MAT 1011

CALCULUS

Digital Assignment - 1

L31+L32

Fall Semester : 2019-20

by

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1| For $f(t) = t \cdot (6-t)^{2/3}$, find critical points, inflexion points, intervals of increasing and decreasing & intervals of concave up and concave down.

$\Rightarrow$ Given : $f(t) = t \cdot (6-t)^{2/3}$

Now : $f'(t) = 1 \cdot (6-t)^{2/3} + t \cdot \frac{2}{3} \cdot (6-t)^{+2/3-1} \cdot (-1)$

$$= (6-t)^{2/3} - \frac{2t}{3} \cdot (6-t)^{+2/3-1}$$

$$= (6-t)^{2/3} - \frac{2t}{3} \cdot \frac{(6-t)^{2/3}}{(6-t)}$$

$$= (6-t)^{2/3} \left[1 - \frac{2t}{3(6-t)} \right]$$

- For critical points, $f'(t) = 0$

$\Rightarrow (6-t)^{2/3} = 0$ & $\left[1 - \frac{2t}{3(6-t)} \right] = 0$

$\Rightarrow 6 - t = 0$

$\Rightarrow \boxed{t = 6}$

$\Rightarrow 2t = 3(6-t)$

$\Rightarrow \boxed{t = 18/5}$

$\therefore$ Critical points are $\frac{18}{5}$ and 6.

- For intervals of increasing & decreasing,

  - For increasing :

$$f'(t) > 0$$

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$$\Rightarrow \quad (6-t)^{2/3} \cdot \left[1 - \frac{2t}{3(6-t)}\right] > 0$$

$$\Rightarrow \quad (6-t)^{2/3} - \frac{2t}{3(6-t)^{1/3}} > 0$$

$$\Rightarrow \quad \frac{3(6-t) - 2t}{3(6-t)^{1/3}} > 0$$

$$\Rightarrow \quad \frac{18 - 5t}{3 \cdot (6-t)^{1/3}} > 0$$

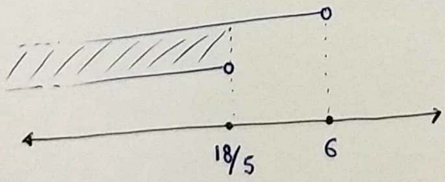
Case-1

$18 - 5t > 0$

$\Rightarrow \quad t < 18/5$

and, $\quad 6 - t > 0$

$\Rightarrow \quad t < 6$



$\Rightarrow \quad t < 18/5$

OR

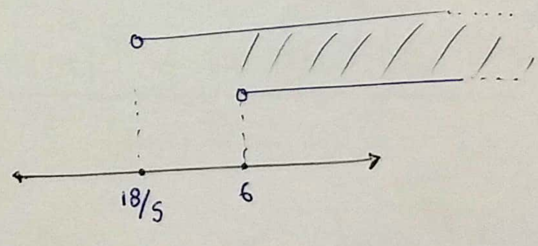
Case-2

$18 - 5t < 0$

$\Rightarrow \quad t > 18/5$

and, $\quad 6 - t < 0$

$\Rightarrow \quad t > 6$



$\Rightarrow \quad t > 6$

$\therefore$ The function is increasing in the following interval:–

$$t \in \left(-\infty, 18/5\right) \cup \left(6, \infty\right)$$

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Similarly, • For decreasing,

$$f'(t) < 0$$

$$\Rightarrow \frac{18-5t}{3(6-t)^{1/3}} < 0$$

Case-1

$(18-5t) < 0$ and $(6-t) > 0$

$\Rightarrow t > 18/5$ & $t < 6$

OR

Case-2

$(18-5t) > 0$ and $(6-t) < 0$

$\Rightarrow t < 18/5$ & $t > 6$,

which is not possible.

∴ The function is decreasing in the following interval:

$$t \in \left(18/5, 6\right)$$

Now, we've, again,

$$f'(t) = \frac{18-5t}{3(6-t)^{1/3}}$$

$$\Rightarrow f''(t) = \frac{(-5)\cdot(3)\cdot(6-t)^{1/3} + (18-5t)\cdot 3\cdot\left(\frac{1}{3}\right)\cdot(6-t)^{-2/3}}{9\cdot(6-t)^{2/3}}$$

$$= \frac{-15\cdot(6-t)^{1/3} + (18-5t)\cdot(6-t)^{-2/3}}{9\cdot(6-t)^{2/3}}$$

$$\therefore f''(t) = \frac{10t-72}{9\cdot(6-t)^{4/3}}$$

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④ For inflection point,

$$f''(t) = 0$$

$$\Rightarrow \quad 10t - 72 = 0 \qquad \left[\because (6-t)^{4/3} > 0\right]$$

$$\Rightarrow \quad t = 72/10 = 36/5$$

$\therefore$ The inflection point of the function is $t = 36/5$.

○ For intervals of concavity,

• For concave up :

$$f''(t) > 0$$

$$\Rightarrow \quad (t - 36/5) > 0$$

$$\Rightarrow \quad t > 36/5$$

$\therefore$ For $t \in (36/5, \infty)$, $f(t)$ is concave up.

• For concave down :

$$f''(t) < 0$$

$$\Rightarrow \quad (t - 36/5) < 0$$

$$\Rightarrow \quad t < 36/5$$

$\therefore$ For $t \in (-\infty, 36/5)$, $f(t)$ is concave down.

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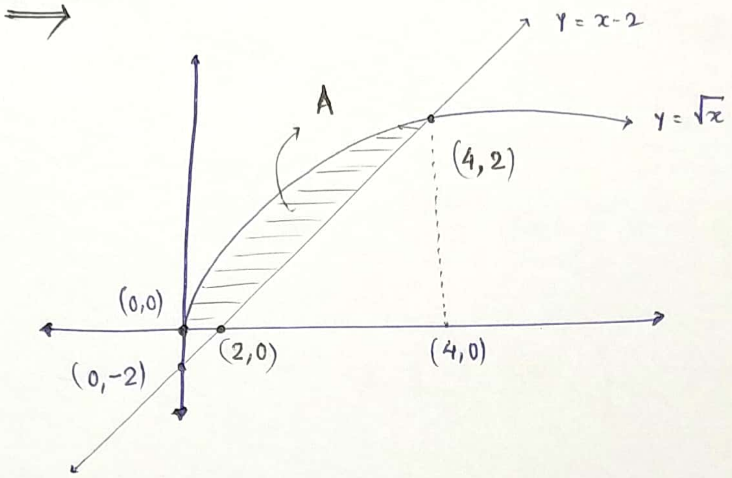
2| Find the area bounded by $y = \sqrt{x}$ and $y = x - 2$ above x-axis.

$\Longrightarrow$



We've : $y = \sqrt{x}$

$\Rightarrow y^2 = x$ ——— ①

putting the value of x from ① in the 2nd line,

$y = y^2 - 2$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow y^2 - 2y + y - 2 = 0$

$\Rightarrow y(y-2) + 1\cdot(y-2) = 0$

$\Rightarrow (y-2)\cdot(y+1) = 0$

$\Rightarrow y = -1$ (or) 2

But for real x, $y \neq -1$

$\therefore y = 2$

$\therefore x = y^2 = 4$

$\therefore$ Area, A

$$= \int_0^2 \sqrt{x}\, dx + \int_2^4 \left(\sqrt{x} - (x-2)\right) dx \text{ sq. units}$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^2 + \int_2^4 \left(\sqrt{x} - (x-2)\right) dx \text{ sq. units}$$

$$= \frac{2}{3}\cdot\left[2^{3/2} - 0\right] + \frac{2}{3}\left[x^{3/2}\right]_2^4 - \frac{1}{2}\left[x^2\right]_2^4 + 2\left[x\right]_2^4 \text{ sq. units}$$

$$= \frac{4\sqrt{2}}{3} + \left[\frac{2}{3}\left(4^{3/2} - 2^{3/2}\right)\right] - \frac{1}{2}\left[4^2 - 2^2\right] + 2\left[4-2\right] \text{ sq. units}$$

$$= \frac{4\sqrt{2}}{3} + \left[\frac{2}{3}\left(8 - 2\sqrt{2}\right)\right] - 6 + 4 \text{ sq. units}$$

$$= \cancel{\frac{4\sqrt{2}}{3}} + \frac{16}{3} - 2 - \cancel{\frac{4\sqrt{2}}{3}} \text{ sq. units}$$

$$= \frac{10}{3} \text{ sq. units.}$$

Ans.

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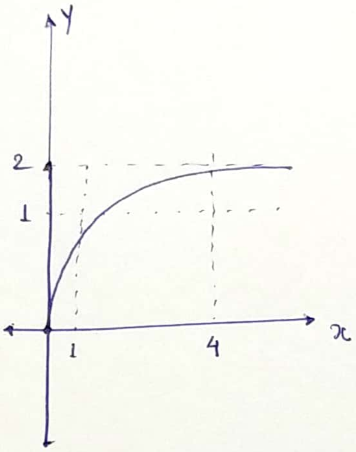
3| Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the line $x = 4$ about the line $x = 1$.

$\Longrightarrow$



We've, Volume of the curve (solid) generated as :-

$$\pi \int_0^1 (4-1)^2 \, dy + \pi \int_1^2 (4-1)^2 \, dy - \pi \int_1^2 (y^2-1)^2 \, dy$$

$$= 9\pi + 9\pi - \pi \int_1^2 (y^4 - 2y^2 + 1) \, dy$$

$$= 18\pi - \pi \left[\left(\frac{32-1}{5}\right) - 2\left(\frac{8-1}{3}\right) + 1 \right]$$

$$= 18\pi - \pi \left[\frac{31}{5} - \frac{14}{3} + 1 \right]$$

$$= \left[\frac{(18 \times 5) - (31 \times 3) + (14 \times 5) - 15}{15} \right] \pi$$

$$= \frac{232}{15} \pi$$ Ans.

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4| (i) Evaluate : $L\left(\int_0^t \frac{e^{-t} \cdot \sin t}{t}\right)$

$\longrightarrow$

We've :

$$L(\sin t) = \frac{1}{s^2+1}$$

$$\Rightarrow L\left(e^{-t} \cdot \sin t\right) = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow L\left(\frac{e^{-t} \cdot \sin t}{t}\right) = \int_s^\infty \frac{1}{(s+1)^2+1} \cdot ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

$$\therefore L\left(\int_0^t \frac{e^{-t} \cdot \sin t}{t}\right) = \frac{1}{s} \cdot \left[\cot^{-1}(s+1)\right]$$ Ans.

4| (ii) Evaluate : $L\left(\int_0^t e^{-t} \cdot \cosh t\right)$

$\Longrightarrow$

We've :

$$L(\cosh t) = \frac{s}{s^2-1}$$

$\left[\because \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}\right]$

$$\Rightarrow L\left(e^{-t} \cdot \cosh t\right) = \frac{(s+1)}{(s+1)^2-1}$$

$$= \frac{(s+1)}{s^2+2s}$$

$$\Rightarrow L\left(\int_0^t e^{-t} \cdot \cosh t \, \cancel{dt}\right) = \frac{1}{s}\left[\frac{s+1}{s^2+2s}\right]$$

$$= \frac{1}{s}\left[\frac{s+1}{s(s+2)}\right]$$

$$\therefore \mathcal{L}\left(\int_0^t e^{-t} \cosh t\right) = \frac{s+1}{s^2(s+2)} \quad \underline{\text{Ans}}.$$

4| (iii) Use step function to evaluate $L(f)$, where :

$$f = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \geq 2\pi \end{cases}$$

$\longrightarrow$

<u>Using step function, we've :</u>

$$\cancel{L(f)}\ f(t) = \sin t\left[u(t-0) - u(t-\pi)\right] + \sin 2t\left[u(t-\pi) - u(t-2\pi)\right] + \sin 3t\left[u(t-2\pi)\right]$$

$$\Rightarrow L(f(t)) = e^{0} L(\sin t) + e^{-\pi s}\left[L(\sin 2t - \sin t)\right] + e^{-2\pi s}\left[L(\sin 3t - \sin 2t)\right]$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} - \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

<u>Ans</u>.

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5 | (i) Use convolution Theorem to evaluate :

$$L^{-1}\left[\frac{s^2}{(s^2+1)\cdot(s^2+4)}\right]$$

⟹ We've :-

$$L^{-1}\left[\frac{s^2}{(s^2+1)\cdot(s^2+4)}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+1}\cdot\frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\int_0^t 2\cdot\cos u\cdot\cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos(u+2t-2u) + \cos(u-2t+2u)\right]du$$

$$= \frac{1}{2}\int_0^t \left[\cos(2t-u) + \cos(3u-2t)\right]du$$

$$= \frac{1}{2}\left(\left[\frac{\sin(2t-u)}{-1}\right]_0^t + \left[\frac{\sin(3u-2t)}{3}\right]_0^t\right)$$

$$= -\frac{1}{2}\cdot(\sin t - \sin 2t) + \frac{1}{6}\cdot(\sin t + \sin t)$$

$$= \frac{-\sin t}{2} + \frac{\sin 2t}{2} + \frac{1}{6}\cdot\sin t + \frac{\sin 2t}{6}$$

$$= \frac{-3\sin t + \sin t}{6} + \frac{3\sin 2t + \sin 2t}{6}$$

$$= \frac{-2\sin t}{6} + \frac{4\cdot\sin 2t}{6}$$

$$= -\frac{\sin t}{3} + \frac{2\cdot\sin 2t}{3}$$ Ans.

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5| (ii) Use Convolution Theorem to evaluate :

$$L^{-1}\left[\frac{s}{(s+1)\cdot(s-3)\cdot(s+5)}\right]$$

$\Longrightarrow$

Let,

$$\frac{s}{(s+1)\cdot(s-3)\cdot(s+5)} \equiv \frac{A}{(s+1)} + \frac{B}{(s-3)} + \frac{C}{(s+5)}$$

Applying partial fractions & cross multiplication, we've :—

$$s = A(s-3)\cdot(s+5) + B(s+1)\cdot(s+5) + C\cdot(s+1)\cdot(s-3)$$

$$\Rightarrow \quad s = A(s^2+2s-15) + B(s^2+6s+5) + C(s^2-2s-3)$$

$$\Rightarrow \quad s = s^2(A+B+C) + s(2A+6B-2C) - 15A + 5B - 3C$$

$\therefore$ we've :-

$$A+B+C = 0; \quad \text{———} (i)$$

$$2A+6B-2C = 1; \quad \text{———} (ii)$$

$$-15A+5B-3C = 0 \quad \text{———} (iii)$$

Solving eqn^s (i), (ii) and (iii), we get :—

$$A = +\frac{1}{16}, \quad B = +\frac{3}{32}, \quad C = -\frac{5}{32}$$

Now, we've :-

$$L^{-1}\left[\frac{s}{(s+1)\cdot(s-3)\cdot(s+5)}\right]$$

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$$= L^{-1}\left[\frac{1/16}{(s+1)} + \frac{3/32}{(s-3)} - \frac{5/32}{(s+5)}\right]$$

$$= \frac{1}{16}\cdot \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{3}{32}\cdot \mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \frac{5}{32}\cdot \mathcal{L}^{-1}\left[\frac{1}{s+5}\right]$$

$$= \frac{1}{16}\cdot e^{-t} + \frac{3}{32}\cdot e^{3t} - \frac{5}{32} e^{-5t}$$ Ans.