G2/ AO~~D~~ Th DA-2 / Fall 2020

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Q1. Solve the given 3 differential equations by using the method of undetermined coefficients.

Sol^n.

(1) $$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 2\sin 3x$$

$$\Rightarrow y'' - y' + y = 2\sin 3x$$

$$\Rightarrow (D^2 - D + 1)y = 2\sin 3x$$

$$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\Rightarrow e^{\frac{1}{2}x}\left(c_1 \cdot \cos \frac{\sqrt{3}}{2}x + c_2 \cdot \sin \frac{\sqrt{3}}{2}x\right)$$

PI of $\frac{2\sin 3x}{D^2 - D + 1}$ (given, $D^2 = -9$)

$$\frac{-2 \cdot \sin 3x\,(D-8)}{(D+8)(D-8)} \Rightarrow \frac{-2\sin 3x\,(D-8)}{-9-64}$$

$$\left(\frac{-2\sin 3x}{73}\right) 8 + \frac{D\,(2\sin 3x)}{73}$$

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$$PI = \left(\frac{-16}{73} \sin 3x + \frac{(-6)\cdot \cos 3x}{73} \right)$$

Hence, C.s of y :-

$$e^{\frac{1}{2}x}\left(c_1 \cdot \cos \frac{\sqrt{3}}{2}x + c_2 \cdot \sin \frac{\sqrt{3}x}{2} \right) - \frac{16}{73} \sin 3x - \frac{6 \cdot \cos 3x}{73}$$

(2) $$\frac{d^2y}{dx^2} + 2 \cdot \frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$$

$$\Rightarrow (D^2 + 2D + 4)y = 2x^2 + 3e^{-x}$$

solving we get, $D = -2, -2$

Hence, $C.F = (c_1 + c_2 x) \cdot e^{-2x}$

$$P.I. \text{ of } = \frac{2x^2 + 3e^{-x}}{(D^2 + 2D + 4)} = \frac{2x^2}{(D^2 + 2D + 4)} + \frac{3e^{-x}}{(D^2 + 2D + 4)}$$

$$D \quad \frac{1}{4} \frac{2x^2}{(D^2 + 2D + 1)} = (1 - D^2 - 2D) \cdot \frac{2x^2}{4_2}.$$

$$\left(\frac{x^2}{2} - 1 - 2x \right) + \frac{3e^{-x}}{(-1)^2 + 2(-1) + 4} \Rightarrow e^{-x}$$

$$C.S = (c_1 + c_2 x) \cdot e^{-2x} + \frac{x^2}{2} - 1 - 2x + e^{-x}$$

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(3) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 e^{-x}$$

$$(D^2 + 2D + 1)y = x^2 \cdot e^{-x}$$

Solving, we get: $D = (-1, -1)$

$$C.F. = (c_1 + c_2 x) \cdot e^{-x}$$

$$P.I. \; f = \frac{x^2 \cdot e^{-x}}{D^2 + 2D + 1} \Rightarrow e^{-x} \cdot \frac{x^2}{(D-1)^2 + 2D - 1}$$

$$\frac{x^2}{D^2} = \frac{(x)^{3+1}}{3 \times 4} \cdot e^{-x} = \frac{e^{-x} \cdot x^4}{12}$$

$$\therefore \; C.S. = (c_1 + c_2 \cdot x) \cdot e^{-x} + \frac{e^{-x} \cdot x^4}{12}$$

Q2. Solve the given differential equations by using the Methods of variation of parameters.

Solⁿ: (1) $$\frac{d^2y}{dx^2} - 2 \cdot \frac{dy}{dx} = e^x \sin x$$

$$\Rightarrow (D^2 - 2D)y = e^x \sin x$$

Solving we get, $D = (0, 2)$

$$C.F. = c_1 \cdot e^{0.x} + c_2 \cdot e^{2x}$$

$$Y_1 = e^0 = 1 \quad \underline{\text{and}} \quad Y_2 = e^{2x}$$

<u>Now,</u>

$$x = e^x \cdot \sin x$$

$$W = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2 \cdot e^{2x} \end{vmatrix} = 2 \cdot e^{2x}$$

$$P.I. = -Y_1 \int Y_2 \cdot \frac{x}{W} + Y_2 \int Y_1 \frac{x}{W}$$

$$\Rightarrow -\int e^{2x} \left(\frac{e^{-x} \sin x}{2} \right) + e^{2x} \int \left(\frac{e^{-x} \cdot \sin x}{2} \right)$$

$$P.I. = \left(\frac{-e^x \sin x + e^x \cos x}{2} \right) - \left(\frac{e^{+x} \cdot \cos x + e^{+x} \cdot \sin x}{2} \right)$$

$$= 0$$

$$C.S = c_1 + c_2 \cdot e^{2x}$$

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(2) $\rightarrow$ $\dfrac{d^2y}{dx^2} - 2 \cdot \dfrac{dy}{dx} + y = x \cdot e^x \cdot \sin 2x$, $y(0) = 0$, $y'(0) = 0$

$$\Rightarrow (D^2 - 2D + 1)\, y = x \cdot e^x \cdot \sin 2x$$

$$\Rightarrow D = (1, 1)$$

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$$C.F. = (c_1 + c_2 . x) \cdot e^x$$

$$Y_1 = e^x, \quad Y_2 = x \cdot e^x, \quad X = x \cdot e^{x} \sin 2x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x \cdot e^x \\ e^x & e^x(1+x) \end{vmatrix}$$

$$= e^{2x}$$

$$P.I. = -y_1 \int y_2 \cdot \frac{X}{W} + y_2 \int y_1 \frac{X}{W}$$

$$\Rightarrow -e^x \int x \cdot e^x \left(x \cdot e^{-x} \sin 2x\right) + x \cdot e^x \int -x \cdot \sin 2x$$

$$\Rightarrow -e^x \left(-x^2 \cdot \frac{\cos 2x}{2} + \frac{x \cdot \sin 2x}{2} + \frac{\cos 2x}{4}\right)$$

$$+ e^x \left(\frac{x^2 \cdot \cos 2x}{2} - \frac{x \cdot \sin 2x}{4}\right)$$

$$\Rightarrow e^x \left(x^2 \cdot \cos 2x - \frac{3x \cdot \sin 2x}{4} - \frac{\cos 2x}{4}\right)$$

$$\therefore \quad C.S. = (c_1 + c_2 x) \cdot e^x + P.I.$$

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Q3. Solve the given differential equations.

Soln: (1) $x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{y}{x} = -ax^2$

$$\Rightarrow x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} - y = -ax^3$$

Let $x = e^z$, where $z = \ln x$.

$$\therefore x^2 \frac{d^2y}{dx^2} = D(D-1)y,$$

$$x \frac{dy}{dx} = D \cdot y.$$

$$\Rightarrow (D^2 - D + D - 1)y = -a \cdot e^{-3z}$$

$$\Rightarrow D = +1, -1$$

$$C.F. = c_1 \cdot e^z + c_2 \cdot e^{-z}$$

$$P.I. \text{ of } = \frac{-a \cdot e^{-3z}}{(D^2 - 1)} = \frac{-a}{8} \cdot e^{-3z}$$

$$\therefore C.S. = c_1 e^z + c_2 \cdot e^{-z} - \frac{a}{8} \cdot e^{-3z}$$

$$\Rightarrow C.S = c_1 x + c_2 x^{-1} - \frac{a}{8} \cdot x^{-3}$$

(2) $4x^2 \cdot \frac{d^2y}{dx^2} + y = \log x.$

let $\longrightarrow$ $x = e^z$ and $z = \ln x.$

$C.F = (4(D(D-1)) + 1) = z$

$\Rightarrow 4D^2 - 4D + 1 = z$

Solving, we get, $D = \frac{1}{2}, \frac{1}{2}$

$\therefore CF = (c_1 + c_2 z) \cdot e^{\frac{1}{2}z}$

$P.I. = \frac{z}{4D^2 - 4D + 1} = \left(1 - (4D^2 - 4D)\right) z$

$(z+4)$

$C.S. = (c_1 + c_2 \cdot \ln x) \cdot e^{\frac{1}{2} \cdot \ln x} + \ln x + 4.$

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(3) $(2x+3)^2 y'' - 2(2x+3) y' - 12y = 6x$

Let, $e^t = (2x+3)$ and $t = \ln(2x+3)$

$\Rightarrow (2(D(D-1)) - 2 \cdot 2(D) - 12) = 6x$

$\Rightarrow 2D^2 - 2D - 4D - 12 = 3(e^t - 3)$

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Solving, we get: $D = 4, -1$.

$$C.F. = c_1 \cdot e^{4t} + c_2 \cdot e^{-t}$$

$$P.I. = \frac{3(e^t - 3)}{2D^2 - 6D - 12}$$

$$\frac{3}{2} \cdot e^t \cdot \frac{1}{(D^2 - 3D - 6)} - \frac{9}{2}\left(\frac{e^{0 \cdot t}}{D^2 - 3D - 6}\right)$$

$$\frac{-3}{16} \cdot e^t + \frac{9}{12} \cdot e^{0 \cdot t}$$

$$\frac{-3}{16} \cdot e^t + \frac{9}{12}$$

$$C.S. = c_1 \cdot (2x+3)^4 + c_2 \cdot (2x+3)^{-1} + \left(\frac{-3}{16}\right)(2x+3) + \frac{9}{16}$$

Q4. Find the eigen values of the given system and hence find general solution by matrix method.

Soln: (1)
$$x'(t) = -3x + 6y + 5z,$$
$$y'(t) = 2x - 12y,$$
$$z'(t) = x + 6y - 5z$$

Hence, the matrix (det.):

$$\begin{vmatrix} -3 & 6 & 5 \\ 2 & -12 & 0 \\ 1 & 6 & -5 \end{vmatrix}$$

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Eigen values $= \alpha\beta\gamma = 0$

$$\alpha+\beta+\gamma = -20$$

$$\lambda_1 = 0$$

$$\lambda_2 = -\sqrt{6}-10$$

$$\lambda_3 = +\sqrt{6}-10$$

Eigen vectors

$$v_1 = \begin{pmatrix} 5/2 \\ 5/12 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{\sqrt{6}-1}{5} \\ \frac{-\sqrt{6}-4}{5} \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} \frac{-\sqrt{6}-1}{5} \\ \frac{\sqrt{6}-4}{5} \\ 1 \end{pmatrix}$$

$$C.F. = c_1 \cdot \begin{pmatrix} 5/2 \\ 5/12 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} \frac{+\sqrt{6}-1}{5} \\ \frac{-\sqrt{6}-4}{5} \\ 1 \end{pmatrix} e^{(-\sqrt{6}-10)t}$$

$$+ c_3 \begin{pmatrix} \frac{-\sqrt{6}-1}{5} \\ \frac{+\sqrt{6}-4}{5} \\ 1 \end{pmatrix} \cdot e^{(\sqrt{6}-10)t}$$

$$x = c_1 \cdot \left(\frac{5}{2}\right) + c_2 \cdot \left(\frac{-\sqrt{6}-1}{5}\right) \cdot e^{(-\sqrt{6}-20)t}$$

$$+ c_3 \cdot \left(\frac{\sqrt{6}-1}{5}\right) \cdot e^{(\sqrt{6}-10)t}$$

$$\rightarrow \frac{5}{2} c_1 + \frac{\sqrt{6}-1}{5} c_2 + \frac{(-\sqrt{6}-1)}{5} \cdot c_3 = x_0$$

$$\Rightarrow \frac{5}{12} \cdot c_1 + \frac{(-\sqrt{6}-4)}{5} \cdot c_2 + \frac{\sqrt{6}-4}{5} \cdot c_3 = 0$$

Also $\Rightarrow$ $c_1 + c_2 + c_3 = 0$

$$\Rightarrow c_3 = -c_1 - c_2$$

$$\frac{5}{12} \cdot c_1 + \frac{-\sqrt{6}-4}{5} c_2 - \left(\frac{\sqrt{6}-4}{5} c_1 + \frac{\sqrt{6}-4}{5} c_2 \right)$$

$$\frac{25 - 12\sqrt{6} + 48}{60} c_1 - \frac{2\sqrt{6}}{5} c_2$$

$$\frac{73 - 12\sqrt{6}}{60} c_1 - \frac{2\sqrt{6}}{5} c_2 = 0 \quad \text{—— (i)}$$

$$\text{Illy}, \quad \frac{27 + 2\sqrt{6}}{10} c_1 + \frac{2\sqrt{6}}{5} c_2 = x_0 \quad \text{—— (ii)}$$

$$\frac{235}{60} c_1 = x_0$$

$$\Rightarrow c_1 = \frac{60}{235} x_0$$

$$c_2 = 1.136\, x_0$$

$$c_3 = -1.4\, x_0$$

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(2) $x_1' = -3x_1 + x_2 - 6 \cdot e^{-2t}$

$x_2' = x_1 - 3x_2 + 2 \cdot e^{-2t}$

$$\text{Matrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$ hence, $\lambda_1 = -2$

$\lambda_2 = (-6+2) = -4$

$$v_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 0$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{C.F.} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot e^{-2t} + c_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot e^{-4t}$$

$$\text{P.I.} \quad f = \begin{bmatrix} -6 \cdot e^{-2t} \\ 2 \, e^{-2t} \end{bmatrix} \rightarrow \begin{bmatrix} A \cdot e^{-2t} \\ B \cdot e^{-2t} \end{bmatrix}$$

$x_1 = A e^{-2t}$ $\qquad x_1' = -2 \cdot A \cdot e^{-2t}$

$x_2 = B \cdot e^{-2t}$ $\qquad x_2' = -2 \cdot B \cdot e^{-2t}$

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$$\therefore -3\left(A\cdot e^{-2t}\right) + B\cdot e^{-2t} - 6\cdot e^{-2t} = -2\cdot A\cdot e^{-2t}$$

$$(B-A) = 6 \quad \text{——— (I)}$$

$$\left(A\cdot e^{-2t}\right) - 3Be^{-2t} + 2\cdot e^{-2t} = -2\cdot B\cdot e^{-2t}$$

$$(A-B) = -2 \quad \text{——— (II)}$$

$$\therefore \text{C.F.} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot e^{-4t}$$

Q5. Find the general solution to given system of equation by method of diagonalization

Sol^n,

$$x_1'' = 6x_1 - x_2$$

$$x_2'' = 4x_1 + 3x_2$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = \frac{9+\sqrt{7}i}{2}, \quad \lambda_2 = \frac{9-\sqrt{7}i}{2}$$

$$v_1 = \left(\frac{-i\sqrt{7}+3}{8}\right) \text{ for } \lambda_1$$

$$v_2 = \left(\frac{\sqrt{7}i+3}{8}\right) \text{ for } \lambda_2$$

$$\therefore \quad D = \begin{pmatrix} \left(\frac{-\sqrt{7}i+9}{8}\right)4 & 0 \\ 0 & \frac{\sqrt{7}i+9}{2} \end{pmatrix}$$

$$A = P.D.\bar{P}^{1}$$

$$(\bar{P}^{1}y)' = D.\bar{P}^{1}.y$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} \frac{9-\sqrt{7}i}{2} & 0 \\ 0 & \frac{9+\sqrt{7}i}{2} \end{pmatrix}$$

$$c_1 \cdot e^{\frac{9-\sqrt{7}i}{2}} + c_2 \cdot e^{\frac{9+\sqrt{7}i}{2}}$$

Q6.

If the governing equation of mass-spring system is given by $my'' + cy' + Ky = r(t)$, where mass $m = 2$ units, damping constant $c = 0$ and spring constant $K = 10$ and $r(t) = 3u(t-12) - 5\delta(t-4)$, then find the displacement y, using Laplace transform method, with initial conditions $y(0) = -1$, $y'(0) = -2$.

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Sol^n: Given mass spring system equation,

$$my'' + cy' + ky = r(t) \quad \text{——— (1)}$$

put $m = 2, \quad c = 0, \quad k = 10$

$$r(t) = 3u(t-12) - 5\delta(t-4)$$

where, $u(t)$ = unit step function

$\delta(t)$ = Dirac delta function

from eq^n (1),

$$2y'' + 10y = 3u(t-12) - 5\delta(t-4) \quad \text{——— (2)}$$

By Laplace transform method,

take laplace transform on both sides of (2),

$$2L(y'') + 10L(y) = 3L(u(t-12)) - 5L(\delta(t-4))$$

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$L\{\delta(t-a)\} = e^{-as}$$

$$2(s^2\bar{y} - sy(0) - y'(0)) + 10\bar{y} = \frac{3e^{-12s}}{s} - 5e^{-4s}$$

$$2(s^2\bar{y} + sr^2) + 10\bar{y} = 3\frac{e^{-12s}}{5} - 5e^{-4s}$$

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$$(2s^2+10)\bar{y} = \frac{3e^{-12s}}{s} - 5e^{-4s}$$

$$\Rightarrow (2s^2+10)\bar{y} = \frac{3e^{-12s}}{s} - 5e^{-4s} - 2(s+2)$$

$$\Rightarrow \bar{y} = \frac{3e^{-12s}}{2(s^2+5)s} - \frac{5e^{-4s}}{2(s^2+5)} - \frac{2(s+2)}{2(s^2+5)}$$

$$\Rightarrow \bar{y} = \frac{3}{2}\cdot\frac{1}{5}\cdot e^{-12s}\left[\frac{1}{s} - \frac{s}{s^2+5}\right] - \frac{5}{2}\cdot e^{-4s}\cdot\frac{1}{s^2+5}$$

$$- \frac{s}{s^2+5} - \frac{2}{s^2+5}$$

$$y(t) = \frac{3}{10} L^{-1}\left\{e^{-12s}\cdot\left[\frac{1}{s} - \frac{s}{s^2+5}\right]\right\}$$

$$- \frac{5}{2}\cdot L^{-1}\left\{\frac{e^{-4s}}{s^2+5}\right\} - L^{-1}\left\{\frac{s}{s^2+5}\right\}$$

$$- 2L^{-1}\left\{\frac{1}{s^2+5}\right\}$$

$$y(t) = \frac{3}{10}\left[1 - \cos\sqrt{5}(t-12)\cdot u(t-12)\right]$$

$$- \frac{5}{2}\left[\frac{1}{\sqrt{5}}\cdot\sin\sqrt{5}(t-4)\cdot u(t-4)\right]$$

$$- \cos\sqrt{5}t - \frac{2}{\sqrt{5}}\cdot\sin\sqrt{5}t$$

$$\therefore \quad y(t) = \frac{3}{10}\left[1 - \cos\sqrt{5}\,(t-12)\right] \cdot u(t-12)$$

$$- \frac{\sqrt{5}}{2} \cdot \sin\sqrt{5}\,(t-4) \cdot u(t-4)$$

$$- \cos\sqrt{5}\,t - \frac{2}{\sqrt{5}} \cdot \sin\sqrt{5}\,t \,.$$

END