State Space Modeling of Roombas for Control

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1 Introduction

This documents covers formulations of Roomba motion through simplifications of the inputs to obtain a state-space description for model-based control.

2 Inputs

The Roomba has two wheels, and our only inputs to the Roomba are the motor inputs to each wheel. We will call the inputs to the two motorss=, u_{left} and u_{right}. However to simplify our analysis, we will instead consider the following parametrization of the inputs:

$$u_{\text{left}} = u_{\text{same}} - u_{\text{diff}} \tag{1}$$

$$u_{\text{right}} = u_{\text{same}} + u_{\text{diff}} \tag{2}$$

for some new set of input variables u_{same} and u_{diff}. Note that this parametrization is not limiting, as we can still attain any u_{left} and u_{right} within saturation limits of the s, through correct choice of u_{same} and u_{diff}. If we want precise angular velocity control of the s, this can be achieve either through identification of the system to choose a correct conversion factor, or PID feedback with the wheel encoders. Therefore we could also consider our inputs to be

$$(v_{\text{same}}, v_{\text{diff}}) \coloneqq g_m(u_{\text{same}}, u_{\text{diff}})$$

For some function g_m which captures the relationship between wheel velocity and the inputs. We assume no slippage between the wheel and the ground, so we need not consider the wheel angular velocity separately. However, we may want to account for saturation and dead-zone effects.

3 States

The important states in our problem are the position in the XY plane, $[x\,y]'$ and the heading, which can be represented as usual by one of three options: an angle θ, a rotation matrix R, or a unit direction vector u. These are all related in the following way:

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$u = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

4 Dynamics

Suppose v is the tangential speed of the vehicle and ω is the rate of change of θ we can write our dynamics as

$$\dot{x} = v\sin(\theta) \tag{3a}$$

$$\dot{y} = v\cos(\theta) \tag{3b}$$

$$\dot{\theta} = \omega, \tag{3c}$$

or in terms of the rotation matrix R,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = R \begin{bmatrix} 1 \\ 0 \end{bmatrix} v \tag{4a}$$

$$\dot{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R\omega. \tag{4b}$$

Either formulation is nonlinear due to the presence of rotations. The equations can be linearized, however it is worth consider the nonlinear formulation since the system is underactuated (imagine the task of parallel parking). From here we must identify the relationship between $(v_{\text{same}}, v_{\text{diff}})$ and (v, ω). The simplest equivalence is to simply assume,

$$v = v_{\text{same}} \tag{5}$$

$$\omega = \frac{v_{\text{diff}}}{r_{\text{Roomba}}}. \tag{6}$$

This certainly hold for situations when exactly one of $(v_{\text{same}}, v_{\text{diff}})$ is zero, consider that for the other cases if $(v_{\text{same}}, v_{\text{diff}})$ stay constant, then the Roomba traces out a circle. If both v_{left} and v_{right} are positive and $v_{\text{left}} < v_{\text{right}}$ then we have the following identities:

$$\frac{r_{\text{left}}}{r_{\text{right}}} = \frac{v_{\text{left}}}{v_{\text{right}}}$$

$$r_{\text{right}} - r_{\text{left}} = 2r_{\text{Roomba}},$$

for r_{right} and r_{left} are the radii of the circles traced out by the outer and inner wheel respectively. Solving for these radii and substituting in equations (1) and (2) we can compute the period of the Roomba completing a circle as

$$T = \frac{2\pi r_{\text{Roomba}}}{v_{\text{diff}}}$$

from which we conclude that equation (6) is correct. Furthermore, since the center of the Roomba traces out a circle with radius $\frac{1}{2}(r_{\text{right}} + r_{\text{left}})$, we can also conclude that (5) is correct as well. Equations (5) and (6) can be similarly verified for all other situations. For a given problem it may be beneficial to limit $(v_{\text{same}}, v_{\text{diff}})$ to some subset of all possible combinations in $\mathbb{R}^2$. For example, we might have a Dubin's vehicle where v_{same} is constant and $v_{\text{diff}} \in \{0, v_{\text{diff,max}}\}$. Or we might want to restrict our vehicle to v_{left} and v_{right} having the same sign as in an automobile.

5 Control

TODO