AOD / Th DA-1 / Fall 2020

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Q1. Verify Cayley-Hamilton theorem for the given matrix and hence find A^{-1}.

Sol^n $\Rightarrow$ $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$

Here $|[A - \lambda I]| = 0$

$$\left| \begin{bmatrix} 2-\lambda & 0 & 1 \\ -2 & 3-\lambda & 4 \\ -5 & 5 & 6-\lambda \end{bmatrix} \right| = 0$$

$\Rightarrow (2-\lambda)(18 + \lambda^2 - 9\lambda - 20) + 1(-10 + 15 - 5\lambda) = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 9\lambda - 2) + (-10) + 15 - 5\lambda = 0$

$\Rightarrow 2\lambda^2 - 20\lambda + 4\lambda - 4 - \lambda^3 + 9\lambda^2 = 0$

$\Rightarrow 11\lambda^2 - \lambda^3 - 16\lambda + 1 - 5\lambda = 0$

$\Rightarrow -\lambda^3 + 11\lambda^2 - 21\lambda + 1 = 0$

According to Cayley-Hamilton Theorem, A should satisfy its characteristic equation

$\therefore \quad -A^3 + 11A^2 - 21A + 1 = 0$

$$\therefore\ A^3 = \begin{bmatrix} -52 & 55 & 67 \\ -288 & 257 & 290 \\ -445 & 390 & 436 \end{bmatrix}, \text{ and}$$

$$A^2 = \begin{bmatrix} -1 & 5 & 8 \\ -30 & 29 & 34 \\ -50 & 45 & 51 \end{bmatrix}$$

Putting these values in the charc. eqn, we get :-

$$\begin{bmatrix} 50 & -55 & -67 \\ 288 & -257 & -290 \\ 445 & -390 & -436 \end{bmatrix} + \begin{bmatrix} -11 & 55 & 88 \\ -330 & 319 & 374 \\ -550 & 495 & 561 \end{bmatrix}$$

$$- \begin{bmatrix} 42 & 0 & 21 \\ -42 & 63 & 84 \\ -105 & 105 & 126 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence, proven

Now, to find A^{-1}, we consider the equation

$$-A^3 + 11A^2 - 21A + I = 0$$

multiplying by A^{-1}, we get: $-A^2 + 11A - 21I + A^{-1} = 0$

$$\Rightarrow A^{-1} = A^2 - 11A + 21I$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 5 & 8 \\ -30 & 29 & 34 \\ -50 & 45 & 51 \end{bmatrix} - \begin{bmatrix} 22 & 0 & 11 \\ -22 & 33 & 44 \\ -55 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

finally, $A^{-1} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$

Q2. Find the eigenvalues and eigenvectors of A and verify the following :-

(i) Sum of the eigenvalues = trace of A.

(ii) product of eigenvalues = Det (A)

Sol^n → We have : $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

To find eigen-values : $|[A - \lambda I]| = 0$

We've :

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0+1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda = 0, 3, 1$

Now, $\quad (1-\lambda)(2+\lambda^2 - 3\lambda - 2) = 0$

$\Rightarrow \quad \lambda^2 - 3\lambda - \lambda^3 + 3\lambda^2 = 0$

$\Rightarrow \quad -\lambda^3 + 4\lambda^2 - 3\lambda = 0$

Eigen-vector for $\lambda = 0$

$\Rightarrow \quad [A - 0I]\,U = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 0$$

$\Rightarrow \quad U_1 = 0$

$U_2 + U_3 = 0$

$\therefore \; a_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

where $a_1 \neq 0$

$$\therefore \; e_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ for } \lambda = 0$$

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Ily,

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 0 \quad \text{for } \lambda = 3$$

$\Rightarrow \quad -2\,U_1 = 0 \qquad \therefore \; U_1 = 0$

$U_2 = 1$

$-2\,U_2 + U_3 = 0 \qquad U_3 = +2$

$\therefore \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Verification.

(i) Trace of $A = 2+1+1 = 4$

Sum of eigen-values $= 3+1+0 = 4$

(ii) Det $(A) = 0$

product of eigen-values $= 3 \times 1 \times 0 = 0$

Hence, proven

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Q3. Find the eigenvalues and eigenvectors of A and verify the following :

(i) A^{-1} does not exist

(ii) eigenvectors are mutually orthogonal or not.

Solⁿ

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

we've

$$|[A - \lambda I]| = 0$$

$$\left| \begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \lambda^3 - 42\lambda - 17\lambda^2 = 0$$

$$\Rightarrow \lambda^2 + 42 - 17\lambda = 0$$

$$\therefore \lambda = 0,\ 14,\ 3$$

Now,

$$e_1 = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad 10u_1 - 2u_2 - 5u_3 = 0$$

$$-2u_1 + 2u_2 + 3u_3 = 0$$

$$-5u_1 + 3u_2 + 5u_3 = 0$$

$$\therefore \quad e_1 = \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

Illy, $e_2 = \begin{bmatrix} 39 \\ 11 \\ 22 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Verification,

(i) A^{-1} does NOT exist, since $\det(A) = 0$

(ii) e_1, e_2, e_3 aren't orthogonal,

since $e_1 \cdot e_2 \neq 0$, and

$e_2 \cdot e_3 \neq 0$

Hence, proven

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Q4. Reduce the quadratic form $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ to the canonical form by an orthogonal transformation.

Sol^n ⟹

let $E = 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

We've $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

$\lambda = (1, -2, 1)$

and, $e_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $e_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

thus, $P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

from normalization of vectors, we've:

$$N = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

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$$AN = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -2/\sqrt{3} & 0 & 2/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

Also, $D = N^T \cdot A \cdot N$

$$\therefore D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Required canonical form $= (y_1, y_2, y_3)\, D \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= -2y_1^2 + y_2^2 + y_3^2$$

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Q5. Reduce the quadratic form Q to the canonical form by an orthogonal transformation. What kind of conic section is described by the equation if Q is altered to = −18.

Sol^n ⟹ Given. $Q = -6x^2 + 6xy - 6y^2$

let. $A = \begin{bmatrix} \frac{1}{2}xy & y^2 \\ x^2 & \frac{1}{2}yx \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$

WK $|[A - \lambda I]| = 0$

$$\begin{vmatrix} -(6+\lambda) & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

$\Rightarrow 36 + \lambda^2 + 12\lambda - 9 = 0$

$\Rightarrow \lambda^2 + 9\lambda + 3\lambda + 27 = 0$

$\therefore \lambda^2 = -9, -3$

Now $e_1 = \begin{bmatrix} -15 & 3 \\ 3 & -15 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$

$\therefore e_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Normalization of vector : $\begin{bmatrix} 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = N$

$$\therefore AN = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Now $\longrightarrow$ $N^T \times \begin{bmatrix} -8.121 & -2.121 \\ 7.242 & -2.121 \end{bmatrix} = D$.

$$\Rightarrow D = \begin{bmatrix} -13.241 & 0 \\ 0 & -3.0 \end{bmatrix}$$

$$(X \quad Y) \; D \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\Rightarrow (-13.241 X \quad -3Y) \begin{pmatrix} X \\ Y \end{pmatrix}$$

Canonical form $\Rightarrow -13.241 \cdot X^2 - 3Y^2 = 0$

for $Q = -18$,

the eq^n becomes $\Rightarrow -\dfrac{13.241}{18} X^2 + \dfrac{Y^2}{6} = 1$,

thus forming an ellipse

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Q6. Determine the eigenvalues of matrix A (using properties) and verify whether it is diagonalizable or not.

Sol^n →

We've

$$A = \begin{bmatrix} 6 & 2\cancel{3} & 3 \\ 0 & 6 & 6 \\ 0 & 0 & 6 \end{bmatrix}$$

WK Eigen value for a triangular matrix is its diagonal value.

$\therefore \quad \lambda = 6, 6, 6$

Now $\quad \det(A) = 6 \times 6 \times 6 = 216.$

$\because \quad \det(A) \neq 0,$

Matrix A is diagonalizable.

Q7. Find the Fourier series for the function $f(x) = |x|$. in the interval $x \in (-\pi, \pi)$.

Sol^n $\Rightarrow$ We've: $f(x) = |x|$

In the interval $(-\pi, \pi)$, $f(x)$ behaves as an even f^n.

Now, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx$

$\frac{2}{\lambda} \int_0^{\pi} x \, dx = \pi$

$a_n = \frac{1}{\pi} 2 \int_0^{\pi} x \cdot \cos xn \, dx$

$\Rightarrow \frac{2}{n^2 \cancel{\pi}} \left((1-1)^n - 1 \right) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-4}{n^2 \pi}, & \text{if } n \text{ is odd} \end{cases}$

$\therefore f(x) = |x| = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n+1)^2 \pi} \cos nx \quad (\because n \text{ is odd})$

When n is even, however,

$f(x) = \frac{\pi}{2}$.

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Q8. Find the Fourier series for the function $f(x)$, and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Sol^n $\Rightarrow$ We've $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 < x \le \pi \end{cases}$

$\Rightarrow$ $f(x)$ is even f^n, since $f(x) = f(-x)$

so that $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi} \int_0^{\pi} f(x)\,dx \quad \left[f(x) = 1 - \frac{2x}{\pi} \right]$

$= \frac{1}{\pi} \left(x - \frac{2x^2}{2\pi} \right)_0^{\pi}$

$= 0$

$||^y$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \,dx$

$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx \,dx$

$$a_n = \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(\frac{2}{\pi}\right)\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \left[\frac{-4}{n^2\pi^2}(-1)^n + \frac{4}{\pi^2 n^2}\right]$$ thus $\Rightarrow$ $\begin{cases} 0 & \text{when } x = \text{even} \\ \dfrac{8}{\pi^2 n^2} & \text{when } x = \text{odd} \end{cases}$

Now, $$f(x) = \sum_{n=1}^{\infty} \frac{8\cos nx}{\pi^2 n^2}$$

$$= \sum_{n=1}^{\infty} \frac{8\cos nx}{(2n-1)^2\pi^2}$$

Putting $x = 0$ at particular instance, we get:

$$\left[1-\frac{2(0)}{\pi}\right] = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(0)}{(2n-1)^2}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Hence, proven

Q9. obtain fourier series of cosine terms for the function $f(x)$.

Sol^n → given $f(x) = \begin{cases} x, & 0 < x \leq \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$

is a Half-consine series.

we've $a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$

$= \frac{1}{\pi} \left(\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx \right)$

$= \frac{1}{\pi} \left(\frac{\pi^2}{8} + \left[\pi x - \frac{x^2}{2} \right]_{\pi/2}^{\pi} \right)$

$= \frac{1}{\pi} \left(\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{3\pi^2}{8} \right)$

$= \frac{\pi}{4}$

lly $a_n = \frac{2}{\pi} \left(\int_0^{\pi/2} x \cdot \cos nx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx \right) dx$

$= \frac{\pi}{2} \sin\left(\frac{n\pi}{2}\right) + \frac{2}{n^2} \cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} - \left(\frac{(-1)^n}{n^2}\right)$

$$\therefore \quad f(x) = \frac{2\pi}{8} + \sum_{n=0}^{\infty} \frac{\pi}{2(-1)^{\frac{(n+1)}{2}}} + 0 + 0 \quad \left\{ \text{for } n = \text{odd} \right\}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=0}^{\infty} \left[\frac{2}{n^2} (-1)^{2n+2} - \frac{2}{n^2} \right]$$

Q10. Find the Fourier series of y upto 2nd harmonic from the given table.

Solⁿ We've : $f(x) = a_0 + a_1 \cdot \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta$

$$\because \quad a_0 = \frac{1}{N} \Sigma y$$

$$a_1 = \frac{2}{N} \Sigma y . \cos\theta$$

$$b_1 = \frac{2}{N} \Sigma y \left(\cos\theta \times \frac{\sin\theta}{\cos\theta} \right)$$

Here, $N = 6$

$$\therefore \quad h = \frac{360^\circ}{6}$$

$$= 60^\circ$$

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$Y \cos 2\theta$	Y	θ	$Y \cos\theta$	$Y \sin\theta$	$Y \sin 2\theta$
4	4	0°	4	0	0
−4	8	60°	4	6.92	6.92
−8.5	17	120°	−8.5	−14.72	−14.72
7	7	180°	−7	0	0
−3	6	240°	−3	−5.19	5.19
−1	2	300°	1	−1.73	−1.732
−5.5			−9.5	14.72	−4.342

Therefore, $a_0 = \frac{44}{6} = \frac{22}{3}$,

$a_1 = \frac{-9.5}{3}$, $a_2 = \frac{1}{3}(-5.5)$

$b_1 = \frac{14.72}{3}$, $b_2 = \frac{1}{3}(-4.342)$

Reqd. fourier series :

$$f(x) = 7.3 + (-3.16 \cos\theta) + 4.9 \sin\theta + (-1.83 \cos 2\theta) + (-1.44 \sin 2\theta)$$

Q 11. obtain the 1st 3 coefficients in the fourier cosine series for y, where y is given in the table.

Sol^n →

First three cosine coefficients : a_1, a_2, a_3 < let's assume >

We've, $f(x) = a_0 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta)$

Y	θ	Y cos θ	Y cos 2θ	Y cos 3θ
0	0°	0	0	0
5224	30°	4524.11	2612	0
8097	60°	4048.5	−4048.5	−8097
7850	90°	0	−7850	0
5499	120°	−2749.5	−2749.5	5499
2626	150°	−2274.18	1313	0
		3548.93	−10723	−2598

Hence,

$$a_1 = \frac{1}{N} \Sigma y \cos\theta = \frac{1}{6}(3548.93) = 591.488$$

$$a_2 = \frac{1}{N} \Sigma y \cos 2\theta = \frac{1}{6}(-10723) = -1787.16$$

$$a_3 = \frac{1}{N} \Sigma y \cos 3\theta = \frac{1}{6}(-2598) = -433$$

—END—