Institute of Data

2023

Data Science and AI

Module 1
Part 1:

Mathematics & Statistics

Agenda: Module 1 Part 1

- Linear algebra
- Calculus
- Multivariable calculus
- Statistics
- Probability

What is Linear Algebra and *why is it important for Data Science*?

- Linear Algebra is the branch of mathematics that deals with ***linear equations*** and their representations
- Linear Algebra is used extensively in science and engineering to ***'model'*** many systems such as in economics, health and finance
- Although many systems are ***'non-linear'***, Linear Models can be effective ***first-order approximation***. This is crucial, because non-linear models are ***very difficult*** to represent and manipulate.
- One excellent way to better understand Linear Algebra is use the **geometrical representations** of its constructs
- Another way to better understand Linear Algebra is through ***programming***. This is crucial for a Data Scientist.
- Key constructs of Linear Algebra are: Scalar, Vector, Matrix and Tensor

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Linear Algebra

- Vectors
  - definitions
  - vector arithmetic (adding, subtracting and multiplying vectors), dot products
- Matrices
  - definitions
  - matrix arithmetic
  - inverses, determinants, transposes
- Solving systems of linear equations
- Eigenvalues and eigenvectors

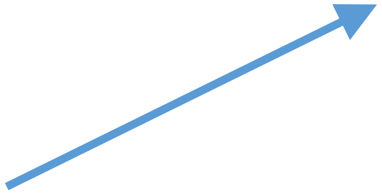
Mapping and usage of Linear Algebra in Data Science

Concept	Definition	Mapping to Data Science	Examples
Scalar	A ‘zero-dimensional’ dataset. A number, value, magnitude. Geometrically, it’s ***a point on on a line***.	A single data point	Age of a customer
Vector	A one-dimension dataset. A two or more values. Geometrically it represent a vector in a plane that has **magnitude and direction**.	A number of data points (usually about a single entity)	Attributes (or **features**) of one customer: Age, income, marital status, postcode, …, etc. In Deep Learning a vector could be the input to a Neural Network.
Matrix	A two-dimensional dataset. Geometrically, it represents a **transformation** of two or more vectors.	A set of observations for multiple entities. A transformation of a dataset from one representation to another.	Information about all customers. In Deep Learning a matrix may represents the mapping and weights on hidden layer.
Tensor	An n-dimensional dataset.	A number of sets of observations	Information about all customers. TensorFlow is built around tensors.

Vectors

def: ?

a directed quantity

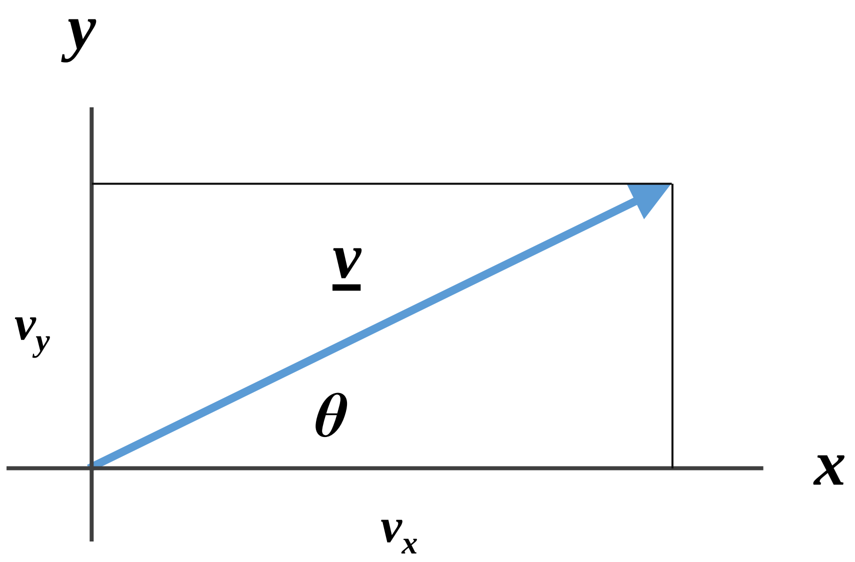


dimensionality > 1

examples: ?

- displacement (*not* length)
- velocity (*not* speed)
- acceleration
- force
- weight (*not* mass)

Vector Decomposition: 2D

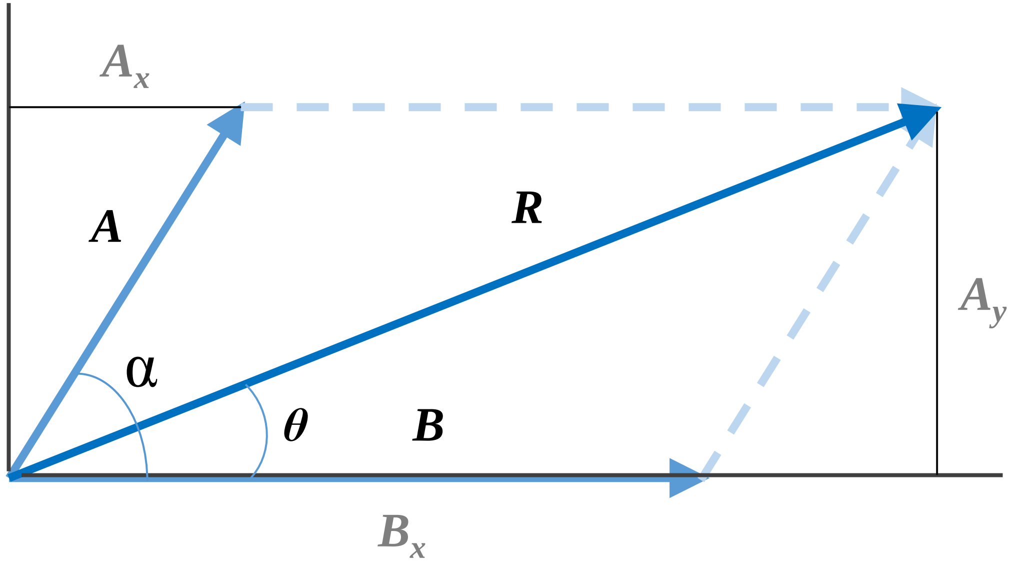


$$v_x = v \cos \theta$$

$$v_y = v \sin \theta$$

$$|\,\underline{v}\,| = (v_x^{\,2} + v_y^{\,2})^{1/2}$$

Vector Addition



in general:

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

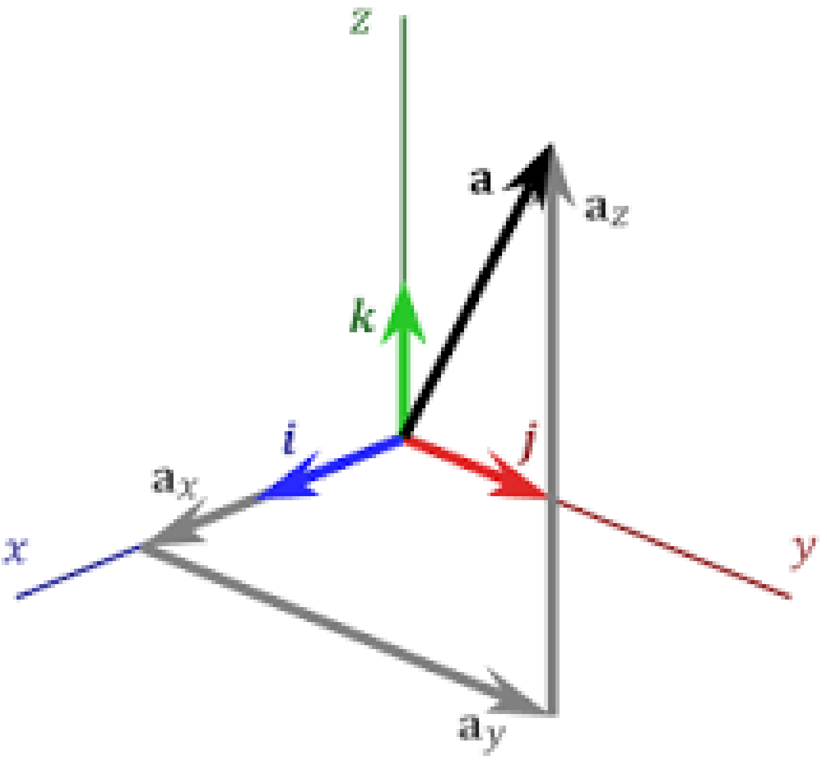
in this example:

$$R_x = A \cos \alpha + B$$

$$R_y = A \sin \alpha$$

$$\theta = \tan^{-1} (R_y / R_x)$$

3D Vectors



$| \underline{a} | = (a_x^2 + a_y^2 + a_z^2)^{1/2}$

Note:

i, j, k are unit vectors

Scalar Multiplication of Vectors

aka inner product, dot product

result is a *scalar*

$$
\begin{aligned}
\boldsymbol{a} \bullet \boldsymbol{b} &= (a_1, a_2, \dots a_n) \bullet (b_1, b_2, \dots \mathrm{b}_n) \\
&= a_1 b_1 + a_2 b_2 + \cdots + a_n \mathrm{b}_n \\
&= |a|\,|b|\,cos(\theta)
\end{aligned}
$$

Vector Operations

Entry-wise multiplication:

aka Hadamard product

result is a *vector*

$$\boldsymbol{a} * \boldsymbol{b} = (a_1 b_1, a_2 b_2, \dots a_n b_n)$$

Transpose

$$(a_1, a_2, \cdots a_n)^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Matrices

def: ?

a rectangular array of numbers

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

Matrix Arithmetic

addition

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Matrix Arithmetic

multiplication

$$A\,B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} & a_{11}\,b_{12} + a_{12}\,b_{22} \\ a_{21}\,b_{11} + a_{22}\,b_{21} & a_{21}\,b_{12} + a_{22}\,b_{22} \end{pmatrix}$$

in general:

A is m x n $\quad$ B is n x p

$A\,B$ is m x p

Matrix Operations3

Transpose

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$$

the rows of A^T are the columns of A

Determinant of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = |A| = a_{11}\,a_{22} - a_{12}\,a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}\,(a_{21}\,a_{33} - a_{23}\,a_{31}) + a_{13}\,(a_{21}\,a_{32} - a_{22}\,a_{31})$$

Identity Matrix

Define the n x n identity I_n :

$$A\, I_n = I_n\, A = A$$

$$\begin{bmatrix} 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & . \\ 0 & 0 & 1 & . & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & 1 \end{bmatrix}$$

Matrix Inversion

Define the inverse A^{-1} of an invertible matrix A:

$$A\,A^{-1} = I_n = A^{-1}A$$

only exists if…

A is n x n

$\det(A) \neq 0$

Methods:

- Gaussian (Gauss-Jordan) elimination
- LU decomposition (orthogonalisation)
- Eigen decomposition

Solving Systems of Linear Equations

Simultaneous linear equations in n unknowns (x_i):

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

in matrix form:

$$A\mathbf{x} = \mathbf{b}$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Solving Systems of Linear Equations – cont'd

Problem:

$$A\,\boldsymbol{x} = \boldsymbol{b}$$

Solution:

$$\boldsymbol{x} = A^{-1}\,\boldsymbol{b}$$

So, we only need to invert the matrix of coefficients A and multiply with vector $\boldsymbol{b}$

Eigenvalues and Eigenvectors

def: A nonzero vector that scales linearly in response to a linear transformation

$$T(\boldsymbol{v}) = \lambda \, \boldsymbol{v}$$

T is a linear transformation

λ is a scalar = 'eigenvalue' (*aka* characteristic value, characteristic root)

$\boldsymbol{v}$ is a vector = 'eigenvector' (*aka* characteristic vector)

eigenbasis: a set of eigenvectors of T that forms a basis of the domain of T

What is bigger than a matrix?

tensor

- represented by an *n*-dimensional array
- examples
  - stress tensor (mechanics)
  - spacetime tensor (general relativity)

Lab 1.1.1: Vector and Matrix Operations

Purpose:

- To apply the definitions of vector and matrix operations by designing code that implements them.

Materials:

- 'Lab 1.1.docx'
- See Notebook:
  - 'Lab 1.1 – Notebook – Linear Algebra'

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Calculus

- Limits and continuity
- Taking derivatives
- Integration
- Sequences and series

What is Calculus and why is it important for Data Scientists?

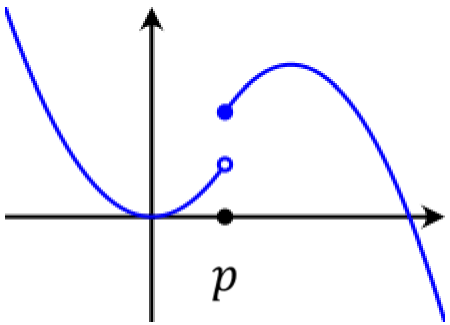
- Calculus is the mathematical study of **continuous change**. It is used extensively in many science and engineering domains such as business, economics and medicine.
- All key concepts in calculus can be mapped directly to **geometrical concepts**. For example, differentiation is the slope of a curve and integration is the area under a curve.
- Calculus is usually used with linear algebra to find the **"best fit"** linear approximation for a set of points in a domain. Therefore it is essential for Data Science as it underpins all model optimisation.

Limits and Continuity

If a function $f(x)$ approaches a value L ('limit') as x approaches p, then

$$\lim_{x \to p} f(x) = L$$

Note: The limits of a discontinuous function are directional



$$\lim_{x \to p^-} f(x) \neq \lim_{x \to p^+} f(x)$$

Limit Theorems

$$\lim_{n\to\infty} k\, a_n = k \lim_{n\to\infty} a_n$$

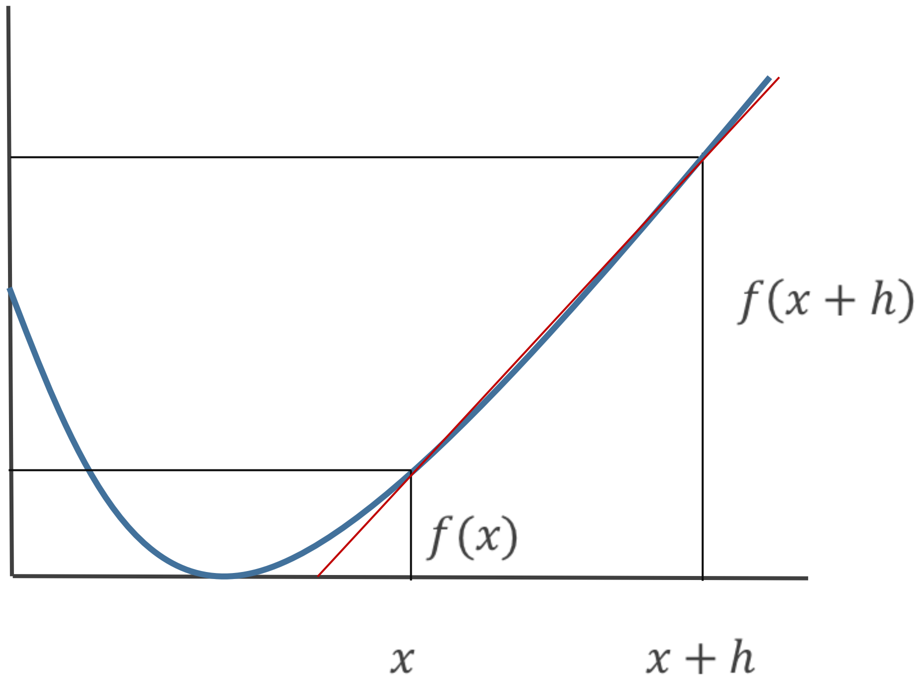
$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

Differentiation

Rate of change of a continuous function $f(x)$:



Derivative of $f(x)$:

$$\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Rules of Differentiation

Let f, g, h be functions of x, and let a, b be constants …

Linearity:

$$\frac{d(af+bg)}{dx} = a\frac{df}{dx} + b\frac{dg}{dx}$$

Product rule:

$$\frac{d(fg)}{dx} = g\frac{df}{dx} + f\frac{dg}{dx}$$

Chain rule:

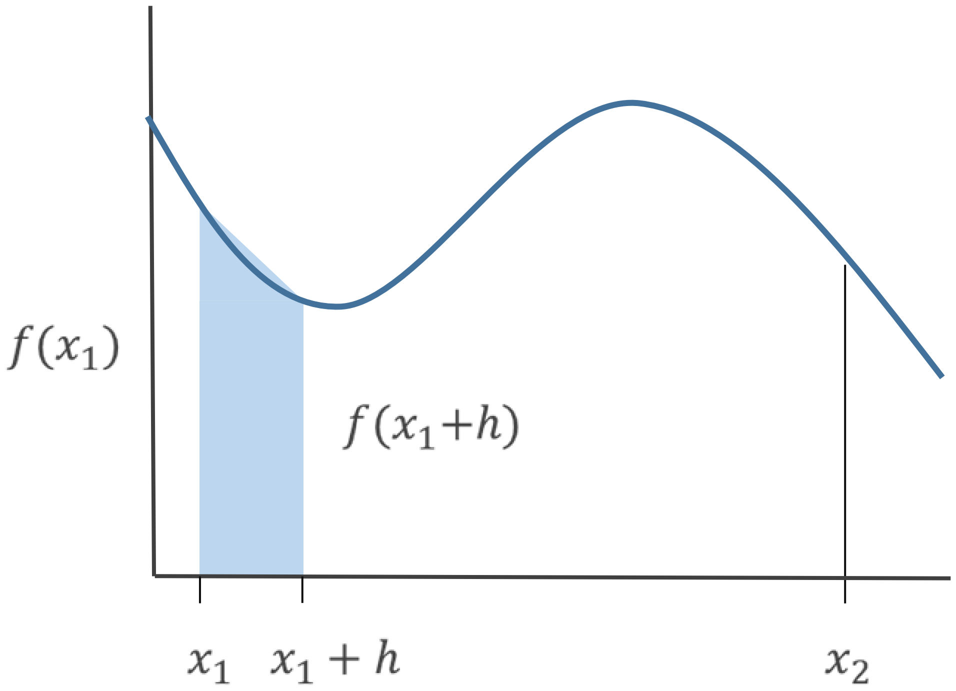
if $$h(x) = f(g(x))$$

then $$\frac{dh}{dx} = \frac{dh}{dg}\frac{dg}{dx}$$

Integration

Area under a continuous function $f(x)$:

$$F(x) = \lim_{h\to 0} \Sigma[f(x_i) + f(x_i + h)]\; h/2$$



If $f(x) = dF/dx$

then the integral of $f(x)$ between x_1 and x_2 is:

$$\int_{x_1}^{x_2} f(x)\, dx = F(x_2) - F(x_1)$$

Sequences and Series

Infinite sequence

$$a_n = f(n)$$

typically: $0 \leq n \leq \infty$

$-\infty \leq n \leq \infty$

examples:

$$a_n = n$$

$$a_n = 1/n$$

$$a_n = 1/n^2$$

$$a_n = (-1)^n$$

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Multivariate Calculus

- Partial derivatives
- Multivariate differentiation
- Optimising multivariate functions

Partial Derivatives

If f is a function of several variables, we can calculate the partial derivative with respect to any single variable by treating the others as constants:

example:

$$f(x, y) = 3x^2 + 2xy$$

$$\frac{\partial f}{\partial x} = 6x + 2y$$

$$\frac{\partial f}{\partial y} = 2x$$

Partial Derivatives – cont’d

For a function operating on a 3-dimensional Euclidean space, the partial derivatives define the ***gradient*** of the function:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

The *del* operator is often written as:

$$\nabla = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \widehat{\boldsymbol{k}}\frac{\partial}{\partial z}$$

Multivariate Optimisation

Given a function $f: A \to \mathbb{R}$

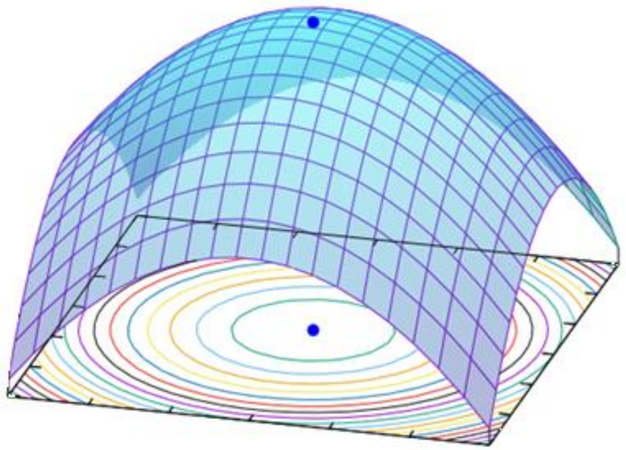
minimisation:

find $x_0 \in A$ such that $f(x_0) \leq f(x) \; \forall \, x \in A$

maximisation:

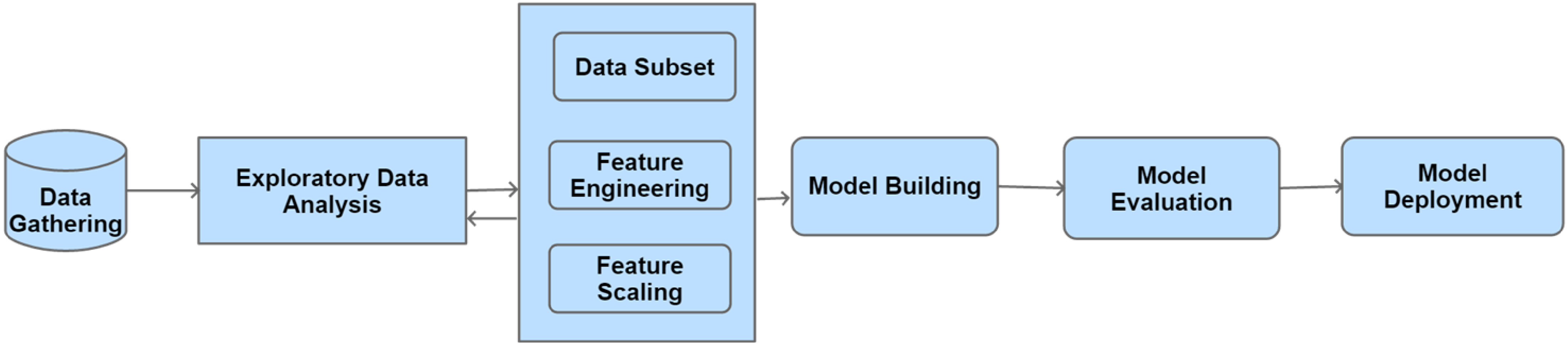
find $x_0 \in A$ such that $f(x_0) \geq f(x) \; \forall \, x \in A$

f = [objective | loss | cost | utility | fitness | energy] function



Discussion

- Why do data scientists need to be proficient at calculus, infinite series, and linear algebra?
- Considering the illustrative Data Science process, where would you use calculus?



Homework: Optimisation

Purpose:

- To introduce several methods of multivariate optimisation via working examples.

Materials:

- An Interactive Tutorial on Numerical Optimization
  - https://www.benfrederickson.com/numerical-optimization/

- Prepare to discuss:
  - trade-off: convergence speed vs accuracy
    - faster convergence requires lower resolution
    - prefer methods with adaptive resolution

Statistics

- Statistical Thinking
- Categorical data
- Continuous variables
- Summarising quantitative data
- Modelling data distributions
- Confidence intervals
- Significance tests and hypothesis testing
- Statistical Inference

Why statistics is important for a Data Scientist?

- **Statistical Thinking** is an an essential component of a data-driven mindset which is crucial for a Data Scientist
  - Statistical analysis must start with the appropriate **data** (sample)
  - Statistical Inference (reasoning) should start with measurement, ideally, via **controlled experiments**
  - Statistics uses samples (a small subset of the population) and therefore always has a degree of **uncertainty**
  - Sampling must be **random, and preferably, independent**

- The best way to learn statistics is by **experimenting with data using Python code and visualisation**

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Statistics – Part 1

- Analysing categorical data

Categorical Variables

Examples

- FALSE / TRUE (alt: 0 / 1)
- colour
- size
- class
  - e.g. species, occupation, degree program, disease category
- tier
  - e.g. age range, income range, frequency range

Analysing Categorical Variables

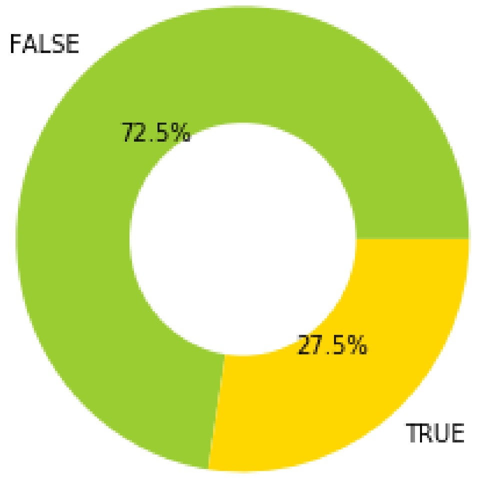
Frequency Tables (*aka* contingency tables)

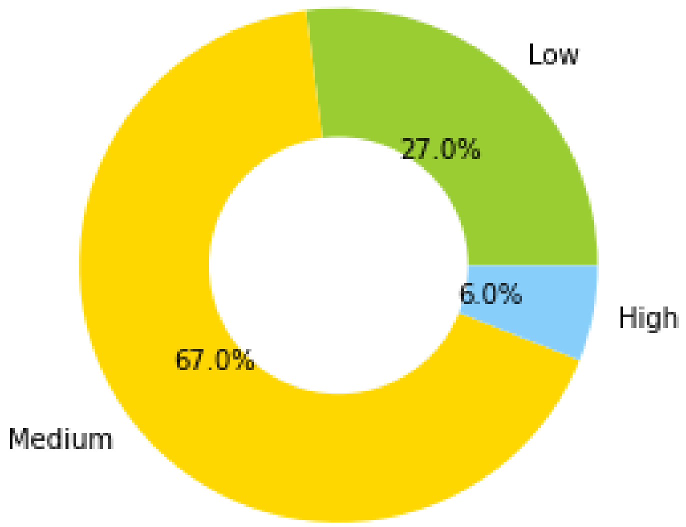
category incidence for a single variable within a population

Passed Exam	
FALSE	37
TRUE	14

Income Bracket	
Low	0.27
Medium	0.67
High	0.06

Donut Charts





Analysing Categorical Variables – cont'd

Two-Way Frequency Tables

- *for a Single Variable within Two Populations*

Income Bracket:	Low	Medium	High	Total
Male	27	75	6	108
Female	32	59	3	94
TOTAL	59	134	9	202

- totals row, column: marginal frequencies (*aka* marginal distribution)

Analysing Categorical Variables – cont'd

Dummy Variables (*aka* dummy coding)

- *allows categorical variables to be treated like continuous variables*

Passed Exam	
0	37
1	14

Treatment	**T1**	**T2**
Control	0	0
Drug 1	1	0
Drug 2	0	1

color	color_red	color_green	color_blue	color_black
red	1	0	0	0
green	0	1	0	0
blue	0	0	1	0
black	0	0	0	1

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Statistics – Part 2

- Continuous variables
- Summarising quantitative data

Continuous Variables

Examples

- height
- dose
- temperature
- concentration
- revenue
- clicks

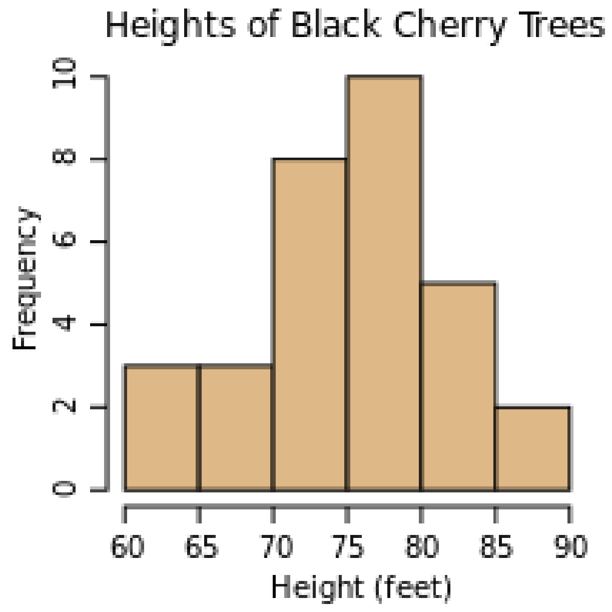
"Continuous"?

- variability is treated as infinite
  - precision is determined by data acquisition methodology
- range usually has practical limits
  - outliers can be defined statistically or heuristically
- *frequency (contingency) is not meaningful*

Analysing Continuous Variables

Distribution of binned data:

- choose an appropriate bin width
- 'cut' the data into bins
- count the number of samples that fall into each bin



what is the resulting plot called?

- histogram

Summarising Quantitative Data

Measuring the centre of the data

mean

the average value of the variable

median

the value that separates the 50% lowest values from the rest

mode

the most frequently occurring value

Summarising Quantitative Data – cont’d

Quantiles

- inverse of binning data for a histogram:
  - specify proportions of samples we want in each bin
  - compute bin boundaries that correspond

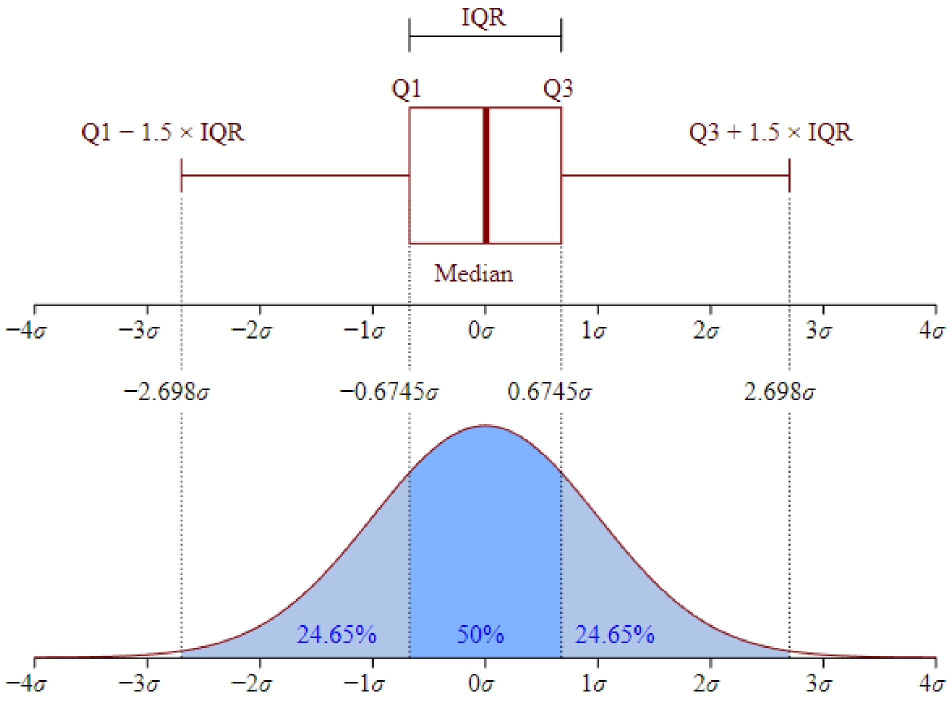
example: 4 quantiles from a random sample (mean = 0, variance = 1):

Quantile	Boundaries	Count
(0 - 0.25]	(-3.135, -1.61]	82
(0.25 - 0.50]	(-1.61, -0.0913]	82
(0.50 - 0.75]	(-0.0913, 1.427]	82
(0.75 - 1.0]	(1.427, 2.946]	82

Summarising Quantitative Data – cont'd

Interquartile range (IQR)

- IQR = [0.25, 0.75]
- box plots are drawn with whiskers extending 1.5 IQR beyond the 0.25 and 0.75 quantiles (i.e. the 1st and 3rd quartiles)
- ***outliers*** are typically defined as lying outside this range



Summarising Quantitative Data – cont'd

Moments of a Sample

mean

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

variance

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

skewness

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^3$$

kurtosis

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^4 - 3$$

Lab 1.1.3: Simple data visualisation

Purpose:

- Use various plot types to visualise statistical observations.

Materials:

- Notebook: 'Lab 1.1.3 Statistics - part 1 Lab'

Note:

- There may not be enough time to complete this lab in the class. Please complete it as a part of you homework.
  This should apply to all labs.

Statistics – Part 3

- Modelling data distributions

Summarising Quantitative Data – cont'd

Summary statistics

standard deviation

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

square root of variance

same units as mean

Modelling Data Distributions

Sample vs Population

μ = mean of population

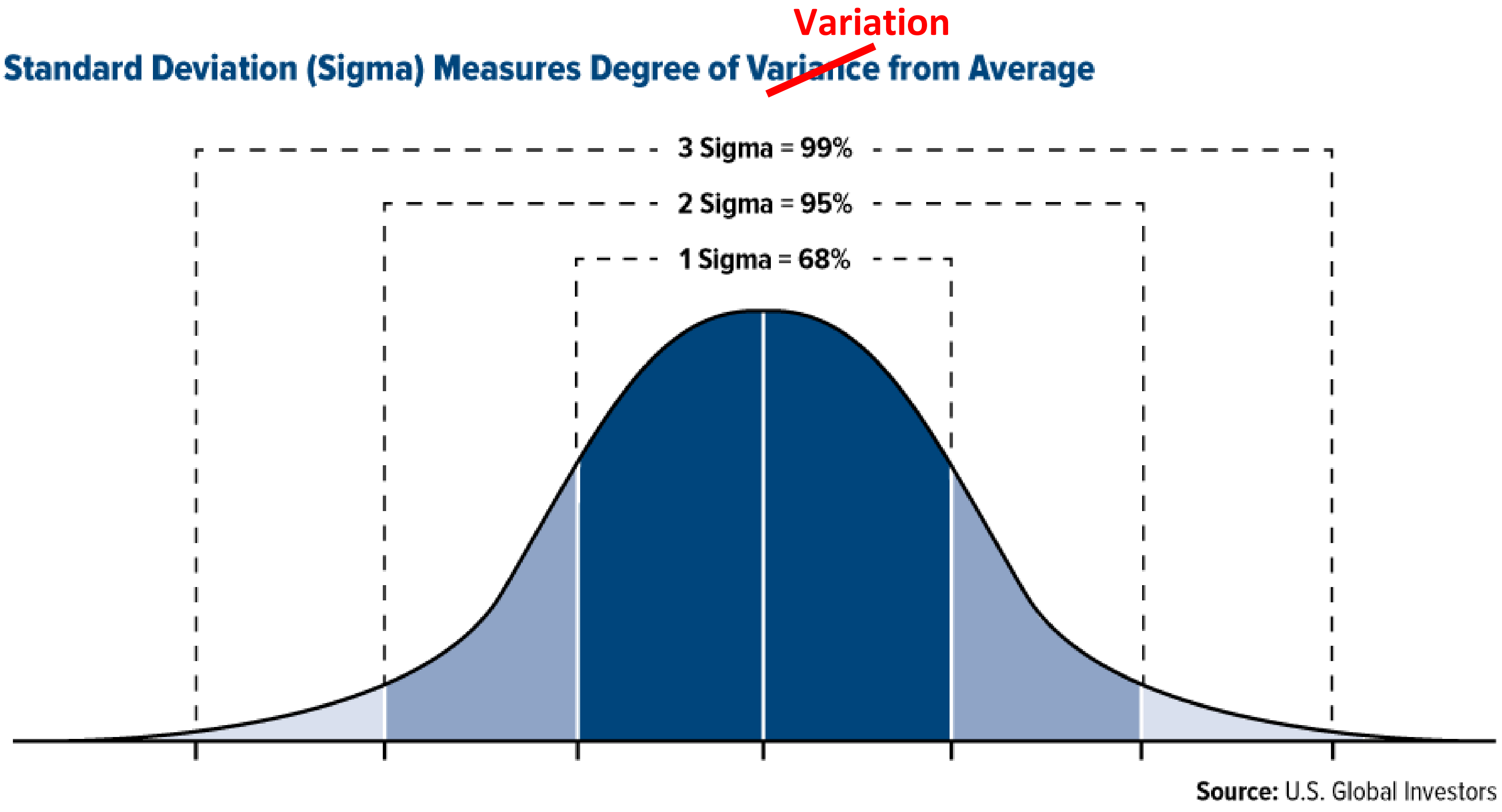
$\bar{x}$ = mean of sample

σ = standard deviation of population

s = standard deviation of sample

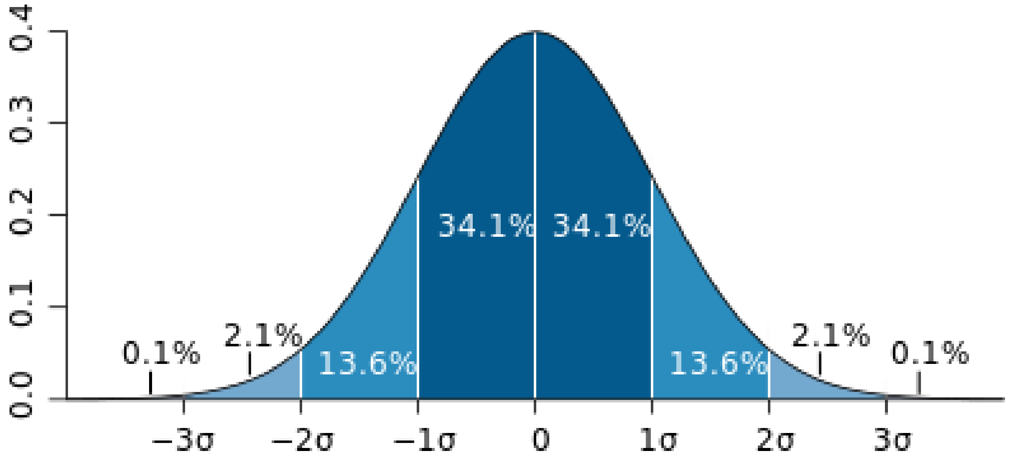
Modelling Data Distributions – cont'd

Mean and Standard Deviation of a Population

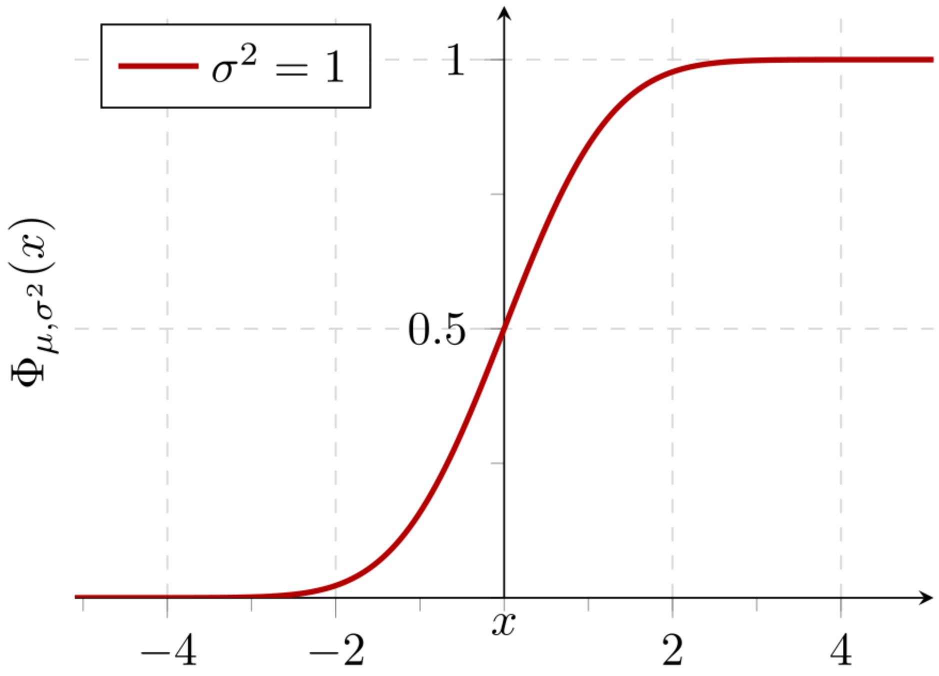


Modelling Data Distributions – cont'd

Probability Density Function



Cumulative Probability

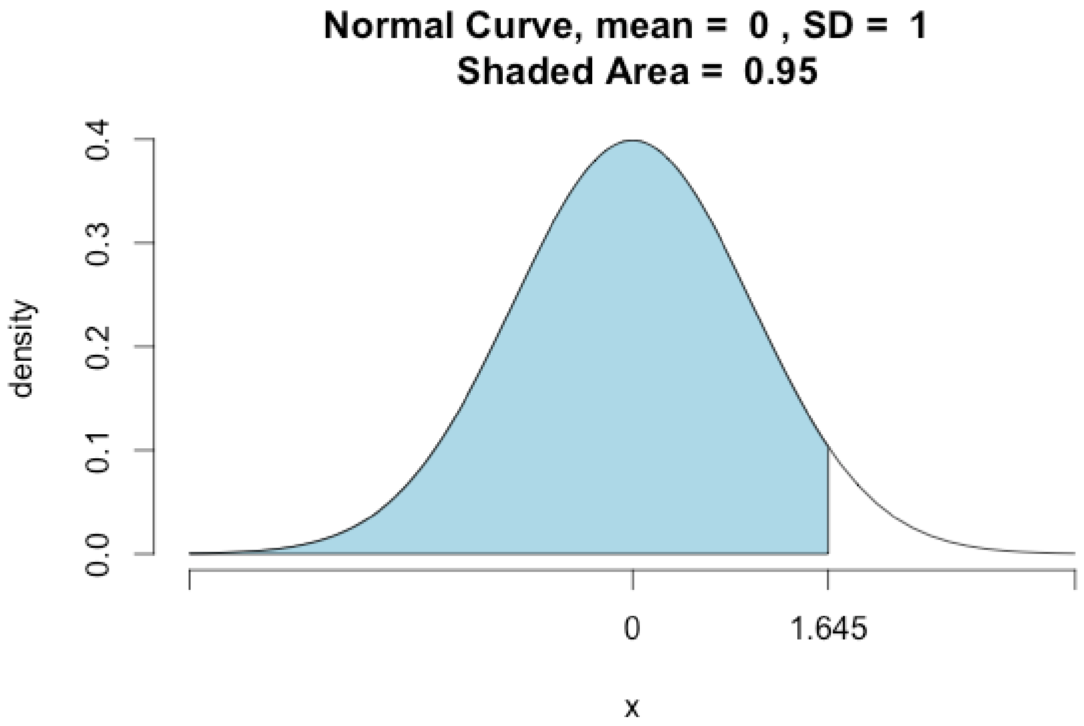


Modelling Data Distributions – cont'd

z-score

measures how far a sample lies from the population mean:

$$z = \frac{x - \mu}{\sigma}$$

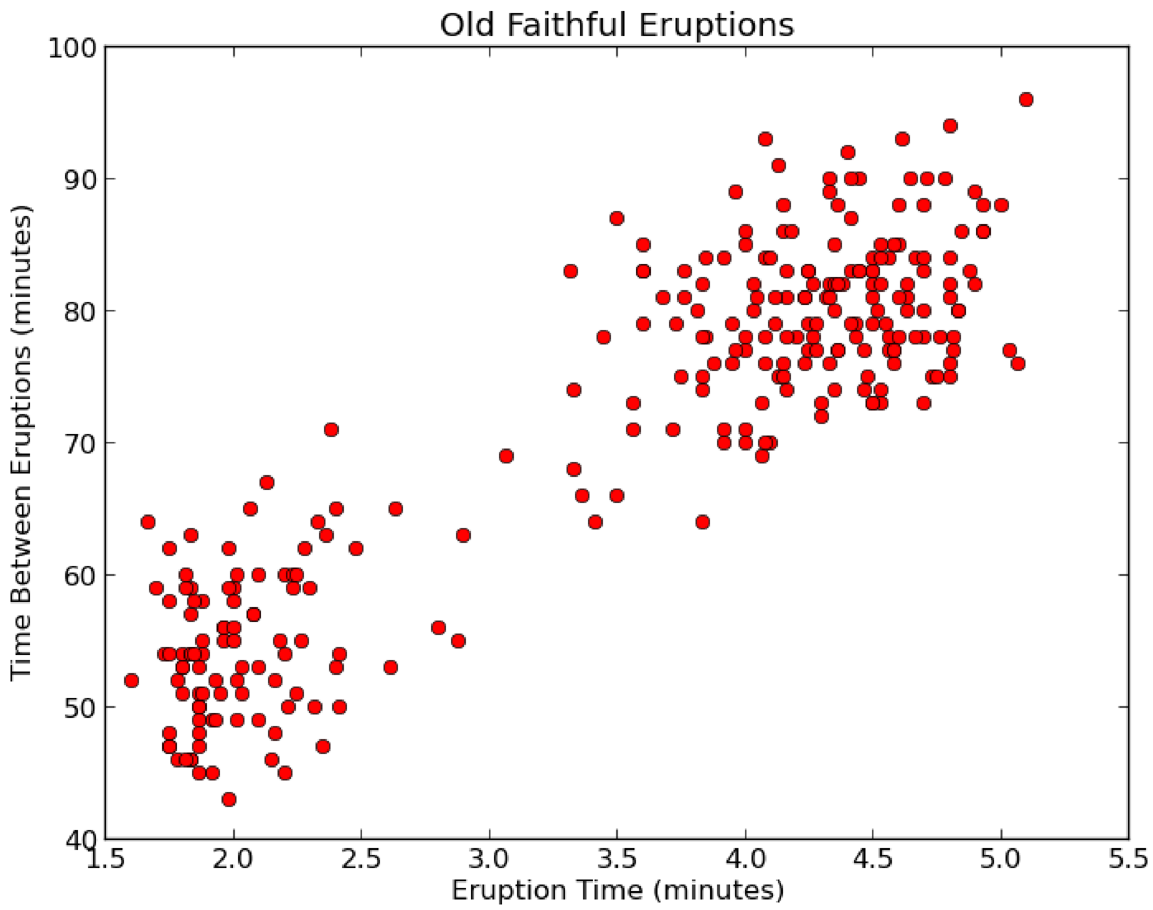


Statistics – Part 4

- Exploring bivariate numerical data

Scatter Plots

- 2D: plots one variable against another
- demonstrates a relation (or lack thereof) between two variables
- *assumption: data pairs are sampled simultaneously*



Correlation

Pearson correlation coefficient

measures strength of covariance between one variable and another:

$$r_{xy} = \frac{1}{(n-1)\, s_x\, s_y} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

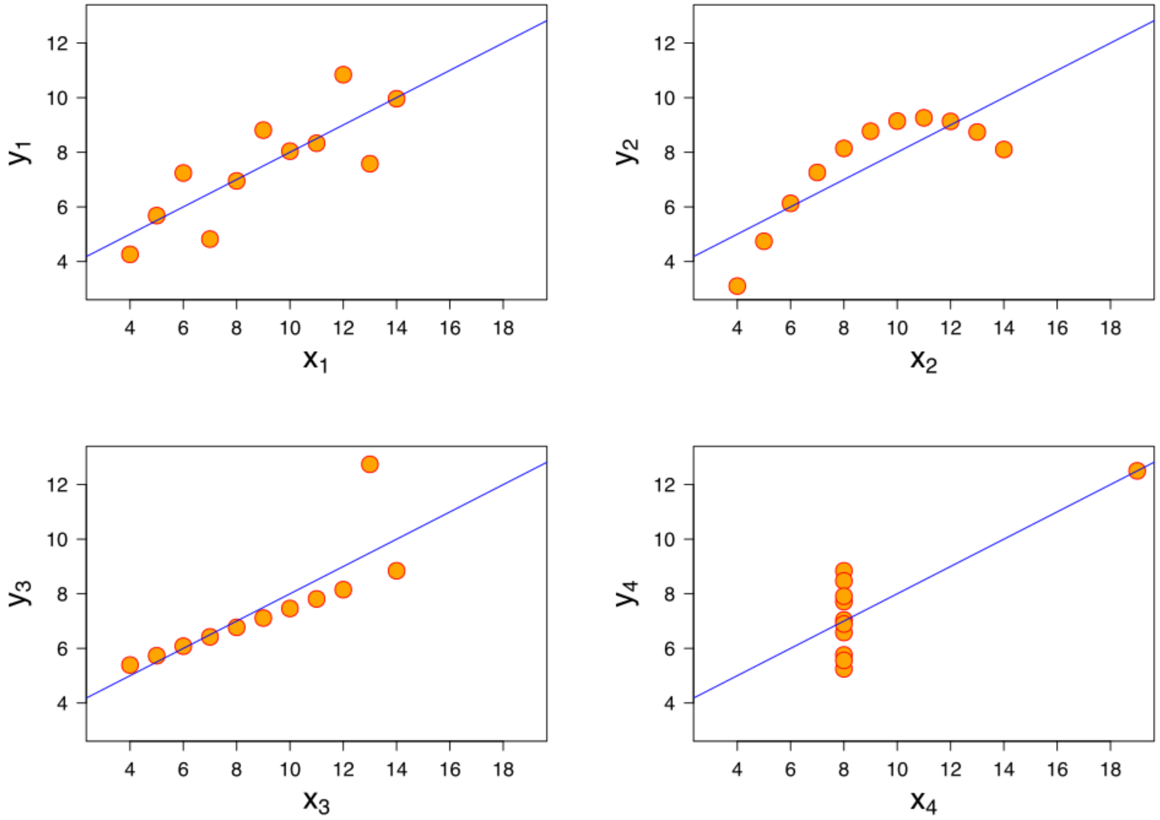
- large when big variations in y correspond to big variations in x
- small when small variations in y cancel out big variations in x (or vice versa)

Correlation

Anscomb’s quartet

four very different sets of 11 data pairs, each with $r_{xy} = 0.816$

- correlation coefficient
    - assumes a *linear* relation
    - does not completely characterise the relationship between x and y



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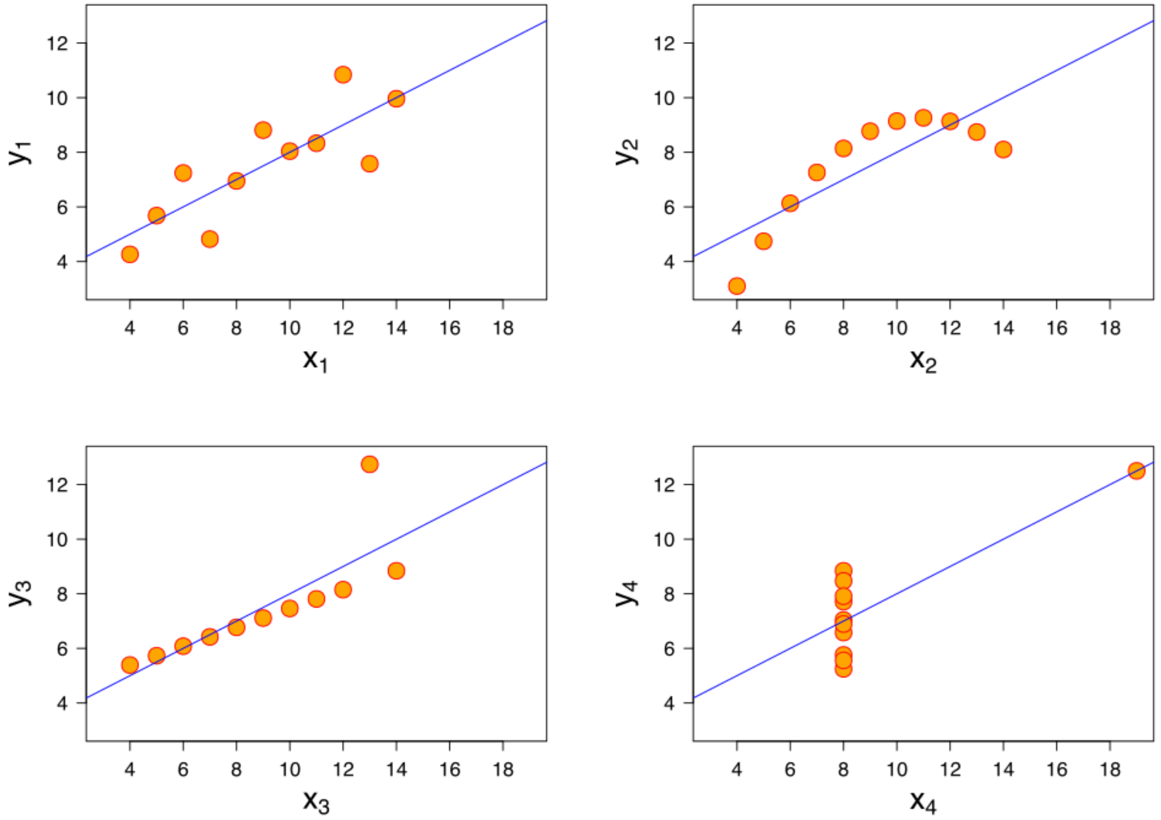
Trend Lines

best (linear) fit to a 2D scatter plot

- the line that minimises error
  by some criterion
- line is specified by
  - slope
  - intercept



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Least-Squares Regression

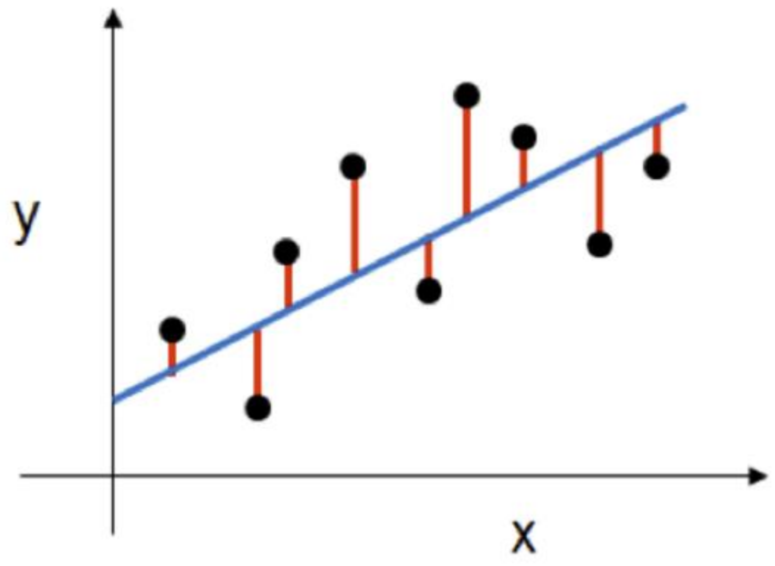
def: ***residual***

the difference between the observed and predicted values:

$$\varepsilon_i = y_i - \hat{y}_i$$

- least-squares criterion:

$$err = \sum_{i=1}^{n} {\varepsilon_i}^2$$



Least-Squares Regression

minimise:

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1\, x_i)\right)^2$$

solution:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Statistics – Part 5

- Random variables

Random Variables

Discrete random variables

- range is finite or countably infinite
- distribution can be described by a ***probability mass function***
  - assigns a probability to each value in the image of X

Continuous random variables

- distribution can be described by a ***probability density function***
  - assigns a probability to each specified interval over the range of X

Transforming random variables

standardising variables for analysis:

- ***centering*** (subtracting the mean)
- ***scaling*** (dividing by SD)
- allows standard tables to be used to compute percentiles of the sample distribution and probabilities of sampled values

$$x' = \frac{x - \bar{x}}{s}$$

recall: z-score $z = \frac{x-\mu}{\sigma}$

Transforming random variables – cont'd

offset

$$x' = x + 1$$

- datasets based on counts (binning) may contain zeros
  - examples:
    - calls received in each minute at a call centre
    - instances of a keyword in a corpus
- if the method relies on the logarithm of the count (which many do), it will blow up for $x_i = 0$

Transforming random variables – cont'd

logarithmic rescaling

$$x' = \log(x)$$

- datasets with large dynamic range
  - examples:
    - lifetime value of customer
- algorithm could be skewed if a small amount data with large values dominates a large amount of data with small values

Transforming random variables – cont'd

Box-Cox transformation

$$x' = x^{\lambda} \qquad \lambda \in \{0, \pm 0.5, \pm 1, \pm 2, \pm 3\}$$

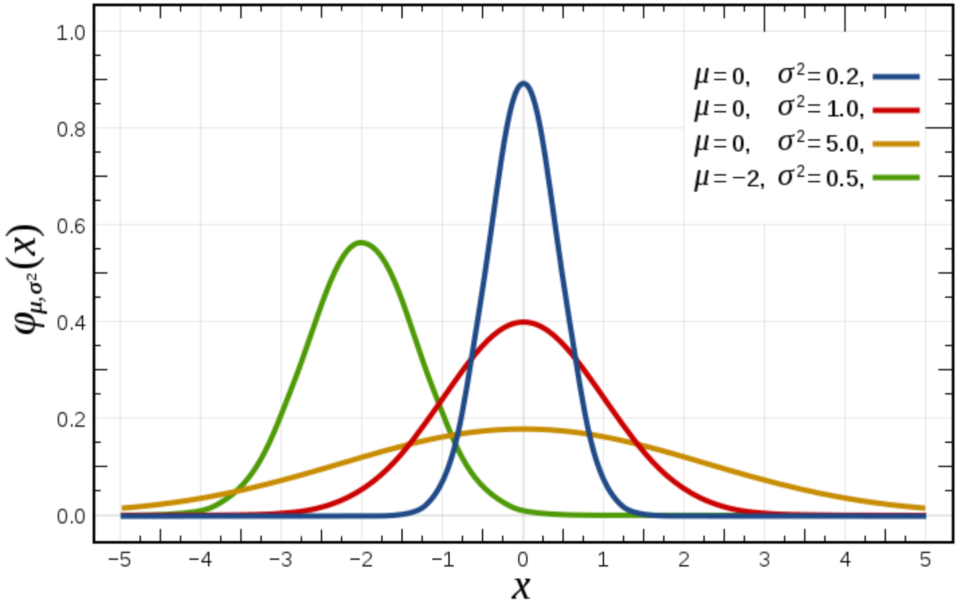
- datasets high skewness or kurtosis
- try different values of λ
  - choose the one that gives the most normal distribution

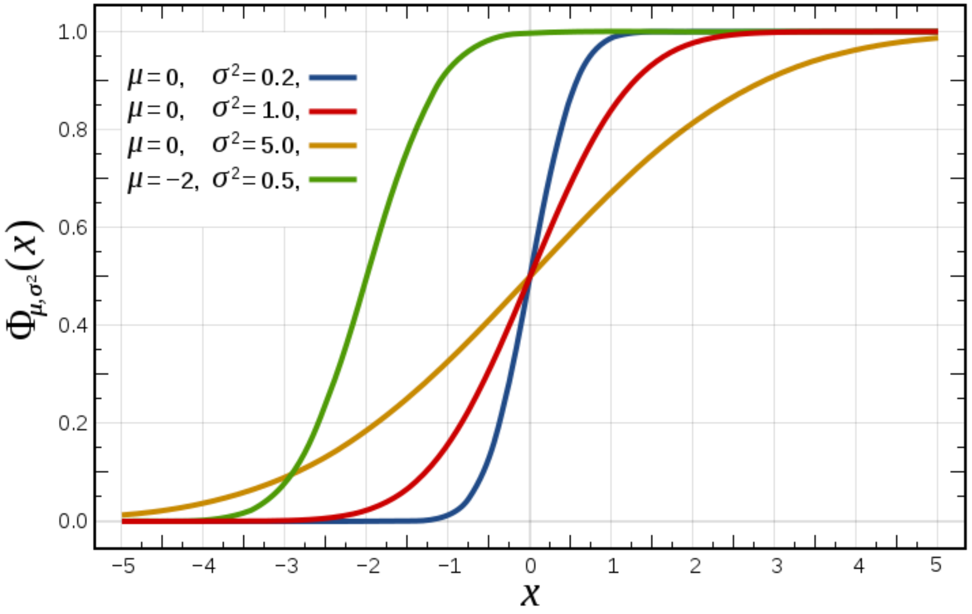
Normal Distribution

aka Gaussian distribution, bell curve

$$PDF = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$CDF$$





By Inductiveload - self-made, Mathematica, Inkscape, Public Domain, https://commons.wikimedia.org/w/index.php?curid=3817954

Other types of Probability Distributions

- Bernoulli distribution
  - The outcome of a single Bernoulli trial (e.g. success/failure, yes/no)
- Binomial distribution
  - The number of "positive occurrences" (e.g. successes, yes votes, etc.) given a fixed total number of independent occurrences
- Geometric distribution
  - Binomial-type observations but where the quantity of interest is the number of failures before the first success; a special case of the negative binomial distribution

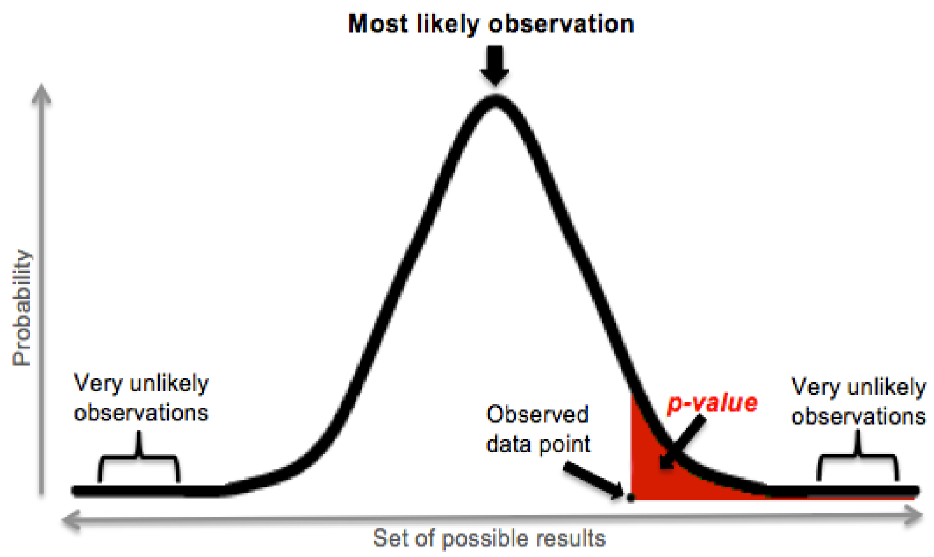
Statistics – Part 6

- Confidence intervals
- Significance tests and hypothesis testing
- Inference
- ANOVA

p-Value

P-value measures the probability that a more extreme-valued sample could be randomly drawn from the distribution.

$$p(x > x') = 1 - \mathrm{p}(\mathrm{x} \leq x')$$



A *p-value* (shaded red area) is the probability of an observed (or more extreme) result arising by chance

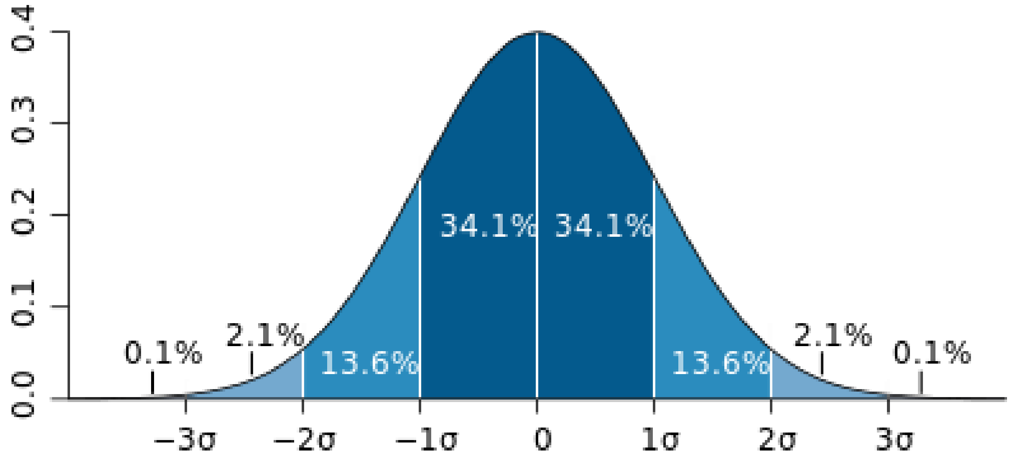
Confidence Intervals

recall: *z-score*

measures how far a sample lies from the population mean:

$$z = \frac{x-\mu}{\sigma}$$

normal distribution:



mean ±	% population
1 σ	68.2
2 σ	95.4
3 σ	99.7

Confidence Intervals – cont’d

We define confidence intervals in terms of target probability bands:

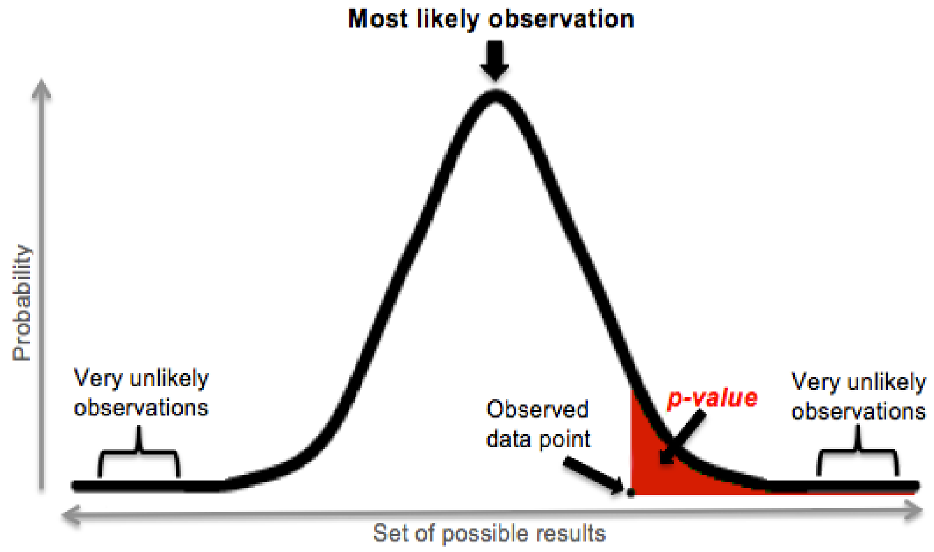
confidence interval	mean ±	p-value
0.68	~1 σ	0.32
0.95	~2 σ	0.05
0.99	~3 σ	0.01

Significance Tests

- Given a specified confidence interval, is a particular sample close enough to the mean that we can confidently presume that comes from the same population?
- Conversely, can we say that the new sample is *different* enough that it probably comes from a different population?

example:

if we choose $p = 0.05$, then
a new sample $x^o > \bar{x} + 2\sigma$
then we can say that x^o is
significantly greater than $\bar{x}$
(for a 95% confidence interval)



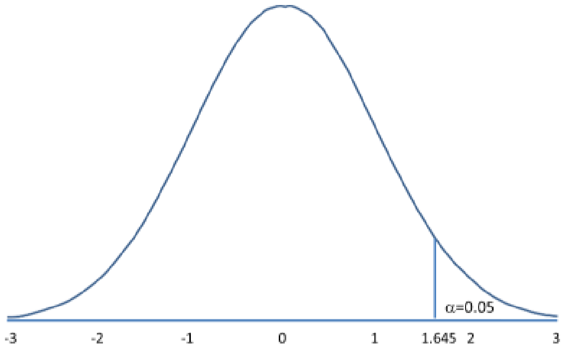
A *p-value* (shaded red area) is the probability of an observed (or more extreme) result arising by chance

One-Tailed Test vs Two-Tailed Test

One-tailed test: is B greater than A?

a 95% confidence interval would mean we are interested in the last 5% of the right tail

Nb. for “Is B less than A” we would be looking at the left tail instead of the right.



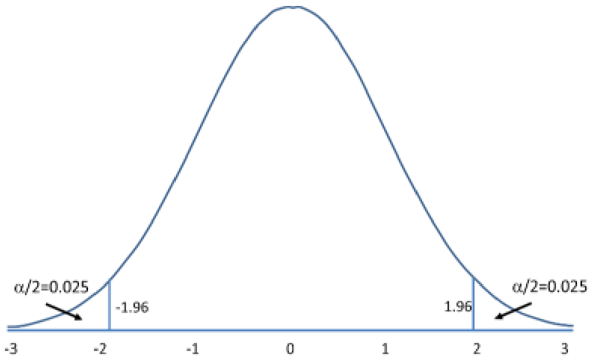
Rejection Region for Upper-Tailed Z Test (H_1: $\mu > \mu_0$) with α=0.05

The decision rule is: Reject H_0 if $Z \geq 1.645$.

standard normal distribution

Two-tailed test: is B different from A?

a 95% confidence interval would mean we are interested in the last 2.5% of *each* tail



Rejection Region for Two-Tailed Z Test (H_1: $\mu \neq \mu_0$) with α =0.05

The decision rule is: Reject H_0 if $Z \leq -1.960$ or if $Z \geq 1.960$.

Standard Error of the Mean

Corrects the standard deviation for a finite population (i.e. an acquired dataset):

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \cong \frac{s}{\sqrt{n}}$$

σ is the population SD
s is the sample SD

Central Limit Theorem:

- as sample size increases, SEM approaches the population mean

Student's *t*-Test

Corrects the z-score for a finite population (i.e. an acquired dataset):

$$t = \frac{z}{\sigma_{\bar{x}}} = \frac{(\bar{x} - \mu)}{\sigma / \sqrt{n}}$$

z is the z-score

$\sigma_{\bar{x}}$ is the standard error of the mean

μ, σ are the population mean, SD

$\bar{x}, s$ are the sample mean, SD

Null Hypothesis

If we want to test whether ***B*** is different from ***A***, we first assume that it is not.

Then we test to see if the difference between ***A*** and ***B*** is likely to occur by random chance.

If the difference between ***A*** and ***B*** exceeds the confidence interval, we reject the null hypothesis and infer that ***B*** is not from the same population.

ANOVA

- For comparing multiple groups, repeated application of the *t*-test would randomly give rise to apparent significance
- ANOVA avoids this error by introducing the *F*-test (analogous to the t-test but for more than 2 groups)
- You can use SciPy to estimate variations between two or more groups

Probability

- Basic theoretical probability
- Bayesian inference
- Probability using sample spaces
- Basic set operations
- Permutations and combinations
- Conditional probability and independence

Probability

If A, B are events, the probability of …

A occurring $= P(A)$

A not occurring $= 1 - P(A)$

both occurring (and) $= P(A \cap B)$

either occurring (or) $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A occurring if B occurs (conditional) $= P(A|B) = \frac{P(A \cap B)}{P(B)}$

If A, B are **independent** events, $P(A \cap B) = P(A)\,P(B)$ and $P(A|B) = P(A)$.

Sample Space

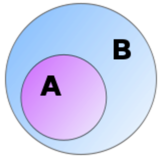
def: The set of all possible outcomes of an experiment

- Ordered: sequence is important
- Unordered: sequence is ignored

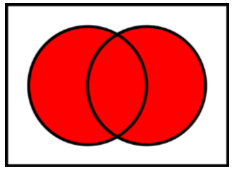
$$\Omega = \{s_1, s_2, \dots\ s_n\}$$

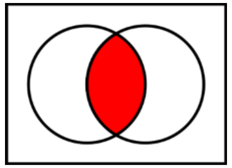
$$P(A) = \frac{\text{number of outcomes in event } A}{\text{number of outcomes in sample space } \Omega}$$

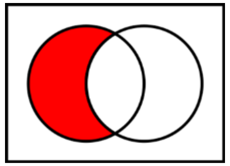
Set Operations

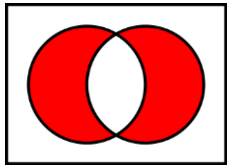
$A \subseteq B$ 

subset

$A \cup B$  union

$A \cap B$  intersection

$A - B$  relative complement of B in A

$A \triangle B$  symmetric difference of A and B

Permutations and Combinations

Permutation: an ordered set

Combination: an unordered set

number of k-permutations of n:

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$

number of k-combinations of n:

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Bayes' inference theorem

- Bayes' inference theorem used to update the probability for a hypothesis as more evidence or information becomes available.
- Theorem:
  - $P(H|E) = P(E|H).P(H)/P(E)$
- Definition:
  - $P(H|E)$: The probability of a event H given E

Lab 1.1.4: Applying statistical thinking using Python

Purpose:

- Explore how to use Python (and related packages) to apply Statistical Thinking on data.

Materials:

- Notebook: 'Lab 1.1.4 Statistics - part 2 Lab'

Note:

- The may not be enough time to complete this lab in the class.
  Please complete it as a part of you homework.
  This should apply to all labs.

Questions?

End of Presentation!