UNIT-2

PERMUTATION & COMBINATION

Permutation $\Rightarrow {}^nP_r = \frac{\lfloor n}{\lfloor n-r}$

(Arrangement)

Keywords:

I) Arrangement
II) Ordering of objects
III) Questions with numbers
IV) Questions with alphabets
V) Linear Arrangement

Combination $\Rightarrow {}^nC_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} = \frac{{}^nP_r}{\lfloor r}$

(Selection)

Keywords:

I) Selection
II) Picking up of the objects
III) Team formation
IV) Committee formation

↳ To arrange n different letters/~~digits~~ items, then

Total no. of ways $= \lfloor n$

↳ If repetition is allowed, then

Total no. of ways $= n^n$

↳ If repetition is not allowed, then

Total no. of ways $= \frac{\lfloor n}{\lfloor n_1 \lfloor n_2 \ldots \lfloor n_k}$

↳ Circular arrangement of n ~~digit~~ distinct objects is $\lfloor n-1$.

↳ If clockwise & anticlockwise to be same then it is $\lfloor n-1 /2$.

→ In case of 'AND' we have to multiply & while in case of 'OR' we have ~~to~~ to add the things.

→ **Permutations:** The no. of permutations of n diff. objects, taken r at a time, $(0 < r \leq n)$, where the objects do not repeats is ${}^nP_r = \frac{\lfloor n}{\lfloor n-r}$.

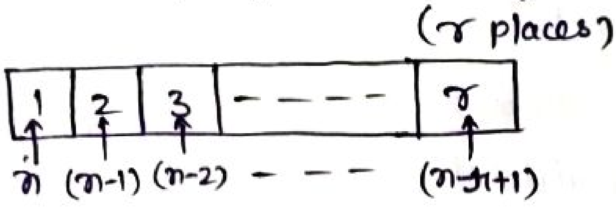
eg: Suppose we have 3 leters (A, B, C) then the possible arrangements of these letters are:

Soln:

ABC, ACB, BAC, BCA, CAB, CBA

No. of permutations $= {}^3P_3 = \frac{\underline{|3}}{\underline{|3-3}} = \frac{\underline{|3}}{\underline{|0}} = \frac{6}{1} = 6$

This is same as filling r places by $n, n-1, ---, n-r+1$ objects.



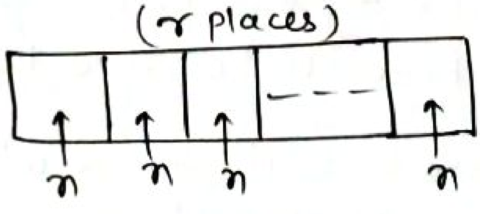
Total arrangements $= n \times (n-1) \times (n-2) \times --- \times (n-r+1)$

$$= {}^nP_r = \frac{\underline{|n}}{\underline{|n-r}}$$

$$= \frac{n(n-1)---(n-r+1)\cancel{\underline{|n-r}}}{\cancel{\underline{|n-r}}}$$

$$= n(n-1)---(n-r+1).$$

→ The no. of permutations of n different objects taken r at a time, $0 < r \leq n$ and repetation is allowed $= n^r$.



Total no. of arrangements

$$= \underbrace{n \times n \times n \times --- \times n}_{r \text{ times}}$$

$$= n^r.$$

→ The no. of permutations of n objects where p_1 objects are of one kinds p_2 are of second kind, ---, p_k objects of k kind

$$= \frac{n!}{p_1!\, p_2! --- p_k!}$$

eg: MATHEMATICS

$n = 11$, 2M, 2A, 2T other letters are different

No. of words that can be form $= \dfrac{\lfloor 11}{\lfloor 2\ \lfloor 2\ \lfloor 2} = \dfrac{11\times 10\times 9\times \cancel{8}\times 7\times \lfloor 6}{\cancel{2\times 2\times 2}}$

$= 110\times 63\times 720 = 49,89,600$

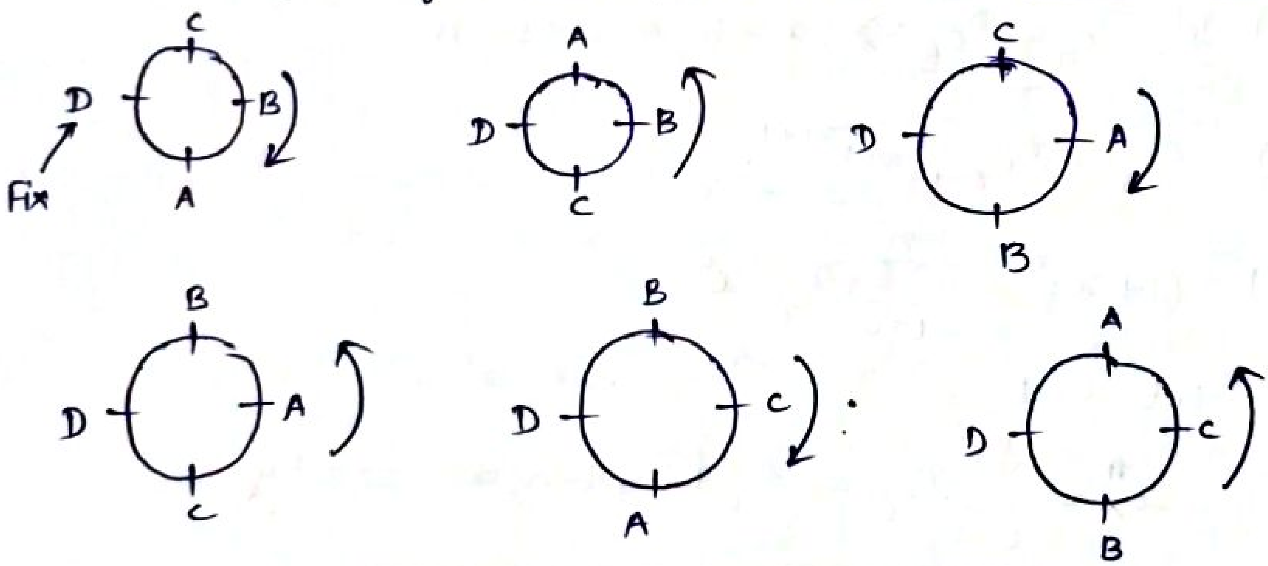
Ans:

→ **Circular Permutation :**

The no. of circular permutation of n distinct objects is $\lfloor n-1$

eg: A, B, C, D are 4 persons, these persons are sit around a circular table.

No. of circular Permutations $= \lfloor 4-1 = \lfloor 3 = 6$



→ **Combinations :**

Combination is the possible arrangements of things, where the order is not important.

⊙ An unordered selection of r objects from n distint objects is called r-combinations of n objection ($r \le n$).

The r-combinations of n objects $= {}^{n}C_{r} = \dfrac{\lfloor n}{\lfloor r\ \lfloor n-r}$

eg: How many ways are there to select 11 players from a team of 23 players for a final?

Soln: $n = 23,\ r = 11$

No. of ways in which 11 players can be selected $= {}^{23}C_{11}$

$$= \frac{\lfloor 23}{\lfloor 11 \lfloor 12}$$

NOTE:

I) ${}^{n}C_{0} = {}^{n}C_{n} = 1$

II) ${}^{n}P_{r} = {}^{n}C_{r} \times r!$; $0 < r \leq n$

III) ${}^{n}C_{r} = {}^{n}C_{n-r}$

IV) If ${}^{n}C_{a} = {}^{n}C_{b} \Rightarrow a = b$ & $a + b = n$

V) ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

VI) $(1+x)^{n} = \sum_{i=0}^{n} {}^{n}C_{i} \cdot x^{i}$

Put $x = 1$

$$2^{n} = \sum_{i=0}^{n} {}^{n}C_{i} = {}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \cdots + {}^{n}C_{n}$$

$$\boxed{{}^{n}C_{1} + {}^{n}C_{2} + \cdots + {}^{n}C_{n} = 2^{n} - 1}$$

→ **Product Rule:** If an activity can be performed in r successive steps and step 1 can be done in n_1 ways

and step 2 ——— n_2 ways

⋮

step r ——— n_r ways.

Total no. of ways to perform the activity $= n_1 \times n_2 \times \cdots \times n_r$.

→ **Addition rule:**

If an activity can be done either in one of n_1 ways or in one of n_2 ways where none of the set of n_1 ways is same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

eg: A student can choose a computer project from one of three lists. The three lists conteins 23, 15 and 19 possible project, respectively. No project is in more than one lists. How many possible projects are there to choose from?

Soln: The no. of ways to choose the projects $= 23 + 15 + 19 = 57$ wa

eg: Five boys and four girls are to be seats for a photograp in a row. It is desired that no two girls sit together. Find the no. of ways in which they can be seated.

Soln:

$$\underset{G_1}{\uparrow} \quad B_1 \quad \underset{G_2}{\uparrow} \quad B_2 \quad \underset{G_3}{\uparrow} \quad B_3 \quad \underset{G_4}{\uparrow} \quad B_4 \quad \underset{G_5}{\uparrow} \quad B_5 \quad \underset{G_6}{\uparrow}$$

Five boys can be seated in 5 places $= {}^5P_5 = \lfloor 5 = 120$

The girls can be seated in 6 places $= {}^6P_4$ ways

$$= \frac{\lfloor 6}{\lfloor 2} = \frac{\cancel{6}^{3} \times \lfloor 5}{\cancel{2}}$$

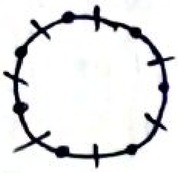
$$= 360$$

By multiple rule the no. of ways 5 boys & 4 girls seated $= \lfloor 5 \cdot {}^6P_4 = 120 \times 360 = 43200$ Ans

eg: In how many ways can a party of seven gentle man and 6 ladies be seated at a round table so that no two ladies are seated together?

Soln; Gentleman can be sit in $\underline{|7-1}$ ways

$= \underline{|6}$ ways



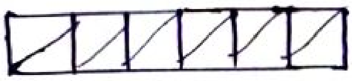
For each ways of sitting of gentlemen the ladies can be seated in $= {}^{7}P_{6}$ ways.

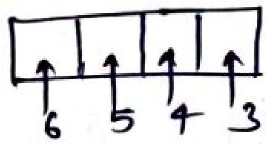
Total no. of ways $= \underline{|6} \times {}^{7}P_{6} = \underline{|6} \times \dfrac{\underline{|7}}{\underline{|7-6}} = \underline{|6} \times \dfrac{\underline{|7}}{\underline{|1}}$

$= 720 \times \underline{|7} = 36,28,800$ Ans

eg: a) Assuming that repetions are not allowed, how many four-digit numbers can be formed from the six digits 1, 2, 3, 5, 7, 8.

Soln;



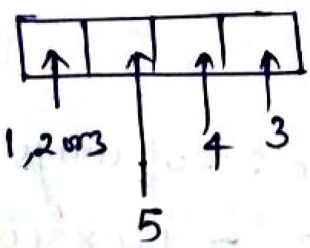


Hence, No. of 4 digit numbers $= {}^{6}P_{4} = \dfrac{\underline{|6}}{\underline{|2}}$

$= 6 \times 5 \times 4 \times 3$

$= 360$ Ans

b) How many of these numbers are less than 4000?



No. of 4 digit numbers less than 4000 $= 3 \times (5 \times 4 \times 3)$

$= 3 \times (5 \times 4 \times 3)$

$= 180$ Ans

c) How many numbers in Part (a) are even?

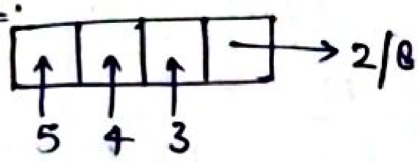
Soln: Required no. of even numbers

$= 2 \times {}^5P_3$

$= 2 \times \frac{5!}{2!}$

$= \underline{120}$ Ans:

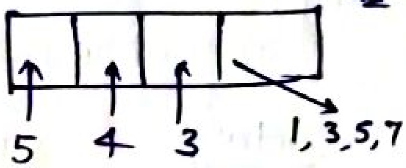
Alter:



$\underline{(5 \times 4 \times 3) \times 2 = 120}$ Ans:

d) How many numbers in Part (a) are odd?

Soln: Required no. of odd numbers $= 4 \times {}^5P_3 = 4 \times \frac{5!}{2!} = \underline{240}$ Ans:



Alter:

$\underline{(5 \times 4 \times 3) \times 4 = 240}$ Ans:

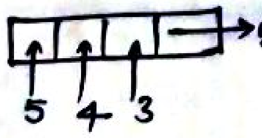
(OR)

Total − Even $= 360 - 120 = \underline{240}$ Ans:

e) How many numbers in Part (a) are multiple of 5?

Soln:

Required numbers which are multiple of 5.



$= 1 \times {}^5P_3$

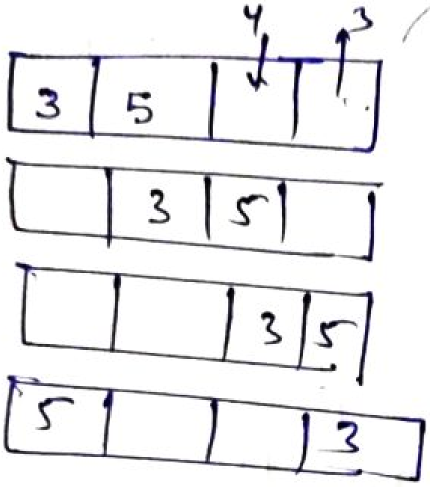
$= 1 \times \frac{5!}{2!} = \underline{60}$ Ans:

(or) $\underline{5 \times 4 \times 3 = 60}$ Ans:

f) How many of the numbers in Part (a) contain both the digits 3 & 5?

Soln:

1, 2, 3, 4, 5, 7, 8



$4 \times 3 \times \lfloor 2$ → 5 और 3 को intersection [illegible]

$\Rightarrow (4 \times 3 \times \lfloor 2) \times 4$

$= 12 \times 2 \times 4$

$= 96$ Ans

eg:

a) In how many ways can 6 boys and 4 girls sit in a row?

Soln: 6 boys + 4 girls = 10 persons

10 can sit in a row $= \lfloor 10$ ways. i.e. ${}^{10}P_{10}$ ways. Ans:

b) In how many ways can they sit in a row if the boys are sit together and girls are sit together?

Soln: $[(B_1, B_2, B_3, B_4, B_5, B_6)\ (G_1, G_2, G_3, G_4)]$

$[(G_1 - - - G_4)\ (B_1 - - - - B_6)]$

The required no. of ways in which they can sit

$= 2 \times \lfloor 6 \times \lfloor 4 = 2 \times 720 \times 24 = 34,560$ Ans:

c) How many ways can they sit in a row if girls are sit together?

Soln:

$\underbrace{B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5 \quad B_6 . (G_1 - - - G_4)}$

Let → 7 objects $= \lfloor 7 \times 4!$ ways Ans:

Example 6.1

(a) Assuming that repetitions are not permitted, how many four-digit numbers can be formed form the six digits 1, 2, 3, 5, 7, 8?

(b) How many of these numbers are less than 4000?

(c) How many of the numbers in part (a) are even?

(d) How many of the numbers in part (a) are odd?

(e) How many of the numbers is part (a) are multiples of 5?

(f) How many of the numbers in part (a) contain both the digits 3 and 5?

(a) The 4-digit number can be considered to be formed by filling up 4 blank spaces with the available 6 digits. Hence, the number of 4-digits numbers

$$= \text{the number of 4-permutations of 6 numbers}$$
$$= P(6, 4) = 6 \times 5 \times 4 \times 3 = 360$$

(b) If a 4-digit number is to be less than 4000, the first digit must be 1, 2, or 3. Hence the first space can be filled up in 3 ways. Corresponding to any one of these 3 ways, the remaining 3 spaces can be filled up with the remaining 5 digits in $P(5, 3)$ ways. Hence, the required number = 3 × $P(5, 3)$

$$= 3 \times 5 \times 4 \times 3 = 180.$$

(c) If the 4-digit number is to be even, the last digit must be 2 or 8. Hence, the last space can be filled up in 2 ways. Corresponding to any one of these 2 ways, the remaining 3 spaces can be filled up with the remaining 5 digits in $P(5, 3)$ ways. Hence the required number of even numbers

$$= 2 \times P(5, 3) = 120.$$

(d) Similarly the required number of odd numbers $= 4 \times P(5, 3) = 240$.

(e) If the 4-digit number is to be a multiple of 5, the last digit must be 5. Hence, the last space can be filled up in only one way. The remaining 3 spaces can be filled up in $P(5, 3)$ ways.

Hence, the required number $= 1 \times P(5, 3) = 60$.

(f) The digits 3 and 5 can occupy any 2 of the 4 places in $P(4, 2) = 12$ ways.

The remaining 2 places can be filled up with the remaining 4 digits in $P(4, 2) = 12$ ways. Hence, the required number $= 12 \times 12 = 144$.

Example 6.2

(a) In how many ways can 6 boys and 4 girls sit in a row?

(b) In how many ways can they sit in a row if the boys are to sit together and the girls are to sit together?

(c) In how many ways can they sit in a row if the girls are to sit together?

(d) In how many ways can they sit in a row if *just* the girls are to sit together?

(a) 6 boys and 4 girls (totally 10 persons) can sit in a row (viz., can be arranged in 10 places) in $P(10, 10) = 10!$ ways.

(b) Let us assume that the boys are combined as one unit and the girls are combined as another unit. These 2 units can be arranged in $2! = 2$ ways.

Corresponding to any one of these 2 ways, the boys can be arranged in a row in 6! ways and the girls in 4! ways.

$\therefore$ Required number of ways $= 2 \times 6! \times 4! = 34{,}560$.

(c) The girls are considered as one unit (object) and there are 7 objects consisting of one object of 4 girls and 6 objects of 6 boys.

These 7 objects can be arranged in a row in 7! ways.

Corresponding to any one of these ways, the 4 girls (considered as one object) can be arranged among themselves in 4! ways. Hence, the required number of ways $= 7!\ 4! = 1{,}20{,}960$.

(d) No. of ways in which girls only sit together

= (No. of ways in which girls sit together)

– (No of ways in which boys sit together and girls sit together)

$= 1{,}20{,}960 - 34{,}560 = 86{,}400$.

Example 6.3 How many different paths in the xy-plane are there from (1, 3) to (5, 6), if a path proceeds one step at a time by going either one step to the right (R) or one step upward (U)?

To reach the point (5, 6) from (1, 3), one has to traverse $5 - 1 = 4$ steps to the right and $6 - 3 = 3$ steps to the up.

Hence, the total number of 7 steps consists of 4 R's and 3 U's.

To traverse the paths, one can take R's and U's in any order.

Hence, the required number of different paths is equal to the number of permutations of 7 steps, of which 4 are of the same type (namely R) and 3 are of the same type (namely U).

Example 6.6 How many permutations of the letters $A\ B\ C\ D\ E\ F\ G$ contain (a) the string BCD, (b) the string $CFGA$, (c) the strings BA and GF, (d) the strings ABC and DE, (e) the strings ABC and CDE, (f) the strings CBA and BED?

(a) Treating BCD as one object, we have the following 5 objets:

$$A, (BCD), E, F, G.$$

These 5 objects can be permuted in

$$P(5, 5) = 5! = 120 \text{ ways}$$

Note B, C, D should not be permuted in the string BCD.

(b) Treating $CFGA$ as one object, we have the following 4 objects: $B, D, E, (CFGA)$.

The no. of ways of permuting these 4 objects = $4! = 24$.

(c) The objects (BA), C, D, E and (GF) can be permuted in $5! = 120$ ways.

(d) The objects (ABC), (DE), F, G can be permuted in $4! = 24$ ways.

(e) Even though (ABC) and (CDE) are two strings, they contain the common letter C. If we include the strings $(ABCDE)$ in the permutations, it includes both the strings (ABC) and (CDE). Moreover we cannot use the letter C twice.

Hence, we have to permute the 3 objects $(ABCDE)$, F and G. This can be done in $3! = 6$ ways.

(f) To include the 2 strings (CBA) and (BED) in the permutations, we require the letter B twice, which is not allowed. Hence, the required no. of permutations = 0.

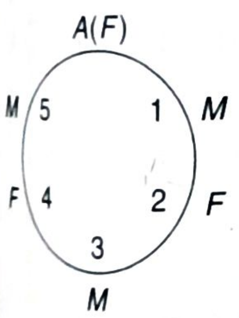
Example 6.7 If 6 people A, B, C, D, E, F are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

If A, B, C are females and the others are males, in how many arrangements do the sexes alternate?

The no. of different circular arrangements of n objects is $(n - 1)!$

$\therefore$ The required no. of circular arrangements = $5! = 120$.

Since rotation does not alter the circular arrangement, we can assume that A occupies the top position as shown in the figure.



Of the remaining places, positions 1, 3, 5 must be occupied by the 3 males. This can be achieved in $P(3, 3) = 3! = 6$ ways.

The remaining two places 2 and 4 should be occupied by the remaining two females. This can be achieved in $P(2, 2) = 2$ ways.

$\therefore$ Total no. of required circular arrangements = $6 \times 2 = 12$.

Example 6.8 From a club consisting of 6 men and 7 women, in how many ways can we select a committee of

(a) 3 men and 4 women?
(b) 4 persons which has at least one woman?
(c) 4 persons that has at most one man?
(d) 4 persons that has persons of both sexes?
(e) 4 persons so that two specific members are not included?

(a) 3 men can be selected from 6 men in $C(6, 3)$ ways.
4 women can be selected from 7 women in $C(7, 4)$ ways.
$\therefore$ The committee of 3 men and 4 women can be selected in $C(6, 3) \times C(7, 4)$ ways. (by the product rule)

i.e., in $\dfrac{6!}{3!3!} \times \dfrac{7!}{4!3!} = 700$ ways.

(b) For the committee to have at least one woman, we have to select 3 men and 1 woman or 2 men and 2 women or 1 man and 3 women or no man and 4 women.
This selection can be done in

$$C(6, 3) \cdot C(7, 1) + C(6, 2) \cdot C(7, 2) + C(6, 1) \cdot C(7, 3) + C(6, 0) \cdot C(7, 4)$$
$$= 20 \times 7 + 15 \times 21 + 6 \times 35 + 1 \times 35$$
$$= 140 + 315 + 210 + 35 = 700 \text{ ways.}$$

(c) For the committee to have at most one man, we have to select no man and 4 women or 1 man and 3 women.
This selection can be done in

$$C(6, 0) \cdot C(7, 4) + C(6, 1) \cdot C(7, 3) = 1 \times 35 + 6 \times 35 = 245 \text{ ways.}$$

(d) For the committee to have persons of both sexes, the selection must include 1 man and 3 women or 2 men and 2 women or 3 men and 1 woman.
This selection can be done in

$$C(6, 1) \times C(7, 3) + C(6, 2) \times C(7, 2) + C(6, 3) \times C(7, 1)$$
$$= 6 \times 35 + 15 \times 21 + 20 \times 7$$
$$= 210 + 315 + 140 = 665 \text{ ways.}$$

(e) First let us find the number of selections that contain the two specific members. After removing these two members, 2 members can be selected from the remaining 11 members in $C(11, 2)$ ways. In each of these selections, if we include those 2 specific members removed, we get $C(11, 2)$ selections containing the 2 members.
The no. of selections not including these 2 members

$$= C(13, 4) - C(11, 2)$$
$$= 715 - 55 = 660.$$

Example 6.9 In how many ways can 20 students out of a class of 30 b selected for an extra-curricular activity, if
(a) Rama refuses to be selected?
(b) Raja insists on being selected?

(c) Gopal and Govind insist on being selected?
(d) either Gopal or Govind or both get selected?
(e) just one of Gopal and Govind gets selected?
(f) Rama and Raja refuse to be selected together?

(a) We first exclude Rama and then select 20 students from the remaining 29 students.
$\therefore$ Number of ways = $C(29, 20)$ = 1, 00, 15, 005.

(b) We separate Raja from the class, select 19 students from 29 and then include Raja in the selections.
$\therefore$ Number of ways = $C(29, 19)$ = 2, 00, 30, 010.

(c) We separate Gopal and Govind, select 18 students from 28 and then include both of them in the selections.
$\therefore$ Number of ways = $C(28, 18)$ = 1, 31, 23, 110

(d) Number of selections which include Gopal = $C(29, 19)$
Number of selections which include Govind = $C(29, 19)$
Number of selections which include both = $C(28, 18)$
$\therefore$ By the principle of inclusion – exclusion, the required number of selections
= $C(29, 19) + C(29, 19) - C(28, 18)$
= 2, 69, 36, 910.

(e) Number of selections including either Gopal or Govind
= (Number of selections including either Gopal or Govind or both)
– (Number of selections including both)
= $[C(29, 19) + C(29, 19) - C(28, 18)] - C(28, 18)$
= 2, 69, 36, 910 – 1, 31, 23, 110 = 1, 38, 13, 800.

(f) Number of ways of selecting 20 excluding Rama and Raja together
= (Total number of selections) – (Number of selections including both Rama and Raja)
= $C(30, 20) - C(28, 18)$ [as in part (c)]
= 3, 00, 45, 015 – 1, 31, 23, 110 = 1, 69, 21, 905.

Example 6.10 In how many ways can 2 letters be selected from the set $\{a, b, c, d\}$ when repetition of the letters is allowed, if (i) the order of the letters matters (ii) the order does not matter?

(i) When the order of the selected letters matters, the number of possible selections = 4^2 = 16, which are listed below:

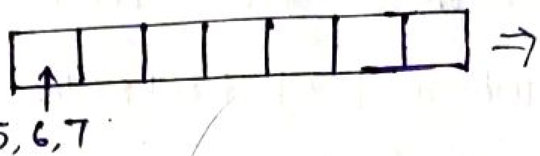
aa, ab, ac, ad
ba, bb, bc, bd
ca, cb, cc, cd
da, db, dc, dd

In general, the number of r-permutations of n objects, if repetition of the objects is allowed, is equal to n^r, since there are n ways to select an object from the set for each of the r-positions.

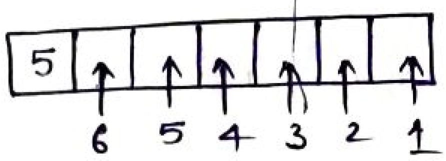
(ii) When the order of the selected letter does not matter, the number of possible selections $C(4 + 2 - 1, 2) = C(5, 2) = 10$, which are listed below:

eg: How many +ve integers n can be formed using the digits 3, 4, 4, 5, 5, 6, 7 if $n > 50,00,000$? (Repetition not allowed).

Soln;

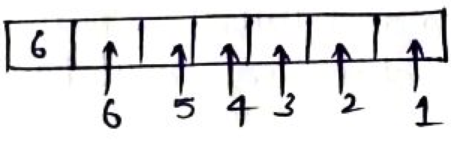


$\Rightarrow$



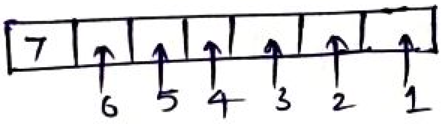
$$\Rightarrow \frac{\lfloor\underline{6}}{\lfloor\underline{2}} = 3 \times 120 = 360$$

(4) $\rightarrow \lfloor\underline{2}$ $\hookrightarrow \lfloor\underline{2}$ [illegible] (5) [illegible] fix [illegible]



$$\Rightarrow \frac{\lfloor\underline{6}}{\lfloor\underline{2}\ \lfloor\underline{2}} = 180$$

[illegible] fix [illegible]



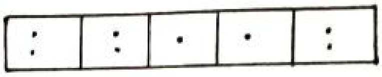
$$\Rightarrow \frac{\lfloor\underline{6}}{\lfloor\underline{2}\ \lfloor\underline{2}} = 180$$

$\therefore$ Total numbers formed $n > 50,00,000 = 360 + 180 + 180$

$= \underline{720}$ Ans!

→ Pigeon Hole Principle

If 'n' pigeons are accommodated in 'm' pigeonholes and $n > m$, then atleast one pigeonhole will contain 2 or more pigeons.



eg: $m = 5$ (pigeon holes)
$n = 7$ (pigeons)

Equivalently, if n objects are put in m boxes and $n > m$, then atleast 1 box will contain two or more objects.

Generalizations of the Pigeonhole Principle

If n pigeons are accommodated in m pigeonholes and $n > m$, then one of the pigeonholes must contain atleast

$\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

eg: $\lfloor 2.3 \rfloor = 2$, eg: $\lfloor 7.59 \rfloor = 7$

eg: Show that if 30 dictionaries in a library contain a total of 61327 pages then one of the dictionary must have atleast 2045 pages.

Soln: No. of dictionaries $(m) = 30$

No. of pages $(n) = 61327$

By the Pigeonhole principle one of the dictionary must have

$$\left\lfloor \frac{61327-1}{30} \right\rfloor + 1 = \lfloor 2044.2 \rfloor + 1 = 2044 + 1$$

$= 2045$ pages.

Proved:

Ans:

= 30,492.

Example 6.16 A man hiked for 10 hours and covered a total distance of 45 km. It is known that he hiked 6 km in the first hour and only 3 km in the last hour. Show that he must have hiked at least 9 km within a certain period of 2 consecutive hours.

Since, the man hiked 6 + 3 = 9 km in the first and last hours, he must have hiked 45 – 9 = 36 km during the period from second to ninth hours.

If we combine the second and third hours together, the fourth and fifth hours together, etc. and the eighth and ninth hours together, we have 4 time periods.

Let us now treat 4 time periods as pigeonholes and 36 km as 36 pigeons.

Using the generalised pigeonhole principal,

the least no. of pigeons accommodated in one pigeonhole

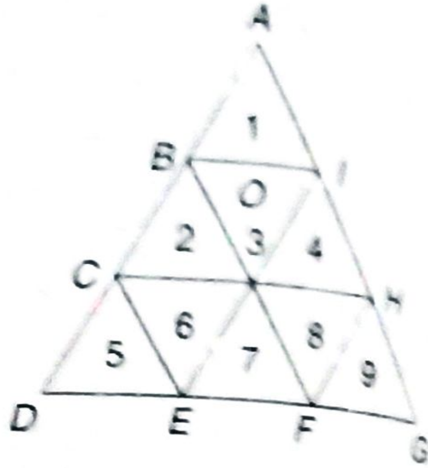
$$= \left\lfloor \frac{36-1}{4} \right\rfloor + 1$$

$$= \lfloor 8.75 \rfloor + 1 = 9$$

viz., the man must have hiked at least 9 km in one time period of 2 consecutive hours.

Example 6.17 If we select 10 points in the interior of an equilateral triangle of side 1, show that there must be at least two points whose distance apart is less than $\frac{1}{3}$.

Let ADG be the given equilateral triangle. The pairs of points B, C; E, F and H, I are the points of trisection of the sides AD, DG and GA respectively. We have divided the triangle ADG into 9 equilateral triangles each of side $\frac{1}{3}$.



The 9 sub-triangles may be regarded as 9 pigeon-holes and 10 interior points may be regarded as 10 pigeons.

Then by the pigeonhole principle, at least one sub triangle must contain 2 interior points.

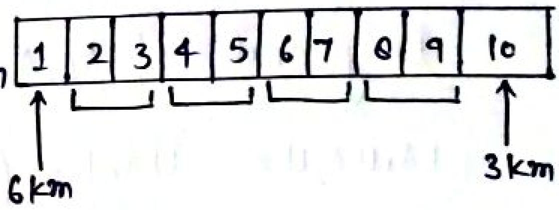
The distance between any two interior points of any sub triangle cannot exceed the length of the side, namely, $\frac{1}{3}$.

Hence the result.

eg: A man hiked for 10 hrs and covered a total distance of 45 km. It is known that he hiked 6 km in the first hour and only 3 km in the last hour. Show that he must have atleast 9 km within a certain period of two consecutive hours.

Soln: Since, the man hiked $6+3=9$ km in the first and last hours.

He must have hiked $45-9=36$ km during the period from second to ninth hours.



If we combine the second & third hours together, the 4^{th} & 5^{th} hrs together, 6^{th} & 7^{th} hrs together and 8^{th} & 9^{th} hrs together then we have 4 periods.

Let us consider 4 time periods as pigeonholes and 36 km as 36 pigeons using the generalised pigeons whole principle.

The man must hiked at least $= \left\lfloor \frac{36-1}{4} \right\rfloor + 1 = \lfloor 8.75 \rfloor + 1$

$= 8+1 = 9$ km in one time period of 2 consecutive hrs.

Proved:

Ans:

→ Inclusion Exclusion Principle

Let A & B be two finite sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

i.e. $|A \cup B| = |A| + |B| - |A \cap B|$

$|\ |$ ⇝ cardinality (no. of elements).

Let A, B & C are finite sets then

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Let A, B, C, D are four finite sets then

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D|$$
$$- |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap C \cap D|$$
$$+ |A \cap B \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

In general, if $A_1, A_2, ---, A_n$ are n finite sets then

$$|A_1 \cup A_2 \cup --- \cup A_n| = \left(\sum_{i=1}^{n} |A_i|\right) - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$
$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + --- + (-1)^{n} |A_1 \cap A_2 \cap A_3 --- \cap A_n|$$

---

eg: There are 250 students in an engineering college. Of these 188 have taken a course in Fortran, 100 have taken a course in C and 35 have taken a course in Java. Further 88 have taken courses in both Fortran and C. 23 have taken courses in both C and Java and 29 have taken courses in both Fortran and Java. If 19 of these students have taken all the three courses, how many of these 250 students have not taken a course in any of these three programming languages?

Soln:

$|F| = 188$, $|C| = 100$, $|J| = 35$, $|F \cap C| = 88$, $|C \cap J| = 23$, $|F \cap J| = 29$, $|F \cap C \cap J| = 19$.

$$|F \cup C \cup J| = |F| + |C| + |J| - |F \cap C| - |C \cap J| - |F \cap J| + |F \cap C \cap J|$$
$$= (188 + 100 + 35) - (88 + 23 + 29) + 19 = 202$$

No. of students who have not taken any of the languages

$= 250 - 202 = 48$ Ans.

eg: Find the number of integers b/w 1 and 250 both inclusive that are not divisible by any of the integers 2, 3, 5 & 7.

Soln: Let A, B, C & D be the sets integers b/w 1 & 250 that are divisible by 2, 3, 5 & 7.

$|A| = \left\lfloor \frac{250}{2} \right\rfloor = 125$, $|B| = \left\lfloor \frac{250}{3} \right\rfloor = 83$

$|C| = \left\lfloor \frac{250}{5} \right\rfloor = 50$; $|D| = \left\lfloor \frac{250}{7} \right\rfloor = 35$

$|A \cap B| = \left\lfloor \frac{250}{2 \times 3} \right\rfloor = 41$, $|A \cap D| = \left\lfloor \frac{250}{2 \times 7} \right\rfloor = 17$

$|A \cap C| = \left\lfloor \frac{250}{2 \times 5} \right\rfloor = 25$, $|B \cap C| = \left\lfloor \frac{250}{15} \right\rfloor = 16$

$|B \cap D| = \left\lfloor \frac{250}{3 \times 7} \right\rfloor = 11$, $|C \cap D| = \left\lfloor \frac{250}{5 \times 7} \right\rfloor = 7$

$|A \cap B \cap C| = \left\lfloor \frac{250}{2 \times 3 \times 5} \right\rfloor = 8$, $|A \cap B \cap D| = \left\lfloor \frac{250}{2 \times 3 \times 7} \right\rfloor = 5$

$|A \cap C \cap D| = \left\lfloor \frac{250}{2 \times 5 \times 7} \right\rfloor = 3$, $|B \cap C \cap D| = \left\lfloor \frac{250}{3 \times 5 \times 7} \right\rfloor = 2$

$|A \cap B \cap C \cap D| = \left\lfloor \frac{250}{2 \times 3 \times 5 \times 7} \right\rfloor = 1.$

By the principle of inclusion & exclusion the no. of integers b/w 1 & 250 that are divisible by at least one of 2,3,5 & 7 is given by

$$|A \cap B \cap C \cap D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - \cdots + |A \cap B \cap C \cap D|$$

$$= (125 + 83 + 50 + 35) - (41 + 25 + 17 + 16 + 11 + 7) + (8 + 5 + 3 + 2) - 1$$

$$= 193$$

No. of integers b/w 1 & 250 that are not divisible by any one of the integers 2,3,5 & 7 $= 250 - 193$

$= 57$

Ans:

Divisibility

When a & b are two integers with $a \neq 0$, a is said to be divisor of b, if there exist an integer c such that $b = ac$ & denoted by $a|b$.

eg: $a = 4,\ b = 14$

$b = 16 = 4 \cdot \underset{c}{4}$

$$\begin{array}{r} 4)\,16\,(4 \\ 16 \\ \hline 0 \\ \hline \end{array}$$

$$\boxed{\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}}$$

NOTE: Let $a, b, c \in \mathbb{Z}$, then

① If $a|b$ and $a|c$, then $a|b+c$

eg: $2|4$ & $2|8$, $2|(4+8)$

② If $a|b$ and $b|c$, then $a|c$

③ If $a|b$, then $a|mb$ for any integer m.

④ If $a|b$ and $a|c$, then $a|(mb+nc)$ for any integers m & n.

eg: $2|6$ & $2|8$ $(m=1, n=-2 \longrightarrow \text{Given})$

$mb+nc = 1\times6+(-2)\times8 = -10$

$\Rightarrow 2|(-10)$

⑤ If $a|b$, then $(-a)|b$

→ **Prime Numbers**

A positive integers $p>1$ is called prime, if the only +ve factors of p are 1 & p.

eg: 2, 3, 5, 7, 11, 13, 17, 19, ----

→ **Composite Numbers**

A positive integer $n>1$ which is not prime is called composite number.

eg: 4, 6, 9, 10, 12, ---

NOTE: 1 is neither prime nor composite.

→ Fundamental Theorem of Arithematic

Every integer $n>1$ can be uniquely expressed as a product of prime numbers.

eg: $n=10 \quad = 2\times 5$

$n=20 \quad = 2\times 2\times 5 = 2^2 \times 5$

$n=200 \quad = 2\times 2\times 2\times 5\times 5 = 2^3\times 5^2$

Product of prime's powers.

→ Note:

I) The unique expression for the integers $n>1$ as a product of primes is called prime factorisation or prime decomposition.

II) If there be k_i prime factors of n, each equal to p_i, where $1<i<r$, then n can be uniquely written as

$\boxed{n = p_1^{k_1}\cdot p_2^{k_2}\cdot \ldots \cdot p_r^{k_r}}$ → Product of prime's power.

where, $p_1, p_2, \ldots, p_r \longrightarrow$ Prime numbers

$k_1, k_2, \ldots, k_r \longrightarrow$ +ve integers.

eg: Find the Prime factorisation of each of the following

a) 6647 b) $\lfloor 10$

Soln:

a) $6647 = 17\times 17\times 23 = (17)^2 \cdot 23$

17	6647
17	391
23	23
	1

b) $\lfloor 10 = 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1$

$= 2\times 5\times 3\times 3\times 2^3\times 7\times 2\times 3\times 5\times 2^2\times 3\times 2\times 1$

$= 2^8\times 3^4\times 5^2\times 7$

Ans:

→ Greatest Common Divisor (Gcd)

When a & b are non-zero integers, then an integer $d \neq 0$ is said to be common divisor of a & b, if $d|a$ & $d|b$.
If d is the higest of all common divisors of a & b, then d is called the greatest common divisor and denoted by $\gcd(a,b)$.

eg: $a = 12$, $b = 36$

$12 = 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$

Common divisors of 12 & 36 : 2, 3, 4, 6, 12

$\gcd(12, 36) = 12 = 2 \times 2 \times 3$.

NOTE:

1) If $\gcd(a,b) = 1$, then a & b are said to be relatively prime or co-prime.

eg: $a = 2$, $b = 3$; $\gcd(2,3) = 1$

2 & 3 are co-prime

eg: Is 0 co-prime to 1 ?

11) If $\gcd(a_i, a_j) = 1$ for $1 \leq i < j \leq n$, then the integers $a_1, a_2, \ldots, a_n$ are said to be pairwise relatively prime.

eg: Let $a_1 = 2$, $a_2 = 3$, $a_3 = 5$

$\gcd(a_1, a_2) = \gcd(a_1, a_3) = \gcd(a_2, a_3) = 1$

$\Rightarrow a_1, a_2, a_3$ are pairwise co-prime.

Theorem:

If $n > 1$ is a composite number and p is a prime factor of n, then $p \leq \sqrt{n}$.

eg: $n = 18$ (composite number)

$= 2 \times 3 \times 3$

Prime factors of 18 are 2 & 3.

$\Rightarrow \quad 2 < \sqrt{18} \Rightarrow 2 < 4.24$

$\Rightarrow \quad 3 < \sqrt{18} \Rightarrow 3 < 4.24$.

→ **Division Algorithm:**

When a & b are any two integers, $b>0$, there exist unique integers q (quotient) & r (remainder) such that

$$a = bq + r, \quad \text{where } 0 \le r < b$$

$b)\,a\,(q$
$\quad\underline{}$
$\quad r$

dividend = divisor × quotient + remainder.

NOTE:

If b is any integer, then above state as $a = bq + r$, where $0 \le r < |b|$.

→ **Euclid's Algorithm for finding gcd(a, b):**

If a & b are any two integers ($a > b$), if r_1 is remainder when a is divided by b, r_2 is the remainder, when b is divided by r_1, r_3 is the remainder, when r_1 is divided by r_2 --- and so on if $r_{k+1} = 0$, then last (previously) non-zero remainder r_k is called the gcd(a, b).

$\because a > b$

$b)\,a\,(q_1$
$r_1)\,b\,(q_2$
$r_2)\,r_1\,(q_3$
$r_3)\,r_2\,(q_4$
r_4
⋮
$\gcd(a,b) \leftarrow r_k$
$r_{k+1} = 0.$

eg: Find gcd (1575, 231) by using Euclid's algorithm.

Soln: By division algorithm

$$1575 = 6 \times 231 + 189$$
$$231 = 1 \times 189 + 42$$
$$189 = 4 \times 42 + 21$$
$$42 = 2 \times 21 + 0$$

$\gcd(231, 1575) = 21.$

231) 1575 (6
1386
189) 231 (1
189
42) 189 (4
168
21) 42 (2
42
0

Theorem:

$\gcd(a,b)$ can be expressed as an integral linear combination of a & b i.e. $\gcd(a,b) = ma + nb$, for some integers m & n.

NOTE: This linear combination is not unique.

eg: Express the gcd (1575, 231) as a linear combination of 1575 & 231.

Soln: We have, by the division algorithm.

$1575 = 6 \times 231 + 189$ ——— (i)

$231 = 1 \times 189 + 42$ ——— (ii)

$189 = 4 \times 42 + 21$ ——— (iii)

$42 = 2 \times 21 + 0$ ——— (iv)

$$
\begin{aligned}
\gcd(1575, 231) &= 21 \\
&= 189 - 4 \times 42 \\
&= 189 - 4(231 - 1 \times 189) \\
&= 5(189) - 4(231) \\
&= 5(1575 - 6 \times 231) - 4(231) \\
&= 5(1575) - 34(231)
\end{aligned}
$$

$$21 = 5 \times \underset{a}{1575} + (-34) \times \underset{b}{231}$$

$\Rightarrow m = 5$ & $n = -34$.

Some Properties of gcd:

1) If $c|ab$ & a and c are co-prime then $c|b$.

eg: $2|3.6 \Rightarrow 2|6$

11) If a & b are co-prime and a & c are co-prime, then a & bc are co-prime.

eg: 2 co-prime to 3

2 ——— 5

2 ——— $3 \times 5 = 15$.

III) If a, b are any integers, which are not simultaneously zero and k is any positive integer, then

$$\gcd(ka, kb) = k.\gcd(a,b)$$

eg: $a = 6$, $b = 8$, $k = 2$

$\gcd(6,8) = 2$

$ka = 12$, $kb = 16$

$\gcd(12,16) = 4 = 2\cdot2 = 2.\gcd(6,8)$.

$\gcd(2\cdot6, 2\cdot8) = 2.\gcd(6,8)$.

IV) If $\gcd(a,b)=1$, then for any integer c, $\gcd(ac, b) = \gcd(a,b) = 1$.

eg: $a = 2$, $b = 3$, $c = 2$

$\gcd(4,3) = 1 = \gcd(2,3)$.

V) If $\gcd(a,b) = d$, then $\gcd(a/d, b/d) = 1$.

VI) If $a_1, a_2, ---, a_n$ is co-prime to b then, the product $a_1.a_2.a_3.----.a_n$ is also co-prime to b.

eg: Use the Euclidean's algorithm to find the gcd (1819, 3587) also expressed the gcd as a linear combination of the given numbers.

Soln: By division algorithm

$3587 = 1\times1819 + 1768$

$1819 = 1\times1768 + 51$

$1768 = 34\times51 + 34$

$51 = 1\times34 + 17$

$34 = 2\times17 + 0$

$\gcd(1819, 3587) = 17$

```
1819)3587(1
     1819
     ----
     1768)1819(1
          1768
          ----
            51)1768(34
               153
               ---
                238
                204
                ---
                 34)51(1
                    34
                    --
                    17)34(2
                       34
                       --
                        0
```

$$
\begin{aligned}
17 &= 51 - 1\times 34 \\
&= 51 - 1\times(1768 - 34\times 51) \\
&= 35(51) - 1(1768) \\
&= 35(1819 - 1768) - 1768 \\
&= 35(1819) - 36(1768) \\
&= 35(1819) - 36(3587 - 1819) \\
&= \underline{71\times 1819 - 36\times 3587} \rightarrow \text{Ans:}
\end{aligned}
$$

eg: Find the integers x & y such that $154x + 260y = 2$.

Soln: By division algorithm

$$
\begin{aligned}
260 &= 1\times 154 + 106 \\
154 &= 1\times 106 + 48 \\
106 &= 2\times 48 + 10 \\
48 &= 4\times 10 + 8 \\
10 &= 1\times 8 + 2 \\
8 &= 4\times 2 + 0
\end{aligned}
$$

154)260(1
154
106)154(1
106
48)106(2
96
10)48(4
40
8)10(1
8
2)8(4
8
0

$\gcd(154, 260) = 2$.

Now, we will express 2 as a linear combination of 154 & 260.

$$
\begin{aligned}
2 &= 10 - 1\times 8 \\
&= 10 - 1\times(48 - 4\times 10) \\
&= 5(10) - 48 \\
&= 5(106 - 2\times 48) - 48 \\
&= 5(106) - 11(48) \\
&= 5(106) - 11(154 - 106) \\
&= 16(106) - 11(154) \\
&= 16(260 - 1\times 154) - 11(154)
\end{aligned}
$$

$\cancel{2 = 16(260) + 5(154)}$

$2 = 16(260) - 27(154)$

$\Rightarrow \underline{x = -27 \;\&\; y = 16} \rightarrow$ A eg: Prove that a) $\log_3 5$ b) $\sqrt{5}$ are irrational numbers.

Soln: Let $\log_3 5$ is an ~~ir~~rational number.

a) Let $\log_3 5 = \frac{p}{q}$, $q \neq 0$; $p, q \in Z$

$\Rightarrow 5 = 3^{p/q}$ $\qquad \because \log_e x = y \Rightarrow x = e^y$

$\Rightarrow 5^q = 3^p = n$ (say)

which is contradiction of fundamental of arithmetic, since n is expressed in two ways as 5^q & 3^p.

$\Rightarrow$ our assumption was wrong.

$\Rightarrow \log_3 5$ is an irrational number.

b) Let $\sqrt{5}$ is a rational number.

$\Rightarrow \sqrt{5} = \frac{p}{q}$, $q \neq 0$ & $p, q \in Z$

No factor common b/w p & q.

$\Rightarrow 5 = \frac{p^2}{q^2} \Rightarrow p^2 = 5.q^2$ —— (1)

$\Rightarrow p^2$ is divisible by 5.

$\Rightarrow p$ is divisible by 5.

$\Rightarrow p = 5m$; for some integer m.

Put $p = 5m$ in eqn (1)

$(5m)^2 = 5q^2$

$\Rightarrow q^2 = 5m^2$

$\Rightarrow q^2$ is divisible by 5

$\Rightarrow q$ ————

$\Rightarrow q = 5n$; for some integer n.

$\Rightarrow p$ & q have a common factor 5.

$\Rightarrow$ Our assumption was wrong.

$\therefore \sqrt{5}$ is an irrational number.

eg: Using prime factorisation, find the gcd & LCM of 231 & 1575.

Soln: We have

$231 = 3 \times 7 \times 11$

$1575 = 3 \times 3 \times 5 \times 5 \times 7$

$\gcd(231, 1575) = 3 \times 7 = \underline{21}$

$\text{LCM}(231, 1575) = 3 \times 7 \times 11 \times 3 \times 5 \times 5 = \underline{17325}$ An

eg: Determine gcd & LCM of 45 and 75, also verify that $\text{lcm}(45,75) \times \gcd(45,75) = 45 \times 75$.

Soln: To prove: $\text{lcm}(45,75) \times \gcd(45,75) = 45 \times 75$

$45 = 3 \times 3 \times 5$

$75 = 3 \times 5 \times 5$

$\gcd(45,75) = 3 \times 5 = 15$

$\text{lcm}(45,75) = 3 \times 5 \times 3 \times 5 = 225$

$\text{lcm}(45,75) \cdot \gcd(45,75) = 15 \times 225 = 3375.$ — (i)

$45 \times 75 = 3375$ — (ii)

Verified An