.14 (2009), pp. 1822–1830.

[16] Ankur Chowdhary et al. "Dynamic Game based Security framework in SDN-enabled Cloud Networking Environments". In: *Proceedings of the ACM International Workshop on Security in Software Defined Networks & Network Function Virtualization - SDN-NFVSec '17*. ACM Press, 2017. DOI: 10.1145/3040992.3040998.

[17] Belton Cobb. *Stand to Arms*. London: Wells Gardner, 1916.

[18] Olivier Compte. "Communication in Repeated Games with Imperfect Private Monitoring". In: *Econometrica* 66.3 (1998), pp. 597–626. ISSN: 00129682, 14680262. URL: http://www.jstor.org/stable/2998576.

[19] The Axelrod project developers. *Axelrod: v4.7.0*. Apr. 2016. DOI: 10.5281/zenodo.3517155. URL: http://dx.doi.org/10.5281/zenodo.3517155.

[20] Oxford English Dictionary. *game theory, n.* Online. Accessed: 08.10.2019. Mar. 2013. URL: https://www.oed.com/view/Entry/319028?redirectedFrom=game+theory&.

[21] Pradeep Dubey and Mamoru Kaneko. "Information patterns and Nash equilibria in extensive games: 1". In: *Mathematical Social Sciences* 8.2 (Oct. 1984), pp. 111–139. DOI: 10.1016/0165-4896(84)90011-8.

[22] Steven N. Durlauf and Lawrence E. Blume. "Repeated Games: Imperfect Monitoring". In: *The New Palgrave Dictionary of Economics*. Ed. by Steven N. Durlauf and Lawrence E. Blume. Springer, 2016, p. 102. ISBN: 9781349588022.

[23] Prajit K. Dutta. "A Folk Theorem for Stochastic Games". In: *Journal of Economic Theory* 66 (1995), pp. 1–32.

[24] Jeffrey C. Ely, Johannes Horner, and Wojciech Olszewski. "Belief-Free Equilibria in Repeated Games". In: *Econometrica* 73.2 (Mar. 2005). cited By 70, pp. 377–415. DOI: 10.1111/j.1468-0262.2005.00583.x. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-27744460948&doi=10.1111%2fj.1468-0262.2005.00583.x&partnerID=40&md5=7db49053936b7d19cd405059a246462e.

[25] Jeffrey C. Ely and Juuso Välimäki. "A Robust Folk Theorem for the Prisoner's Dilemma". In: *Journal of Economic Theory* 102.1 (Jan. 2002). cited By 95, pp. 84–105. DOI: 10.1006/jeth.2000.2774. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-0036167111&doi=10.1006%2fjeth.2000.2774&partnerID=40&md5=b8cc560b170e8796d5239cbd7ef72ef9.

[26] James W Friedman. "A non-cooperative equilibrium for supergames". In: *The Review of Economic Studies* 38.1 (1971), pp. 1–12.

[27] James W. Friedman. "A Non-cooperative Equilibrium for Supergames". In: *The Review of Economic Studies* 38.1 (Jan. 1971), p. 1. DOI: 10.2307/2296617.

[28] Drew Fudenberg and David K. Levine. "The Nash-threats folk theorem with communication and approximate common knowledge in two player games". In: *A Long-Run Collaboration on Long-Run Games*. Vol. 132. 1. WORLD SCIENTIFIC, Dec. 2008, pp. 331–343. DOI: http://dx.doi.org/10.1016/j.jet.2005.08.006.

[29] Drew Fudenberg and Eric Maskin. "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information". In: *Econometrica* 54.3 (May 1986), p. 533. DOI: 10.2307/1911307.

[30] Takako Fujiwara-Greve and Yosuke Yasuda. "The Folk Theorem in Repeated Games with Endogenous Termination". In: *SSRN Electronic Journal* (2018). DOI: 10.2139/ssrn.3267427.

[31] Nikoleta E. Glynatsi and Vincent A. Knight. *A bibliometric study of research topics, collaboration and influence in the field of the Iterated Prisoner's Dilemma.* 2019. arXiv: `1911.06128 [physics.soc-ph]`.

[32] Nikoleta E. Glynatsi and Vincent A. Knight. *A meta analysis of tournaments and an evaluation of performance in the Iterated Prisoner's Dilemma.* 2020. arXiv: `2001.05911 [cs.GT]`.

[33] Olivier Gossner. "Overlapping Generation Games with Mixed Strategies". In: *Mathematics of Operations Research* 21.2 (May 1996), pp. 477–486. DOI: `10.1287/moor.21.2.477`.

[34] Yuval Heller. "Instability of belief-free equilibria". In: *Journal of Economic Theory* 168 (Mar. 2017). cited By 1, pp. 261–286. DOI: `10.1016/j.jet.2017.01.001`. URL: `https://www.scopus.com/inward/record.uri?eid=2-s2.0-85008894373&doi=10.1016%2fj.jet.2017.01.001&partnerID=40&md5=e6021d15598968bf7d198f9c0619e9b7`.

[35] M. Henle. *A Combinatorial Introduction to Topology.* A series of books in mathematical sciences. W. H. Freeman, 1979. ISBN: 9780716700838. URL: `https://books.google.co.uk/books?id=TQSmQgAACAAJ`.

[36] Johannes Hörner and Wojciech Olszewski. "The Folk Theorem for Games with Private Almost-Perfect Monitoring". In: *Econometrica* 74.6 (2006), pp. 1499–1544. URL: `https://www.jstor.org/stable/4123082`.

[37] K. Ikeda. "Foundation of quantum optimal transport and applications". In: *Quantum Information Processing* 19.1 (2020). cited By 0. DOI: `10.1007/s11128-019-2519-8`. URL: `https://www.scopus.com`.

[38] M. Jurišić, D. Kermek, and M. Konecki. "A review of iterated prisoner's dilemma strategies". In: *2012 Proceedings of the 35th International Convention MIPRO*. 2012, pp. 1093–1097.

[39] M. Kandori. "Randomization, communication, and efficiency in repeated games with imperfect public monitoring". English. In: *Econometrica* 71.1 (2003). Cited By :13, pp. 345–353. URL: `www.scopus.com`.

[40] Michihiro Kandori and Hitoshi Matsushima. "Private Observation, Communication and Collusion". In: *Econometrica* 66.3 (1998), pp. 627–652. URL: `https://EconPapers.repec.org/RePEc:ecm:emetrp:v:66:y:1998:i:3:p:627-652`.

[41] Michihiro Kandori and Ichiro Obara. “Less is more: an observability paradox in repeated games”. In: *International Journal of Game Theory* 34.4 (Sept. 2006). cited By 5, pp. 475–493. DOI: 10.1007/s00182-006-0032-7. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-33750923312&doi=10.1007%2fs00182-006-0032-7&partnerID=40&md5=d8b2b09a452a7522691b58f57a8abf89.

[42] Mamoru Kaneko. “Some remarks on the folk theorem in game theory”. In: *Mathematical Social Sciences* 3.3 (Oct. 1982), pp. 281–290. DOI: 10.1016/0165-4896(82)90075-0.

[43] Vincent Knight. *Chapter 10 - Infinitely Repeated Games*. [Accessed on: 15/01/2020]. 2017. URL: https://vknight.org/Year_3_game_theory_course/Content/Chapter_10_Infinetely_Repeated_Games/.

[44] Vincent Knight. *Chapter 2 - Normal Form Games*. Online. [Accessed on: 13/10/2019]. 2017. URL: https://vknight.org/Year_3_game_theory_course/Content/Chapter_02-Normal_Form_Games/.

[45] Vincent Knight. *Chapter 3 - Rationalisation*. Online. Accessed on: 23/10/2019. 2019. URL: https://vknight.org/gt/chapters/03/.

[46] Vincent Knight. *Chapter 8 - Repeated Games*. [Accessed on: 08/01/2020]. 2019. URL: https://vknight.org/gt/chapters/08/.

[47] Vincent Knight. *Chapter 9 - Finitely Repeated Games*. Online. Accessed on 06/01/2020. 2017. URL: https://vknight.org/Year_3_game_theory_course/Content/Chapter_09_Finitely_Repeated_Games/.

[48] Jiawei Li and Graham Kendall. “On Nash Equilibrium and Evolutionarily Stable States That Are Not Characterised by the Folk Theorem”. In: *PLOS ONE* 10.8 (Aug. 2015). Ed. by Pablo Brañas-Garza, e0136032. DOI: 10.1371/journal.pone.0136032.

[49] Rui Li. “Sufficient communication in repeated games with imperfect private monitoring”. In: *Economics Letters* 108.3 (Sept. 2010), pp. 322–326. DOI: 10.1016/j.econlet.2010.06.005.

[50] Xiaoxi Li. “The Folk Theorem for Repeated Games With Time-Dependent Discounting”. In: *SSRN Electronic Journal* (2019). DOI: 10.2139/ssrn.3407197.

[51] Xiannuan Liang and Yang Xiao. “Game theory for network security”. In: *IEEE Communications Surveys & Tutorials* 15.1 (2012), pp. 472–486.

[52] Michael Maschler, Eilon Solan, and Shmuel Zamir. *Game Theory*. Cambridge University Press, 2013. DOI: 10.1017/CBO9780511794216.

[53] Eric Maskin, D. Fudenberg, and D. Levine. “The Folk Theorem with Imperfect Public Information”. In: *Econometrica* 62.5 (1994). Reprinted in E. Maskin (ed.), Recent Developments in Game Theory, London: Edward Elgar, 1999, pp. 345-387. Also reprinted in D. Fudenberg and D. Levine (eds.), A Long-Run Collaboration on Games with Long-Run Patient Players, World Scientific Publishers, 2009, pp. 231-274, pp. 997–1039.

[54] J. Masso and R. W. Rosenthal. “More on the "anti-folk theorem"”. In: *Journal of Mathematical Economics* 18.3 (1989). Cited By :7, pp. 281–290. URL: www.scopus.com.

[55] Hitoshi Matsushima. “Repeated Games with Private Monitoring: Two Players”. In: *Econometrica* 72.3 (May 2004). cited By 40, pp. 823–852. DOI: 10.1111/j.1468-0262.2004.00513.x. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-2642524464&doi=10.1111%2fj.1468-0262.2004.00513.x&partnerID=40&md5=a972047919e3b64a6e8a254e603c11e7.

[56] John Nash. “Non-cooperative games”. In: *Annals of mathematics* (1951), pp. 286–295.

[57] John F Nash et al. “Equilibrium points in n-person games”. In: *Proceedings of the national academy of sciences* 36.1 (1950), pp. 48–49.

[58] Colm O’Riordan. *Iterated Prisoner’s Dilemma: A review*. Tech. rep. Department of Information Technology, National University of Ireland, Galway, 2001. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.136.4521&rep=rep1&type=pdf.

[59] Ichiro Obara. “Folk theorem with communication”. In: *Journal of Economic Theory* 144.1 (Jan. 2009), pp. 120–134. DOI: 10.1016/j.jet.2007.08.005.

[60] Juan Parras and Santiago Zazo. “A distributed algorithm to obtain repeated games equilibria with discounting”. In: *Applied Mathematics and Computation* 367 (Feb. 2020), p. 124785. DOI: 10.1016/j.amc.2019.124785.

[61] Marcin Pęski. “An anti-folk theorem for finite past equilibria in repeated games with private monitoring”. In: *Theoretical Economics* 7.1 (Jan. 2012), pp. 25–55. DOI: 10.3982/te760.

[62] W. H. Press and F. J. Dyson. “Iterated Prisoner’s Dilemma contains strategies that dominate any evolutionary opponent”. In: *Proceedings of the National Academy of Sciences* 109.26 (May 2012), pp. 10409–10413. DOI: 10.1073/pnas.1206569109.

[63] Ariel Rubinstein. "Equilibrium in supergames with the overtaking criterion". In: *Journal of Economic Theory* 21.1 (Aug. 1979), pp. 1–9. DOI: 10.1016/0022-0531(79)90002-4.

[64] V. Srivastava et al. "Using game theory to analyze wireless ad hoc networks". In: *IEEE Communications Surveys Tutorials* 7.4 (Apr. 2005), pp. 46–56. DOI: 10.1109/COMST.2005.1593279.

[65] Christopher Stover. *Quasi-Concave Function*. [Accessed on: 15/01/2020]. Wolfram Mathworld. URL: http://mathworld.wolfram.com/Quasi-ConcaveFunction.html.

[66] John Von Neumann, Oskar Morgenstern, and Harold William Kuhn. *Theory of games and economic behavior (commemorative edition)*. Princeton university press, 2007.

[67] Jin Wang et al. "Infinitely repeated game based real-time scheduling for low-carbon flexible job shop considering multi-time periods". In: *Journal of Cleaner Production* 247 (Feb. 2020), p. 119093. DOI: 10.1016/j.jclepro.2019.119093.

[68] Shuyin Wang and Limei Jiang. "Study of Agent Cooperation Incentive Strategy Based on Game Theory in Multi-Agent System". In: *Lecture Notes in Electrical Engineering*. Vol. 463. cited By 0. Springer Singapore, June 2018, pp. 1871–1878. DOI: 10.1007/978-981-10-6571-2_227. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-85048687335&doi=10.1007%2f978-981-10-6571-2_227&partnerID=40&md5=6988d25bb46e794e764ed6761aa9a99e.

[69] James N. Webb. *Game Theory: Decisions, Interaction and Evolution*. Springer Undergraduate Mathematics Series. Springer, 2007. ISBN: 978-1-84628-423-6.

[70] Eric W. Weisstein. *Compact Space*. [Accessed on: 06/01/2020]. Wolfram Mathworld. URL: http://mathworld.wolfram.com/CompactSpace.html.

[71] Eric W. Weisstein. *Convex*. [Accessed on: 06/01/2020]. Wolfram MathWorld. URL: http://mathworld.wolfram.com/Convex.html.

[72] Q. Wen. "A folk theorem for repeated sequential games". In: *Review of Economic Studies* 69.2 (2002). Cited By :29, pp. 493–512. URL: www.scopus.com.

[73] Yuichi Yamamoto. "A limit characterization of belief-free equilibrium payoffs in repeated games". In: *Journal of Economic Theory* 144.2 (Mar. 2009). cited By 15, pp. 802–824. DOI: 10.1016/j.jet.2008.07.005. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-59749087963&doi=10.1016%2fj.jet.2008.07.005&partnerID=40&md5=7d88a156fde28ffdf65ea6b55

[74] Yuichi Yamamoto. "Characterizing belief-free review-strategy equilibrium payoffs under conditional independence". In: *Journal of Economic Theory* 147.5 (Sept. 2012). cited By 3, pp. 1998–2027. DOI: 10.1016/j.jet.2012.05.016. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-84864804226&doi=10.1016%2fj.jet.2012.05.016&partnerID=40&md5=51959f6f0a44c992e657870ad82e701b.

[75] Kiho Yoon. "An Anti-folk Theorem in Overlapping Generations Games with Limited Observability". In: *Review of Economic Dynamics* 4.3 (July 2001), pp. 736–745. DOI: 10.1006/redy.2000.0127. URL: www.scopus.com.