Coding Theory
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Coding Theory:- The process of communication involves transmitting information through noisy channels. Since information carrying signals (messages) must be encoded by a sender in accordance. During the transmission, due to the noise produced in channel, any mistake that occurs during the receiver side. Coding theory provides the tools that necessary to encode and detect and correct the errors occurs in distributed messages.

Encoder:- The encoder is a device which transforms the incoming messages in form a way that the chances of error in the transformed message is reduced.

Decoder:- Decoder is device that transforms the received message into original form.

Group Code:- Let $F_2 = \{1, 0\}$, $+ = \oplus$

Then $(F_2, +_2)$ is an abelian group

$F_2^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in F_2, 1 \le i \le n\}$

If $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in F_2^n$

$x + y = (x_1 +_2 y_1, x_2 +_2 y_2, \ldots, x_n +_2 y_n)$

Check that $(F_2^n, +)$ is an abelian group.

Eg $F_2^3 = \{000, 111, 100, 010, 001, 110, 011, 101\}$

$000 + 111 = 111$

$111 + 111 = 000$

$100 + 010 = 110$

$010 + 001 = 011$

$001 + 101 = 100$

$110 + 011 = 101$

$011 + 101 = 110$

$101 + 110 = 011$

i.e. $(F_2^3, +)$ is an abelian group.

Any code which is group is said is called group code.

Eg $F_2^3 \to F_2^5$

$000 \to 00000$

$001 \to 00111$

$010 \to 01010$

$011 \to 01101$

$100 \to 10011$

$101 \to 10100$

$110 \to 11001$

$111 \to 11110$

$C(110) = C(100) + C(010)$

$C(011) = C(001) + C(010)$

$C(101) = C(100) + C(001)$

$C(111) = C(100) + C(010) + C(001)$

$\Rightarrow$ Code is group code

$\Rightarrow$ Decode $C(11111) = 111$

C is group code iff $(C, \oplus)$ is a group.

Hamming Code:- The codes obtained by introducing additional digits (parity check digits) to the original message are called Hamming code.

If original message is k digits of length m, Hamming encoded message is of length n (n>m). Then n digits are called information bits & remaining $n-m$ bits are parity check bits that helped to correct the errors.

Weight of a codeword $w(C)$ be a code of length n, $C \in F^n$

If $x = (x_1, x_2, \ldots, x_n) \in C$. Then weight of x is denoted by $wt(x)$ & defined as

$wt(x)$ = Number of 1's present in x

Eg $x = 1010101 \in F^{10}$

$wt(x) = 3$

Hamming distance b/w codewords:- Let $C \subseteq F^n$

If $x, y \in C$. Then Hamming dist b/w x & y is the # places at which they differ i.e. $d(x,y) = |\{i \mid x_i \neq y_i, 1 \le i \le n\}|$

or

$d(x,y) = wt(x \oplus y)$

Eg $x = 1010101 \in F^{10}$

$y = 0110101 \in F^{10}$

$d(x,y) = |\{1, 5, 6, 7\}| = 4$

$x \oplus y = 1000100110$

$d(x,y) = wt(x \oplus y) = 4$

Hamming distance of a code:- Let $C \subseteq F^n$. Then min d of C is denoted by $d(C)$ & defined as

$d(C) = \min\{wt(x \oplus y) \mid x \neq y, x, y \in C\}$

$= \min\{d(x,y) \mid x \neq y, x, y \in C\}$

If C is a group code, $(\oplus$ = Binary addition)

If $x, y \in C \Rightarrow x \oplus y \in C$

$d(C) = \min\{wt(x) \mid x \in C, x \neq 0\}$

Eg $C = \{11111, 00000, 10101, 01010\}$

$d(C) = \min\{5, 3, 2\} = 2$

Thm If d is the minimum distance of a code, then C can detect almost $(d-1)$ error & correct $\left\lfloor \frac{d-1}{2} \right\rfloor$ error in any transmission.

Eg If $d(C) = 5 \Rightarrow C$ can detect almost $(5-1) = 4$ errors & can correct $\left\lfloor \frac{5-1}{2} \right\rfloor = 2$ errors

Generator matrix:- Let $e: F^m \to F^n$ be an encoding function. Then e can be given by a matrix G, called generator matrix of order $m \times n$.

$G = [I_m | A]_{m \times n}$ where I_m is identity matrix & $A_{m \times (n-m)}$

If $w \in F^m$ is a message. Then $e(w) = wG \in F^n$ is a codeword

$C = \{e(F^m)\} \subseteq F^n$. C is group code iff $(C, \oplus)$ is a group.

Eg Let $e: F^2 \to F^5$ be an encoding function given by the generator matrix $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Show that $e(F^2) = C$ is a group code. Find also find min d of C. How many error this code can detect & correct.

Soln $F^2 = \{00, 11, 10, 01\}$

$e(00) = [00]G = [00] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = 00000$

$e(10) = [10]G = [10]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = 10110$

$e(01) = [01]G = [01]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = 01011$

$e(11) = [11]G = [11]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = 11101$

$C = \{00000, 10110, 01011, 11101\}$ is group

If $(C, \oplus)$ is a group

$d(C) = \min\{3, 3, 3, 4\} = 3$

$\Rightarrow$ C can detect $(3-1) = 2$ error & correct $\left\lfloor \frac{3-1}{2} \right\rfloor = 1$ error (almost)

Parity Check Matrix:- Let $e: F^m \to F^n$ be an encoding function. Then e is given by generator matrix $G = [I_m | A]_{m \times n}$. Then the parity check matrix is given by

$H = \begin{bmatrix} A^T & I_{n-m} \end{bmatrix}_{(n-m) \times n}$

Parity check matrix is useful to decode a message.

How to decode a message:-

If x is received vector. Then find

(i) $HxT = 0 \Rightarrow$ There is no error in transmission & received vector is correct codeword (which has been sent) and find an bits are original message.

(ii) HxT = column of H matrix H. Then there is error at that place of the received vector. By changing this bit place, we get the correct codeword and find its bits contained original message.

(iii) HxT = multiple column of H $\Rightarrow$ there are two errors occur. We can not correct the error.

Qu Let $e: F^2 \to F^5$ be an encoding function given by a G.M. $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Find by using G decode the received vector

(i) 11010 (ii) 11101 (iii) 01011 (iv) 10101

Soln Here $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

Parity check matrix $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5 \times 3}$

$H^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{3 \times 5}$

(i) $x = 11010$

$HxT = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 2H + 0H^4 \text{ or } = \text{Error at } 2H \text{ & at 4th coordinate}$

$= 01011 = \text{any one}$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = D^{H} + 2H^{H} \text{ or } = \text{error at } 2H \text{ & } 4H \text{ place}$

$= 10110 = \text{any one}$

(ii) $x = 11101$

$HxT = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ = that is x is error in transmission

& original msg is 11

(iii) $x = 01011$

$HxT = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = No error in transmission & correct one is 2

(iv) $x = 11011$

$HxT = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ = there is an error at 3H place of $x = 11011$. So correct codeword is 01011 & original msg = 01

Qu Let $e: F^2 \to F^5$ be an encoding function given by $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Find by using G decode

(i) 11010 (ii) 00111 (iii) 11101

Soln $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(i) $x = 01011$

$HxT = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ There is an error at 3H place of $x = 01011$. So correct codeword is 01011 & original msg = 01

Qu Let $e: F^2 \to F^5$ be an encoding function given by $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Find by using G decode

(i) 11010 (ii) 00111 (iii) 11101

Soln $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(i) $x = 01011$

$HxT = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ There is an error at 3H place of $x = 01011$. So correct codeword is 01011 & original msg = 01

Qu Let $e: F^2 \to F^5$ be an encoding function given by $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Find by using G decode

(i) 11010 (ii) 00111 (iii) 11101

Soln $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(i) $x = 01011$

$HxT = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ There is an error at 3H place of $x = 01011$. So correct codeword is 01011 & original msg = 01