ECE147A
Homework 3 Solutions

September 27, 2017

1 Problem 1

To find our operating points, we must first define our input vector u, state vector x, and output vector y, and our functions $\dot{x} = f(x,u)$ and $y = g(x,u)$.

$$\text{Let, } x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \omega \end{bmatrix}$$
$$u = \tau$$

$$y = \tilde{y}$$

Then,

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\omega} \end{bmatrix} = f(x,u) = \begin{bmatrix} \dot{\theta} \\ \omega^2 \sin(\theta)\cos(\theta) - \frac{\nu_1}{m}\dot{\theta} - \frac{g}{R}\sin(\theta) \\ -\nu_2\omega + \tau \end{bmatrix}$$

and

$$y = g(x,u) = R\left(1 - \cos(\theta)\right)$$

Then operating points are those such that $\theta^* = \cos^{-1}\left(1 - \frac{y^*}{R}\right)$ and $\dot{x}^* = 0$

When $y^* = \frac{R}{3}$, then $\theta^* = \cos^{-1}\left(\frac{2}{3}\right) \approx \pm 0.8412$ radians.

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \omega^2 \frac{2\sqrt{5}}{9} - \frac{\nu_1}{m}\dot{\theta} - \frac{g}{R}\frac{\sqrt{5}}{3} \\ -\nu_2\omega + \tau \end{bmatrix} = 0$$

Therefore

$$\dot{\theta}^* = 0$$

and,

$$\omega^* = \sqrt{\frac{3g}{2R}}$$
$$u^* = \nu_2\sqrt{\frac{3g}{2R}}$$

or

$$\omega^* = -\sqrt{\frac{3g}{2R}}$$
$$u^* = -\nu_2\sqrt{\frac{3g}{2R}}$$

There are, in total, four possible operating points (mathematically, of course, physically they aren't particularly different).

You should get the following Jacobian matrices:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ \omega^2(2\cos^2(\theta)-1) - \frac{g}{R}\cos(\theta) & -\frac{\nu_1}{m} & 2\omega\sin(\theta)\cos(\theta) \\ 0 & 0 & -\nu_2 \end{bmatrix}$$
$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\frac{\partial g}{\partial x} = \begin{bmatrix} R\sin(\theta) & 0 & 0 \end{bmatrix}$$
$$\frac{\partial g}{\partial x} = 0$$

Then our locally linear state space representation for the equillibrium $\left(\begin{bmatrix} 0 \\ +cos^{-1}(\frac{2}{3}) \\ +\sqrt{\frac{3g}{2R}} \end{bmatrix}, \nu_2\sqrt{\frac{3g}{2R}}\right)$ is:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^*,u^*} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5g}{6R} & -\frac{\nu_1}{m} & \frac{2\sqrt{30\frac{g}{R}}}{9} \\ 0 & 0 & -\nu_2 \end{bmatrix}$$
$$B = \left.\frac{\partial f}{\partial u}\right|_{x^*,u^*} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \left.\frac{\partial g}{\partial x}\right|_{x^*,u^*} = \begin{bmatrix} R\frac{\sqrt{5}}{3} & 0 & 0 \end{bmatrix}$$
$$D = \left.\frac{\partial g}{\partial u}\right|_{x^*,u^*} = 0$$

Next to calculate the transfer function, we use the formula:$G(s) = C(sI - A)^{-1}B + D$, where we have the fact that $A \cdot \text{adj}(A) = \det(A) \cdot I$ where:

$$\det(A) \triangleq \sum_{j=1}^{n} a_{ij} \cdot \text{cof}_{ij}(A) \cdot (-1)^{i+j}$$

$$\text{adj}_{ij}(A) \triangleq \text{cof}_{ji}(A) \cdot (-1)^{i+j} \quad (\text{ie} \ \ \text{adj}(A) = \text{cof}^T(A))$$

and $\text{cof}_{ij}(A)$ (the ijth cofactor of A) is defined as the determinant of the matrix formed by removing the ith row and jth column of A (ie. the determinant of the ijth minor of A). Then:

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \text{adj}(sI - A)$$

Which you may recognize as the adjoint of $(sI - A)$ divided by the characteristic function of A,

$$sI - A = \begin{bmatrix} s & -1 & 0 \\ \frac{5g}{6R} & s + \frac{\nu_1}{m} & -2\frac{\sqrt{30g}}{9\sqrt{R}} \\ 0 & 0 & s + \nu_2 \end{bmatrix}$$

$$\det(sI - A) = 0 - 0 + (s(s + \frac{\nu_1}{m}) - (-1)\frac{5g}{6R})(s + \nu_2) = (s^2 + \frac{\nu_1}{m}s + \frac{5g}{6R})(s + \nu_2)$$

Note that since C and B will select only the first row last column element of $(sI - A)^{-1}$ and multiply it by $R\frac{\sqrt{5}}{3}$, we only need to calculate that element of the adjoint, which is:

$$\text{cof}_{3,1}(sI - A) = 2\frac{\sqrt{30g}}{9\sqrt{R}}$$

Then the transfer function of our linearized system is:

$$\frac{2\frac{\sqrt{30g}}{9\sqrt{R}}}{(s + \nu_2)(s^2 + \frac{\nu_1}{m}s + \frac{5g}{6R})}$$

$(s + \nu_2)$ is certainly in the LHP when $\nu_2 > 0$, and remembering our Routh-Hurwitz rule for a second order polynomial, $(s^2 + \frac{\nu_1}{m}s + \frac{5g}{6R})$, will have stable poles iff its coefficients are both positive, which happens when each of ν_1, m, g, R are positive. All poles are in the LHP, therefore the equillibrium is stable.

We will also have operating points with the same θ^*, ω^*, and τ^* when $\theta^* = 0$ or $\theta^* = \pi$ (the top and bottom of the hoop). At these values of θ^*, any torque/angular velocity pair is an equillibrium (there are infinitely many), though not stable in either case unless $\omega \to 0$ or $\omega \to \infty$ respectively. This makes sense physically since perturbations will cause the bead to either move up to the previous stable equillibrium or fall (and do some cool stuff - `http://www.mathworks.com/matlabcentral/fileexchange/33425-bead-on-a-rotating-hoop`). For those who were interested in this problem, as well as many others in nonlinear dynamics, the book "Nonlinear Dynamics and Chaos" by Stephen Strogatz is a good reference.

2 Problem 2

$$\frac{\partial T}{\partial P} = \frac{(1+PC)\cdot C - PC\cdot C}{(1+PC)^2} = \frac{C}{(1+PC)^2}$$

$$\frac{\partial T}{\partial P}\frac{P(1+PC)}{PC} = \frac{1}{1+PC}$$

Ok, so this one was trivial, BUT, there is a fundemental meaning to this equation. Recall that we are defining Sensitivity as how much our total system is altered by small changes or miscalculations in the plant transfer function. This could happen if, for example, we are using measurements with error such as approximate values for the mass of a cart or the spring coefficient, or if the system was slightly time varying (as is often the case with physical systems when the effect of heat is factored in). $\frac{\partial T}{\partial P}$ is how much the feedback system changes with changes in P, which we then scale by the value of the plant with respect to the entire system. Additionally, the sensitivity function conveniently gives us information about several aspects of our system performance like reference reproduction, disturbance attenuation, and noise attenuation.

3 Problem 3

Let us first compute the transfer function, $\frac{Y}{U}$ of the above system:

$$\frac{Y}{U} = \frac{PC}{1+PC} = \frac{\frac{50(s+3)}{s(s+2)(s+18)}}{1+\frac{50(s+3)}{s(s+2)(s+18)}} = \frac{50(s+3)}{s(s+2)(s+18)+50(s+3)} = \frac{50(s+3)}{s^3+20s^2+86s+150}$$

Since we'll be computing these a lot, it may be useful to remember this as: $\frac{\mathrm{num}(P)\mathrm{num}(C)}{\mathrm{den}(P)\mathrm{den}(C)+\mathrm{num}(P)\mathrm{num}(C)}$

Two equally valid ways of checking stability are to use the Routh-Hurwitz citerion which for a monic third-order denominator polynomial, $s^2 + c_2s^2 + c_1s + c_0$, states that if the coefficients, c_0, c_1, c_2 are all positive and $c_2 \cdot c_1 > c_0$, then the transfer function has stable poles. We will go more in depth on why this is true later, so using 'roots([1 20 86 150])' in Matlab and checking that they all have negative real parts would have sufficed.

Looking at our diagram again, we can see that our transfer function $\frac{Y}{W}$ is:

$$\frac{P}{1+PC} = \frac{s+18}{s^3+20s^2+86s+150}$$

Then using the FVT for a step input:

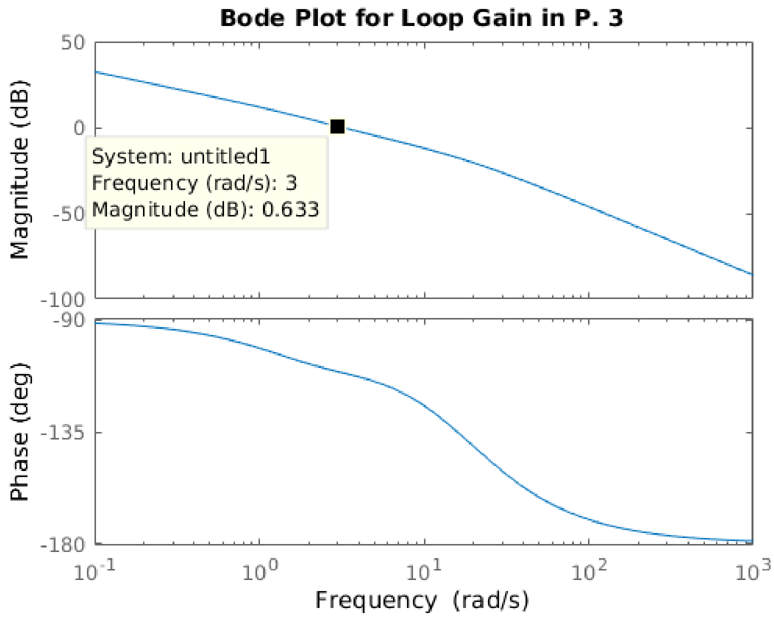
$$\lim_{s\to\infty}\frac{s+18}{s^3+20s^2+86s+150} = \frac{18}{150}$$

Which is our steady state gain to a step disturbance.

Similarly, to compute our steady state gain to a step input:

$$\lim_{s\to\infty}\frac{50s+150)}{s^3+20s^2+86s+150} = 1$$

Next our Bode Plot:



We can see that our high gain at low frequency and low gain at high frequency are definitely satisfied (and will be for any strictly proper loop gain with an integrator). Our crossover point occurs after passing two poles and one zero, so we can reasonably expect our slope at crossover to be close to -20dB/dec which is also desirable.

If we calculate the Sensitivity of our system at steady state we look at:

$$\lim_{s\to\infty} \frac{s(s+3)(s+18)}{s^3+20s^2+86s+150} = 0$$

This is expected, since we already saw that $\frac{Y}{U}$ at steady state did not depend on one of the plant pole locations (if it was at $-2+\epsilon$ the DC gain would still be 1). The limitation of sensitivity is it doesn't take into account if the pole at zero (integrator) in the plant was slightly off.

Recall that we saw in the notes that mathematically:

$$\frac{Y}{U} = 1 - S(s) \quad \text{and} \quad \frac{Y}{W} = P(s)S(s)$$

Therefore if $S(0) = 0$ then the output with respect to our step reference is exactly $y = u$, and our output with respect to a step disturbance is as small as possible (note: not zero because $P(0) \to \infty$ so will be some nonzero value, but it will be the minimum of possible $\frac{Y}{W}(0)$).

4 Problem 4

Our loop gain, $C(s)P(s)$ is now $\dfrac{K(s+2)}{s(s+2)(s+18)} = \dfrac{K}{s(s+18)}$

When this happens, we call it a **pole-zero cancellation**, and we will see these again many times throughout the course, in situations where they can be either good or bad.

Now, our closed loop transfer function is: $\dfrac{K}{s^2+18s+K}$, which is stable for all positive K.

Recall that a second order system (SOS), can be written in terms of its damping (ζ) and natural frequency (ω_n) as follows:

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+w_n^2}$$

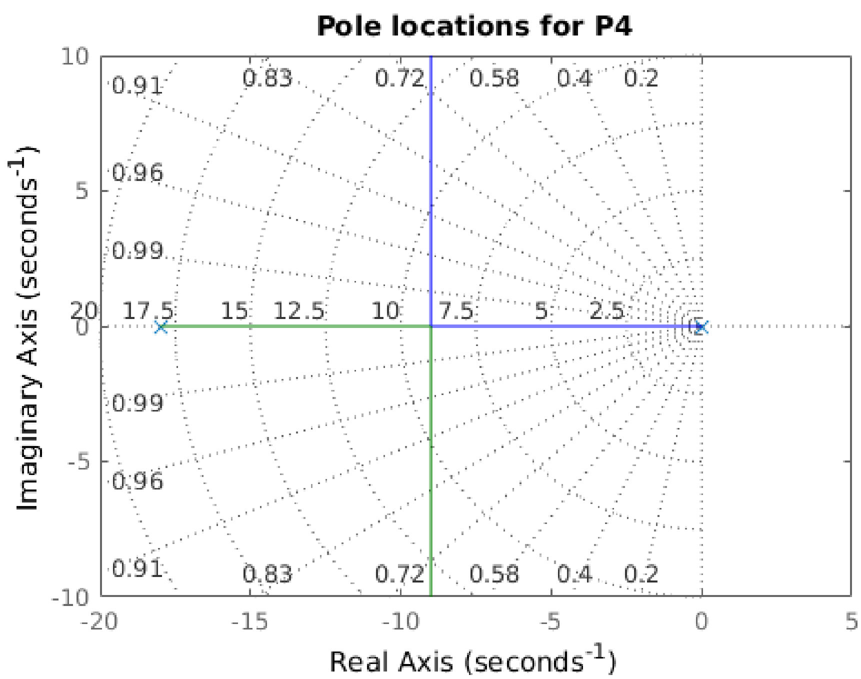
We also know that our step response parameters can be written in terms of ζ and ω_n as follows:

$$t_r = \frac{1.8}{\omega_n^2}, \qquad t_s = \frac{4.6}{\zeta\omega_n}, \qquad \%OS = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

The book uses 4 instead of 4.6 for t_s. Then, noting we always accept conservative results:

$$\omega_n^2 \geq \frac{1.8}{0.2} = 9 \qquad \sigma = \zeta\omega_n \geq \frac{4.6}{0.5} = 9.2 \qquad \zeta \geq \frac{-ln(0.15)}{\sqrt{\pi^2+ln^2(0.15)}} = 0.517$$

Looking at our root locus plot:



We see that two of our specifications, $t_r = 0.2$ and $\%OS = 15$ can be met simultaneously for $9 \leq K \leq 303.1$ but that our t_s specification cannot be met by both poles anywhere. The minimal relaxation of t_s that allows us to meet all three specs at once is $t_s = \dfrac{4.6}{9} \approx 0.511$.