Folk Theorem and, in particular, at the thresholds where cooperation is more beneficial than defection in the game of a Prisoners' Dilemma.

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SUMMARY

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Chapter 1

Introduction

Game Theory is the study of interactive decision making and developing strategies through mathematics [6]. It analyses and gives methods for predicting the choices made by players (those making a decision), whilst also suggesting ways to improve their 'outcome' [11]. Here, the abstract notion of utility is what the players wish to maximise (see Chapter 2 in [11] for a detailed discussion on the topic of utility theory or Section 1.3 [16] for a more introductory explanation). One of the earliest pioneers of game theory is mathematician, John von Neumann who, along with economist Oskar Morgenstern, published *The Theory of Games and Economic Behaviour* in 1944 [11]. This book discusses the theory, developed in 1928 and 1940-41, by von Neumann, regarding "games of strategy" and its applications within the subject of economics [15]. Following this, several advancements have been made in the area, including, most notably, John Nash's papers on the consequently named Nash Equilibria in 1950/51 [13, 12]. Due to the "context-free mathematical toolbox" [11] nature of this subject, it has been applied to many areas, from networks [10, 14] to biology [5, 1]. In this project, the main focus is on a particular class of theorems, within game theory, known as "Folk Theorems" with application to the game of A Prisoner's Dilemma. These will be defined and discussed in the subsequent sections.

1.1 An Introduction to Games

Consider the following scenario:

Two convicts have been accused of an illegal act. Each of these prisoners, separately, have to decide whether to reveal information (defect) or stay silent (cooperate). If they both cooperate then the convicts are given a short sentence whereas if they both defect then a medium sentence awaits. However, in the

situation of one cooperation and one defection, the prisoner who cooperated has the consequence of a long term sentence, whilst the other is given a deal. [7]

This is one of the standard games in game theory known as A Prisoner's Dilemma. It has four distinct outcomes, for the given two player version, which can be represented as a table (see Table 1.1).

height column player	cooperate	defect
row player cooperate	(3, 3)	(0, 5)
defect	(5, 0)	(1, 1)

Each coordinate (a, b) in the table represents the utility values obtained for each player, where a is the utility value obtained by the row player and b is the utility gained by the column player. These utility values are as given in [2] and are used throughout this project. More formally, the game can be represented as the following matrix:

$$\begin{array}{c} \\ coop \\ defect \end{array} \begin{array}{c} \begin{array}{cc} coop & defect \end{array} \\ \begin{pmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{pmatrix} \end{array}$$

which is known as a *normal form* representation of the game. The following definition is adapted from [11].

In general a *normal form* or *strategic form* game is defined by an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where:

- $N = \{1, 2, ..., n\}$ is a finite set of players;
- $S = S_1 \times S_2, \times ... \times S_n$ is the set of strategies for all players in which each vector $(S_i)_{i \in N}$ is the set of strategies for player i [1]; and

[1] Since the game of A Prisoner's Dilemma has a finite strategy set for each player $S_i =$

- $u_i : S \to \mathbb{R}$ is a payoff function which associates each strategy vector, $\mathbf{s} = (s_i)_{i \in N}$, with a utility [2] $u_i (i \in N)$.

Yet another way of representing this game is as a pair of matrices, A, B, defined as follows:

$$A = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \text{ and } B = A^T \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$$

This way of defining games allow for the use of calculating payoffs for each player (see Section 1.2).

Before continuing the discussion into the key notions of game theory, it needs to be highlighted that there is an important assumption, which is central to most studies of game theory, entitled *Common Knowledge of Rationality*. This, more formally, is an infinite list of statements which claim:

- The players are rational;
- All players know that the other players are rational;
- All players know that the other players know that they are rational; etc.

Assuming Common Knowledge of Rationality allows for the prediction of rational behaviour through a processes entitled *rationalisation* [8] (see section 4.5 in [11] for an alternative explanation of this assumption).

A strategy for player i, s_i, is *strictly dominated* if there exists another strategy for player i, say $\bar{s}_i$, such that for all strategy vectors $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) < u_i(\bar{s}_i, s_i).$$

In this case we say that s_i is *strictly dominated* by $\bar{s}_i$. Here, $s_{-i} = \{s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n\}$, i.e. the ith player's strategy has been omitted. The set, S_{-i}, is defined similarly. Looking at the row player's matrix of a Prisoner's Dilemma 1.1 (the first entries in the ordered tuples), it is clear that cooperation is a strictly dominated strategy. Due to the symmetricity of the game, this is also true for the column player. [11]

So far, only the pure strategies, $S_i = \{\text{coop}, \text{defect}\}$, have been discussed, thus the notion of a probability distribution over S_i is now introduced, giving the so-called *mixed strategies* as defined in [11]: Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game (with each S_i finite), then a *mixed strategy* for player i is a probability

$\{\text{cooperate}, \text{defect}\} (i \in N)$, in this project only finite strategy spaces are considered.

[2] 'Utility' is referred to as a player's 'payoff' throughout the remainder of this report.

distribution over their strategy set S_i. Define:

$$\Sigma_i := \{\sigma_i : S_i \to [0,1] : \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

to be the set of mixed strategies for player i. Hence, observe that the pure strategies are specific cases of mixed strategies, with $\sigma_i = (1,0)$ for cooperation and $\sigma_i = (0,1)$ for defection, in the example of a Prisoner's Dilemma.

This leads onto the following definition of a *mixed extension* of a game, taken from [11]: Let G be a finite normal form game as above, with $S = S_1 \times S_2 \times \cdots \times S_N$ defining the pure strategy vector set and each pure strategy set, S_i being non-empty and finite. Then the *mixed extension* of G is denoted by

$$\Gamma = (N, (\Sigma_i)_{i \in N}, (U_i)_{i \in N}),$$

and is the game in which, Σ_i is the ith player's strategy set and $U_i : \Sigma \to \mathbb{R}$ is the corresponding payoff function, where each $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in \Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_N$ is mapped to the payoff

$$U_i = \mathbb{E}_\sigma(u_i(\sigma)) = \sum_{(s_1, s_2, \ldots, s_N) \in S} u_i(s_1, s_2, \ldots, s_N)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_N(s_n)$$

for all players $i \in N$.

1.2 Nash Equilibrium for Normal Form Games

As mentioned above, mathematician, John Nash, introduced the concept of an equilibrium point and proved the existence of mixed strategy Nash Equilibria in all finite games. These notions are central to the study of game theory [11] and hence, in this section, Nash's concepts will be defined and proved in detail.

Firstly, before the definition of a Nash equilibrium, the idea, as given in [11] of a *best response* is introduced: For a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, the strategy, s_i, of the ith player is considered a *best response* to the strategy vector s_{-i} if $u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i})$.

This leads onto the main definition of the section:

Definition 1.2.1. Given a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, the vector of strategies $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a *Nash equilibrium* if, for all players $i \in N$, s_i^* is a best response to $s_i^* \in N$. [11]

In other words, s^* is a Nash equilibrium if and only if no player has any reason to deviate from their current strategy s_i^*.

Recall that, in Section 1.1, for any player in A Prisoner's Dilemma, defection dominated cooperation. This leads to the following observation:

The strategy pair (Defect, Defect), is the unique Nash equilibrium for A Prisoner's Dilemma, with a payoff value of 1 for each player. s[11]

This can be visualised as followed: Assume the row player uses the following mixed strategy, $\sigma_r = (x, 1-x)$, i.e. the probability of cooperating is x and the probability of defecting is $1-x$. Similarly, assume the column player has the strategy, $\sigma_c = (y, 1-y)$. The payoff obtained for the row and column player, respectively, is then:

$$A\sigma_c^T = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 3y \\ 4y+1 \end{pmatrix},$$

$$\sigma_r B = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 1-x \end{pmatrix} = \begin{pmatrix} 3x & 4x+1 \end{pmatrix}$$

Plotting these gives the following: From Figure 1.1 it is clear that, regardless of

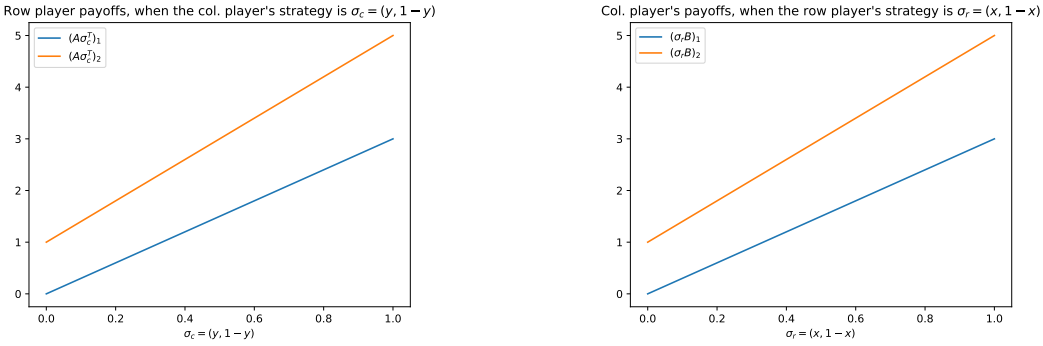

Figure 1.1: Graphs to show the row and column players' payoffs against a mixed strategy.

the strategy played by the opponent, defection is indeed the only rational move for one to play. Thus, both players have no incentive to deviate if and only if both play the strategy $\sigma = (0, 1)$, i.e. defection for every single game of A Prisoner's Dilemma.

On the other hand, consider, for example, a game with no dominated strate-

gies [3]:

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\mathrm{B} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Are Nash Equilibria guaranteed to exist? This result is given in the next theorem, taken from [12], Nash's second paper on equilibria in games.

Theorem 1.2.1. Every finite game has an equilibrium point.

The proof of Theorem includes the use of a*fixed point theorem* and thus, a short sub-section regarding one such result is given, for completeness, before providing a formal proof of 1.2.

1.2.1 Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem is a result from the theory of topology. Named after the Dutch mathematician, L.E.J. Brouwer, it was proven in 1912 [4]. However, before stating this notion, a few conditions regarding the properties of sets are recalled.

The following three definitions appear as in [18, 3, 17] respectively:

Definition 1.2.2. A set $X \subseteq \mathbb{R}^d$ is called *convex* if it contains all line segments connecting any two points $\mathbf{x}_1, \mathbf{x}_2 \in X$.

Definition 1.2.3. An *open cover* of a set $S \subset X$, a topological space, is a collection of open sets $A_1, A_2, \ldots \subset X$ such that $A_1 \cup A_2 \cup \ldots \supset S$, that is, the union of the open sets contain S.

Definition 1.2.4. A subset $S \subseteq X$, a topological space, is called *compact* if, for each open cover of S, there is a finite sub-cover of S.

The presentation of Brouwer's Fixed Point Theorem is now given as in [11]

Theorem 1.2.2. Let $X \subseteq \mathbb{R}^n$ be a non-empty convex and compact set, then each continuous function $f : X \to X$ has a fixed point.

[3]The game highlighted here is another standard used in game theory entitled *Matching Pennies*. Moreover, it is what is defined as a *zero-sum* game. Interested readers are encouraged to read Example 4.21 of [16] for an introduction to the game and Section 4.12 in [11] for an explanation of zero-sum games.

In other words if X and f satisfy the conditions given above then there exists a point $x \in X$ such that $f(x) = x$.

Since this project is regarding game theory, rather than topology, the proof to the above theorem is omitted. However, the interested reader is referred to [] for an in-depth consideration into the theory of topology.

1.2.2 Proof of Nash's Theorem (Theorem 1.2)

The proof provided here is adapted from the original, as presented in [12].

Proof. NEED TO DO!!!!!! □

1.3 Repeated Games

The folk theorems studied in this project are a consequence of games which are repeated several times (not just once). Thus, before discussing the main ideas, the theory of both finitely- and infinitely- repeated games is presented.

However, firstly, a couple of alterations to the terminology used in previous sections is redefined for conciseness and to be consistent with the literature [].

What was known as a 'game' will become known as a *stage game* to highlight the fact that a one-off game is being considered. Also, what was defined previously as a 'strategy' will now be referred to as an *action* to differentiate it from a strategy of a repeated game (see Section 1.3.1).

1.3.1 Finite Repeated Games

According to [9], a *T-stage repeated game*, $T < \infty$ is when the stage game, G, is played T times, over discrete time intervals. Each ith player has a strategy based on previous 'rounds' of the game and the payoff of a repeated game is calculated as the total sum of the stage game payoffs.

Prior to giving the notion of a strategy in a repeated game, the idea of *history*, within the context of repeated games, is provided. The *history*, $H(t)$ of a repeated game is the knowledge of previous actions of all players up until the tth stage game, assumed to be known by player i for all $i \in 1, \ldots, N$. Note that, when $t = 0$, $H(0) = \underbrace{(\emptyset, \emptyset, \ldots, \emptyset)}_{N\text{times}}$, since no stage games have yet been played.

As given in [], a *strategy* of a T-stage repeated game is defined to be a mapping

from the complete history so far to an action of the stage game, that is

$$\tau_i : \cup_{t=0}^{T-1} Ht \rightarrow a_i.$$

Here, $H(t)$ is the history of play as defined above and $a_i is the i$th player's action of the stage game.

Consider, for example, the repeated game in which the stage game of a Prisoner's Dilemma is played each time. This is known as the *Iterated Prisoner's Dilemma* (IPD) and has been a popular topic of research for many years see Chapter **??**.

Chapter 2

Initial Investigation

There exist three algorithms for computing the Nash Equilibria of a game: Support Enumeration, Vertex Enumeration and Lemke Howson algorithm. Thus, firstly an exploration into the properties and execution times was needed to verify which algorithm would be the most appropriate. This investigation, along with a description of each algorithm (in the two-player case), is detailed in the next few sections.

2.1 Support Enumeration Algorithm

For a non-degenerate two player game

$$G = (2, (S_i)_{i=1,2}, (u_i)_{i=1,2})$$

, the following algorithm yields all Nash equilibria:

- For all $1 \leq k \leq min(|S_1|, |S_2|)$,
- and each $I, J \subset S_1, S_2$, respectively, with $|I| = |J| = k$,
- solve $\sum_{i \in I} \sigma_i b_{i,j} = v$, $\sum_{j \in J} a_{i,j} \sigma_j = v$ for all $j \in J$, $i \in I$ respectively
- such that $\sum_{i \in I} \sigma_i = 1$ and $\sum_{j \in J} \sigma_j = 1$ with $\sigma_i, \sigma_j \geq 0$
- and the best response condition in section **??** is satisfied.

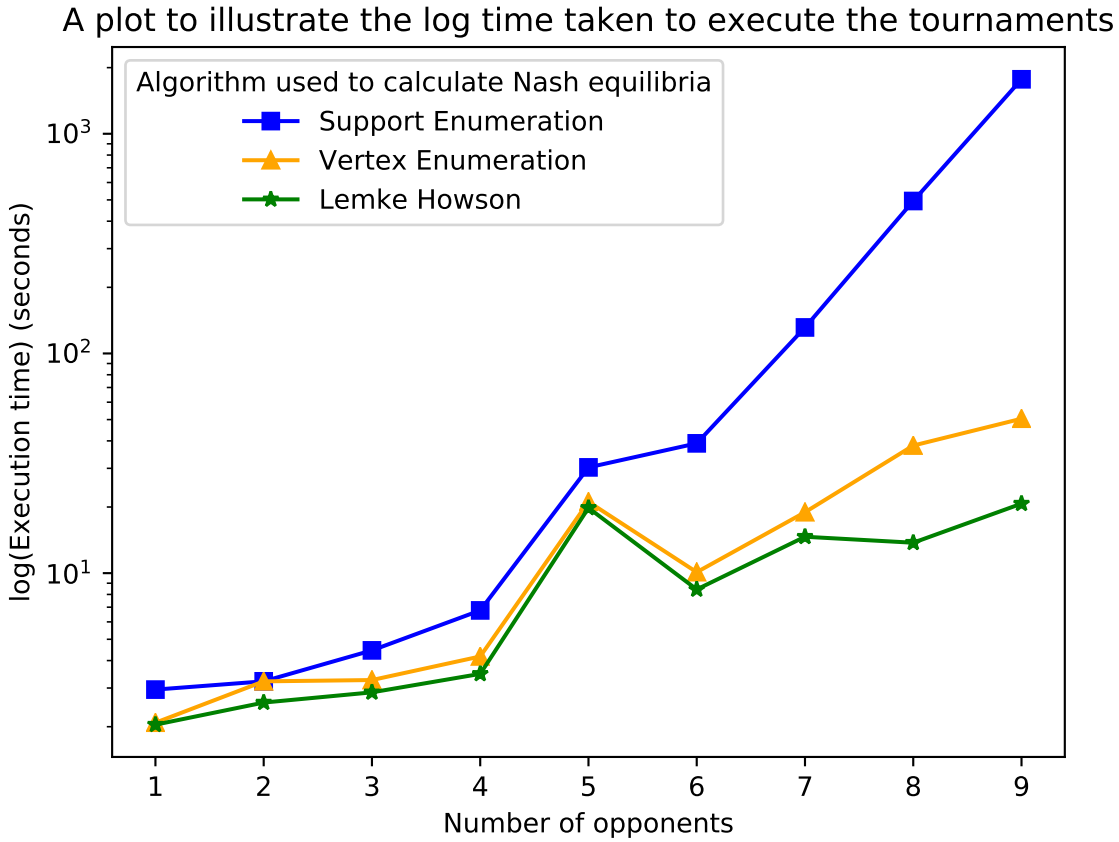

Figure 2.1: A graph to show the results of the timed experiments for each of the three stated algorithms.

2.2 Vertex Enumeration Algorithm

2.3 Lemke Howson Algorithm

2.4 Execution Times

The code used to obtain the results in this section can be found in Appendix ...

From figure **??**, it can be seen that until the Defector has five opponents, each algorithm has approximately the same running time.

After this, as the number of opponents increases it is clear that the support enumeration algorithm blows up significantly quicker exponentially in comparison to boththe Vertex Enumeration and Lemke Howson algorithm. The latter two algorithms seem to have a similar execution time until the number of opponents reaches eight and nine. This is when the Vertex Enumeration algorithm's

execution time starts to grow faster than the Lemke Howson algorithm.

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Appendix A

APPENDIX A TITLE

Appendix B

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