UNIT-3

MATHEMATICAL LOGIC

Logic is a field of study that deals with the method of reasoning. Logic provides rule by which we can determine whether a given argument or reasoning is valid (correct) or not. Logical reasonings used in Mathematics to prove theorems. In computer science, logic is used to verify the correctness of programs.

Proposition (or Statement):

A Statement or proposition is a declarative ~~statement~~ sentence which is either True or False but not both.

These 2 values 'True & False' denoted by the symbol 'T & F' respectively. Sometimes it is also denoted by '1 & 0'.

Sentences which are exclamatory, interrogative in nature are not proposition.

eg:

1. New Delhi is the capital of India. ✓/T
2. $8 > 10$. ✓/F
3. Blood is green. ✓/F
4. It is raining. ✓/T or F.
5. The sun will come out tomorrow. ✓/T or F
6. $x+y-z$ ✗ / bcoz $x, y \& z$ are not assigned.
7. How beautiful is rose? ✗ / bcoz it is a question.
8. Who are you? ✗ / "
9. Close the door. ✗ / bcoz it is command.

Truth Values of a statement:

A proposition (or statement) can have one and only one of 2 possible values namely True & False. These 2 values T & F are called Truth values of a statement.

↳ Connectives :

The words or symbol which are used to make a sentence by 2 sentences are called logical connectives.

Let us define the following basic & fundamental connective -

		WORD	SYMBOL
i)	Negation	(NOT)	$\sim$ or $\neg$
ii)	Conjuction	(AND)	$\wedge$
iii)	Disjunction	(OR)	$\vee$
iv)	conditional (Implication)	(IF --- THEN)	$\longrightarrow$
v)	Biconditional	(If and only If)	$\longleftrightarrow$

1. Negation (NOT): If a proposition is denoted by the symbol P then its negation denoted by $\sim P$ or $\neg P$. If P is True then $\neg P$ is FALSE and if P is false then $\neg P$ is true.

Truth Table

P	$\sim P$
T	F
F	T

-eg: P: Delhi is a city.

$\neg P$: Delhi is not a city.

(OR)

It is false that Delhi is a city.

2. Conjuction (AND): If p and q are any two proposition then conjuction (AND) of p & q is also a proposition and it is denoted by $p \wedge q$.

Truth table

p	q	$p \wedge q$	
T	T	T	$1 \cdot 1 = 1$
T	F	F	$1 \cdot 0 = 0$
F	T	F	$0 \cdot 1 = 0$
F	F	F	$0 \cdot 0 = 0$

-eg: p: Ram is an honest man.

q: Rahim ________________.

$p \wedge q$: Ram and Rahim are honest man.

3. Disjunction (OR): If p & q are two proposition then the disjunction of p & q is denoted by $p \vee q$.

Truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

-eg: p: I will watch the game on television.

q: I shall go to the home.

$p \vee q$: I shall watch the game or go to the home.

4. Conditional Proposition (If --- Then) :

If p & q are two proposition then the compound statement 'If p then q' denoted as $p \rightarrow q$ is called a conditional proposition or implication.

Here p is called Hypothesis or premise & q is called the consequence or conclusion.

Truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

eg: p: I am hungry.
q: I will eat.

$p \rightarrow q$: If I am hungry then I will eat.

5. Biconditional proposition (If and only if) :

If p and q are two propositions then the compound proposition "p if and only if q" written as $p \leftrightarrow q$ is called biconditional proposition.

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

eg: p: He swims.
q: Water is warm.

$p \leftrightarrow q$: He swims iff water is warm.

NOTE:

Compound Proposition: A proposition obtained from the combination of 2 or more proposition by means of logical operations is called compound proposition.

Truth Table: A Truth Table is a table that shows the truth values of compound proposition for all possible case.

eg: Construct the truth table for

I) $p \wedge \sim p$ II) $p \vee \sim q$ III) $(p \vee q) \vee \sim p$ IV) $(p \vee q) \wedge r$ V) $(p \vee q) \rightarrow (p \wedge q)$

VI) $(p \rightarrow q) \rightarrow (q \rightarrow p)$ VII) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

Soln:

I) $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

II) $p \vee \sim q$

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

III) $(p \vee q) \vee \sim p$

p	$\sim p$	q	$(p \vee q)$	$(p \vee q) \vee \sim p$
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T

~~IV) $(p \vee q) \wedge r$~~

IV) $(p \vee q) \wedge r$

p	q	r	$(p \vee q)$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

V) $(p \vee q) \longrightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

VI) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

p	q	$(p \rightarrow q)$	$(q \rightarrow p)$	$(p \rightarrow q) \longrightarrow (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

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vii) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\neg p$	$(q \to \neg p)$	$(p \leftrightarrow q)$	$(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

eg: What is the negation of the following proposition?

1) p: No one wants to buy my house.
$\sim$p: ~~Everyone~~ Someone wants to buy my house.

2) p: Some people have no scooter.
$\sim$p: Every person have a scooter.

3) p: All students are intelligent.
$\sim$p: No student is intelligent.

4) p: No student is intelligent.
$\sim$p: All students are intelligent.

eg: Using the statements
R: Mark is rich.
H: Mark is happy.
write the following statements in symbolic form.

a) Mark is poor and happy. $\sim R \wedge H$

b) Mark is rich or unhappy. $R \vee \neg H$

c) Mark is neither rich nor happy. $\sim R \vee \sim H$

d) Mark is poor or he is both rich & unhappy. $\sim R \vee (R \wedge \sim H)$

Tautology and Contradiction

A proposition which is True for all possible truth values of its propositional variable is called a Tautology.

A proposition is called contradiction if it contains only false value.

eg: Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

Soln:

p	q	$(p \rightarrow q)$	$(p \wedge (p \rightarrow q))$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since, All the entries in last column is T, So, it is a Tautology.

eg: Show that the statement $p \wedge \neg p$ is a contradiction.

Soln:

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Since, all entries in last column are F, So, it is a contradiction.

Equivalence of Proposition

Two compound proposition $A(p_1, p_2, ---, p_n)$ and $B(p_1, p_2, ---, p_n)$ are said to be logically equivalent if they have identical truth values. It is denoted by $\equiv$ or $\Leftrightarrow$

i.e. $A(p_1, p_2, ---, p_n) \equiv B(p_1, p_2, ---, p_n)$

or $A \Leftrightarrow B$

↳ If a proposition is neither Tautology nor a Contradiction, is called a contingency.

eg: Prove that $p \to q \equiv \sim p \vee q$

Soln:

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T ✓	T	T ✓

Daulity Law :

Two statements are said to be daul of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F & F by T.

eg: Find the dual of the following:

	Dual
I) $(p \vee q) \wedge r$	I) $(p \wedge q) \vee r$
II) $(p \wedge q) \vee T$	II) $(p \vee q) \wedge F$
III) $\neg(p \vee q) \wedge (p \vee \sim(p \wedge s))$	III) $\neg(p \wedge q) \vee (p \wedge \sim(p \vee s))$

Algebra of Propositions

S.No:	Name of the Law	Primal Form	Dual Form
1.	Commulative Law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
2.	Associative Law	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
3.	Distributive Law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
4.	Idempotent Law	$p \vee p \equiv p$	$p \wedge p \equiv p$
5.	Identity Law	$p \vee F \equiv p$	$p \wedge T \equiv p$
6.	Dominant Law	$p \vee T \equiv T$	$p \wedge F \equiv F$
7.	Complement Law	$p \vee \sim p \equiv T$	$p \wedge \sim p \equiv F$
8.	Absorption Law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9.	Demorgan's Law	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$

↳ Equivalence involving Conditionals

1. $p \to q \equiv \sim p \vee q$
2. $p \to q \equiv \sim q \to \sim p$
3. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
4. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
5. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
6. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

↳ Equivalence involving Biconditionals

1. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
2. $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

eg: Without using truth table, prove that

1) $(\sim p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

Soln:

LHS: $(\sim p \vee q) \wedge (p \wedge (p \wedge q))$

$= (\sim p \vee q) \wedge ((p \wedge p) \wedge q)$	Associative Law
$= (\sim p \vee q) \wedge (p \wedge q)$	Idempotent Law
$= ((\sim p \vee q) \wedge p) \wedge q$	Associative Law
$= (p \wedge (\sim p \vee q)) \wedge q$	Commutative Law
$= ((p \wedge \sim p) \vee (p \wedge q)) \wedge q$	Distributive Law
$= (F \vee (p \wedge q)) \wedge q$	Complement law
$= (p \wedge q) \wedge q$	Identity law
$= p \wedge (q \wedge q)$	Associative law
$= p \wedge q$	Idempotent law
$=$ RHS	

Proved

Ans:

II) $p \rightarrow (q \rightarrow p) \equiv \sim p \rightarrow (p \rightarrow q)$

Soln:

LHS: $p \rightarrow (q \rightarrow p)$

$= p \rightarrow (\sim q \vee p)$

$= \sim p \vee (\sim q \vee p)$

$= \sim (q \vee p) \vee \sim p$ (commutative)

$= \sim q \vee (p \vee \sim p)$ (Associative)

$= \sim q \vee T$ (complement)

$= T$ (Dominant law)

RHS:

$\sim p \rightarrow (p \rightarrow q)$

$= \sim(\sim p) \vee (\sim p \vee q)$

$= p \vee (\sim p \vee q)$

$= (p \vee \sim p) \vee q$ (Associative law)

$= T \vee q$ (complement law)

$= T$ (Dominant law)

LHS = RHS

Proved

III) $\sim (p \leftrightarrow q) \equiv (p \vee q) \wedge \sim (p \wedge q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

Soln:

LHS: $\sim (p \leftrightarrow q)$

$= \sim ((p \rightarrow q) \wedge (q \rightarrow p))$

$= \sim ((\sim p \vee q) \wedge (\sim q \vee p))$

$= \sim (((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p))$ Distributive law

$= \sim (((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p)))$ Distributive law

$= \sim (((\sim p \wedge \sim q) \vee F) \vee (F \vee (p \wedge q)))$ Complement law

$= \sim ((\sim p \wedge \sim q) \vee (p \wedge q))$ Identity law

$= \sim(\sim p \wedge \sim q) \wedge \sim(p \wedge q)$ Demorgan law

$= \boxed{(p \vee q) \wedge \sim(p \wedge q)}$ ''

$= (p \vee q) \wedge (\sim p \vee \sim q)$ ''

$= ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q)$ Distributive law

$= ((p \wedge \sim p) \vee (q \wedge \sim p)) \vee ((p \wedge \sim q) \vee (q \wedge \sim q))$ ''

$= (F \vee (q \wedge \sim p)) \vee ((p \wedge \sim q) \vee F)$ Complement law

$= (q \wedge \sim p) \vee (p \wedge \sim q)$ Identity law

$= \boxed{(p \wedge \sim q) \vee (\sim p \wedge q)}$ Commutative law

eg: Prove that

i) $\sim p \to (q \to r) \equiv q \to (p \vee r)$

Soln:

LHS: $\sim p \to (q \to r)$

$= \sim p \to (\sim q \vee r)$

$= \sim(\sim p) \vee (\sim q \vee r)$

$= p \vee (\sim q \vee r)$

$= (\sim q \vee r) \vee p$ Commutative law

$= \sim q \vee (r \vee p)$ Associative law

$= q \to (p \vee r)$

$=$ RHS

Proved Ans:

ii) $p \to (q \to r) \equiv p \to (\sim q \vee r) \equiv (p \wedge q) \to r$

Soln: $p \to (q \to r)$

$= \boxed{p \to (\sim q \vee r)}$

$= \sim p \vee (\sim q \vee r)$

$= (\sim p \vee \sim q) \vee r$ Associative law

$= \sim(p \wedge q) \vee r$ Demorgan's law

$= \boxed{(p \wedge q) \to r}$

Proved Ans:

III) $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ is a tautology.

Soln:
$= ((p \vee q) \wedge (p \vee (q \wedge r)) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ — Demorgan Law

$= ((p \vee q) \wedge ((p \vee q) \wedge (p \vee r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ — Distributive law

$= (((p \vee q) \wedge (p \vee q)) \wedge (p \vee r)) \vee \sim(p \vee q) \vee \sim(p \vee r)$ — Asso. & Demorgan law

$= ((p \vee q) \wedge (p \vee r)) \vee \sim((p \vee q) \wedge (p \vee r))$ — Idempotent & Demorgan law

$= T$ — ($\because x \vee \sim x \equiv T$) Dominant law.

Proved Ans:

eg: Prove the following equivalences by proving the equivalences of the duals.

1) $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

Soln: The dual of the given equivalences:

$\sim((\sim p \vee q) \wedge (\sim p \vee \sim q)) \wedge (p \vee q) \equiv p$ —— (1)

LHS of (1) is

$\sim((\sim p \vee q) \wedge (\sim p \vee \sim q)) \wedge (p \vee q)$

$= \sim((\sim p \vee (q \wedge \sim q))) \wedge (p \vee q)$ — Distributive law

$= \sim((\sim p \vee F)) \wedge (p \vee q)$ — Complement law

$= \sim(\sim p) \wedge (p \vee q)$

$= p \wedge (p \vee q)$

$= p$ — Absorption law

Proved Ans:

II) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

? Soln: Dual is

$= \sim(p \vee q) \vee r \equiv (\sim p \vee r) \wedge (\sim q \vee r)$

$= \sim(p \vee q) \wedge r \equiv (\sim p \wedge r) \vee (\sim q \wedge r)$ —— (1)

LHS: of eqn (1) is $\sim(p \wedge q) \wedge r$

$= (\sim p \vee \sim q) \wedge r$ — Demorgan's law

$= (\sim p \wedge r) \vee (\sim q \wedge r)$ — Distibutive law

$=$ RHS Proved Ans.

iii) $(p \wedge (p \leftrightarrow q)) \rightarrow q \equiv T$

Soln:

$= p \wedge ((p \rightarrow q) \wedge (q \rightarrow p)) \rightarrow q \equiv T$

$= p \wedge ((\sim p \vee q) \wedge (\sim q \vee p)) \longrightarrow q \equiv T$

$= \sim (p \wedge ((\sim p \vee q) \wedge (\sim q \vee p))) \vee q \equiv T$ —— (i)

Dual of (i) is

$= \sim (p \vee ((\sim p \wedge q) \vee (\sim q \wedge p))) \wedge q \equiv F$ —— (ii)

LHS:

$\sim (p \vee ((\sim p \wedge q) \vee (\sim q \wedge p))) \wedge q$

$= \sim (p \vee ((\sim q \wedge p) \vee (\sim p \wedge q))) \wedge q$ Commutative law

$= \sim ((p \vee (\sim q \wedge p)) \vee (\sim p \wedge q)) \wedge q$ Associative law

$= \sim (((p \vee (p \wedge \sim q)) \vee (\sim p \wedge q)) \wedge q$ Commutative law

$= \sim (p \vee (\sim p \wedge q)) \wedge q$ Absorption law

$= \sim ((p \vee \sim p) \wedge (p \vee q)) \wedge q$ Distributive law

$= \sim (T \wedge (p \vee q)) \wedge q$ Complement law

$= \sim (p \vee q) \wedge q$ Identity law

$= (\sim p \wedge \sim q) \wedge q$ Demorgan's law

$= \sim p \wedge (\sim q \wedge q)$ Associative law

$= \sim p \wedge F$ Complement law

$= F$ Demenant law

Proved

Ans:

Tautological Implication

A compound proposition $A(p_1, p_2, \cdots, p_n)$ is said to be tautologically imply or simply the compound proposition $B(p_1, p_2, \cdots, p_n)$, if B is true whenever A is true.

or equivalently if and only if $A \rightarrow B$ is a tautology.

This is denoted by $A \Rightarrow B$ read as "A implies B".

eg: $p \Rightarrow p \vee q$

Now we shall show that $p \to (p \vee q)$ is a tautology.

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$$p \Rightarrow p \vee q$$

NOTE:

$\to$ Conditional

$\leftrightarrow$ Biconditional

$\equiv$ or $\Leftrightarrow$ Equivalence

$\Rightarrow$ Tautological Implication.

eg: Prove the following implications without using truth table

1) $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$

Soln: We shall show that

$(p \vee q) \wedge (p \to r) \wedge (q \to r) \longrightarrow r$ is a tautology

$= (p \vee q) \wedge [(p \to r) \wedge (q \to r)] \longrightarrow r$

$= (p \vee q) \wedge [(p \vee q) \longrightarrow r] \longrightarrow r$

$= (p \vee q) \wedge [\sim(p \vee q) \vee r] \longrightarrow r$

$= [(p \vee q) \wedge \sim(p \vee q)] \vee [(p \vee q) \wedge r] \longrightarrow r$ Distributive law

$= [F \vee (p \vee q) \wedge r] \longrightarrow r$ Complement law

$= [(p \vee q) \wedge r] \longrightarrow r$ Identity law

$= \sim[(p \vee q) \wedge r] \vee r$

$= [\sim(p \vee q) \vee \sim r] \vee r$ Demorgan's law

$= \sim(p \vee q) \vee (\sim r \vee r)$ Asso. law

$= \sim(p \vee q) \vee T$ Complement law

$= T$ Dominant law

ii) $((p \vee \sim p) \rightarrow q) \rightarrow ((p \vee \sim p) \rightarrow r) \Rightarrow q \rightarrow r$

Soln:
$$((p \vee \sim p) \rightarrow q) \rightarrow ((p \vee \sim p) \rightarrow r) \rightarrow (q \rightarrow r)$$
$$(T \rightarrow q) \rightarrow (T \rightarrow r) \rightarrow (q \rightarrow r)$$
$$(\sim T \vee q) \rightarrow (\sim T \vee r) \rightarrow (q \rightarrow r)$$
$$(F \vee q) \rightarrow (F \vee r) \rightarrow (q \rightarrow r)$$
$$(q \rightarrow r) \rightarrow (q \rightarrow r)$$
$$\sim (q \rightarrow r) \vee (q \rightarrow r)$$
$$T$$

Proved:

Predicate Calculus

The propositional calculus does not allow us to represent many of the statements that we use in mathematics, computer science and in every day life. Predicate calculus is the generalization of the proposition calculus.

A part of a declarative sentence ~~deciding~~ describing the properties of an object or relation among objects is called a predicate.

eg: consider 2 proposition.

P_1: Ram is a bachelor.

P_2: Shyam is a bachelor.

Both Ram & Shyam has the property of being bachelor. There is no relation b/w P_1 & P_2 but have some common part. These two proposition can be repeated by a single statement. "x is a bachelor" and we can get both proposition P_1 & P_2 by replacing x by Ram & Shyam.

The sentence "x is a bachelor" is symbolised as $P(x)$, where x is a predicate variable & $P(x)$ is called propositional functn.

Arguments

An argument is a sequence of statements. All statements (propositions) except the final one are called premises (or assumption / hypothesis) and final statement is called conclusion.

Sometimes an argument is written in the following form

$$\left.\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \end{array}\right\} \text{Premises}$$

$\underline{\quad C \quad}$ ← Conclusion.

An argument is said to be logically valid argument iff the conjunction of the premises implies the conclusion i.e. If all the premises are true, the conclusion must also be true, the argument which yield a conclusion C from the premises $p_1, p_2, \ldots p_n$ is valid iff the statement

$(p_1 \wedge p_2 \wedge p_3 \ldots \wedge p_n) \longrightarrow C$ is a Tautology. { $A \Rightarrow B$ / $A \to B$ is Tautology.

Theory of influence

Influence theory is concerned with the inferring of a conclusion from certain hypothesis or premises by applying certain principles of reasing, called rules of influence.

Rules of Influence

Rule P: A premises may be introduced at any step in the derivation.

Rule T: A formula S may be introduced in the derivation.

Rules of Influence	Tautological Form	Name of the Rule
1. (a) $\dfrac{p}{p \vee q}$ (b) $\dfrac{q}{p \vee q}$	a) $p \rightarrow p \vee q$ b) $q \rightarrow p \vee q$	Addition
2. (a) $\dfrac{p \wedge q}{p}$ (b) $\dfrac{p \wedge q}{q}$	a) $p \wedge q \rightarrow p$ b) $p \wedge q \rightarrow q$	Simplication
3. $\begin{array}{c} p \\ q \\ \hline p \wedge q \end{array}$	$(p) \wedge (q) \rightarrow p \wedge q$	Conjunction
4. $\begin{array}{c} p \rightarrow q \\ p \\ \hline q \end{array}$	$[(p \rightarrow q) \wedge p] \rightarrow q$	Modus Ponens
5. $\begin{array}{c} p \rightarrow q \\ \sim q \\ \hline \sim p \end{array}$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$	Modus tollens
6. $\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
7. $\begin{array}{c} p \vee q \\ \sim p \\ \hline q \end{array}$	$(p \vee q) \wedge \sim p \rightarrow q$	Disjunctive syllogism
8. $\begin{array}{c} (p \rightarrow q) \wedge (r \rightarrow s) \\ p \vee r \\ \hline \therefore q \vee s \end{array}$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow q \vee s$	Constructive dilemma
9. $\begin{array}{c} (p \rightarrow q) \wedge (r \rightarrow s) \\ \sim q \vee \sim s \\ \hline \sim p \vee \sim r \end{array}$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\sim q \vee \sim s)] \rightarrow (\sim p \vee \sim r)$	Destructive dilemma

Inconsistent Premises

A set of premises $p_1, p_2, \ldots, p_n$ is said to be inconsistent if their conjunction implies a contradiction.

i.e. $p_1 \wedge p_2 \ldots \wedge p_n \Rightarrow R \wedge \sim R$ for some formula R.

Indirect method of Proof

In this method we assume that C is false and include $\sim C$ as an additional premices.

CP Rule

If the conclusion is of the form $r \to s$, we will take r as an additional premise and derive s using the given premises and r.

eg ①: Find whether the conclusion C follows from the premises H_1, H_2, H_3 in the following cases, using truth table technique.

1) $H_1 : \sim p$, $H_2 : p \vee q$, $C : p \wedge q$

Soln:

		H_1	H_2	C		
p	q	$\sim p$	$p \vee q$	$p \wedge q$	$H_1 \wedge H_2$	$H_1 \wedge H_2 \to C$
T	T	F	T	T	F	T
T	F	F	T	F	F	T
F	T	T	T	F	T	F
F	F	T	F	F	F	T

Since, $H_1 \wedge H_2 \to C$ is not a Tautology.

$\therefore$ C does not follow from H_1 & H_2.

OR

H_1, H_2 & $H_1 \wedge H_2$ are true in 3rd row but C is false.
So C does not follow from H_1 & H_2.

Ans:

eg②: Show that rule modus ponens is valid.

Soln:

$$\begin{array}{l} p \to q \\ p \\ \hline q \end{array}$$

We have to prove that $[(p\to q)\land p]\to q$ is a Tautology.

p	q	$p\to q$	$(p\to q)\land p$	$[(p\to q)\land p]\to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since, all are true.

$\therefore$ Modus Ponens is valid.

Proved :

eg③: Applying the rule of influence, check the validity of the argument.

1)

$$\begin{array}{l} p \to r \\ \sim p \to q \\ q \to s \\ \hline \therefore \sim r \to s \end{array}$$

Soln:

S.No:	Statement	Reason
1.	$\sim p \to q$	P
2.	$q \to s$	P
3.	$\sim p \to s$	T, 1, 2, Hypothetical Syllogism
4.	$p \to r$	P
5.	$\sim r \to \sim p$	T, 4, $A\to B \equiv \sim B \to \sim A$
6.	$\sim r \to s$	T, 4, 6, Hy. Sy.

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11) $p \rightarrow q$
$r \rightarrow \sim q$
$\overline{\quad p \rightarrow \sim r \quad}$

Soln:

S.No:	Statement	Reason.
1.	$p \rightarrow q$	P
2.	$r \rightarrow \sim q$	P
3.	$q \rightarrow \sim r$	T, 2, $A \rightarrow B \equiv \sim B \rightarrow \sim A$
4.	$p \rightarrow \sim r$	T, 1, 3, Hy. Sy

eg (4): Show that $(t \wedge s)$ can be derived from the premises $p \rightarrow q$, $q \rightarrow \sim r$, r, $p \vee (t \wedge s)$

Soln:

S.No:	Statement	Reason
1.	$p \rightarrow q$	P
2.	$q \rightarrow \sim r$	P
3.	$p \rightarrow \sim r$	T, 1, 2, Hy. Sy
4.	r	P
5.	$\sim p$	T, 3, 4, modus tollens
6.	$p \vee (t \wedge s)$	P
7.	$t \wedge s$	T, 5, 6, Disjuctive syllogism.

eg (5): Show that $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \rightarrow \sim r$, $\sim r \rightarrow (s \wedge \sim t)$ and $(s \wedge \sim t) \rightarrow (a \vee b)$.

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Soln:

S.No:	Statement	Reason
1.	$(p \vee q) \rightarrow \sim r$	P
2.	$\sim r \rightarrow (s \wedge \sim t)$	P
3.	$(p \vee q) \rightarrow (s \wedge \sim t)$	T, 1, 2, Hy. sy
4.	$(s \wedge \sim t) \rightarrow (a \vee b)$	P
5.	$(p \vee q) \rightarrow (a \vee b)$	T, 3, 4, Hy. sy.
6.	$p \vee q$	P
7.	$a \vee b$	T, 5, 6, Modus Ponens

eg 6: Show that $(p \rightarrow q) \wedge (r \rightarrow s)$, $(q \rightarrow t) \wedge (s \rightarrow u)$, $\sim(t \wedge u)$ and $(p \rightarrow r) \Rightarrow \sim p$.

Soln:

S.No:	Statement	Reason
1.	$(p \rightarrow q) \wedge (r \rightarrow s)$	P
2.	$p \rightarrow q$	T, 1, simplification
3.	$r \rightarrow s$	T, 1, simplification
4.	$(q \rightarrow t) \wedge (s \rightarrow u)$	P
5.	$q \rightarrow t$	T, 4, simplification
6.	$s \rightarrow u$	T, 4, "
7.	$p \rightarrow t$	T, 2, 5, Hy. sy
8.	$r \rightarrow u$	T, 3, 6, Hy. sy
9.	$p \rightarrow r$	P
10.	$p \rightarrow u$	T, 8, 9, Hy. sy
11.	$(p \rightarrow t) \wedge (p \rightarrow u)$	T, 7, 10, conjuction
12.	$p \rightarrow (t \wedge u)$	T, 11, $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
13.	$\sim (t \wedge u)$	P
14.	$\sim p$	T, 12, 13, modus tollens

eg (7): Show that $(a \to b) \wedge (a \to c), \sim(b \wedge c), (d \vee a) \Rightarrow d$.

Soln:

S.No:	Statements	Reason
1.	$(a \to b) \wedge (a \to c)$	P
2.	$a \to (b \wedge c)$	T, 1, $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
3.	$\sim(b \wedge c)$	P
4.	$\sim a$	T, 2, 3, Modus tollens
5.	$d \vee a$	P
6.	d	T, 4, 5, Disjunctive syllogism.

eg (8): Give a direct proof for the implication $p \to (q \to s), (\sim r \vee p), q \Rightarrow (r \to s)$.

Soln:

S.No:	Statements	Reason
1.	$p \to (q \to s)$	P
2.	$\sim r \vee p$	P
3.	$r \to p$	T, 2, $A \to B \equiv \sim A \vee B$
4.	$r \to (q \to s)$	T, 1, 3, Hy. sy
5.	$\sim r \vee (\sim q \vee s)$	T, 4
6.	$(\sim q \vee s) \vee \sim r$	T, 5, commutative
7.	$\sim q \vee (s \vee \sim r)$	T, 6, Associative
8.	$q \to (s \vee \sim r)$	T, 7
9.	q	P
10.	$s \vee \sim r$	T, 8, 9, Modus Ponens
11.	$\sim r \vee s$	T, 10, Associative
12.	$r \to s$	T, 11

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Indirect Method

eg(9): Use indirect method to show that $r \to \sim q$, $r \vee s$, $s \to \sim q$, $p \to q \Rightarrow \sim p$

Soln: Let us include $\sim(\sim p) = p$ as an additional premises.

S.No:	Statement	Reason
1.	$p \to q$	P
2.	$7(\sim p)$	P (Additional)
3.	q	T, 1, 2, Modus Ponens
4.	$s \to \sim q$	P
5.	$\sim s$	T, 3, 4, Modus tollens
6.	$r \vee s$	P
7.	r	T, 5, 6, Disjunctive sy.
8.	$r \to \sim q$	P
9.	$\sim q$	T, 7, 8 Modus Ponens
10.		
11.	F	T, 10, complement

(OR)

S.No:	Statement	Reason
1.	$r \to \sim q$	P
2.	$s \to \sim q$	P
3.	$(r \vee s) \to \sim q$	T, 1, 2 equivalence
4.	$r \vee s$	P
5.	$\sim q$	T, 3, 4, Modus Ponens
6.	$p \to q$	P

7.	$\sim p$	T, 5, 6, Modus tollens
8.	p	P (additional)
9.	$p \wedge \sim p$	T, 7, 8 Conjunction
10.	F	T, 9, complement.

eg (10): Show that b can be derived from the premises $a \to b$, $c \to b$, $d \to (a \vee c)$, d by indirect method.

Soln: Let us include $\sim b$ as an additional premises.

S.No:	Statement	Reason
1.	$a \to b$	P
2.	$c \to b$	P
3.	$(a \to b) \wedge (c \to b)$	T, 1, 2, Conjunction
4.	$(a \vee c) \to b$	T, 3, equivalence
5.	$d \to (a \vee c)$	P, 4
6.	$d \to b$	T, 4, 5, Hy. sy.
7.	d	P
8.	b	T, 6, 7, Modus Ponens
9.	$\sim b$	P (Additional)
10.	$b \wedge \sim b$	T, 8, 9, conjunction
11.	F	T, 10, complement.

eg (11): Prove that the premises $p \to q$, $q \to r$, $s \to \sim r$ and $p \wedge s$ are inconse .

Soln:

S.No:	Statement	Reason
1.	$p \to q$	P
2.	$q \to r$	P
3.	$p \to r$	T, 1, 2, Hy.sy
4.	$s \to \sim r$	P
5.	$r \to \sim s$	T, 4, $A \to B \equiv \sim B \to \sim A$
6.	$p \to \sim s$	T, 3, 5, Hy.sy
7.	$p \wedge s$	P
8.	p	T, 7, simplification
9.	s	"
10.	$\sim s$	T, 6, 8 Modus Ponens
11.	$s \wedge s$	T, 9, 10 conjunction
12.	F	T, 11, complement

eg (12): Using indirect method of pf derive $p \to \sim s$ from the premises $p \to (q \vee r)$, $q \to \sim p$, $s \to \sim r$, p

Soln: Let us include $\sim(p \to \sim s) = \sim(\sim p \vee \sim s) = p \wedge s$ as ~~are~~ an additional premises.

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S.No:	Statement	Reason
1.	$p \to (q \vee r)$	P
2.	p	P
3.	$q \vee r$	T, 1, 2, Modus Ponens
4.	$\sim q \to r$	T, 3, $A \to B \equiv \sim A \vee B$
5.	$s \to \sim r$	P
6.	$r \to \sim s$	T, 5, $A \to B \equiv \sim B \to \sim A$
7.	$\sim q \to \sim s$	T, 4, 6, Hy. sy
8.	$q \to \sim p$	P
9.	$s \to q$	T, 7, $A \to B \equiv \sim B \to \sim A$
10.	$s \to \sim p$	T, 8, 9, Hy. sy
11.	$p \wedge s$	P (additional)
12.	p	T, 11, simplification
13.	s	"
14.	$\sim p$	T, 10, 13, Modus Ponens
15.	$p \wedge \sim p$	T, 12, 14, conjunction
16.	F	T, 15, complement.