Folk Theorem and, in particular, at the thresholds where cooperation is more beneficial than defection in the game of a Prisoners' Dilemma.

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SUMMARY

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This project looked into the Folk Theorem for the IPD and, in particular, the threshold at which the probability of defection is less than that given in the stage game or the finitely repeated game. The three aims of the study were stated in Chapter **??**. These are provided here for convenience:

1. To look thoroughly into the recent (and past) literature of research already produced in the area of Folk Theorems;
2. To execute a large experiment involving many tournaments of the IPD with different types and numbers of players to obtain graphs similar to those in Figure **??**; and
3. To perform analyses on where the p-threshold seem to lie and whether it is affected by the different environments of the tournaments (that is, differing number of players, stochasticity within the tournament itself and the players etc.).

Therefore, in this chapter, a brief section discussing the results of each aim is provided. Then, a few shortcomings of the project are mentioned, before highlighting areas in which the research could be expanded further.

1.1 Conclusions

In this section, the results of each aim is explored with regards to what was found and how well they were addressed. Overall, the first two aims were successfully achieved with a well rounded knowledge in the recent research of folk theorems and a successful empirical experiment carried out. However, maximal results were not obtained for the final aim due to time limitations and the unforeseen non-triviality of the proportion of degeneracy and randomness.

1.1.1 Aim 1

The first aim was addressed in Chapter **??**. Here, a literature search on "folk theorem" yielded a vast range of results which spanned the last fifty years. It was discussed that the true origin of the 'folk theorem' is unknown however its statement and proofs appear in written work since the 1970s. These papers focused on infinitely repeated games with subgame perfect equilibria. Following this, many generalisations and refinements of the folk theorem were explored. Varying types of games, for which folk theorems have been considered, were discussed. These included: games with complete information, games with imperfect private monitoring and games with communication, among others. Within these papers, it was also discovered that other equilibria besides the more common subgame perfect equilibria have been considered. Some examples are: sequential equilibria, belief-free equilibria and type-k equilibria. Throughout, there appeared to be two main assumptions required for a folk theorem to exist. These regarded full dimensionality and the number of signals in proportion to the number of actions. Since these assumptions are quite restricting, a few papers have attempted to find situations in which they can be weakened, whilst still obtaining a folk theorem. On the other hand, an area of research has developed which show the instability of equilibria and / or the violation of folk theorems. The chapter is concluded by a discussion on recent applications of the folk theorem to computing, multi-agent systems and quantum transportation. Finally, to the best of the author's knowledge, no folk theorem experiments of the same size, as executed in this study, have been computed before.

1.1.2 Aim 2

Chapter **??** discusses the set-up and execution of the IPD experiment referred to in Aim 2. This included justifications for the use of certain software and methods as well as a detailed view of the data collection algorithm. Within this, reasons were given as to why the particular attributes of the tournaments were collected. Following this an exploration into the varying file types available for data collection was provided. After considering the positives and drawbacks it was decided that a binary file, in particular a relational database file, would be the most appropriate format. The chapter then continues with an explanation of the software implementation and execution of the aforementioned algorithm. It provides details as to how good software development principles were followed and the programs utilised (for a more detailed overview which is not as involved as the chapter, see **??**). The actual experiment was executed remotely over a few weeks and how this was achieved was explained, before the issues, how to compute the Nash equilibria and degeneracy, were addressed. Out of the three algorithms: Support Enumeration, Vertex Enumeration and Lemke-Howson; it was decided that Support Enumeration would be the main method used in calculating Nash equilibria, due to its robustness. The execution timings of each algorithm were also explored but did not yield any significant differences. To conclude, the difficulties which could be faced due to degeneracy were briefly acknowledged and possible solutions discussed.

1.1.3 Aim 3

The final aim regarding the analysis of p-threshold was addressed in Chapter **??**. Firstly, an initial analysis was carried out which identified the main 'shapes' of the graphs obtained and detailed the summary statistics. This included looking at characteristics of the strategies as well as the number of equilibria obtained from each tournament. The majority of strategies chosen were deterministic - this is because of the ratio of deterministic to stochastic strategies implemented in the Axelrod library. When analysing the p-threshold, the minimum, mean, median and maximum potential values of the threshold were taken, due to the varying sources of noise obscuring a clear threshold. Here, the focus was on the effects of the number of players and the effects of the tournament noise level. However, this was concluded as a non-trivial task due to the proportion of tournaments which were potentially degenerate and the many sources of randomness within them. As a result of this, significant conclusions were unable to be drawn with the obtained plots not revealing many (if any) trends. Thus, suggestions are provided on how further work regarding this study may reveal more significant information. These are restated in Section 1.3. In conclusion, however, the experiment was successful in providing clear visualisations for the folk theorem and gives insight into a wide range of directions for future research.

1.2 Limitations

In this section a few of the shortcomings of the study are highlighted along with justifications.

Firstly, this project included a very large empirical study on the folk theorem which required the development and implementation of an algorithm into code. This inevitably took a significant amount of time; to ensure the functions were accurate, clear and gave the required information. Therefore, one of the main drawbacks was the time constraint of completing the project. Indeed, after the successful implementation of the data collection algorithm into Python, there was a certain time frame before analysis could start, to ensure enough data had been collected. Then the analysis took another significant portion of the time, especially due to the data's non-triviality.

As a result of this, the amount of data collected was much less than what was required for a large-scale statistical analysis on extremely random data. Also, parameters inputted into the algorithm were restricted by this. Only 500 repetitions of each tournament were performed but this was only able to 'stabilise' some of the payoff matrices, whilst the rest resulted in less clear threshold graphs. Thus, higher repetitions were probably needed but unrealistic for the time frame. Moreover, the eighteen strategies which are classified with a long run-time had to be omitted but, in an ideal situation would have been included to obtain more information.

Furthermore, the analysis which was carried out only touched briefly on some of the potential effects of the environment on the p-threshold. The 'randomness' of the data obtained meant that trends were harder to find and key conclusions could not be drawn. Ideally, many more possible environmental changes, in addition to number of players and level of tournament noise, would have been discussed. Another factor which may have improved the situation, but had to be omitted due to time, was the packaging of the code. Creating a fully working Python package, with all the required functions in, may have made the remote computing easier.

Finally, degeneracy was a major limitation due to its uncertainty. Nashpy only identifies potential degeneracy if it is 'strange'. That is, if the algorithm detects division by zero or 'peculiar' equilibria, for example. Indeed, some games may still be degenerate even if some correct equilibria are identified. This implies that all the discussions regarding degeneracy in Chapter **??** are based on the assumption that support enumeration detects all degeneracy, which is potentially unrealistic.

1.3 Recommendations

Throughout this study, many interesting questions have been raised as potential directions for further work. Hence, in this section, recommendations on how this research could be extended, to provide more insightful results, are given.

Firstly, during analysis (Chapter **??**), a significant proportion of the tournament graphs showed no threshold, appearing constant at zero or one. This indicates that the threshold must lie in the intervals (0, 001) or (0.999, 1) and hence the precision level was not fine enough. Therefore, it is suggested that these tournament sets are rerun within these intervals with a fine precision in order to obtain more information. Moreover, an analysis of the strategies involved and level of tournament noise in these sets might provide a clearer insight into reasons for this. In addition, rerunning the whole experiment, with a finer precision in the game ending probabilities, may give more accurate thresholds.

In order to explore the effects of stochastic players on the threshold, it is recommended that those tournament sets which contain one (many) stochastic player(s) are rerun without them. If the threshold is different in both of the runs, this could indicate that stochasticity does indeed have an effect.

Recall, vertex enumeration was also used in a separate run of the experiment however, unfortunately, it was omitted from the analysis due to the time restrictions mentioned previously. Having said this, it is highly recommended that these results are looked into (or rerun) as a way of checking the reliability of the collected data. In particular, it could be used in comparison with the support enumeration data to identify whether the same Nash equilibria are yielded or, more importantly, whether the same games are identified as potentially degenerate. However, care is needed when analysing vertex enumeration as it is less robust, as discussed in Chapter **??**. Furthermore, increasing the number of tournament repeats is suggested to observe whether this 'smooths' the payoff matrices with more success, perhaps, resulting in clearer p-thresholds.

Regarding multivariate data analysis of this large empirical study, it may be insightful to perform a regression or clustering algorithm. This is motivated by the possibility of being able to predict approximately the threshold of a tournament by its characteristics: strategies, level of tournament noise etc. Finally, the experiment could potentially be extended to consider the different 'types' of folk theorem discussed in Chapter **??**.

Appendix A

APPENDIX A TITLE

Appendix B

APPENDIX B TITLE