UNIT-1

(sets, Relations and functions)

Set

A Set is well-defined Collection of objects. The object of set is called elements or members of the set. By 'well-defined' Means that we should able to determine if a given element is Contained in the set or not. Capital Letters A, B, C --- are used to denote sets and Lower Case Letters a, b, c --- to denote elements of the set.

e.g. $N = \{1, 2, 3 --\} \rightarrow$ set of Natural number

Representation of sets

Usually a set is Represented in two ways
1. Roster form (Tabular form)
2. Builder form

1. Roster form In Roster notation, all the elements of sets are listed. If possible, seperated by commas and enclosed within braces.

e.g.
1. The set of Binary digits ie $A = \{0, 1\}$
2. The set of all Vowels in English alphabet: $V = \{a, e, i, o, u\}$
3. The set of positive Integer less than 50
   $A = \{1, 2, 3, 4, 5 ---, 49\}$

2. Builder form In Builder Notation, we define the elements of the set by specifying a property that they have in common.

e.g. The set $A = \{x : x$ is a positive integer not exceeding $49\}$ is the same as $A = \{1, 2, 3 -- 49\}$

Types of sets

Universal set: The set which contains all the objets under consideration is called the universal set and denoted as U

Null set (Empty set) A set which contain no elements at all is called the Null set or Empty set and is denoted by ϕ or $\{\}$

eg. (1) $A = \{x : x^2+1 = 0, x \text{ is real}\} \Rightarrow A = \{\phi\}$

(2) $A = \{x : 9 < x < 10, x \text{ is a Natural Number}\}$

Singleton set A set which contain only one element is called Singleton set.

eg. $A = \{0\}$ is singleton set

finite and Infinite Set A set which contains a finite numbers of elements is called finite Set :-

eg. (i) $A = \{1, 4, 9, 11\}$

(ii) $B = \{x : x^2 < 100, x \in Z^+\}$

A set which contains Infinite numbers of elements is called Infinite set

eg. (i) $A = \{x : x \text{ is an even positive integer}\}$ is an Infinite set

(ii) $B = \{1, 3, 5, 7 \text{ ---}\}$

Equal set Two set A and B are said to be equal set If they have same elements

eg. $A = \{1, 2, 3\}$ and $B\{3, 2, 1\}$ are same set

Subset The set A is said to be a subset of B, if every element of A is also an element of B and it is denoted as $A \subseteq B$

e.g. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4\}$

Here $A \subseteq B$

Proper Subset If every element of A is in B but B has more elements then the set is called proper subset

e.g. Let $A = \{3, 4\}$, $B = \{3, 4, 5\}$

Here $A \subset B$

Note

1. If A is not subset of B then we write $A \not\subseteq B$
2. Every set A is subset of itself ie $A \subseteq A$
3. $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
4. Null set ϕ is subset of any set ie $\phi \subseteq A$
5. If A is subset of B $\Rightarrow$ B is called super set of A
   ie $A \subseteq B \Rightarrow B \supseteq A$

Power set The set of all subsets of the set S is called the power set of S and is denoted by $P(S)$

e.g. if $S = \{a, b, c\}$

Then $P(S) =$ set of all subsets of $\{a, b, c\}$

ie $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

Note:- If the set has n elements then its power set has 2^n elements.

ie If $|S| = n$
Then $|P(S)| = 2^n$

Cartesian Product

Let $A = \{1, 2, 3\}$, $B = \{3, 2\}$

$A \times B = \{(1,3), (1,2), (2,3), (2,2), (3,3), (3,2)\}$

$B \times A = \{(3,1), (3,2), (3,3), (2,1), (2,2), (2,3)\}$

$\therefore$ we Observe That $A \times B \neq B \times A$

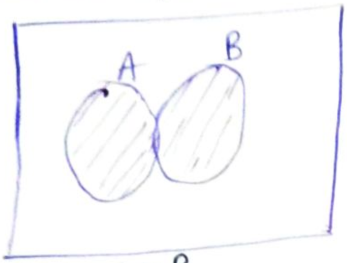
$A \times B = \{(a, b) : a \in A, b \in B\}$

Operation on sets

① **Union** The union of two sets A and B, denoted by $A \cup B$, is the set of elements belong to A or to B or both

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

Venn Diagram



$A \cup B$

The union of A and B is represented by the shaded area, shown in the figure.

e.g. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5\}$

then

$A \cup B = \{1, 2, 3, 4\}$, $A \cup C = \{1, 2, 3, 4, 5\}$

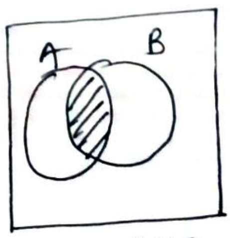
② **Intersection** The Intersection of two sets A and B denoted by $A \cap B$, is the set of elements that belong to both A and B

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

e.g. $A = \{2, 3\}$, $B = \{3, 4\}$

Then $A \cap B = \{3\}$

Venn Diagram



$A \cap B$

Note:- If A and B do not have any element in common then the sets A & B are said to be disjoint.

e.g $A = \{1, 2, 3\}$, $B = \{4, 5\}$ Then $A \cap B = \phi$

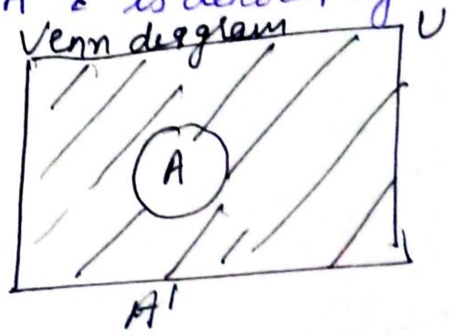
③ **Complement:-** If U is the universal set and A is any set then the set of elements which belongs to A but which donot belong to A is called the complement of A & is denoted by A' or A^c or $\bar{A}$

$A' = \{x : x \in U \text{ and } x \notin A\}$

e.g If $U = \{1, 2, 3, 4, 5\}$ & $A = \{1, 3, 5\}$

Then $A' = \{2, 4\}$

Venn diagram



A'

(4) <u>Difference</u> If A and B are any two sets then the set of elements that belong to A but do not belong to B is called the difference of A and B or relative component of B with respect to A and is denoted by $A-B$ or $A \setminus B$

$$A-B = \{x : x \in A \text{ and } x \notin B\}$$

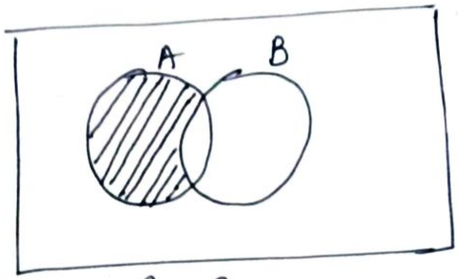
<u>e.g.</u> If $A = \{1,2,3\}$

$B = \{1,3,5,7\}$

Then $A-B = \{2\}$

& $B-A = \{5,7\}$

<u>Venn Diagram</u>



$A-B$

(5) <u>Symmetric difference</u>

If A and B are any two sets, The set of elements that belong to A or B, but not both is called the symmetric difference of A and B and is denoted by $A \oplus B$ or $A \Delta B$ or $A + B$.

$$A \oplus B = (A-B) \cup (B-A)$$

<u>e.g.</u> If $A = \{1,2,3,4\}$

& $B = \{3,4,5,6\}$

Then

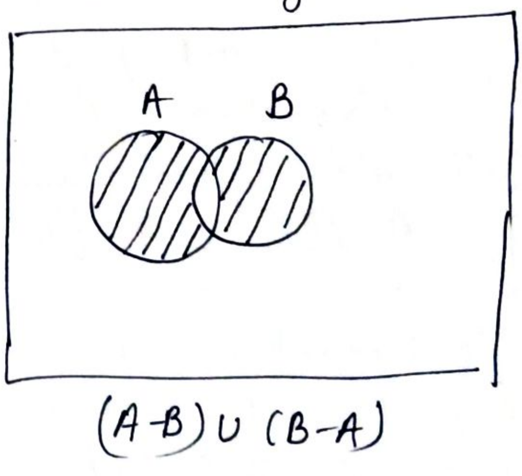
$A-B = \{1,2\}$ & $B-A = \{5,6\}$

$\therefore A \oplus B = (A-B) \cup (B-A)$

$= \{1,2\} \cup \{5,6\}$

$= \{1,2,5,6\}$

Venn Diagram



$(A-B) \cup (B-A)$

The Algebraic Law of set Theory

S. No	Name of the Law	Rule/Identity
1.	Commutative Law	(a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$
2.	Associative Law	(a) $A \cup (B \cup C) = (A \cup B) \cup C$ (b) $A \cap (B \cap C) = (A \cap B) \cap C$
3.	Distributive Law	(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4.	Identity Law	(a) $A \cup \phi = A$ (b) $A \cap U = A$
5.	Idempotent Law	(a) $A \cup A = A$ (b) $A \cap A = A$
6.	Dominant Law	(a) $A \cup U = U$ (b) $A \cap \phi = \phi$
7.	Complement Law (or Inverse Law)	(a) $A \cup \overline{A} = U$ (b) $A \cap \overline{A} = \phi$
8.	Double Complement Law (or Involution Law)	$\overline{\overline{A}} = A$
9.	Demorgan's Law	(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$
10.	Absorption Law	(a) $A \cup (A \cap B) = A$ (b) $A \cap (A \cup B) = A$

Ques 2.1 Prove that $(A-C) \cap (C-B) = \phi$ analytically, where A, B & C are sets, verify graphically.

Solution L.H.S $(A-C) \cap (C-B)$

$= \{x : x \in (A-C) \text{ and } x \in (C-B)\}$

$= \{x : x \in A \text{ and } x \notin C \text{ and } x \in C \text{ and } x \notin B\}$

$= \{x : x \in A \text{ and } (x \notin C \text{ and } x \in C) \text{ and } x \notin B\}$

$= \{x : x \in A \text{ and } (x \in \bar{C} \text{ and } x \in C) \text{ and } x \in \bar{B}\}$

$= \{x : x \in A \text{ and } (x \in \bar{C} \cap C) \text{ and } x \in \bar{B}\}$

$= \{x : x \in A \text{ and } x \in \phi \text{ and } x \in \bar{B}\}$

$= \{x : (x \in A \text{ and } x \in \phi) \text{ and } x \in \bar{B}\}$

$= \{x : x \in (A \cap \phi) \text{ and } x \in \bar{B}\}$

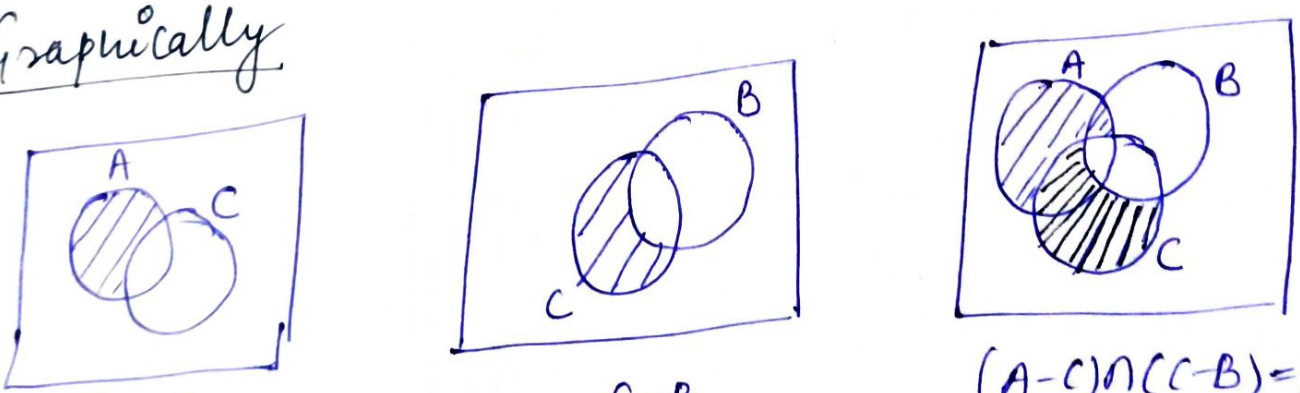
$= \{x : x \in \phi \text{ and } x \in \bar{B}\}$

$= \{x : x \in \phi \cap \bar{B}\}$

$= \{x : x \in \phi\}$

$= \phi$

Graphically



$A-C$ $\quad$ $C-B$ $\quad$ $(A-C) \cap (C-B) = \phi$

e.g 2.2 If A, B and C are sets, prove both analytically and graphically, that $A-(B\cap C) = (A-B)\cup(A-C)$

Solution Analytically

L.H.S $A-(B\cap C) = \{x \in A \text{ and } x \notin (B\cap C)\}$

$= \{x \in A \text{ and } (x \notin B \text{ or } x \notin C)\}$

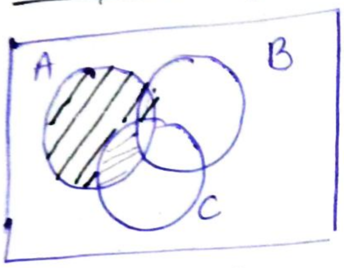
$= \{(x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)\}$

$= \{x : x \in (A-B) \text{ or } x \in (A-C)\}$

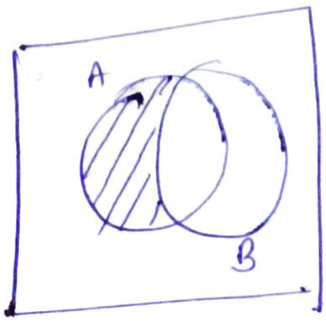
$= \{x : x \in (A-B)\cup(A-C)\}$

$=$ R.H.S

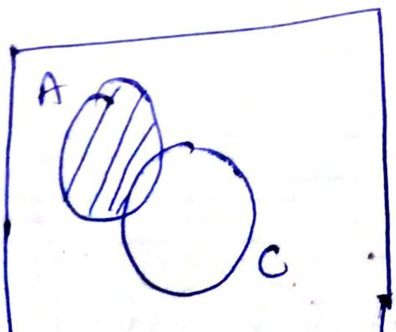
Graphically



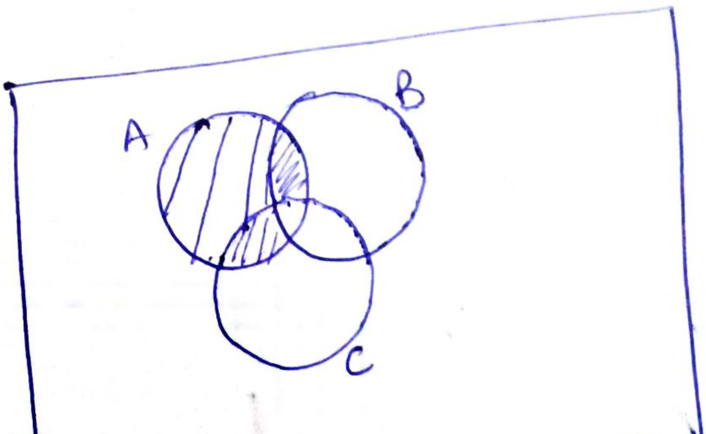
$A-(B\cap C)$



$A-B$



$A-C$



$(A-B)\cup(A-C)$

Example 2.3 If A, B and C are sets, prove, both analytically and graphically, that $A \cap (B - C) = (A \cap B) - (A \cap C)$.

$$
\begin{aligned}
A \cap (B - C) &= \{x \mid x \in A \text{ and } x \in (B - C)\} \\
&= \{x \mid x \in A \text{ and } (x \in B \text{ and } x \notin C)\} \\
&= \{x \mid x \in A \text{ and } (x \in B \text{ and } x \in \bar{C})\} \\
&= \{x \mid x \in (A \cap B \cap \bar{C})\} \\
&= A \cap B \cap \bar{C}
\end{aligned}
$$

Now

$$
\begin{aligned}
(A \cap B) - (A \cap C) &= \{x \mid x \in (A \cap B) \text{ and } x \in \overline{A \cap C}\} \\
&= \{x \mid x \in (A \cap B) \text{ and } x \in (\bar{A} \cup \bar{C}\}, \text{ by De Morgan's law} \\
&= \{x \mid x \in (A \cap B) \text{ and } (x \in \bar{A} \text{ or } x \in \bar{C})\} \\
&= \{x \mid [x \in (A \cap B) \text{ and } x \in \bar{A}] \text{ or } [x \in (A \cap B) \text{ and } x \in \bar{C}]\} \\
&= \{x \mid x \in (A \cap \bar{A} \cap B) \text{ or } x \in (A \cap B \cap \bar{C})\} \\
&= \{x \mid x \in \phi \text{ or } x(A \cap B \cap \bar{C})\} \\
&= \{x \mid x \in A \cap B \cap \bar{C}\} \\
&= A \cap B \cap \bar{C}
\end{aligned}
$$

Hence the result.

(ii) $A \cap B = B \cap A$

Let us use the set builder notation to establish this identity.

$$\begin{aligned} A \cap B &= \{x | x \in A \cap B\} \\ &= \{x | x \in A \text{ and } x \in B\} \\ &= \{x | x \in B \text{ and } x \in A\} \\ &= \{x | x \in B \cap A\} \\ &= B \cap A \end{aligned}$$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$\begin{aligned} A \cup (B \cap C) &= \{x | x \in A \text{ or } x \in (B \cap C)\} \\ &= \{x | x \in A \text{ or } (x \in B \text{ and } x \in C)\} \\ &= \{x | (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\} \\ &= \{x | x \in A \cup B \text{ and } x \in A \cup C\} \\ &= \{x | x \in (A \cup B) \cap (A \cup C)\} \\ &= (A \cup B) \cap (A \cup C) \end{aligned}$$

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let us use Venn diagram to establish this identity.

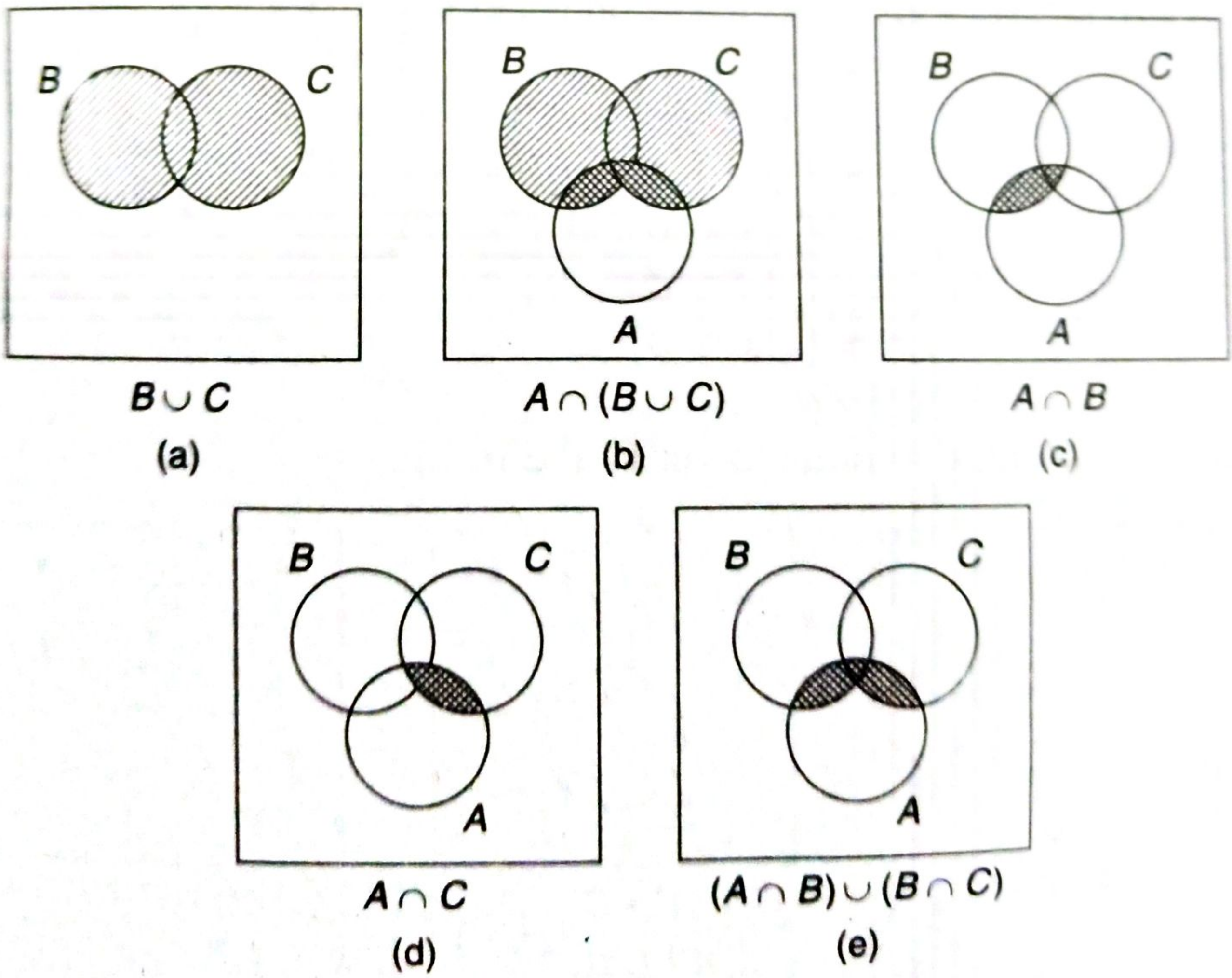


Fig. 2.6

(v) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$$
$$= \{x \mid x \notin A \text{ or } x \notin B\}$$
$$= \{x \mid x \in \bar{A} \text{ or } x \in \bar{B}\}$$
$$= \{x \mid x \in \bar{A} \cup \bar{B}\}$$
$$= \bar{A} \cup \bar{B}$$

(vi) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\therefore \quad \overline{A \cup B} = \bar{A} \cap \bar{B}$

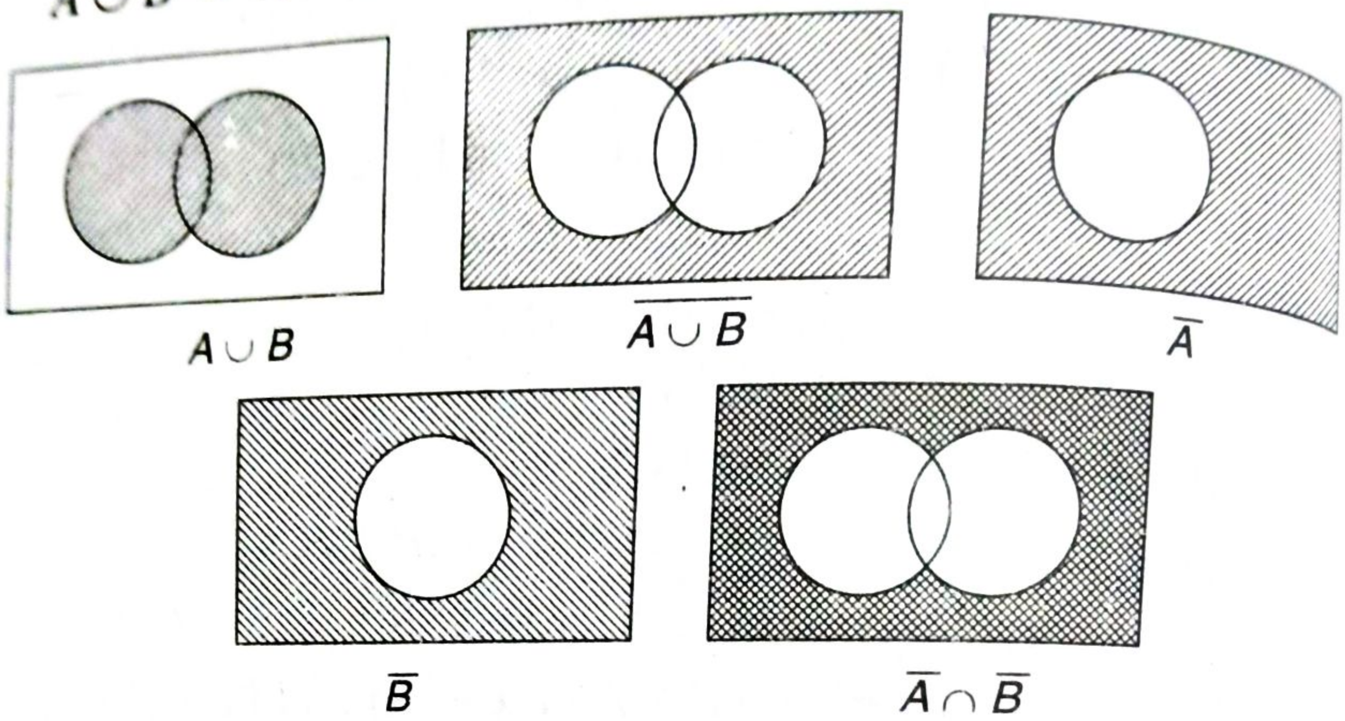


Fig. 2.7

Example 2.4 If A, B and C are sets, prove that

$\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$, using set identities

$$
\begin{aligned}
\text{L.S.} = \overline{A \cup (B \cap C)} &= \bar{A} \cap \overline{(B \cap C)}, \text{ by De Morgan's law} \\
&= \bar{A} \cap (\bar{B} \cup \bar{C}), \text{ by De Morgan's law} \\
&= (\bar{B} \cup \bar{C}) \cap \bar{A}, \text{ by Commutative law} \\
&= (\bar{C} \cup \bar{B}) \cap \bar{A}, \text{ by Commutative law} \\
&= \text{R.S.}
\end{aligned}
$$

$\overline{A \cap C} =$ & $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Example 2.5 If A, B and C are sets, prove algebraically that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$\therefore A \times B =$

$$
\begin{aligned}
A \times (B \cap C) &= \{(x, y) | x \in A \text{ and } y \in (B \cap C)\} \\
&= \{(x, y) | x \in A \text{ and } (y \in B \text{ and } y \in C)\} \\
&= \{(x, y) | (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\} \\
&= \{(x, y) | (x, y) \in A \times B \text{ and } (x, y) \in A \times C\} \\
&= \{(x, y) | (x, y) \in (A \times B) \cap (A \times C)\} \\
&= (A \times B) \cap (A \times C)
\end{aligned}
$$

Example 2.6 If A, B, C and D are sets, prove algebraically that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$. Give an example to support this result.

$$
\begin{aligned}
(A \cap B) \times (C \cap D) &= \{(x, y) | x \in (A \cap B) \text{ and } y \in (C \cap D)\} \\
&= \{(x, y) | (x \in A \text{ and } x \in B) \text{ and } (y \in C \text{ and } y \in D)\} \\
&= \{(x, y) | (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in D)\} \\
&= \{(x, y) | (x, y) \in (A \times C) \text{ and } (x, y) \in (B \times D)\} \\
&= \{(x, y) | (x, y) \in (A \times C) \cap (B \times D)\} \\
&= (A \times C) \cap (B \times D)
\end{aligned}
$$

Example Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{5, 6, 7\}$ and $D = \{6, 7, 8\}$.

Then $A \cap B = \{2, 3\}$ and $C \cap D = \{6, 7\}$

$\therefore \quad (A \cap B) \times (C \cap D) = \{(2, 6), (2, 7), (3, 6), (3, 7)\}$

Now $A \times C = \{(1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7)\}$

$B \times D = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (3, 8), (4, 6), (4, 7), (4, 8)\}$

$\therefore \quad (A \times C) \cap (B \times D) = \{(2, 6), (2, 7), (3, 6), (3, 7)\}$

Hence $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

Example 2.7 Use Venn diagram to prove that $\oplus$ is an associative operation, viz., $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Instead of shading or hatching the regions in the Venn diagram, let us label the various regions as follows:

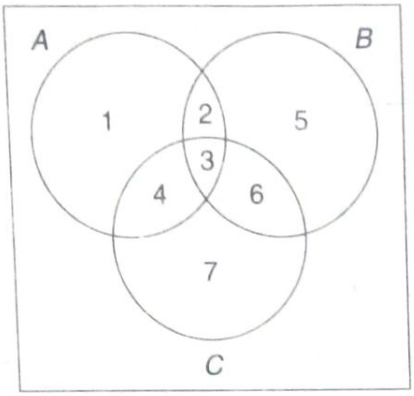


Fig. 2.11

Set A consists of the points in the regions labeled 1, 2, 3, 4; set B consists of the points in the region labeled 2, 3, 5, 6; set C consists of the points in the region labeled 3, 4, 6, 7.

Now $\quad A \oplus B = (A \cup B) - (A \cap B)$

$= \{R_1, R_2, R_3, R_4, R_5, R_6\} - \{R_2, R_3\}$,

where R_i represents the region labeled i

$= \{R_1, R_4, R_5, R_6\}$

$(A \oplus B) \oplus C = \{R_1, R_3, R_4, R_5, R_6, R_7\} - \{R_4, R_6\}$

$= \{R_1, R_3, R_5, R_7\}$

Now $\quad B \oplus C = \{R_2, R_3, R_4, R_5, R_6, R_7\} - \{R_3, R_6\}$

$= \{R_2, R_4, R_5, R_7\}$

$A \oplus (B \oplus C) = \{R_1, R_2, R_3, R_4, R_5, R_7\} - \{R_2, R_4\}$

$= \{R_1, R_3, R_5, R_7\}$

Hence $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

Example 2.8 Use Venn diagram to prove that $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$, where A, B, C are sets.

Using the same assumptions about A, B, and C and the Fig. 2.11 in the Example (8), we have $A \oplus B = \{R_1, R_4, R_5, R_6\}$.

$$\begin{aligned}
(A \oplus B) \times C &= \{R_1, R_4, R_5, R_6\} \times \{R_3, R_4, R_6, R_7\} \\
&= \{R_1 \times R_3, R_1 \times R_4, \ldots, R_6 \times R_7\} \\
A \times C &= \{R_1, R_2, R_3, R_4\} \times \{R_3, R_4, R_6, R_7] \\
&= \{R \times R_3, R_1 \times R_4, \ldots, R_4 \times R_7\} \\
B \times C &= \{R_2, R_3, R_5, R_6\} \times \{R_3, R_4, R_6, R_7\}
\end{aligned}$$

It is easily verified that

$$\begin{aligned}
(A \oplus B) \times C &= (A \times C) \oplus (B \times C) \\
&= \{(R_1 \times R_i), (R_4 \times R_i), (R_5 \times R_i), (R_6 \times R_i)\}
\end{aligned}$$

where $\qquad i = 3, 4, 6, 7$

Example 2.9 Simplify the following sets, using set identities:

(a) $\bar{A} \cup \bar{B} \cup (A \cap B \cap \bar{C})$

(b) $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))]$

(a) $\bar{A} \cup \bar{B} \cup (A \cap B \cap \bar{C}) = (\overline{A \cap B}) \cup [(A \cap B) \cap \bar{C}]$, by De Morgan's law

$= [(\overline{A \cap B}) \cup (A \cap B)] \cap [\overline{A \cap B} \cup \bar{C}]$, by distributive law

$= U \cap \overline{A \cap B} \cup \bar{C}$, by inverse law

$= \overline{A \cap B} \cup \bar{C}$, by identity law

$= \bar{A} \cup \bar{B} \cup \bar{C}$, by De Morgan's law

(b) $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))]$

$= (A \cap B) \cup [B \cap \{C \cap (D \cup \bar{D})\}]$, by distributive law

$= (A \cap B) \cup [B \cap (C \cap U)]$, by inverse law

$= (A \cap B) \cup [B \cap C]$, by identity law

$= (B \cap A) \cup (B \cap C)$, by commutative law

$= B \cap (A \cup C)$, by distributive law

Hence $A \oplus (B \cap C) \neq (A \oplus B) \ldots$

Example 2.12 Find the sets A and B, if

(a) $A - B = \{1, 3, 7, 11\}$, $B - A = \{2, 6, 8\}$ and $A \cap B = \{4, 9\}$

(b) $A - B = \{1, 2, 4\}$, $B - A = \{7, 8\}$ and $A \cup B = \{1, 2, 4, 5, 7, 8, 9\}$

(a) From the Venn diagram, it is clear that

$$A = \{1, 3, 4, 7, 9, 11\}$$

and $$B = \{2, 4, 6, 8, 9\}$$

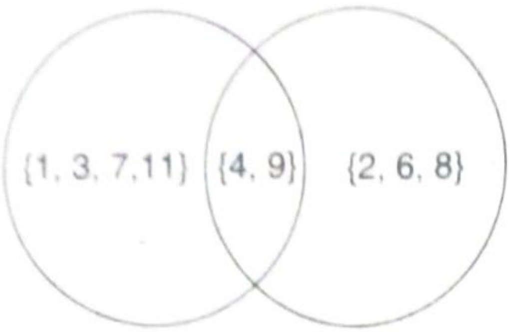


Fig. 2.12

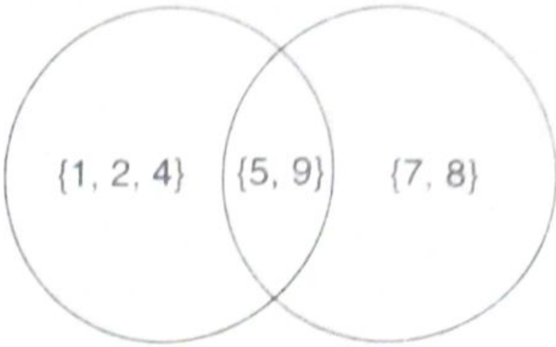


Fig. 2.13

(b) From the Venn diagram, it is clear that

$$A = \{1, 2, 4, 5, 9\}\text{s}$$

and $$B = \{5, 7, 8, 9\}$$

EXERCISE 2(A)

Partition of a set

IF S is a non-empty set a collection of disjoint non empty subset of S whose union S is called a partition of S. In other words, the collection of subsets A_i is a partition of S if and only if

(i) $A_i \neq \phi \ \forall i$

(ii) $A_i \cap A_j = \phi$ for $i \neq j$

(iii) $\bigcup_i A_i = S$ where $\bigcup_i A_i$ represents union of subsets $A_i \ \forall i$

Example let $A = \{1, 2, 3, 4, \ldots 10\}$

$A_1 = \{1, 3, 5\}$, $A_2 = \{2, 4, 6, 8\}$, $A_3 = \{7, 9\}$, $A_4 = \{10\}$

Then $\bigcup_i A_i = A$

therefore A_1, A_2, A_3 and A_4 form a partition.

Example let $A = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 3, 5\}$, $A_2 = \{2, 4\}$ then $\bigcup_i A_i \neq A$

therefore A_1 & A_2 is not a partition since 6 is missing in the union of subsets.

Relation

Let A and B are two non empty sets then a binary relation from A to B is a subset of $A \times B$

ie

$$R \subseteq A \times B$$

If R is a relation from A to B, we use the notation $a R_b$ where $a \in A$, $b \in B$, read it as "a is related by b" by R

If $(a,b) \notin R$ then it is denoted by $a \not R_b$

e.g.

(1) $A = \{1, 2, 3, 4\}$, $B = \{1, 7, 8\}$

R is defined such that "Less than"

Then $R = \{(1,7)(1,8), (2,7)(2,8), (3,7), (3,8), (4,7), (4,8)\}$

(2) Let $A = \{0, 1, 2, 3, 4\}$, $B = \{0, 1, 2, 3\}$ & $a R_b$ iff $a+b=4$

Then $R = \{(1,3)(2,2), (3,1), (4,0)\}$

Domain of $R = \{1, 2, 3, 4\}$

Range of $R = \{0, 1, 2, 3\}$

(3) Let $A = \{1, 2, 3, 4\}$ & $a R_b$ if $a \leq b$, $a, b \in A$

Then $R = \{(1,1), (1,2)(1,3)(1,4), (2,2)(2,3)(2,4), (3,3), (3,4), (4,4)\}$

Here

Domain of R = Range of R

Types of Relations

1) **Universal Relation:-** A relation R on set A is called a universal relation if $R = A \times A$

e.g. If $A = \{1, 2, 3\}$
Then $R = A \times A = \{(1,1)(1,2)(1,3), (2,1), (2,2)(2,3), (3,1).(3,2)(3,3)\}$
is the universal relation on set A.

(2) **Void Relation:-** A Relation R on set A is called a void relation if R is the null set ϕ

e.g. $A = \{3, 4, 5\}$ & aRb iff $a+b > 10$ Then
$R = \{\phi\}$

(3) **Identity Relation:-** A Relation R on set A is called a identity relation if $R = \{(a,a) : a \in A\}$
and it is denoted by I_A.

e.g. If $A = \{1, 2, 3\}$ Then $R = \{(1,1), (2,2), (3,3)\}$ is the identity relation on set A.

(4) **Inverse relation:-** Let R be any relation from A to B Then the inverse of R, denoted by R^{-1} is the relation from B to A.

ie $R^{-1} = \{(b,a) : (a,b) \in R\}$
ie If aRb Then $bR^{-1}a$

e.g. If $A = \{2, 3, 5\}$ $B = \{6, 8, 10\}$ and aRb iff $a \in A$ divides $b \in B$

Then $R = \{(2,6), (2,8), (2,10)\ (3,6)\ (5,10)\}$

Now $R^{-1} = \{(6,2), (8,2), (10,2), (6,3)\ (10,5)\}$

Note that
$D(R) = \{2, 3, 5\} = R(R^{-1})$
$R(R) = \{6, 8, 10\} = D(R^{-1})$

Properties of Relations

Reflexive :- A relation R on set A is said to be reflexive if $aRa \ \forall a \in A$

e.g. If R is the relation on $A = \{1, 2, 3\}$ defined by $(a,b) \in R$ if $a \le b$ where $a, b \in A$

Then $R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

Now R is reflexive since each element of A is related to itself.

Symmetric :- A relation R on set A is said to be symmetric if aRb Then bRa ie if $(a,b) \in R$ Then (b,a) also $\in R$

e.g. If $A = \{1, 2, 3\}$, $B = \{2, 3\}$ & aRb if $a+b=3$

Then $R = \{(1,2), (2,1)\}$

Now R is symmetric

Antisymmetric :- A relation R on set A is said to be Antisymmetric whenever (a,b) & $(b,a) \in R$ ~~but $a \ne b$~~

Then $a = b$

e.g The Relation $R = \{(1,1) (2,2)\}$ is antisymmetric

Transitive :- A Relation R on set A is said to be transitive if whenever aRb & bRc Then aRc

e.g. if $A \subseteq B$ & $B \subseteq C$ Then $A \subseteq C$

Equivalence Relation :- A relation R on set A is called an equivalence relation if R is reflexive, symmetric and transitive.

Ques 2.1 $A = \{0,1,2,3,4\}$, $B = \{0,1,2,3\}$ find aRb
iff
(i) $a > b$
(ii) $\text{lcm}(a,b) = 2$

Solution (i) $R = \{(1,0), (2,0)(2,1), (3,0), (3,1)(3,2), (4,0)(4,1)(4,2)(4,3)\}$

(ii) $R = \{(1,2), (2,1), (2,2)\}$

Ques 2.2 The relation R on set $A = \{1,2,3,4,5\}$ defined by the rule $(a,b) \in R$, if 3 divides $a-b$
(i) List the elements of R & R^{-1}
(ii) find the domain & Range of R & R^{-1}
(iii) List the elements of complement of R

Solution $A \times A = \{(1,1)(1,2)(1,3)(1,4)(1,5)(2,1)(2,2)(2,3)(2,4)(2,5), (3,1)(3,2)(3,3)(3,4)(3,5), (4,1)(4,2), (4,3)(4,4), (4,5), (5,1), (5,2)(5,3)(5,4)(5,5)\}$

Since $(a,b) \in R$, if 3 divides $(a-b)$

$\therefore$ (i) $R = \{(1,1), (1,4), (2,2)(2,5), (3,3), (4,1)(4,4), (5,2)(5,5)\}$

& $R^{-1} = \{(1,1), (4,1)(2,2)(5,2)(3,3), (1,4), (4,4), (2,5), (5,5)\}$

(ii) Domain of $R = \{1,2,3,4,5,\} =$ Range of R^{-1}
Range of $R = \{1,2,3,4,5\} =$ Domain of R^{-1}

(iii) $R' = (A \times A) - R$
$= \{(1,2)(1,3)(1,5), (2,1), (2,3)(2,4)(3,1)(3,2), (3,4)(3,5), (4,2)(4,3)(4,5), (5,1)(5,3)(5,4)\}$

Example 2.1 List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if (i) $a = b$, (ii) $a + b = 4$, (iii) $a > b$, (iv) $a|b$ (viz., a divides b), (v) $gcd(a, b) = 1$ and (vi) $lcm\ (a, b) = 2$.

(i) Since $a \in A$ and $b \in B$ and a R b when $a = b$, $R = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$.

(ii) Since a R b if and only if $a + b = 4$, $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$.

(iii) Since a R b, if and only if $a > b$, $R = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)\}$.

(iv) Since a R b, if and only if $a|b$, $R = \{(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)\}$.

Note $\frac{0}{0}$ is indeterminate and so 0 does not divide 0.

(v) Since a R b, if and only if $gcd\ (a, b) = 1$, $R = \{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)\}$.

(vi) Since a R b, if and only if $lcm\ (a, b) = 2$, $R = \{(1, 2), (2, 1), (2, 2)\}$.

Example 2.2 The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is defined by the rule $(a, b) \in R$, if 3 divides $a - b$.

(i) List the elements of R and R^{-1}.

(ii) Find the domain and range of R.

(iii) Find the domain and range of R^{-1}.

(iv) List the elements of the complement of R.

The Cartesian product $A \times A$ consists of $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), \ldots, (2, 5), (3, 1), (3, 2) \ldots, (3, 5), (4, 1), (4, 2), \ldots, (4, 5), (5, 1), (5, 2), \ldots, (5, 5)\}$

(i) Since $(a, b) \in R$, if 3 divides $(a - b)$, $R = \{(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)\}$

R^{-1} (the inverse of R) $= \{(1, 1), (4, 1), (2, 2), (5, 2), (3, 3), (1, 4), (4, 4), (2, 5), (5, 5)\}$

We note that $R^{-1} = R$

(ii) Domain of R = Range of $R = \{1, 2, 3, 4, 5\}$

(iii) Domain of R^{-1} = Range of $R^{-1} = \{1, 2, 3, 4, 5\}$

(iv) R' (the complement of R) = the elements of $A \times A$, that are not in R = $\{(1, 2), (1, 3), (1, 5), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 1), (5, 3), (5, 4)\}$

Example 2.3 If $R = \{(1, 2), (2, 4), (3, 3)\}$ and $S = \{(1, 3), (2, 4), (4, 2)\}$, find (i) $R \cup S$, (ii) $R \cap S$, (iii) $R - S$, (iv) $S - R$, (v) $R \oplus S$. Also verify that dom $(R \cup S)$ = dom$(R) \cup$ dom (S) and range $(R \cap S) \subseteq$ range $(R) \cap$ range (S).

(i) $R \cup S = \{(1, 2), (1, 3), (2, 4), (3, 3), (4, 2)\}$

(ii) $R \cap S = \{(2, 4)\}$

(iii) $R - S = \{(1, 2), (3, 3)\}$

(iv) $S - R = \{(1, 3), (4, 2)\}$

(v) $R \oplus S = (R \cup S) - (R \cap S)$

$= \{(1, 2), (1, 3), (3, 3), (4, 2)\}$

dom $(R) = \{1, 2, 3\}$; dom $(S) = \{1, 2, 4\}$

Now dom $(R) \cup$ dom $(S) = \{1, 2, 3, 4\}$

= domain $(R \cup S)$

Range $(R) = \{2, 3, 4\}$; Range $(S) = \{2, 3, 4\}$

Range $(R \cap S) = \{4\}$

Clearly $\{4\} \subseteq \{2, 3, 4\} \cap \{2, 3, 4\}$

i.e., Range $(R \cap S) \subseteq$ Range $(R) \cap$ Range (S).

e.g. 2.6

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric and/or transitive, where aRb iff (i) $a \neq b$ (ii) $ab \geq 0$

Solution (i) $a \neq b$

Reflexive :- aRa i.e. $a \neq a$ it is not True

$\therefore$ R is not Reflexive

Symmetric :- $aRb \Rightarrow a \neq b$

$\Rightarrow b \neq a$

$\Rightarrow bRa$ R is symmetric

Transitive :- aRb & bRc

i.e. $a \neq b$ & $b \neq c$

It does not imply That $a \neq c$

$\therefore$ R is not Transitive

(ii) $ab \geq 0$

Reflexive $aRa \Rightarrow aa \geq 0$

$\Rightarrow a^2 \geq 0$ which is True

$\therefore$ R is Reflexive

Symmetric

$aRb \Rightarrow ab \geq 0$

$\Rightarrow ba \geq 0$

$\Rightarrow bRa$

$\therefore$ R is Symmetric

Transitive

aRb & $bRc \Rightarrow ab \geq 0$ & $bc \geq 0$

e.g. Consider $(\underset{a}{2}, \underset{b}{0})$ & $(\underset{b}{0}, \underset{c}{-3})$ but $(2, -3) \notin R$

Since $2(-3) < 0$

$\therefore$ R is not Transitive

$R_6 = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3)\}$

Example 2.6 Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric and/or transitive, where a R b if and only if (i) $a \neq b$, (ii) $ab \geq 0$, (iii) $ab \geq 1$, (iv) a is a multiple of b, (v) $a \equiv b$ (mod 7), (vi) $|a - b| = 1$, (vii) $a = b^2$, (viii) $a \geq b^2$.

(i) '$a \neq a$' is not true. Hence R is not reflexive
$a \neq b \Rightarrow b \neq a$. $\therefore$ R is symmetric
$a \neq b$ and $b \neq c$ does not necessarily imply that $a \neq c$. $\therefore$ R is not transitive
Hence R is symmetric only.

(ii) $a^2 \geq 0$. $\therefore$ R is reflexive.
$ab \geq 0 \Rightarrow ba \geq 0$. $\therefore$ R is symmetric.
Consider $(2, 0)$ and $(0, -3)$, that belong to R. But $(2, -3) \notin R$, as $2(-3) < 0$. $\therefore$ R is not transitive.
$\therefore$ R is reflexive, symmetric and not transitive.

(iii) '$a^2 \geq 1$' need not be true, since a may be zero. $\therefore$ R is not reflexive.
$ab \geq 1 \Rightarrow ba \geq 1$ $\therefore$ R is symmetric.
$ab \geq 1$ and $bc \geq 1 \Rightarrow$ all of $a, b, c > 0$ or < 0
If all of $a, b, c > 0$, least a = least b = least $c = 1$
$\therefore$ $ac \geq 1$
If all of $a, b, c < 0$, greatest a = greatest b = greatest $c = -1$
$\therefore$ $ac \geq 1$. Hence R is transitive.
$\therefore$ R is symmetric and transitive.

(iv) a is a multiple of a. $\therefore$ R is reflexive. If a is a multiple of b, b is not a multiple of a in general. But if a is a multiple of b and b is a multiple of a, then $a = b$.
$\therefore$ R is antisymmetric.
When a is a multiple of b and b is a multiple of c, then a is a multiple of c.
$\therefore$ R is transitive.
Thus R is reflexive, antisymmetric and transitive.

(v) $(a - a)$ is a multiple of 7 $\therefore$ R is reflexive. When $(a - b)$ is a multiple of 7, $(b - a)$ is also a multiple of 7. $\therefore$ R is symmetric.
When $(a - b)$ and $(b - c)$ are multiples of 7, $(a - b) + (b - c) = (a - c)$ is also a multiple of 7.
$\therefore$ R is transitive.
Hence R is reflexive, symmetric and transitive.

(vi) $|a - a| \neq 1$. $\therefore$ R is not reflexive
$|a - b| = 1 \Rightarrow |b - a| = 1$. $\therefore$ R is symmetric.
$|a - b| = 1 \Rightarrow a - b = 1$ or -1 (1)
$|b - c| = 1 \Rightarrow b - c = 1$ or -1 (2)

(1) + (2) gives $a - c = \pm 2$ or 0

i.e. $|a - c| = 2$ or 0

i.e. $|a - c| \neq 1$

Hence R is symmetric only.

(vii) '$a = a^2$' is not true for all integers.

$\therefore$ R is not reflexive.

$a = b^2$ and $b = a^2$, for $a = b = 0$ or 1

$\therefore$ R is antisymmetric.

$a = b^2$ and $b = c^2$ does not simply $a = c^2$

$\therefore$ R is not transitive

Hence R is antisymmetric only.

(viii) '$a \geq a^2$' is not true for all integers.

$\therefore$ R is not reflexive.

$a \geq b^2$ and $b \geq a^2$ imply that $a = b$

$\therefore$ R is antisymmetric

When $a \geq b^2$ and $b \geq c^2$, $a \geq c^2$

$\therefore$ R is transitive

Hence R is antisymmetric and transitive.

Example 2.7 Which of the following relations on $\{0, 1, 2, 3\}$ are equivalence relations? Find the properties of an equivalence relation that the others lack.

(a) $R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

(b) $R_2 = \{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)$

(c) $R_3 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

(d) $R_4 = \{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(e) $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

(a) R_1 is reflexive, symmetric and transitive.

$\therefore$ R_1 is an equivalence relation.

(b) R_2 is reflexive

R_2 is symmetric, but not transitive, since $(3, 2)$ and $(2, 0) \in R_2$, but $(3, 0) \notin R_2$

$\therefore$ R_2 is not an equivalence relation.

(c) R_3 is reflexive, symmetric and transitive. $\therefore$ R_3 is an equivalence relation.

(d) R_4 is reflexive and symmetric, but not transitive, since $(1, 3)$ and $(3, 2) \in R_4$, but $(1, 2) \notin R_4$. $\therefore$ R_4 is not an equivalence relation.

(e) R_5 is reflexive, but not symmetric since $(1, 2) \in R$, but $(2, 1) \notin R$. Also R_5 is not transitive, since $(2, 0)$ and $(0, 1) \in R$. but $(2, 1) \notin R$.

$\therefore$ R_5 is not an equivalence relation.

Ques If R is the relation on the set of ordered pairs of two integers s.t. $(a,b),(c,d) \in R$ whenever $ad = bc$, show that R is an equivalence relation

Sol (i) $(a,b) R (a,b)$ since $ab = ba$

$\therefore$ R is reflexive

(ii) If $(a,b) R (c,d) \Rightarrow ad = bc$

$\Rightarrow \cancel{da = cb}$

$cb = da$

$\Rightarrow d$

$(c,d) R (a,b)$

$\therefore$ R is symmetric

(iii) $(a,b) R (c,d) \Rightarrow ad = bc$ — (1)

& $(c,d) R (e,f) \Rightarrow cf = de$ — (2)

(1) × (2)

$\Rightarrow (a\cancel{d})(\cancel{c}f) = (b\cancel{c})(\cancel{d}e)$

$\Rightarrow af = be$

$\therefore (a,b) R (e,f)$

$\therefore$ R is an equivalence relation

Worshall Algorithm

Q Using worshall Algorithm find All Transitive closure of the Relation $R = \{(4,4),(4,10),(6,6),(6,8),(8,10)\}$ on the set $A = \{4,6,8,10\}$

Sol) set $W_0 = M_R =$

$$\begin{array}{c|cccc} & 4 & 6 & 8 & 10 \\ \hline 4 & 1 & 0 & 0 & 1 \\ 6 & 0 & 1 & 1 & 0 \\ 8 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 0 \end{array}$$

1 Compute W_1 (i) Transfer all 1's from W_0 to W_1.

(II) Location of non zero enteries in C_1 : 4

(III) " " " " " " R_1 : 4, 10

Mark entry 1 in the location of (4,4), (4,10) of W_0 If it is not already there.

$\therefore$ $W_1 =$

$$\begin{array}{c|cccc} & 4 & 6 & 8 & 10 \\ \hline 4 & 1 & 0 & 0 & 1 \\ 6 & 0 & 1 & 1 & 0 \\ 8 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 0 \end{array}$$

2. Compute W_2 (i) Transfer all 1's from W_1 to W_2

(II) Location of non zero enteries in C_2: 6

(III) " " " " R_2 : 6, 8

Mark entry 1 in the location of (6,6) & (6,8)

$\therefore$ $W_2 = W_0 =$

$$\begin{array}{c|cccc} & 4 & 6 & 8 & 10 \\ \hline 4 & 1 & 0 & 0 & 1 \\ 6 & 0 & 1 & 1 & 0 \\ 8 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 0 \end{array}$$

3. Compute w_3

(i) Transfer all 1's from w_2 to w_3

(ii) Location of non zero enteries in C_3 : 6

(iii) " " " " " R_3 : 10

Mark entry 1 in the location (6,10)

$$w_3 = \begin{array}{c|cccc} & 4 & 6 & 8 & 10 \\ \hline 4 & 1 & 0 & 0 & 1 \\ 6 & 0 & 1 & 1 & \boxed{1} \\ 8 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 0 \end{array}$$

4. Compute W_4

(i) Transfer all 1's from w_3 to W_4

(ii) Location of non zero enteries in C_4 : 4, 8, 6

(iii) " " " " " R_4 : —

No new entry so $W_4 = w_3$

$$W_4 = \begin{array}{c|cccc} & 4 & 6 & 8 & 10 \\ \hline 4 & 1 & 0 & 0 & 1 \\ 6 & 0 & 1 & 1 & 1 \\ 8 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 0 \end{array}$$

Hence the Transitive Closure of R, $R^+ = \{(4,4), (4,10), (6,6), (6,8), (6,10), (8,10)\}$.

Note ~~tt~~ w solve upto the last column

ie Has 4 values in the set ∴ solve W_4

Representation of Relation by Graphs (Diagraph)

Let R be a relation on a set A. To represent R Graphically each element of A is represented by a point are called nodes or vertices & Draw a directed line from vertex a_i to vertex a_j if $a_i R a_j$. These directed lines are called edges. The obtain diagram is called directed graph or digraph of R

Note: The digraph of R^{-1}, has exactly the same edges of the diagraph of R, but the direction of the edges are reversed

The digraph representing a relation can be used to determine whether the relation has following properties-

① Reflexive:- A relation R is reflexive Iff there is a loop at every vertex.

② Symmetric: A Relation R is said to be symmetric iff for every edge b/w distinct vertices there is an edge in the opposite direction

③ Antisymmetric: A Relation R is said to be Anti-symmetric Iff these are never two edges in opposite direction b/w distinct vertices

$a \rightarrow b$

$a \leftrightarrow b$

④ Transitive: A Relation R is transitive if there a an edge from a vertex a to b & from b to c then there is an edge from a to c

Hasse diagrams for partial orderings: The simplified form of the diagraph of partial ordering on a finite set that contains sufficient information about the partial ordering is called a **Hasse diagram**.

The simplification of diagraph as a Hasse diagram is achieved in three ways:

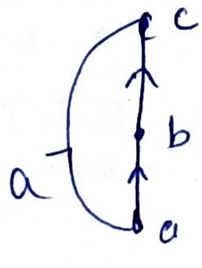
1. Since the partial ordering is a reflexive relation so remove the loops at all vertices.

2. If aRb Then b appears above the element a & element a is connected to b by an edge with arrow upward. Remove all arrows.

becomes

3. Remove all edges whose existence is implied by the Transitive property

$$aRb \ \& \ bRc \Rightarrow aRc$$

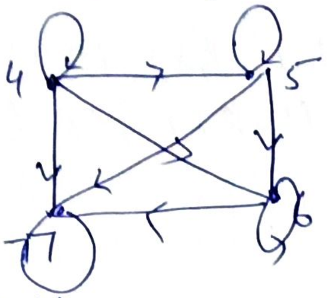


become

[ie No triangle form]

Ques $A = \{4, 5, 6, 7\}$. Let R be the relation $\leq$ on A. Draw the directed graph and Hasse diagram of R

Solution $R = \{(4,4), (4,5), (4,6) (4,7), (5,5) (5,6) (5,7), (6,6), (6,7), (7,7)\}$



Digraph

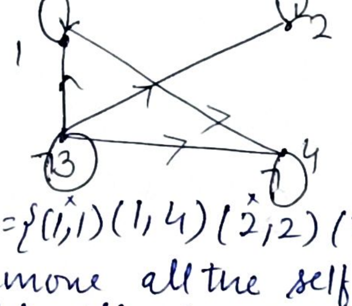
To Draw Hasse digram

1. Delete all the self loop ie $(4,4)\ (5,5), (6,6)\ (7,7)$
2. Delete all edges implied by Transitive property
   ie $(4,6)\ (4,7)\ (5,7)$
3. Arranging all the edges to point upward and deleting all arrows we get.

[Now we have only
$R = \{(4,5)\ (5,6)\ (6,7)\}$

Hasse diagram

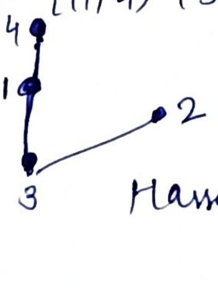
Ques Draw a Hasse diagram from the given directed graph G for a partial order relation on set $A = \{1, 2, 3, 4\}$



Soln $R = \{(1,1) (1,4) (2,2) (3,1) (3,2) (3,3) (3,4) (4,4)\}$

1. Remove all the self loops ie $(1,1)\ (2,2)\ (3,3)\ (4,4)$
2. Delete all edges implied by Transitive property
   ie $(3,4)$

we get $\{(1,4)\ (3,1)\ (3,2)\}$



Hasse diagram

3-1
1-4
3-4

Ques 17 List the ordered pairs in the relation on {1, 2, 3, 4} corresponding to the following matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Also draw the directed gp representing this relation. Use the graph to find if the relation is reflexive, symmetric and/or transitive.

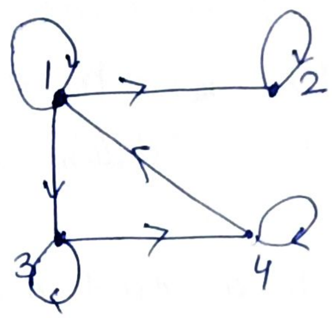
Solution

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Relation

$R = \{(1,1), (1,2), (1,3), (2,2), (3,3) (3,4) (4,1), (4,4)\}$

Directed graph

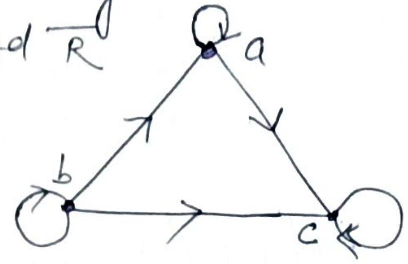


Reflexive Since There is a loop at every vertex of the digraph, The relation is reflexive

Symmetric The Relation is not symmetric
for example There is an edge from 1 to 2
but " " no " " 2 to 1

Transitive The Relation is not Transitive because
There is an edge from 1 to 3
& " " " 3 to 4
but There is no edge from 1 to 4

Ques 18 List the ordered pairs in the relation represented by the digraph in given figure. Also use the graph to prove that the relation is a partial ordering. Also draw the directed graphs representing R^{-1} and $\overline{R}$



Solution

Relation $R = \{(a,a)\ (a,c),\ (b,a),\ (b,b),(b,c),\ (c,c)\}$

Reflexive

Since there is a loop at every vertex of the graph, Therefore the relation is reflexive.

Antisymmetric

Though there are edges from a to c, b to c and b to a & the edges c to a, c to b, a to b are not present in the digraph. Hence the relation is antisymmetric

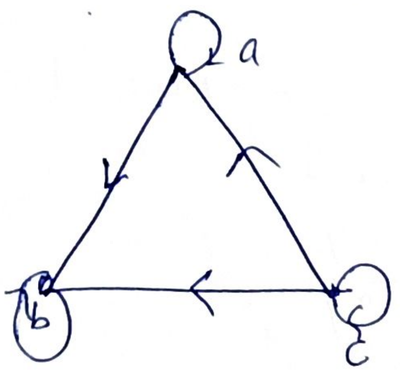
Transitive

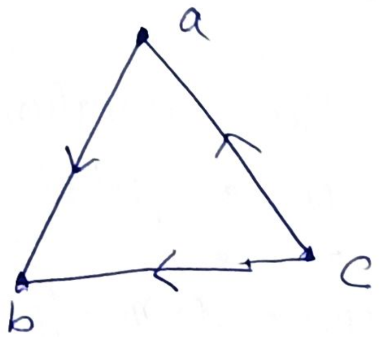
When edges b-a and a-c are present in the digraph, the edges b-c is also present. Hence the relation is transitive.

Since R is Reflexive, Antisymmetric and Transitive Hence the Relation is partial ordering.

$R^{-1} = \{(a,a)\ (c,a)\ (a,b)\ (b,b)\ (c,b)\ (c,c)\}$ (Reversing the direction of edges)

$\overline{R} = \{(a,b)\ (c,a),\ (c,b)\}$





R^{-1} graph

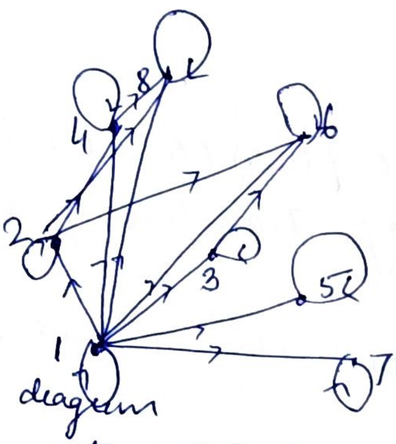
$\overline{R}$ graph

Ques 2.17 List the ordered pairs in the relation on {1, 2, 3, 4}

Ques 2.19 Draw the digraph representing the partial ordering {(a, b) | a divides b} on the set {1, 2, 3, 4, 5, 6, 7, 8}. Reduce it to the Hasse diagram representing the given partial ordering.

Sol^n R = { (1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (1,7) (1,8), (2,2) (2,4) (2,6) (2,8) (3,3) (3,6), (4,4) (4,8) (5,5) (6,6) (7,7) (8,8) }

Digraph

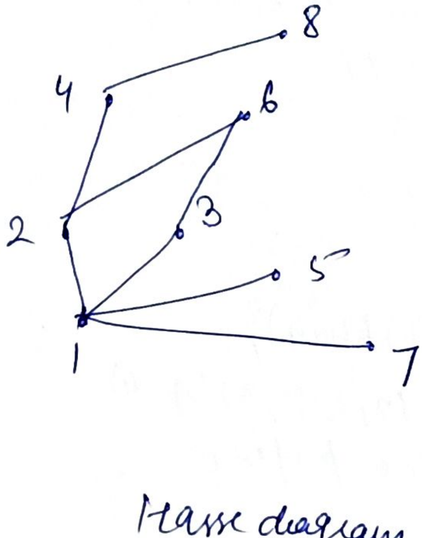


To Draw Hasse diagram

1. Remove all self loops ie (1,1) (2,2) (3,3) - - - (8, 8)
2. Delete all the edges implies Transitive property ie (1,4) (1,6) (1,8) (2,8)

finally we have

R = { (1,2) (1,3) (1,5) (1,7) (2,4) (2,6) (3,6) (4,8) }



Hasse diagram

Example 2.20 Draw the Hasse diagram representing the partial ordering $\{(A, B) | (A \subseteq B)\}$ on the power set $P(S)$, where $S = \{a, b, c\}$. Find the maximal, minimal, greatest and least elements of the poset.

Find also the upper bounds and LUB of the subset $(\{a\}, \{b\}, \{c\})$ and the lower bounds and GLB of the subset $(\{a, b\}, \{a, c\}, \{b, c\})$.

Here $P(S) = (\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

By using the usual procedure (as in the previous example), the Hasse diagram is shown, as shown in Fig. 2.24.

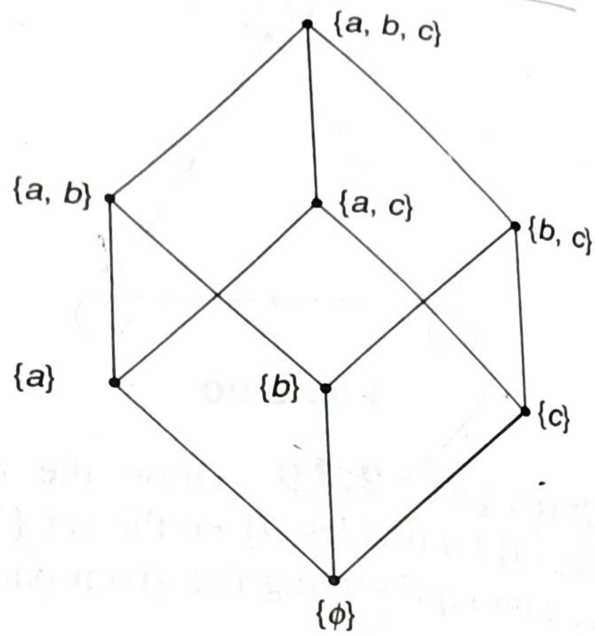


Fig. 2.24

The element $\{a, b, c\}$ does not precede any element of the poset and hence is the only maximal element of the poset.

The element $\{\phi\}$ does not succeed any element of the poset and hence is the only minimal element.

All the elements of the poset are related to $\{a, b, c\}$ and precede it. Hence $\{a, b, c\}$ is the greatest element of the poset.

All the elements of the poset are related to $\{\phi\}$ and succeed it. Hence $\{\phi\}$ is the least element of the poset. The only upper bound of the subset $(\{a\}, \{b\}$, $\{c\})$ is $\{a, b, c\}$ and hence the LUB of the subset.

Function

function is a special kind of relation.
Let X and Y are any two sets. A relation f from X to Y is Called a function if for every $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in f$.

It is represented as $f : X \longrightarrow Y$ or $X \xrightarrow{f} Y$

Sometimes the Terms 'transformation', 'mapping' or 'correspondence' are also used in place of 'function'.
If $y = f(x)$ Then x is called an argument or preimage & y is called the image of x under f or the value of the function f at x.

If $f : X \longrightarrow Y$
Then X is called the domain of f denoted by D_f and Y is called the Co-domain of f.

The set of images of all elements of X is called the range of f denoted by R_f

Clearly $R_f \subseteq Y$

Note:- Every function is a relation but every relation is not a function

eg $\{(1,1)\ (3,1)\ (3,3)\}$

It is a relation but not a function

Representation of function

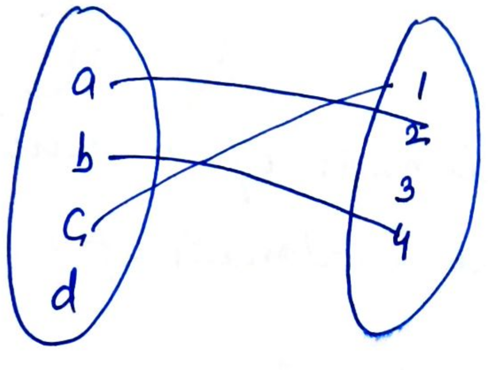
A function can be expressed by means of a Mathematical rule or formula, such as $y = f(x)$ or a relation matrix (since a function is a relation) or a graph.

f can be represented Pictorially as given below:

If $D_f = \{a, b, c, d\}$ & co-domain $= \{1, 2, 3, 4\}$

and $f(a) = 2$, $f(b) = 4$, $f(c) = 1$ and $f(d) = 2$

Then the Pictorial representation of f will be as follows.



Here $R_f = \{1, 2, 4\}$, which is subset of co-domain of f.

TYPES OF FUNCTIONS

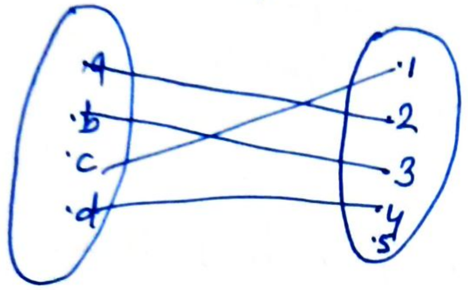
One-one:- A function $f: X \to Y$ is called one-one or injective or injection if distinct elements of X are mapped into distinct elements of Y.

In other words, f is one to one Iff

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

or $f(x_1) = f(x_2)$ whenever $x_1 = x_2$

e.g. $f: X \to Y$



$f(a) = 2,\ f(b) = 3,\ f(c) = 1$

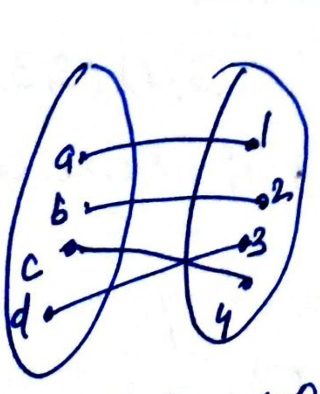
$f(d) = 4$

Onto function and Into function

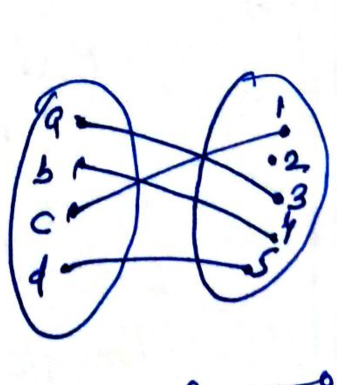
A function $f: X \to Y$ is called onto or surjective or surjection if $R_f = Y$, otherwise it is called into.

In other words, A function f is said to be onto Iff for every element $y \in Y$ There is an element $x \in X$ s.t. $f(x) = y$

e.g.



Onto function



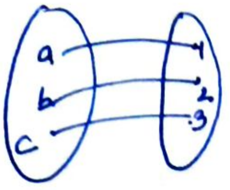
Into function

One to one onto or bijective function

A function $f: X \to Y$ is called one to one onto or bijective or bijection if it is both one to one and onto.

Obviously if $f: X \to Y$ is bijective then X & Y have the same no. of elements.

e.g.

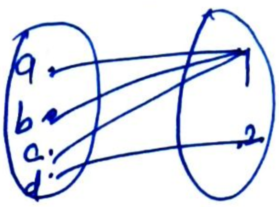


$f(a) = 1,\ f(b) = 2,\ f(c) = 3$

many-one

A function $f: X \to Y$ is said to be many-one if there ~~any~~ are 2 or more elements of X connected with a single element of B.

e.g.

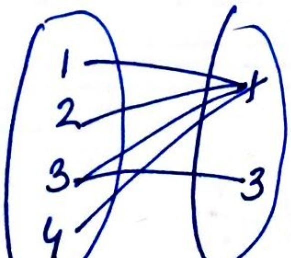


$f(a) = f(b) = f(c) = 1,\ f(d) = 2$

Ques Determine whether the given Relation is function or not

$R = \{(1,1), (2,1)\ (3,1)\ (4,1), (3,3)\}$

Ans:-



It is not a function because one elements cannot have 2 images. Here $(3,1)\ (3,3)$ are 2 pairs in which 3 is connected to 1 & 3

$\therefore$ Image of 3 is not unique

$\therefore$ It is not a function

CLASSIFICATION OF FUNCTIONS

Functions can be classified mainly into two groups (1) Algebraic functions and (2) Transcendental functions.

1. Algebraic Function

A function which consists of a finite number of terms involving integral and/or fractional powers of the independent variable (argument) x, connected by the four operators +, −, ×, and ÷ is called an *algebraic function*. Three particular cases of algebraic functions are the following:

(i) Polynomial function A function of the form $a_0x^n + a_1x^{n-1} + \cdots + a_n$ where n is a positive integer and $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$ is called a *polynomial* in x of degree n.

For example, $2x^4 - 3x^3 + 2x - 4$ is a polynomial in x of degree 4.

(ii) Rational function A function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x) \neq 0$ are polynomials is called a *rational function*.

For example, $F(x) = \dfrac{x^3 - 2x^2 + 3x + 4}{x^2 - 3x + 1}$ is a rational function.

(iii) Irrational function A function involving radicals, viz., fractional powers of polynomials is called an *irrational function*.

For example, $F(x) = \dfrac{\sqrt{x^2 - 1} - x}{\sqrt[3]{x + 1} + x}$ is an *irrational function*.

2. Transcendental Function

A function which is not algebraic is called a *transcendental function*.

For example, the circular (trigonometric) functions, inverse circular functions, exponential functions, logarithmic functions, hyperbolic functions and inverse hyperbolic functions are all transcendental functions.

Apart from these two classes of functions, a few more mathematical functions that occur frequently in computer science are defined as follows.

1. Identity Function

The function $f: A \rightarrow A$ where $f(x) = x$, where $x \in A$ is called the *identity function* on A.

In other words, the identity function is the function that assigns each element of A to itself and is denoted by I_A or simply I. The function I_A is a bijection

2. Floor and Ceiling Functions

If x is a real number, the function that assigns the largest integer that is less than or equal to x is called the *floor function* of x or simply the *floor of* x and denoted by $\lfloor x \rfloor$.

The floor of x is also called the *greatest integer function*.

If $\lfloor x \rfloor = n$, where n is an integer, then $n \le x < n + 1$.

For example, $\lfloor 4.23 \rfloor = 4$, $\lfloor -8.35 \rfloor = -9$, $\lfloor 5 \rfloor = 5$ and $\lfloor -3 \rfloor = -3$.

If x is a real number, the function that assigns the smallest integer that is greater than or equal to x is called the *ceiling function* of x or simply the *ceiling of* x and denoted by $\lceil x \rceil$.

If $\lceil x \rceil = n$, where n is an integer, then $n - 1 < x \le n$.

For example, $\lceil 6.5 \rceil = 7$, $\lceil -4.25 \rceil = -4$, $\lceil 8 \rceil = 8$ and $\lceil -9 \rceil = -9$.

It is obvious that if x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$.

Also it is clear that the floor of x rounds x down, while the ceiling of x rounds x up.

3. Integer Value and Absolute Value Functions

The *integer value* of x, where x is a real number, converts x into an integer by truncating or deleting the fractional part of the number and is denoted by INT (x).

For example, INT $(3.25) = 3$, INT $(-8.54) = -8$ and INT $(6) = 6$.

Note INT $(x) = \lfloor x \rfloor$ or $\lceil x \rceil$, according as x is positive or negative.

The *absolute value* of x, where x is a real number is defined as the greater of x or $-x$ and denoted by ABS (x) or $|x|$.

If x is positive, ABS $(x) = x$; if x is negative, ABS $(x) = -x$ and ABS $(0) = 0$.

Note $|x| = |-x|$, since $|x|$ is positive.

Determine whether each of the following functions is an Injection and/or a Surjection

(i) $f: Z^+ \to Z^+$, defined by $f(x) = x^2+2$

Solution

One-one

Let $f(x_1) = f(x_2)$

$\Rightarrow x_1^2+2 = x_2^2+2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow (x_1^2 - x_2^2) = 0$

$\Rightarrow (x_1-x_2)(x_1+x_2) = 0$

$\Rightarrow x_1 = x_2 \quad (\because x_1+x_2 \neq 0$ as $x_1, x_2 \in Z^+)$

$\therefore$ f is one one

Onto

$y = f(x) = x^2+2$

$x^2 = y-2$

$x = \sqrt{y-2}$

when $y=1$, $x = \sqrt{(-1)} \notin Z^+$

$\therefore f(x)$ is not surjective

(ii) $f: R \to R$ defined by $f(x) = -4x^2+12x-9$

$f(x) = -4x^2+12x-9 = -(2x-3)^2$

One-one

$f(x) = -(2x-3)^2$

$f(1) = -1 = f(2)$

but $1 \neq 2$

$\therefore f(x)$ is not Injective

Onto:-

$y = f(x) = -(2x-3)^2$

$(2x-3)^2 = -y$

$x = \frac{1}{2}(3 \pm \sqrt{-y}) \notin Z^+$

$\therefore f(x)$ is not surjective

(iii) $f: Z \to Z$, $f(x) = x^2 + 14x - 51$

$f(x_1) = f(x_2)$

$x_1^2 + 14x_1 - 51 = x_2^2 + 14x_2 - 51$

$\Rightarrow (x_1^2 - x_2^2) + 14(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 14) = 0$

$\Rightarrow$ if we assume $x_1 \neq x_2$, then $f(x_1) = f(x_2)$

provided $x_1 + x_2 = -14$

for eg when $x_1 = -10$, $x_2 = -4$

ie $x_1 \neq x_2$

$\therefore$ $f(x_1) = f(x_2) = -91$

$\therefore$ $f(x)$ is not injective

If $y = f(x) = x^2 + 14x - 51$

$y = \underbrace{x^2 + 14x + 49}_{} - 49 - 51$

$y = (x+7)^2 - 100$

$y + 100 = (x+7)^2$

when $y \in Z$ & < 100 (for eg $y = -101$)

Then $x \notin Z$

$\therefore$ $f(x)$ is not onto

$\therefore$ $f(x)$ is neither one one nor onto

Ques

If $f: Z \to N$ is defined by

$$f(x) = \begin{cases} 2x-1, & \text{if } x > 0 \\ -2x & \text{if } x \le 0 \end{cases}$$

(a) Prove that f is one to one and onto

(b) Determine f^{-1}

Solution :- $f: Z \to N$

(a) one-one Let $x_1, x_2 \in Z$ and $f(x_1) = f(x_2)$

Then either $f(x_1)$ and $f(x_2)$ are both odd or both even

If they are both odd, then

$$2x_1 - 1 = 2x_2 - 1$$
$$\Rightarrow x_1 = x_2$$

If they are both even, then

$$-2x_1 = -2x_2$$
$$\Rightarrow x_1 = x_2$$

Thus whenever $f(x_1) = f(x_2)$ we get $x_1 = x_2$

then $f(x)$ is one to one

onto Let $y \in N$, if y is odd, its preimage is $\frac{y+1}{2}$

$$f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y \quad , \quad \text{as } \left(\frac{y+1}{2}\right) > 0$$

If y is even its preimage is $\frac{-y}{2}$

$$f\left(-\frac{y}{2}\right) = -2\left(\frac{-y}{2}\right) = y, \quad \text{as } \frac{-y}{2} \le 0$$

Thus for any $y \in N$, the preimage $\frac{y+1}{2}$ or $\frac{-y}{2} \in Z$

$\therefore$ $f(x)$ is onto

Consequently f is invertible

(b) Let $y = f(x) = \begin{cases} 2x-1, & x > 0 \\ -2x, & x \le 0 \end{cases}$

$\therefore f^{-1}(y) = x = \begin{cases} \frac{y+1}{2} & \text{if } y = 1, 3, 5 \dots \\ \frac{-y}{2} & \text{if } y = 0, 2, 4, 6 \dots \end{cases}$

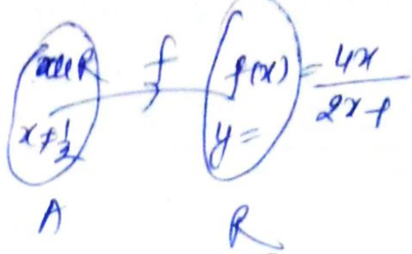
or $f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & \text{if } x = 1, 3, 5 \dots \\ \frac{-x}{2} & \text{if } x = 0, 2, 4, 6 \dots \end{cases}$

eg 4.12 If $A = \{x \in R : x \neq \frac{1}{2}\}$ and $f : A \to R$ is defined by $f(x) = \frac{4x}{2x-1}$

(i) find the range (f)

(ii) Show that f is invertible

(iii) find $dom(f^{-1})$, $range(f^{-1})$ & a formula for f^{-1}



(i) Let $y = f(x) = \frac{4x}{2x-1}$

$2xy - y = 4x$

$x(2y-4) = y$

$x = \frac{y}{2y-4} = \frac{1}{2 - \frac{4}{y}}$

or

$y = \frac{4x}{2x-1}$

$= \frac{4}{2 - \frac{1}{x}}$

when $y = 2$; $x = \infty$

$y \neq 2$, $x \neq \infty$

$y = 2$, $x = \infty$

$y \neq 2$ $x \neq \infty$

$\therefore$ $\boxed{range\ f = \{y \in R : y \neq 2\}}$

(ii) we have to prove that f is invertible

ie f is one one onto

Let $f(a_1) = f(a_2)$

$\frac{4a_1}{2a_1 - 1} = \frac{4a_2}{2a_2 - 1}$

$\Rightarrow$ $8a_1a_2 - 4a_1 = 8a_1a_2 - 4a_2$

$\Rightarrow$ $a_1 = a_2$

$\therefore$ f is one one

Since $y = \frac{4x}{2x-1}$ $y \in R$

$\Rightarrow x = \frac{y}{2y-4}$

Hence for any $y (\neq 2)$ $\exists$ $x \in A$

$\therefore$ f is onto

$\therefore$ f is invertible

(iii) $dom(f^{-1}) = Range(f) = \{y \in R : y \neq 2\}$

$range(f^{-1}) = dom(f) = \{x \in R : x \neq \frac{1}{2}\}$

formula for f^{-1}

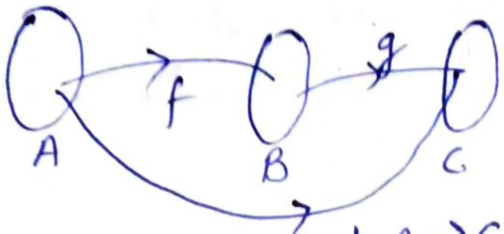
$$f(x) = y = \frac{4x}{2x-1}$$

$$f^{-1}(y) = x = \frac{y}{2y-4}$$

$$\boxed{\therefore f^{-1}(x) = \frac{x}{2x-4}}$$

Composition of functions: If $f: A \to B$ and $g: B \to C$ then the composition of f and g is a new function from A to C denoted by $g \circ f$, is given by

$$(g \circ f)x = g\{f(x)\} \quad \forall\, x \in A$$



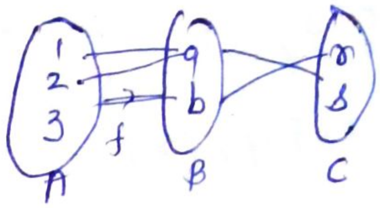
$g \circ f : A \to C$

eg: Let

$A = \{1,2,3\}$, $B = \{a,b\}$ & $C = \{r, s\}$

& $f: A \to B$, defined by $f(1) = a$, $f(2) = a$, $f(3) = b$

& $g: B \to C$ defined by $g(a) = s$, $g(b) = r$



$g \circ f : A \to C$

$g \circ f(1) = g[f(1)] = g(a) = s$

$g \circ f(2) = g[f(2)] = g(a) = s$

$g \circ f(3) = g[f(3)] = g(b) = r$

eg: $f: R \to R$ & $g: R \to R$ defined by

$f(x) = x+2 \;\forall\, x \in R$ & $g(x) = x^2 \;\forall\, x \in R$

Then $g \circ f(x) = g[f(x)] = g[(x+2)] = (x+2)^2 = x^2+4x+4$

& $f \circ g(x) = f[g(x)] = f(x^2) = x^2+2$

$\therefore \quad f \circ g \neq g \circ f$

& $g \circ f(1) = 1^2+4+4 = 9$,

$f \circ g(1) = 3$

(Associative Law) property.

Composition of function is Associative, viz; if $f: A \to B$, $g: B \to C$ & $h: C \to D$ are function then
$$h \circ (g \circ f) = (h \circ g) \circ f$$

Pf

Since $f: A \to B$, $g: B \to C$ Then $g \circ f : A \to C$

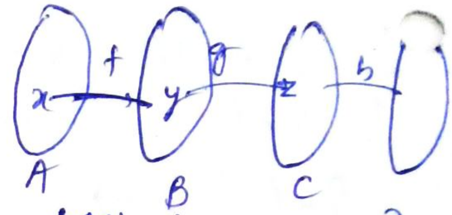
Since $g \circ f : A \to C$ & $h: C \to D$ Then $h \circ (g \circ f) : A \to D$

Now $g: B \to C$ & $h: C \to D$ Then $(h \circ g): B \to D$

Since $(h \circ g): B \to D$ and $f: A \to B$ Then $(h \circ g) \circ f : A \to D$

Let $x \in A$, $y \in B$, $z \in C$

so that $y = f(x)$, $g(y) = z$



$f(x) = y$, $g(y) = z$

L.H.S

$$(g \circ f)(x) = g[f(x)] = g[y] = z$$
$$h \circ (g \circ f)(x) = h(z)$$

R.H.S $$[(h \circ g) \circ f](x) = (h \circ g)[f(x)] = (h \circ g)(y)$$
$$= h[g(y)]$$
$$= h(z)$$

L.H.S = R.H.S

$$\therefore h \circ (g \circ f) = (h \circ g) \circ f$$

Q If $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 8, 9\}$ and the (19)
functions $f: A \to B$ and $g: A \to A$ are defined by
$f = \{(1,8), (3,9), (4,3), (2,1), (5,2)\}$ and
$g = \{(1,2), (3,1), (2,2), (4,3), (5,2)\}$
find fog, gof, fof and gog if they exist.

$(fog)(1) = f(g(1)) = f(2) = 1$
$(fog)(2) = f(g(2)) = f(2) = 1$
$(fog)(3) = f(g(3)) = f(1) = 8$
$(fog)(4) = f(g(4)) = f(3) = 9$
$(fog)(5) = f(g(5)) = f(2) = 1$

Now $\text{rang}(f) = \{1, 2, 3, 8, 9\}$
$\text{dom}(g) = \{1, 2, 3, 4, 5\}$
$\text{range}(f) \not\subseteq \text{dom}(g)$
Hence gof is not defined.
Again $\text{range}(f) = \{1, 2, 3, 8, 9\} \not\subseteq \text{dom}(f) = \{1, 2, 3, 4, 5\}$
Hence fof is not defined.
$\text{range}(g) = \{1, 2, 3\} \subseteq \text{dom}(g) = \{1, 2, 3, 4, 5\}$
Hence gog is defined
Now $(gog)(1) = g[g(1)] = = g(2) = 2$
Hence $gog = \{(1,2), (2,2), (3,2), (4,1), (5,2)\}$

Q. If $S = \{1,2,3,4,5\}$ and if the functions $f, g, h : S \to S$ are given by

$f = \{(1,2),(2,1),(3,4),(4,5),(5,3)\}$

$g = \{(1,3),(2,5),(3,1),(4,2),(5,4)\}$

$h = \{(1,2),(2,2),(3,4),(4,3),(5,1)\}$

(a) Verify $f \circ g = g \circ f$

(b) Explain why f and g have inverses but h does not

(c) Find f^{-1} and g^{-1}

(d) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

Solⁿ

$(f \circ g)(1) = f(g(1)) = f(3) = 4$

$(f \circ g)(2) = f(g(2)) = f(5) = 3$

$(f \circ g)(3) = f(g(3)) = f(1) = 2$

$(f \circ g)(4) = f(g(4)) = f(2) = 1$

$(f \circ g)(5) = f(g(5)) = f(4) = 5$

$f \circ g = [(1,4),(2,3),(3,2)\ (4,1),(5,5)]$ — (1)

Similarly

$g \circ f = [(1,5),(2,3),(3,2),(4,4),(5,1)]$ — (2)

Hence from (1) & (2)

$f \circ g \neq g \circ f$

(b) Both f and g are one-one and onto

Hence they are invertiable

$h(1) = h(2) = 2$ but $1 \neq 2$

$\therefore h$ is not one-to one.

Also range $(h) = [1,2,3,4] \neq S$

h is also not ~~one~~ onto

Hence the Inverse of h does not exists

(c) $f^{-1} = [(2,1), (1,2), (4,3), (5,4), (3,5)]$ — (3)

It is easily verified that

$f \circ f^{-1} = f^{-1} \circ f = [(1,1), (2,2), (3,3), (4,4), (5,5)]$

Similarly

$g^{-1} = [(3,1), (5,2), (1,3), (2,4), (4,5)]$ — (4)

(d) from (1)

$(f \circ g)^{-1} = [(4,1), (3,2), (2,3), (1,4), (5,5)]$ — (5)

from (3) & (4)

$g^{-1} \circ f^{-1} = \{ (2,3), (1,4), (4,1), (5,5), (3,2) \}$ — (6)

Again from (3) & (4)

$f^{-1} \circ g^{-1} = \{ (3,2), (5,1), (1,5), (2,3), (4,4) \}$ — (7)

from (5), (6), (7)

$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

Q If $f, g, h : R \to R$ are defined by $f(x) = x^3 - 4x$,
$g(x) = \frac{1}{x^2+1}$ and $h(x) = x^4$, find $[(f \circ g) \circ h](x)$ and $[f \circ (g \circ h)](x)$,
check if they are equal.

$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2+1}\right) = (x^2+1)^{-3} - 4(x^2+1)^{-1}$

$h(x) = x^4$

$(f \circ g) \circ (h(x)) = (x^8+1)^{-3} - 4(x^8+1)^{-1}$

Now ~~$(g \circ h)(x) =$~~

$(g \circ h)(x) = g[h(x)]$

$= g(x^4)$

$= \frac{1}{x^8+1}$

$[f \circ (g \circ h)(x)] = f\{(x^8+1)^{-1}\}$

$= (x^8+1)^{-3} - 4(x^8+1)^{-1}$

From (1) & (2)

$(f \circ g) \circ h = f \circ (g \circ h)$

Property

When $f : A \to B$ and $g : B \to C$ are functions, then $g \bullet f : A \to C$ is an injection, surjection or bijection according as f and g are injections, surjections or bijections.

Proof

(i) Let $a_1, a_2 \in A$.

Then $(g \bullet f)(a_1) = (g \bullet f)a_2 \Rightarrow g\{f(a_1)\} = g\{f(a_2)\}$

$\Rightarrow f(a_1) = f(a_2)$ ($\because$ g is injective)

$\Rightarrow a_1 = a_2$ ($\because$ f is injective)

$\therefore$ $g \bullet f$ is injective.

(ii) Let $c \in C$.

Since g is onto, there is an element $b \in B$ such that $c = g(b)$.

Since f is onto, there is an element $a \in A$ such that $b = f(a)$.

Now $(g \bullet f)(a) = g\{f(a)\} = g(b) = c$

This means that $g \bullet f : A \to C$ is onto.

(iii) From (i) and (ii), it follows that $g \bullet f$ is bijective when f and g are bijective.

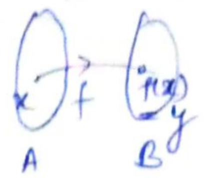
Inverse of a function

If $f: A \to B$ and $g: B \to A$, the the function g is called the inverse of the function f if $g \circ f = I_A$ and $f \circ g = I_B$

Pf Let $x \in A$, $y \in B$

then $(g \circ f)(x) = I_A(x)$

$\Rightarrow g[f(x)] = x$ — (1)



Also $f \circ g(y) = I_B(y)$

$\Rightarrow f[g(y)] = y$ — (2)

From (1) & (2) we see that if $y = f(x)$ Then $x = g(y)$ & vice versa

$\therefore$ $g: B \to A$ is called The inverse of $f: A \to B$

The inverse of f ie g is denoted by f^{-1}

Property

Th. The Inverse of a function f, if exists, is unique.

Pf Let g & h be inverses of f

then by def

$g \circ f = I_A$ and $h \circ f = I_A$

Also $f \circ g = I_B$ and $f \circ h = I_B$

$$\begin{aligned} \therefore\ h &= h \circ I_B \\ &= h \circ (f \circ g) \\ &= (h \circ f) \circ g \\ &= I_A \circ g \\ &= g \end{aligned}$$

$\therefore$ The Inverse of a function is unique

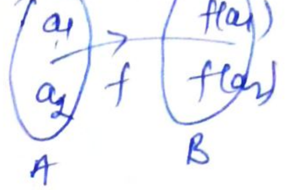
Th. The Necessary and sufficient condition for the function $f: A \to B$ to be investible (viz, for f^{-1} to be exist) is that f is one to one and onto

Pf First we assume That $f: A \to B$ is investible and we will show That f is one-one onto.

Let $f: A \to B$ be investible

Then $\exists$ a function $g: B \to A$

s.t $\quad g \circ f = I_A$ and $f \circ g = I_B$ — (1)



(i) Let $a_1, a_2 \in A$

Let $f(a_1) = f(a_2)$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$ [Since $g: B \to A$ is a function]

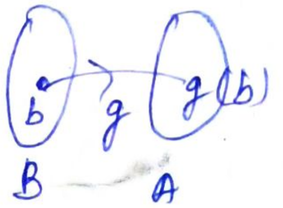
$\Rightarrow g \circ f(a_1) = g \circ f(a_2)$

$\Rightarrow I_A(a_1) = I_A(a_2)$ [from (1)]

$\Rightarrow a_1 = a_2$

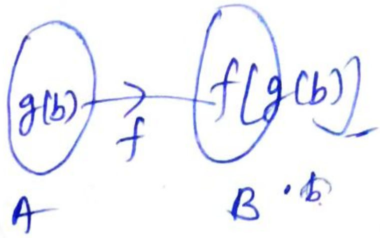
$\therefore$ f is one-one

(ii) Let $b \in B$ Then $g(b) \in A$



Now $b = I_B(b) = f \circ g(b)$ [from (1)]

$= f[g(b)]$



Thus corresponding to every $b \in B$

$\exists$ an element $g(b) \in A$

s.t $f[g(b)] = b$

$\therefore$ f is onto

Conversely we assume That f is one-one onto & we shall show That f is invertible (ie $g \cdot f = I_A$ & $f \cdot g = I_B$ & Uniqueness)

Property

If $f : A \to B$ and $g : B \to C$ are invertible functions, then $g \bullet f : A \to C$ is also invertible and $(g \bullet f)^{-1} = f^{-1} \bullet g^{-1}$

viz., the inverse of the composition of two functions is equal to the composition of the inverses of the functions in the reverse order.

Proof

Since $f : A \to B$ and $g : B \to C$ are invertible, they are bijective.

Hence, $(g \bullet f) : A \to C$ is also bijective (by an earlier property)

$\therefore$ $g \bullet f$ is also invertible. viz., $(g \bullet f)^{-1} : C \to A$ exists

Since $g^{-1} : C \to B$ and $f^{-1} : B \to A$, $f^{-1} \bullet g^{-1} : C \to A$ can be formed.

Thus, both $(g \bullet f)^{-1}$ and $f^{-1} \bullet g^{-1}$ are functions from C to A.

Now for any $a \in A$, let $b = f(a)$ and $c = g(b)$ (1)

$$\therefore \quad (g \bullet f)(a) = g\{f(a)\} = g(b) = c$$

$$\therefore \quad (g \bullet f)^{-1}(c) = a \tag{2}$$

By the assumption (1), $a = f^{-1}(b)$ and $b = g^{-1}(c)$

$$\therefore \quad (f^{-1} \bullet g^{-1})(c) = f^{-1}\{g^{-1}(c)\} = f^{-1}(b) = a \tag{3}$$

From (2) and (3), it follows that

$$(g \bullet f)^{-1} = f^{-1} \bullet g^{-1}, \text{ since } f^{-1}, g^{-1} \text{ and } (g \bullet f)^{-1} \text{ are bijective.}$$