PHYSICS

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by

Sharadindu A.

19BCE2105

Q1. Find the (a) divergence and (b) curl of a gradient for a scalar field $F = e^x \sin(y) \ln(z)$.

$\longrightarrow$

$$F = e^x \cdot \sin(y) \cdot \ln(z)$$

(a) $$\vec{\nabla} F = e^x \cdot \sin(y) \cdot \ln(z)\ \hat{i} + e^x \cdot \cos(y) \cdot \ln(z)\ \hat{j} + e^x \cdot \sin(y) \cdot \frac{1}{z}\ \hat{k}$$

$$\nabla \cdot (\vec{\nabla} F) = e^x \cdot \sin(y) \cdot \ln(z) + e^x \ln(z) \cdot (-\sin y) + e^x \sin y \left(-\frac{1}{z^2}\right)$$

$$= e^x \cdot \sin(y) \cdot \ln(z) - e^x \ln(z) \cdot \sin y - \frac{e^x \sin y}{z^2}$$

(b) $$\nabla \times (\vec{\nabla} F) \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cdot \sin y \cdot \ln(z) & e^x \cdot \cos y \cdot \ln(z) & e^x \sin y \, (1/z) \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(\cancel{e^x \cdot \cos y \cdot \frac{1}{z}} - \cancel{e^x \cdot \cos y \cdot \frac{1}{z}}\right) - \hat{j}\left(\cancel{e^x \cdot \sin y \cdot 1/z} - \cancel{e^x \sin y \cdot 1/z}\right) + \hat{k}\left(\cancel{e^x \cos y \cdot \ln(z)} - \cancel{e^x \cos y \cdot \ln(z)}\right)$$

$$\Rightarrow 0$$

Q2. The vector field given by $\vec{F} = y^2\,\hat{i} + (2xy + z^2)\,\hat{j} + 2yz\,\hat{k}$,

is $\vec{F}$: irrotational and / or solenoidal ?

$\longrightarrow$

$$\vec{F} = y^2\,\hat{i} + (2xy + z^2)\,\hat{j} + 2yz\,\hat{k}$$

Divergence, $\vec{\nabla}\cdot\vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(y^2\,\hat{i} + (2x+z^2)\,\hat{j} + 2yz\,\hat{k}\right)$$

$$= 0 + 2x + 2y$$

$$= 2(x+y)$$

Curl, $\vec{\nabla}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix}$$

$$= \hat{i}\,(\cancel{2z} - \cancel{2z}) - \hat{j}\,(\cancel{0} - \cancel{0}) + \hat{k}\,(\cancel{2y} - \cancel{2y})$$

$$= 0$$

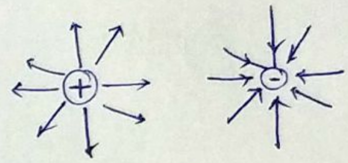
$\therefore$ $\vec{F}$ is conservational / irrotational, but not solenoidal.

Q3. Write the Maxwell's equation in differential and integral form. Explain significance of each equation.

$\longrightarrow$ • Maxwell's equations :—

1. **Gauss' Law** :-

(a) 2 types of charges : $\oplus$ and $\ominus$



↳ curl = 0

$\therefore \nabla \times \vec{F} = 0$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \quad \text{[differential form]}$$

Applying Gauss' Theorem, we've :

$$\oiint_S \vec{E} \cdot d\vec{S} = \iiint_V \frac{\rho}{\epsilon}\, dV \quad \text{[Integral form]}$$

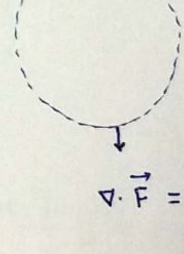
Significance : It states that the net flux of an electric field in a closed surface is directly proportional to the enclosed electric charge.

2. **No one's Law** :-

[Gauss' Law for magnetism]

(a) No magnetic monopole exist

(b) All magnetic field lines form a loop.



$\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{B} = 0 \quad [\text{differential form}]$$

Applying Gauss' Theorem, we've :

$$\oiint_S \vec{B} \cdot d\vec{s} = 0 \quad [\text{integral form}]$$

Significance: It is a general law applying to any closed surface. It permits to calculate the field of an enclosed charge by mapping the field on a surface outside the magnetic charge distribution.

3. Faraday's Law :—

change of magnetic flux induces an electric field along a closed loop.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad [\text{differential form}]$$

Applying Stokes' Theorem, we've :

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot \partial \vec{S} \quad [\text{integral form}]$$

Significance : This law predicts how a magnetic field will interact with an electric circuit to produce an emf.

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4. Ampere's Law :-

Magnetic field lines curl around electric current.

$$\nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{J}$$ [differential form]

Applying Stokes' theorem, we've :

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \mu\epsilon \iint_S \frac{\partial \vec{E}}{\partial t} \cdot \partial \vec{s} + \mu \iint_S \vec{J} \cdot d\vec{s}$$ [integral form]

Significance : This law relates the integrated magnetic field around a closed loop to the electric ~~efectric~~ current passing through the loop.

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Q4. Build the electromagnetic wave equation in conducting media using Maxwell equations and prove the velocity of EM wave is "c".

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From the Maxwell's eqn's, we've :

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} \times \mu \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla} \times (\nabla \times \vec{B}) = \vec{\nabla} \times \mu \vec{J} + \mu\epsilon \left[\vec{\nabla} \times \frac{\partial \vec{E}}{\partial t} \right]$$

$$\Rightarrow (\vec{\nabla} \cdot \vec{B}) \vec{\nabla} - \nabla^2 \vec{B} = \mu (\vec{\nabla} \times \sigma \vec{E}) + \mu \epsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow 0 - \nabla^2 \vec{B} = \mu \sigma (\vec{\nabla} \times \vec{E}) + \mu \epsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow - \nabla^2 \vec{B} = \mu \sigma \left(-\frac{\partial \vec{B}}{\partial t} \right) + \mu \epsilon \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$\Rightarrow \quad \nabla^2 \vec{B} = \mu \cdot \sigma \frac{\partial \vec{B}}{\partial t} + \mu \epsilon \cdot \frac{\partial^2 \vec{B}}{\partial t^2}$$

is the reqd. EM wave eq^n in conducting media.

Now, from the eq^n of wave in classical mechanics,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 y}{\partial t^2} \quad \text{——— (i)}$$

Eq^n of EM wave in vacuum :

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{——— (ii)} \quad \left[\because \sigma = 0 \text{ in vacuum, } \mu \longrightarrow \mu_0 \text{ \& } \epsilon \longrightarrow \epsilon_0 \right]$$

Comparing equations (i) and (ii),

$$\mu_0 \epsilon_0 = 1/v^2$$

$$\Rightarrow \quad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \ ms^{-1} = c$$

$\therefore$ Velocity of EM wave in vacuum is c. proved

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Q5. Establish the relationship between phase & group velocity.
Phase velocity of ocean waves is $\sqrt{\lambda g / 2\pi}$. Here, 'g' is the acceleration due to gravity, and 'λ' is the wavelength.
Find the group velocity of ocean waves.

→ Let the wave equation ① be $y_1 = A \sin(k_1 x - \omega_1 t)$

& ② be $y_2 = A \sin(k_2 x - \omega_2 t)$

Superimposing these 2 waves, we've :

$$y = y_1 + y_2$$

$$= 2A \left(\left[\sin\left(\left(\frac{k_1 + k_2}{2} \right) x - \left(\frac{\omega_1 + \omega_2}{2} \right) t \right) \right] \cdot \left[\cos\left(\left(\frac{k_1 + k_2}{2} \right) x - \left(\frac{\omega_1 - \omega_2}{2} \right) \cdot t \right) \right] \right)$$

$$= 2A \cdot \left(\left[\sin(kx - \omega t) \right] \cdot \left[\cos\left(\frac{\partial k}{2} \cdot x - \frac{\partial \omega}{2} \cdot t \right) \right] \right)$$

Phase velocity ⟶ $kx - \omega t = \text{constant} = c$

$$\Rightarrow k\,dx - \omega\,dt = 0$$

$$\Rightarrow k \cdot dx = \omega\,dt$$

$$\Rightarrow \frac{k}{\omega} = \frac{dt}{dx} \Rightarrow \frac{dx}{dt} = \frac{\omega}{k}$$

⟶ V_p = phase velocity

Group velocity ⟹ $\frac{d\omega}{dk} = V_g$

- Relationship between V_p and V_g :-

$$\omega = k \cdot V_p$$

$$V_g = \frac{d}{dk}(k \cdot V_p) = V_p + k \cdot \frac{d}{dk}(V_p).$$

But, $k = \frac{2\pi}{\lambda}$ ∴ $dk = -\frac{1}{\lambda^2}(2\pi) \cdot d\lambda$

$$\therefore V_g = V_p + \frac{2\pi}{\lambda} \cdot \frac{\lambda^2}{2\pi\,d\lambda} \cdot dV_p$$

$$\therefore V_g = V_p - \lambda \cdot \frac{d}{d\lambda}(V_p)$$

Now, given, $v_p = \sqrt{\dfrac{\lambda g}{2\pi}}$

$$\therefore v_g = \sqrt{\frac{\lambda g}{2\pi}} - \lambda \cdot \sqrt{\frac{g}{2\pi}} \left(2\sqrt{\lambda} \right)^{-1}$$

$$= v_p - \frac{v_p}{2}$$

$$\therefore \boxed{v_g = v_p/2}$$

Q6. The electric field of an EM wave travelling in vacuum is described by the following wave function: $\vec{E} = (5.0\ V/m) \cdot \cos\left[kx - (6 \times 10^6\ s^{-1})\, t\right] \hat{j}$, where k is the wave number in rad/m, x is in meter, t is in second. Find out the following quantities: (i) amplitude, (ii) frequency, (iii) wavelength, (iv) associated magnetic field, & (v) direction of wave propagation.

$\longrightarrow$ $$\vec{E} = (5.0\ Vm^{-1}) \cdot \cos\left[kx - (6 \times 10^6\ s^{-1}) \cdot t\right] \hat{j}$$

(i) Amplitude, $E_0 = 5.0\ Vm^{-1}$

(ii) Frequency, $1/f = 2\pi/\omega$ ~~$= 6\times10$~~

$$\Rightarrow f = \frac{\omega}{2\pi} = \frac{6 \times 10^6}{2\pi}\ Hz$$

$$= 9.55 \times 10^5\ Hz$$

(iii) Wavelength. $\lambda = \frac{c}{f} = \frac{3\times10^{8}\ ms^{-1}}{9.55\times10^{5}\ Hz} = 314.136\ m.$

(iv) associated magnetic field :

We know : $E_0 = B_0 \cdot c \Rightarrow B_0 = \frac{E_0}{c} = \frac{5}{3\times10^{8}}\ T$

$= 1.66\times10^{-8}\ T$

$\therefore\ B = (1.66\times10^{-8}) \cdot \cos\left[(2\times10^{-2})x - (6\times10^{6})\cdot t\right]\hat{k}$

(v) Direction of wave propagation :

$= \hat{j}\times\hat{k} = \hat{i}$ (+ve x-direction)

Q7. Establish the relation between Einstein's A and B coefficients & explain when can stimulated emission predominate over spontaneous emission & vice versa.

→ 1st part : Absorption = Spontaneous emission + stimulated emission

$\Rightarrow \Gamma_{12} = U_{21} + \Gamma_{21}$

$\Rightarrow B_{12} N_1 u(\omega) = A_{21} N_2 + B_{21}\cdot N_2 \cdot u(\omega)$

$\Rightarrow A_{21} = \frac{(B_{12}\cdot N_1 - B_{21}\cdot N_2)\cdot u(\omega)}{N_2}$ —— (1)

We know,

$$\frac{N_1}{N_2} = e^{(E_2 - E_1)/k_B T}$$

$$= e^{\hbar\omega/k_B T} \quad \text{——— (2)}$$

From equation (1),

$$u(\omega) = \frac{A_{21} N_2}{B_{12} N_1 - B_{21} N_2} = \frac{A_{21}}{B_{21} \cdot \frac{N_1}{N_2} - B_{21}}$$

$$\Rightarrow u(\omega) = \frac{A_{21}}{B_{12} e^{\hbar\omega/k_B T} - B_{21}} = \frac{A_{21}}{B_{12}\left(e^{\hbar\omega/k_B T} - \frac{B_{21}}{B_{12}}\right)} \quad \text{——— (3)}$$

From the black-body radiation formula,

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 \cdot c^3} \cdot \frac{1}{e^{\hbar\omega/k_B T} - 1} \quad \text{——— (4)}$$

Comparing equations (3) & (4), we get,

$$\frac{B_{21}}{B_{12}} = 1 \Rightarrow B_{21} = B_{12} = B \quad (\text{say})$$

and, $A_{21} = A$ (say)

$$\therefore \boxed{\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}}$$

IInd part

At thermal equilibrium,

$$\frac{\text{No. of spontaneous emission}}{\text{No. of stimulated emission}} \; (R) :$$

$$R = \frac{A_{21} \cdot N_2}{B_{21} \cdot N_2 \cdot u(\omega)} = e^{\hbar\omega/K_B T} - 1$$

- If $\omega >> \frac{K_B T}{\hbar}$, number of spontaneous emissions will predominate over stimulated emissions.
- If $\omega << \frac{K_B T}{\hbar}$, no. of stimulated emissions will predominate over spontaneous emissions.

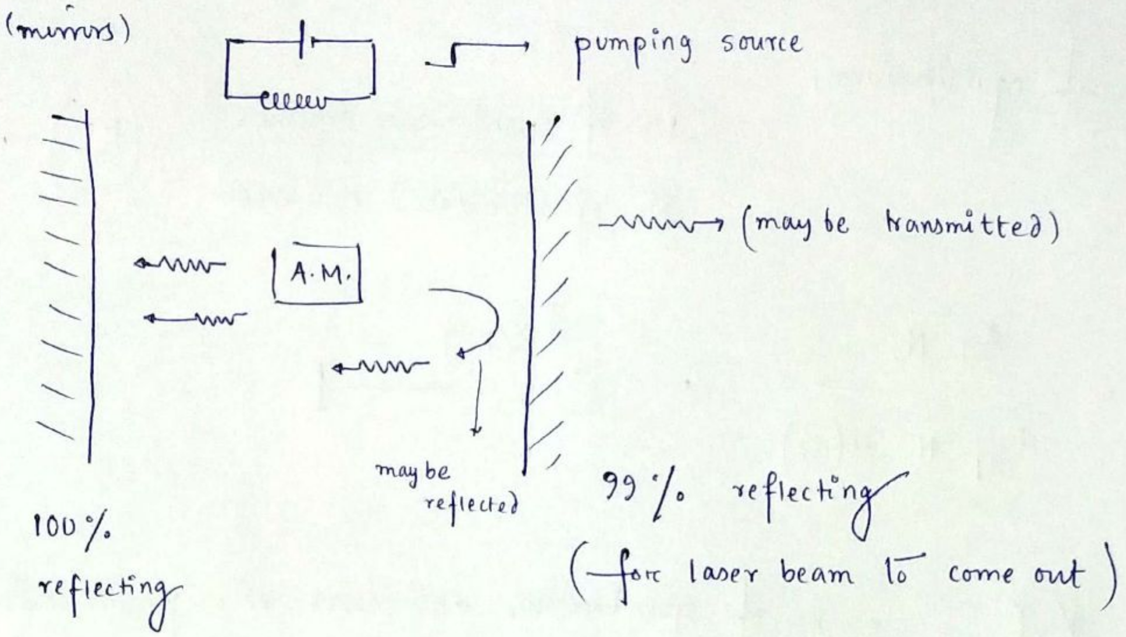
Q8. Explain with the diagram how does the light amplifications occur in a Laser? What is threshold gain. drive expression of gain coefficient for a longitudinal cavity of length. L and attenuation, α. Use R_1 and R_2 as the reflection coefficients of the mirrors

⟶ Elements required for Laser amplification :-

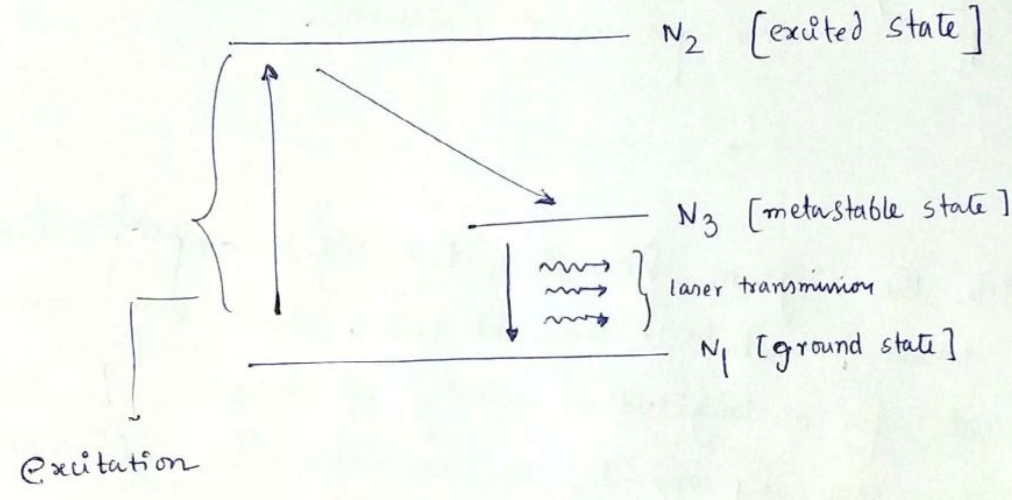
(i) pumping source ⟶ to exite electrons

(ii) Active media ⟶ material used for laser, decides wavelength of the laser.

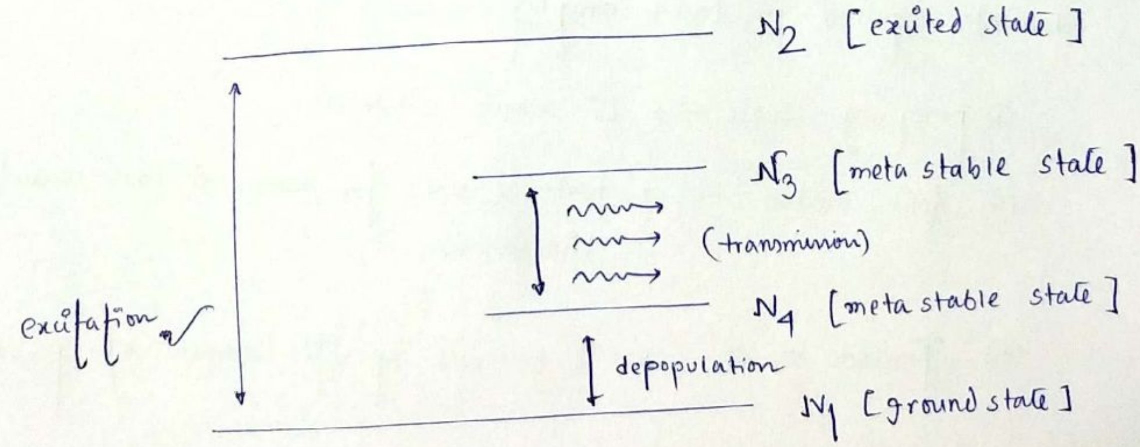
(iii) Feedback circuit ⟶ to generate specific amount of photons.



(i) 3-level-Laser system :-



(ii) 4-level-Laser system :-



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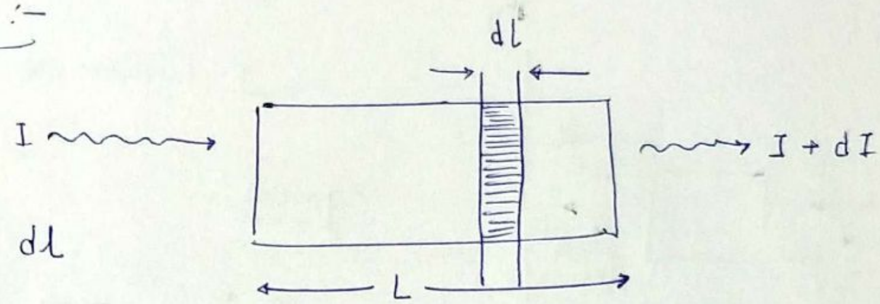
Optical gain:

It is the process by which the intensity of the light produced gets further increased.

Derivation is as follows :-



$$dI = (\Gamma_{21} - \Gamma_{12})\, dl$$

$$= (N_2 B_{21} - N_1 B_{12}) \cdot u(\omega) \cdot dl$$

$$= [(N_2 B_{21}) - (N_1 B_{12})] \cdot I \cdot \frac{h\nu}{c} \cdot dl$$

[$u(\omega)$ = energy density $= I \times \frac{h\nu}{c}$]

$$\frac{dI}{dl} = B_{21}(N_2 - N_1) \cdot \frac{I}{c}(h\nu)$$

[As, we know, $B_{21} = B_{12}$]

$$\int_{I_0}^{I} \frac{dI}{I} = \int_{0}^{L} \frac{B_{21}(N_2 - N_1)}{c} h\nu \cdot dl$$

$$\log_e (I) - \log_e (I_0) = \gamma L$$

[let $\gamma = \frac{B_{21}(N_2 - N_1) h\nu}{c}$]

$$\ln\left(\frac{I}{I_0}\right) = \gamma L$$

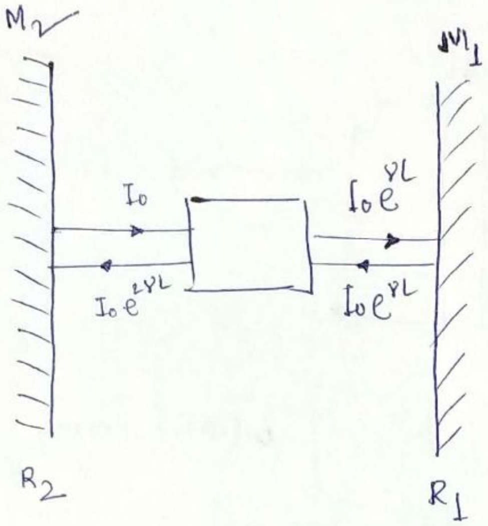
$$\frac{I}{I_0} = e^{\gamma L}$$

$$\therefore \boxed{I = I_0 \cdot e^{\gamma L}}$$

For oscillation,

Gain $\geq$ Losses

positive feedback [to maintain continuous cycle]



I increases as $I_0 e^{n\gamma L}$ $(n \geq 1)$

Process :-

$R_1 I_0 \cdot e^{\gamma L} \longrightarrow R_1 \cdot I_0 \cdot e^{2\gamma L} \rightarrow R_1 R_2 I_0 \cdot e^{2\gamma L}$

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If we consider loss due to angle deviation, the coefficient of attenuation $= \alpha$.

$$\Rightarrow I = I_0 \cdot e^{(\gamma - \alpha) L}$$

Hence, if we proceed towards the threshold condition,

$$R_1 R_2 \cdot e^{2(\gamma - \alpha) L} \geq 1$$

$$\Rightarrow 2(\gamma - \alpha) \cdot L \geq \ln\left(\frac{1}{R_1 \cdot R_2}\right)$$

$$\Rightarrow 2(\gamma - \alpha) L \geq -\ln(R_1 \cdot R_2)$$

$$\Rightarrow (\gamma - \alpha) \geq -\frac{\ln(R_1 R_2)}{2L}$$

$$\therefore \boxed{\gamma_{threshold} \geq \alpha - \frac{\ln(R_1 R_2)}{2L}}$$

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Q9. A laser beam of wavelength 740 nm has coherence time 4×10^{-5} seconds. Calculate its coherence length and spectral half-width.

→

$\lambda = 740$ nm

$\tau_c = 4 \times 10^{-5}$ s.

Coherence length, $= c \cdot \tau_c = (3 \times 10^8 \times 4 \times 10^{-5})$ m

$= 12 \times 10^3$ m

Spectral width, $\Delta\lambda = \dfrac{\lambda^2}{L_c} = \dfrac{(740 \times 10^{-9})^2}{12 \times 10^3}$ m

$= 4.563 \times 10^{-17}$ m

∴ Spectral half-width $= \dfrac{\Delta\lambda}{2} = 2.2815 \times 10^{-17}$ m

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Q10. Consider a 2-level laser proposed to be implemented with optical pumping. Under thermal equilibrium, show that the population of the upper level can never exceed the lower level.

→

we know that,

$$\frac{N_1}{N_2} = e^{(E_2 - E_1)/k_B T}$$

where,

N_1 = no. of atoms at ground state

N_2 = no. of atoms at excited state

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$E_2 - E_1 > 0$ (always) ; $E_2 \longrightarrow$ higher energy level

$E_1 \longrightarrow$ Lower energy level

$K_B T > 0$

$\therefore$ RHS > 1 (always)

$\Longrightarrow$ LHS > 1

$\Longrightarrow \frac{N_1}{N_2} > 1 \Rightarrow N_1 > N_2$ (always),

at thermal equilibrium.

END