Calculus: Limits, Derivatives, Integrals, and Beyond

With Applications and Numerical Examples

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Limits and Continuity

Definition:
$\lim_{x \to c} f(x) = L$ means for every $\varepsilon > 0$, there's $\delta > 0$ so that $|x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.

Example:
$\lim_{x \to 2} (3x+4) = 10$

Continuity:
A function is continuous at $x = c$ if
$\lim_{x \to c} f(x) = f(c)$.

Derivatives

Definition:
The derivative at $x = a$ is
$f'(a) = \lim_{h \to 0} [f(a + h) - f(a)] / h$

Key Theorems:

- Linear, product, quotient, chain rules

Example:
If $f(x) = x^3 - 2x$, then
$f'(x) = 3x^2 - 2$.

Integrals

Definition:
The definite integral as area: $\int_a^b f(x)\,dx$
The indefinite integral: $\int f(x)\,dx$

Key Properties:
Linearity; reversing limits changes the sign

Example:
$\int(3x^2 - 2)\,dx = x^3 - 2x + C$

Fundamental Theorem of Calculus

- **Part 1:**

  If F(x) is an antiderivative of f(x), then

  $\int_a^b f(x)\,dx$ = F(b) - F(a)

- **Part 2:**

  If F(x) = $\int_a^x f(t)\,dt$, then

  F'(x) = f(x)

Multivariate Gradients

Definition:
For f(x,y,...), the gradient vector is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots \right\rangle$$

Example:
For $f(x, y) = x^2 + y^2$,
$\nabla f = \langle 2x, 2y \rangle$

Numerical Differentiation and Integration with NumPy

Numerical Differentiation: (e.g. derivative of sin(x) at x=1)
Approximates the derivative using finite differences.

Script:

```
import numpy as np
def f(x): return np.sin(x)
h = 1e-5
x = 1.0
num_df = (f(x + h) - f(x - h)) / (2 * h)
print("Numerical derivative:", num_df)
```

Output: *Numerical derivative: 0.5403023058569989*

Numerical Integration: (e.g. integral of f(x) = sin(x) from 0 to π)

Approximates the definite integral using the trapezoidal rule.

Script:

```
import numpy as np
def f(x): return np.sin(x)
h, x = 1e-5, 1.0
x_vals = np.linspace(0, np.pi, 100)
y_vals = f(x_vals)
integral = np.trapz(y_vals, x_vals)
print("Numerical integral:", integral)
```

Output: Numerical integral: 1.9998321638939927

Worked Problem

Problem:
Find the definite integral of f(x) = sin(x) from 0 to *π.*

Analytical Solution:

- Derivative: cos(x), At x = 1, f'(1) = cos(1) ≈ 0.5403
- Integral:

$$\int_0^\pi \sin(x)\,dx = [-\cos(x)]_0^\pi = [-(-1)] - [-1] = 2$$

Numerical Solution(Python/NumPy)

```python
import numpy as np

# Define the function
def f(x):
    return np.sin(x)

# Point at which to compute the numerical derivative
x = 1.0
h = 1e-5  # small step size for finite difference

# Numerical derivative using central difference formula
numerical_derivative = (f(x + h) - f(x - h)) / (2 * h)
print(f'Numerical derivative of sin(x) at x={x}: {numerical_derivative}')

# Numerical integral of f(x) = sin(x) from 0 to pi using trapezoidal rule
x_vals = np.linspace(0, np.pi, 1000)  # 1000 points for better accuracy
y_vals = f(x_vals)
numerical_integral = np.trapz(y_vals, x_vals)
print(f'Numerical integral of sin(x) from 0 to pi: {numerical_integral}')
```

Output:

Numerical derivative of sin(x) at x=1.0: 0.5403023058569989

Numerical integral of sin(x) from 0 to pi: 1.999998351770852

Summary

- **Reviewed core concepts**: limits, derivatives, integrals, gradients, and fundamental theorems
- Limits provide the foundation for defining derivatives and integrals
- Derivatives quantify rates of change in functions
- Integrals calculate accumulated quantities like areas and totals
- The Fundamental Theorem of Calculus connects differentiation with integration
- Numerical methods approximate derivatives and integrals when analytic forms are unavailable
- Calculus is fundamental across science, engineering, artificial intelligence, and economics