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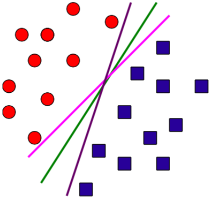

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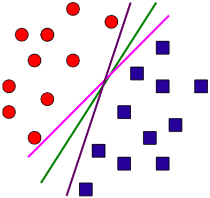

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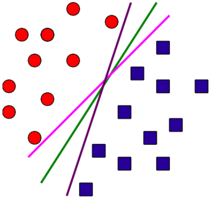

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- Large margin leads to good generalization on the test data
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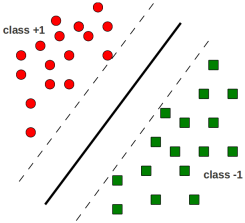
- *Additionally* uses the Maximum Margin Principle
  - Finds the hyperplane with *maximum separation margin* on the training data



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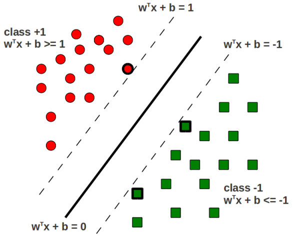

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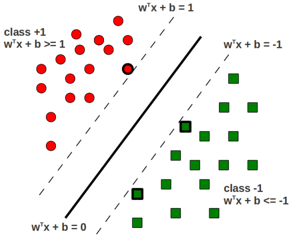

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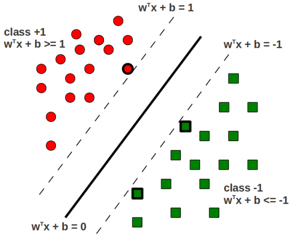

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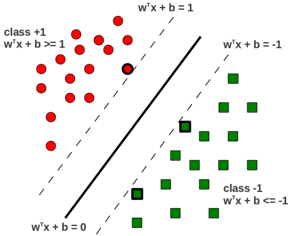

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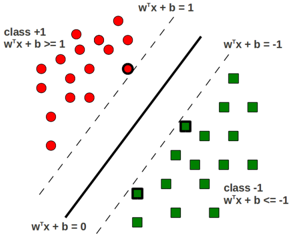

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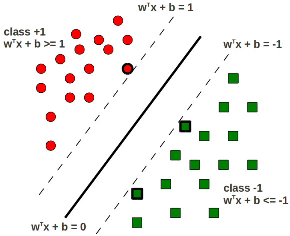

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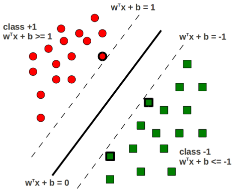

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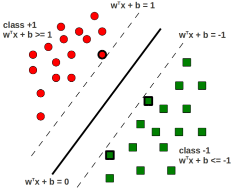

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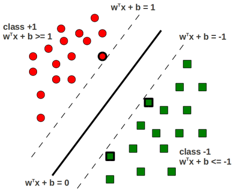

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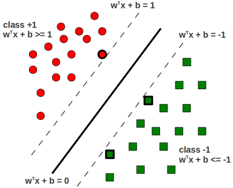

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- This is a Quadratic Program (QP) with N linear inequality constraints

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- We can now solve this Lagrangian
  - i.e., optimize $L(\mathbf{w}, b, \alpha)$ w.r.t. $\mathbf{w}$, b, and α
  - .. making use of the Lagrangian Duality theory..

Next class..

- Solving the SVM optimization problem
- Allowing misclassified training examples (**non-zero loss**)
- Introduction to kernel methods (nonlinear SVMs)