An Empirical Study on the Folk Theorem

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Summary

"Under what conditions will cooperation emerge in a world of egoists without central authority?" Robert Axelrod provides justification for this study in the first section of his book [8]. Today, many examples can be given in which cooperation has evolved in situations where, in the short term, it may be preferable to counteract. Due to this, a class of theorems have emerged over the past fifty years, providing explanation for the unintuitive phenomena.

This project consists of an empirical study into these theorems, entitled 'Folk Theorems', which are key in the repeated games theory. The aims of the project include: an in-depth review of academic literature regarding the theorems; the execution of a large experiment based on the 'original' folk theorem of Friedman [33] with the Iterated Prisoner's Dilemma; and an analysis of the effects of different tournament characteristics on the p-threshold[1] described in the folk theorems. These ideas are extended from a third year assignment, completed by the author, in Game Theory.

Firstly, after an introduction to the theory of one-shot and repeated games in Chapter 1, a search of the folk theorem literature is provided in Chapter 2. This reveals the vast directions of research in the area for the past fifty years. Many generalisations and refinements of the folk theorem have been analysed since the first written papers of the 1970s. The games for which the notion has been applied to range from complete information games to games with imperfect private monitoring. However, in certain cases, the strategies used in the proof of these are unstable. Also, in situations where an individual deviator cannot be identified, a much smaller set of payoffs is achieved, yielding the so-called 'Anti-Folk Theorems'. More recently, focus has been on the application of the folk theorem to different scenarios including: computing and quantum transportation. Finally, it is concluded that, to the best of the author's knowledge, this is the first study to execute an experiment of this size on the folk theorem.

[1]The p-threshold is defined as the probability of the tournament ending for which the least probability of defection in Nash equilibria becomes zero.

Following this, a detailed description of how the experiment was set-up and executed is given in Chapter 3, with justifications to the specific methods and software chosen. After considering the benefits and drawbacks of varying file formats for storing data, it is stated that a relational database, in particular SQLite, would be the most appropriate. This is due to: the existence of libraries in Python enabling easier access of the data; the general robustness of databases; and their ability to perform out of memory operations. Due to the pre-existing game theoretic libraries, Axelrod and Nashpy, Python was chosen as the language of implementation and ensuring good software development principles were followed is highlighted as a priority. The volume of data which was intended to be collected meant that remote computing was required and this is explained here. The choice of support enumeration for the calculation of Nash equilibria and the potential issues which may be faced due to degeneracy is also discussed.

The main analysis of the data[2] collected is provided in Chapter 4. An initial analysis discussing the characteristics of the strategies used and the overall summary statistics is detailed, before the p-thresholds are explored. The tournament characteristics focused on are: the number of opponents the Defector had, and the level of standard Iterated Prisoner's Dilemma noise included. However, this is concluded as a non-trivial task due to the uncertainty of degeneracy. Also, the inevitability of randomness within the tournaments meant that a lot more data is required. Indeed, there are three sources of noise impacting the tournaments. On the other hand, the graphs that were yielded are successful in visualising the folk theorem.

Finally, Chapter 5 details the conclusions of the research prior to giving recommendations for future work. Amongst these are suggestions on how further characteristics of the tournament set could be studied, and the potential to predict the p-threshold via regression analysis.

[2]The code used to obtain this data is available in the GitHub repository (https://github.com/shapperzsm/final-project) and the data is archived at (https://doi.org/10.5281/zenodo.3784594)

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Last but most certainly not least, I want to say a huge thank you to my Mum, Dad and Zoe for their continual love and support. To say these have been a challenging four years is an understatement. My Mum, thank you for keeping me fuelled with coffee and biscuits through the long days and for being a shoulder to cry on when necessary. I will give you enough hair dye to cover all those grey ones I caused! Dad, thank you for making me laugh with your terrible jokes, and finally Zoe, (yes, you are getting a mention because, yes, you did play a key part) thank you for keeping me sane and being a great sister.

To those I have forgotten to mention: my apologies and thank you.

Contents

List of Figures

Chapter 1

Introduction

World War I, a time of harsh conflict and battle, provided an example of how cooperation need not evolve from friendship. Indeed, [8] states that small units of common soldiers on the Western Front were able to execute a "live and let live" system, even against the will of the officers. They knew that "if the British shelled the Germans, the Germans replied; and the damage was equal" (quoted from [25] as given in [8]). Moreover, this was achieved without a direct truce as officers forbade it. The analysis of such circumstances, and other situations involving choice, is covered by an area of mathematics entitled *game theory*.

According to [78], *game theory* is the study of interactive decision making and the development of strategies through mathematics. It analyses and gives methods for predicting the choices made by players (those making a decision), whilst also suggesting ways to improve their 'outcome' [68]. Here, the abstract notion of utility is the outcome players wish to maximise. For further information on the topic of utility theory, readers are referred to Chapter 2 in [68] for a detailed discussion or Section 1.3 in [102] for a more introductory explanation.

One of the earliest pioneers of game theory is mathematician, John von Neumann who, along with economist Oskar Morgenstern, published *The Theory of Games and Economic Behaviour* in 1944 [68]. This book [74] discusses the theory, developed in 1928 and 1940, by von Neumann, regarding "games of strategy" and its applications within the subject of economics. Following this, several advancements have been made in the area including, most notably, John Nash's papers on the consequently named Nash Equilibria in 1950-51 [71, 72]. Due to the "context-free mathematical toolbox" [68] nature of this subject, it has been applied to many areas, from networks [65, 92] to biology [2, 22].

In this project, the main focus is a class of theorems within game theory, known

C, C	C, D
D, C	D, D

Table 1.1: Outcomes for a game of the Prisoner's Dilemma.

as "Folk Theorems". These ideas assist in the analysis of long-term behaviour and evolution of cooperative strategies. In particular, the theory will be applied to the game of a Prisoner's Dilemma which is introduced in subsequent sections. The structure of this report is as follows: Chapter 2 provides a literature review on the topic of folk theorems, before Chapter 3 discusses the development of a large experiment regarding these notions in the Prisoner's Dilemma. Chapter 4 analyses the results obtained whilst Chapter 5 provides the conclusions and recommendations for further study. However, first, the remainder of Chapter 1 is dedicated to the key definitions and theorems required for a study in game theory.

Unless stated otherwise, the definitions and notation in this chapter have been adapted from [68].

1.1 An Introduction to Games

Consider the following scenario:

> Two convicts have been accused of an illegal act. Each of these prisoners, separately, have to decide whether to reveal information (defect) or stay silent (cooperate). If they both cooperate then the convicts are given a short sentence whereas if they both defect then a medium sentence awaits. However, in the situation of one cooperation and one defection, the prisoner who cooperated has the consequence of a long term sentence, whilst the other is given a deal [55, 68].

This is one of the standard games in game theory known as the Prisoner's Dilemma (PD). It has four distinct outcomes, for the given two player version, which can be viewed in Table 1.1. Note, following the standard literature, cooperation and defection is indicated by C and D, respectively.

More formally, the game can be represented as the following matrix:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (3,3)(R,R) & (0,5)(S,T) \\ (5,0)(T,S) & (1,1)(P,P) \end{pmatrix} \end{array} \tag{1.1}$$

where each coordinate (a, b) in the table represents the utility values obtained for each player, where a is the utility value obtained by the row player and b is the utility gained by the column player. These utility values (payoffs)[1] are as given in [7] and used throughout this project. In general, the PD payoffs are constrained by the two conditions:

$$T > R > P > S \tag{1.2}$$

and

$$2R > T + S \tag{1.3}$$

where (1.2) ensures that D is preferable to C and yet (1.3) ensures that mutual C is best [56, 85]. The matrix given in (1.1) is known as a *normal form* representation of the game.

Definition 1.1.1. In general a *normal form* or *strategic form* game is defined by an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where:

- $N = \{1, 2, \ldots, n\}$ is a finite set of players;
- $S = S_1 \times S_2, \times \cdots \times S_n$ is the set of strategies for all players in which each vector $(S_i)_{i \in N}$ is the set of strategies for player i[2]; and
- $u_i : S \to \mathbb{R}$ is a payoff function which associates each strategy vector, $s = (s_i)_{i \in N}$, with a utility $u_i(i \in N)$.

Yet another way of representing this game is as a pair of matrices, A, B, defined as follows:

$$A = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \text{ and } B = A^T = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} \tag{1.4}$$

This way of defining games allows for the use of linear algebraic expressions in the calculation of utilities (see Section 1.2).

[1] 'Utility' is referred to as a player's 'payoff' throughout the remainder of this report.

[2] Since the game of the PD has a finite strategy set for each player $S_i = \{\mathrm{C}, \mathrm{D}\}(i = 1, 2, \ldots, n)$, in this project only finite strategy spaces are considered.

Before continuing the discussion into the key notions of game theory, it needs to be highlighted that there is an important assumption, which is central to most studies of game theory, entitled *Common Knowledge of Rationality.* This, more formally, is a recurring list of beliefs which claim:

- The players are rational;
- All players know that the other players are rational;
- All players know that the other players know that they are rational;
- etc.

Assuming Common Knowledge of Rationality allows for the prediction of rational behaviour through a process known as *rationalisation* [56]. See Section 4.5 in [68] for an alternative explanation of this assumption.

Thus far, only the pure strategies, $S_i = \{C, D\}$, have been discussed, hence the notion of a probability distribution over S_i is now introduced, giving the so-called *mixed strategies.*

Definition 1.1.2. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game, then a *mixed strategy* for player i is a probability distribution over their strategy set S_i. The set of mixed strategies for player i is defined by

$$\Sigma_i = \left\{ \sigma_i : S_i \to [0, 1] \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}. \tag{1.5}$$

Hence, observe that the pure strategies are specific cases of mixed strategies, with $\sigma_i = (1, 0)$ for cooperation and $\sigma_i = (0, 1)$ for defection, in the PD.

This leads onto the following definition of a *mixed extension* for a game.

Definition 1.1.3. Let G be a finite normal form game as above, with $S = S_1 \times S_2 \times \cdots \times S_n$ defining the pure strategy vector set and each pure strategy set, S_i being non-empty and finite. Then the *mixed extension* of G is denoted by

$$\Gamma = (N, (\Sigma_i)_{i \in N}, (U_i)_{i \in N}), \tag{1.6}$$

and is the game in which, Σ_i is the ith player's strategy set and $U_i : \Sigma \to \mathbb{R}$ is the corresponding payoff function, where each $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ is mapped to the payoff:

$$U_i = \mathbb{E}_\sigma(u_i(\sigma)) = \sum_{(s_1, s_2, \ldots, s_n) \in S} u_i(s_1, s_2, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n) \tag{1.7}$$

for all players $i \in N$.

1.2 Nash Equilibrium for Normal Form Games

As mentioned above, mathematician, John Nash, introduced the concept of an equilibrium point and proved the existence of mixed strategy Nash Equilibria in all finite games. These notions are central to the study of game theory [68] and hence, in this section, Nash's concepts will be defined and proved in detail.

Firstly, the idea of a *best response* is introduced.

Definition 1.2.1. For a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, the strategy, s_i, of the ith player is considered a *best response* to the strategy vector s_{-i} if $u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i})$.

This leads onto the main definition of the section.

Definition 1.2.2. Given a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ and its mixed extension, Γ, the vector of strategies $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* if, for all players $i \in N$, σ_i^* is a best response to $\sigma_{-i}^* \in N$.

In other words, σ^* is a Nash equilibrium if and only if no player has any reason to deviate from their current strategy σ_i^*.

The following observation is highlighted as an example.

The strategy pair (D, D), is the unique Nash equilibrium for the PD, with a payoff value of 1 for each player.

Assume the row player uses the following mixed strategy, $\sigma_r = (x, 1-x)$, that is, the probability of cooperating is x and the probability of defecting is $1 - x$.

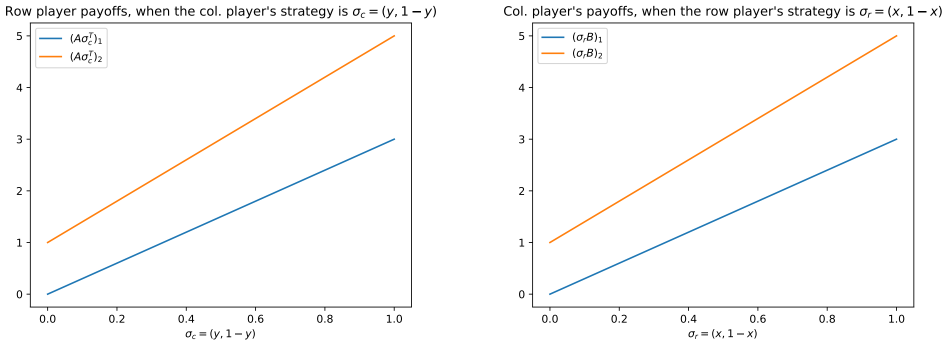


Figure 1.1: Graphs to show the row and column players' payoffs against a mixed strategy.

Similarly, assume the column player has the strategy, $\sigma_c = (y, 1-y)$. The payoff obtained for the row and column player, respectively, is then:

$$A\sigma_c^T = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 3y \\ 4y+1 \end{pmatrix},$$

$$\sigma_r B = \begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3x & 4x+1 \end{pmatrix}.$$

Plotting these gives the graphs as seen in Figure 1.1.

From Figure 1.1 it is clear that, regardless of the strategy played by the opponent, defection is indeed the only rational move. Thus, the players have no incentive to deviate if and only if both play the strategy $\sigma = (0, 1)$, that is, defection for every one-shot game of the PD.

This next result is taken from [72], Nash's second paper on equilibria in games. The notion obtained here is fundamental to many areas of game theory, including the folk theorems.

Theorem 1.2.1. Every finite game has an equilibrium point.

The proof of Theorem 1.2.1 includes the use of a *fixed point theorem*. Thus, a short sub-section regarding one such result is given for completeness, before providing a formal proof of Theorem 1.2.1.

1.2.1 Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem is a result from the field of topology. Named after the Dutch mathematician, L.E.J. Brouwer, it was proven in 1912 [20]. However, before stating this notion, a few conditions regarding the properties of sets are recalled.

The following three definitions appear as in [10, 103, 104] for Definitions 1.2.4, 1.2.5, and 1.2.3, respectively.

Definition 1.2.3. A set $X \subseteq \mathbb{R}^d$ is called *convex* if it contains all line segments connecting any two points $x_1, x_2 \in X$.

Definition 1.2.4. An *open cover* of a set $S \subset X$, a topological space, is a collection of open sets $A_1, A_2, \ldots \subset X$ such that $A_1 \cup A_2 \cup \ldots \supset S$, that is, the union of the open sets contain S.

Definition 1.2.5. A subset $S \subseteq X$, a topological space, is called *compact* if, for each open cover of S, there is a finite sub-cover of S.

The presentation of Brouwer's Fixed Point Theorem is now given as in [68].

Theorem 1.2.2. Let $X \subseteq \mathbb{R}^n$ be a non-empty convex and compact set, then each continuous function $f : X \to X$ has a fixed point.

In other words if X and f satisfy the conditions given above then there exists a point $x \in X$ such that $f(x) = x$.

Since this project is regarding game theory, rather than topology, the proof to the above theorem is omitted. However, the interested reader is referred to [44] for an in-depth consideration into the theory of topology.

1.2.2 Proof of Nash's Theorem

The proof provided is adapted from the original, as presented in [72], with extra notes from [68]. According to [68], the general idea is to define a function, which satisfies the conditions required for Theorem 1.2.2, by using the payoff functions on the set of mixed strategies. Then, through identifying each equilibrium point with a fixed point of the function, the required result is obtained.

Firstly, a brief restatement of the notation needed is provided for clarity. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a finite game with mixed extension $\Gamma = (N, (\Sigma_i)_{i \in N}, (U_i)_{i \in N})$. Here, $N = \{1, \ldots, n\}$ denotes the set of players; $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of pure strategies for all players, with $(S_i)_{i \in N}$ the pure strategy set for player i; Σ is defined similarly but relating to mixed strategies; and $U_i : \Sigma \to \mathbb{R}$ are the payoff functions as given in (1.7).

Proof. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a tuple of mixed strategies and $U_{i,t}(\sigma)$ be the ith player's payoff if they changed to their s_i^tth pure strategy and all other players continue to use their mixed strategy. Now, define function $f : \Sigma \to [0, \infty)$ such that

$$f_{i,t}(\sigma) = \max\left(0, U_{i,t}(\sigma) - U_i(\sigma)\right) \tag{1.8}$$

and also let

$$\sigma_i' = \frac{\sigma_i + \sum_t f_{i,t}(\sigma) s_i^t}{1 + \sum_t f_{i,t}(\sigma)} \tag{1.9}$$

be a modification of each $\sigma_i \in \sigma$, with $\sigma' = (\sigma_1', \sigma_2', \ldots, \sigma_n')$. In words, this modification increases the proportion of the pure strategy s_i^t used in σ_i if the payoff gained by the ith player is larger when they replace their mixed strategy by s_i^t. Else, it remains the same if doing this decreases their payoff as $f_{i,t}(\sigma) = 0$ in this case. Note, the denominator ensures that the ending vector is still a probability distribution by standardising.

The aim is to apply Theorem 1.2.2 to the mapping $T : \sigma \to \sigma'$ and show that its fixed points correspond to Nash equilibria. Thus, firstly compactness and convexity of the set Σ is shown along with continuity of the function f.

Claim 1: The set Σ is compact and convex. Observe that each σ_i can be represented by a point in a simplex in a real vector space with the vertices given by the pure strategies, s_i^t. Therefore, it follows that the set Σ_i is convex and compact. Using the result, *If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are convex compact sets then the set $A \times B$ is a convex compact subset of* $\mathbb{R}^{n+m}$ (highlighted in [68]), gives the convexity and compactness of the set Σ, the cross product of all Σ_is.

Claim 2: The function f is continuous. The continuity of the function f depends upon the continuity of the payoff functions U_i. As given in [68], this is shown by first proving that the U_i are multilinear functions in the variables $(\sigma_i)_{i \in N}$ and then applying the fact that any multilinear function over Σ is a continuous function[3]. The result then follows.

Hence, by Theorem 1.2.2, the mapping T must have at least one fixed point. The proof is concluded by showing that any fixed points of T are Nash equilibria and vice versa.

Claim 3: Any fixed point of T is a Nash equilibrium. Suppose σ is such that $T(\sigma) = \sigma$. Then the proportion of s_i^t used in the mixed strategy σ_i must not be altered by T. Therefore, in σ_i', the sum $\sum_t f_{i,t}(\sigma)$ in the denominator must equal zero, otherwise the total sum of the denominator will be greater than one, decreasing the proportion of s_i^t. This implies that for all pure strategies s_i^q, $f_{i,q}(\sigma) = 0$. That is, player i can not improve their payoff by adopting any of the

[3] For a detailed consideration of the continuity of the payoff functions see [68], pages 148–149.

pure strategies. Note, this is true for all i and s_i^q by definition of $T(\sigma) = \sigma$ and thus no player is able to improve their payoff. By Definition 1.2.2, this is exactly the conditions of a Nash equilibrium.

Claim 4: Any Nash equilibrium is a fixed point of T. Assume σ is a Nash equilibrium. Then, by definition, it must be that $f_{i,q}(\sigma) = 0$ for all pure strategies q for all players, $i \in N$. Note, if $f_{i,q}(\sigma) \neq 0$, then the ith player would benefit from changing their strategy to the pure strategy s_i^q, which violates the condition for a Nash equilibrium. From this it follows that $T(\sigma) = \sigma$, that is, σ is a fixed point of T. This concludes the proof. $\square$

1.3 Repeated Games

The folk theorems studied in this project are a consequence of games which are repeated several times. Indeed, repeated games provide more insight into how and why cooperation can evolve. Moreover, there are cases in which further equilibria become supported when compared to the one-shot equivalent. It could also be argued that repeated games provide more realistic results regarding interactions, since the majority of situations are faced on a regular basis. Thus, before discussing the main statements of the study, the theory of both finitely and infinitely repeated games is presented.

Firstly, a couple of alterations to the terminology used in previous sections is redefined, to be consistent with the literature. The notion of a 'game' will become known as a *stage game* to highlight the fact that a one-off game is being considered. Also, what was defined previously as a 'strategy' will now be referred to as an *action* to differentiate it from a strategy of a repeated game, see Section 1.3.1.

1.3.1 Finite Repeated Games

According to [55], a *T-stage repeated game*, $T < \infty$ is when the stage game, G, is played T times, over discrete time intervals. Each player has a strategy based on previous 'rounds' of the game and the payoff of a repeated game is calculated as the total sum of the stage game payoffs.

Prior to giving a formal description of a strategy in a repeated game, the idea of *history*, within the context of repeated games, is provided.

Definition 1.3.1. The *history*, $H(t)$ of a repeated game is the knowledge of previous actions of all players up until the tth stage game, assumed to be known by all players. Note that, when $t = 0$, $H(0) = \underbrace{(\emptyset, \emptyset, \ldots, \emptyset)}_{N\text{times}}$, since no stage games have yet been played.

Definition 1.3.2. As given in [56, 68], a *strategy* of a T-stage repeated game is defined to be a mapping from the complete history so far to an action of the stage game, that is

$$\tau_i : \bigcup_{t=0}^{T-1} H(t) \to a_i. \tag{1.10}$$

Here, $H(t)$ is the history of play as defined in Definition 1.3.1 and a_i is the ith player's action of the stage game.

Consider, for example, the environment in which the stage game PD is repeated each time. This is known as the *Iterated Prisoner's Dilemma* (IPD) and has been a popular topic of research for many years[4]. Note that the objective here is to maximise payoff. The player:

No matter what my opponents play, I will always defect,

commonly known as the 'Defector' has the following strategy mapping:

$$\tau_i : \bigcup_{t=0}^{T-1} H(t) \to a_i, \tag{1.11}$$

where $a_i = D$ for all time periods $\tau \geq 0$. Other common IPD strategies include:

- Cooperator — *No matter what my opponents play, I will always cooperate*;
- Random — *I will either cooperate or defect with a probability of 50%*; and
- Tit For Tat — *I will start by cooperating but then will duplicate the most recent decision of my opponent.*

Figure 1.2 shows the possible payoffs obtained in a 2-stage repeated IPD with two players.

Now, a discussion on Nash equilibria in repeated games is provided. It can be proven that there exist many equilibria in repeated games [33]. The next result, adapted from [56, 68] guarantees at least one.

[4] The interested reader is referred to the following papers [39, 50, 76] for good reviews regarding the IPD.

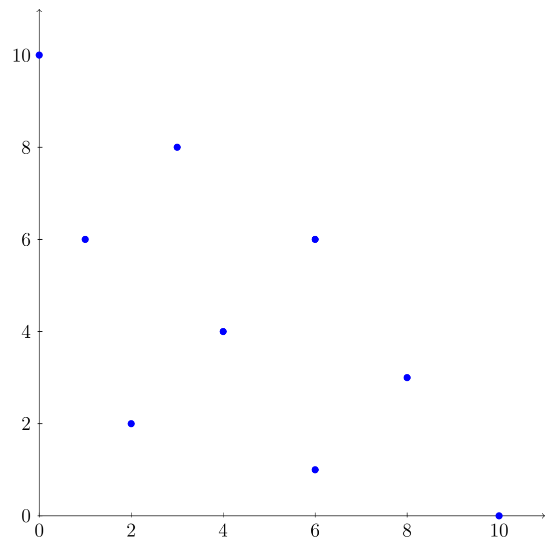


Figure 1.2: A plot to show the possible payoffs of the game between two players in which the PD is repeated twice.

Theorem 1.3.1. Consider a T-stage repeated game with $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ as the stage game, $0 < T < \infty$. Define by $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$, a stage Nash equilibrium of G. Then the sequence in which σ^* is continuously played is a Nash equilibrium of the T-stage repeated game.

Proof. Since σ^* is a stage Nash equilibrium, it is, in particular, a Nash equilibrium of the Tth stage game. Thus, no player has any reason to deviate here. However, σ^* was also played at the $(T-1)$th stage, meaning there is still no reason to deviate. Therefore, continuing via backwards induction gives the required result. $\square$

Hence, for the T-stage IPD, all players executing the Defector strategy yields a Nash equilibrium. However, it could be argued that this does not explain why cooperation evolves in many situations.

1.3.2 Infinite Repeated Games

This section discusses the case when $T \to \infty$ and results linked to *infinitely repeated games*. These provide a more realistic framework for analysing behaviours.

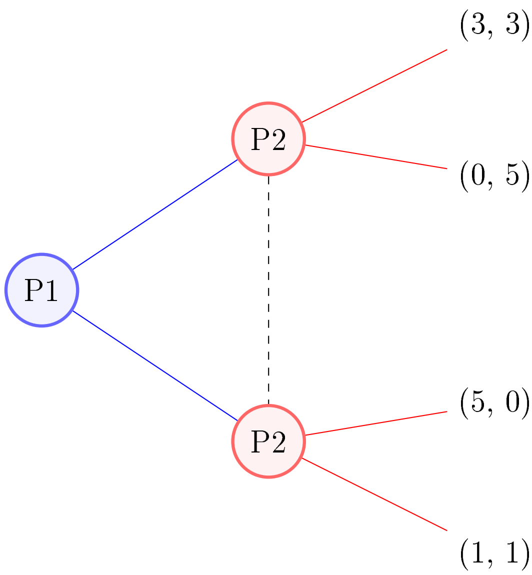


Figure 1.3: The extensive form representation of the PD.

Extensive Form Games

The Folk Theorem discussed in Section 1.4 considers a stronger refinement of Nash equilibria, for repeated games, known as *subgame perfect equilibria.* In order to fully understand this notion, a new representation of games is introduced.

Definition 1.3.3. An *extensive form game* is given by the ordered vector $\Gamma = (N, V, E, x_0, (V_i)_{i \in N}, O, u)$ where $N = \{1, 2, \dots, n\}$ is a finite set of players; (V, E, x_0) is a *game tree*[5]; $(V_i)_{i \in N}$ is a partition of the set $V \setminus L$, where L is the set of all leaves, or terminal points, of the game tree; O is the set of outcomes for the game; and u is a function which maps each leaf in L to an outcome in O.

This leads on to the following definition, adapted from [102].

Definition 1.3.4. A player's *information set* is a subset of the nodes in a game tree where:

- Only the player concerned is deciding;
- This player is not aware of which node has been reached, except that it is definitely one of the elements found in this set.

In Figure 1.3, the extensive form representation of the PD is provided. Here, only two players are considered and any information sets are represented by a dashed line. Any normal form game can be represented as an extensive form game.

[5]The triple (V, E, x_0) is defined as a *tree* if the set of vertices, V, and the set of edges, E, create a *directed graph*, that is, each element in E is an ordered tuple. The root, or starting node, of the graph is represented by x_0

Definition 1.3.5. According to [102], a *subgame* is a sub-graph of the game tree such that:

- The sub-graph begins at a decision node, say x_i;
- This node, x_i, is the only element contained in its information set; and
- The sub-graph contains all of the decision nodes which follow x_i.

This leads to the following definition of *subgame perfect equilibria*, also adapted from [102].

Definition 1.3.6. A *subgame perfect equilibrium* is a Nash equilibrium which satisfies the condition that the strategies played define a Nash equilibrium in every subgame.

Hence the strategy defined in Theorem 1.3.1 is a subgame perfect equilibrium. A few final definitions are now highlighted before introducing the Folk Theorem.

Final Definitions Needed

Now, in order to be able to discuss the payoffs of strategies in infinite games, a few final definitions are required.

Definition 1.3.7. In [55] a *discounted payoff* is defined as:

$$V_i(\sigma) = \sum_{t=1}^{\infty} \delta^{t-1} U_i(\sigma), \tag{1.12}$$

where the discount factor, δ, can be thought of as the probability that the game continues. That is, the probability that another stage game will be played.

Definition 1.3.7 can be used to define *average payoffs.*

Definition 1.3.8. According to [55], the *average payoffs* per stage game, are given by:

$$\frac{1}{\overline{T}} V_i(\sigma) = (1-\delta) V_i(\sigma), \tag{1.13}$$

where $\overline{T} = \frac{1}{1-\delta}$ is the average length of a game.

Finally, Figure 1.4 shows those payoffs which are individually rational for a two player version of PD. In general, an *individually rational payoff* is an average payoff which exceeds those obtained in the stage Nash equilibria for all players [55]. Often the Nash equilibrium payoff is not the optimal payoff players could achieve.

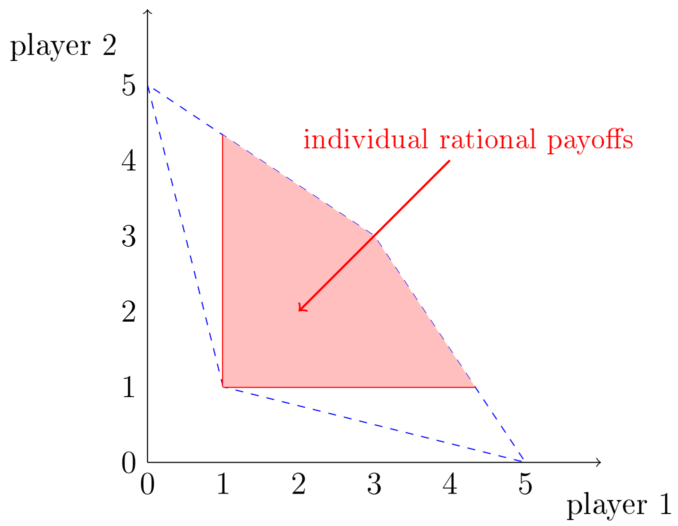


Figure 1.4: A plot highlighting the individually rational payoffs for the PD.

1.4 Folk Theorem

This section contains the statement and proof of the main theorem in this project.

According to [102], the Folk Theorems are so-called because their results were well-known before a formal proof was provided. In general, these theorems state that players can achieve a better payoff than the Nash equilibrium (if the Nash equilibrium payoff is not optimal) when the stage game is repeated many times and the probability of the game continuing is high enough.

It is believed that [33] was one of the first to provide a formal proof to the widely accepted Folk Theorem [1, 102]. Thus, the presentation of the statement and proof given here is adapted from [33] as well as [55].

Theorem 1.4.1. Assume the conditions provided in Section 1.4.1 are satisfied for the given infinite repeated game. Then, for any individually rational payoff V_i, there exists a discount parameter δ^* such that for all δ_i, $0 < \delta^* < \delta_i < 1$ there is a subgame perfect Nash equilibria with payoffs equal to V_i.

1.4.1 Assumptions

Here, the assumptions which Friedman [33] requires the infinite repeated game to satisfy in order for Theorem 1.4.1 to hold are listed.

1. The mixed action sets, Σ_i are compact and convex for all $i \in N$.
2. The payoff functions, $U_i : \Sigma \to \mathbb{R}$, are continuous and bounded for all $i \in N$.

3. The $U_i(\sigma)$s are quasi-concave[6] functions of $sigma_i$ for all $i \in N$.

4. If $U_i' \leq U_i''$, for all $i \in N$ and $U_i', U_i'' \in \mathcal{U}$, then, for all $U_i' \leq U \leq U_i''$, $U \in \mathcal{U}$. Here, $\mathcal{U}$ is defined to be the set of feasible payoffs, $\{U(\sigma) : \sigma \in \Sigma\}$, where $U(\sigma) = (U_1(\sigma), U_2(\sigma), \ldots, U_N(\sigma))$.

5. $\mathcal{U}^*$ is concave, where $\mathcal{U}^* \subset \mathcal{U}$ denotes the set of all Pareto optimal payoffs[7].

6. All stage games are identical in the infinitely repeated game.

7. The discount parameter, δ, is equal in all time periods.

8. The stage game has a unique Nash equilibrium.

9. The Nash equilibrium is not Pareto optimal[8].

Note [33] later goes on to prove that assumptions six to nine can be removed with only a small effect on the result. However, since the game being studied in this project is the IPD (which satisfies all the above assumptions), this generalisation will be omitted. Only the proof of the original theorem will be provided.

1.4.2 Proof of the Folk Theorem

Proof. Consider the set of all actions which yield greater payoffs than the Nash equilibrium, denoted by:

$$B = \{\sigma : \sigma \in \Sigma, U_i(\sigma) > U_i(\sigma^*), i \in N\} \tag{1.14}$$

where σ^* is the Nash equilibrium strategy. Define the following trigger strategy:

$$\sigma_{i1} = \sigma_i'; \quad \sigma_{it} = \begin{cases} \sigma_i', & \text{if } \sigma_{j\tau} = \sigma_j' \ j \neq i, \tau = 1, 2, \ldots, t-1, t = 2, 3, \ldots \\ \sigma_i^*, & \text{otherwise,} \end{cases} \tag{1.15}$$

where $\sigma_i' \in B$. In words, the ith player will choose σ_i' unless any other player does not play σ_j', in which case they continue by playing their Nash equilibrium action, σ_i^*.

Now, by definition, the strategy in (1.15) is an equilibrium of the repeated game if

$$\sum_{\tau=0}^{\infty} \delta_i^\tau U_i(\sigma_i') > U_i(\sigma_{-i}', t_i) + \sum_{\tau=1}^{\infty} \delta_i^\tau U_i(\sigma^*), \quad i \in N, \tag{1.16}$$

[6]According to [94], a real-valued function f, defined on a convex subset $C \subset \mathbb{R}^n$, is *quasi-concave* if for all $\alpha \in \mathbb{R}$, the set $\{x \in C : f(x) \geq a\}$ is convex.

[7]The paper [33] defines a *Pareto optimal payoff* as a point in the payoff space $U_i(\sigma^*)$ which satisfies the conditions: $\sigma^* \in \Sigma$ and $U_i(\sigma^*) > U_i(\sigma)$ for all $i \in N$

[8]That is, the payoff yielded from the Nash equilibrium is not a Pareto optimal payoff.

which can be rearranged to

$$\frac{\delta_i}{1-\delta_i}[U_i(\sigma') - U_i(\sigma^*)] > U_i(\sigma'_{-i}, t_i) - U_i(\sigma\prime), \quad i \in N, \tag{1.17}$$

where $U_i(\sigma'_{-i}, t_i) = \max_{\sigma_i \in \Sigma_i} U_i(\sigma'_{-i}, \sigma_i)$, $t_i \in \Sigma_i$. Note, $\sigma_{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$ is the strategy vector with the ith player removed.

To check if this strategy is indeed a best response to all others players, who are executing the same strategy in (1.15), consider their alternatives. The ith player has two options. Either they execute the strategy (1.15), or they play the strategy in which $\sigma_{i1} = t_i$. The latter implies $\sigma_{i\tau} = \sigma^*$ will be the best response as every other player will convert to $\sigma_{j\tau} = \sigma^*$, for all $\tau > 1$. Note that any other strategy is weakly dominated by one of these two, since playing t_i in any other stage $\tau \neq 1$ will yield less gains due to increased discounting.

Now if, from playing the Nash equilibria, the discounted loss

$$\frac{\delta_i}{1-\delta_i}[U_i(\sigma') - U_i(\sigma^*)], \tag{1.18}$$

is greater than the gain achieved by playing t_i against σ'_{-i}, then the rational strategy choice for player i, assuming all other players are executing (1.15), is to play (1.15).

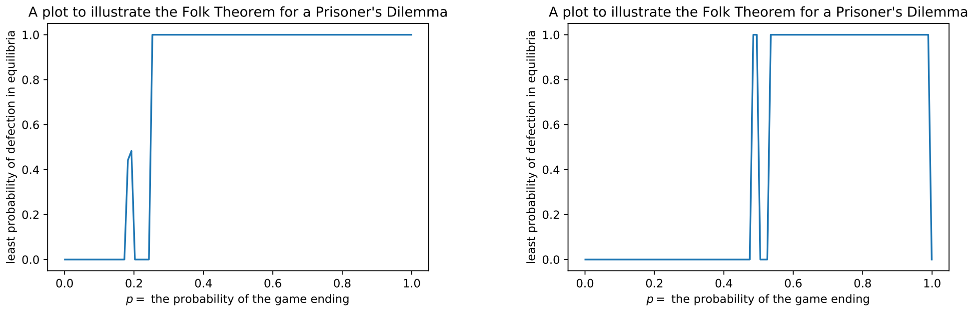
Observe, as the discount parameter, $\delta \to 1$ from below, the discounted loss in (1.18) tends to infinity. However, the gain obtained from playing t_i, that is, $U_i(\sigma'_{-i}, t_i) - U_i(\sigma\prime)$ is finite. Thus, for all $\sigma'_i \in B$ there exists a $\delta^* \in (0, 1)$ such that for all $\delta_i > \delta^*$, the strategy (1.15) is optimal against the same strategy for all players $j \neq i$. Therefore, if the conditions are true for all players $i = 1, 2, \dots, n$, the strategy $(\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_n)$, where $\bar{\sigma}_i$ denotes (1.15), yields a Nash equilibrium.

Finally, by construction, the strategy (1.15) is indeed a subgame perfect equilibrium. $\square$

1.5 Aims of the Project

This project stemmed from an initial idea presented in a game theory assignment completed by the author. The topic of this coursework was Nash equilibria of repeated games and the two graphs, as presented in Figure 1.5, were obtained.

Figure 1.5 was obtained from repeating IPD tournaments which are implemented in Python via the package Axelrod [96]. In general, these tournaments are a group of strategies, who all compete in a variety of round-robin, two-player IPDs with the aim of achieving the largest payoffs. The implementation in Axelrod allows



(a) A plot of the least probabilities of defection against the strategies: *Cooperator*, *TitForTat* and *Random*. Here, the p-threshold is approximately 0.25.

(b) A plot of the least probabilities of defection against the strategies: *Winner21*, *AntiTitForTat* and *OmegaTFT*. Here, the p-threshold is around 0.5.

Figure 1.5: Original plots obtained which influenced the subject of this project.

for the simulation and analysis of IPD tournaments under different environments. For example:

- It is possible to vary the number of strategies making the groups competing in the tournament[9].
- Both finite and infinite IPD can be considered. The infinite IPD is simulated through the scenario of a probabilistic ending, denoted throughout this study as p_e[10]. Note, p_e is related to the discount parameter, δ, introduced in Definition 1.3.7, by $p_e = 1 - \delta$.
- Varying levels of noise can be introduced. This is referred to as *standard PD noise*[11] within this report. According to [40], *standard PD noise* is the probability, p_n, of an action being altered within any particular round. That is, the probability of a C being seen as a D and vice versa.

The two plots seen in Figure 1.5 were yielded from setting $p_n = 0$, the player set size equal to four, p_e taking 100 distinct values within $(0, 1)$ and each tournament to repeat 100 times. The graphs show the least probability of defection obtained

[9]In this project, the group of strategies will be referred to as a *player set* for clarity. Consider Figure 1.5a, the player set here consists of the strategies: *Cooperator*, *TitForTat*, *Random* and *Defector*.

[10]The probability of the game ending, p_e, will also be referred to as a game-ending probability, in this project.

[11]Note, there are three potential sources of noise within an IPD tournament. In order to differentiate between them the following terms are used. *Standard PD noise* to refer to the probability of an action being altered; *stochastic player noise* to refer to the noise induced by a stochastic strategy; and *unexpected noise* to refer to noise which is expected from running numerical experiments.

in the Nash equilibria of the corresponding game. Note, the 'corresponding game' here refers to the matrix of mean payoff values which can be calculated from the tournament results of each strategy (see Chapter 3 for further explanation of this).

From Figure 1.5, it can be seen that there is a clear game-ending probability p_e for which the least probability of defection goes to zero. In this project, this probability is defined as a *p-threshold*. However, what was most intriguing was that the two different games obtained had a different p-threshold. This is approximately 0.25 for Figure 1.5a but for Figure 1.5b the threshold appears at around 0.5. This initiated the idea to investigate whether there are any specific characteristics of an IPD tournament that affect the value of the p-threshold.

Therefore, the aims of this project are as follows:

1. To provide a review of past and present literature already published in the field of folk theorems;
2. To develop a program which executes a large experiment involving tournaments of the IPD with differing environments to obtain graphs similar to those in Figure 1.5; and
3. To perform analyses on where the p-thresholds seem to lie and whether it is affected by the change in the number of players, levels of standard PD noise, etc.

Chapter 2

Literature Review

The folk theorems are a class of results which generally state that for repeated games, any feasible and individually rational payoff vector can be achieved as a subgame perfect equilibrium if the players are patient enough [64]. The origin of these theorems is unknown, however written proof and research involving these ideas first appeared in [6, 33, 88][1] in the 1970s. Since then, many generalisations and refinements of the ideas have been explored, for different games, including: games with private monitoring [46, 70, 83], sequential games [15, 41, 105] and games of complete information [1, 13, 14], to name a few. Due to the identification of further equilibria (as compared with the stage game), which are key in predicting future behaviour, these theorems have been commented on as 'fundamental' in the theory of non-cooperative games by [46, 62]. On the other hand, the majority of the strategies used in the proofs assume the identification of individual deviators [69] which may not be realistic in certain situations. Hence, an area of research on the so-called 'anti-folk theorems' was introduced [69, 83, 109]. Folk theorems appear to still be an active area of research today [47, 82, 100] with many differing applications. Therefore, in this chapter, a review of the literature on this topic is provided with papers ranging from the 'original' ideas in the 1970s to applications of the theorems in 2020.

2.1 First Papers

According to [1], the earliest work on Folk-type theorems is [33]. In [33], infinitely repeated games with discounting are considered. In particular, focus is on a class of strategies, now known as *grim trigger*. These are used to prove that, for any

[1] The paper [6] referenced here was published in 1994. However, as implied in [1], this is a more recent version of a paper written in 1976.

feasible and individually rational payoff vector, there exists a discount parameter such that a subgame perfect equilibrium with payoffs equal to that vector exist. This is first shown under the constraints of: identical stage games, constant discount parameter and unique Nash equilibrium that is not Pareto optimal. This is in addition to those made on the strategy and payoff spaces. However, a more general result is then given which removes these restrictive conditions. The application of oligopoly is used throughout [33]. Moreover, he introduces the notion of 'temptation'. He motivates this through the explanation that 'threat' is no longer "credible" since players are unable to communicate in non-cooperative games. On the other hand, 'temptation' is said to be analogous to 'threat'.

In contrast to this, [6] presents folk theorems for infinite games without discounting, assuming payoffs take a "limiting average form". This choice is justified through the statement that, as the discount rate approaches zero, the limit of the discounted sum behaves similarly to the limiting average payoff form. However, trigger strategies are still required in the proof and, for simplicity, two player games are assumed. Two folk theorems are proven in this paper with the latter one discussing subgame perfect equilibria similar to [33], whilst the former is a more generic version of the theorem. In [6] it is discussed that, though the generic version of the theorem does exist, the subgame perfect equilibrium points allow for more 'believable' behaviour. They conclude by considering an example in which payoffs are discounted. A similar approach is taken in [88], with the statement of a folk theorem for infinite games with no discounting and the existence of subgame perfect equilibria. The only difference being the use of an "overtaking criterion" instead of a "limit of the mean criterion" as in [6].

2.2 Games with Complete Information

Folk theorems of games assuming complete information are studied in [1, 14]. That is, all players have common knowledge of payoff functions and strategies. Paper [1] focuses on the necessary and sufficient conditions required for a folk theorem proof. They state that feasibility and individual rationality of the payoff vector are necessary, and also approximately sufficient, conditions for a payoff to be in the equilibrium. This is followed by a discussion on the full dimensionality constraint first introduced by [35] and often used in proving folk theorems. It is highlighted in [1] that the equality between the dimension of the convex hull of feasible payoffs and the number of players is a sufficient condition. This provides motivation for the main result of the paper; the introduction of a 'non equivalent utilities' condition, which is proved to be sufficient and almost necessary for a folk theorem. The condition, as indicated here, is weaker than the aforementioned full dimensionality condition. According to [1], it only requires “no pair of players have equivalent utility functions”.

Similarly, a folk theorem for complete information games with subgame perfect equilibria is proved in [14]. However, instead of exponential discounting as in [1, 35], he assumes discounting is present-biased. That is, the discount function implies a player is more willing to alter an event in the future than altering a current event. The folk theorem is proved in cases of where the player is time consistent (prefers to maximise their initial preferences) and time inconsistent (prefers to maximise current preferences). In [14], the notion of ‘patience’ is taken to be the sum of all discount factors. Also considering time consistent and inconsistent players, [64] discusses folk theorems with respect to time-dependent discounting. In contrast to [14], the long-term characterisation of ‘patience’ is taken as the discount factors at all stages uniformly converge to one. Motivation for the study of time-dependent discounting is explained by empirical studies which seem to suggest that a player’s time is unstationary, rather than stationary which is assumed in most discounted game models.

2.3 Games with (Imperfect) Private Monitoring

In the late 1990s attention turned to games with imperfect monitoring. That is, a player’s actions can no longer be observed accurately, instead public or private signals are detected [29]. According to [70], games with (imperfect) public monitoring were the first to be considered. More recent studies in this area include [21, 53]. Repeated games with public monitoring are stated in [53] to give a suitable model for studying long term relationships. They say intuition suggests that cooperation becomes easier to sustain as action observability improves. However, [53] continues to show that this intuition is false, by considering a repeated public monitored game which satisfies “the limit perfect public equilibrium payoff set can achieve full efficiency asymptotically as public information becomes less sensitive to hidden actions”. They give an example which violates the sufficient condition given in [36] yet a folk theorem can still be obtained. This paper mainly focuses on the PD but, similarly to [1], they state an example which satisfies the folk theorem without the full dimensionality condition.

The notion of a robust equilibrium to incomplete information is considered in [21]. That is, if the equilibrium yielded is near the original equilibrium for all perturbed games consisting of a small independent and identically distributed information shock. Conversely to [33], they discuss the implication of grim trigger strategies not being robust when the aim is to sustain cooperation. A folk theorem is proved in robust equilibria for games of public monitoring, however [21] highlights that a much stronger condition is needed in comparison to the full dimensionality requirement in [36].

As of 2004, according to [70], repeated games with private monitoring was a relatively new area of study. In this paper, conditional independence of signals is assumed in order to show a folk theorem for the IPD. Applications to duopolies are then described. Another study to consider a folk theorem for the IPD is [32]. Using a similar definition of robustness to [21], [32] proves that for a discounted PD with private monitoring technologies, a folk theorem with robust equilibrium strategies can be obtained. In particular, they consider almost-perfect private monitoring and a limit folk theorem (for sequential equilibria) follows. In a similar paper, [31] introduce the idea of "belief-free" equilibrium strategies; a property which implies the belief of an opponent's history is not required when obtaining a best response. They use these strategies in proving a folk theorem for the two-player PD however, for general games, the set of belief-free payoffs is not large enough to provide a folk theorem. Moreover, [31] highlight that, for a larger number of players, the calculation of the payoff set becomes significantly harder.

In contrast to this, [46] discusses a folk theorem using strategies that, although are not belief-free, still make beliefs "irrelevant" at the start of each T-block of stage games. Their result is much more generic than [31] since it applies to N-player finite games under the assumption of full dimensionality with private, but almost-perfect, monitoring. The T-block strategies mentioned above are modified in [107, 108] such that they "can support any vector in the belief-free equilibrium payoff set". This modification results in belief-free equilibrium strategies. The results in [107, 108] are generalisations from the two-player PD in [31] to the corresponding N-player game.

2.4 Games with Communication

Academics introduced games with communication to deal with the complications faced with imperfect monitoring. For example, according to [51], games with public monitoring can obtain a folk theorem under weaker assumptions than those given in [36], if communication is introduced. In this paper, communication is a message, taken from the set of possible actions, which the players give simultaneously after choosing an action and observing a signal. He proves a folk theorem for symmetric games with four or more players, without the assumption of the number of signals relevant to the number of actions.

Regarding private monitoring, [77] claims repeated games are "very difficult to analyse without communication". Thus examples of papers, proving folk theorems for these games with communication, include [34, 63, 77]. A 'Nash threats' folk theorem is proved in [34], similar to [33], in the case of almost public information games, without independent signals, for two players. In this paper, communication is defined in the form of announcements where each set of announcements is the same for all players. The decision to only consider two player games is justified in [34] by highlighting that, although the results can be generalised, in certain cases it can be seen as advantageous to have additional players. Similarly, [77] proves a 'Nash threats' folk theorem for private monitoring games with communication. He increases the number of environments where the folk theorem is applicable through developing further the idea of "delayed communication", as given in [27]. In addition, [77] uses the assumptions of correlated private signals; and each player's deviation from strategy is statistically identifiable from the other players' signals. A model of private monitoring and communication within games is also considered in [63]. He has the aim of increasing the number of applicable environments for [52] frequent communication folk theorem. The paper states that [52] assumed only private signals were publicised in this theorem but if other information was useful and communication was free and legal then players would also want to share their actions. This motivates the reasoning behind the paper. However, assumptions of full dimensionality, and the number of actions and signals, are still required.

Another paper which considers communication is [16], regarding self referential games with codes of conduct. These codes are descriptions of how the players and opponents should play and an application to computer algorithms is provided. Two folk theorems are proved: one assuming common knowledge of the codes of conduct, and the other where only certain players observe certain codes. The latter is the main result of the paper and is motivated by the fact that, often, individuals have good knowledge of those closest to them but not the whole community. Moreover, [16] obtain the sufficient condition that, with public communication, if every player is observed by another two opponents, then a folk theorem is yielded.

2.5 Finite Horizon Games

Another area of interest is the existence of folk theorem-type results for games of finite repetition. In [13], the case of finitely repeated games of complete information and the associated subgame perfect equilibria is explored. Despite the existence of games which, when repeated finitely, "produce no non-cooperative equilibrium outcomes"; they state there may be subgame perfect equilibria of finite-repeated games, when the corresponding single game has multiple equilibria. Indeed, using their "three phase punishment", [13] prove that "any rational and feasible payoff vector can be obtained in the limit". This is assuming the feasible payoff region has dimension equal to the number of players and each player has two Nash equilibrium payoff values. In a similar manner, [5] considers an alternative version of the PD in which an additional strategy is included. This gives a second pure-strategy equilibrium and a folk theorem result for the finitely repeated version of this game.

Although not strictly a finite game, [37] study a repeated game in which players may "strategically terminate" it. In particular, this involves the incorporation of a voting-step, at the start of each repetition, where a certain number of players decide whether or not to keep interacting. This is motivated by the increasing possibilities of ending business partnerships due to more technology and knowledge. A general folk theorem for any stage game (with the additional voting), which is satisfied "for all majority rules except the unanimous ending" is proved by [37]. Indeed, for the unanimous ending rule, they show that the theorem may not hold but sufficient conditions are provided for when it is satisfied.

2.6 Stochastic and Sequential Games

Other game types to have associated folk theorem results include stochastic [30] and overlapping generation [15, 41].

It is explained in [30] that often the standard assumption of "a completely unchanging environment", within the theory of repeated games, is not reliable in applications. This reasoning is the motivation for studying, the more generic, stochastic games. These games may not have a pre-decided stage game, instead a 'state variable' is used to represent its environment which alters according to"initial conditions, player's actions, and the transition law". The paper [30] discusses equilibrium payoffs in the case of very patient players, without the need for the Markovian property. Specifically, perfect monitoring is assumed, along with asymptotic state independence and either of payoff asymmetry or full dimensionality. Two folk theorems are proved in [30]: one with unobservable mixed strategies (in which case, fully dimensionality is required) and also, similar to [1], one with the slightly weaker condition of payoff asymmetry (here, mixed strategies have to be observed).

Both [15, 41] provide insight into folk theorems associated with overlapping generation games. These are similar to the repeated normal form games except that the players are considered to be finite. That is, each player is involved in a certain number of stage games before they are replaced by another, identical player. A variety of folk theorems are proved in [41] both with and without discounting and / or observable mixed strategies. He shows that the full dimensionality assumption is not required in these games since players are assumed to not end simultaneously. On the other hand, although [15] does provide a folk theorem, his main result is an anti-folk theorem, see Section 2.7. He studies games with imperfect public monitoring and states that, in such overlapping generational games, cooperation becomes impossible in contrast to repeated games. Furthermore, [15] constructs a mixed strategy equilibrium folk theorem. However, he goes on to show that these strategies are unstable to perturbations, resulting in an anti-folk theorem.

A similar study by [4] looks at dynastic repeated games. These differ from the overlapping generation games in two aspects: perfect observation of the past is assumed, and the payoffs obtain no dynastic component. Under the assumptions of full dimensionality and the existence of a payoff vector which strictly Pareto dominates the stage game equilibrium, [4] proves a folk theorem for private communication games, with greater than three players, in sequential equilibria.

Considering now sequential games, in which players do not choose their action simultaneously, [105] introduces a concept of effective minimax values before proving a corresponding folk theorem. In his model, players pick actions in groups. Thus the effective minimax value is defined to be the lowest equilibrium payoff a player will receive, even if none of their opponents have equivalent utilities. According to [105], his folk theorem can be applied to other game models as it is a "uniform characterisation".

2.7 Anti-Folk Theorems

A common theme in the proofs of folk theorems is the use of strategies which "identify and punish" deviators [69]. However, as soon as the game contains incomplete / imperfect information, deviators cannot necessarily be identified. This yields a much smaller equilibrium set and these results are termed "Anti-Folk Theorems". This description of anti-folk theorems is adapted from [69], who state the original term was given in [28, 54]. An anti-folk theorem is proved in [69] using the "long-run average criterion" instead of the discounted criterion.

Considering similar models to [15, 41], an anti-folk theorem for a limited-observability overlapping generations model is obtained in [109]. They show that cooperation cannot be sustained when new players can only observe recent history. This is in contrast to the folk theorems obtained under the assumption of common knowledge of all past actions. Though the results in [109] are restrictive in certain cases, they justify the work by stating it is suitable for modelling "high turnover" rates.

Another game model where an anti-folk theorem has resulted is a repeated game with private monitoring [83]. In this paper the following three assumptions are made: infinite and connected private monitoring (that is, infinitely many connected signals); finite past; and independent and identically distributed shocks affect the payoffs. Under these assumptions, [83] shows the violation of the folk theorem. That is, the equilibria of the repeated game consist only of the equilibria of the single game.

2.8 Evolutionary Stability

Recently, there has been research into the results of the folk theorem with respect to the evolutionary game theoretic paradigm. Indeed, [62] state that the folk theorem is often used to characterise evolutionary stable strategies. This is since exact solutions using theoretical results from evolutionary game theory are hard to obtain. However, they show the assumption that the folk theorem yields all Nash equilibria is misleading. This is achieved by defining "type-k equilibria" which are a refinement of the Nash equilibria. The set of type-k equilibria is proved in [62] to be contained within the set of repeated-game Nash equilibria using "reactive strategies".

In contrast to [31, 107, 108], who discuss folk theorems using belief-free strategies, [43] discusses the instability in an evolutionary sense. He shows that the belief-free equilibria are not robust to small perturbations in games with private monitoring and, in certain cases, this is extreme. Similar to [62], he states that Nash equilibria are used to predict evolutionary behaviour since they are thought of as stable. However, [43] goes on to show that only the choice of repeated stage-game Nash equilibria satisfy evolutionary stability.

2.9 Recent Applications

In recent years, studies have been applying results of the folk theorem in various areas, for example, to create algorithms. This section briefly discusses a few of these.

The folk theorem is used in [24] for the creation of a model which aims to suppress the effects of distributed denial of service attacks. They claim that all networks suffer from attacks to infrastructure and services. Thus, [24] use the programmability of software defined network environments to perform a game theoretic analysis. An algorithm is created for reward and punishment based on the Nash folk theorem. Similarly, [100] make use of the cooperative equilibrium solution from the folk theorem in [33] to create an algorithm suggested to optimise a 'multi-period production planning based real-time scheduling method', for a job shop. Also, [101] uses the result of the folk theorem in the IPD to study the cooperation rates of varying agent strategies in a multi-agent system.

Another application of the folk theorem is an algorithm used to obtain equilibria of a discounted repeated game [82]. A new algorithm, entitled "Communicate & Agree", is introduced in [82] to find equilibria in incomplete information, but perfect monitoring, games. Using the folk theorem in the algorithm enables the payoffs obtained to be potentially higher than those achieved by repeating the Nash equilibria of the single game. However, [82] go on to highlight that the algorithm is not always guaranteed to find equilibria. They say it is dependent on: the discount factor, sampling density, and whether it is a zero-sum game or not. Finally, in a different area, [47] discuss the potential of using game theoretic ideas in quantum optimal transport. In particular, he defines the Quantum PD and explores the possibility of a quantum folk theorem in relation to the corresponding repeated game.

2.10 Conclusion

In this section, an overview into research regarding the folk theorem has been provided. The research history of the folk theorem spans from the 1970s until now, with many different models being considered. Examples include: games with complete information, games with imperfect private monitoring and finite-horizon games. However, there have also been studies into situations where the folk theorem does not hold, or the equilibrium strategies used in proving the theorems are unstable and / or not robust.

Chapter 3

Methodology and Experiment Setup

In this chapter the methods used to collect the data is provided along with justifications. This study required the execution of several IPD tournaments and thus appropriate software needed to be implemented and tested for accuracy. All the appropriate code, written for this project, is made available in the GitHub repository (https://github.com/shapperzsm/final-project).

3.1 Data Collection Algorithm

This section describes the overall algorithm used to obtain the data and the attributes collected. Firstly, the aim of this exercise was to illustrate the Folk Theorem and analyse the p-thresholds via a large experiment. It was expected that plots similar to Figure 3.1 would be yielded. This clearly shows that there eventually exists a p_e where defection is not a rational decision.

Therefore, to observe whether any environmental settings of the tournament do affect the p-threshold, a large amount of data was needed. This was in order for any observations made to be statistically significant. Figure 3.2 shows a pictorial representation of the collection method used. Each step visible is explained in detail throughout this chapter with further references to appropriate sections.

From Figure 3.2, it can be seen that the first step was to set up an empty database ready to input each tournament result into. The specific details of implementation into the algorithm is discussed in Section 3.2. However, the choices made on the attributes to collect are described here.

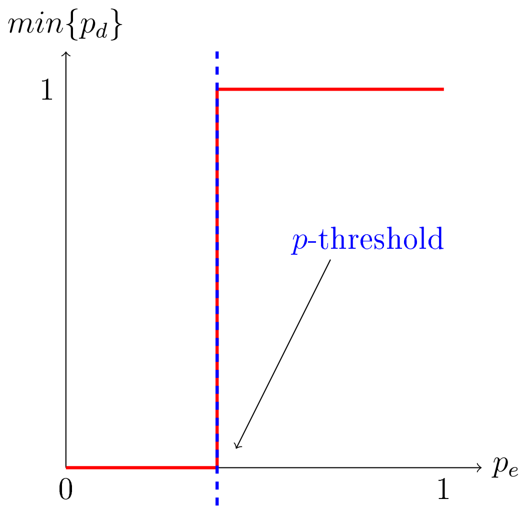


Figure 3.1: An example plot illustrating the p-threshold as indicated by the Folk Theorem. The minimum probability of defection in equilibria is denoted $\min\{p_d\}$.

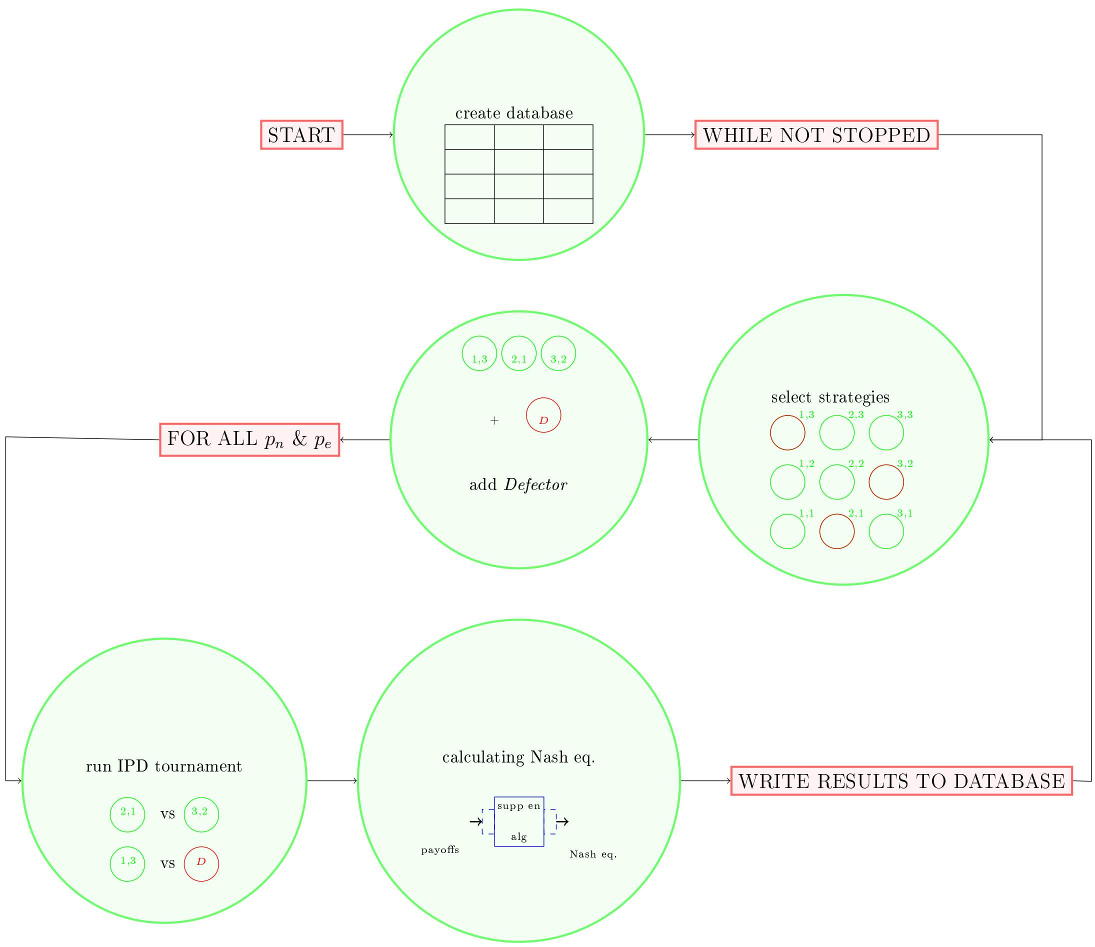


Figure 3.2: Representation of the algorithm used to collect the data.

Attribute	**Description**
experiment_number	A unique seed for each tournament run.
number_of_players	The number of strategies including the Defector.
tournament_player_set	A unique number for a particular set of strategies.
player_strategy_name	The strategy name as given in [96].
is_long_run_time	Characteristic from [96].
is_stochastic	Characteristic from [96].
memory_depth_of_strategy	Characteristic from [96].
prob_of_game_ending	The value of p_e.
payoff_matrix	A string of the matrix of mean payoff values from the tournament.
num_of_repetitions	A value indicating how many iterations of the tournament is required.
num_of_equilibria	The number of equilibria yielded as output from the algorithm.
nash_equilibria	A string containing the list of equilibria.
least_prob_of_defection	The lowest probability of playing the *Defector* obtained from the Nash equilibria.
greatest_prob_of_defection	The highest probability of playing the *Defector* obtained from the Nash equilibria.
noise	Level of standard PD noise, p_n.
warning_message	This column contained a string if the algorithm detected potential degeneracy.

Table 3.1: Table of attributes collected.

Table 3.1, shows each attribute chosen to be observed in this study. The attributes experiment_number and tournament_player_set were used to provide a unique identification for each tournament and set of strategies. This was to ensure clear separation during analysis. The three characteristics of the strategies were collected with the aim of analysing the different strategies' effect on the p-threshold. The attributes least_prob_of_defection, warning_message, number_of_players and noise were the most important attributes to this study. These were the main effects analysed with regards to the p-threshold and key descriptors of the game. The rest of the attributes were retained: for evaluation of degeneracy; in order to replicate the tournaments for validity; and for further research.

Following this came the choice of opponents. The number of opponents selected ranged from one to eight and were randomly selected from the appropriate collection of strategies in the Axelrod library [96]. Out of the 235 strategies currently implemented in Axelrod, only 216 were valid for this experiment. Since this prroject was considering the Folk Theorem for the IPD, research strategies which 'cheated' (that is, those which return False when entered into the 'obey_axelrod' function) were omitted. The *Defector* strategy was also removed as it would later be added to all sets of players. Furthermore, due to time constraints, the 18 strategies which are classified as having a long execution time were omitted.

The actual IPD tournament was run using the original Axelrod tournament setup [2] as implemented in the Axelrod library [96]. This is a round-robin type tournament where each strategy plays every other strategy once [7]. Each round robin was repeated 500 times to obtain 'smoother' estimates of the mean payoff values. Moreover, each strategy set was run in a tournament for 100 values of p_e within the range $[0.001, 0.999]$. Note, zero was not included as this implies the tournament would never end and one was omitted as the tournament would immediately end after the first turn. However, it is possible for the probabilities within the range $[0.999, 1]$ to be included in this range; and this is recommended for future research. These tournaments were also repeated for values of p_n in the set, $\{0.1, 0.2, \ldots, 1\}$. The main output of interest from each tournament is the payoff matrix, which is then implemented as the game matrix in the Nashpy library [97]. According to [96], each entry $a_{i,j}$ gives the mean payoff of player i

against player j. Consider the following example:

$$\begin{pmatrix} 2.990 & 2.996 & 0.487 \\ 2.996 & 3.000 & 0.989 \\ 3.053 & 1.042 & 1.000 \end{pmatrix} \tag{3.1}$$

The matrix in (3.1) is yielded from a three player tournament with $p_e = 0.001$ and $p_n = 0$. The strategies in this case were *Colbert*; *Tideman and Chieruzzi* and *Defector*. In this case, the entry $a_{1,2} = 2.996$ is interpreted as the mean payoff between *Colbert* and *Tideman and Chieruzzi*.

For the calculation of the Nash equilibria, the support enumeration algorithm was used. This algorithm as well as the justification of its use is provided in Section 3.5.

Finally, the values obtained were written to the database file. One record is inserted for each strategy in order to retain all characteristics. Here, any entries that were not integers or floats had to be converted to strings, in order to be stored. For example, the payoff matrix given in (3.1) is a list and hence could not be inserted into the database in its current format.

In summary, pseudo code for the overall algorithm is provided in Algorithm 1.

Algorithm 1: Folk Theorem Exploration

```
input  : maximum number of opponents, number of strategy sets for each
         number of opponents, set of p_n, set of p_e, number of tournament
         repetitions, the database file path, and whether or not support
         enumeration should be used to calculate the Nash equilibria.
output: a database containing the results as detailed previously.
while True do
    for each number of opponents do
        for each repetition with the same number of strategies do
            Randomly select a set of opponents and add in the Defector.
            for each p_n do
                for each p_e do
                    Run the IPD tournament.
                    Obtain the Nash equilibria and the corresponding
                      probabilities of defection, using the algorithm indicated.
                    for each player in the current set do
                        Write the required information to a record in the
                          database file.
                    end
                end
            end
        end
        Repeat
    end
end
```

3.2 Databases

There are many different options regarding types of file available for storage of data. For example, csv, json, tex, txt, db extensions. These are generally split into two types: plain text and binary files. In this section, the justification for using an SQLite database is provided, along with how this was implemented. However, firstly the advantages and drawbacks of plain text and binary files are discussed.

Plain text, or flat file, is a format which stores data entries in a single table with columns separated by delimiters such as commas or tabs [95]. The contents are comprehensible by humans. Examples of these include csv and txt extensions. The advantages of plain text formats include: a simple structure, less disk space used and portable [95]. However, there are also drawbacks. Flat files are not scalable and are protected by less security [98]. Only one user can edit the file at any one time and, when wanting to search through the file, it has to be fully loaded in the system [19]. Moreover, the columns must all contain the same data type [95].

On the other hand, binary file is a format in which the sequences of zeros and ones are unconstrained, compared to plain text files (where the binary codes have to represent character sets) [91]. Databases, executables and media files are examples of these [91]. The benefits of using binary file formats include: uses less storage space, is less effort computationally and more secure (it is not understood by humans) [9]. Although this also a disadvantage as it makes a file harder to edit [9].

3.2.1 Types of Database

Using the reasons provided above, it was decided that a binary file, in particular a database file, would be the most appropriate format to use. Primarily, this was due to the fact that databases are generally more robust and support out of memory operations. Indeed, a csv file could have been used however every entry would have needed to be a string and, if this contained commas, would break the column structure. Research into the ideal file format for database collection resulted in the identification of two main types of databases: relational and noSQL (Not Only SQL).

Relational databases are a file format which store data according to the relational model described in [26]. Examples of relational databases management systems include: SQLite, MySQL, PostgreSQL, Oracle and Db2. Briefly, this model involves structuring the data into a table, where each row is an observation with unique ID, or key, and each column is an attribute. This provides an ideal way to identify relationships between the varying records. The model was developed in the 1970s and was motivated by the reason that, originally, structures of databases varied with the application used. There are many advantages to a relational database format, including: data consistency — the data is immediately available across several instances of the database, no 'catch-up' time is needed; commitment — strict rules regarding permanent changes within the database; stored procedures are allowed — blocks of code which can be repeatedly accessed; SQL (Structured Query Language), which has been developed for ease of query performance using mathematics; and data locking / concurrency — allows many users to query the database simultaneously without conflicts. A good paper on the model and benefits of relational databases is [79]. On the other hand, one major drawback to this format is its performance in handling extremely large data sets; which have become increasingly popular. Once the data goes beyond a certain size, a relational database has to be distributed across many servers. Also, this model does not support high scalability, that is, relational models are unable to support large volumes of workloads. Furthermore, the strict structure required for this format means that, if data cannot be easily transformed into this structure, the complexity of the model increases. The article [48], provides a more detailed description of these disadvantages.

Alternatively, noSQL databases were created with the motivation to be more efficient with large volumes of data. There are several different types of noSQL databases. Key-value store databases, such as RIAK, store the data as a simplistic key-value pair and are similar to hash tables. Column-oriented databases, for example Cassandra, are hybrid row-column databases and document-oriented databases, such as MongoDB, store 'records' in the form of documents with a unique key for representation. Finally, graph databases and object-oriented databases, such as Neo4j and Db4o, respectively, store the data as graphs (in the former case) and as objects, similar to those in Object Oriented Programming (in the latter case). Advantages of noSQL databases include: more flexibility with a wide range of models available; supports scalability; and are more efficient. However, these models are relatively new in comparison with relational models and there is no standard querying language. Also, some of these databases are not as effective as relational databases with regards to consistency and commitment; and maintenance is challenging. For a more detailed approach to noSQL, see [73].

Using this information, it was decided that a relational database would be the most appropriate. Although it was intended that a large amount of data would be collected, due to time limitations, the database was unlikely to become too big for the system. Moreover, the structure and consistency of a relational model was ideal for comparison of the IPD experiments. The database management system decided upon was SQLite. This was due to the fact that there exist Python libraries, for example SQLite3 and SQLAlchemy, for accessing the database and its contents. Also, it is portable, with the entire database stored in a single file, meaning it could be transferred easily from the varying computers being used. According to [80], other benefits include: its ease of use, with no configuration files, and the fact it is self-contained.

```python
database_management_sys = sa.create_engine(
    "sqlite:///" + database_filepath + "main.db"
)
connect_dbms_to_db = database_management_sys.connect()

read_into_sql = """
    INSERT into folk_theorem_experiment
        (experiment_number, number_of_players, tournament_player_set,
        player_strategy_name, is_long_run_time, is_stochastic,
        memory_depth_of_strategy, prob_of_game_ending, payoff_matrix,
        num_of_repetitions, num_of_equilibria, nash_equilibria,
        least_prob_of_defection, greatest_prob_of_defection, noise,
        warning_message)
    VALUES
        (?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?)
"""

record = (
    experiment_number, number_of_players, tournament_player_set,
    str(player_strategy_name), is_long_run_time, is_stochastic,
    memory_depth_of_strategy, prob_of_game_ending, payoff_matrix_as_string,
    num_of_repetitions, num_of_equilibria, nash_equilibria_as_string,
    least_prob_of_defection, greatest_prob_of_defection, noise,
    warning_message,
)

connect_dbms_to_db.execute(read_into_sql, record)
```

Listing 1: Python code used to record the results for a single strategy into the database.

3.2.2 Implementation of the Database

The ability to import results straight from Python was through the library SQLAlchemy [12]. This allowed for the creation of the database through to accessing the results, via Python functions and expressions. For example, consider the code in Listing 1 which was used to insert a record of results for one strategy into the database.

Moreover, to ensure records were being inserted into the database correctly, the graphical user interface, DB Browser [84] was utilised. It is implemented with a “familiar spreadsheet-like interface” for ease of use and is compatible with SQLite databases which made it ideal for this use [84]. See Figure 3.3, for a screenshot of the interface.

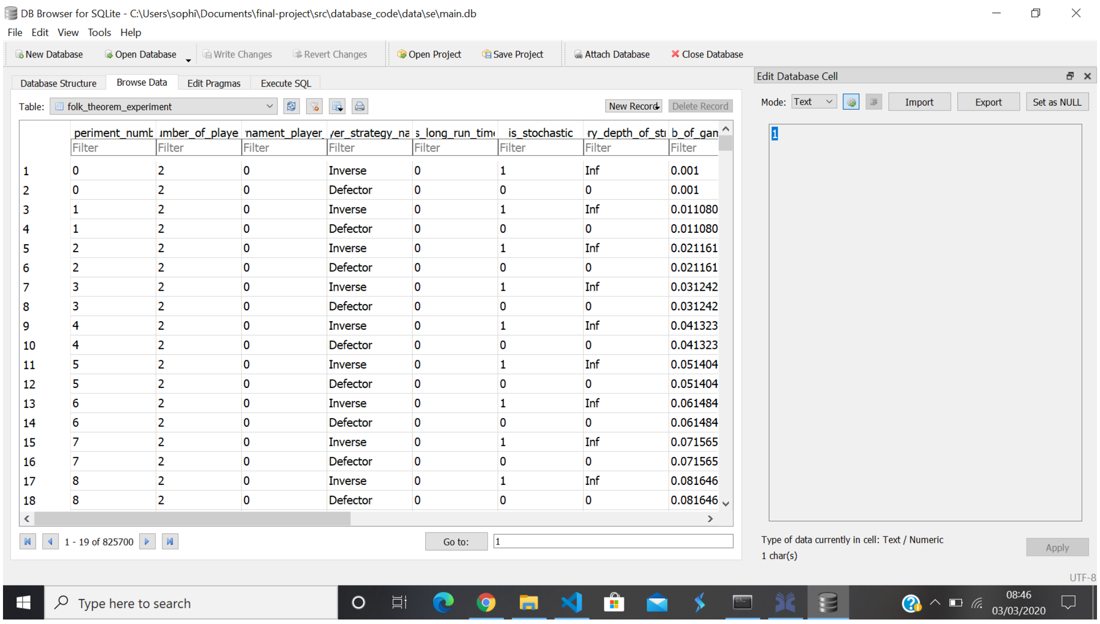


Figure 3.3: DB Browser, a database graphical user interface.

3.3 Software Implementation

There were many potential choices of language for the execution of this experiment. However, Python had the added advantage of pre-existing libraries, Axelrod [96] and Nashpy [97], which enabled running the IPD as well as calculation of the Nash equilibria. Thus, Python was used for the collection and analysis of data.

Throughout the implementation of this experiment into Python, good software development principles were followed [49, 89, 106]. Self-documenting code was ensured through the careful naming of variables, as well as the use of docstrings to fully describe function parameters and usage. Python libraries Black [61] and Blackbook [57] assisted in improving the readability of the code, through formatting according to the guidelines of PEP-8 [87]. This gave consistency to all code files created during the study. Moreover, modularity was ensured through the creation of several smaller functions, focusing on one task. This not only assisted with debugging, it also allows for future usability of the code in newer developments.

Testing is another key part of software development to guarantee the durability of the code. Thus unit tests were implemented, using Pytest [60], to assist with the identification of bugs in the functions created for data collection. However, further work in this area is needed to provide a fully tested program. Indeed, from executing the Python library Coverage [11], a coverage of 59% was identified (for the 'experiment_functions.py' file), yielding Table 3.2. This could be

Module	statements	missing	excluded	coverage
src/database_code/__init__.py	0	0	0	100%
src/database_code/experiment_functions.py	87	36	0	59%
src/database_code/run-experiment-support-enumeration.py	6	6	0	0%
src/database_code/run-experiment-vertex-enumeration.py	6	6	0	0%
Total	99	48	0	52%

Table 3.2: A screenshot of the html report produced when utilising Coverage.

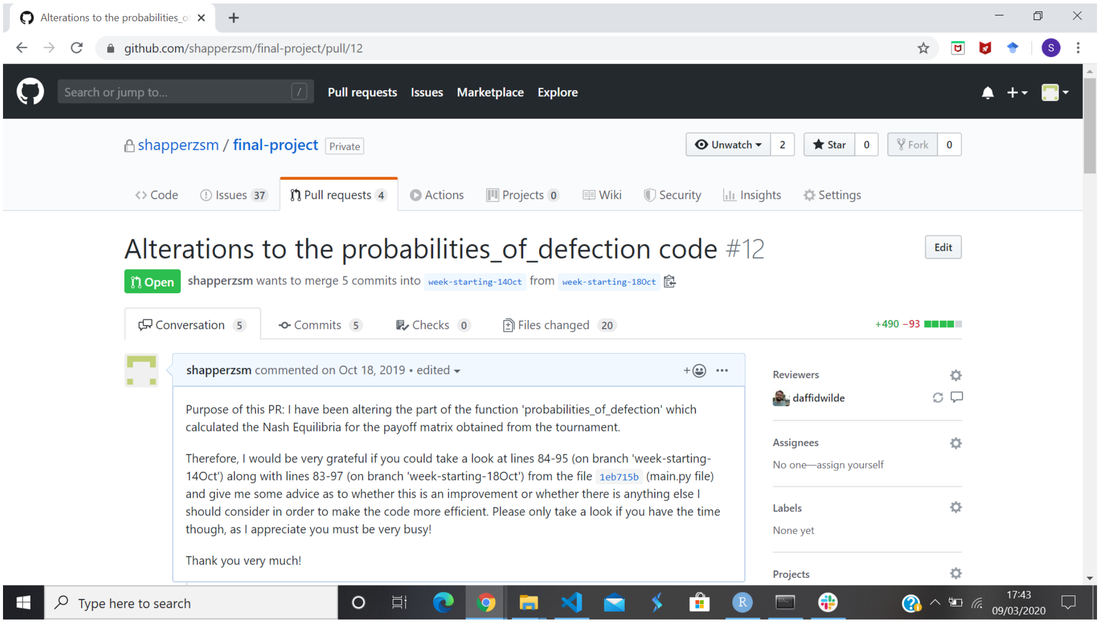


Figure 3.4: A screenshot of a GitHub pull request which allowed for collaboration between supervisors and author.

improved through the creation of integration tests, between the database and the experiment results, or the implementation of functional tests, to confirm the end result of the full algorithm.

The use of version control is key for keeping track of past changes to the system. In this study the software Git (https://git-scm.com/) was used. This allowed for the adaption of code from several different function attempts and, through GitHub (https://github.com/), enabled collaboration between the author and supervisors as seen in Figure 3.4. The use of GitHub ensured that a back-up copy of all project files were available, should the current system happen to fail. Moreover, it acted as an intermediate step between the author's laptop and the remote server, which was used for running the main experiment (see Section 3.4).

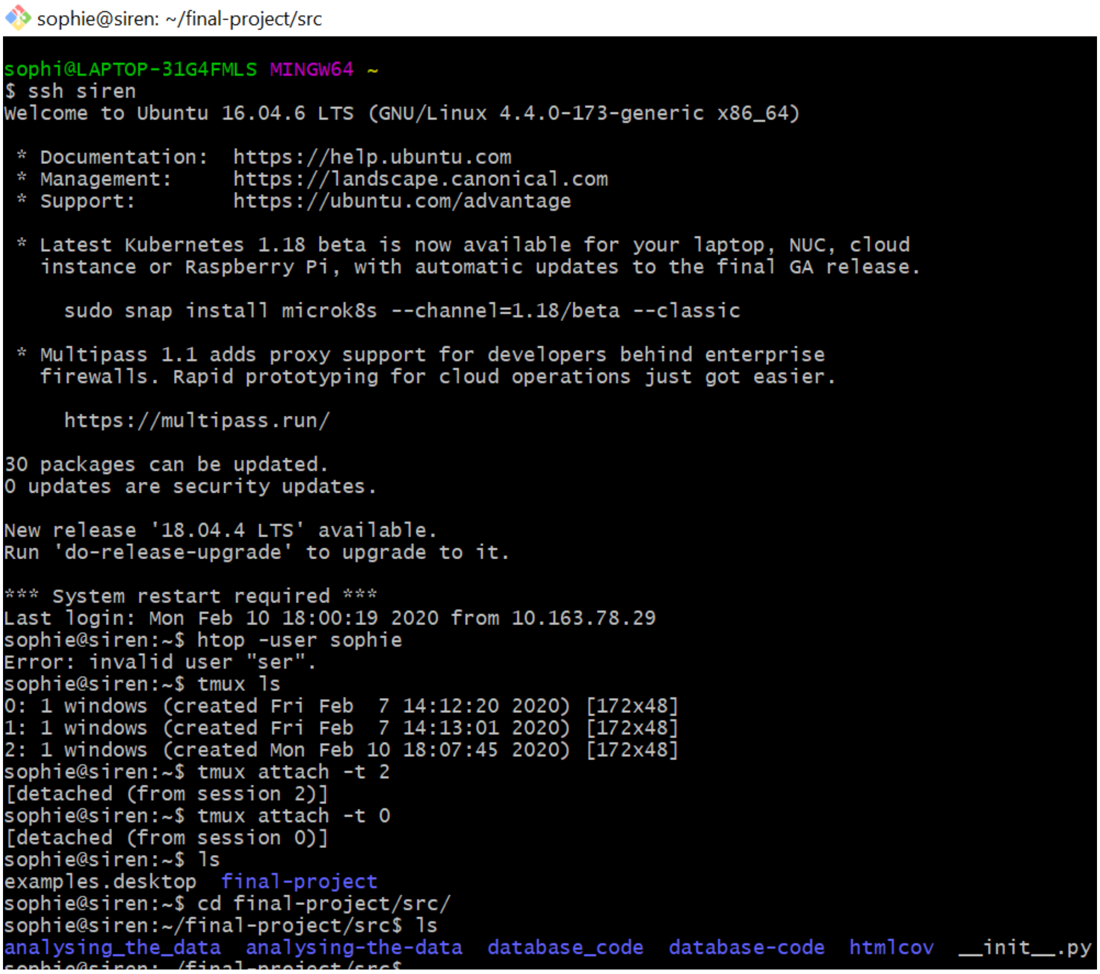


Figure 3.5: A screenshot of the authors connection to Siren via an SSH tunnel.

3.4 Remote Computing

This section describes the execution of the experiments via a remote computer and the reasons for doing so.

Firstly, due to the volume of data that was planned on being collected, it was decided that running the data remotely would be ideal, in order to allow the code to run uninterrupted for several weeks. Thus, Cardiff University School of Mathematics' computer Siren, a headless server with large storage, was used. However, when a trial was executed, it was decided that the run time was 'quick enough' and hence parallel processing would not be necessary. Yet, for future runs of the code, parallel processing is recommended, especially if the player set sizes being trialled are 'large'. See Section 3.5.3 for an explanation.

To connect to Siren, an SSH tunnel was used. An SSH (or Secure Shell) tunnel is used for sending and receiving network data over an encrypted connection. It adds a layer of network security, to those applications that do not natively

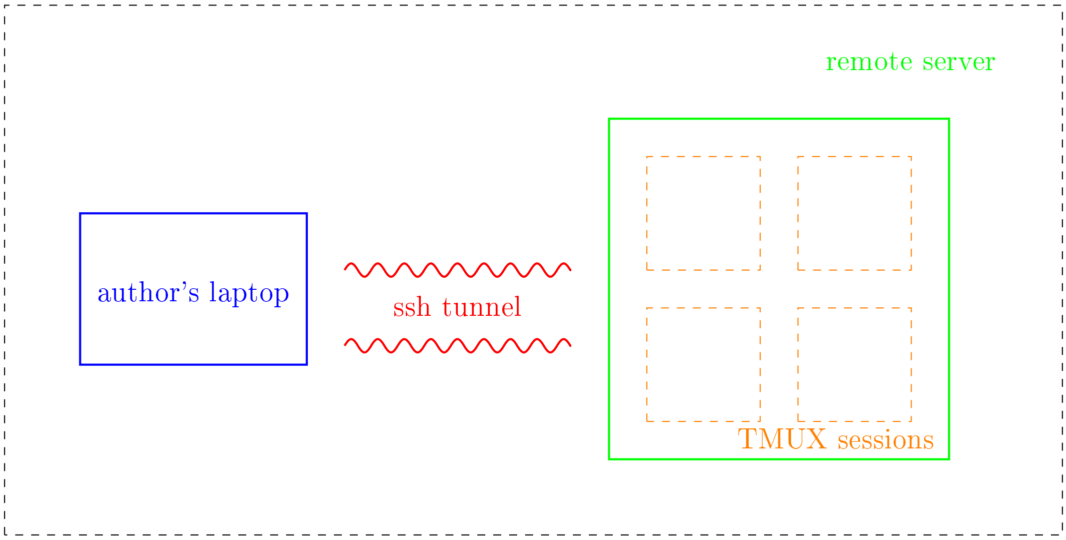


Figure 3.6: Representation of how the experiments were run remotely. Note, 'tmux sessions' correspond to emulators of terminals.

support encryption, and lowers the risk of interception. For a more detailed description of SSH, see [93]. Figure 3.5 shows the author's connection to Siren via SSH.

In order for the experiment to keep running, whilst disconnected from Siren, a terminal emulator was required, as Siren does not have a job scheduler. Specifically, the terminal multiplexer, TMUX [67] was used. This allowed for the creation and execution of several terminals from one screen. Moreover, a user could detach from a terminal and reattach later without execution being halted. Figure 3.6, shows a diagram of how the remote server, ssh tunnel, and user's laptop, were all in connection.

Once the experiment was running, the database had to be copied from Siren in order for analysis to begin. This was also achieved via SSH. Note, the code was written in a way which enabled the results to be written to the database concurrently, after every tournament. This was done to ensure that, if the system were to break, data would still have been available and retained within the database. This meant the database file was almost always 'open' resulting in the integrity check failing once transferred over SSH and an OperationalError in SQLAlchemy. Thus, in order to to load the database, with no failures, into a Jupyter Notebook, it needed to be compressed before transferring. This was achieved using the command line code as seen in Figure 3.7.

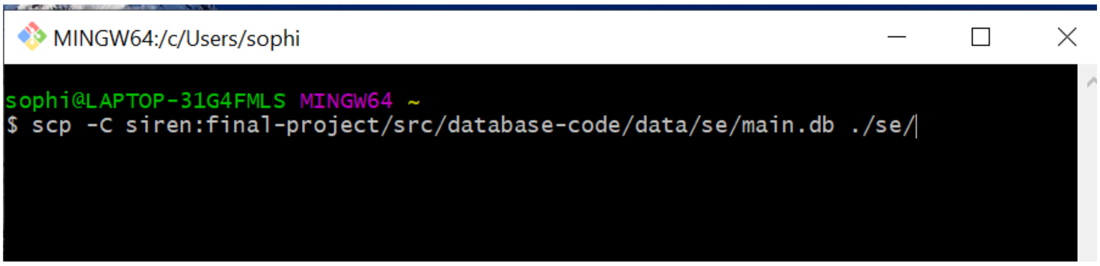


Figure 3.7: Command line code used to retrieve the database from the remote server.

3.5 Calculating Nash Equilibria

Recall Nash's Theorem, Theorem 1.2.1, explains that there exists at least one equilibrium in every finite game. However, it does not indicate how to obtain them. The proof of the theorem relies on finding the fixed point of the defined mapping but the proof of Brouwer's Fixed Point Theorem is an existence, and not a constructive, proof. That is, it does not give a method for obtaining the fixed point. Indeed, although not NP-complete[1], finding Brouwer fixed points has been shown to be a hard problem [45, 81]. Thus, defining algorithms which obtain Nash equilibria to some degree of efficiency has been a large research topic for many years, particularly in the 2000s. Example papers include [17, 23, 38, 42, 58, 59, 66]. Within the Python Library Nashpy [96], three such algorithms have been implemented: Support Enumeration, Vertex Enumeration and Lemke-Howson. However, the Lemke-Howson Algorithm will only find *one* equilibria and hence is not suitable for this study. Therefore the definitions of the first two algorithms, in the case of a two player game, are provided. Unless specified otherwise, this section (Section 3.5) is adapted from [75].

3.5.1 Support Enumeration

Before stating the method for the support enumeration algorithm, a few extra theoretical ideas are needed.

Recall, a *mixed strategy*, σ, is a probability distribution over the pure strategies.

[1]Both finding Brouwer fixed points and Nash equilibria cannot be NP-complete since existence of a solution is guaranteed [75]. Most problems in the set NP-complete are situations in which a solution might not exist [75]. However, [81] has shown that these two problems belong to a alternative complexity class, PPAD or Polynomial Parity Argument (Directed case). For a discussion into this, readers are referred to [81].

Definition 3.5.1. The *support* of σ is the set of all pure strategies, $s_i \in \sigma$, such that $s_i > 0$. That is, all pure strategies which have a positive probability within the mixed strategy.

Definition 3.5.2. A game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is called *non-degenerate* if no mixed strategy of support size $1 \leq k \leq |S_i|$ has more than k pure best responses.

For example, consider the following 2-player normal form game:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} \tag{3.2}$$

In (3.2), if the column player was playing the strategy $\sigma_2 = (0.2, 0.8, 0)$ then the support is {strategy1, strategy2}. Also, observe that, if the column player picked strategy1 then the row player could choose either their first or second strategy and hence this game is *degenerate*.

The method of support enumeration is given in Algorithm 2.

Algorithm 2: Support Enumeration

input : A *nondegenerate* two-player normal form game, where A, B are the row and column player's payoff matrices and σ_1, σ_2 are their strategy vectors, respectively.

output: Every Nash equilibrium of the input game.

for *all* $k = 1, \dots, \min\{m, n\}$ **do**

  for *all support pairs* I, J, *with* $I \in M, J \in N$ *and* $|I| = |J| = k$ **do**

    Solve

$$\sum_{i \in I} \sigma_{1i} B_{i,j} = v, \text{ such that } \sum_{i \in I} \sigma_{1i} = 1, \sigma_{1i} \geq \mathbf{0} \text{ for all } j \in J \tag{3.3}$$

    and

$$\sum_{j \in J} A_{i,j} \sigma_{2j} = u, \text{ such that } \sum_{j\ inJ} \sigma_{1i} = 1, \sigma_{1i} \geq \mathbf{0} \text{ for all } i \in I \tag{3.4}$$

    Check the best response conditions

$$\sigma_{1i} > 0 \implies (A\sigma_2)_i = \max\{(A\sigma_2)_k | k \in M\} \tag{3.5}$$

    and

$$\sigma_{2j} > 0 \implies (\sigma_1 B)_j = \max\{(\sigma_1 B)_l | l \in N\} \tag{3.6}$$

  end

end

Note, in Algorithm 2, solutions are not guaranteed to exist for (3.3) and (3.4). In this case, the support does not yield a Nash equilibrium.

An example

Here the computation of Nash equilibria via support enumeration is considered for a payoff matrix obtained from a two-player IPD tournament[2].

Consider the following normal form game:

$$A = \begin{pmatrix} 3.000 & 0.829 \\ 1.686 & 1.000 \end{pmatrix}; \quad B = A^T \tag{3.7}$$

Firstly, take $k = 1$, that is, looking for any pure best responses:

$$A = \begin{pmatrix} \underline{3.000} & 0.829 \\ 1.686 & \underline{1.000} \end{pmatrix} \quad B = \begin{pmatrix} \underline{3.000} & 1.686 \\ 0.829 & \underline{1.000} \end{pmatrix}.$$

Thus, two pairs of pure best responses are visible, giving the following two Nash equilibria:

$$\sigma = \{(1, 0), (1, 0)\} \text{ and } \sigma = \{(0, 1), (0, 1)\}.$$

Since this is a two-player game, the only other support that needs checking is $I = J = \{1, 2\}$.

Here, (3.3) and (3.4) become,

$$3\sigma_{r1} + 0.829\sigma_{r2} = 1.686\sigma r1 + \sigma_{r2} \quad \text{and} \quad 3\sigma_{c1} + 0.829\sigma_{c2} = 1.686\sigma c1 + \sigma_{c2}.$$

Rearranging and checking the constraint $\sum \sigma_r = 1, \quad \sum \sigma_c = 1$, yields

$$\sigma_{r1} = \sigma_{c1} = \frac{19}{165} \quad \text{and} \quad \sigma_{r2} = \sigma_{c2} = \frac{146}{165}.$$

Note, checking the best response condition for two players with the same number of pure strategies is trivial [56]. However, it is included here for completeness.

$$A\sigma_c^T = \begin{pmatrix} 3 & 0.829 \\ 1.686 & 1 \end{pmatrix} \begin{pmatrix} \frac{19}{165} \\ \frac{146}{165} \end{pmatrix} = \begin{pmatrix} 1.079 \\ 1.079 \end{pmatrix} \quad \text{and}$$

[2]Note, the *Defector*'s opponent in this tournament was stochastic player, *Inverse*. Here, $p_e = 0.0514, \quad p_n = 0$. This game was not identified as degenerate.

$$\sigma_r B = \begin{pmatrix} \frac{19}{165} & \frac{146}{165} \end{pmatrix} \begin{pmatrix} 3 & 1.686 \\ 0.829 & 1 \end{pmatrix} = \begin{pmatrix} 1.079 & 1.079 \end{pmatrix}$$

Therefore, the best response condition holds for both players. Hence, the third and final Nash equilibrium is given by:

$$\sigma = \left\{ (\frac{19}{165}, \frac{146}{165}), (\frac{19}{165}, \frac{146}{165}) \right\}.$$

Advantages and Drawbacks

Support enumeration is known to be a robust method for obtaining Nash equilibria. That is, given a non-degenerate game, it is guaranteed to return all equilibria and, even in the case of degeneracy, it will find some equilibria (see Section 3.6). However, this method essentially compares all pairs of supports and thus has an exponential complexity [86]. This implies that support enumeration is computationally expensive and the larger the game, the slower it will become.

3.5.2 Vertex Enumeration

Vertex enumeration is based on a geometric representation of games and hence, in this section, a brief introduction to this is provided. The reader is referred to [75] for a more detailed approach to this topic.

Definition 3.5.3. Let A, B be *positive* payoff matrices for the row and column player; that is, each element $a_{ij}, b_{ij} > 0$, for all $i = 1, \dots, M, j = 1, \dots, N$. Then the row, column *best response polytopes*[3], denoted P, Q are given respectively by

$$P = \{x \in \mathbb{R}^M | x \geq \mathbf{0},\ xB \leq \mathbf{1}\} \quad Q = \{y \in \mathbb{R}^N | Ay^T \leq \mathbf{1},\ y \geq \mathbf{0}\}. \tag{3.8}$$

It is assumed that the utility values which appear in (3.3) and (3.4) have been normalised to 1. This means that the vertices are no longer probabilities and hence scaling will be required to find the Nash equilibria. Note, the strictly positive payoffs is not a constraint since a constant can be added to each with no effect.

For example, consider the payoff matrix

$$A = \begin{pmatrix} 1 & 5 \\ 4 & 1 \end{pmatrix} \tag{3.9}$$

[3] In general, a *polytope* is defined as a bounded set $\{z \in \mathbb{R}^d | Cz^T \leq q\}$ where C is a $k \times d$ matrix and z is a $1 \times d$ vector [75].

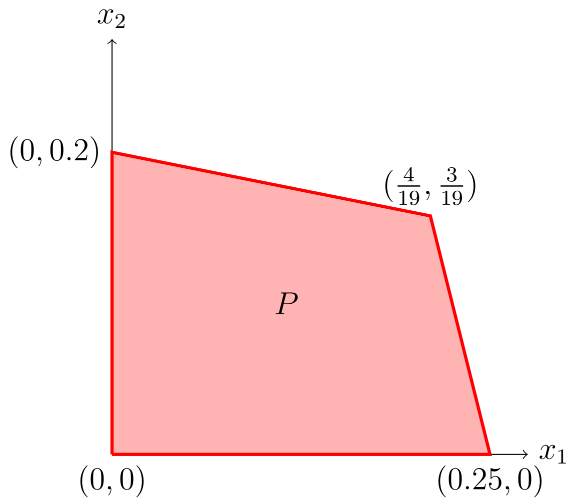


Figure 3.8: Best response polytope, P, obtained from the payoff matrix given in (3.9).

Then the best response polytope, P here is given by the inequalities:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + 5x_2 \leq 1, \quad 4x_1 + x_2 \leq 1$$

which yields the following polytope as given in Figure 3.8.

Two algorithms are given, Algorithm 3 'prepares' the polytope for Algorithm 4 which returns the Nash equilibria.

Algorithm 3: Vertex Labelling

input : A polytope, $P \in \mathbb{R}^n$.
output: A labelled-vertex polytope.

enumerate each of the defining inequalities of P, $c_1, \dots, c_k$
for *each vertex* $v_i \in P$ **do**
  find the inequalities of P which are *binding* at v_i, that is the defining equations are equalities.
  the label of v_i is given by $\{c_{i1}, \dots, c_{il}\}$, where c_{ij} is in the label if and only if equation c_j is binding for v_i.
end

A pair of labels $v_i \in P$, $u_j \in Q$ are called *fully labelled* if every inequality 'number' of $P \cup Q$ appears in either the label of v_i or the label of u_j. Then there is a notion which states that each fully labelled pair, when normalised, corresponds to a Nash equilibrium. Note, this does not include the vertex pair $(\mathbf{0}, \mathbf{0})$ since, although this is a fully labelled pair, it corresponds to neither opponent playing any strategies.

Algorithm 4: Vertex Enumeration

input : Best response polytopes, P, Q, as defined in 3.5.3, for a *nondegenerate* game.

output: All Nash equilibria of the corresponding game.

```
for each polytope, P, Q do
    Execute Algorithm 3.
    for each pair of vertices {u_i, v_j} in P, Q respectively, except (0, 0) do
        check if they are fully labelled.
        if they are fully labelled then
            add to the list of equilibria.
        end
        else
            continue
        end
    end
end
```

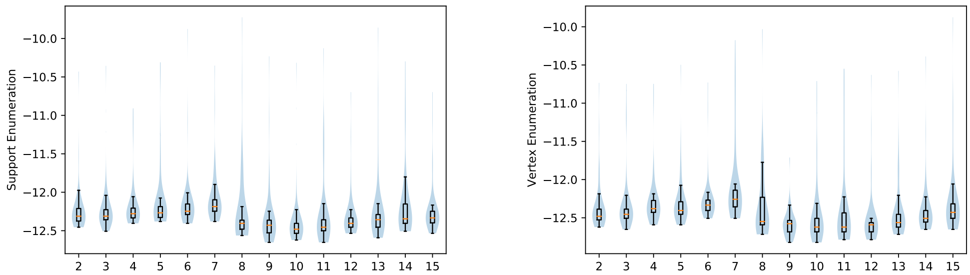
Advantages and Drawbacks

Vertex enumeration is more efficient than support enumeration, according to [75], since there are more supports in a game than there are vertices. Indeed, consider the example of $M = N$ where M, N are the number of binding inequalities in the best response polytopes, P, Q, respectively. Using the support enumeration algorithm, approximately 4^n support pairs will need to be observed but, according to the 'Upper Bound Theorem' for polytopes [3, 18, 90], P and Q have less than 2.6^n vertices. Thus, given there exists an efficient method for enumerating vertices, this implies less further computational complexity. However, if the game is degenerate, this algorithm may not return any Nash equilibria.

3.5.3 Algorithm Execution Timings

Due to the robustness of the support enumeration, it was decided that this would be the main method of calculating Nash equilibria. Having said this, timings for each of the two algorithms were obtained, to see if their computational times were significantly different. For the purposes of this trial run, the following parameters were used: 100 tournament repetitions, $p_e = 0.2$ and $p_n = 0$. The results can be seen in Figure 3.9.

From Figure 3.9, it can be seen that there is not a significant difference in the execution times of the algorithms. Hence, although support enumeration has a greater computational complexity than vertex enumeration, it is not going to have any recognisable impact on the number of experiments executed.



(a) Timing the support enumeration algorithm.

(b) Timing the vertex enumeration algorithm.

Figure 3.9: Violinplots of the log timings obtained for the experiment and calculation of Nash equilibria using the two algorithms discussed.

3.6 Degeneracy

Recall, the definition of non-degeneracy, given in Definition 3.5.2. This implies a *degenerate* game is one in which there exists a support of size k where the number of pure best responses is greater than k. For example, consider the following payoff matrix:

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 4 & 2 \end{pmatrix} \tag{3.10}$$

Then, if the column players picks their second strategy (support of size 1), the row player can pick either of their strategies (that is, two pure best responses). Thus, this game is degenerate.

Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a degenerate game. Then, recall that support enumeration may not return all Nash equilibria. This is due to the fact that if solutions to (3.3) and (3.4) of Algorithm 2 exist, they may not be unique. Indeed, the number of Nash equilibria in a degenerate game may be infinite [75]. Considering degeneracy in terms of vertex enumeration implies that a vertex of the best response polytope $P = \{x \in \mathbb{R}^M | x \geq \mathbf{0},\ xB \leq \mathbf{1}\}$ may have more than M labels leading to a 'badly defined' polytope.

Within the library Nashpy, the algorithms have been implemented such that if potential degeneracy is identified, for example, by the reasons given above, then a warning is issued. Thus, in order to retain whether a game is possibly degenerate, the algorithm was required to 'catch' the given warnings. This was achieved using the code seen in Listing 2, with the Python warnings module.

Note, warnings for potential degeneracy are only highlighted if the Nash equilibria

```
with warnings.catch_warnings(record=True) as w:
    warnings.simplefilter("always")

    if support_enumeration is True:
        nash_equilibria = list(game.support_enumeration())

    else:
        highlight_numpy_warning = np.seterr(all="warn")
        nash_equilibria = list(game.vertex_enumeration())

if len(w) == 0:
    warning_message = None

else:
    warning_message = str([w[i].message for i in range(len(w))])
```

Listing 2: Python code used to ‘catch’ potential degeneracy.

obtained do not make sense, that is, are not a probability distribution. On the other hand, it is possible for the algorithm to obtain some correct Nash equilibria, even if the game is degenerate. This means that, any results, regarding degeneracy of the games, obtained during the experiment (see Chapter 4) need to be inferred with caution. These are only the games caught by the algorithms in Nashpy and are not necessarily all of them.

3.7 Conclusion

In this chapter, the creation and execution of the large experiment was detailed in its entirety. Justifications were given as to why certain methods and software were used or not. Also, how the Nash equilibria are calculated was explained, with an example given. Moreover, the potential problems which could be faced as a result of degeneracy are highlighted.

Chapter 4

Analyses

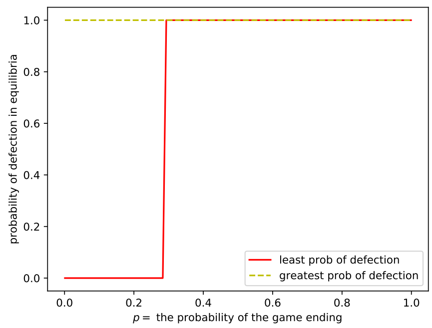
In this chapter, an analysis of the data collected via the methods described in the previous chapter (Chapter 3) is given. Firstly, a brief initial overview is provided, where descriptive statistics of the equilibria obtained and the overall characteristics of the strategies used are discussed. Following this, a critical analysis of the p-thresholds obtained is carried out. Here, the environmental effects on the outcome of the game discussed include: number of opponents and level of added noise. Note, the final database has 825700 entries (rows) and a total number of 159 player sets. This dataset has been archived at (https://doi.org/10.5281/zenodo.3784594).

Before this, a reminder that any analysis stated, in this chapter, on degeneracy is purely from those identified by Nashpy. This implies that there may be further degeneracy that was unidentified.

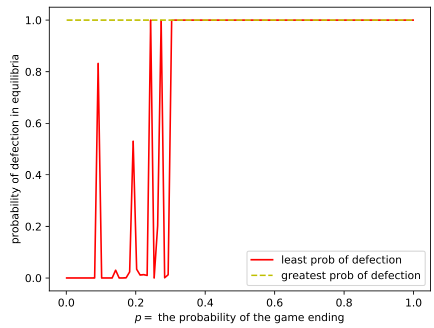
4.1 Initial Analysis

In this section, all the games (including those identified as degenerate) are considered. Taking a brief look at the graphs produced for each set of tournaments, it can be seen that the main 'shapes' obtained are as seen in Figure 4.1.

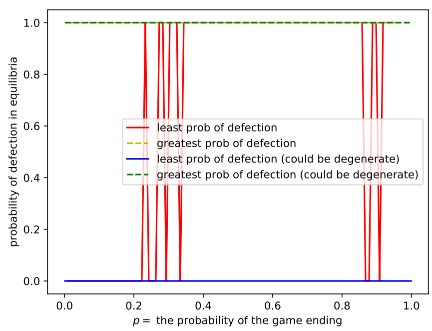
In Figure 4.1a, a clear p-threshold of approximately 0.28 is apparent, clearly visualising the Folk Theorem. In this tournament set the opponent was *Inverse*, a stochastic player, indicating that perhaps the stochasticity of a player does not affect the accuracy of the p-threshold. In Figure 4.1b, the precise value of the p-threshold is less clear, lying approximately in the range (0.1, 0.3). This could be due to the potential degeneracy identified or just an element of numerical experiment noise, that appears within each tournament, possibly magnified by



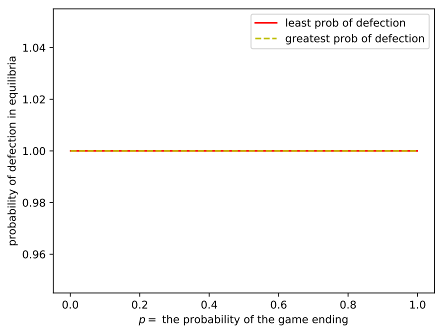
(a) 2-player tournament set with $p_n = 0$, one stochastic player and no degeneracy identified.



(b) 6-player tournament set with $p_n = 0$, three stochastic players and potential degeneracy was yielded from 10 of the tournaments.



(c) 5-player tournament set with $p_n = 0$, two stochastic players and five tournaments played yielded potentially degenerate games.



(d) 4-player game with $p_n = 0$, one stochastic player and no degeneracy identified.

Figure 4.1: Example graphs obtained from the experiment.

	experiment_number	number_of_players	tournament_player_set	num_of_equilibria	least_prob_of_defection	greatest_prob_of_defection
mean	107364.481228	5.435509	97.105850	1.913727	0.342275	0.459722
std	46880.538807	1.726832	42.619612	2.022014	0.469061	0.489564
min	0.000000	2.000000	0.000000	1.000000	0.000000	0.000000
25%	72231.000000	4.000000	65.000000	1.000000	0.000000	0.000000
50%	114641.000000	6.000000	104.000000	1.000000	0.000000	0.000000
75%	147396.000000	7.000000	133.000000	3.000000	1.000000	1.000000
max	175399.000000	8.000000	159.000000	39.000000	1.000000	1.000000

Table 4.1: Summary statistics from the data.

the number of stochastic players. The opponents within this set were: *Feld: 1.0, 0.5, 200*; *Cooperator*; *EvolvedLookerUp2_2_2*; *Tullock: 11*; and *ZD-GEN-2: 0.125, 0.5, 3*[1]. Figure 4.1c shows a potential problem with the visualisation of the Folk Theorem when degenerate games are involved. It becomes unclear as to what is happening in the graph, especially regarding where the p-threshold lies. The opponents, in this case, were: *Random: 0.5*; *Grumpy: Nice, 10, -10*; *Fortress3*; and *Negation*. Finally, Figure 4.1d gives an example of a tournament set for which there was always a non-zero probability of defection, regardless of the p_e value. In this case, the precision of game-ending probabilities chosen was not accurate enough to identify the p-threshold. It implies that the tournament has to have an ending probability of almost zero, within the interval (0, 0.001), in order for a zero probability of defection to be rational. Here, the opponents of the *Defector* were: *AntiCycler*; *e*; and *Stalker: (D,)*. Similarly, it is observed that some graphs obtained are constant at zero. Again this indicates that the precision of game-ending probabilities was not fine enough to highlight the p-threshold. The ending probability of these tournaments has to lie within the interval (0.999, 1). That is, almost immediately, the decision to defect is no longer rational. Further research into these tournaments is highly recommended.

The summary statistics are given in Table 4.1. From this, it can be seen that the number of opponents the *Defector* played against ranged from one to seven, with an average of four opponents. Also, as expected, the mean p_e value was 0.5. Observe that, overall, there were $175,399$ distinct tournaments played with a total of 159 distinct player sets. Looking now at the statistics for Nash equilibria, it can be seen that a total of $823,823$ equilibrium points were calculated in this experiment, with an average of $1.914 \approx 2$ equilibria per game. However, observe, at least one game obtained 39 equilibria which will be explored into later on in this section. Considering the probabilities of defection within these equilibria, notice that both the greatest and the least probabilities of defection ranged from

[1]Note, these are the names of the strategies as implemented in the Axelrod library and hence includes their corresponding parameters.

```
len(axl.filtered_strategies(filterset={"stochastic": True})),
len(axl.filtered_strategies(filterset={"stochastic": False}))

(86, 156)
```

zero to one inclusive with a 50th percentile of zero. But, looking at the average values, the least probability has a mean of 0.342 and only just above this, the greatest probability has a mean of 0.460.

Next, further descriptive statistics were calculated for the strategies. This was to obtain a more in-depth view on the types of strategies randomly chosen to play and their characteristics. It is observed that the player which appeared the most times (9 times) is *ZD-GEN-2: 0.125, 0.5, 3*; followed closely by *Tideman and Chieruzzi* with 7 sets of tournaments. On the other hand 38 out of the 200 strategies playing in this experiment appeared only once. Considering the memory depths of the strategies found the majority of strategies to have an infinite memory depth. On the other hand, strategies having no memory or a depth equal to one were also significant. Looking at the frequency of stochastic players, it is clear that there is a bias towards deterministic strategies. However, this is to be expected as running the code in Listing 4.1 highlights that over half the strategies implemented in the Axelrod library are classed as deterministic.

The number of Nash equilibria obtained for each game was also analysed and their distributions with respect to the number of players was plotted (see Figure 4.2). Obtaining counts on the number of equilibria gave the conclusion that the majority of games (131773) yielded one Nash equilibria with 28793 games obtaining 3 equilibria. Furthermore, the maximum number of equilibria yielded, 39, was for a game which, doing a search on the database, was found to be a six player game with $p_n = 0.1$. The opponents of the *Defector* were: *Inverse Punisher*; *Prober*; *PSO Gambler 2_2_2 Noise 05*; *Handshake*; and *More Tideman and Chieruzzi*. This game is likely to be degenerate however, when taking a closer look, no degeneracy was identified. This could be worth looking at in futer investigation of the dataset.

Figure 4.2 shows the distributions of the number of Nash equilibria per number of players. This visualisation turns out to not be extremely revealing, possibly as an effect of degeneracy, however some insights can be found here. Observe, all of the distributions in this plot have a clear modal value of one. That is, irrespective of the number of players, the majority of games yielded only one equilibria. Moreover, there also seems to be an increase in density around 3

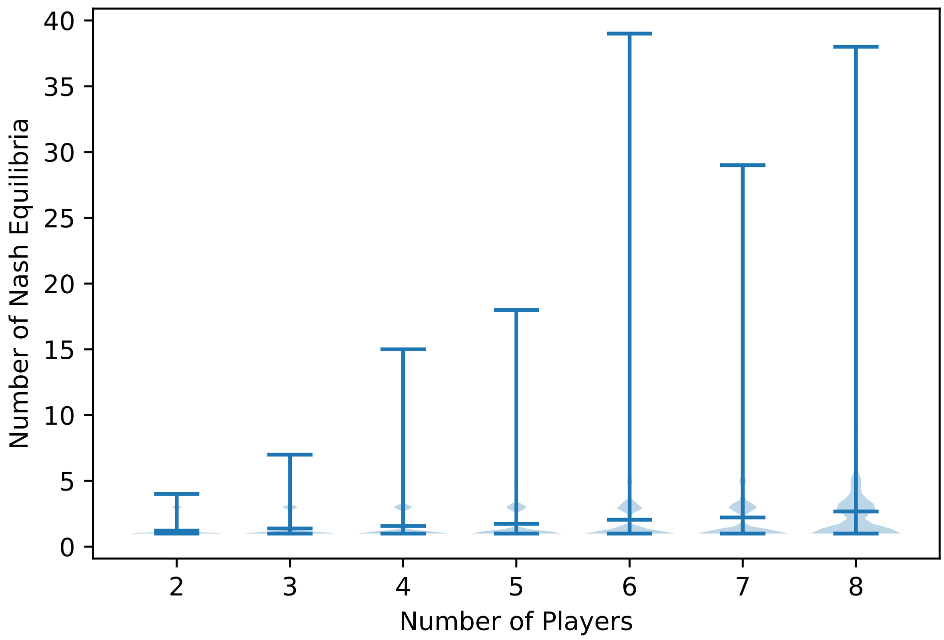


Figure 4.2: A violinplot showing the distribution of the number of equilibria obtained for varying numbers of players.

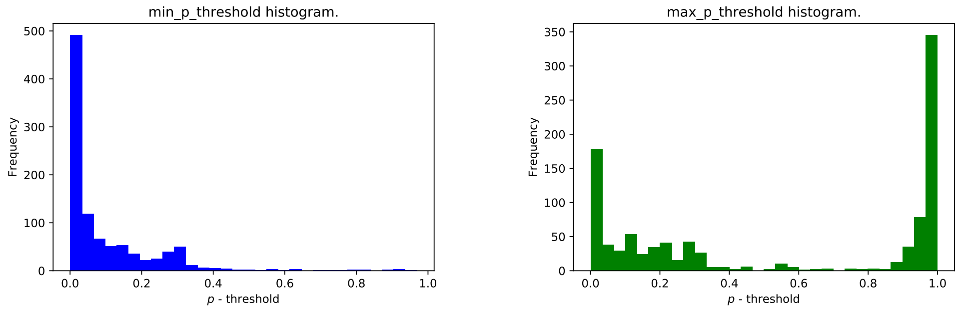
equilibria which becomes more prominent as the number of players increases. As can be seen from the plot, the variance in the number of equilibria increases with the number of players, apart from when there were 6 players, where the spread is maximum. This could be due to the 39 equilibria gained for one game as previously discussed. Considering the mean of the distributions, these are also slightly increasing as the number of players increase.

4.2 Analysis of the p-Threshold

Firstly, for clarity, here is a restatement of the definition of a *p-threshold*: The value of p_e for which the least probability of defection in Nash equilibria becomes zero.

In order to analyse the p-thresholds of the tournaments, a csv file was created[2] containing the minimum, mean, median and maximum p-threshold probabilities for each tournament set. This was in order to gain as much information as possible from tournaments which gave graphs such as in Figure 4.1b described above.

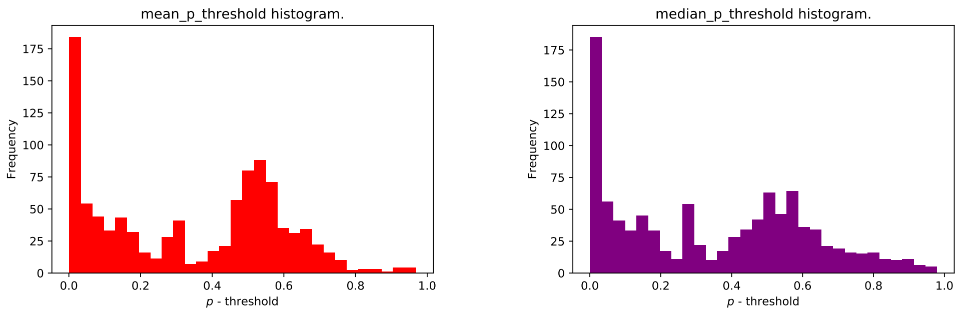
[2]Please see the GitHub Repository (https://github.com/shapperzsm/final-project) for the code used to obtain this file. The csv file has also been archived at (https://doi.org/10.5281/zenodo.3784594).



(a) A plot to show the minimum p-thresholds.

(b) A plot to show the maximum p-thresholds.

Figure 4.3: Plots to show the p-thresholds for all 1001 sets of tournaments.



(a) A plot to show the mean p-thresholds.

(b) A plot to show the median p-thresholds.

Figure 4.4: Plots to show the p-thresholds for all 1001 sets of tournaments.

Within this file, other than the varying thresholds, the information about the number of players, tournament set and p_n were retained. Moreover, it contains a column which identifies whether any of the strategy sets led to possible degenerate games. An exploration into the overall p-thresholds is now given.

Observe, in Figure 4.3a, the majority of minimum p-thresholds were less than 0.4, with a clear modal value of zero. That is, in a significant proportion of the tournaments, there was no p_e identified for which the probability of defection was zero. Now considering Figure 4.3b, it can be seen that the modal value for the maximum p-threshold is one. Also, in comparison with the minimum p-threshold, there is a larger spread in the density. Yet, there is still a peak around zero as with the minimum p-threshold data.

Looking at the mean and median p-threshold data, Figure 4.4, observe that the distributions obtained are similar with, again, a modal value of zero, indicating that, for many games, p_e has to be almost zero before defection becomes irrational. That is, the probability of the game continuing is high. However, in

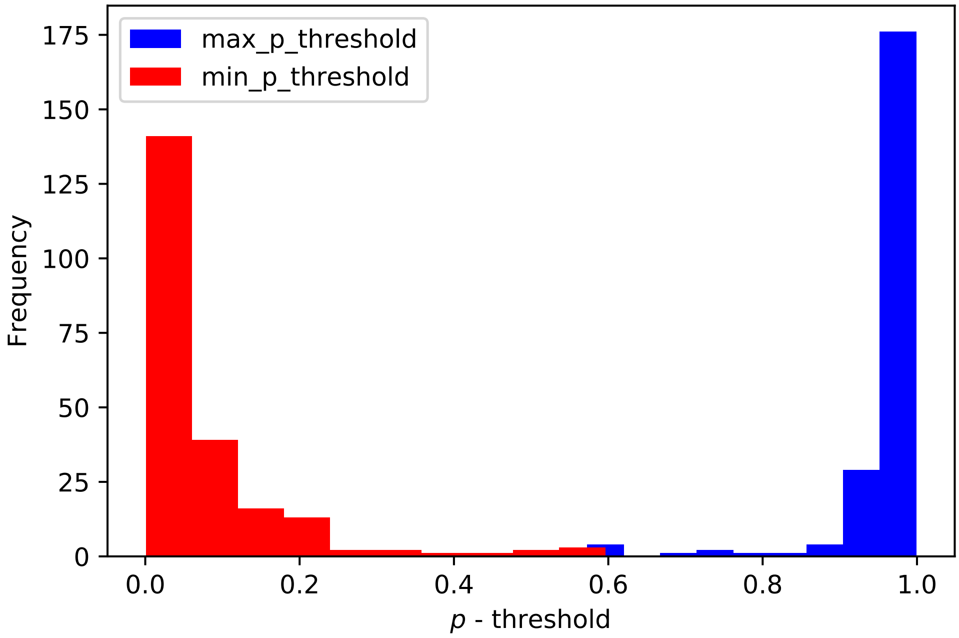
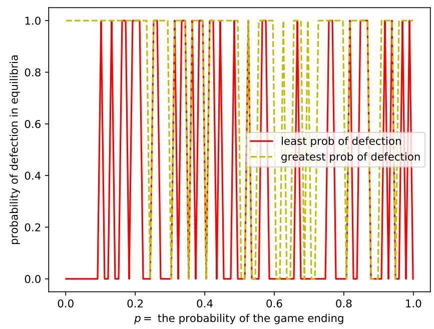


Figure 4.5: A plot to show the minimum and maximum p-thresholds for those tournaments which had an mean threshold within the range $[0.5, 0.6]$.

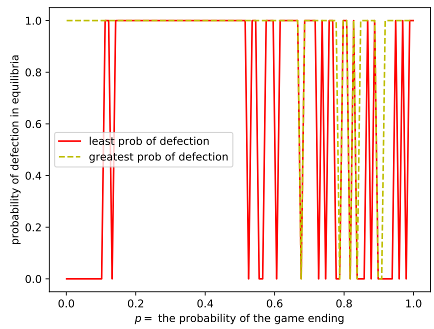
these histograms there is also a clear peak around a p-threshold of 0.5. This suggests that some of the tournaments may have several p-thresholds, perhaps due to stochasticity, degeneracy or just numerical experiment noise that appears in the tournaments. Indeed, obtaining the minimum and maximum p-thresholds for those tournaments where the mean p-threshold was within the range $[0.5, 0.6]$, it can be seen that, from Figure 4.5 for a significant proportion, their minimum threshold was around zero and their maximum around one. Moreover, looking closer at three of the tournaments which satisfied the above, there is clear noise within the corresponding graphs, Figure 4.6.

The plots contained in Figure 4.6 were sampled using the random library in Python. What is interesting here is that they all contain varying amounts of standard PD noise and further analysis into this would be beneficial. This could yield whether this is one of the main causes for the inaccuracy of the thresholds. For example, looking at how many of the tournaments with this property had $p_n > 0$.

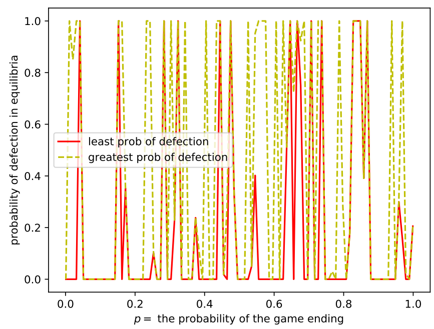
Regarding degeneracy, out of all 1754 tournaments, 372 were identified as potentially leading to degenerate games. These are omitted from the following plots in an attempt to focus on non-degenerate games.



(a) 6-player tournament set with $p_n = 0.2$, two stochastic players and no degeneracy was identified.

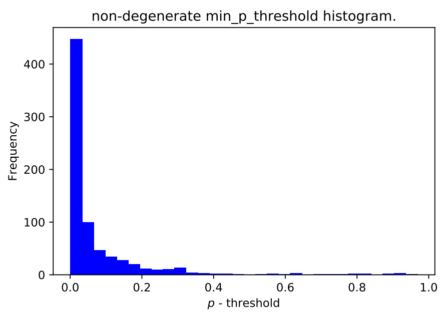


(b) 3-player tournament set with $p_n = 0.1$, one stochastic player and no degeneracy was identified.

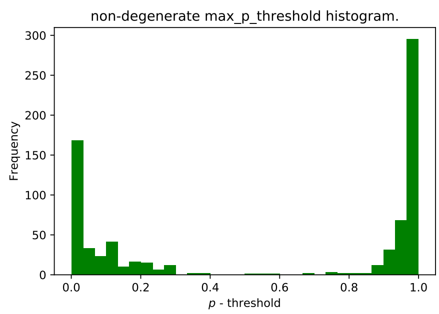


(c) 3-player tournament set with $p_n = 0.5$, one stochastic player and only one tournament identified as degenerate, with $p_e = 0.9788$

Figure 4.6: Example plots of the tournaments where the mean p-threshold was within the range $[0.5, 0.6]$.

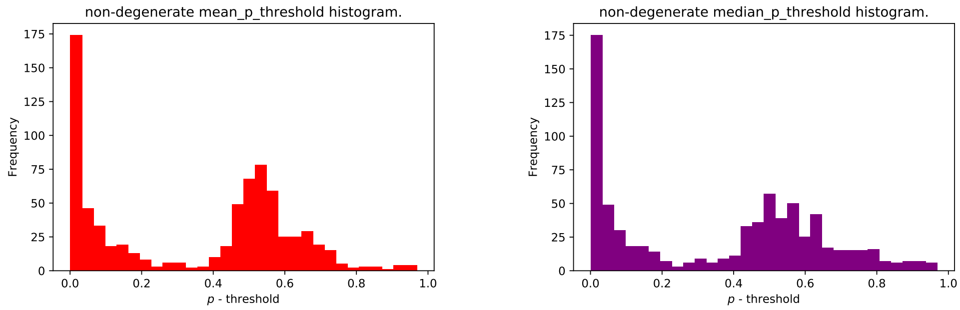


(a) A plot to show the minimum p-thresholds.



(b) A plot to show the maximum p-thresholds.

Figure 4.7: Plots to show the p-thresholds for all tournaments which were not identified as degenerate.



(a) A plot to show the mean p-thresholds.

(b) A plot to show the median p-thresholds.

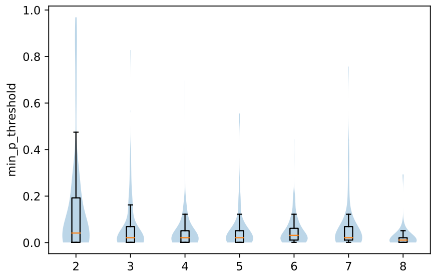
Figure 4.8: Plots to show the p-thresholds for all tournaments which led to games not identified as degenerate.

Comparing Figures 4.7a, 4.7b, 4.8a and 4.8b with Figures 4.3a, 4.3b, 4.4a and 4.4b, respectively, it can be seen that, in general, there is no significant change in the distributions of the thresholds. However, there is a more prominent peak in Figure 4.4a around 0.3 than in the corresponding potentially non-degenerate plot of Figure 4.8a. Further work regarding the effects of degeneracy is advised as the identification of games which are degenerate is non-trivial.

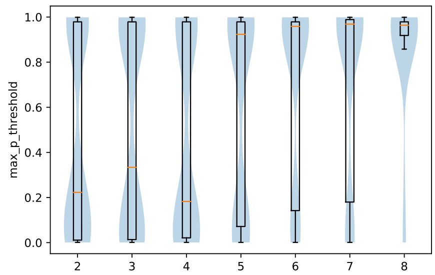
4.2.1 Effects of the Number of Players

In this section, the p-thresholds will be analysed with respect to the number of opponents the *Defector* played against. Note, in this section, any games identified as degenerate are omitted.

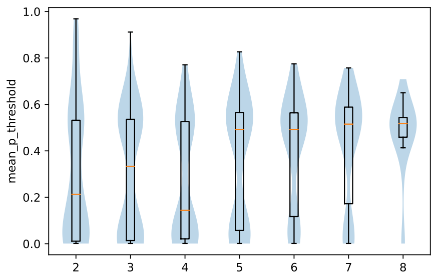
From Figure 4.9a, it can be seen that the distributions of the minimum p-thresholds, with respect to the number of players, all have a modal value around zero. However, apart from the seven-player tournaments, the spread of the distributions decrease, along with the mean values. Considering the maximum p-thresholds, Figure 4.9b, the distributions become bimodal with mode values around zero and one. Yet, as the number of players increases the modal value at zero becomes less prominent with the eight-player tournament distribution not appearing to have a mode at zero. The variance of the distributions is similar and, apart from four-player tournaments, the means increase with the number of players. Looking now to the mean and median violinplots, Figure 4.9c and Figure 4.9d respectively, there is no significant difference between the two plots, other than the spread is a little larger in the median thresholds. Within the mean plot, Figure 4.9c, it can be seen that the distributions also start off bimodal, with modal values at zero and approximately 0.5. However again, the



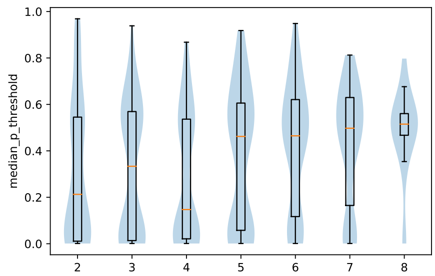
(a) Minimum p-threshold violinplot.



(b) Maximum p-threshold violinplot.



(c) Mean p-threshold violinplot.



(d) Median p-threshold violinplot.

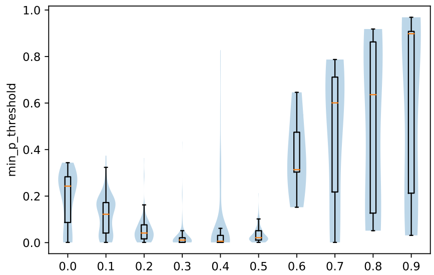
Figure 4.9: Violinplots of the thresholds for each number of opponents.

mode at zero becomes less distinct, with the eight-player tournament distribution being unimodal. The modal value at around 0.5 is a consequence of the reason stated previously. Moreover, observe that, apart from four-player tournaments, the variance of the distributions seem to decrease with the size of the player set whilst the means increase. These take a value of approximately 0.3 for two-player tournaments to around 0.5 for eight-players. Looking at Figure 4.9 as a whole, it is implied that the number of players has no significant effect on the value of the p-threshold.

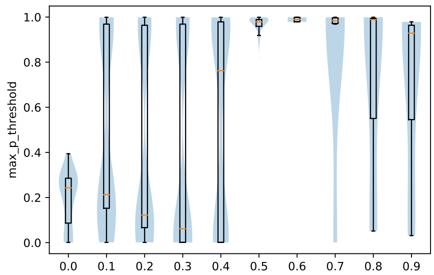
4.2.2 Effects of Noise

Here, an analysis on the affect values of p_n have on the p-threshold is provided. According to [40], the addition of noise to a tournament indicates that, with a certain probability, the action of a particular strategy is altered. That is, an action of C changes to D and vice versa.

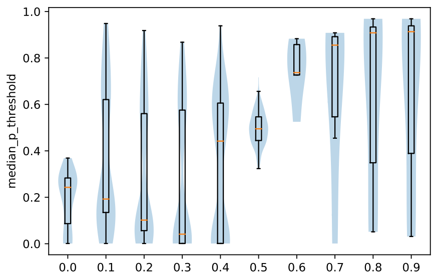
Figure 4.10a, shows the distribution of the minimum p-thresholds for each p_n value used in the tournaments. It can be noted that the distributions of p_n values at least 0.6 have a large variance, indicating that using a large value of p_n is highly random and thus no conclusions can be drawn. On the other hand,



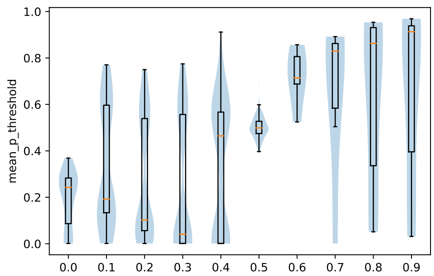
(a) Minimum p-threshold violinplot.



(b) Maximum p-threshold violinplot.



(c) Median p-threshold violinplot.



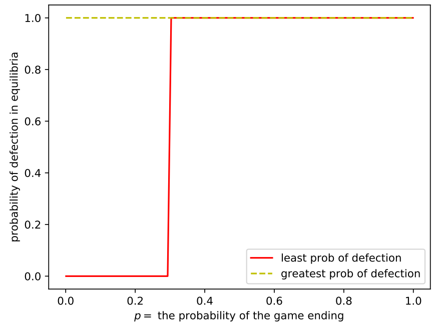
(d) Mean p-threshold violinplot.

Figure 4.10: Violinplots of the thresholds for each value of p_n.

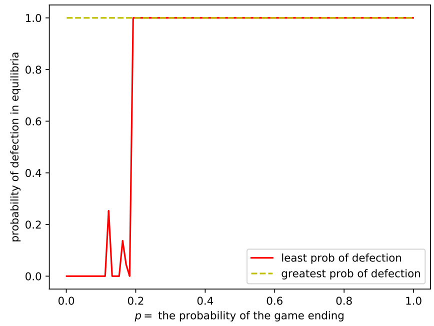
considering the noise levels less than 0.6, observe that the mean p-thresholds decrease from around 0.25 to approximately zero as the noise increases. These distributions are also clearly unimodal. Looking now at Figure 4.10b, it can be seen that the majority of distributions have a much larger spread here, varying over the full spectrum of game-ending probabilities. Similar observations can be made from Figure 4.10c and Figure 4.10d.

Figure 4.11 shows an example of the same player set through a variety ofp_n values. Indeed, here it can clearly be seen that the amount of noise does effect the p-threshold. However, this is to be expected since, by definition, as tp_n increases, the *Defector* will be observed as similar to the *Cooperator* when $p_n = 0$.

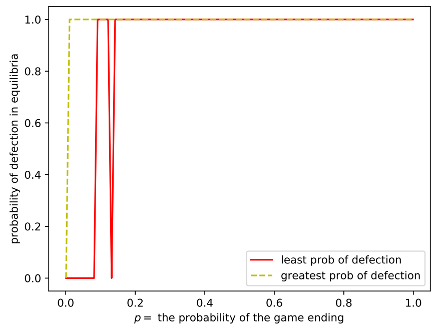
Therefore, on the whole, there are not many significant conclusions that can be made here. This implies that the addition of noise to an already random tournament obscures any possible visions of the p-threshold, especially as p_n increases.



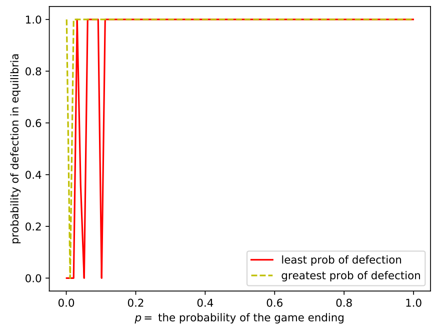
(a) $p_n = 0$



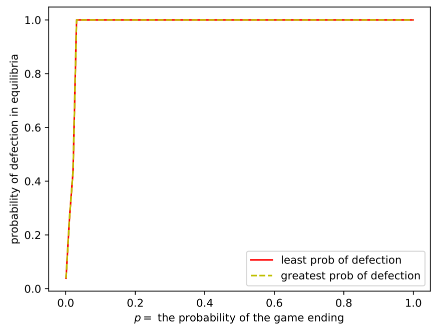
(b) $p_n = 0.1$



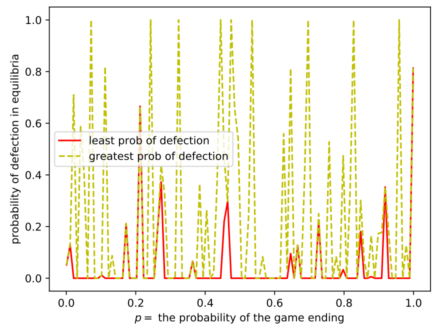
(c) $p_n = 0.2$



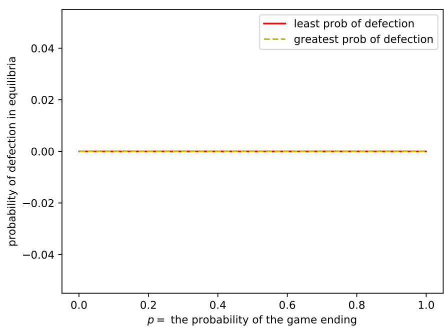
(d) $p_n = 0.3$



(e) $p_n = 0.4$



(f) $p_n = 0.5$



(g) $p_n \geq 0.6$

Figure 4.11: Observation of one 6-player tournament set through the varying values of p_n. There was one stochastic player and 13 out of the 1100 tournaments played yielded potential degenerate games. The opponents here were: *Getzler*; *Punisher*; *Forgiver*; *GrudgerAlternator*; and *GraaskampKatzen*.

4.3 Conclusions and Further Work

In this section, the beginnings of an analysis into the p-thresholds was discussed. The effects of the number of players and level of standard PD noise on these p-thresholds were the main focus, with a brief discussion regarding the degeneracy of games also given. This turned out to be a non-trivial task due to the three sources of noise potentially affecting the tournaments (standard PD noise, stochasticity of players and numerical experiment noise) as well as the difficulty in identifying degeneracy. However, a few points of interest were highlighted and these are summarised here.

Firstly, the potential degeneracy of games yielded by the tournaments, at first glance appears to not create much effect. The histograms of the p-thresholds, when the tournaments including potential degeneracy were omitted, did not have any significant changes when compared to the original histograms of all tournaments. However, this could be due to the inclusion of unidentified degenerate games in the non-degenerate plots and thus further exploration is advised here. Regarding the number of players in a tournament, it was initially thought that this would be a key factor in the variability of the p-threshold. Yet, on obtaining the distributions of the p-thresholds for player sets of size two to eight, it was implied that the number of players does not have a significant impact. Finally, the effect of different values of p_n on the p-threshold was analysed. Here, it was observed that, as expected, the level of additional noise did affect the p-threshold however there were no significant trends appearing out of the randomness.

As stated above, this is only the very start of an analysis into the p-thresholds described by the 'original' Folk Theorem [33] and the effects of the varying environmental factors in tournaments of the IPD. Thus, many questions regarding this are still to be researched and a few recommendations regarding further work are now summarised. Firstly, it was observed that there were a significant proportion of tournaments for which the graph remained constant at zero or one. That is, the p-threshold was not identified using the precision of game-ending probabilities chosen and therefore must lie within the intervals (0, 0.001) or (0.999, 1), respectively. Hence, these tournament sets could be rerun with a much finer precision, within the appropriate intervals, to highlight exactly what is happening here. Also, an analysis into the characteristics of the strategies involved, and the value of p_n included, in these tournaments could provide a clearer insight into potential reasons. Moreover, with regards to analysing the characteristics of players, it is suggested that those player sets in which stochastic players were included could be repeated with the stochastic players removed. This could help in revealing whether the stochasticity of the player has any effect on the p-threshold.

To check the reliability of the data collected, a second experiment is recommended using a different algorithm for calculating the Nash equilibria, for example vertex enumeration. This could be used in comparison with the data already collected to identify whether the algorithms are producing the same Nash equilibria or, more importantly, whether they identify the same games as degenerate. Furthermore, it is suggested that this experiment be executed with a larger number of tournaments repeats (greater than 500) to observe whether this 'smooths' the payoff matrices with greater success to enable for a clearer visualisation of the p-thresholds. Finally, some multivariate data analysis of the results, for example regression, could provide some more insights into this topic.

Overall, this chapter has been successful in visualising the Folk Theorem using the data collection setup as explained in Chapter 3 and using the plots obtained as in Figure 4.1a.

Chapter 5

Conclusions and Recommendations

This project looked into the Folk Theorem for the IPD and, in particular, at the game-ending probabilities for which defection is no longer rational. The three aims of the study, as given in Chapter 1, are restated here for convenience:

1. To provide a review of past and present literature already published in the field of folk theorems;
2. To develop a program which executes a large experiment involving tournaments of the IPD with differing environments to obtain graphs similar to those in Figure 1.5; and
3. To perform analyses on where the p-thresholds seem to lie and whether it is affected by the change in the number of players, levels of standard PD noise, etc.

Therefore, in this chapter, a brief section discussing the results of each aim is provided. Then, a few shortcomings of the project are mentioned and areas in which the research could be expanded further are highlighted.

5.1 Conclusions

In this section, each aim is explored with regards to results found and how well they were addressed. Overall, the first two aims were successfully achieved. A well rounded knowledge in the recent research of folk theorems was obtained and a successful experiment carried out. However, maximal results were not obtained for the final aim due to time limitations and the non-triviality of degeneracy.

5.1.1 Aim 1: Reviewing the Literature

The first aim was addressed in Chapter 2. Here, a literature search on "folk theorem" yielded a vast range of results which spanned the last fifty years. It was discussed that the true origin of the 'folk theorem' is unknown however its statement and proofs appear in written work since the 1970s. These papers focused on infinitely repeated games with subgame perfect equilibria. Following this, many generalisations and refinements of the folk theorem were explored. Varying types of games, for which folk theorems have been considered, were discussed. These included: games with complete information, games with imperfect private monitoring and games with communication, among others. Within these papers, it was also discovered that other equilibria (besides the more common subgame perfect equilibria) have been considered. Some examples are: sequential equilibria, belief-free equilibria and type-k equilibria. Throughout, there appeared to be two main assumptions required for a folk theorem to exist. These were full dimensionality and the number of signals in proportion to the number of actions. Since these assumptions are restricting, a few papers have attempted to find situations in which they can be weakened whilst still obtaining a folk theorem. In contrast, an area of research has developed which shows the instability of equilibria and the violation of folk theorems. The chapter is concluded by a discussion on recent applications of the folk theorem to computing, multi-agent systems and quantum transportation. Finally, to the best of the author's knowledge, no folk theorem experiments, of the same size as executed in this study, have been computed before.

5.1.2 Aim 2: Software Development

Chapter 3 discusses the set-up and execution of the IPD experiment referred to in Aim 2. This included justifications for the use of certain software and methods, as well as a detailed view of the data collection algorithm. Within this, reasons were given on why the particular attributes of the tournaments were collected. Following this, an exploration into the file types available for data collection was provided. After considering the advantages and drawbacks, it was decided that a binary file, in particular a relational database file, would be the most appropriate. The chapter then continues with an explanation of the software implementation of the aforementioned algorithm. It provides details on how good software development principles were followed and on the programs utilised. For a more detailed overview, which is not as involved as the chapter, see Chapter A in the Appendix. The main experiment was executed remotely over a few weeks and how this was achieved is explained here. Following this, how to compute the Nash equilibria and the notion of degeneracy were addressed. Out of the three algorithms: Support Enumeration, Vertex Enumeration and Lemke-Howson; it was decided that Support Enumeration would be the method used for calculating Nash equilibria, due to its robustness. The execution timings of each algorithm were also explored but did not yield any significant differences. To conclude, the difficulties which could be faced due to degeneracy were briefly acknowledged and possible solutions discussed.

5.1.3 Aim 3: Analysing the Thresholds

The final aim, regarding the analysis of p-thresholds, was addressed in Chapter 4. Firstly, an initial analysis identified the main 'shapes' of the graphs obtained and detailed the summary statistics. This included looking at characteristics of the strategies, as well as the number of equilibria obtained from each tournament. The majority of strategies chosen were deterministic - this is because of the ratio between deterministic and stochastic strategies implemented in the Axelrod library. When analysing the thresholds, the minimum, mean, median and maximum values of the p-threshold were taken, due to varying sources of noise obscuring a clear threshold. Here, the focus was on the effects of the number of players and standard PD noise. However, this was concluded as a non-trivial task due to the difficulty of identifying degenerate games and the many sources of noise. As a result of this, significant conclusions were unable to be drawn with the obtained plots not revealing many trends. Thus, suggestions are provided on how further work regarding this study could reveal more significant information. These are restated in Section 5.3. In conclusion however, the experiment was successful in providing clear visualisations for the folk theorem and it gives insight into a wide range of directions for future research.

5.2 Limitations

In this section a few shortcomings of the study are highlighted along with justifications.

Firstly, this project included a very large empirical study on the folk theorem. This required the development and implementation of an algorithm. Inevitably, to ensure the functions were accurate, clear and gave the required information, a significant amount of time was taken. Therefore, one of the main drawbacks was the time constraint of completing the project. Indeed, after the successful implementation of the data collection algorithm into Python, there was a certain time frame before analysis could start. This was to ensure enough data had been collected. The analysis then took a portion of the time, especially due to the data's non-triviality mentioned previously.

As a result of this, the amount of data collected was insufficient for a large-scale statistical analysis on extremely random data. Also, the parameters used in the algorithm were restricted by this. Only 500 repetitions of each tournament were performed but this was only able to 'stabilise' some of the payoff matrices. The rest resulted in threshold graphs which were less clear. Thus, higher repetitions were needed but unrealistic for the given time-frame. Moreover, the eighteen strategies classified as 'long-run' had to be omitted but, in an ideal situation, they would have been included to obtain more information.

Furthermore, the analysis carried out only briefly considered some of the potential environmental effects on the p-threshold. The 'randomness' of the data meant that trends were hard to infer and key conclusions could not be drawn. Ideally, further environmental changes, in addition to number of players and level of standard PD noise, would have been discussed. Another factor which could have improved the situation, but had to be omitted due to time, was the packaging of the code. Creating a fully working Python package, with all the required functions implemented, may have improved the remote computing stage.

Finally, degeneracy was a major limitation due to its uncertainty. Nashpy only identifies potential degeneracy if the equilibria are 'strange'. That is, if the algorithm detects division by zero or non-probability distributions, for example. Indeed, some games may still be degenerate even if equilibria are identified. This implies that discussions regarding degeneracy in Chapter 4 assume that Nashpy can detect all degeneracy, which is unrealistic.

5.3 Recommendations

Throughout this study, many interesting questions have been raised as potential directions for future work. Hence, in this section, recommendations on how this research could be extended, to provide more insightful results, are given.

Firstly, during analysis (Chapter 4), a significant proportion of the graphs showed no p-threshold, appearing constant at zero or one. This indicates that the threshold must lie in the intervals (0, 001) or (0.999, 1), implying the precision level was not fine enough. Therefore, it is suggested that these tournaments are rerun, within these intervals, using a greater precision to obtain more information. Also, an analysis of the strategies involved and values of standard PD noise might provide a clearer insight into reasons for this. Moreover, re-running the whole experiment, with a finer precision could yield more accurate thresholds.

In order to explore the effects of stochastic players on the threshold, it is recommended that tournament sets which contain stochastic players are rerun without them. If the threshold is different, this could indicate that stochasticity does indeed have an effect.

It is highly recommended that a separate run of the experiment is performed, using vertex enumeration to calculate the equilibria. This could then be used in comparison with the support enumeration data to check the reliability of the results obtained. However, care is needed when implementing vertex enumeration as it is less robust, (see Chapter 3). Furthermore, increasing the number of tournament repetitions is suggested to observe whether clearer p-thresholds are obtained.

Regarding multivariate data analysis of this large empirical study, it may be insightful to perform a regression or clustering algorithm. This is motivated by the possibility of being able to approximately predict the p-threshold of a tournament by its characteristics: strategies, level of standard PD noise etc. Finally, the experiment could potentially be extended to consider the different 'types' of folk theorem discussed in Chapter 2.

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Appendix A

Computer Report

This short report summarises the computer work involved in the study. A key aim of this project was the development and execution of a large experiment regarding the Iterated Prisoner's Dilemma and the Folk Theorem. As a result of this, much focus was given to ensuring good software development principles [49, 89, 106] were followed and hence only a brief overview will be provided here. The interested reader is encouraged to read Chapter 3 of the main report, where full details are given.

A.1 The Overall Program

Firstly, the pseudo-code of the program written is provided to give a general idea of its purpose. This can be found in Algorithm 5.

The main functions created were implemented in Python. This was due to its versatility and due to the two game theoretic libraries, Axelrod [96] and Nashpy [97], where key functions are already implemented. Aside from these, other dependencies of the program include: NumPy [99], Random, Warnings, os and SQLAlchemy [12], plus additional libraries for the analysis. This program was implemented through the creation of eight functions. This was to ensure the code was easier to read, debug and test for any potential user. Furthermore, variables and functions were given descriptive names in order to produce self-documenting code.

Algorithm 5: Folk Theorem Exploration

```
input  : maximum number of opponents, number of strategy sets for each number of
         opponents, noise levels, game ending probabilities, number of repetitions of the
         tournament, the path to the database file, and whether or not support enumeration
         should be used to calculate the Nash equilibria.
output: a database containing the results as detailed above.
while True do
    for each number of opponents do
        for each repetition with the same number of strategies do
            Randomly select a set of opponents and add in the Defector
            for each noise level do
                for each game ending probability do
                    Run the IPD tournament
                    Obtain the Nash equilibria and the corresponding probabilities of
                      defection using the algorithm indicated
                    for each player in the current set do
                        Write the required information to a record in the database file.
                    end
                end
            end
        end
        Repeat
    end
end
```

Observe, Algorithm 5 states the use of a database to retain the information collected. This file type was chosen because it supports out of memory operations and is known to be robust. In particular, the relational database SQLite was utilised due to the existence of Python libraries for accessing the file, its structure and its consistency [26, 80].

A.2 Code Development and Remote Computing

Version control software Git (https://git-scm.com/) and associated platform GitHub (https://github.com/) allowed the author to keep track of how the program was evolving. It also ensured a back-up copy was always available, should the system fail and meant that past versions of functions could be easily accessed. Moreover, GitHub acted as an intermediate step between between the author's laptop and remote server, used for the main data collection exercise.

Due to the volume of data which was to be collected, it was decided that the program would be executed remotely. This allowed the code to run uninterrupted for several weeks. However, parallel processing was not required, as the computational time was deemed 'quick enough'. Figure A.1 provides a visualisation of how the author's laptop was connected to the School of Mathematics' headless server, Siren, via a secure shell (SSH) tunnel [93] and terminal multiplexer

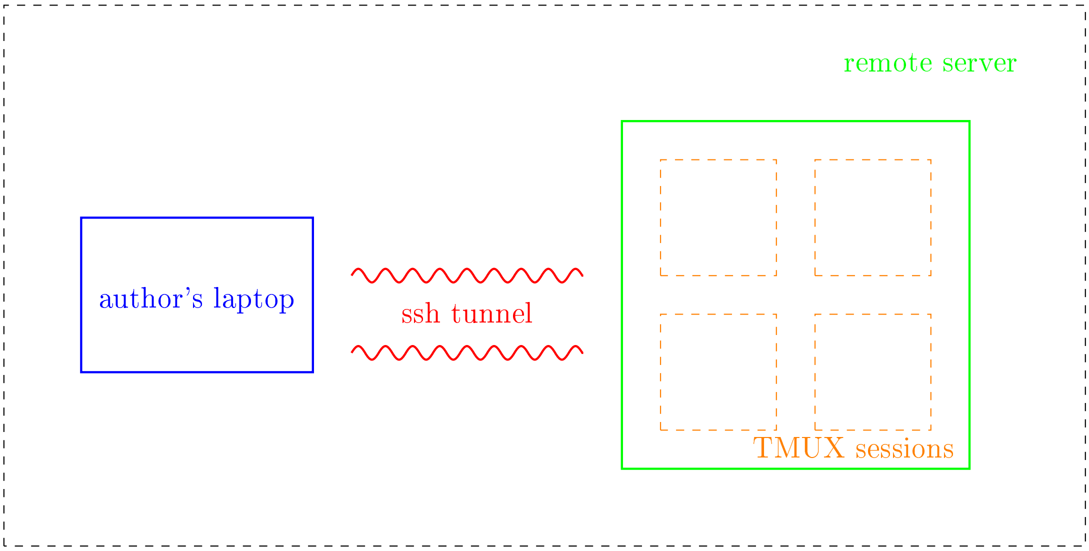

Figure A.1: Representation of how the experiments were run remotely. Note, 'tmux sessions' correspond to emulators of terminals.

TMUX [67]. The latter ensured the code kept running whilst the author was disconnected from Siren. Finally, in order to access the database for analysis, the file was compressed and securely transferred from Siren.

A.3 Further Details

In conclusion, there were several components to the computing elements of this study. The reader is referred to the GitHub repository of this project (https://github.com/shapperzsm/final-project/), where all the relevant code can be accessed. Also, the data collected during the study is archived at (https://doi.org/10.5281/zenodo.3784594) for future reference.