UNIT-5
GRAPH THEORY

Graph:

A Graph $G = G(V, E)$ consists of 2 components

1) The finite set of vertices V, also called points (nodes or junction) and
11) The finite set of edges E, also called lines or arcs connecting pair of vertices.

(OR)

A graph $G = (V, E)$ consits of a non-empty set V, called the set of vertices (nodes, points) and a set E of ordered or unordered pairs of elements of V, called the set of edges, such that there is a mapping from the set E to the set of ordered or unordered pairs of elements of V.

$$G = G(V, E)$$

Graph

Vertex

Edges.

If an edge $e \in E$ is associated with an ordered pair (u, v) or an unordered pair (u, v), where $u, v \in V$, then e is said to connect or join the nodes u and v. The edge e that connects the nodes u and v is said to be incident on each of the nodes. The pair of nodes that are connected by an edge are called adjacent nodes.

A vertex having no edge incident on it is called isolated node. A graph containing only isolated nodes (no edges) is called a null graph.

If in graph $G = (V, E)$, each edge $e \in E$ is associated with an ordered pair of vertices, then G is called a directed graph or digraph.

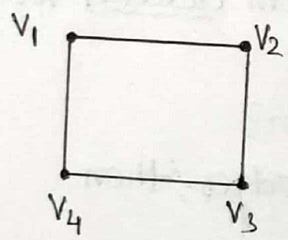
If each edge is associated with an unordered pairs of vertices, then G is called an undirected graph.

An edge of a graph that joins a vertex to itself is called a loop.

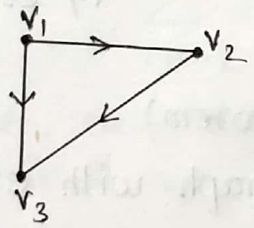
If, in a directed or undirected graph, certain pairs of vertices are joined by more than one edge, such edges are called parallel edges.

A graph which contains some parallel edges is called a multigraph.

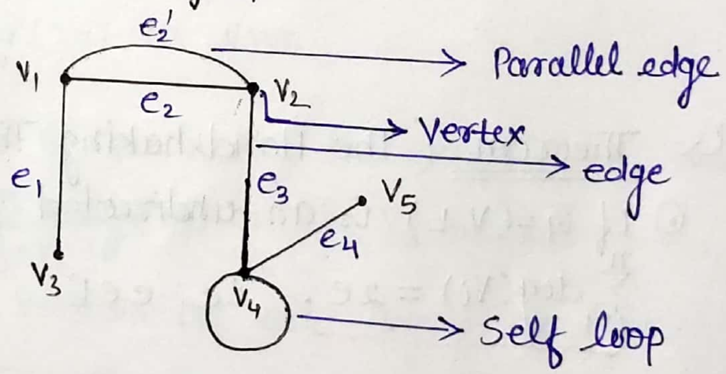
A graph, in which there is only one edge b/w a pair of vertices, is called a simple graph.

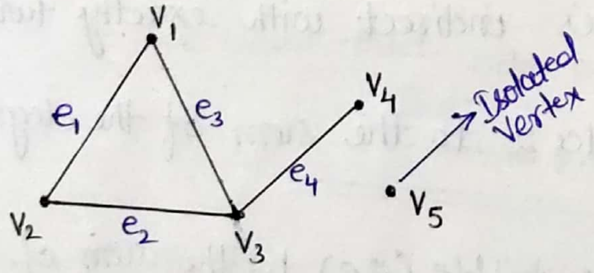


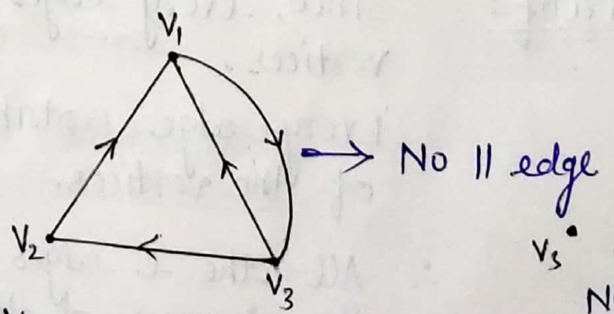
Undirected Graph

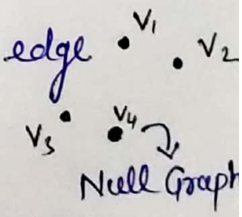


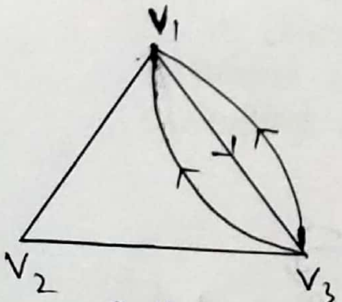
Directed Graph



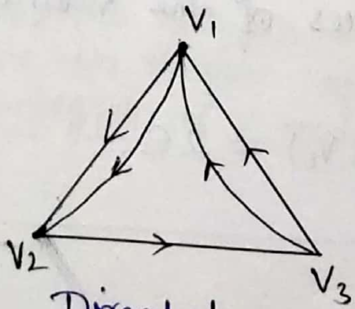




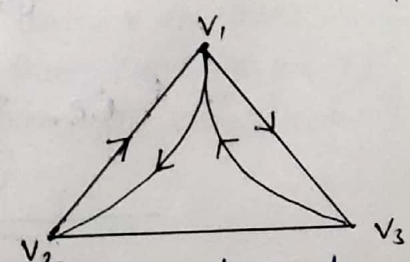




2 || edge



Directed graph with || edges.



Directed graph with distinct edges.

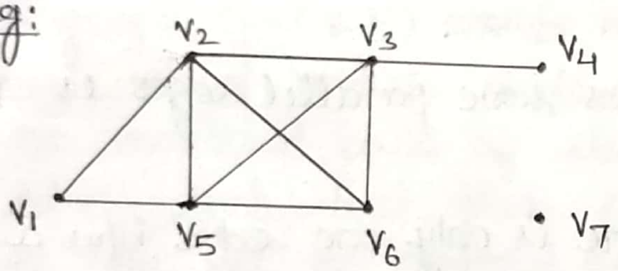
Degree of a Vertex

The degree of a vertex is an undirected graph is the number of edges incident with it, with the exception that a loop at a vertex contributes twice to the degree of that vertex.

The degree of a vertex v is denoted by deg(V). Clearly the degree of an isolated vertex is zero.

If the degree of a vertex is one, it is called a pendant vertex.

eg:



$deg(V_1) = 2$
$deg(V_2) = 4$
$deg(V_3) = 4$
$deg(V_4) = 1$

$d(V_5) = 4$
$d(V_6) = 3$
$d(V_7) = 0$

Here, V_4 is pendant vertex & V_7 is an isolated vertex.

Theorem (The Handshaking Theorem)

If $G = (V, E)$ is an undirected graph with e edges, then $\sum_{i=1}^{n} deg(V_i) = 2e$. i.e $e \in E$

Proof: Since, every edges is indirect with exactly two vertices.

Every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the e edges contribute (2e) to the sum of the degrees of the vertices.

i.e. $$\sum_{i=1}^{n} deg(V_i) = 2e.$$

Proved:

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↳ Theorem

⊙ The number of vertices of odd degree in an undirected graph is even.

Proof:

Let $G = (V, E)$ be the undirected graph.

Let V_1 & V_2 be the sets of vertices of G of even and odd degrees respectively.

Then, by the previous theorem,

$$2e = \sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg(v_j) \quad \text{——— (1)}$$

∵ Each $\deg(v_i)$ is even, $\sum_{v_i \in V_1} \deg(v_i)$ is even.

As the LHS of (1) is even, we get

$$\sum_{v_j \in V_2} \deg(v_j) \text{ is even.}$$

Since, each $\deg(v_j)$ is odd, the number of terms contained in $\sum_{v_j \in V_2} \deg(v_j)$ or in V_2 is even, i.e., the number of vertices of odd degree is even.

Proved:

We know that odd + odd = even

But

odd + odd + odd = odd.

→ Definitions

In a directed graph, the number of edges with v as their terminal vertex (i.e. the number of edges that converge at v) is called the in-degree of v and is denoted as $\deg^{-}(v)$.

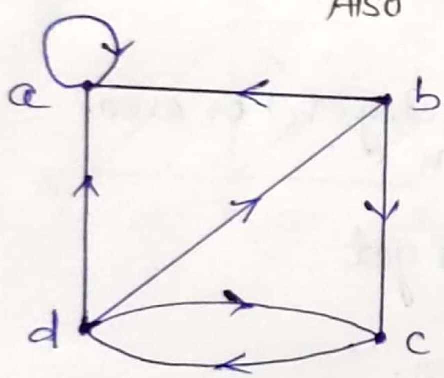
The number of edges with v as their ~~first~~ initial vertex, (i.e., the no. of edges that emanate from v) is called the out-degree of v and is denoted as $deg^+(v)$.

A vertex with zero in degree is called a source and a vertex with zero out-degree is called a sink.

Let us consider the following directed graph.
We note that $deg^-(a) = 3$, $deg^-(b) = 1$, $deg^-(c) = 2$, $deg^-(d) = 1$ and $deg^+(a) = 1$, $deg^+(b) = 2$, $deg^+(c) = 1$, $deg^+(d) = 3$.

Also we ~~k~~ note that $\sum deg^-(v) = \sum deg^+(v)$

$=$ the number of edges $= 7$



This property is true for any directed graph

$G = (V, E)$, viz., $\sum_{v \in V} deg^-(v)$

$$= \sum_{v \in V} deg^+(v) = e.$$

This is obvious, bcoz each edge of the graph converges at one vertex and emanates from one vertex and hence contributes 1 each to the sum of the in-degrees and to the sum of the out-degrees.

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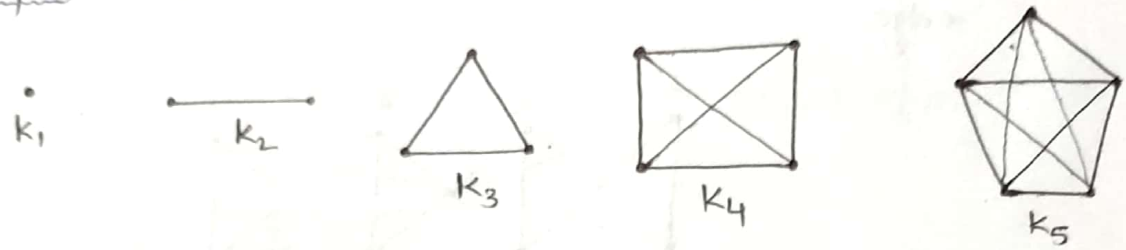
↳ Some Special Simple Graphs

→ Complete Graph

A simple graph, in which there is exactly one ~~degree~~ edge b/w each pair of distinct vertices, is called a complete graph.

eg:



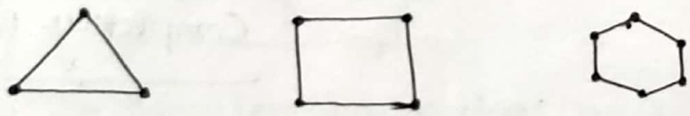
↳ NOTE: The number of edges in K_n is '${}^nC_2$' or $\frac{n(n-1)}{2}$.

→ Regular Graph

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.

If every vertex in a regular graph has degree n, then the graph is called n-regular.

eg:



⟶ 2-regular graphs

⟶ 3-regular graphs

$\boxed{\text{Regular} \subset \text{Simple Graph}}$

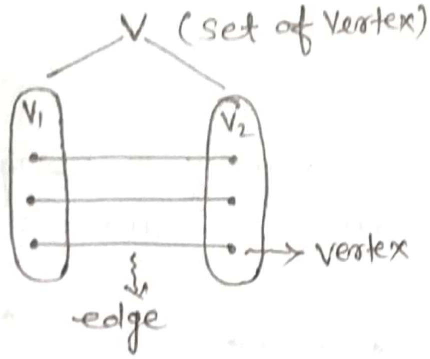
→ Bipartite Graph

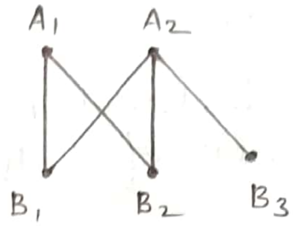
A graph $G=(V,E)$ is called bipartite if its vertex set V can be partitioned into two subsets V_1 & V_2 such that every edge of G connects a vertex in V_1 & a vertex V_2.

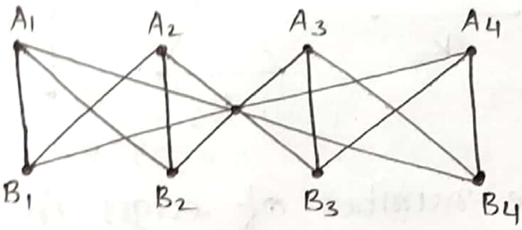
If each vertex of V_1 is connected with every vertex of V_2 by an edge, then G is called a completely bipartite graph.

If V_1 contains m vertices and V_2 contains n vertices, the completely bipartite graph is denoted by $K_{m,n}$.

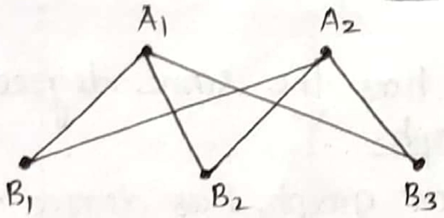
eg:



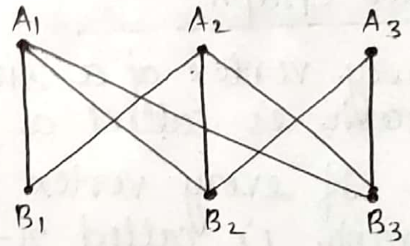




Bipartite Graphs

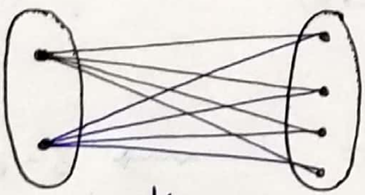


$K_{2,3}$ graph



$K_{3,3}$ graph

Completely Bipartite Graph



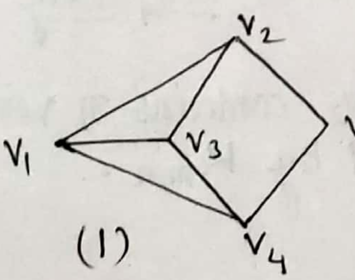
$K_{2,4}$

→ Subgraphs

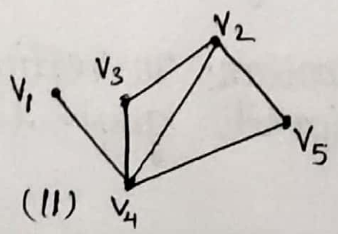
A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$.

If $V' \subset V$ & $E' \subset E$, then H is called a proper subgraph of G.

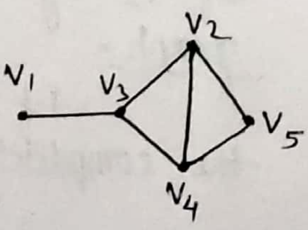
eg:



(I)



(II)



(III)

Any subgraph of a graph G can be obtained by removing certain vertices and edges from G and it is noted that removal of an edge doesn't go with the removal of its adjacent vertices, whereas the removal of a vertex goes with the removal of any edge incident on it.

eg: $A = \{1,2,3,4,5\}$

$B = \{1,2,3,4\} \longrightarrow$ subset of A.

→ **Spanning Subgraph**

A subgraph H of a graph G is said to be spanning subgraph if all the vertices of G are present in the subgraph H.

→ **Isomorphic Graphs**

Two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are said to be isomorphic if $\exists$ a one-one correspondence b/w their vertex sets (V_1 & V_2) & b/w their edge sets (E_1 & E_2).

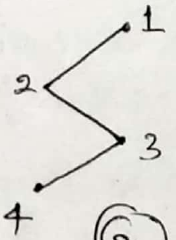
i.e.

I) Same no. of edges

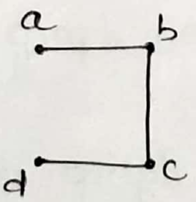
II) ——— vertex

III) The corresponding vertices with the same degree.

eg:



G_1



G_2

no. of vertex = 4,
edge = 3

$d(1) = 1$
$d(2) = 2$
$d(3) = 2$
$d(4) = 1$

no. of vertex = 4,
edge = 3

$d(a) = 1$
$d(b) = 2$
$d(c) = 1$
$d(d) = 2$

$\Rightarrow$

$f(1) = a$
$f(2) = b$
$f(3) = d$
$f(4) = c$

$\because$ Satisfy all condition.

$\therefore$ It is an isomorphic graph.

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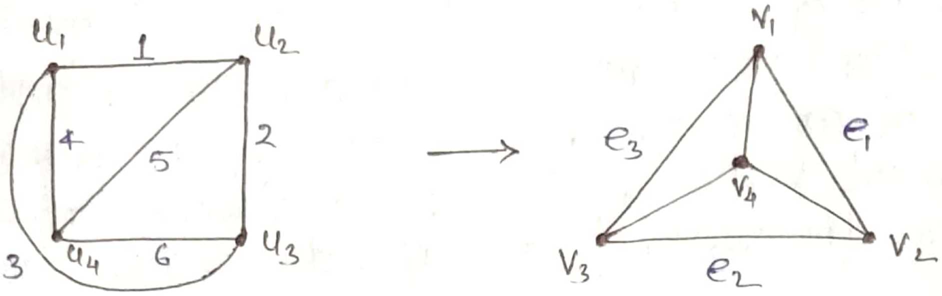
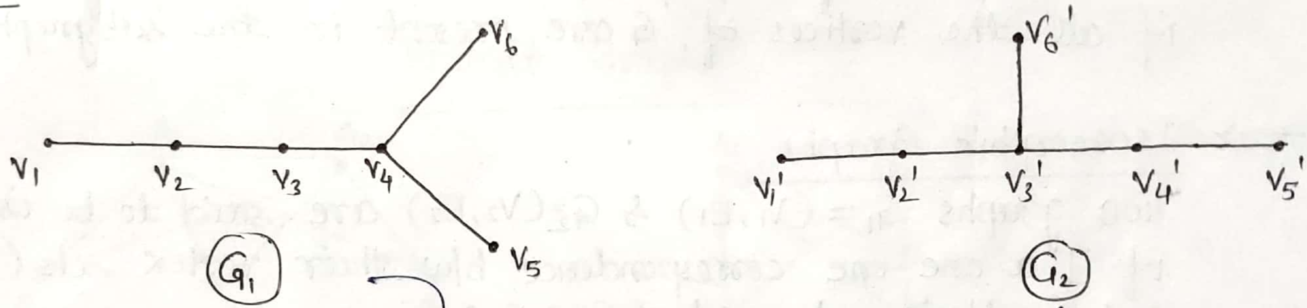
eg:



eg:



इसमें v_2 & v_3 adjacent हैं।
∴ इसमें degree
$\cancel{v_2 \neq v_3}$
$v_2 = v_3$

इसमें v_2' & v_3' adjacent नहीं है।
∴ इसमें degree
$v_2 \neq v_3$.

Correspondence:

$f(v_1) = v_1'$

$f(v_2) = v_2'$

$f(v_3) = v_4'$

$f(v_4) = v_3'$

$f(v_5) = v_5'$

$f(v_6) = v_6'$

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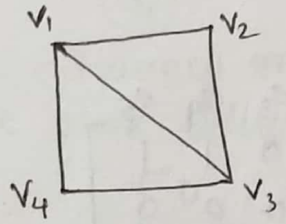
↳ Matrix Representation of Graphs

→ **Adjacency matrix**: Let G be a simple graph with n vertices $v_1, v_2, v_3, ---, v_n$, the matrix A (or A_G) $\equiv [a_{ij}]$.

where, $a_{ij} = \begin{cases} 1 ; \text{ if } v_i v_j \text{ is an edge of } G \\ 0 ; \text{ otherwise.} \end{cases}$

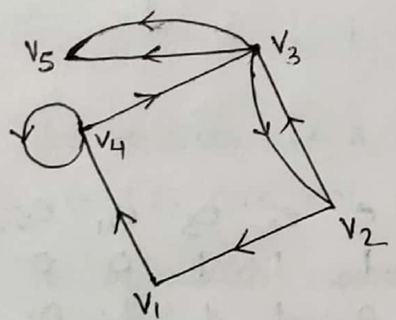
is called adjacency matrix.

eg:



$$a_{ij} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

eg:



$$a_{ij} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

→ **Properties of adjacency matrix**

1. Since, a simple graph has no loops, each diagonal entry of A $\forall i j.$, $a_{ij} = 0$.
2. If G is a simple graph with parallel edges then the a_{ij} is symmetric i.e., $\forall i j.$, $a_{ij} = a_{ji}$
3. $\deg(v_i)$ is equal to the no. of 1's in the i^{th} row & i^{th} column.

→ **Incidence matrix**: If $G = (V, E)$ is an undirected graph with n-vertices $v_1, v_2, ---, v_n$ & ~~edges~~ m edges $e_1, e_2, ---, e_m$, then the $(n \times m)$ matrix $B = [b_{ij}]$

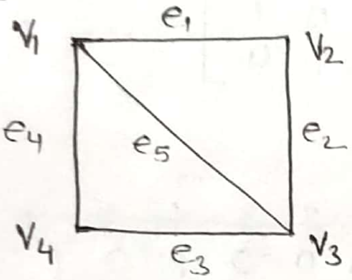
where $b_{ij} = \begin{cases} 1 ; \text{ when edge } e_j \text{ is incident on } v_i \\ 0 ; \text{ otherwise.} \end{cases}$

is called the incidence matrix of G.

→ Properties

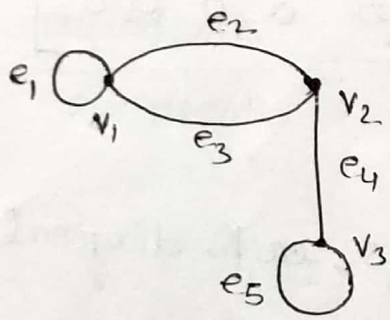
① Each column of B contains exactly 2 unit entries.
② A row with all 0 entries corresponds to an isolated matrix.
③ A row with a single unit entry corresponds to a pendent vertex.
④ $deg(v_i)$ is equal to the no. of 1's in the i^{th} row.

eg:



$$\Rightarrow \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

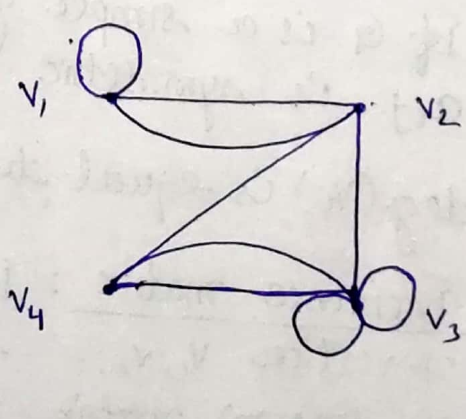
eg:



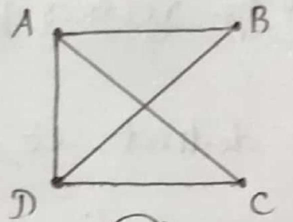
$$\Rightarrow \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

eg:

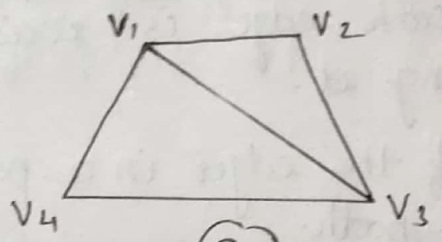
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array} \Rightarrow$$



eg: Establish the isomorphism of the two graphs given in fig. by considering their adjacency matrices.



G_1



G_2

The adjacency matrices A_1 & A_2 of G_1 & G_2 respectively are given below:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Soln: The matrices A_1 & A_2 are not same.

$\therefore$ G_1 & G_2 are not isomorphic.

To establish isomorphism b/w G_1 & G_2 we have to find a permutation matrix P s.t $PA_1P^T = A_2$

Since, A_1 & A_2 are fourth order matrices, P is a 4^{th} order matrix got by permuting the rows of the unit matrix I_4. Thus, there are $\underline{|4} = 24$ diff. forms for P.
It is difficult to find the appropriate P from among the 24 matrices by trial that will satisfy $PA_1P^T = A_2$.

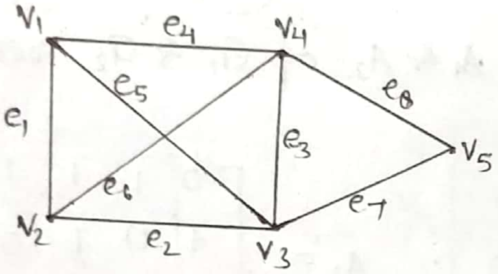
$Deg(A) = 3 = Deg(V_1)$ i.e. I_4 can be taken as first row. ?
$deg(B) = 2 = deg(V_2)$ or $deg(B) = 2 = deg(V_4)$
$deg(C) = 2 = deg(V_4)$ or $deg(C) = 2 = deg(V_2)$
$deg(D) = 3 = deg(V_3)$

? $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Path

- A path in a graph is a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident on the vertices preceding and following it.

If the edges in a path are distinct, it is called a simple path.



$V_1 e_1 V_2 e_2 V_3 e_5 V_1 e_1 V_2$ is a path, since it contains the e_1 twice.

$V_1 e_4 V_4 e_6 V_2 e_2 V_3 e_7 V_5$ is a simple path, as no edge appears more than once.

The number of edges in a path (simple or general) is called the length of the path.

The length of both the paths given above is 4.

Circuit

- If the initial & final vertices of a path (of non-zero length) are same, the path is called a circuit or cycle.

$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_6 V_2 e_1 V_1$ is a circuit of length 5.

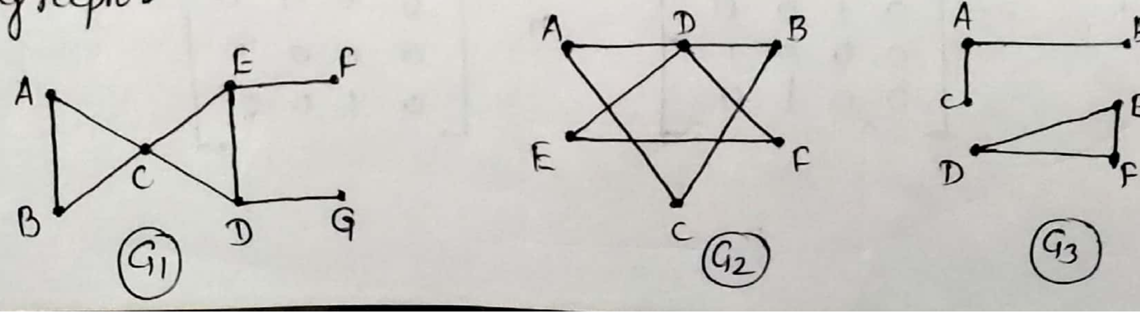
$V_1 e_5 V_3 e_7 V_5 e_8 V_4 e_4 V_1$ is a simple circuit of length 4.

Connected Graph

- A graph G is said to be connected if there exists a path b/w every vertices in G.

A graph which is not connected is called a disconnected graph.

eg:



Clearly a disconnected graph is the union of two or more connected subgraphs, each pair of which has no vertex is common. These disjoint connected subgraphs are called the connected components of the graph.

Theorem

⊙ If a graph G (either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

Proof:

Case (I) Let G be connected.

Let V_1 & V_2 be the only vertices of G which are of odd degree. But we have already proved that the number of odd vertices is even.

Clearly there is a path connecting V_1 & V_2, since G is connected.

Case (II) Let G be disconnected.

Then the components of G are connected. Hence, V_1 & V_2 should belong to the same component of G.

Hence, there is a path b/w V_1 & V_2.

Proved:

Theorem

⊙ The max. no. of edges in a simple disconnected graph G within n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$.

Proof: Let the no. of vertices in the i^{th} component of G be n_i $(n_i \geq 1)$.

Then $n_1 + n_2 + \cdots + n_k = n$ or $\sum_{i=1}^{k} n_i = n$ —— (I)

Hence, $\sum_{i=1}^{k} (n_i - 1) = n - k$

$$\therefore \left\{ \sum_{i=1}^{k} (n_i - 1) \right\}^2 = n^2 - 2nk + k^2$$

i.e., $\sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i \neq j} (n_i - 1)(n_j - 1) = n^2 - 2nk + k^2$ —— (II)

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i.e., $\sum_{i=1}^{k}(n_i-1)^2 \le n^2-2nk+k^2$

[$\because$ the 2nd member in the L.S. of (2) is ≥ 0, as each $n_i \geq 1$]

i.e., $\sum_{i=1}^{k}(n_i^2-2n_i+1) \le n^2-2nk+k^2$

i.e., $\sum_{i=1}^{k} n_i^2 \le n^2-2nk+k^2+2n-k$ ——— (iii)

We know that the max. no. of edges if n vertices in complete graph $= \frac{n(n-1)}{2}$

Now, the max. no. of edges in the ith component of $G = \frac{1}{2}n_i(n_i-1)$

$\therefore$ Max. no. of edges of G $= \frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$

$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n$ from (1)

$\le \frac{1}{2}(n^2-2nk+k^2+2n-k)-\frac{1}{2}n$ from (iii)

$\le \frac{1}{2}(n^2-2nk+k^2+n-k)$

$\le \frac{1}{2}\{(n-k)^2+(n-k)\}$

$\le \frac{1}{2}(n-k)(n-k+1)$

Proved

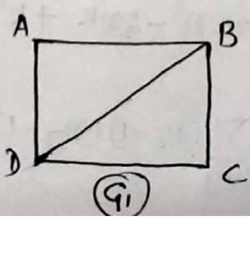
Eulerian And Hamiltonian Graphs

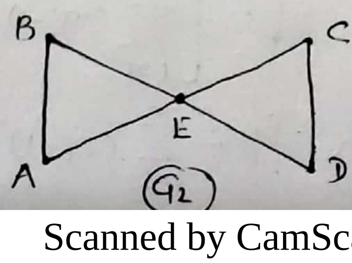
Eulerian Graph : A path of graph G is called an Eulerian path, if it includes each edge of G exactly once.

A ckt of a graph G is called an Eulerian circuit, if it includes each edge of G exactly once.

A graph containing an Eulerian circuit is called an Eulerian graph.

eg:





Graph G_1 contains an Eulerian path b/w B & D namely, B – D – C – B – A – D, since it includes each of the edges exactly once.

Graph G_2 contains an Eulerian ckt, namely, A – E – C – D – E – B – A, since it includes each of the edges exactly once.

G_2 is an Euler graph, as it contains an Eulerian ckt.

Theorem (I)

- A connected graph contains an Euler ckt, if and only if each of its vertices is of even degree.

Theorem (II)

- A connected graph contains an Euler path, iff it has exactly two vertices of odd degree.

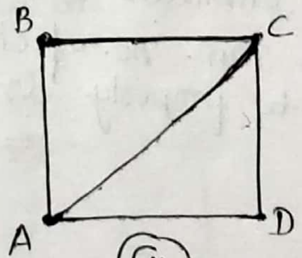
↳ Hamiltonian Path

- A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.

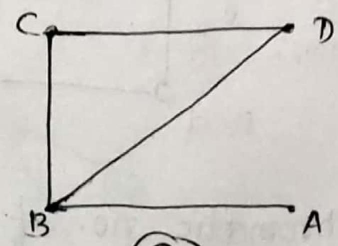
A ckt of a graph G is called a Hamiltonian ckt, if it includes each vertex of G exactly once, except the starting and end vertices which appear twice.

A graph containing a Hamiltonian circuit is called a Hamiltonian graph.

eg:



G_1



G_2

The graph G_1 has a Hamiltonian ckt namely, A – B – C – D – A. In this ckt all the vertices appear once.

The graph G_2 has a Hamiltonian path, namely, A – B – C – D, but not a Hamiltonian ckt.

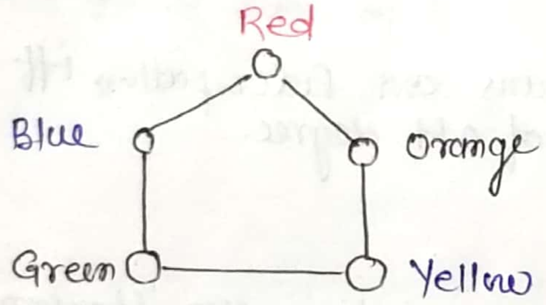
NOTE: A compete graph K_n will always have a Hamiltonian ~~path~~ circuit (ckt), when $n \geq 3$.

→ **Graph Coloring**

⊙ Graph coloring is process of assigning colors to the vertices of a graph s.t no 2 adjacent vertices of it are assigned same color. Such a graph is called as a properly colored grapher vertex colouring.

It ensures that ∃ no edge in the graph whose end vertices are coloured with the same colour.

eg:



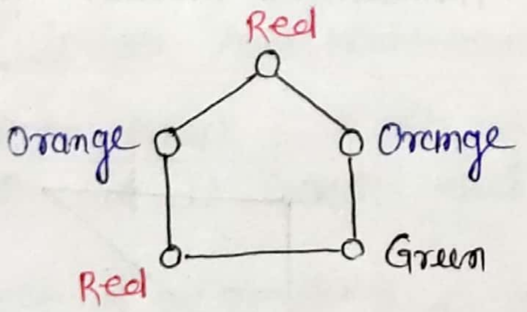
In this graph no 2 vertices are coloured with the same colour. So it is properly coloured graph.

↳ **Chromatic Number**

⊙ Chromatic no. is the min. no. of clrs required to properly clr any graph & it is denoted by $\gamma(G)$.

If graph G is K-chromatic then $\gamma(G) = K$.

eg:



Here, $\gamma(G) = 3$

So, chromatic no. = 3

i.e. min. no. of clrs required to properly clr the vertices = 3.

Chromatic no. of Graphs

① **Cycle Graph**

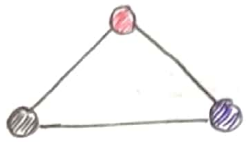
⊙ A simple graph of n vertices ($n \geq 3$) and n edges forming a cycle of length n is called a cycle graph.

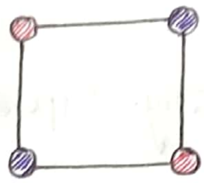
⊙ In a cycle graph, all the vertices are of degree 2.

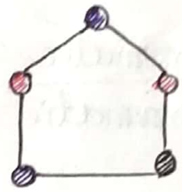
Chromatic no.

1) If the no. of vertices in cycle graph is even then $\chi(G) = 2$.
11) If the ——— odd then $\chi(G) = 3$.

eg:


$\chi(G) = 3$


$\chi(G) = 2$


$\chi(G) = 3$

② Planar Graphs

⊙ A planar graph is a graph that can be drawn in a plane s.t none of its edge cross each other.

Chromatic no.

1) Chromatic no. of any planar graph ≤ 4

11) All the above cycle graphs are also planar graph. Chromatic no. of each graph ≤ 4

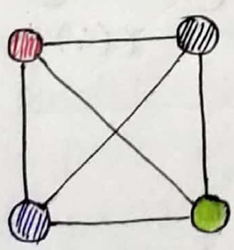
③ Complete Graphs

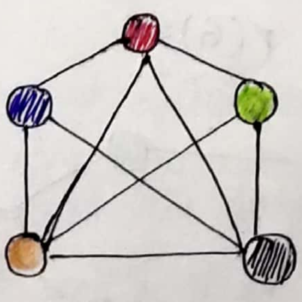
⊙ A complete graph is a graph in which every 2 distinct vertices are joined by exactly one edge.

Chromatic no.

Chromatic no. of any complete graph = no. of vertices in that complete graph.

eg:


$\chi(G) = 4$
= no. of vertices


$\chi(G) = 5$
= no. of vertices.

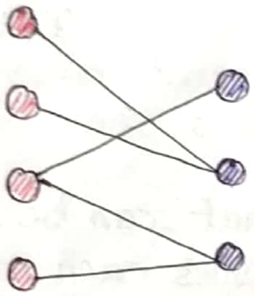
④ Bipartite Graph

⊙ A bipartite graph consists 2 set of vertices X, Y, the edge only join vertices in X to vertices in Y, not vertices within a set.

Chromatic no.

Chromatic no. of any Bipartite graph = 2.

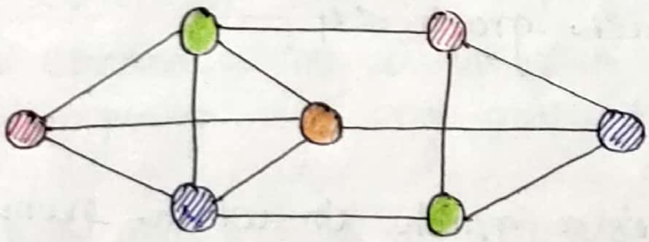
eg:



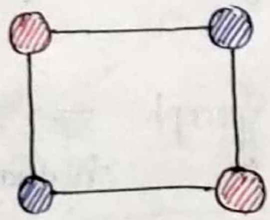
$\chi(G) = 2$

Four colour Theorem

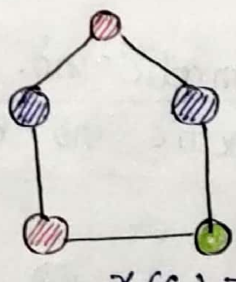
Any planar graph is atmost 4-colorable.



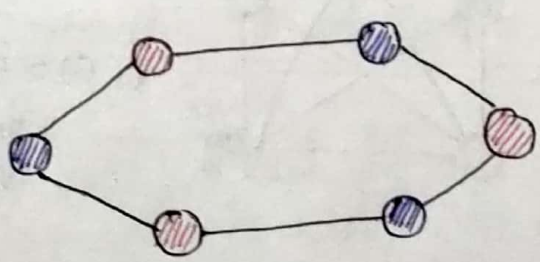
$\chi(G) = 4$



$\chi(G) = 2$



$\chi(G) = 3$

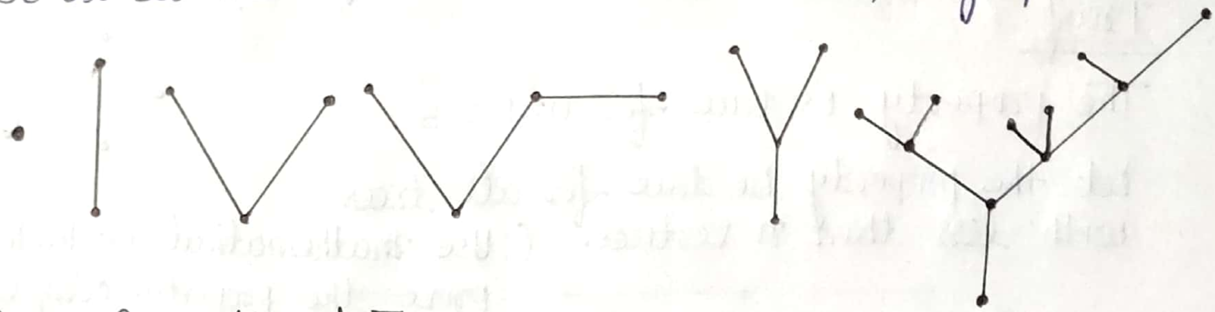


$\chi(G) = 2$

↳ Trees

⊙ A connected graph without any circuits is called a tree.
So it is clear that a tree has a simple graph.

eg:



Some Properties of Trees

Property 1

⊙ An undirected graph is a tree, iff there is a unique simple path b/w every pair of vertices.

Proof:

1) Let the undirected graph T be a tree.
Then, by definition of a tree, T is connected.
Hence, there is a simple path b/w any pair of vertices, say v_i & v_j.
If possible, let there be two paths b/w v_i & v_j one from v_i to v_j and the other from v_j to v_i. Combination of these two paths would contain a circuit.
But T cannot have a ckt, by definition.
Hence, there is a unique simple path b/w every pair of vertices in T.

II) Let a unique path exist b/w every pair of vertices in the graph T.
Then, T is connected.
If possible, let T contain a ckt. This means that there is a pair of vertices v_i & v_j b/w which two distinct paths exist, which is against the data.
Hence, T cannot have a ckt and so T is a tree.

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Property 2

⊙ A tree with n vertices has $(n-1)$ edges.

Proof:

The property is true for $n = 1, 2, 3$.

Let the property be true for all trees with less than n vertices. (Use mathematical induction to prove the property completely.)

Let us now consider a tree T with n vertices.

Let e_k be the edge connecting the vertices v_i & v_j of T.

Then, by property (1), e_k is the only path b/w v_i & v_j.

If we delete the edge e_k from T, T becomes disconnected and $(T - e_k)$ consists of exactly two components, say, T_1 & T_2 which are connected.

Since T did not contain any ckt T_1 & T_2 also will not have circuits.

Hence, both T_1 & T_2 are trees, each having less than n vertices, say r and $n - r$ respectively.

∴ By the induction assumption, T_1 has $(r-1)$ edges and T_2 has $(n-r-1)$ edges.

∴ T has $(r-1) + (n-r-1) + 1 = n - 1$ edges.

Thus, a tree with n vertices has $(n-1)$ edges.

Proved:

Property 3

⊙ Any connected graph with n vertices and $(n-1)$ edges is a tree.

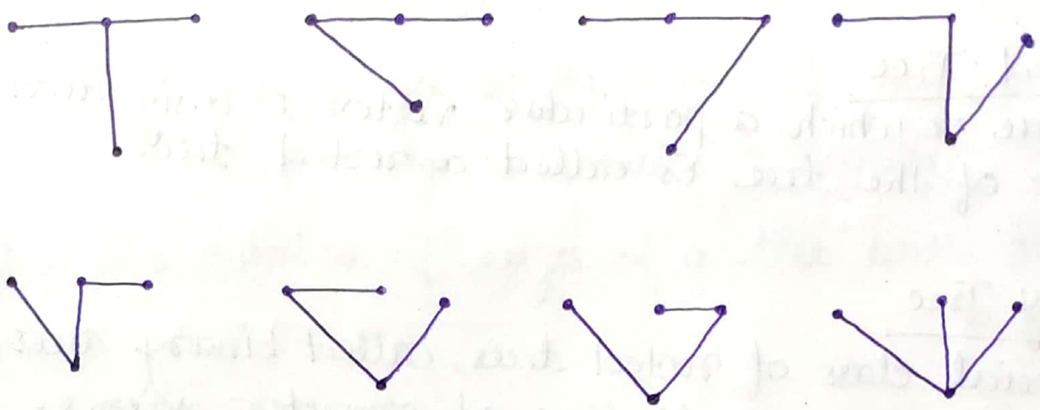
Property 4

⊙ Any circuitless graph with n vertices and $(n-1)$ edges is a tree.

Spanning Trees

- If the subgraph T of a connected graph G is a tree containing all the vertices of G, then T is called a spanning tree of G.

G



Every connected graph has atleast one spanning tree.

↳ Minimum Spanning Tree

- If G is a connected weighted (length of edge) graph (a graph in which a non-negative number is assigned to each edge), the spanning tree of G with the smallest total weight (viz., the sum of the weights of its edges) is called the minimum spanning tree of G.

↳ Krushal's Algorithm

Step 1

The edges of the given graph G are arranged in the order of increasing weights.

Step 2

An edge G with min. weight is selected as an edge of the required spanning tree.

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Step 3

Edges with min. weight that do not from a ckt with the already selected edges are successively added.

Step 4

The procedure is stopped after $(n-1)$ edges have been selected.

vertex

Rooted Tree

- A tree in which a particular vertex is designated as the root of the tree is called a rooted tree.

Binary Tree

- A special class of rooted trees, called binary trees, is of importance in applications of computer science.

- In every internal vertex of a rooted tree has exactly / at most 2 children, the tree is called a full binary tree / a binary tree.

Properties of Binary Trees.

Property 1.

- The number n of vertices of a full binary tree is odd and the number of pendant vertices (leaves) of the tree is equal to $\frac{(n+1)}{2}$.

Proof:

In a full binary tree, only one vertex, namely, the root is of even degree (namely 2) and each of the other $(n-1)$ vertices is of odd degree (namely 1 or 3).

Since the no. of vertices of odd degree in an undirected graph is even, $(n-1)$ is even.

$\therefore$ n is odd.

Now, let p be the number of pendant vertices of the full binary tree.

$\therefore$ The number of vertices of degree $3 = n-p-1$.

$\therefore$ The sum of the degrees of all the vertices of the tree.

$$= 1\times 2 + p\times 1 + (n-p-1)\times 3$$

$$= 3n-2p-1.$$

$\therefore$ Number of edges of the tree $= \frac{1}{2}(3n-2p-1)$

($\because$ each edge contributes 2 degrees)

But the number of edges of a tree with n vertices $= (n-1)$.

$$\therefore \quad \frac{1}{2}(3n-2p-1) = n-1$$

$$3n-2p-1 = 2n-2$$

$$2p = n+1$$

$$\boxed{p = \frac{n+1}{2}}$$

Proved:

Property 2

- The min. height of a n-vertex binary tree is equal to $\lceil \log_2(n+1) - 1 \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Proof: Let h be the height of the binary tree.

i.e., the max. level of any vertex of the tree is h.

If n_i represents the no. of vertices at level i, then

$$n_0 = 1;\ n_1 \le 2^1;\ n_2 \le 2^2, ---, n_h \le 2^h$$

$$n = n_0 + n_1 + --- + n_h \le 1 + 2^1 + 2^2 + --- + 2^h.$$

$$n \le 2^{h+1} - 1$$

$$2^{h+1} \ge n+1$$

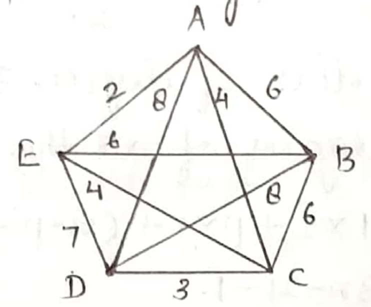
$$h+1 \ge \log_2(n+1) \text{ or } h \ge \log_2(n+1) - 1$$

$\therefore$ Minimum value of $h = \lceil \log_2(n+1) - 1 \rceil$.

Proved:

29-04-022

eg: Find the minimum spanning tree for the weighted graph by using kruskal's algorithm.



Soln: We first arrange the edges in the increasing order of the edges and proceed as per kruskal's algorithm.

Edge	Weight	Included in the spanning tree or not	If not included, circuit formed
AE	2	Yes	—
CD	3	Yes	—
AC	4	Yes	—
EC	4	No	A–E–C–A / ACEA
AB	6	Yes	—
BC	6	No	A–B–C–A
BE	6	—	—
ED	7	—	—
AD	8	—	—
BD	8	—	—

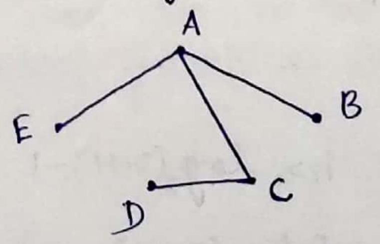
∵ $n = 5$ (no. of ~~ede~~ vertex)

∴ We stop the procedure at $(n-1)$ i.e 4^{th} step (means 4 बार yes आने तक) edge

So, the required minimum spanning tree consist 4 edges. AE, CD, AC, AB.

Therefore, length of min. spanning tree $= 2+3+4+6$

$= 15$ Ans:

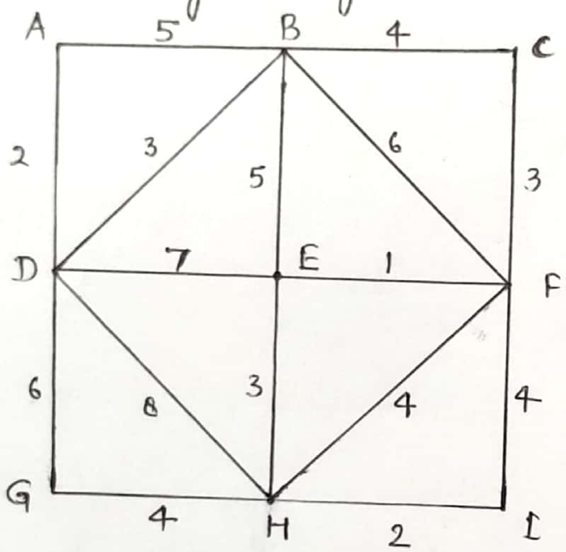


There are 5 other alternative min. spanning trees of total length 15 whose edges are

I) AE, CD, AC, BC II) AE, CD, AC, BE III) AE, CD, CE, AB

IV) AE, CD, CE, BC IV) AE, CD, CE, BE

eg: Use Kruskal's algorithm to find a min. spanning tree for the weighted graph.



Soln:

Weight	Edge	Included in the spanning tree or not	If not included, circuit formed
1	EF	YES	—
2	AD	Yes	—
2	HI	Yes	—
3	BD	Yes	—
3	CF	Yes	—
3	EH	Yes	—
4	BC	Yes	—
4	FH	No	E-F-H-E
4	GH	Yes	—
4	FI	—	—
5	BE	—	—
5	AB	—	—
6	BF	—	—
6	DG	—	—
7	ED	—	—
8	HD	—	—

The required min. spanning tree consists of 8 edges EF, AD, HI, BD, CF, EH, BC & GH.

The total length of the min. spanning tree $= 1+2+2+3+3+3+4+4$

$= 22$ Ans :

धन्यवाद !