# Introduction to Linear Algebra

*Vector Spaces, Linear Transformations, Eigenvalues/Eigenvectors, SVD, & PCA*

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# Motivation & Application

- High-dimensional data (images, sounds, etc.) is hard to analyze.
- Principal Component Analysis (PCA) uses eigenvectors of the data covariance matrix to identify key directions (“principal components”).
- Why? Reduce dimensionality while preserving maximal variance-enables compression, denoising, and visualization.

# Vector Spaces

**Definition**: A vector space V over a field F is a set with two operations (addition and scalar multiplication), such that:

- For all u, v $\in$ V, u+v $\in$ V
- For all a $\in$ F, av $\in$ V
- The zero vector exists; addition is commutative and associative; distribution, identity, and inverse laws hold

**Examples**:

- R^n: vectors of n real numbers
- Polynomials of degree $\leq 3$: $a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$

# Linear Transformation

**Definition**: A function $T : V \rightarrow W$ is a linear transformation if:

$T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ and $T(a\mathbf{v}) = a.T(\mathbf{v})$ for all $a \in F$.

**Examples**:

- Rotation of vectors in $R^2$
- Scaling: $T(v) = 2.v$
- Matrix multiplication: $T_A(x) = A \cdot x$

**Matrix Representation**:
Every linear transformation can be represented by a matrix

# Eigenvalues & Eigenvectors

**Definition**: For a square matrix A, $v \neq 0$ is an eigenvector if $A.v = \lambda.v$
Here, $\lambda$ is the eigenvalue.

**Key Theorem**:
An $n \times m$ matrix has $\leq n$ eigenvalues (some may be complex/repeated).

**Geometric Meaning**:
Eigenvectors are invariant directions under transformation A; eigenvalues scale them.
**Example**: For A = [ [2, 0], [0, 3] ]
The standard basis vectors are eigenvectors with eigenvalues 2 and 3.

# Singular Value Decomposition (SVD)

**Theorem(SVD):**

Every real m x n matrix A can be decomposed:

$A = U\Sigma V^T$

where U and V are orthogonal, and Σ is diagonal (with non-negative numbers “singular values”).

**Interpretation:**

- Columns of U: output directions
- Columns of V: input directions
- Σ: scaling factors

**Application:** Used in PCA, image compression, latent semantic analysis (LSA).

# Worked Problem

**Problem**: Find the eigenvalues and vectors of

$A = [[2, 1], [1, 2]]$

**Solution**:

1. **Characteristic equation:**

   $\det(A - \lambda I) = 0$

   $\Rightarrow |2 - \lambda \quad 1|$

   $|1 \quad 2 - \lambda| = 0$

   $(2 - \lambda)^2 - 1 = 0 \Rightarrow (2 - \lambda)^2 = 1 \Rightarrow 2 - \lambda = \pm 1$

   Thus, $\lambda_1 = 1$, $\lambda_2 = 3$

## 2. Eigenvectors:

For $\lambda 1 = 1$: $(A - I)v = 0 \Rightarrow v1 = [[1], [-1]]$

For $\lambda 2 = 3$: $(A - 3I)v = 0 \Rightarrow v2 = [[1], [1]]$

# Summary

- Core concepts: vector spaces, linear transformations, eigenvalue & eigenvector, SVD
- Eigenvectors key to engineering use cases (PCA, stability analysis)
- SVD as a powerful generalization to all matrices