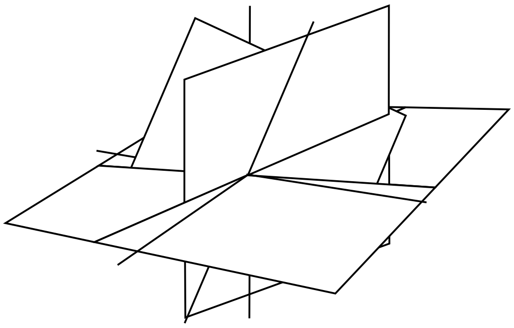
MAT 3004

APPLIED LINEAR ALGEBRA



Digital Assignment – 1

A2 | MB224

WINTER SEMESTER 2020-21

by

SHARADINDU ADHIKARI

19BCE2105

  sharadindu.adhikari2019@vitstudent.ac.in

Answer Any Five ($5 \times 2 = 10$ Marks)

Question 1

1. Let V and W be the subspaces of the vector space $\mathbb{R}^4$ spanned by $v_1 = (3,-1,4,1)$, $v_2 = (5,0,5,1)$, $v_3 = (5,-5,10,3)$ and $w_1 = (9,-3,3,2)$, $w_2 = (5,-1,2,1)$, $w_3 = (6,0,4,1)$, respectively. Find the bases and dimensions for $V + W$ nd $V \cap W$, and hence prove that $dim(V + W) = dim(V) + dim(W) - dim(V \cap W)$.

Solution:

A1.

$V = [v_1 \ v_2 \ v_3]$

we've to check if v_1, v_2, and v_3 are the bases of V or not.

Converting V into row-echelon form:

$$V = \begin{bmatrix} 3 & 5 & 5 \\ -1 & 0 & -5 \\ 4 & 5 & 10 \\ 1 & 1 & 3 \end{bmatrix} \quad R_1 \leftarrow R_1 + 2R_2$$

$$= \begin{bmatrix} 1 & 5 & -5 \\ -1 & 0 & -5 \\ 4 & 5 & 10 \\ 1 & 1 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 4R_1 \\ R_4 \leftarrow R_4 - R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 5 & -5 \\ 0 & 5 & -10 \\ 0 & -15 & 30 \\ 0 & -4 & 8 \end{bmatrix} \quad R_2 \leftarrow R_2 + R_4$$

$$= \begin{bmatrix} 1 & 5 & -5 \\ 0 & 1 & -2 \\ 0 & -15 & 30 \\ 0 & -4 & 8 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow R_1 - 5R_2 \\ R_3 \leftarrow R_1 + 15R_2 \\ R_4 \leftarrow R_1 + 4R_2 \end{matrix}$$

 © Sharadindu Adhikari, 19BCE2105 sharadindu.adhikari2019@vitstudent.ac.in

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As we can clearly see from the matrix, the 3rd vector, i.e., the v_3 is not basis, only v_1 and v_2 are.

Hence, $\dim(V) = 2$.

Now, $W = [w_1 \quad w_2 \quad w_3]$

we've to check if w_1, w_2, w_3 are bases of W or not.

Converting W into row-echelon form:

$$W = \begin{bmatrix} 9 & 5 & 6 \\ -3 & -1 & 0 \\ 3 & 2 & 4 \\ 2 & 1 & 1 \end{bmatrix} \quad R_1 \leftarrow R_1 - 4R_4$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ -3 & -1 & 0 \\ 3 & 2 & 4 \\ 2 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \leftarrow R_2 + 3R_1 \\ R_3 \leftarrow R_3 - 3R_1 \\ R_4 \leftarrow R_4 - 2R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 6 \\ 0 & -1 & -2 \\ 0 & -1 & -3 \end{bmatrix} \quad R_2 \leftarrow R_2 + R_3$$

  sharadindu.adhikari2019@vitstudent.ac.in

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & -3 \end{bmatrix} \quad \begin{aligned} R_1 &\leftarrow R_1 - R_2 \\ R_3 &\leftarrow R_3 + R_2 \\ R_4 &\leftarrow R_4 + R_2 \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} R_3 &\leftarrow R_3/2 \\ R_1 &\leftarrow R_1 + 2R_3 \\ R_2 &\leftarrow R_2 - 4R_3 \\ R_4 &\leftarrow R_4 - R_3 \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

As we can clearly see from the matrix, all the vectors are linearly independent to each other, & hence they are all bases.

$\therefore$ $\dim(W) = 3$.

again, we've: $Q = [v_1 \ \ v_2 \ \ w_1 \ \ w_2 \ \ w_3]$

Gotta reduce [Q] into row-echelon form and see for ourselves its dependencies.

< p.t.o. >

$$Q = \begin{bmatrix} 3 & 5 & 9 & 5 & 6 \\ -1 & 0 & -3 & -1 & 0 \\ 4 & 5 & 3 & 2 & 4 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix} \quad R_1 \leftarrow R_1 + 2R_2$$

$$= \begin{bmatrix} 1 & 5 & 3 & 3 & 6 \\ -1 & 0 & -3 & -1 & 0 \\ 4 & 5 & 3 & 2 & 4 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 4R_1 \\ R_4 \leftarrow R_4 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 5 & 3 & 3 & 6 \\ 0 & 5 & 0 & 2 & 6 \\ 0 & -15 & -9 & -10 & -20 \\ 0 & -4 & -1 & -2 & -5 \end{bmatrix} \quad R_2 \leftarrow R_2 + R_4$$

$$= \begin{bmatrix} 1 & 5 & 3 & 3 & 6 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & -15 & -9 & -10 & -20 \\ 0 & -4 & -1 & -2 & -5 \end{bmatrix} \quad \begin{array}{l} R_1 \leftarrow R_1 - 5R_2 \\ R_3 \leftarrow R_3 + 15R_2 \\ R_4 \leftarrow R_4 + 4R_2 \end{array}$$

  sharadindu.adhikari2019@vitstudent.ac.in

$$= \begin{bmatrix} 1 & 0 & 8 & 3 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & -24 & -10 & -5 \\ 0 & 0 & -5 & -2 & -1 \end{bmatrix} \quad R_3 \leftarrow R_3 - 5R_4$$

$$= \begin{bmatrix} 1 & 0 & 8 & 3 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & -2 & -1 \end{bmatrix} \quad \begin{array}{l} R_1 \leftarrow R_1 - 8R_3 \\ R_2 \leftarrow R_2 + R_3 \\ R_4 \leftarrow R_4 + 5R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{array}{l} R_1 \leftarrow R_1 + R_4 \\ R_4 \leftarrow \left(-\frac{1}{2}\right) R_4 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \quad R_1 \leftarrow R_1 - R_4$$

  sharadindu.adhikari2019@vitstudent.ac.in

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right]$$

As we can clearly see from the matrix, there are 4 leading ~~0s~~ 1s, and so we have 4 bases.

So, The bases of Q will be the bases of $(V+W)$.

They're:

$$(3,-1,4,1);\ (5,0,5,1);\ (9,-3,3,2);\ (5,-1,2,1).$$

$$\dim(Q) = \dim(V) = \dim(V+W) = 4$$

Finally, we've:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

  sharadindu.adhikari2019@vitstudent.ac.in

$$\Rightarrow \quad x_1 - \frac{x_5}{2} = 0 \quad \Rightarrow \quad x_1 = \frac{x_5}{2}$$

$$x_2 + x_5 = 0 \quad \Rightarrow \quad x_2 = -x_5$$

$$x_3 = 0 \quad \Rightarrow \quad x_3 = 0$$

$$x_4 + \frac{x_5}{2} = 0 \quad \Rightarrow \quad x_4 = \frac{-x_5}{2}$$

we've to take 1 independent variable (or free variable, i.e.), so I'm taking x_5 here.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 1/2 \\ -1 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}$$

So the bases of $N(Q) = N(V)$ is $\left(1/2, -1, 0, -1/2, 1\right)$.

$$\dim(N(Q)) = \dim(N(V)) = \dim(V \cap W) = 1$$

Bases of $N(Q)$ = Bases of $(N \cap W)$.

$$\therefore \quad \dim(V+W) = 4$$

$$\dim(V) = 2$$

$$\dim(W) = 3$$

$$\dim(V \cap W) = 1$$

clearly, $4 = 2 + 3 - 1$

$$\Rightarrow \dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W).$$

Hence, proven

Question 2

2. (a) Let A and B be two $k \times k$ matrices. Show that if $AB = 0$, then the column space of B is a subspace of the null space of A.
(b) Find the right inverse of the matrix $\begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, if exists.

Solution:

A2. (a) Let the columns of B be $b_1, b_2, \ldots\ldots, b_n$.

Then, $B = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots\cdots & b_n \end{bmatrix}$

$$AB = 0$$

$$A\begin{bmatrix} b_1, b_2, \ldots, b_n \end{bmatrix} = \begin{bmatrix} Ab_1, Ab_2, \ldots\ldots, Ab_n \end{bmatrix} = 0$$

$\Longleftrightarrow$ $Ab_1 = Ab_2 = \cdots\cdots = Ab_n = 0$

$\Longleftrightarrow$ All the elements of column space of B must be contained in the Null space of A.

Hence, proven.

(b) First, we've to see for ourselves, whether the right inverse of the given matrix exists or not.

For that, we'll take its transpose and then find its Nullity. If the variables come out to be 0, then it has right inverse matrix.

We've:

$$\begin{bmatrix} 6 & 3 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \quad R_1 \leftarrow R_1 - R_2$$

$$= \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \quad R_1 \leftarrow R_1/2$$

$$= \begin{bmatrix} 1 & 1/2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 4R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \\ 0 & -1/2 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow R_1 + R_3 \\ R_2 \leftarrow R_2 + (-2) R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

As we can clearly see, both of them are independent of each other,

Then,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

  sharadindu.adhikari2019@vitstudent.ac.in

$$\Rightarrow \quad x_1 = 0 \,, \quad x_2 = 0$$

As both x_1 and x_2 are 0, we can say that they've right inverse matrix.

Now, for getting the right matrix, we can write the inverse as : $A A^T \cdot (A A^T)^{-1} = I$

So basically,

$$A \cdot \underbrace{A^T \cdot (A \cdot A^T)^{-1}}_{\text{this part is our right matrix}} = I$$

we've :

$$A \cdot A^T = \begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 3 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 61 & 29 \\ 29 & 14 \end{bmatrix}$$

we know that,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{61 \times 14 - 29 \times 29} \begin{bmatrix} 14 & -29 \\ -29 & 61 \end{bmatrix}$$

  sharadindu.adhikari2019@vitstudent.ac.in

$$\Rightarrow \left[A \cdot A^T\right]^{-1} = \frac{1}{13}\begin{bmatrix} 14 & -29 \\ -29 & 61 \end{bmatrix}$$

$$A^T \cdot \left(A \cdot A^T\right)^{-1} = \begin{bmatrix} 6 & 3 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \cdot \frac{1}{13} \cdot \begin{bmatrix} 14 & -29 \\ -29 & 61 \end{bmatrix}$$

$$= \frac{1}{13} \cdot \begin{bmatrix} -3 & 9 \\ -2 & 6 \\ 13 & -26 \end{bmatrix}$$

$$\therefore \text{reqd. Right matrix} = \begin{bmatrix} -\frac{3}{13} & \frac{9}{13} \\ -\frac{2}{13} & \frac{6}{13} \\ 1 & -2 \end{bmatrix}$$

  sharadindu.adhikari2019@vitstudent.ac.in

Question 4

4. (a) Let V and W be two vector spaces over $\mathbb{R}$. Prove that if V and W are isomorphic, then $dim\, V = dim\, \mathrm{W}$.
(b) Find the transition matrix from the standard ordered basis α to anot er basis β for $\mathbb{R}^3$, where $\beta = \{(1,1,0), (1,1,1), (0,1,1)\}$.

Solution:

A4. (a)

Let $T: V \to W$ be an isomorphism, and let $[v_1\ v_2 \cdots v_n]$ be a basis of V. Then, we show that the set $[T(v_1)\ T(v_2) \cdots T(v_n)]$ is a basis for W, so that

$$\dim(W) = n = \dim(V).$$

(1) It is linearly independent: Since T is one-to-one, the equation $0 = c_1 \cdot T(v_1) + \cdots + c_n \cdot T(v_n)$

$$= T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

implies that $0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$.

since the V_i's are linearly independent, we've:

$C_i = 0$ for all $i = 1, 2, \ldots, n$.

(2) It spans W: Since T is onto, for any $y \in W$, there exists an $x \in V$, such that $T(x) = y$,

$$x = \sum_{i=1}^{n} a_i v_i.$$

Then,
$$y = T(x) = T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$$
$$= a_1 \cdot T(v_1) + a_2 \cdot T(v_2) + \cdots + a_n \cdot T(v_n)$$

i.e. y is a linear combination of $T(v_1), T(v_2), \ldots\ldots, T(v_n)$.

Conversely, if $\boxed{\dim V = \dim W = n}$,

Then one can choose bases $\{v_1, v_2, \ldots\ldots, v_n\}$ and $\{w_1, w_2, \ldots\ldots, w_n\}$ for V and W respectively.

There exists a linear transformation $T: V \to W$ and $S: W \to V$, such that:

$$T(v_i) = w_i \quad \text{and} \quad S(w_i) = v_i \quad \forall\, i = 1, 2, \ldots, n.$$

clearly, $(S \circ T)(v_i) = v_i$
and $(T \circ S)(w_i) = w_i$, $\forall\, i = 1, 2, \ldots\ldots, n.$

which implies that $S \circ T$ and $T \circ S$ are the identity transformations on V and W, respectively.

Hence, T and S are isomorphisms, and consequently, V and W are isomorphic.

(b) Standard ordered basis $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

and given $\beta = \{(1,1,0), (1,1,1), (0,1,1)\}$ for $\mathbb{R}^3$.

We've to find the transition matrix from α to β.

$$T(v_1) = a_{11}(1,1,0) + a_{21}(1,1,1) + a_{31}(0,1,1)$$
$$T(v_2) = a_{12}(1,1,0) + a_{22}(1,1,1) + a_{32}(0,1,1)$$
$$T(v_3) = a_{13}(1,1,0) + a_{23}(1,1,1) + a_{33}(0,1,1)$$

Hence the transition matrix $= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

also,

$$(1,0,0) = (a_{11}+a_{21},\ a_{11}+a_{21}+a_{31},\ a_{21}+a_{31})$$
$$(0,1,0) = (a_{12}+a_{22},\ a_{12}+a_{22}+a_{32},\ a_{22}+a_{32})$$
$$(0,0,1) = (a_{13}+a_{23},\ a_{13}+a_{23}+a_{33},\ a_{23}+a_{33})$$

$\Rightarrow a_{11}+a_{21} = 1$; $a_{11}+a_{21}+a_{31} = 0$; $a_{21}+a_{31} = 0$

$\boxed{a_{21} = 1} = 1-0$; $a_{11} + (-a_{31}) + a_{31} = 0$; $a_{21} = -a_{31}$

$\boxed{a_{11} = 0}$; $\boxed{a_{31} = -1}$

sharadindu.adhikari2019@vitstudent.ac.in

$a_{12} + a_{22} = 0$; $a_{12} + a_{22} + a_{32} = 1$; $a_{22} + a_{32} = 0$

$a_{12} = -a_{22}$ ✓ ; $-\cancel{a_{22}} + \cancel{a_{22}} + a_{32} = 1$; $a_{32} = -a_{22}$ ✓

$\boxed{a_{12} = 1}$; $\boxed{a_{32} = 1}$; $\boxed{a_{22} = -1}$

$a_{13} + a_{23} = 0$; $a_{13} + a_{23} + a_{33} = 0$; $a_{23} + a_{33} = 1$

$a_{13} = -a_{23}$; $-a_{23} + a_{23} + a_{33} = 0$; $\boxed{a_{23} = 1}$

$\boxed{a_{13} = -1}$; $\boxed{a_{33} = 0}$

$$\text{Transition matrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$ ✓

$$= \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Question 5

5. Let T: $P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be a transformation defined as: $T(f(x)) = f'(x)\ \forall\ f(x) \in P_2(\mathbb{R})$, and S: $P_1(\mathbb{R}) \to P_2(\mathbb{R})$ be transformation defined as: $S(g(x)) = xg(x)\ \forall\ g(x) \in P_1(\mathbb{R})$. Prove that S and T are linear transformations and also find the matrix representation of ToS w.r.t. standard bases $\alpha = \{1, x, x^2\}$ of $P_2(\mathbb{R})$ and $\beta = \{1, x\}$ of $P_1(\mathbb{R})$, i.e., find$[ToS]_\beta$.

Solution:

A5. For T, let the 2 different polynomials be:

$P_1 = a_0x^2 + a_1x + a_2$

$P_2 = b_0x^2 + b_1x + b_2$

we have: $T(P_1) = 2a_0x + a_1$

$T(P_2) = 2b_0x + b_1$

$\therefore\ T(P_1+P_2) = T(a_0 \cdot x^2 + a_1x + a_2 + b_0x^2 + b_1x + b_2)$

$= T((a_0+b_0)x^2 + (a_1+b_1)x + (a_1+a_2))$

$= 2(a_0+b_0)x + (a_1+b_1)$

$= 2a_0x + a_1 + 2b_0x + b_1$

$= T(P_1) + T(P_2)$

Let K be any constant

Then, $T(K(P_1)) = T(K(a_0x^2 + a_1x + a_2))$

$= T(Ka_0x^2 + K \cdot a_1x + K \cdot a_2)$

$= 2K \cdot a_0x + K \cdot a_1$

$= K(2a_0x + a_1)$

$= K \cdot T(P_1)$

  sharadindu.adhikari2019@vitstudent.ac.in

Hence, T is a linear transformation.

Now, $S : P_1(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

let $p_1 = a_0 x + a_1$

$p_2 = b_0 x + b_1$

$S(p_1) = x(a_0 x + a_1) = a_0 x^2 + a_1 x$

$S(p_2) = x(b_0 x + b_1) = b_0 x^2 + b_1 x.$

$$
\begin{aligned}
S(p_1 + p_2) &= S(a_0 x + a_1 + b_0 x + b_1) \\
&= S((a_0 + b_0)x + (a_1 + b_1)) \\
&= x(a_0 + b_0)x + x(a_1 + b_1) \\
&= a_0 x^2 + b_0 x^2 + a_1 x + b_1 x \\
&= a_0 x^2 + a_1 x + b_0 x^2 + b_1 x \\
&= S(p_1) + S(p_2)
\end{aligned}
$$

again

Let K be any constant.

Then, $S(K p_1) = S(K(a_0 x + a_1))$

$= S(a_0 K x + a_1 K)$

  sharadindu.adhikari2019@vitstudent.ac.in

$$= x\left(a_0 \cdot Kx\right) + x\left(a_1 \cdot K\right)$$

$$= a_0 \cdot Kx^2 + a_1 \cdot Kx$$

$$= K\left(a_0 x^2 + a_1 x\right)$$

$$= K \cdot S\left(p_1\right)$$

Hence, S is also linear transformation. proven.

Now, we've been given, $\alpha = \{1, x, x^2\}$ of $P_2(\mathbb{R})$

and $\beta = \{1, x\}$ of $P_1(\mathbb{R})$

we've to find out $[T \circ S]_\beta$

$$[T \circ S]_\beta = [T]_\alpha^\beta \cdot [S]_\beta^\alpha$$

let $v_1 = 1$

$v_2 = x$

$v_3 = x^3$

$T(v_1) = T(1) = 0$

$T(v_2) = T(x) = 1$

$T(v_3) = T(x^2) = 2x$

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$[T(v_1)]_\beta = 0 \cdot 1 + 0 \cdot x$

$[T(v_2)]_\beta = 1 \cdot 1 + 0 \cdot x$

$[T(v_3)]_\beta = 0 \cdot 1 + 2 \cdot x$

  sharadindu.adhikari2019@vitstudent.ac.in

$S(w_1) = S(1) = x$

$S(w_2) = S(x) = x^2$ Let $W_1 = 1$, $W_2 = x$

$[S(w_1)]_\alpha = 0.1 + 1.x + 1.x^2$

$[S(w_2)]_\alpha = 0.1 + 0.x + 1.x^2$

$$[S]_\alpha^\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[T \circ S]_\beta = [T]_\alpha^\beta \cdot [S]_\beta^\alpha$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$ ans.

Question 6

6. Find the general formula for $T: \mathbb{R}^3 \to \mathbb{R}^3$, if $T(1,0,1) = (1,2,0)$, $T(1,-2,1) = (0,1,0)$ and $T(0,0,1) = (0,2,-1)$. Also find $T(2,-3,1)$.

Solution:

A6.

To find: General formula for $T: \mathbb{R}^3 \to \mathbb{R}^3$.

given:

$$T(1,0,1) = (1,2,0)$$
$$T(1,-2,1) = (0,1,0)$$
$$T(0,0,1) = (0,2,-1)$$
$$T(2,-3,1) = ?$$

We've:

Basis vector $= \{(1,0,1), (1,-2,1), (0,0,1)\}$

So, $T(v_1) = (1,2,0)$, $T(v_2) = (0,1,0)$; $T(v_3) = (0,2,-1)$

let $(x,y,z) \in \mathbb{R}^3$.

$$(x,y,z) = \alpha(1,0,1) + \beta(1,-2,1) + \gamma(0,0,1)$$
$$= (\alpha+\beta, -2\beta, \alpha+\beta+\gamma)$$

$\alpha+\beta = x$

$-2\beta = y$

$z = \alpha+\beta+\gamma$

$\boxed{\beta = \frac{-y}{2}}$ $\boxed{\alpha = x + \frac{y}{2}}$

$$\gamma = z - \alpha - \beta$$
$$= z - x - \cancel{\tfrac{y}{2}} + \cancel{\tfrac{y}{2}}$$

$$\boxed{\gamma = z - x}$$

  sharadindu.adhikari2019@vitstudent.ac.in

So now, we've found the values of α, β, γ.

$$(x, y, z) = \left(\frac{2x+y}{2}\right)(1,0,1) + \left(-\frac{y}{2}\right)(1,-2,1) + (z-x)(0,0,1).$$

Taking T on both sides,

$$T(x,y,z) \cancel{= \left(\frac{2x+y}{2}\right) T(1,2,0) + \left(-\frac{y}{2}\right)(0,1,0)}$$

$$= \left(\frac{2x+y}{2}\right) \cdot T(1,0,1) + \left(-\frac{y}{2}\right) \cdot T(1,-2,1) + (z-x) \cdot T(0,0,1).$$

$$= \left(\frac{2x+y}{2}\right) \cdot (1,2,0) + \left(-\frac{y}{2}\right) \cdot (0,1,0) + (z-x) \cdot (0,2,-1)$$

  sharadindu.adhikari2019@vitstudent.ac.in

$$\therefore \; T(x, y, z) = \left(\frac{2x+y}{2}, \; \frac{y+4z}{2}, \; x-z \right)$$

<u>again</u>,

$$T(2, -3, -1)$$

$$= \left(\frac{2\times 2 + (-3)}{2}, \; \frac{(-3) + 4(1)}{2}, \; 2-1 \right)$$

$$= \left(\frac{4-3}{2}, \; \frac{-3+4}{2}, \; 2-1 \right)$$

$$\therefore \; T(2, -3, -1) = \left(\frac{1}{2}, \frac{1}{2}, 1 \right)$$

<u>ans</u>.