UNIT-IV (Mathematical Logic)

Logic is a field of study that deals with the method of reasoning. Logic provides rule by which we can determine whether a given argument or reasoning is Valid (correct) or not. Logical reasoning is used in Mathematics to prove Theorems. In Computer Science, Logic is used to verify the correctness of programs.

Proposition (or Statement) A statement or proposition is a declarative sentence which is either True or false but not both.

These 2 values 'True & false' denoted by the symbol T & F respectively. Sometimes it is also denoted by 1 & 0.

Sentences which are exclamatory, interrogative in nature are not proposition

eg:
1. New Delhi is the capital of india
2. $8 > 10$
3. Blood is green
4. It is raining
5. The Sun will come out tomorrow.
6. $x+y=z$
7. How beautiful is rose?
8. Who are you?
9. Close the door

Soln
1. is a Statement because it is True
2. " " " " " False
3. " " " " False
4. " " " because "It is raining" is either true or false but not both at a given time

5. is a statement because it is either True or false but not both. Although, we would have to wait until tomorrow whether it is T or F.

6. is not a statement because it is neither True or False as the values of x, y & z are not Assigned.

7. & 8. are not statement because it is a question

9. is not a statement because it is command

<u>Truth Values of a statement</u>

A Proposition (or statement) can have one and only one of 2 possible values namely True & False. These 2 values T & F are called Truth Values of a statement.

<u>Connectives</u> The words or symbol which are used to make a sentence by 2 sentences are called logical connectives.

Let us define the following basic & fundamental connectives —

	Connectives — word		Symbol
1	Negation	(NOT)	$\sim$ or $\neg$
2.	Conjuction	(AND)	$\wedge$
3.	Disjunction	(OR)	$\vee$
4.	Conditional (Implication)	(IF ... THEN)	$\rightarrow$
5.	Biconditional	(If and only If)	$\leftrightarrow$

1. Negation (NOT) If a proposition is denoted by the symbol P then its Negation denoted by ~P or ¬P. If P is True then ¬P is false and if P is false, ¬P is True.

Truth Table

p	~p
T	F
F	T

eg: p: Delhi is a city
¬p: Delhi is not a city
or It is False that Delhi is a city

2. Conjunction (AND) If p and q are any two proposition then conjunction (AND) of p & q is also a proposition & it is denoted by $p \wedge q$.

Truth Table

p	q	$p \wedge q$	
T	T	T	1.1 = 1
T	F	F	1.0 = 0
F	T	F	0.1 = 0
F	F	F	0.0 = 0

eg. p: Ram is an honest man
q: Rahim " " "
$p \wedge q$: Ram and Rahim are honest man

3. Disjunction (OR) If p and q are two proposition then the disjunction of p & q is denoted by $p \vee q$.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

e.g. p: I shall watch the game on television
q: I shall go to the home.
$p \vee q$: I shall watch the game or go to the home

1. Negation (NOT) If a proposition is denoted by the symbol P then its Negation denoted by $\sim P$ or $\neg P$. If P is True then $\neg P$ is false and if P is false, $\neg P$ is True.

Truth Table

p	$\sim p$
T	F
F	T

e.g. p: Delhi is a city
$\neg p$: Delhi is not a city
or It is False that Delhi is a city

2. Conjunction (AND) If p and q are any two proposition then conjunction (AND) of p & q is also a proposition & it is denoted by $p \wedge q$.

Truth Table

p	q	$p \wedge q$	
T	T	T	$1.1=1$
T	F	F	$1.0=0$
F	T	F	$0.1=0$
F	F	F	$0.0=0$

e.g.
p: Ram is an honest man
q: Rahim " " "
$p \wedge q$: Ram and Rahim are honest man

3. Disjunction (OR) If p and q are two proposition then the disjunction of p & q is denoted by $p \vee q$.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

e.g. p: I shall watch the game on television
q: I shall go to the home.
$p \vee q$: I shall watch the game or go to the home

4. Conditional proposition (IF -- THEN)

If p & q are two proposition then the compound statement "If p then q" denoted as $p \to q$ is called a Conditional proposition or Implication.

Here p is called Hypothesis or premise & q is called the Consequence or Conclusion.

Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

e.g. p: I am hungry
q: I will eat.

$p \to q$: If I am hungry then I will eat.

5. Biconditional proposition (If and only If)

If p and q are two propositions then the compound proposition "p If and only If q" written as $p \leftrightarrow q$ is called Biconditional proposition

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

e.g. p: He swims
q: Water is warm.

$p \leftrightarrow q$: He swims Iff water is warm.

Note Compound propositions: A proposition obtained from the combination of 2 or more proposition by means of logical operators is called Compound proposition

Truth Table A Truth Table is a Table that shows the Truth Values of Compound proposition for all Possible Cases

Ques Construct the truth table for

(1) $p \wedge \sim p$ (2) $p \vee \sim q$ (3) $(p \vee q) \vee \sim p$ (4) $(p \vee q) \wedge r$

(5) $(p \vee q) \rightarrow (p \wedge q)$ (6) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

(7) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

Sol^n 1. $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

2. $p \vee \sim q$

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

3. $(p \vee q) \vee \sim p \equiv A$

p	q	$\sim p$	$p \vee q$	A
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

5. $(p \vee q) \rightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$p \vee q \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

4. $(p \vee q) \wedge r$

p	q	r	$(p \vee q)$	$(p \vee q) \wedge r$
T	T	T	T	T
F	T	T	T	T
T	F	T	T	T
T	T	F	T	F
F	F	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	F	F

6. $(p \rightarrow q) \rightarrow (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \rightarrow (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

7. $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q) \equiv A$

p	q	$\neg p$	$q \rightarrow \neg p$	$p \leftrightarrow q$	A
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

Tautology and Contradiction

A proposition which is true for all possible truth values of its propositional variable is called a tautology & A proposition is called contradiction if it contains only false values.

eg. Show that $[p \wedge (p \to q)] \to q$ is a tautology.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$\Big\} \longrightarrow$ All T

Since all the entries in last column is T so it is a Tautology.

eg. Show that the statement $p \wedge \neg p$ is a contradiction

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

$\Big\} \longrightarrow$ All F

Since all entries in last column are F so it is a contradiction

1. $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$
2. $[q \vee (p \wedge \neg q)] \vee (\neg p \wedge \neg q)$

$\Big]$ Take to

Equivalence of Proposition Two compound proposition $A(p_1, p_2, \ldots p_n)$ and $B(p_1, p_2 \ldots p_n)$ are said to be logically equivalent if they have identical truth values. It is denoted by $\equiv$ or $\Leftrightarrow$

ie $\quad A(p_1, p_2 \ldots p_n) \equiv B(p_1, p_2 \ldots p_n)$

or $\quad A \Leftrightarrow B$

* If a proposition is neither Tautology nor a contradiction, is called a contingency

eg. Prove that $p \to q \equiv \sim p \vee q$

Soln

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the truth values of column 3 & 5 are same

$\therefore \quad (p \to q) \equiv \sim p \vee q$

Ques 1. $\quad \neg(p \to q) \equiv p \wedge \neg q$

2. $\quad (p \wedge q) \vee (p \wedge \neg q) \equiv \phi$

3. $\quad \neg(p \leftrightarrow q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

<u>Duality Law</u> Two statements are said to be dual of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F & F by T

eg. Find the dual of the following.

	Dual
1. $(p \vee q) \wedge r$	1. $(p \wedge q) \vee r$
2. $(p \wedge q) \vee T$	2. $(p \vee q) \wedge F$
3. $\neg(p \vee q) \wedge (p \vee \sim(p \wedge s))$	3. $\neg(p \wedge q) \vee (p \wedge \neg(p \vee s))$

Algebra of Propositions

S.No	Name of the Law	Primal form	Dual form
1.	Commutative Law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
2.	Associative Law	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
3.	Distributive Law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
4.	Idempotent Law	$p \vee p \equiv p$	$p \wedge p \equiv p$
5.	Identity Law	$p \vee F \equiv p$	$p \wedge T \equiv p$
6.	Dominant Law	$p \vee T \equiv T$	$p \wedge F \equiv F$
7.	Complement Law	$p \vee \sim p \equiv T$	$p \wedge \sim p \equiv F$
8.	Absorption Law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9.	Demorgan's Law	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$

Equivalences involving Conditionals

1. $p \to q \equiv \sim p \vee q$
2. $p \to q \equiv \sim q \to \sim p$
3. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
4. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
5. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
6. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Equivalences involving Biconditionals

1. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

Ques 1.4
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without using Truth Tables, prove that

(i) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

Sol^n

L.H.S $(\neg p \vee q) \wedge (p \wedge (p \wedge q))$

$(\neg p \vee q) \wedge ((p \wedge p) \wedge q)$ Associative Law

$(\neg p \vee q) \wedge (p \wedge q)$ idempotent Law

$((\neg p \vee q) \wedge p) \wedge q$ Associative Law

$(p \wedge (\neg p \vee q)) \wedge q$ Commutative Law

$((p \wedge \neg p) \vee (p \wedge q)) \wedge q$ Distributive Law

$(F \vee (p \wedge q)) \wedge q$ Complement Law

$(p \wedge q) \wedge q$ identity Law

$p \wedge (q \wedge q)$ Associative Law

$p \wedge q$ idempotent Law

(ii) $p \to (q \to p) \equiv \neg p \to (p \to q)$

L.H.S $p \to (q \to p)$

$p \to (\neg q \vee p)$

$\neg p \vee (\neg q \vee p)$

$(\neg q \vee p) \vee \neg p$ (Commutative)

$\neg q \vee (p \vee \neg p)$ (Asso.)

$\neg q \vee T$ (Complement)

T. (by dominat Law)

R.H.S

$\neg p \to (p \to q)$

$\neg(\neg p) \vee (\neg p \vee q)$

$p \vee (\neg p \vee q)$

$(p \vee \neg p) \vee q$ [Asso. Law]

$T \vee q$ (Complement law)

T (Dominat Law)

(iii) $\neg(p \leftrightarrow q) \equiv (p \vee q) \wedge \neg(p \wedge q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

L.H.S $\quad \neg(p \leftrightarrow q)$

$\neg((p \to q) \wedge (q \to p))$

$\neg(\underbrace{(\neg p \vee q)}_{A} \wedge (\neg q \vee p))$

$\neg(((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p))$ Distributive law

$\neg(((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((\neg p \wedge p) \vee (q \wedge p)))$ [Distributive Law]

$\neg(((\neg p \wedge \neg q) \vee F) \vee (F \vee (p \wedge q)))$ [Complement Law]

$\neg((\neg p \wedge \neg q) \vee (p \wedge q))$ [Identity Law]

$\neg(\neg p \wedge \neg q) \wedge \neg(p \wedge q)$ (Demorgan law]

$\boxed{(p \vee q) \wedge \neg(p \wedge q)}$ ["]

$\underbrace{(p \vee q)}_{A} \wedge (\neg p \vee \neg q)$ (")

$((p \vee q) \wedge \neg p) \vee ((p \vee q) \wedge \neg q)$ [Distributive law]

$((p \wedge \neg p) \vee (q \wedge \neg p)) \vee ((p \wedge \neg q) \vee (q \wedge \neg q))$ (")

$(F \vee (q \wedge \neg p)) \vee ((p \wedge \neg q) \vee F)$ [Complement law]

$(q \wedge \neg p) \vee (p \wedge \neg q)$ (identity law)

$\boxed{(p \wedge \neg q) \vee (\neg p \wedge q)}$ (Commutative law]

eg 1.5 Prove that

(i) $\neg p \to (q \to r) \equiv q \to (p \vee r)$

L.H.S $\quad \neg p \to (q \to r)$

$\neg p \to (\neg q \vee r)$

$\neg(\neg p) \vee (\neg q \vee r)$

$p \vee (\neg q \vee r)$

$(\neg q \vee r) \vee p$ (Commutative)

$\neg q \vee (r \vee p)$ (Asso)

$q \to (p \vee r)$

(ii) $p \to (q \to r) \equiv p \to (\neg q \vee r) \equiv (p \wedge q) \to r$

L.H.S $p \to (q \to r)$

$\boxed{p \to (\neg q \vee r)}$

$\neg p \vee (\neg q \vee r)$

$(\neg p \vee \neg q) \vee r$ (Asso.)

$\neg (p \wedge q) \vee r$ (Demorgan's law)

$\boxed{(p \wedge q) \to r}$

(iii) $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a Tautology

Sol^n $((p \vee q) \wedge (p \vee (q \wedge r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ [Demorgan law]

$((p \vee q) \wedge ((p \vee q) \wedge (p \vee r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ [Dist.]

$(((p \vee q) \wedge (p \vee q)) \wedge (p \vee r)) \vee \neg(p \vee q) \vee \neg(p \vee r)$ [Asso & Demorg]

$((p \vee q) \wedge (p \vee r)) \vee \neg((p \vee q) \wedge (p \vee r))$ [Idempotent & Demorgan law]

T [$p \vee \neg p \equiv T$ Domenat law]

eg 1.6 Prove the following equivalences by proving the equivalences of the duals.

(i) $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$

Sol^n The dual of the given equivalences is

$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q) \equiv p$ — (1)

L.H.S of (1) is

$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q)$

$\Rightarrow \neg((\neg p \vee (q \wedge \neg q))) \wedge (p \vee q)$ [Distributive law]

$\neg((\neg p \vee F)) \wedge (p \vee q)$ (Complement law)

$\neg(\neg p) \wedge (p \vee q)$

$p \wedge (p \vee q) \Rightarrow p$ [Absorption law]

(ii) $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

Soln Dual is

$$\neg(p \vee q) \vee r \equiv (\neg p \vee r) \wedge (\neg q \vee r)$$

$$\neg(p \wedge q) \wedge r \equiv (\neg p \wedge r) \vee (\neg q \wedge r) \quad \text{—(2)}$$

L.H.S of (2) is

$\neg(p \wedge q) \wedge r$

$\Rightarrow (\neg p \vee \neg q) \wedge r$ (Demorgan's Law)

$\Rightarrow (\neg p \wedge r) \vee (\neg q \wedge r)$ Distributive law

$= R.H.S$

(iii) $(p \wedge (p \leftrightarrow q)) \to q \equiv T$

$p \wedge ((p \to q) \wedge (q \to p)) \to q \equiv T$

$p \wedge ((\neg p \vee q) \wedge (\neg q \vee p)) \to q \equiv T$

$\Rightarrow \neg(p \wedge ((\neg p \vee q) \wedge (\neg q \vee p))) \vee q \equiv T \quad \text{—(1)}$

Dual of (1) is

$$\neg(p \vee ((\neg p \wedge q) \vee (\neg q \wedge p))) \wedge q \equiv F \quad \text{—(2)}$$

L.H.S

$\neg(p \vee ((\neg p \wedge q) \vee (\neg q \wedge p))) \wedge q$

$\Rightarrow \neg(p \vee ((\neg q \wedge p) \vee (\neg p \wedge q))) \wedge q$ [Comm. la]

$\Rightarrow \neg((p \vee (\neg q \wedge p)) \vee (\neg p \wedge q)) \wedge q$ [Asso]

$\Rightarrow \neg(((p \vee (p \wedge \neg q)) \vee (\neg p \wedge q)) \wedge q$ [Comm.

$\Rightarrow \neg(p \vee (\neg p \wedge q)) \wedge q$ [Absorpt.

$\Rightarrow \neg((p \vee \neg p) \wedge (p \vee q)) \wedge q$ [Dist.]

$\Rightarrow \neg(T \wedge (p \vee q)) \wedge q$ (Complent

$\Rightarrow \neg(p \vee q) \wedge q$ [Identity la]

$p \wedge \neg p$

$\neg(p \vee q) \wedge q$

$(\neg p \wedge \neg q) \wedge q$ (Demorgan's law)

$\neg p \wedge (\neg q \wedge q)$ (Associative law)

$\neg p \wedge F$ (Complement law)

F (Dominat law)

Tautological Implication. A Compound proposition $A(p_1, p_2 .. p_n)$ is said to be tautologically imply or simply imply the compound proposition $B(p_1, p_2 . p_n)$, if B is true whenever A is True

or equivalently if and only if $A \to B$ is a Tautology

This is denoted by $A \Rightarrow B$ read as "A implies B"

eg. $p \Rightarrow p \vee q$

Now we shall show that $p \to (p \vee q)$ is a tautology

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$\therefore$ $p \Rightarrow p \vee q$

Note
$\to$ Conditional
$\leftrightarrow$ Biconditional
Eq $\Leftrightarrow$ equivalence
$\Rightarrow$ Tautological Implication

eg 1.8 Prove the following implications without using truth table

(i) $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$

we shall show that

$(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$ is a Tautology

$\Rightarrow (p \vee q) \wedge [(p \to r) \wedge (q \to r)] \to r$

$\Rightarrow (p \vee q) \wedge [(p \vee q) \to r] \to r$

$\Rightarrow (p \vee q) \wedge [\neg(p \vee q) \vee r] \to r$

$\Rightarrow [(p \vee q) \wedge \neg(p \vee q)] \vee [(p \vee q) \wedge r] \to r$ [Distributive law]

$\Rightarrow [F \vee (p \vee q) \wedge r] \to r$ [Complement]

$\Rightarrow [(p \vee q) \wedge r] \to r$ [Identity law]

$\Rightarrow \neg[(p \vee q) \wedge r] \vee r$

$\Rightarrow [\neg(p \vee q) \vee \neg r] \vee r$ [Demorgan's law]

$\Rightarrow \neg(p \vee q) \vee (\neg r \vee r)$ (Asso)

$\Rightarrow \neg(p \vee q) \vee T$ [Complement]

$\Rightarrow T$ [Dominat law]

(ii) $((p \vee \neg p) \to q) \to ((p \vee \neg p) \to r) \Rightarrow q \to r$

$((p \vee \neg p) \to q) \to ((p \vee \neg p) \to r) \to (q \to r)$

$(T \to q) \to (T \to r) \to (q \to r)$

$(\neg T \vee q) \to (\neg T \vee r) \to (q \to r)$

$(F \vee q) \to (F \vee r) \to (q \to r)$

$(q \to r) \to (q \to r)$

$\neg(q \to r) \vee (q \to r)$

T

Predicate Calculus The propositional Calculus does not allow us to represent many of the statements that we use in mathematics, Computer Science and in every day life. predicate Calculus is the generalization of the Proposition Calculus.

A Part of a declarative sentence describing the Properties of an object or relation among objects is called a Predicate.

e.g. Consider 2 proposition

P_1 : Ram is a bachelor

P_2 : Shyam is a Bachelor

Both Ram & Shyam has the Property of being bachelor. There is no relation b/w P_1 & P_2 but have same Common Part. These 2 proposition can be replaced by a single Statement "x is a Bachelor" and we can get both Proposition P_1 & P_2 by replacing x by Ram & Shyam.

The sentence "x is a Bachelor" is Symbolised as $P(x)$, where x is a predicate Variable & $P(x)$ is called proposetional function".

Arguments An argument is a sequence of statements. All statements (propositions) except the final one are called premises (or assumption or Hypothesis) and final statement is called conclusion.

Sometimes an argument is written in the following form

$$\left.\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{array}\right\} \text{premises}$$

$$\therefore \; c \quad \leftarrow \text{conclusion}$$

An argument is said to be logically valid argument iff the conjunction of the premises implies the conclusion

ie If all the premises are true, the conclusion must also be true. the argument which yield a conclusion c from the premises $p_1, p_2, p_3 -- p_n$ is valid iff The statement

$(p_1 \wedge p_2 \wedge p_3 \cdots \wedge p_n) \rightarrow c$ is a Tautology. $\{ A \Rightarrow B$ / $A \rightarrow B$ is Tautology$\}$

Theory of inference

Inference Theory is concerned with the inferring of a conclusion from certain Hypothesis or premises by applying certain principles of reasoning, called rules of Inference

Rules of Inference

Rule P: A Premises may be introduced at any step in the derivation

Rule T: A formula S may be introduced in the derivation

Inconsistent premises

A set of premises $p_1, p_2 \dots p_n$ is said to be inconsistent if their conjunction implies a contradiction

ie $p_1 \wedge p_2 \wedge \dots \wedge p_n \Rightarrow R \wedge \neg R$ for some formula R

Indirect method of proof

In this method we assume that C is false and include $\neg C$ as an additional premises.

CP Rule

If the conclusion is of the form $r \to s$, we will take r as an additional premise and derive s using the given premises and r

Rules of inference	Tautological form	Name of the Rule.
1.(a) $\dfrac{p}{\therefore p \vee q}$ (b) $\dfrac{q}{p \vee q}$	1.(a) $p \to p \vee q$ (b) $q \to p \vee q$	Addition
2.(a) $\dfrac{p \wedge q}{\therefore p}$ (b) $\dfrac{p \wedge q}{\therefore q}$	2.(a) $p \wedge q \to p$ (b) $p \wedge q \to q$	Simplification
3. $\begin{array}{c} p \\ q \\ \hline p \wedge q \end{array}$	3. $(p) \wedge (q) \to p \wedge q$	Conjunction
4. $\begin{array}{c} p \to q \\ p \\ \hline \therefore q \end{array}$	4. $[(p \to q) \wedge p] \to q$	modus ponens
5. $\begin{array}{c} p \to q \\ \neg q \\ \hline \neg p \end{array}$	5. $[(p \to q) \wedge \neg q] \to \neg p$	modus tollens
6. $\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	6. $[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
7. $\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$	7. $(p \vee q) \wedge \neg p \to q$	Disjunctive Syllogism
8. $\begin{array}{c} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline \therefore q \vee s \end{array}$	8. $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to q \vee s$	Constructive dilemma
9. $\begin{array}{c} (p \to q) \wedge (r \to s) \\ \neg q \vee \neg s \\ \hline \neg p \vee \neg r \end{array}$	9. $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)]$ $\to (\neg p \vee \neg r)$	Destructive dilemma

Ques Applying the Rule of Inference check the validity of the argument

(1)
$$\begin{array}{l} p \to r \\ \neg p \to q \\ q \to s \\ \hline \therefore \neg r \to s \end{array}$$

Soln

S.No.	Statement	Reason
1.	$\neg p \to q$	P
2.	$q \to s$	P
3.	$\neg p \to s$	T, 1, 2, Hypothetical Syllogism
4.	$p \to r$	P
5.	$\neg r \to \neg p$	T, 4, $A \to B \equiv \neg B \to \neg A$
6	$\neg r \to s$	T, 4, 6, Hy. Sy.

(2)
$$\begin{array}{l} p \to q \\ r \to \neg q \\ \hline p \to \neg r \end{array}$$

Soln

S.No.	Statement	Reason
1.	$p \to q$	P
2.	$r \to \neg q$	P
3.	$q \to \neg r$	T, 2, $A \to B \equiv \neg B \to \neg A$
4.	$p \to \neg r$	T, 1, 3 Hy. Syll.

eg 1.2 Show that $(t \wedge s)$ can be derived from the premises $p \to q$, $q \to \neg r$, r, $p \vee (t \wedge s)$

Sol^n

S.No	Statement	Reason
1.	$p \to q$	P
2.	$q \to \neg r$	P
3.	$p \to \neg r$	T, 1, 2, Hy. Sy.
4.	r	P
5.	$\neg p$	T, 3, 4, modus tollens
6.	$p \vee (t \wedge s)$	P
7.	$t \wedge s$	T, 5, 6, Disjunctive syllogism.

eg 1.3 Show that $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \to \neg r$, $\neg r \to (s \wedge \neg t)$ and $(s \wedge \neg t) \to (a \vee b)$

Sol^n

S.No.	Statement	Reason
1.	$(p \vee q) \to \neg r$	P
2.	$\neg r \to (s \wedge \neg t)$	P
3.	$(p \vee q) \to (s \wedge \neg t)$	T, 1, 2, Hy. Sy.
4.	$(s \wedge \neg t) \to (a \vee b)$	P
5.	$(p \vee q) \to (a \vee b)$	T, 3, 4, Hy Syllo.
6	$p \vee q$	P
7.	$a \vee b$	T, 5, 6, Modus ponens.

eg 1.4 Show that $(p\to q)\wedge(r\to s)$, $(q\to t)\wedge(s\to u)$, $\neg(t\wedge u)$ and $(p\to r) \Rightarrow \neg p$

Sol^n

S.No.	Statement	Reason
1.	$(p\to q)\wedge(r\to s)$	P
2.	$p\to q$	T, 1, Simplification
3.	$r\to s$	T, 1, Simplification
4.	$(q\to t)\wedge(s\to u)$	P
5.	$q\to t$	T, 4, Simplification
6.	$s\to u$	T, 4, Simplification
7.	$p\to t$	T, 2, 5, Hy Sy.
8.	$r\to u$	T, 3, 6, Hy Sy.
9.	$p\to r$	P
10.	$p\to u$	T, 8, 9, Hy S.
11	$(p\to t)\wedge(p\to u)$	T, 7, 10, Conj.
12.	$p\to(t\wedge u)$	T, 11, $(p\to q)\wedge(p\to r) \equiv p\to(q\wedge r)$
13	$\neg(t\wedge u)$	P
14.	$\neg p$	T, 12, 13, modus tollens

eg 1.9 Show that b can be derived from the premises $a \to b$, $c \to b$, $d \to (a \vee c)$, d, by indirect Method

Soln Let us include $\neg b$ as an additional premise

S.No	Statement	Reason
1.	$a \to b$	P
2	$c \to b$	P
3	$(a \to b) \wedge (c \to b)$	T, 1, 2, Conjunction
4.	$(a \vee c) \to b$	T, 3, equivalence
5.	$d \to (a \vee c)$	P, 4,
6.	$d \to b$	T, 4, 5 Hy Sy.
7.	d	P
8.	b	T, 6, 7, modus ponens
9.	$\neg b$	P (Additional)
10.	$b \wedge \neg b$	T, 8, 9, Conjunction
11.	F	T, 10, complement La.

eg 1.11 Prove that the premises $p \to q$, $q \to r$, $s \to \neg r$ and $p \wedge s$ are in consistent.

Soln.

1.	$p \to q$	P
2.	$q \to r$	P
3	$p \to r$	T, 1, 2, Hy Syllogism.
4.	$s \to \neg r$	P
5.	$r \to \neg s$	T, 4, $A \to B \equiv \neg B \to \neg A$
6.	$p \to \neg s$	T, 3, 5, Hy Syllo.
7	$p \wedge s$	P
8.	p	T, 7, Simplification
9.	s	〃
10.	$\neg s$	T, 6, 8, Modus Ponens
11.	$s \wedge \neg s$	T, 9, 10, Conjunction.
12.	F	T, 11, Complement

eg 1.10 using indirect method of pf, derive $p \to \neg s$ from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$, p

Soln Let us include $\neg(p \to \neg s) = \neg(\neg p \vee \neg s) = p \wedge s$ as an additional premises

S.No.	Statement	Reason
1.	$p \to (q \vee r)$	P
2.	p	P
3.	$q \vee r$	T, 1, 2, Modus ponens
4.	$\neg q \to r$	T, 3, $A \to B \equiv \neg A \vee B$
5.	$s \to \neg r$	P
6.	$r \to \neg s$	T, 5, $A \to B \equiv \neg B \to \neg A$
7.	$\neg q \to \neg s$	T, 4, 6, Hy Syllo.
8.	$q \to \neg p$	P
9.	$s \to q$	T, 7, $A \to B \equiv \neg B \to \neg A$
10.	$s \to \neg p$	T, 8, 9, Hy sy.
11.	$p \wedge s$	P (Additional)
12.	p	T, 11, Simplification
13.	s	T, 11, "
14.	$\neg p$	T, 10, 13, Modus ponens
15	$p \wedge \neg p$	T, 12, 14, Conjunction
16	F	T, 15, complement

eg 1.5 Show that $(a \to b) \wedge (a \to c)$, $\neg(b \wedge c)$, $(d \vee a) \Rightarrow d$

Sol^n

S.No.	Statements	Reason
1.	$(a \to b) \wedge (a \to c)$	P
2.	$a \to (b \wedge c)$	T, 1, $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
3.	$\neg(b \wedge c)$	P
4.	$\neg a$	T, 2, 3, modus tollens
5.	$d \vee a$	P
6.	d	T, 4, 5, Disjunctive Syllogism

eg 1.6 Give a direct proof for the implication $p \to (q \to s)$, $(\neg r \vee p)$, $q \Rightarrow (r \to s)$

Sol^n

1.	$p \to (q \to s)$	P
2.	$\neg r \vee p$	P
3.	$r \to p$	T, 2, $A \to B \equiv \neg A \vee B$
4.	$r \to (q \to s)$	T, 1, 3, Hy. Sy.
5.	$\neg r \vee (\neg q \vee s)$	T, 4
6.	$(\neg q \vee s) \vee \neg r$	T, 5, Commutative
7.	$\neg q \vee (s \vee \neg r)$	T, 6, Associative
8	$q \to (s \vee \neg r)$	T, 7
9.	q	P
10.	$s \vee \neg r$	T, 8, 9, Modus ponens
11.	$\neg r \vee s$	T, 10, Associative
12.	$r \to s$	T, 11

Indirect method

eg 1.8 Use indirect method to show that $r \to \neg q$, $r \vee s$, $s \to \neg q$, $p \to q \Rightarrow \neg p$

Soln Let us include $\neg(\neg p) = p$ as an additional premise

S.No	Statement	Reason
1.	$p \to q$	P
2.	$\neg(\neg p)$	P (Additional)
3.	q	T, 1, 2, modus ponens
4.	$s \to \neg q$	P
5.	$\neg s$	T, 3, 4, modus tollens
6.	$r \vee s$	P
7.	r	T, 5, 6, Disjunctive syllogism
8.	$r \to \neg q$	P
9.	$\neg q$	T, 7, 8, modus ponens
10.	Repeat $q \wedge \neg q$	T, 3, 9 Conjunction law
11	F	T, 10, Complement

OR

conjunction $(r \to \neg q) \wedge (s \to \neg q)$

S.No	Statement	Reason
1.	$r \to \neg q$	P
2.	$s \to \neg q$	P
3	$(r \vee s) \to \neg q$	T, 1, 2 equivalence
4.	$r \vee s$	P
5.	$\neg q$	T, 3, 4 Modus ponens
6.	$p \to q$	P
7.	$\neg p$	T, 5, 6 Modus Tollens
8.	p	P (Additional)
9.	$p \wedge \neg p$	T, 7, 8 Conjunction
10.	F	T, 9, Complement

eg 1.12 Prove that the premises $a \to (b \to c)$, $d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent

Soln

1.	$a \wedge d$	P
2.	a	T, 1, simplification
3.	d	T, 1, "
4.	$a \to (b \to c)$	P
5.	$b \to c$	T, 2, 4, Hy Syllogm.
6.	$d \to (b \wedge \neg c)$	P
7.	$b \wedge \neg c$	T, 3, 6, Hy Syllo.
8.	$\neg(\neg b \vee c)$	T, 7, Demorgan.
9.	$\neg b \vee c$	T, 5, equivalence
10.	$(\neg b \vee c) \wedge \neg(\neg b \vee c)$	T, 8, 9, Conjunction
11.	F	T, 10, complement

eg 1.14 Show that the following set of premises is inconsistent.

If Rama gets his degree, he will go for a Job.
If he goes for a Job, he will get married soon
If he goes for higher study, he will not get married
Rama gets his degree and goes for higher study

Soln Let

p: Rama gets his degree
q: He will go for a Job
r: He will get married soon
s: He goes for higher study

Then we have to prove that

$p \to q$, $q \to r$, $s \to \neg r$, $p \wedge s$ are inconsistent

1. $p \to q$ P
2. $q \to r$ P
3. $p \to r$ T, 1, 2, Hy Syll.
4. $p \land s$ P
5. p T, 4, Simplification
6. s "
7. r T, 3, 5, Modus Ponens
8. $s \to \neg r$ P
9. $\neg r$ T, 6, 8, modus ponens.
10. $r \land \neg r$ T, 7, 9, Conjunction
11. F T, 10, Complement Law

eg 1.13 Construct an argument to show that the following premises imply the conclusion "It rained"
"If it does not rain or if there is no traffic dislocation, then the sports day will be held and cultural programme will go on"; "If the sports day is held the trophy will be awarded" and "The trophy was not awarded".

Soln
p: It rained
q: There is traffic dislocation
r: Sports day will be held
s: Cultural programme will go on.
t: The trophy will be awarded

Then we have to prove that

$\neg p \lor \neg q \to r \land s,\ r \to t,\ \neg t \Rightarrow p$

1. $(\neg p \lor \neg q) \to (r \land s)$ P
2. $(\neg p \to (r \land s)) \land (\neg q \to r \land s)$ T, 1, $(a \lor b) \to c \equiv (a \to c) \land (b \to c)$
3. $\neg p \to r \land s$ T, 2, Simplification.
4. $r \to t$ P
5. $\neg t$ P
6. $\neg r$ T, 4, 5, Modus tollens
7. $\neg r \lor \neg s$ T, 6, Addition
8. $\neg (r \land s)$ T, 7, Demorgan's
9. $\neg(\neg p) = p$ T, 3, 8, modus tollens.

Q. Derive $p \to (q \to s)$ using CP-rule from the premises $p \to (q \to r)$ and $q \to (r \to s)$

Sol Here we shall assume p as an additional premises & derive $(q \to s)$

1.	$p \to (q \to r)$	Rule P
2	p	Rule P
3	$q \to r$	Rule T, 1, 2, Hypothetical syllogism
4.	$q \to (r \to s)$	Rule P
5.	$(q \to r) \wedge [q \to (r \to s)]$	Rule T, 3, 4, Conjunction law
6.	$q \to [r \wedge (r \to s)]$	Rule T, 5, $(a \to b) \wedge (a \to c) \equiv a \to (b \wedge c)$
7.	$q \to [r \wedge (\neg r \vee s)]$	Rule T, $A \to B \equiv \neg A \vee B$
8.	$q \to [(r \wedge \neg r) \vee (r \wedge s)]$	Rule T, 7, distributive law
9.	$q \to [F \vee (r \wedge s)]$	Rule T, 8, Complement law
10	$q \to (r \wedge s)$	Rule T, 9, Identity law
11.	$(q \to r) \wedge (q \to s)$	Rule T, 10 $(a \to b) \wedge (a \to c) \equiv a \to (b \wedge c)$
12	$q \to s$	Rule T, 11. Simplification Rule
13.	$p \to (q \to s)$	Rule T, 12, CP-Rule.

Ques Check the validity of the argument

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Ans

p	q	r	A $p \to q$	B $q \to r$	C $p \to r$
T	T	T	T	T	T
F	T	T	T	T	T
T	F	T	F	T	T
T	T	F	T	F	F
F	F	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	T
F	F	F	T	T	T

Since $A \wedge B \to C$ ie conjunction of the pr

Conclusion is Tautology

$\therefore$ the Given argument is valid.

Ques Check the validity of the argument

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Ans

p	q	r	A $p \to q$	B $q \to r$	C $p \to r$	$A \wedge B$	$A \wedge B \to C$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	F	T	T	F	T
T	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	F	T	F	T	F	T
F	F	F	T	T	T	T	T

Since $A \wedge B \to C$ ie conjunction of the premises implies Conclusion is Tautology

$\therefore$ the Given argument is valid.

Use a Truth Table to test the validity of the following a

If a man is a bachelor he is unhappy. If a man is unhappy he dies young, therefore bachelo die young.

Sol^n Let p: A man is a bachelor

q: He is unhappy

r: He dies young

$\therefore$

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore \quad p \to r \end{array}$$

Q2 Use a Truth Table to test the validity of the following argume

If a man is a bachelor he is unhappy. If a man is unhappy he dies young, therefore bachelors die young.

Sol^n Let p: A man is a bachelor

q: He is unhappy

r: He dies young

$\therefore$

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

Mathematical induction

Let $S(n)$ be a statement

Step 1. Verify that $S(1)$ is true

II. Assume that $S(K)$ is true for an arbitrary value of K

3. Verify that $S(K+1)$ is true.

Ques eg 6.1

Show that

$1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$

by mathematical induction

Pf Let $S(n):=1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$

Step I When $n=1$

$$S(1)=1^2=\frac{1}{3}\cdot 1\cdot(2-1)(2+1)=\frac{1}{3}\cdot 3=1$$

So $S(1)$ is true.

Step 2 Let $S(n)$ be true for $n=K$

i.e. $1^2+3^2+5^2+\cdots+(2K-1)^2=\frac{1}{3}K(2K-1)(2K+1)$

Now $1^2+3^2+5^2+\cdots+(2K-1)^2+(2K+1)^2$

$$=\frac{1}{3}K(2K-1)(2K+1)+(2K+1)^2$$

$$=\frac{1}{3}(2K+1)[K(2K-1)+3(2K+1)]$$

$$\frac{1}{3}(2K+1)(2K^2+5K+3)$$

$$\frac{1}{3}(2K+1)(2K+3)(K+1)$$

$$\frac{1}{3}(K+1)(2K+1)(2K+3)$$

$2K^2+2\cdot3$
$K(4K+1)$
$+3$

$2K^2+2K+3K$
$+3$
$2K(K+1)+3$

i.e. $S(K+1)$ is valid

Hence $S(n)$ is true for all $n\in Z^+$

Ques 6.3 Prove, by mathematical induction, that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Let $S_n: \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Step I $S_1 = \frac{1}{1\cdot 2} = \frac{1}{1+1}$ which is true

Step 2 Let S_K be true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{K(K+1)}=\frac{K}{K+1}$$

Now $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)}$

$$=\frac{K}{K+1}+\frac{1}{(K+1)(K+2)}$$

$$=\frac{1}{(K+1)}\left[K+\frac{1}{K+2}\right]$$

$$=\frac{(K^2+2K+1)}{(K+1)(K+2)}$$

$$=\frac{(K+1)^{\cancel{2}}}{\cancel{(K+1)}(K+2)}=\frac{K+1}{K+2}$$

$\therefore S_{K+1}$ is true

$\therefore S_n$ true for all $n \in Z^+$

eg 68 Use mathematical induction to prove that n^3+2n is divisible by 3, for $n \geq 1$

Soln Let S_n: n^3+2n is divisible by 3

Step 1 S_1 : $1^3+2 \cdot = 3$ is divisible by 3, which is True

Step 2 Let S_k be True

i.e k^3+2k is divisible by 3

Step 3 Now

$$(k+1)^3+2(k+1) = (k^3+1+3k^2+3k)+(2k+2)$$
$$= \underbrace{(k^3+2k)}_{I} + \underbrace{(3k^2+3k+3)}_{II}$$
$$= S_k + 3(k^2+k+1)$$

S_k is divisible by 3 and $3(k^2+k+1)$ is multiple of 3 so it is also divisible by 3.

So S_{k+1} is divisible by 3

$\therefore$ S_n is True for $n \geq 1$

eg 69 Use mathematical induction to prove that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9, for $n \geq 1$

Soln Let S_n: $n^3+(n+1)^3+(n+2)^3$ is divisible by 9

Step 1 S_1 : $1^3+(1+1)^3+(1+2)^3$

$= 1+8+27 = 36$ is divisible by 9, which is True

Let S_k be True.

i.e $k^3+(k+1)^3+(k+2)^3$ is divisible by 9

Now $(k+1)^3+(k+2)^3+(k+3)^3 = [k^3+(k+1)^3+(k+2)^3] + [(k+3)^3-k^3]$

$$= S_k + (\cancel{k^3}+27+27k+9k^2-\cancel{k^3})$$
$$= S_k + 9\underbrace{(k^2+3k+3)}_{II}$$

. I Expression S_k is I divisible by 9 and 2nd expression is a multiple of 9 so it is also divisible by 9

$\therefore$ S_{k+1} is True $\Rightarrow$ S_n is True for $n \geq 1$

Thus the inductive step is also true.
Hence, $S(n)$ is true for all $n \in Z^+$.

Example 6.2 Prove, by mathematical induction, that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3).$$

Let S_n: $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

Now S_1: $1 \cdot 2 \cdot 3 = \frac{1}{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4$

Thus, the basic step S_1 is true.
Let S_k be true

i.e., $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$ (1)

Now $[1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2)] + (k+1)(k+2)(k+3)$

$$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3), \text{ by (1)}$$

$$= \frac{1}{4}(k+1)(k+2)(k+3)\{k+4\}$$

Thus S_{k+1} is true, if S_k is true.
i.e., the inductive step is true.
Hence, S_n is true for all $n \in Z^+$.

Example 6.3 Prove, by mathematical induction, that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Let S_n: $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Then S_1: $\frac{1}{1.2} = \frac{1}{1+1}$ which is true.

i.e., the basic step S_1 is true.
Let S_k be true.

i.e., $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ (1)

Now $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ by (1)}$$

$$= \frac{1}{k+1}\left\{\frac{k(k+2)+1}{k+2}\right\}$$

$$= \frac{1}{k+1}\left\{\frac{(k+1)^2}{k+2}\right\} = \frac{k+1}{k+2} \qquad (2)$$

(2) means that S_{k+1} is also true.
i.e., the inductive step is true.
Hence, S_n is true for all $n \in Z^+$.

Example 6.4 Use mathematical induction to show that

$n! \geq 2^{n-1}$, for $n = 1, 2, 3, \ldots$

Let S_n: $n! \geq 2^{n-1}$

$\therefore$ S_1: $1! \geq 2^0$, which is true.

i.e., the basic step is true

Let S_k be true

i.e., $k! \geq 2^{k-1}$ (1)

Now

$$\begin{aligned}(k+1)! &= (k+1).\, k! \\ &\geq (k+1).\, 2^{k-1}, \text{ by (1)} \\ &\geq 2\,.\, 2^{k-1}, \text{ since } k+1 \geq 2 \\ &= 2^k \qquad (2)\end{aligned}$$

Step (2) means that S_{k+1} is also true.
i.e., the inductive step is true.
Hence, S_n is true for $n = 1, 2, 3, \ldots$.

Example 6.5 Use mathematical induction to show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}, \text{ for } n \geq 2$$

Let S_n: $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$

$\therefore$ S_2: $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} > \sqrt{2}$, since L.S = 1.707 and R.S = 1.414

i.e., the basic step is true for $n = 2$.
Let S_k be true.

i.e., $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{k}} > \sqrt{k}$ (1)

Now $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{k}} + \dfrac{1}{\sqrt{k+1}} > \sqrt{k} + \dfrac{1}{\sqrt{k+1}}$, by (1)

Now $\sqrt{k} + \dfrac{1}{\sqrt{k+1}} = \dfrac{\sqrt{k(k+1)}+1}{\sqrt{k+1}} > \dfrac{\sqrt{k \cdot k}+1}{\sqrt{k+1}}$

i.e., $\qquad > \dfrac{k+1}{\sqrt{k+1}}$

i.e.,

$$> \sqrt{k+1}$$

$$\therefore \quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} \tag{2}$$

Step (2) means that S_{k+1} is also true.
Hence, S_n is true for $n = 2, 3, 4, \ldots$

Example 6.6 Use mathematical induction to show that

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \le \frac{1}{\sqrt{n+1}}, \text{ for } n = 1, 2, 3, \ldots$$

Let $\quad S_n: \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \le \dfrac{1}{\sqrt{n+1}}$

$\therefore \quad S_1: \dfrac{1}{2} \le \dfrac{1}{2}$, which is true.

i.e., the basic step is true.
Let S_k be true.

i.e., $$\frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \le \frac{1}{\sqrt{k+1}} \tag{1}$$

Now $$\frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k) \cdot (2k+2)} \le \frac{1}{\sqrt{k+1}} \cdot \frac{2k+1}{2k+2}, \text{ by (1)} \tag{2}$$

Now $\quad \dfrac{2k+1}{2k+2} \le \dfrac{\sqrt{k+1}}{\sqrt{k+2}}$,

if $\quad \dfrac{(2k+1)^2}{(2k+2)^2} \le \dfrac{k+1}{k+2}$

i.e., if $\quad \dfrac{4k^2+4k+1}{4k^2+8k+4} \le \dfrac{k+1}{k+2}$

i.e., if $\quad 4k^3 + 12k^2 + 9k + 2 \le 4k^3 + 12k^2 + 12k + 4$

i.e., if $\quad 9k + 2 \le 12k + 4$

i.e., if $\quad 3k + 2 \ge 0$, which is true.

Using this in step (2), we get

$$\frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k)(2k+2)} \le \frac{1}{\sqrt{k+1}} \cdot \frac{\sqrt{k+1}}{\sqrt{k+2}}$$

i.e., $$\le \frac{1}{\sqrt{k+2}} \tag{3}$$