

# A CANONICAL SYSTEMS REALIZATION OF THE RIEMANN $\Xi$ FUNCTION

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**ABSTRACT.** We prove that there exists an entire function  $E$  with the Hermite–Biehler (HB) property on  $\mathbb{H}$  such that

$$\Xi(z) = \frac{1}{2}(E(z) + E^*(z)), \quad |E^*(z)/E(z)| < 1 \quad (\text{Im } z > 0).$$

Hence all zeros of  $\Xi$  are real, and by the de Branges Wronskian identity they are simple. As consequences, the de Bruijn–Newman constant satisfies  $\Lambda(\Xi) = 0$  (together with Rodgers–Tao’s lower bound) and the Keiper–Li coefficients are nonnegative.

The argument works entirely at a fixed height  $\eta_0 > 0$ . We discretize a canonical system on  $[0, L]$  into  $N \asymp L^2$  constant blocks and apply gauge and intrinsic normalizations to obtain *height equivalence*. An outer factor  $W_{\eta_0}^+$  controls sizes on the seams, while a strict Schur margin is created near the base and propagated by  $SU(1, 1)$  pseudohyperbolic control. A square seam extractor has  $\Theta(\sqrt{N})$  conditioning and an  $O(L^2)$  global Lipschitz bound, so the inverse function theorem yields radius  $r \asymp L^{-1}$  and a finite-hop continuation to the weighted boundary data of  $\Xi/W_{\eta_0}^+$ . A terminal tail, compactness of Hamiltonians, and stability of transfer matrices produce a limiting system with endpoint HB function  $E$ ; a two-seam continuation then propagates boundary identities to the interior, giving the stated representation.

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## INTRODUCTION

Let

$$\Xi(z) := \xi\left(\frac{1}{2} + iz\right)$$

be the even entire function that encodes the nontrivial zeros of the Riemann zeta function on the critical line. Inside de Branges’ framework of canonical systems [dB68], we prove that  $\Xi$  can be realized as the real part of a Hermite-Biehler entire function  $E$

$$\Xi = \frac{E + E^*}{2} \quad \text{with} \quad |E^*/E| < 1 \quad \text{on } \mathbb{H}, \tag{1}$$

by using a *fixed-height* canonical-systems scheme. (Notation: For any function  $F$  on  $\mathbb{C}$  we write  $F^*(z) := \overline{F(\bar{z})}$  for Schwarz reflection across  $\mathbb{R}$ .)

We fix a height  $\eta_0 > 0$  once and for all and run the entire analysis on a racetrack contour  $\partial\Omega_{\eta_0}(L) \subset \{\eta_0 \leq \operatorname{Im} z \leq 2\eta_0\}$  of length  $\asymp L$ . The system’s Hamiltonian on  $[0, L]$  is discretized into  $N \asymp L^2$  constant blocks of size  $\ell = L/N$ . We gauge-normalize boundary traces so that the canonical tail has unit amplitude at any height. An *intrinsic amplitude normalizer* removes the height-dependent gain on each two-row Jacobian block, producing a uniform *height-equivalence*: after normalization, the  $\eta_0$ -blocks are uniformly comparable to their  $\eta = 0$  counterparts. In parallel, an outer factor  $W_{\eta_0}^+$  placed only on  $\operatorname{Im} z \geq \eta_0$  makes the boundary *size* estimates independent of  $L$  (via Lemma 8.11); the strict–Schur margins themselves come from the local ball and per-hop control (Lemmas 9.20 and 8.7).

Our main result in this paper is that by working entirely at a fixed height, we construct a sequence of real, PSD Hamiltonians  $H_L$  ( $\operatorname{tr} H_L \equiv 1$ ) whose endpoint HB

functions  $E_L$  satisfy

$$\sup_{z \in \Sigma_{\eta_0}(L)} \left| \frac{A_{H_L}(z)}{W_{\eta_0}^+(z)} - \frac{\Xi(z)}{W_{\eta_0}^+(z)} \right| \xrightarrow[L \rightarrow \infty]{} 0,$$

and, for every compact  $K \Subset \Omega_{\eta_0}(L)$ ,

$$\sup_{z \in K} \left| \frac{A_{H_L}(z)}{W_{\eta_0}^+(z)} - \frac{\Xi(z)}{W_{\eta_0}^+(z)} \right| \xrightarrow[L \rightarrow \infty]{} 0.$$

Here  $\Sigma_{\eta_0}(L) := \partial\Omega_{\eta_0}(L) \cap \{\operatorname{Im} z = \eta_0, \operatorname{Im} z = 2\eta_0\}$ . On the vertical arcs  $\Gamma_{\eta_0}(L) := \partial\Omega_{\eta_0}(L) \setminus \Sigma_{\eta_0}(L)$  we retain the uniform bound

$$\sup_{z \in \Gamma_{\eta_0}(L)} \left| \frac{A_{H_L}(z)}{W_{\eta_0}^+(z)} \right| \leq C(\eta_0).$$

while maintaining a uniform strict Schur bound  $|E_L^*/E_L| \leq \rho_\star < 1$  on  $\partial\Omega_{\eta_0}(L)$ . By enforcing a terminal Hamiltonian on  $[L - \Lambda, L]$ , we ensure a clean limiting process. Local compactness of  $H_L$  and a stability theorem for transfer matrices yield a limiting canonical system with an HB function  $E$  satisfying (1). This construction allows us to obtain three key corollaries:

- **All zeros of  $\Xi$  are real and simple.** (Corollary 10.10)
- **De Bruijn-Newman.** We prove  $\Lambda(\Xi) \leq 0$ ; combined with Rodgers-Tao's  $\Lambda \geq 0$  [RT20], this yields  $\Lambda(\Xi) = 0$  (Corollary 10.11).
- **Li-positivity.** All Keiper-Li coefficients  $\{\lambda_n\}$  of  $\xi$  are nonnegative (see [Li97] for their origin) (Corollary 10.12).

The proof is specifically based on four key devices tailored to our fixed-height setting:

- *Height-aware gauge normalization and intrinsic amplitude normalization.* For cell phase  $u_k = \pi k/N$  and  $\alpha = \eta_0 \ell/2$ , the intrinsic amplitude normalizer  $E_{\text{amp}}(\alpha, u_k)$  normalizes the two odd-frequency packets arising from Duhamel's formula. This produces a blockwise transform  $T_k$  with uniformly controlled singular values, establishing a rigorous *height-equivalence* between Jacobians at height  $\eta_0$  and height zero (Sections 2–5).
- *Trimmed DST-I conditioning independent of  $L$ .* After trimming the ends of the sampling grid, the normalized boundary map has singular values of order  $\Theta(\sqrt{N})$ , exactly as at height zero. This follows from the classic DST Gram identity and the height-equivalence (Section 6).
- *Uniform strict Schur bound via block "kicks" and Schwarz–Pick.* We decompose the left block into constant pieces of size  $\leq \ell_{\max}$  and use the *exact* tail contraction on  $\operatorname{Im} z = \eta_0$  together with nonexpansion in the disk to obtain an  $L$ -independent

bound on the top seam (Lemma I.10 and Remark I.11):

$$|v_L(x+i\eta_0)| \leq \tanh\left(\frac{C(\varepsilon)|z|X}{1-r^2}\right) =: \rho_{\text{sch}}(\eta_0) < 1, \quad r := C(\varepsilon) (\max_{|z| \leq Y} |z|) \ell_{\max} < 1.$$

A margin-aware pseudohyperbolic control (Proposition I.15) then yields

$$\beta(v(z;\theta), v(z;\tilde{\theta})) \leq \frac{C_{\text{lin}}}{1-\rho_0^2} \sqrt{K} \|\theta - \tilde{\theta}\|_2,$$

and the bound transfers to  $v = E^*/E$  via the explicit  $\text{SU}(1, 1)$  relation (Appendix I).

- *Square, length-stable inversion at the boundary.* We form a *square* extractor that keeps the first  $K \sim \kappa L$  DST modes from the boundary data of  $A_L/W_{\eta_0}^+$ . Its Jacobian has  $\Theta(\sqrt{N})$  conditioning and a global Lipschitz bound of  $O(L^2)$ , justified by the strict Schur bound. This yields a quantitative inverse-function radius  $r \asymp L^{-1}$  and a finite-hop continuation to the boundary data of  $\Xi/W_{\eta_0}^+$  (Section 9).

The overall idea of the proof is that at the canonical base  $H \equiv \frac{1}{2}I$ , linearization produces two odd-frequency packets per block. Gauge normalization and intrinsic normalization remove the height-mixing, making the Jacobian at height  $\eta_0$  comparable to the height-zero case. The trimmed DST-I Riesz bound then yields  $\Theta(\sqrt{N})$  singular values for the core linear map. On the nonlinear side, we first create a strict Schur margin near the base and keep the continuation inside this ball; the margin-aware pseudohyperbolic control (Proposition I.15) then yields quantitative bounds in the hyperbolic metric  $\beta = \operatorname{arctanh} \rho$ . This preserves a uniform denominator  $1 - |v|^2 \geq 1 - \rho_0^2$  in the oscillatory two-insertion kernel estimate, leading to an  $O(L^2)$  Lipschitz bound for the stacked Jacobian. We use a square mode extractor; the inverse function theorem then provides a local inverse of radius  $r \asymp L^{-1}$ . A finite-hop homotopy reaches the weighted boundary data of  $\Xi$  while staying inside the same strict-Schur ball. Finally, a terminal tail and translation invariance yield a clean limit, and local compactness plus stability of transfer matrices produce a limiting HB function  $E$  with  $\Xi = \frac{1}{2}(E + E^*)$  and  $|E^*/E| < 1$  on  $\mathbb{H}$ . Figure 1 outlines the proof.

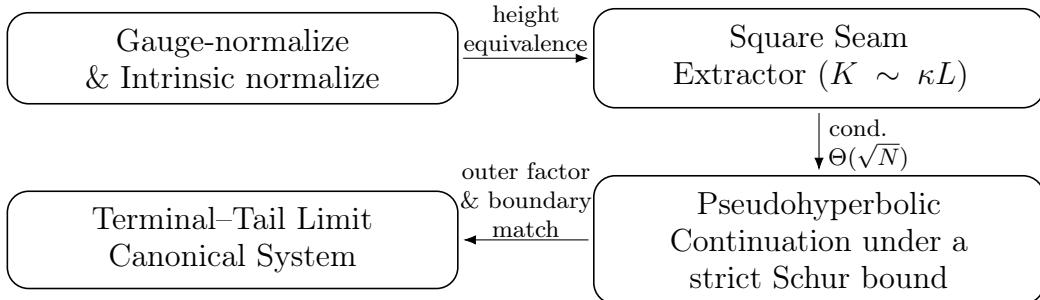


FIGURE 1. An outline of the proof.

One may ask "Why fixed height?". Working strictly above the real axis avoids height-zero identification and associated star-limit subtleties; the only place where the star operation commutes with limits here is for the limiting entire function, handled cleanly after we pass to the limit (Appendix H). The outer factor  $W_{\eta_0}^+$  makes all boundary inequalities insensitive to the growing racetrack length, and the gauge normalization / intrinsic normalization "removes the height" from the bulk linear algebra without ever taking  $\eta \downarrow 0$  inside the construction.

The paper is organized as follows: Section 1 reviews canonical systems and HB normalization. Sections 2–5 introduce the gauge normalization and intrinsic normalization and prove the height-equivalence. Section 6 establishes DST conditioning. Sections 7–8 develop the fixed-height boundary geometry and the global  $O(L^2)$  Lipschitz bound on the stacked Jacobian. Section 9 constructs the square extractor and carries out the finite-hop continuation. Section 10 proves compactness for Hamiltonians, stability for transfer matrices, and the limiting identification (1).

Unless stated otherwise, absolute constants  $C, c$  and functions  $C(\cdot)$  may change from line to line. They depend only on the fixed parameters  $\eta_0, \delta, \varepsilon$  and on the  $C^{1,1}$  geometry of  $\partial\Omega_{\eta_0}(L)$ ; in particular they are independent of  $L$ , of  $N \asymp L^2$ , and of the continuation hop index. We write

$$B_r(\theta_{\text{base}}) := \{ \theta : \|\theta - \theta_{\text{base}}\|_2 \leq r \}$$

for the bootstrap ball in parameter space used throughout.

Symbol	Meaning / where defined
$c_{\text{amp}}, C_{\text{amp}}$	normalization bounds, Lemma 3.2
$c_{\text{mix}}, C_{\text{mix}}$	mixer bounds (height-equivalence), Prop. 5.2
$\rho_{\text{sch}}(\eta_0)$	Weyl-side strict Schur bound, Lemma. 8.6
$C_{\text{Lip}}$	Lipschitz const. in Thm. 8.8
$r$	inverse-function radius, §9 ("Quantitative radius")

TABLE 1. Constants used throughout; all are independent of  $L$ .

## 1. CANONICAL SYSTEMS: SETUP, BASE FLOW, AND ENDPOINT ENTIRE FUNCTIONS

We fix

$$\Xi(z) := \xi \left( \frac{1}{2} + iz \right),$$

so that  $z \in \mathbb{R}$  corresponds to the critical line  $\text{Re}(s) = \frac{1}{2}$ .

For every  $\eta > 0$ , we have

$$|\Xi(x + i\eta)| \ll_\eta (1 + |x|)^{A(\eta)} e^{-\pi|x|/4} \quad (|x| \rightarrow \infty),$$

hence  $\lambda_\eta(x) := \log(1 + |\Xi(x + i\eta)|) \in L^1(dx/(1 + x^2))$ , justifying the outer-function construction in §8.1.

Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A (left) half-line canonical system is

$$JY'(s, z) = z H(s) Y(s, z), \quad s \geq 0, \quad z \in \mathbb{C}, \quad (2)$$

with measurable Hamiltonian  $H : [0, \infty) \rightarrow \mathbb{R}^{2 \times 2}$ , positive semidefinite (PSD),  $\text{tr } H(s) \equiv 1$  a.e. For  $0 \leq t \leq s$  let  $\Phi(s, t; z)$  be the transfer matrix with  $\Phi(t, t; z) = I$ ; write  $\Phi(s; z) = \Phi(s, 0; z)$ .

**Lemma 1.1** (Propagator bounds). *If  $\text{tr } H \equiv 1$ , then for any  $z \in \mathbb{C}$  and  $s \geq t$ ,*

$$\|\Phi(s, t; z)\| \leq e^{|z|(s-t)}, \quad \|\Phi(t, s; z)\| \leq e^{|z|(s-t)}.$$

*Proof.* Write  $Y' = M(u)Y$  with  $M(u) = -zJH(u)$ . Since  $H(u) \succeq 0$  with  $\text{tr } H(u) = 1$ ,  $\|H(u)\| \leq 1$ . Also  $\|J\| = 1$ . Hence  $\|M(u)\| = \|zJH(u)\| \leq |z|$ . Grönwall yields the bounds.  $\square$

Let  $\Phi(L; z) = \begin{pmatrix} A^\Phi & B^\Phi \\ C^\Phi & D^\Phi \end{pmatrix}$ . We use the Hermite-Biehler normalization

$$E_L(z) := A^\Phi(z) - i B^\Phi(z), \quad A_L = \frac{1}{2}(E_L + E_L^*), \quad B_L = \frac{1}{2i}(E_L^* - E_L), \quad (3)$$

where  $F^*(z) := \overline{F(\bar{z})}$ . With this choice,  $E_L$  is Hermite-Biehler.

We write  $\Phi(L; z) = \begin{pmatrix} A^\Phi & B^\Phi \\ C^\Phi & D^\Phi \end{pmatrix}$  for transfer entries and reserve  $A_L := \frac{1}{2}(E_L + E_L^*)$  for the endpoint entire function (HB side). Note that  $A_L(z) \equiv A^\Phi(z)$ . We never use bare  $A(z)$  or  $B(z)$ : transfer entries are always  $A^\Phi, B^\Phi$ , while HB real/imaginary parts are denoted  $A_L, B_L$ .

**Lemma 1.2** (HB normalization; endpoint and Weyl Schur ratios). *Assume  $H$  is real, PSD with  $\text{tr } H \equiv 1$  on  $[0, \infty)$  and equals  $\frac{1}{2}I$  on  $[L, \infty)$ . Let*

$$\Phi(L; z) = \begin{pmatrix} A^\Phi & B^\Phi \\ C^\Phi & D^\Phi \end{pmatrix},$$

*and set  $E_L := A^\Phi - iB^\Phi$ . Then  $E_L$  is Hermite-Biehler:  $|E_L^*(z)| < |E_L(z)|$  for  $z \in \mathbb{H}$ , hence  $E_L$  has no zeros in  $\mathbb{H}$ . Define the endpoint ratio*

$$v_L(z) := \frac{E_L^*(z)}{E_L(z)} \quad (z \in \mathbb{H}),$$

*which is Schur on  $\mathbb{H}$  (i.e.  $|v_L(z)| < 1$ ). If  $m_L$  is the Weyl function, its Schur transform*

$$v_L(z) := \frac{m_L(z) - i}{m_L(z) + i}$$

is also Schur on  $\mathbb{H}$ . We do not identify  $v_L$  and  $v_L$ ; they play different roles below. (From now on, the symbol  $v$  is reserved for the Cayley transform of the Weyl-Titchmarsh function; the outer-weight boundary datum is denoted  $\lambda_\eta$ .)

*Proof.* By de Branges [dB68, Ch. VI], for real PSD  $H$  with  $\text{tr } H \equiv 1$  and tail  $H \equiv \frac{1}{2}I$  on  $[L, \infty)$ : (i) the system is limit-point at  $+\infty$ , so the Weyl function  $m_L$  is Herglotz on  $\mathbb{H}$ , hence its Cayley transform  $v_L = (m_L - i)/(m_L + i)$  is Schur on  $\mathbb{H}$ ; and (ii) with  $E_L := A^\Phi - iB^\Phi$  one has the Hermite-Biehler property  $|E_L^*(z)| < |E_L(z)|$  on  $\mathbb{H}$ , hence  $v_L := E_L^*/E_L$  is Schur. Finally, the identity  $A_L = \frac{1}{2}(E_L + E_L^*)$  is a direct calculation.  $\square$

**Lemma 1.3** (Translation invariance of the canonical system). *Let the Hamiltonian on  $[X, L]$  be the terminal Hamiltonian  $H \equiv \frac{1}{2}I$ . Then the endpoint HB function at  $L$  is related to the one at  $X$  by*

$$E_L(z) = e^{-iz(L-X)/2} E_X(z) \quad (\text{all } z \in \mathbb{C}).$$

Consequently, the gauge-normalized endpoint function is independent of  $L$  for  $L > X$ :

$$e^{+izL/2} E_L(z) = e^{+izX/2} E_X(z).$$

*Proof.* On the terminal segment, the transfer matrix is a rotation:  $\Phi(L, X; z) = R(-\frac{z}{2}(L-X))$  [dB68]. Let  $\Theta(X, z)$  be the first column of  $\Phi(X, z)$ . Then  $\Theta(L, z) = \Phi(L, X; z)\Theta(X, z)$ . Using the identity  $\ell_+R(\theta) = e^{i\theta}\ell_+$  from Lemma 2.1, where  $\ell_+ = [1 \ i]$ , we have

$$E_L(z) = \ell_+ \Theta(L, z) = \ell_+ \Phi(L, X; z) \Theta(X, z) = e^{-iz(L-X)/2} \ell_+ \Theta(X, z) = e^{-iz(L-X)/2} E_X(z).$$

The second identity follows by multiplying both sides by  $e^{+izL/2}$ .  $\square$

**Lemma 1.4** (Finite-length Weyl formula). *If  $\Phi(L; z) = \begin{pmatrix} A^\Phi & B^\Phi \\ C^\Phi & D^\Phi \end{pmatrix}$ , then for  $z \in \mathbb{H}$ ,*

$$m_L(z) = (iA^\Phi - C^\Phi)(D^\Phi - iB^\Phi)^{-1}.$$

*Proof.* At  $s = L$  the Weyl boundary condition is  $C^\Phi + mD^\Phi = i(A^\Phi + mB^\Phi)$ , so  $(D^\Phi - iB^\Phi)m = iA^\Phi - C^\Phi$  and hence  $m = (iA^\Phi - C^\Phi)(D^\Phi - iB^\Phi)^{-1}$ .

For  $z \in \mathbb{H}$  the matrix  $D^\Phi - iB^\Phi$  is invertible by limit-point at  $+\infty$  and the Herglotz property of  $m_L$  (see [dB68, Ch. VI]).  $\square$

In all our continuations, we attach the canonical tail  $H(s) \equiv \frac{1}{2}I$  on  $[L, \infty)$ . With this choice,  $\Phi_0(s, t; z) = R(-\frac{z}{2}(s-t))$  and, with the HB choice,  $E_{0,L}(z) := \cos \frac{zL}{2} - i \sin \frac{zL}{2} = e^{-izL/2}$ . This fixes the base normalization [dB68, Rom14].

In addition to the right tail on  $[L, \infty)$ , we also attach the canonical tail on the left:

$$H(s) \equiv \frac{1}{2}I \quad \text{on } (-\infty, 0] \quad \text{and} \quad [L, \infty).$$

This defines the right and left Weyl functions for the half-line problems  $(s, \infty)$  and  $(-\infty, s)$ :

$$m_R(s; z), \quad m_L(s; z), \quad v_{\text{in}}(s; z) := \frac{m_R(s; z) - i}{m_R(s; z) + i}, \quad v_{\text{pre}}(s; z) := \frac{m_L(s; z) - i}{m_L(s; z) + i}.$$

The base normalization from Appendix E for  $E_{0,L}$  and the gauge normalization is unchanged. This auxiliary left tail is only used for the Weyl/endpoint comparison in Appendix I; it does not alter the HB endpoint normalization of  $E_L$  at length  $L$ .

Before going further, we retain some standard assumptions for reference: we work with real, PSD Hamiltonians  $H$  with  $\text{tr } H \equiv 1$  on  $[0, L]$ , and attach the canonical tails  $H \equiv \frac{1}{2}I$  on  $(-\infty, 0]$  and  $[L, \infty)$ . All fixed-height analyses are performed for some  $\eta_0 > 0$  and on the racetrack  $\partial\Omega_{\eta_0}(L)$  of §7. Bulk trimming uses  $\mathcal{K}_\delta$  with  $\delta \in (0, \frac{1}{4})$ , and intrinsic normalization is the factor  $E_{\text{amp}}(\alpha, u)$  from (10).

## 2. ALGEBRA AND BASE GAUGE NORMALIZATION

We record identities used repeatedly and fix the gauge (phase) normalization so that the base row equals 1 at any height.

**Lemma 2.1** (Elementary identities). *With  $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ ,  $\text{diag}(1, -1)$ ,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\ell_+ = [1 \ i]$ ,*

$$\ell_+ R(\theta) = e^{i\theta} \ell_+, \quad R(\theta) e_1 = (\cos \theta, \ \sin \theta)^\top, \quad J \text{diag}(1, -1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\text{diag}(1, -1).$$

Moreover,

$$\begin{aligned} R(\theta) \text{diag}(1, -1) R(\theta) &= \cos(2\theta) \text{diag}(1, -1) + \sin(2\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ R(\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(\theta) &= -\sin(2\theta) \text{diag}(1, -1) + \cos(2\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

*Proof.* Direct computation. □

We now fix the base gauge normalization: take the canonical base  $H_0 \equiv \frac{1}{2}I$ , so  $\Phi_0(s, t; z) = R\left(-\frac{z}{2}(s-t)\right)$  and

$$E_{0,L}(z) = A(z) - iB(z) = e^{-izL/2}.$$

Let  $x \in \mathbb{R}$ ,  $\eta_0 \geq 0$ . Define the gauge-normalized seam rows by

$$\mathfrak{s}_k^+(E) := \text{Re}\left(e^{+ix_k L/2} E(x_k + i\eta_0)\right) e^{-\eta_0 L/2}, \quad \tilde{\mathfrak{s}}_k^+(E) := \text{Im}\left(e^{+ix_k L/2} E(x_k + i\eta_0)\right) e^{-\eta_0 L/2}, \quad (4)$$

and

$$\mathfrak{s}_k^-(E) := \text{Re}\left(e^{+ix_k L/2} E(x_k - i\eta_0)\right) e^{+\eta_0 L/2}, \quad \tilde{\mathfrak{s}}_k^-(E) := \text{Im}\left(e^{+ix_k L/2} E(x_k - i\eta_0)\right) e^{+\eta_0 L/2}. \quad (5)$$

Throughout, ‘gauge-normalized top seam’ for Weyl-side functions  $(m, v)$  means the scalar factor  $e^{-ixL/2} e^{+\eta_0 L/2}$ ; for endpoint-side functions  $(E, A)$  we use  $e^{+ixL/2} e^{\pm\eta_0 L/2}$

as in (4)–(5). This makes the base endpoint rows have real part 1 and imaginary part 0. All rowwise estimates are norm-based and unaffected by these fixed constants.

Here  $x_k = \frac{2\pi k}{L}$ . At the base,

$$e^{+ix_k L/2} E_{0,L}(x_k \pm i\eta_0) = e^{\pm\eta_0 L/2},$$

and the seam factors yield  $\mathfrak{s}_k^\pm(E_{0,L}) = 1$  and  $\tilde{\mathfrak{s}}_k^\pm(E_{0,L}) = 0$ .

For  $A_L$  we use the same convention and define the gauge-normalized seam rows by

$$\mathfrak{a}_k^+(A_L) := \operatorname{Re}(e^{-ix_k L/2} A_L(x_k + i\eta_0)) e^{+\eta_0 L/2}, \quad \tilde{\mathfrak{a}}_k^+(A_L) := \operatorname{Im}(e^{-ix_k L/2} A_L(x_k + i\eta_0)) e^{+\eta_0 L/2}, \quad (6)$$

and

$$\mathfrak{a}_k^-(A_L) := \operatorname{Re}(e^{-ix_k L/2} A_L(x_k - i\eta_0)) e^{-\eta_0 L/2}, \quad \tilde{\mathfrak{a}}_k^-(A_L) := \operatorname{Im}(e^{-ix_k L/2} A_L(x_k - i\eta_0)) e^{-\eta_0 L/2}. \quad (7)$$

On the bulk set  $\mathcal{K}_\delta$  we normalize both by

$$\mathcal{A}_k^{\text{top/bot}} := \frac{\mathfrak{a}_k^\pm(A_L)}{E_{\text{amp}}(\alpha, u_k)}, \quad \tilde{\mathcal{A}}_k^{\text{top/bot}} := \frac{\tilde{\mathfrak{a}}_k^\pm(A_L)}{E_{\text{amp}}(\alpha, u_k)}.$$

### 3. BLOCK DISCRETIZATION, PARAMETER BOX, GRID, AND NORMALIZERS

Partition  $[0, L]$  into  $N = \lceil L^2 \rceil$  equal blocks of length  $\ell = L/N$ . On block  $j \in \{0, \dots, N-1\}$  set

$$H_j := \frac{1}{2}I + s_j \operatorname{diag}(1, -1) + t_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (8)$$

with the admissible parameter box

$$\mathcal{P} := \left\{ (s_j, t_j) : \sqrt{s_j^2 + t_j^2} \leq \frac{1}{2} - \varepsilon \right\}, \quad 0 < \varepsilon < \frac{1}{2},$$

so that each  $H_j$  is symmetric PSD with  $\operatorname{tr} H_j \equiv 1$ . Let  $H(s) \equiv H_j$  on  $s \in [j\ell, (j+1)\ell)$ . Denote  $\theta = (s_0, t_0, \dots, s_{N-1}, t_{N-1})$ .

**Lemma 3.1.** *Each  $H_j$  in (8) is symmetric, PSD, and  $\operatorname{tr} H_j \equiv 1$ . Moreover,*

$$\frac{\partial H_j}{\partial s_j} = \operatorname{diag}(1, -1), \quad \frac{\partial H_j}{\partial t_j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \left\| \frac{\partial H_j}{\partial s_j} \right\| = \|\operatorname{diag}(1, -1)\| = 1, \quad \left\| \frac{\partial H_j}{\partial t_j} \right\| = \left\| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| = 1.$$

*Proof.* Immediate from (8) and the definitions of  $\operatorname{diag}(1, -1)$ ,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . PSD and unit trace follow from  $\sqrt{s_j^2 + t_j^2} \leq \frac{1}{2} - \varepsilon$ .  $\square$

Next we specify the seam grid and the associated normalization. Set:

$$u_k := \frac{\pi k}{N}, \quad x_k := \frac{2u_k}{\ell} = \frac{2\pi k}{L}, \quad z_k^\pm := x_k \pm i\eta_0, \quad k = 1, \dots, N-1. \quad (9)$$

We then trim to the bulk and introduce the intrinsic amplitude normalizer. Fix any  $\delta \in (0, \frac{1}{4})$  independent of  $L$ , and keep the bulk index set

$$\mathcal{K}_\delta := \{ k \in \{1, \dots, N-1\} : \lceil \delta N \rceil \leq k \leq \lfloor (1-\delta)N \rfloor \},$$

so that  $u_k = \frac{\pi k}{N} \in [\delta\pi, \pi - \delta\pi]$  on  $\mathcal{K}_\delta$ . Let  $\alpha = \eta_0\ell/2$ . Define the *intrinsic amplitude normalizer* at height  $\eta_0$  by

$$\begin{aligned} E_{\text{amp}}(\alpha, u) &= \left\| \begin{pmatrix} \text{Re} \sin(2(u - i\alpha)) \\ \text{Re}(1 - \cos(2(u - i\alpha))) \\ \text{Im} \sin(2(u - i\alpha)) \\ \text{Im}(1 - \cos(2(u - i\alpha))) \end{pmatrix} \right\|_2 \\ &= \left\| \begin{pmatrix} \sin 2u \cosh 2\alpha \\ 1 - \cos 2u \cosh 2\alpha \\ -\cos 2u \sinh 2\alpha \\ -\sin 2u \sinh 2\alpha \end{pmatrix} \right\|_2. \end{aligned} \tag{10}$$

For each  $k \in \mathcal{K}_\delta$ , normalize *both* real/imag rows at  $u_k$  by dividing by  $E_{\text{amp}}(\alpha, u_k)$ :

$$\begin{aligned} \mathcal{S}_k^{\text{bottom}}(E) &:= \frac{\mathfrak{s}_k^-(E)}{E_{\text{amp}}(\alpha, u_k)}, & \widetilde{\mathcal{S}}_k^{\text{bottom}}(E) &:= \frac{\widetilde{\mathfrak{s}}_k^-(E)}{E_{\text{amp}}(\alpha, u_k)}, \\ \mathcal{S}_k^{\text{top}}(E) &:= \frac{\mathfrak{s}_k^+(E)}{E_{\text{amp}}(\alpha, u_k)}, & \widetilde{\mathcal{S}}_k^{\text{top}}(E) &:= \frac{\widetilde{\mathfrak{s}}_k^+(E)}{E_{\text{amp}}(\alpha, u_k)}. \end{aligned}$$

On the bulk set  $\mathcal{K}_\delta$  (with  $\alpha = \eta_0\ell/2$  fixed), there exist constants  $0 < c_1(\delta, \eta_0) \leq C_1(\eta_0) < \infty$  such that  $c_1(\delta, \eta_0) \leq E_{\text{amp}}(\alpha, u_k) \leq C_1(\eta_0)$  for all  $k \in \mathcal{K}_\delta$ .

**Lemma 3.2** (Intrinsic amplitude: exact identity and bulk bounds). *For  $\alpha = \eta_0\ell/2$  and real  $u$ ,*

$$E_{\text{amp}}(\alpha, u)^2 = 2 \cosh(2\alpha) (\cosh(2\alpha) - \cos(2u)).$$

Consequently, for  $u \in [\delta\pi, \pi - \delta\pi]$ ,

$$\sqrt{2} \sin(\delta\pi) \leq E_{\text{amp}}(\alpha, u) \leq \sqrt{2 \cosh(2\alpha) (\cosh(2\alpha) + 1)}.$$

*Proof.* Compute

$$\sin(2(u - i\alpha)) = \sin 2u \cosh 2\alpha - i \cos 2u \sinh 2\alpha, \tag{11}$$

$$1 - \cos(2(u - i\alpha)) = 1 - \cos 2u \cosh 2\alpha - i \sin 2u \sinh 2\alpha. \tag{12}$$

adding the squares of the four real components gives

$$\begin{aligned} |\sin(2(u - i\alpha))|^2 + |1 - \cos(2(u - i\alpha))|^2 \\ = \sin^2(2u) \cosh^2(2\alpha) + \cos^2(2u) \sinh^2(2\alpha) \end{aligned}$$

$$\begin{aligned} & + (1 - \cos(2u) \cosh(2\alpha))^2 + \sin^2(2u) \sinh^2(2\alpha) \\ & = 2 \cosh(2\alpha) (\cosh(2\alpha) - \cos(2u)) =: E_{\text{amp}}(\alpha, u)^2. \end{aligned}$$

On the bulk,  $\cos 2u \leq 1 - 2 \sin^2(\delta\pi)$  gives the lower bound;  $\cos 2u \geq -1$  gives the upper bound.  $\square$

#### 4. EXACT KERNEL INTEGRAL AT $\eta = 0$ AND TWO-FREQUENCY STRUCTURE

Let  $\Phi_0(s, t; x) = R(\omega(s-t))$  be the base flow with  $\omega = x/2$ . For a parameter  $\vartheta \in \{s_j, t_j\}$  supported on block  $j$ , Duhamel gives

$$\partial_\vartheta \Phi(L, 0; x) \Big|_{H_0} = \Phi_0(L, (j+1)\ell; x) K_\vartheta(x) \Phi_0(j\ell, 0; x), \quad (13)$$

where

$$K_\vartheta(x) := x \int_0^\ell R(\omega s) J \partial_\vartheta H_j R(\omega s) ds, \quad \partial_\vartheta H_j = a_\vartheta \text{diag}(1, -1) + b_\vartheta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (14)$$

**Lemma 4.1** (Kernel integral). *Let  $u_{\text{cell}} := \omega\ell = x\ell/2$ . Then*

$$K_\vartheta(x) = \sin(2u_{\text{cell}}) A_\vartheta + (1 - \cos(2u_{\text{cell}})) B_\vartheta,$$

with

$$A_\vartheta := a_\vartheta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - b_\vartheta \text{diag}(1, -1), \quad B_\vartheta := -a_\vartheta \text{diag}(1, -1) - b_\vartheta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

*Proof.* By Lemma 2.1,

$$\begin{aligned} R(\omega s) J \partial_\vartheta H_j R(\omega s) &= J \left[ a_\vartheta (\cos 2\omega s \text{diag}(1, -1) + \sin 2\omega s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \right. \\ &\quad \left. + b_\vartheta (-\sin 2\omega s \text{diag}(1, -1) + \cos 2\omega s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \right] \\ &= a_\vartheta (\cos 2\omega s J \text{diag}(1, -1) + \sin 2\omega s J \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \\ &\quad + b_\vartheta (-\sin 2\omega s J \text{diag}(1, -1) + \cos 2\omega s J \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \\ &= a_\vartheta (\cos 2\omega s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \sin 2\omega s \text{diag}(1, -1)) \\ &\quad + b_\vartheta (-\sin 2\omega s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \cos 2\omega s \text{diag}(1, -1)). \end{aligned}$$

and the stated integrals yield the result.  $\square$

Set  $\phi := \omega j\ell = ju_{\text{cell}}$ .

**Lemma 4.2** (Two odd frequencies with explicit mixing). *Let  $\vartheta \in \{s_j, t_j\}$  and set  $u_{\text{cell}} = x\ell/2$ ,  $\phi = ju_{\text{cell}}$ . At the base  $H_0 \equiv \frac{1}{2}I$ ,*

$$e^{+ixL/2} \partial_\vartheta E_L(x) \Big|_{H_0} = F_0(u_{\text{cell}}) \left( c_\vartheta^- \sin(\phi - u_{\text{cell}}) + c_\vartheta^+ \sin(\phi + u_{\text{cell}}) \right), \quad F_0(u_{\text{cell}}) = 2|\sin u_{\text{cell}}|,$$

with real constants

$$(\mathbf{c}_{s_j}^-, \mathbf{c}_{s_j}^+) = (+1, 0), \quad (\mathbf{c}_{t_j}^-, \mathbf{c}_{t_j}^+) = (0, 1).$$

*Proof.* From (13) and Lemma 4.1, with  $\partial_{s_j} H_j = \text{diag}(1, -1)$  and  $\partial_{t_j} H_j = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ , we have  $K_{s_j}(x) = \sin(2u_{\text{cell}})(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) - (1 - \cos 2u_{\text{cell}})\text{diag}(1, -1)$  and  $K_{t_j}(x) = -\sin(2u_{\text{cell}})\text{diag}(1, -1) - (1 - \cos 2u_{\text{cell}})(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ . Multiplying  $\ell_+ K_\vartheta(x) R(\phi) e_1$  by  $e^{i(\phi+u)}$  produces only  $\sin(\phi \pm u)$  with the stated coefficients; the common factor  $\sqrt{\sin^2(2u) + (1 - \cos 2u_{\text{cell}})^2} = 2|\sin u|$  gives  $F_0(u)$ .  $\square$

The coefficient vectors are  $(\mathbf{c}_{s_j}^-, \mathbf{c}_{s_j}^+) = (+1, 0)$  and  $(\mathbf{c}_{t_j}^-, \mathbf{c}_{t_j}^+) = (0, 1)$ , hence the two packets are selected without mixing at the base.

**Lemma 4.3** (Two odd frequencies for  $A_L$ ). *With  $\vartheta \in \{s_j, t_j\}$ ,  $u_{\text{cell}} = x\ell/2$ ,  $\phi = ju_{\text{cell}}$ , and  $F_0(u_{\text{cell}}) = 2|\sin u_{\text{cell}}|$ ,*

$$e^{\frac{ixL}{2}} \partial_\vartheta A_L(x) \Big|_{H_0} = \frac{1}{2} e^{\frac{ixL}{2}} \left( \partial_\vartheta E_L(x) + \overline{\partial_\vartheta E_L(x)} \right) \Big|_{H_0} \quad (15)$$

$$= F_0(u_{\text{cell}}) \begin{pmatrix} \mathbf{a}_\vartheta^- \sin(\phi - u_{\text{cell}}) \\ + \mathbf{a}_\vartheta^+ \sin(\phi + u_{\text{cell}}) \end{pmatrix}. \quad (16)$$

with

$$(\mathbf{a}_{s_j}^-, \mathbf{a}_{s_j}^+) = (+1, 0), \quad (\mathbf{a}_{t_j}^-, \mathbf{a}_{t_j}^+) = (0, 1).$$

*Proof.* Apply Lemma 4.2 and take the real part; the coefficients are unchanged.  $\square$

The coefficient vectors are  $(\mathbf{a}_{s_j}^-, \mathbf{a}_{s_j}^+) = (+1, 0)$  and  $(\mathbf{a}_{t_j}^-, \mathbf{a}_{t_j}^+) = (0, 1)$ .

**Lemma 4.4** (Two odd frequencies for  $v$  at the base). *With  $\vartheta \in \{s_j, t_j\}$ ,  $u_{\text{cell}} = x\ell/2$ ,  $\phi = ju_{\text{cell}}$ , and  $\alpha = \eta_0 \ell/2$ , for  $z = x + i\eta_0$  one has*

$$e^{-ixL/2} e^{+\eta_0 L/2} \partial_\vartheta v(x + i\eta_0) \Big|_{H_0} = \kappa(\eta_0, u_{\text{cell}}) \left( \mathbf{c}_\vartheta^- \sin(\phi - u_{\text{cell}}) + \mathbf{c}_\vartheta^+ \sin(\phi + u_{\text{cell}}) \right),$$

with  $(\mathbf{c}_{s_j}^-, \mathbf{c}_{s_j}^+) = (1, 0)$ ,  $(\mathbf{c}_{t_j}^-, \mathbf{c}_{t_j}^+) = (0, 1)$ . After bulk trimming and dividing both rows by  $E_{\text{amp}}(\alpha, u)$ , the prefactor  $\kappa(\eta_0, u)$  is bounded above and below uniformly on the bulk. Consequently  $DG_K(\theta_{\text{base}})$  has singular values  $\Theta(\sqrt{N})$  uniformly in  $L$ .

*Proof.* Recall  $m = (iA^\Phi - C^\Phi)(D^\Phi - iB^\Phi)^{-1}$  (Lemma 1.4). At the base  $H_0 \equiv \frac{1}{2}I$  and for  $z = x + i\eta_0$ ,

$$A^\Phi = D^\Phi = \cos \frac{zL}{2}, \quad B^\Phi = \sin \frac{zL}{2}, \quad C^\Phi = -\sin \frac{zL}{2},$$

hence  $m_{\text{base}}(z) = i$ , and

$$\begin{aligned} D^\Phi - iB^\Phi &= e^{-izL/2} &= e^{-ixL/2} e^{+\eta_0 L/2}, \\ (D^\Phi - iB^\Phi)^{-1} &= e^{+ixL/2} e^{-\eta_0 L/2}, \\ iA^\Phi - C^\Phi &= i e^{-izL/2}. \end{aligned}$$

Differentiate:

$$\partial_\vartheta m = (i\partial_\vartheta A^\Phi - \partial_\vartheta C^\Phi)(D^\Phi - iB^\Phi)^{-1} - (iA^\Phi - C^\Phi)(D^\Phi - iB^\Phi)^{-2}(\partial_\vartheta D^\Phi - i\partial_\vartheta B^\Phi).$$

Using  $iA^\Phi - C^\Phi = i(D^\Phi - iB^\Phi)$  at the base, this simplifies to

$$\partial_\vartheta m = (i\partial_\vartheta A^\Phi - \partial_\vartheta C^\Phi - i\partial_\vartheta D^\Phi - \partial_\vartheta B^\Phi)(D^\Phi - iB^\Phi)^{-1}.$$

By Lemma 4.3 (applied blockwise at the base), each  $\partial_\vartheta A^\Phi, \partial_\vartheta B^\Phi, \partial_\vartheta C^\Phi, \partial_\vartheta D^\Phi$  is a real linear combination of the two odd packets  $\sin(\phi \pm u_{\text{cell}})$  with  $u_{\text{cell}} = x\ell/2$ ,  $\phi = ju_{\text{cell}}$ , and with selector coefficients  $(1, 0)$  or  $(0, 1)$  depending on  $\vartheta \in \{s_j, t_j\}$ . Therefore

$$e^{-ixL/2} e^{+\eta_0 L/2} \partial_\vartheta m(x + i\eta_0) \Big|_{H_0} = \kappa_1(\eta_0, u_{\text{cell}}) \left( c_\vartheta^- \sin(\phi - u_{\text{cell}}) + c_\vartheta^+ \sin(\phi + u_{\text{cell}}) \right),$$

where the factor  $e^{-ixL/2} e^{-\eta_0 L/2}$  would be needed to cancel  $(D^\Phi - iB^\Phi)^{-1}$ . Instead, we multiply by  $e^{-ixL/2} e^{+\eta_0 L/2}$  to define the gauge-normalized derivative. Now  $v = f(m) = (m - i)/(m + i)$  so  $f'(m) = 2i/(m + i)^2$  and  $f'(i) = -i/2$ . Thus, applying the same gauge-normalization factor,

$$\begin{aligned} e^{-ixL/2} e^{+\eta_0 L/2} \partial_\vartheta v(x + i\eta_0) \Big|_{H_0} &= f'(i) (e^{-ixL/2} e^{+\eta_0 L/2} \partial_\vartheta m) \Big|_{H_0} \\ &= \kappa(\eta_0, u_{\text{cell}}) \left( c_\vartheta^- \sin(\phi - u_{\text{cell}}) + c_\vartheta^+ \sin(\phi + u_{\text{cell}}) \right). \end{aligned}$$

Finally, after intrinsic amplitude normalization by  $E_{\text{amp}}(\alpha, u_{\text{cell}})$  on the bulk (Lemma 3.2),  $\kappa(\eta_0, u_{\text{cell}})$  is bounded above/below uniformly in  $u_{\text{cell}} \in [\delta\pi, \pi - \delta\pi]$ . Hence the DST Gram comparison (Lemma 6.1) gives  $\Theta(\sqrt{N})$  singular values for  $DG_K(\theta_{\text{base}})$ .  $\square$

*Remark 4.5* (Seam gauge-normalization convention). On the top seam  $z = x + i\eta_0$ , the phase factor is  $e^{-ixL/2} e^{+\eta_0 L/2}$  for  $m$  and it persists for  $v = f(m)$ ; for scalars we use  $\tilde{A}(z) := e^{-\frac{iz}{2}L} A(z)$ .

## 5. HEIGHT MIXING AT FIXED $\eta_0 > 0$ : EQUIVALENCE AFTER NORMALIZATION

Throughout §5–§8 we keep  $N \asymp L^2$  and  $\ell = L/N$ . On the racetrack  $\partial\Omega_{\eta_0}(L)$  we have  $|z| \lesssim L$  and thus  $|z|\ell \leq C(\eta_0)$ ; write  $\omega = |x|/2$  and gap variable  $\omega d\ell$ .

Let  $\alpha = \eta_0\ell/2$ . Replacing  $x$  by  $x + i\eta_0$  multiplies  $\sin(2u_{\text{cell}})$  and  $1 - \cos(2u_{\text{cell}})$  by height-dependent combinations of  $\cosh 2\alpha$  and  $\sinh 2\alpha$ .

**Lemma 5.1** (Quantitative height equivalence on the bulk). *Fix  $\eta_0 > 0$  and  $\delta \in (0, \frac{1}{4})$ . Let  $\alpha = \eta_0\ell/2$  and  $u_k = \pi k/N$ , and keep  $k \in \mathcal{K}_\delta$ . After the gauge normalization (4)–(5) and intrinsic amplitude normalization (10), for each  $k \in \mathcal{K}_\delta$  there exists a real  $2 \times 2$  matrix  $T_k = T_k(\alpha, u_k, s_j, t_j)$  such that the height- $\eta_0$  normalized two-row block equals  $T_k$  applied to the  $\eta = 0$  normalized two-row block. Moreover, uniformly over the PSD parameter box  $\sqrt{s_j^2 + t_j^2} \leq \frac{1}{2} - \varepsilon$ ,*

$$\|T_k - I_2\| \leq C(\eta_0, \delta, \varepsilon) \alpha, \quad c_*(\eta_0, \delta, \varepsilon) \leq \sigma_{\min}(T_k) \leq \sigma_{\max}(T_k) \leq C_*(\eta_0, \delta, \varepsilon).$$

*Proof.* Work on a single block at  $z = x + i\eta_0$  with  $u_{\text{cell}} = x\ell/2$  and  $\alpha = \eta_0\ell/2$ . Using

$$\sin(2(u_{\text{cell}} - i\alpha)) = \sin 2u \cosh 2\alpha - i \cos 2u_{\text{cell}} \sinh 2\alpha,$$

$$1 - \cos(2(u_{\text{cell}} - i\alpha)) = 1 - \cos 2u_{\text{cell}} \cosh 2\alpha - i \sin 2u_{\text{cell}} \sinh 2\alpha,$$

the two odd-frequency coefficients in Lemma 4.3 pick up a real  $2 \times 2$  mixing which is analytic in  $\alpha$  and smooth in  $(u, s_j, t_j)$ . Splitting into Re / Im and dividing both rows by

$$E_{\text{amp}}(\alpha, u_{\text{cell}}) = \left( 2 \cosh(2\alpha) (\cosh(2\alpha) - \cos 2u_{\text{cell}}) \right)^{1/2}$$

(cf. (10)) cancels the dominant height factor. On the bulk, where the cell phase takes discrete values  $u_k = \pi k/N$ , we have  $u_k \in [\delta\pi, \pi - \delta\pi]$ ,  $E_{\text{amp}}$  is uniformly bounded above/below (Lemma 3.2). Since  $|z|\ell = O(1)$  on  $\partial\Omega_{\eta_0}(L)$  and  $\partial H$  is uniformly bounded on the PSD box, a first-order Duhamel estimate shows the resulting block is  $T_k = I + R_k$  with  $\|R_k\| \leq C(\eta_0, \delta, \varepsilon) \alpha$ . The stated singular-value bounds follow.  $\square$

**Proposition 5.2** (Height-equivalence on the bulk with intrinsic normalization). *With bulk trimming  $k \in \mathcal{K}_\delta$  and the intrinsic normalizer (10), each height- $\eta_0$  Jacobian column equals the corresponding  $\eta = 0$  column composed with a block-diagonal transform  $T = \text{diag}(T_k)$  where  $T_k \in \mathbb{R}^{2 \times 2}$  satisfies*

$$\|T_k - I_2\| \leq C(\eta_0, \delta, \varepsilon) \alpha, \quad 1 - C\alpha \leq \sigma_{\min}(T_k) \leq \sigma_{\max}(T_k) \leq 1 + C\alpha.$$

*Proof.* As in Lemma 5.1, applied independently at each bulk node  $k$ .  $\square$

**Lemma 5.3** (Explicit mixer entries). *Let  $u_{\text{cell}} = x\ell/2$ ,  $\alpha = \eta_0\ell/2$ , and set*

$$S := \sin 2u, \quad C := \cos 2u_{\text{cell}}, \quad C_h := \cosh 2\alpha, \quad S_h := \sinh 2\alpha,$$

$$X_1 := S C_h, \quad Y_1 := -C S_h, \quad X_2 := (1 - C) C_h, \quad Y_2 := -S S_h.$$

*In the gauge-normalized, intrinsically normalized basis, the height- $\eta_0$  two-row block equals  $T_k(\alpha, u)$  applied to the  $\eta = 0$  block with*

$$T_k(\alpha, u_{\text{cell}}) = I_2 + \alpha R(\alpha, u_{\text{cell}}), \quad R(\alpha, u) = \frac{1}{\sqrt{2C_h(C_h - C)}} \begin{pmatrix} X_1 - S & X_2 - (1 - C) \\ Y_1 & Y_2 \end{pmatrix},$$

and  $\|R(\alpha, u_{cell})\| \leq C(\eta_0, \delta, \varepsilon)$  uniformly for cell phases  $u_{cell} \in [\delta\pi, \pi - \delta\pi]$ . In particular,  $1 - C\alpha \leq \sigma_{\min}(T_k) \leq \sigma_{\max}(T_k) \leq 1 + C\alpha$ .

**Lemma 5.4** (Uniform non-degeneracy of the height mixing). *Fix  $\delta \in (0, \frac{1}{4})$ ,  $\eta_0 > 0$ , and the parameter box  $\sqrt{s_j^2 + t_j^2} \leq \frac{1}{2} - \varepsilon$ . For bulk indices  $k \in \mathcal{K}_\delta$ , the block map*

$$(\alpha, u_{cell}, s_j, t_j) \mapsto T_k(\alpha, u_{cell}, s_j, t_j) \in \mathbb{R}^{2 \times 2}$$

is continuous on the compact set

$$\left\{ 0 \leq \alpha \leq \eta_0 \ell/2, \quad u_{cell} \in [\delta\pi, \pi - \delta\pi], \quad (s_j, t_j) \in \mathcal{P} \right\},$$

and satisfies  $T_k(0, u_{cell}, s_j, t_j) = I_2$ . Hence there exist constants  $0 < c_* \leq C_* < \infty$  (depending only on  $\delta, \eta_0, \varepsilon$ ) such that

$$c_* \leq \sigma_{\min}(T_k) \leq \sigma_{\max}(T_k) \leq C_* \quad \text{for all } k \in \mathcal{K}_\delta,$$

uniformly in  $N, L$  and the parameter box.

Using Lemma 3.2,  $T_k(\alpha, u) = I_2 + \alpha R(\alpha, u)$  with  $R$  bounded uniformly on  $u \in [\delta\pi, \pi - \delta\pi]$  and  $(s_j, t_j)$  in the PSD box; in particular  $T_k \rightarrow I_2$  as  $\alpha \rightarrow 0$  uniformly on the compact parameter set.

*Proof.* As in Proposition 5.2, with continuity in  $(\alpha, u_{cell}, s_j, t_j)$ ; at  $\alpha = 0$ ,  $T_k = I_2$ . Compactness yields the uniform bounds.  $\square$

## 6. WEIGHTED-DST RIESZ BOUNDS AND SEAM-CORE CONDITIONING

We work on the bulk index set  $\mathcal{K}_\delta$  from §3. For  $k \in \mathcal{K}_\delta$  set  $u_k = \pi k/N$ . We consider the sine packets  $\sin((j \pm 1)u_k)$  indexed by  $j \in \{2, \dots, N-2\}$ .

**Lemma 6.1** (Bulk Gram lower bound). *Let  $\mathcal{S}$  be the DST-I synthesis matrix with entries  $\mathcal{S}_{k,m} = \sin(mu_k)$  for  $m \in \{1, \dots, N-1\}$  and  $u_k = \pi k/N$ . Then  $\mathcal{S}^\top \mathcal{S} = \frac{N}{2} I$ . Let  $P$  be the diagonal projector onto the bulk rows  $k \in \mathcal{K}_\delta$ . There exists  $C(\delta) \in (0, 1)$  with  $C(\delta) \rightarrow 0$  as  $\delta \downarrow 0$  such that*

$$\frac{N}{2} (1 - C(\delta)) I \preceq \mathcal{S}^\top P \mathcal{S} \preceq \frac{N}{2} I.$$

(The projector  $P$  effectively drops the first and last  $\lceil \delta N \rceil - 1$  rows, where the DST basis functions are small.)

*Proof.* Write  $\widehat{\mathcal{S}} = \mathcal{S}/\sqrt{N/2}$ , so  $\widehat{\mathcal{S}}^\top \widehat{\mathcal{S}} = I$  and  $\widehat{\mathcal{S}}^\top P \widehat{\mathcal{S}} = I - \widehat{\mathcal{S}}^\top (I - P) \widehat{\mathcal{S}}$ . For any  $m$ ,

$$\sum_{k \notin \mathcal{K}_\delta} \sin^2(mu_k) \leq \sum_{u_k \leq \delta\pi} (mu_k)^2 + \sum_{u_k \geq \pi - \delta\pi} (\pi - mu_k)^2 \leq C(\delta) N,$$

uniformly in  $m$ , by comparing the Riemann sums with  $\int_0^{\delta\pi} t^2 dt$  and its symmetric counterpart. Hence  $\|\widehat{\mathcal{S}}^\top(I - P)\widehat{\mathcal{S}}\| \leq C(\delta)$ , which yields the stated bounds after rescaling by  $N/2$ .  $\square$

**Theorem 6.2** (Bulk DST conditioning with intrinsic normalizer). *On the bulk index set  $\mathcal{K}_\delta$  and after dividing each two-row block by  $E_{\text{amp}}(\alpha, u_k)$ , there exist  $0 < c(\eta_0, \delta) \leq C(\eta_0, \delta) < \infty$  such that*

$$c(\eta_0, \delta) \sqrt{N} \leq \sigma_{\min}(J_{\text{seam}}^{\eta_0}) \leq \sigma_{\max}(J_{\text{seam}}^{\eta_0}) \leq C(\eta_0, \delta) \sqrt{N}.$$

*Proof.* Write  $S$  for the (trimmed) DST-I synthesis on the two selected seams at height 0, and let  $P$  denote the row-selector that keeps the first  $K$  DST-I modes on each selected stream. At  $\eta = 0$  the (untrimmed) DST-I Gram is  $\frac{N}{2}I$ ; by the bulk deletion estimate (Lemma 6.1), trimming to the first  $K \leq c_\delta N$  preserves a uniform fraction of this Gram:

$$S^\top P^\top P S \succeq c_{\text{bulk}} N I_{2K} \quad (c_{\text{bulk}} = c_{\text{bulk}}(\delta) > 0).$$

At height  $\eta_0$  the “real/imag split” and gauge normalization are orthogonal on mode space, and the intrinsic normalizer is a bounded isomorphism with constants depending only on  $(\eta_0, \delta)$  (Appendix D, Lemma D.7). Thus the base Jacobian on the two chosen streams is of the form

$$J_0 = R D P S,$$

where  $R$  is orthogonal (row mixing of real/imag parts) and  $D$  is block-diagonal satisfying  $cI \preceq D \preceq CI$  with  $c, C > 0$  depending only on  $(\eta_0, \delta)$ . Hence

$$J_0^\top J_0 = S^\top P^\top D^\top R^\top R D P S \succeq c^2 S^\top P^\top P S \succeq (c^2 c_{\text{bulk}}) N I_{2K}.$$

This gives  $\sigma_{\min}(J_0) \geq \sqrt{c^2 c_{\text{bulk}}} \sqrt{N}$  and, since the same bounded isomorphisms control the rows, also  $\sigma_{\max}(J_0) \lesssim \sqrt{N}$  with constants depending only on  $(\eta_0, \delta)$ . The claimed  $\Theta(\sqrt{N})$  conditioning follows.  $\square$

**Lemma 6.3** (Bulk Riesz lower bound for the base linearized map). *Fix  $\eta_0 > 0$  and  $\delta \in (0, \frac{1}{4})$ . On the bulk set  $\mathcal{K}_\delta$  with intrinsic normalization (10), let*

$$\mathcal{L}_K : \mathbb{R}^{2K} \rightarrow \mathbb{R}^{2K}$$

*map the truncated DST coefficients*

$$\theta^{(K)} = (\widehat{s}_1, \dots, \widehat{s}_K, \widehat{t}_1, \dots, \widehat{t}_K)$$

*to the first  $K$  DST-I coefficients (orders  $1, \dots, K$ ) of the normalized, gauge-normalized top-seam streams (real and imaginary parts) of  $\partial_\vartheta A_L$  at the base  $H_0 \equiv \frac{1}{2}I$ . Then there exists  $c_0 = c_0(\delta, \eta_0, \varepsilon) > 0$  such that*

$$\|\mathcal{L}_K v\|_2 \geq c_0 \|v\|_2 \quad \forall v \in \mathbb{R}^{2K},$$

uniformly for all  $K \leq c_\delta N$  and all  $L, N$  with  $N \asymp L^2$ .

*Proof.* Apply Lemma 4.3 with  $\vartheta \in \{s_j, t_j\}$ . Each two-row block is an invertible mixing of  $\sin(\phi \pm u)$  with unit determinant. Projecting to the first  $K$  DST-I modes gives a Gram comparable to the bulk DST Gram (Lemma 6.1), hence the uniform lower bound.  $\square$

**Proposition 6.4** (Seam core selection). *Work on the bulk set  $\mathcal{K}_\delta$ . Build a square selection by keeping the first  $K$  DST-I modes from two seam streams (e.g., top seam real and top seam imaginary) of the normalized  $A_L$ -data; this gives exactly  $2K = p_K$  rows. Then, at the base and throughout the ball of Lemma 9.12,*

$$c\sqrt{N} \leq \sigma_{\min}(DF_K(\theta)) \leq \sigma_{\max}(DF_K(\theta)) \leq C\sqrt{N},$$

with constants depending only on  $(\delta, \eta_0)$  (and not on  $L, N$ ).

*Proof.* At the base, write  $J_{\text{seam}} = RDPS$  as in the proof of Theorem 6.2, restricted to the two chosen streams. Then

$$J_{\text{seam}}^\top J_{\text{seam}} = S^\top P^\top D^\top D P S \succeq c^2 S^\top P^\top P S \succeq (c^2 c_{\text{bulk}}) N I,$$

with  $c > 0$  from the normalizer bounds and  $c_{\text{bulk}} > 0$  from bulk deletion. Thus  $\sigma_{\min}(DF_K(\theta_{\text{base}})) \gtrsim \sqrt{N}$ ; the corresponding upper bound is immediate from  $D$  bounded and  $\|PS\|^2 \lesssim N$ .

Finally, by Theorem 8.8,  $\|DF_K(\theta) - DF_K(\theta_{\text{base}})\| \leq CL^2\|\theta - \theta_{\text{base}}\|$ , so in the small ball of Lemma 9.12 the same  $\Theta(\sqrt{N})$  bounds persist by Weyl's inequality.  $\square$

## 7. RACETRACK GEOMETRY STRICTLY ABOVE $\mathbb{R}$ AND FIXED-HEIGHT SCHUR MARGIN

Fix  $\eta_0 > 0$ . For each  $L \geq 1$ , define

$$X(L) = \pi L,$$

((Inequalities of the form  $|z| \lesssim L$  hide an absolute multiple of  $\pi$ .)

and let  $\Gamma_{\eta_0}(L)$  be a  $C^{1,1}$  racetrack: the two horizontal segments  $\{x \in [-X(L), X(L)], y = \eta_0 + \varepsilon_0\}$  and  $\{x \in [-X(L), X(L)], y = 2\eta_0\}$  joined by two rounded arcs contained in  $\{(x, y) : x = \pm X(L), y \in [\eta_0, 2\eta_0]\}$ . Its interior is  $\Omega_{\eta_0}(L)$ . The boundary has uniform interior rolling disks of radius  $\varrho_{\text{roll}}(\eta_0) > 0$  independent of  $L$ . (Uniformity in  $L$  is preserved under  $C^{1,1}$  corner roundings; harmonic measure and boundary constants change only by universal factors.)

Throughout this section we use that  $E_L$  is Hermite-Biehler, so

$$v(z) := \frac{E_L^*(z)}{E_L(z)}$$

is Schur with  $|v(z)| < 1$  on  $\mathbb{H}$  (Lemma 1.2).

**Lemma 7.1** (Mesh-to-boundary via tangential control). *Let  $R$  be continuous on  $\partial\Omega_{\eta_0}(L)$  and differentiable along each boundary component  $\gamma : [0, S] \rightarrow \partial\Omega_{\eta_0}(L)$  parametrized by arclength. Let  $\{t_r\}$  be a mesh with spacing  $h > 0$  (i.e.  $|t - t_r| \leq h$  for every  $t \in [0, S]$ ). If*

$$|R(\gamma(t_r))| \leq \delta \quad \text{and} \quad \left| \frac{R(\gamma(t_{r+1})) - R(\gamma(t_r))}{h} \right| \leq M$$

for all valid  $r$ , then  $\sup_{\partial\Omega_{\eta_0}(L)} |R| \leq \delta + Mh$ .

*Proof.* For any  $t$  choose the nearest mesh point  $t_r$ ; then  $|R(\gamma(t)) - R(\gamma(t_r))| \leq Mh$  by the mean value theorem along  $\gamma$ . Combine with  $|R(\gamma(t_r))| \leq \delta$ .  $\square$

**Lemma 7.2** (Endpoint bound from Schur). *Let  $\rho_* := \sup_{\zeta \in \partial\Omega_{\eta_0}(L) \cap \mathbb{H}} |v(\zeta)|$ , where  $v = E_L^*/E_L$ . Then*

$$|E_L| \leq \frac{2}{1 - \rho_*} |A_L| \quad \text{on } \partial\Omega_{\eta_0}(L) \cap \mathbb{H}.$$

*Proof.* Let  $\rho_* := \sup_{\partial\Omega_\eta(L) \cap \mathbb{H}} |v|$ . If  $\rho_* < 1$  (as will hold inside the bootstrap ball used later), then since  $A_L = \frac{1}{2}(1 + v)E_L$ , we have  $|A_L| \geq \frac{1}{2}(1 - |v|)|E_L| \geq \frac{1}{2}(1 - \rho_*)|E_L|$ .  $\square$

**Corollary 7.3** (Weighted endpoint bound). *Let  $\tilde{E} := E_L/W_\eta^+$  and  $\tilde{A} := A_L/W_\eta^+$  on  $\text{Im } z \geq \eta$ . If  $\rho_* := \sup_{\partial\Omega_\eta(L) \cap \mathbb{H}} |v|$  with  $v = E_L^*/E_L$ , then*

$$|\tilde{E}| \leq \frac{2}{1 - \rho_*} |\tilde{A}| \quad \text{on } \partial\Omega_\eta(L) \cap \mathbb{H}.$$

*Proof.* Since  $E_L$  is Hermite-Biehler,  $v = E_L^*/E_L$  is Schur on  $\mathbb{H}$ , so  $|v| < 1$  on  $\mathbb{H}$ . As  $\partial\Omega_\eta(L) \subset \mathbb{H}$  is compact,  $\rho_* := \sup_{\partial\Omega_\eta(L) \cap \mathbb{H}} |v| < 1$ .

From  $A_L = \frac{1}{2}(1 + v)E_L$  we have  $|A_L| \geq \frac{1}{2}(1 - \rho_*)|E_L|$  on  $\partial\Omega_\eta(L) \cap \mathbb{H}$ . Divide both sides by  $|W_\eta^+| \geq 1$  on  $\text{Im } z \geq \eta$  (Lemma 8.11).  $\square$

**Lemma 7.4** (Tangential bounds for the weighted trace). *Fix  $\eta > 0$  and the racetrack family  $\partial\Omega_\eta(L)$ . Let  $\tilde{A}_L := A_L/W_\eta^+$  with  $W_\eta^+$  from §8.1. With  $\varrho_{\text{roll}}(\eta) > 0$  the uniform interior rolling radius from Lemma D.5, there exists  $C = C(\eta)$  (independent of  $L$ ) such that*

$$\|\partial_\tau \tilde{A}_L\|_{L^\infty(\partial\Omega_\eta(L))} \leq \frac{C(\eta)}{\varrho_{\text{roll}}(\eta)} \sup_{\partial\Omega_\eta(L)} |\tilde{A}_L|, \quad \text{Lip}_{\partial\Omega_\eta(L)}(\partial_\tau \tilde{A}_L) \leq \frac{C(\eta)}{\varrho_{\text{roll}}(\eta)^2} \sup_{\partial\Omega_\eta(L)} |\tilde{A}_L|.$$

*Proof.* For each boundary point  $\zeta$ , take the interior disk  $B_\zeta \subset \Omega_\eta(L)$  of radius  $\varrho_{\text{roll}}(\eta)$  tangent at  $\zeta$  (Lemma D.5). Since  $|W_\eta^+| \geq 1$  on  $\text{Im } z \geq \eta$ , the maximum principle gives  $\sup_{B_\zeta} |\tilde{A}_L| \leq \sup_{\partial\Omega_\eta(L)} |\tilde{A}_L|$ . Cauchy's estimate on  $B_\zeta$  yields the derivative and Lipschitz bounds with constants depending only on  $\eta$  and the geometry.  $\square$

## 8. OSCILLATORY SECOND-DERIVATIVE BOUND AND GLOBAL LIPSCHITZ OF THE STACKED JACOBIAN

Throughout this section, constants  $C(\cdot)$  may change from line to line and depend only on the indicated arguments and on the fixed  $C^{1,1}$  racetrack geometry; in particular, they are independent of  $L$ .

We work throughout inside a fixed bootstrap ball  $B_r(\theta_{\text{base}}) := \{\theta : \|\theta - \theta_{\text{base}}\|_2 \leq r\}$  on which the Weyl-Schur transform  $v$  has a uniform strict Schur bound  $|v| \leq \rho_{\text{sch}}(\eta) < 1$  along  $\partial\Omega_\eta(L)$ ; all constants are taken relative to this ball.

**Lemma 8.1** (Bootstrap strict Schur bound (Weyl side)). *Fix  $\eta > 0$  and assume  $K \leq \kappa L$ . Define the parameter ball*

$$B_r := \left\{ \theta : \sqrt{K} \|\theta - \theta_{\text{base}}\|_2 \leq r \right\}.$$

*Then for every  $r > 0$ ,*

$$\sup_{\zeta \in \partial\Omega_\eta(L)} |v(\zeta; \theta)| \leq \tanh(C(\eta)r) \quad (\theta \in B_r).$$

*In particular, choosing  $r_0 = r_0(\eta) > 0$  small enough gives*

$$\sup_{\zeta \in \partial\Omega_\eta(L)} |v(\zeta; \theta)| \leq \rho_\eta < 1 \quad (\theta \in B_{r_0}).$$

*Proof.* By Proposition I.15 (pseudohyperbolic  $L^2$  control),

$$\sup_{\partial\Omega_\eta(L)} \rho(v(\cdot; \theta), v(\cdot; \theta_{\text{base}})) \leq C(\eta) \sqrt{K} \|\theta - \theta_{\text{base}}\|_2 \leq C(\eta)r.$$

Hence  $\operatorname{arctanh}|v(\zeta; \theta)| \leq \operatorname{arctanh}|v(\zeta; \theta_{\text{base}})| + C(\eta)r$ . At the base  $v(\cdot; \theta_{\text{base}}) \equiv 0$ , so  $\operatorname{arctanh}|v(\zeta; \theta)| \leq C(\eta)r$ , i.e.  $|v(\zeta; \theta)| \leq \tanh(C(\eta)r)$ .  $\square$

**Definition 8.2** (Stacked Jacobian). We write  $D\mathcal{F}(\theta)$  for the Jacobian obtained by stacking the two seam rows on  $\Gamma_{\eta_0}$  with the normalization described above.

**Lemma 8.3** (gauge-normalized two-insertion kernel with oscillatory decay). *Let  $z = x + i\eta_0$  and  $\omega := x/2$ . For the top seam, define the (scalar) gauge-normalized functional*

$$\mathcal{L}_{\text{top}}(M) := e^{-ixL/2} e^{+\eta_0 L/2} \ell_+ M e_1,$$

*and analogously for the bottom seam with  $e^{-\eta_0 L/2}$ . (This choice differs from (4) by a fixed multiplicative factor; all rowwise estimates below are norm-based and thus unaffected.)*

*Let  $\rho_{\eta_0} := \sup_{\zeta \in \partial\Omega_{\eta_0}(L)} |v(\zeta)| < 1$ . There exists  $C = C(\eta_0, \rho_{\eta_0})$  such that for any two blocks  $I_{j_1}, I_{j_2}$  (length  $\ell$ ), any  $s_2 \in I_{j_2}$ ,  $s_1 \in I_{j_1}$  with  $s_2 \geq s_1$ , and any admissible*

*Hamiltonian in the parameter box,*

$$\left| \int_{I_{j_2}} \int_{I_{j_1}} \mathcal{L}_{\text{top}} \begin{pmatrix} \Phi(L, s_2; z) (z J \partial H_{j_2}) \Phi(s_2, s_1; z) \\ (z J \partial H_{j_1}) \Phi(s_1, 0; z) \end{pmatrix} ds_1 ds_2 \right| \leq \frac{C(\eta_0)}{1 - \rho_{\eta_0}^2} \frac{\ell^2}{1 + \omega |j_2 - j_1| \ell}. \quad (17)$$

The same bound holds for the bottom seam, and (on the bulk  $k \in \mathcal{K}_\delta$ ) after applying the intrinsic normalization (up to a factor depending only on  $\eta_0, \delta$ ).

*Proof.* Write the interaction picture with respect to the real phase:  $\Phi(s, t; z) = R(\omega(s - t)) U(s, t; z)$ , where  $U'(s, t; z) = z \tilde{K}(s) U(s, t; z)$  and  $\tilde{K}(s) = R(-\omega(s - t)) J H(s) R(\omega(s - t))$ , so  $\|\tilde{K}(s)\| \leq 1$ . Insert this at  $(L, s_2), (s_2, s_1), (s_1, 0)$  and use  $\ell_+ R(\theta) = e^{i\theta} \ell_+, R(\theta) e_1 = (\cos \theta, \sin \theta)^\top$  to factor out the only gap-dependent term:

$$\begin{aligned} & \mathcal{L}_{\text{top}} \left( \Phi(L, s_2) (z J \partial H_{j_2}) \Phi(s_2, s_1) (z J \partial H_{j_1}) \Phi(s_1, 0) \right) \\ &= \underbrace{\left( e^{-ix(L-s_2)/2} e^{+\eta_0(L-s_2)/2} \ell_+ U(L, s_2; z) \right)}_{=:r(s_2)^\top} \left( z \tilde{K}(s_2) \right) R(\omega(s_2 - s_1)) \\ &\quad \times \left( z \tilde{K}(s_1) \right) \underbrace{\left( U(s_1, 0; z) e^{+ixs_1/2} e^{+\eta_0 s_1/2} e_1 \right)}_{=:c(s_1)}. \end{aligned}$$

*Seam scalars in  $SU(1, 1)$  form.* Let  $(\alpha_s, \beta_s)$  be the  $SU(1, 1)$  Cayley data of  $\Phi(L, s; z)$  and set  $D_s := \overline{\beta_s} v_{\text{in}}(s; z) + \overline{\alpha_s}$ . Define  $r_N(s; z) := T_s(v_{\text{in}}(s; z))$ , where  $T_s(w) := \frac{\alpha_s w + \beta_s}{\overline{\beta_s} w + \overline{\alpha_s}}$ . Then

$$e^{-ix(L-s)/2} e^{+\eta_0(L-s)/2} \ell_+ U(L, s; z) \begin{pmatrix} 1 \\ m_R(s; z) \end{pmatrix} = \frac{r_N(s; z)}{D_s(z)},$$

and analogously on the left with  $\tilde{D}_s$  and  $c_N(s; z) := \tilde{T}_s(v_{\text{pre}}(s; z))$ . The  $SU(1, 1)$  identity yields

$$1 - |r_N|^2 = \frac{1 - |v_{\text{in}}|^2}{|D_s|^2}, \quad 1 - |c_N|^2 = \frac{1 - |v_{\text{pre}}|^2}{|\tilde{D}_s|^2},$$

hence  $|r_N|, |c_N| \leq 1$  and, writing  $\rho := \sup_{\partial\Omega_{\eta_0}(L)} |v|$ ,

$$|D_s|, |\tilde{D}_s| \geq \sqrt{1 - \rho^2}.$$

Therefore each seam contributes at most  $(1 - \rho^2)^{-1/2}$ ; together they contribute  $(1 - \rho^2)^{-1}$ .

Each short-block insertion also satisfies  $\left\| \int_{I_j} z \tilde{K}(s) ds \right\| \leq C_1(\eta_0)$  since  $|z| \ell \lesssim 1$  on  $\partial\Omega_{\eta_0}(L)$ . Therefore, after integrating in  $s_1 \in I_{j_1}, s_2 \in I_{j_2}$ , the magnitude of the

double integral is bounded by

$$C_2(\eta_0) \left| \int_{I_{j_2}} \int_{I_{j_1}} e^{i\omega(s_2 - s_1)} ds_1 ds_2 \right| \leq \frac{C_3(\eta_0) \ell^2}{1 + \omega |j_2 - j_1| \ell},$$

where we used Lemma C.1 on the difference variable  $t = s_2 - s_1$ . This gives the top-seam bound. The bottom seam is identical (replace  $e^{+\eta_0(\cdot)/2}$  by  $e^{-\eta_0(\cdot)/2}$ ); intrinsic normalization rescales by a factor bounded above/below on the bulk (Lemma 3.2).  $\square$

**Lemma 8.4** (Pair counting at fixed gap). *Let the interval  $[0, L]$  be partitioned into  $N$  blocks of length  $\ell = L/N$ , indexed by  $j = 0, \dots, N - 1$ . For a nonnegative integer  $d$ , the number of ordered pairs  $(j_1, j_2)$  with  $0 \leq j_1 \leq j_2 \leq N - 1$  and  $j_2 - j_1 = d$  equals  $N - d \leq N$ .*

**Proposition 8.5** (Second derivative bound for any normalized seam row). *Fix  $\eta_0 > 0$  and  $\delta \in (0, \frac{1}{4})$ . Along any path in parameter space contained in the PSD box and with endpoint-side strict Schur bound  $|v| \leq \rho_0 < 1$  on  $\partial\Omega_{\eta_0}(L)$ , every normalized seam row  $\mathbf{r}$  (gauge + intrinsic) satisfies*

$$\|D^2\mathbf{r}(\theta)\| \leq \frac{C(\eta_0, \delta, \varepsilon)}{(1 - \rho_0^2)^2} L^2.$$

*Proof.* A generic second derivative is a double insertion over blocks  $j_1, j_2$ . Each insertion contributes a uniformly bounded factor because  $\|z \partial H\| \lesssim |z|\ell$  with  $|z| \leq CL$  on  $\Gamma_{\eta_0}(L)$  and  $\ell = L/N$ , while  $\partial H$  is uniformly bounded over the parameter box; hence  $|z|\ell \lesssim 1$ .

By Lemma 8.3, for a pair at gap  $d = |j_2 - j_1|$ , the (gauge-normalized, intrinsically normalized) two-insertion contribution of that pair is

$$\lesssim \frac{\ell^2}{1 + |\omega| d \ell}, \quad \omega := x/2,$$

uniformly over the parameter box.

Using Lemma 8.4, there are  $N - d \leq N$  pairs at gap  $d$ . Summing over all gaps,

$$\sum_{d=0}^{N-1} (N - d) \frac{\ell^2}{1 + \omega d \ell} \leq N \sum_{d=0}^{N-1} \frac{\ell^2}{1 + \omega d \ell} = L \ell \sum_{d=0}^{N-1} \frac{1}{1 + \omega d \ell} \leq L \cdot \frac{1}{\omega} \log(1 + \omega L)$$

where we used that  $N \asymp L^2$  and the function  $t \mapsto \frac{1}{t} \log(1 + t)$  is bounded on  $[0, \infty)$  with maximum at  $t = 0$ . This yields the stated  $O(L^2)$  bound uniformly in  $x \in [-\pi L, \pi L]$ .

*Remark.* For  $|x| \geq L^{-\gamma}$  (any fixed  $\gamma \in (0, 1)$ ) the sum is  $O(L \log(1 + L))$ . We keep the uniform  $O(L^2)$  bound to cover the window  $|x| \ll L^{-1}$ .  $\square$

**Lemma 8.6** (Local strict Schur bound (Weyl side)). *There exist constants  $C_0 = C_0(\eta_0, \delta)$  and a radius  $\rho_* = (2C_0\sqrt{K})^{-1}$  such that for every parameter vector  $\theta$  in the ball  $B_* := \{\theta : \|\theta - \theta_{\text{base}}\|_2 \leq \rho_*/\ell\}$ , the Cayley transform of the Weyl-Titchmarsh*

function satisfies

$$\sup_{z \in \partial\Omega_{\eta_0}(L)} |v(z; \theta)| \leq \tanh(C_0 \sqrt{K} \|\theta - \theta_{\text{base}}\|_2) \leq \tanh(1/2) =: \rho_0 < 1.$$

*Proof.* The result is a direct consequence of the pseudohyperbolic control established in Lemma 9.21. That lemma shows that for any  $\theta$ ,

$$\sup_{z \in \partial\Omega_{\eta_0}(L)} \rho(v(z; \theta), v(z; \theta_{\text{base}})) \leq C_0 \sqrt{K} \|\theta - \theta_{\text{base}}\|_2.$$

At the canonical base,  $m_{\text{base}}(z) \equiv i$ , so the Cayley transform of the Weyl-Titchmarsh function is  $v(z; \theta_{\text{base}}) \equiv 0$ . Since  $\rho(w, 0) = |w|$  and  $\text{dist}_{\mathbb{D}}(w, 0) = \operatorname{arctanh}|w|$ , integrating the differential form of the Schwarz-Pick control along a straight path from  $\theta_{\text{base}}$  to  $\theta$  gives (see [Dur70, Gar07, Bea83])

$$\operatorname{arctanh}|v(z; \theta)| = \text{dist}_{\mathbb{D}}(v(z; \theta), 0) \leq C_0 \sqrt{K} \|\theta - \theta_{\text{base}}\|_2.$$

Taking the hyperbolic tangent of both sides yields the first inequality. Choosing the ball radius  $\rho_* = (2C_0 \sqrt{K})^{-1}$  ensures that for any  $\theta \in B_*$ , the argument of  $\tanh$  is at most  $1/2$ , which provides the uniform strict Schur bound with parameter  $\rho_0 = \tanh(1/2) < 1$ .  $\square$

**Lemma 8.7** (Per-hop invariance of the strict-Schur ball). *Work on a ball where Lemma 8.6 gives  $\sup_{\partial\Omega_{\eta_0}(L)} |v| \leq \rho_0 < 1$ . There exists  $c > 0$  (independent of  $L$ ) such that if  $\|\theta^{(j)} - \theta^{(j-1)}\| \leq r$  with*

$$\frac{2C_0}{1 - \rho_0^2} \sqrt{K} r \leq c(1 - \rho_0),$$

*then  $\sup_{\partial\Omega_{\eta_0}(L)} |v(\cdot; \theta^{(j)})| \leq \rho_0 + \frac{1}{2}(1 - \rho_0) < 1$ ; hence the entire  $M$ -step path remains in the same strict-Schur ball, uniformly in  $L$ .*

*Proof.* By Lemma 8.6, on the strict-Schur ball  $\sup_{\partial\Omega} \rho(v(\cdot; \theta), v(\cdot; \tilde{\theta})) \leq C_0 \sqrt{K} \|\theta - \tilde{\theta}\|_2$ . On  $\{|v| \leq \rho_0\}$ ,  $|w - w'| \leq \frac{2}{1 - \rho_0^2} \rho(w, w')$ , hence  $\|\Delta v\|_{L^\infty} \leq \frac{2C_0}{1 - \rho_0^2} \sqrt{K} \|\Delta \theta\|_2$ . Choose  $r$  so the RHS  $\leq \frac{1}{2}(1 - \rho_0)$ ; then per hop  $\sup |v|$  grows by at most  $\frac{1}{2}(1 - \rho_0)$ . With  $K = \kappa L$  and  $r \asymp L^{-1}$ , the condition is uniform for large  $L$ ; for small  $L$  subdivide hops.  $\square$

**Theorem 8.8** (Local Lipschitz of  $DF_K$  on a ball with a uniform strict Schur bound). *If  $\sup_{\partial\Omega_{\eta_0}(L)} |v(\cdot; \theta)| \leq \rho_0 < 1$  on a convex ball in parameter space, then for all  $\theta, \tilde{\theta}$  in that ball*

$$\|DF_K(\theta) - DF_K(\tilde{\theta})\|_2 \leq \frac{C(\eta_0, \delta, \varepsilon)}{(1 - \rho_0^2)^2} L^2 \|\theta - \tilde{\theta}\|_2.$$

*Proof.* Work inside the strict-Schur ball  $\sup_{\partial\Omega_{\eta_0}(L)} |v(\cdot; \theta)| \leq \rho_0 < 1$ . The extractor  $\Pi_K^{\text{mix}}$  is linear and parameter-independent. Division by  $W_{\eta_0}^+$  is parameter-independent

and  $|W_{\eta_0}^+| \geq 1$  on  $\text{Im } z \geq \eta_0$  (Lemma 8.11). Thus it suffices to bound the change of the (gauge-normalized, intrinsically normalized) boundary streams.

Rowwise, each selected coefficient is an  $L^2$ -bounded functional of the boundary traces, and its Jacobian difference is obtained by a two-insertion variation of the transfer matrix. The associated oscillatory kernel has the explicit factor

$$\frac{1}{(1 - |v|^2)^2} K_{\text{osc}},$$

with  $K_{\text{osc}}$  depending only on  $(\eta_0, \delta, \varepsilon)$  and supported on a set of size  $\asymp L$  in each seam index. Hence the rowwise change is bounded by

$$\frac{C(\eta_0, \delta, \varepsilon)}{(1 - \rho_0^2)^2} \left( \sum_{n \asymp L} \sum_{m \asymp L} |K_{\text{osc}}(n, m)| \right) \|\theta - \tilde{\theta}\| \lesssim \frac{C(\eta_0, \delta, \varepsilon)}{(1 - \rho_0^2)^2} L^2 \|\theta - \tilde{\theta}\|,$$

where the double sum contributes the  $L^2$  factor (two insertions along seams of length  $\asymp L$ ). Taking the operator norm across the  $2K$  rows and using that the row-mixing and normalizer act as bounded isomorphisms (Appendix D) gives

$$\|DF_K(\theta) - DF_K(\tilde{\theta})\|_2 \leq \frac{C(\eta_0, \delta, \varepsilon)}{(1 - \rho_0^2)^2} L^2 \|\theta - \tilde{\theta}\|.$$

□

### 8.1. Outer factor normalization at fixed height.

*Remark 8.9* (What is used about  $W_\eta^+$ ). All properties needed in this paper are collected in Lemma 8.11; we will cite that lemma when invoking the outer factor.

*Remark 8.10* (Uniformity on racetracks). The racetracks  $\partial\Omega_{\eta_0}(L)$  have  $C^{1,1}$  geometry with curvature and rolling radius bounded independent of  $L$ . Hence Poisson kernels, Cauchy estimates, Jackson/Fejér constants, and boundary layer potentials depend only on  $(\eta_0, Y)$ , not on  $L$  [Pom92, McL00, Ran95].

Fix  $\eta > 0$ . Set

$$\lambda_\eta(x) := \log(1 + |\Xi(x + i\eta)|) \quad (x \in \mathbb{R}).$$

By Stirling for  $\Gamma(\cdot/2)$  and vertical-strip bounds for  $\zeta$  (see [THB86, IK04]), there is  $A(\eta) \geq 0$  such that

$$|\Xi(x + i\eta)| \ll_\eta (1 + |x|)^{A(\eta)} e^{-\pi|x|/4} \quad (|x| \rightarrow \infty).$$

Consequently  $\lambda_\eta \in L^1\left(\frac{dx}{1+x^2}\right)$ . Hence there exists an outer (Nevanlinna) function  $\widetilde{W}_\eta$  on  $\mathbb{H}$  with boundary values

$$\log |\widetilde{W}_\eta(x + i0)| = \lambda_\eta(x) \quad \text{a.e. } x \in \mathbb{R}.$$

Define the *shifted* upper weight by

$$W_\eta^+(z) := \widetilde{W}_\eta(z - i\eta) \quad (\operatorname{Im} z > \eta),$$

and define the lower weight by the reflection rule

$$W_\eta^-(\bar{z}) = \overline{W_\eta^+(z)} \quad (z \in \mathbb{H}). \quad (18)$$

Then  $W_\eta^+$  is analytic and zero-free on  $\{\operatorname{Im} z > \eta\}$  and  $W_\eta^-$  is analytic and zero-free on  $\{\operatorname{Im} z < -\eta\}$ .

Existence and these properties of  $\widetilde{W}_\eta$  follow from the outer (Nevanlinna) factor construction in the upper half-plane [Lev96, §II.4]. Since  $\lambda_\eta \geq 0$ , its Poisson integral  $\log |W_\eta^+(z)|$  is nonnegative for  $\operatorname{Im} z \geq \eta$  by the positivity of the Poisson kernel. Exponentiating gives  $|W_\eta^+(z)| \geq 1$  in that half-plane (and similarly  $|W_\eta^-(z)| \geq 1$  for  $\operatorname{Im} z \leq -\eta$ ).

**Lemma 8.11** (outer factor bounds used in the paper). *Let  $\eta > 0$ ,  $\lambda_\eta(x) := \log(1 + |\Xi(x + i\eta)|)$ , and let  $\widetilde{W}_\eta$  be the outer function on  $\mathbb{H}$  with boundary modulus  $e^{\lambda_\eta}$ . Define  $W_\eta^+(z) := \widetilde{W}_\eta(z - i\eta)$  for  $\operatorname{Im} z \geq \eta$ . Then*

$$\left| \frac{\Xi(x + i\eta)}{W_\eta^+(x + i\eta)} \right| \leq 1 \quad (x \in \mathbb{R}), \quad |W_\eta^+(z)| \geq 1 \quad (\operatorname{Im} z \geq \eta).$$

*Proof.* By definition  $|W_\eta^+(x + i\eta)| = e^{\lambda_\eta(x)} = 1 + |\Xi(x + i\eta)|$ , giving the first inequality. The Poisson extension of  $\lambda_\eta \geq 0$  to  $\operatorname{Im} z \geq \eta$  is nonnegative, hence  $|W_\eta^+| \geq 1$  there.  $\square$

If one prefers to work exactly on  $y = \eta_0$ , mollify the boundary modulus  $\lambda_{\eta_0}$  by  $\lambda_{\eta_0, \sigma} := e^{\phi_\sigma * \log \lambda_{\eta_0}}$  with a symmetric mollifier  $\phi_\sigma$  on  $\mathbb{R}$ , and let  $W_{\eta_0, \sigma}^+$  be the outer with boundary modulus  $\lambda_{\eta_0, \sigma}$ .

As a standing convention we work strictly above the lower seam: fix  $\varepsilon \in (0, \eta_0/4)$  and set  $y_1 = \eta_0 + \varepsilon$ ,  $y_2 = 2\eta_0$ . All boundary approximation, coefficient solves, and seam→interior steps are carried out on  $\operatorname{Im} z \in \{y_1, y_2\}$  and the same vertical caps as in  $\partial\Omega_{\eta_0}(L)$ . On the racetrack we use the weighted streams  $\tilde{A}_L = A_L/W_\eta^+$  and  $\tilde{E}_L = E_L/W_\eta^+$ .

*Remark 8.12* (Uniform constants in §8). All size bounds below that come from Cauchy/maximum-principle or Jackson/Fejér/DST depend only on  $\eta_0$  and the racetrack geometry (via  $\|1/W_{\eta_0}^+\|_{L^\infty}$ ), not on  $L$  or  $K$  (see Lemma 8.11 and Appendix D). Any strict-Schur margin is provided by Lemma 9.20 and Lemma 8.7.

On the racetrack  $\partial\Omega_\eta(L) \subset \{\operatorname{Im} z \geq \eta\}$  we work with the *weighted streams*

$$\tilde{A}_L := \frac{A_L}{W_\eta^+}, \quad \tilde{E}_L := \frac{E_L}{W_\eta^+}. \quad (19)$$

Since  $|W_\eta^+(z)| \geq 1$  for  $\text{Im } z \geq \eta$ , the Cauchy/maximum-principle and Jackson/Fejér/DST size constants we use are uniform in  $L$  (depending only on  $\eta$  and the racetrack geometry). Strict–Schur margins come from Lemma 9.20 and Lemma 8.7. We will not use  $W_\eta^-$  on the racetrack. All divisions by  $W_\eta^+$  are performed only on the strip  $\{\text{Im } z \geq \eta\}$ .

## 9. SQUARE-MODE INVERSION AT FIXED HEIGHT AND BOUNDARY CONTROL

Fix a height  $\eta > 0$  and racetrack  $\partial\Omega_\eta(L)$  as in §7. We keep the gauge normalization (4) and intrinsic normalization (10) on the bulk index set  $\mathcal{K}_\delta$ .

To extract square coefficients, we parameterize the Hamiltonian by the first  $K$  DST-I modes of  $(s_j)$  and  $(t_j)$ :

$$\widehat{s}_m := \frac{2}{N} \sum_{j=1}^{N-1} s_j \sin \frac{\pi m j}{N}, \quad \widehat{t}_m := \frac{2}{N} \sum_{j=1}^{N-1} t_j \sin \frac{\pi m j}{N}, \quad m = 1, \dots, K,$$

with DC parts fixed. The parameter vector is

$$\theta^{(K)} = (\widehat{s}_1, \dots, \widehat{s}_K, \widehat{t}_1, \dots, \widehat{t}_K) \in \mathbb{R}^{p_K}, \quad p_K = 2K.$$

On the seams and arcs we now use the *weighted* streams  $\tilde{A}_L := A_L/W_\eta^+$  (see §8.1), still with the same gauge normalization and intrinsic normalization as in §2–§3. From those weighted boundary traces we keep their first  $K$  DST-I coefficients for each of the real and imaginary parts; this yields exactly  $2K = p_K$  equations, giving a square system.

**Definition 9.1** (Weyl-side square map). With the same gauge normalization, intrinsic normalization, and mixed selection  $\Pi_K^{\text{mix}}$  as for  $F_K$ , define

$$G_K(\theta) := \Pi_K^{\text{mix}}(\text{Re } v(\cdot; \theta), \text{Im } v(\cdot; \theta)) \in \mathbb{R}^{2K}, \quad p_K = 2K.$$

**Lemma 9.2** (Base conditioning for  $G_K$ ). *For  $K \leq c_\delta N$ ,*

$$c(\delta, \eta_0) \sqrt{N} \leq \sigma_{\min}(DG_K(\theta_{\text{base}})) \leq \sigma_{\max}(DG_K(\theta_{\text{base}})) \leq C(\delta, \eta_0) \sqrt{N}.$$

*Proof.* At the base  $H_0 \equiv \frac{1}{2}I$ , Lemma 4.4 shows each Jacobian column of  $DG_K(\theta_{\text{base}})$  is the two-row block  $\kappa(\eta_0, u_k)(\mathbf{c}_\vartheta^- \sin(\phi - u_k) + \mathbf{c}_\vartheta^+ \sin(\phi + u_k))$  with  $u_k = \pi k/N$  and  $(\mathbf{c}_\vartheta^-, \mathbf{c}_\vartheta^+) \in \{(1, 0), (0, 1)\}$ . On the bulk  $k \in \mathcal{K}_\delta$  and after intrinsic normalization (Lemma 3.2), this is an invertible  $2 \times 2$  mixing of the DST-I synthesis columns. Hence the Gram of  $DG_K(\theta_{\text{base}})$  is comparable to the bulk DST Gram from Lemma 6.1, whose eigenvalues are  $\asymp N$ . Taking square roots yields the claimed  $\Theta(\sqrt{N})$  bounds.  $\square$

**Lemma 9.3** (Local Jacobian Lipschitz for  $G_K$ ). *For any  $\theta, \tilde{\theta}$  in the ball  $B_*$  from Lemma 8.6,*

$$\|DG_K(\theta) - DG_K(\tilde{\theta})\| \leq \frac{C(\eta_0, \delta, \varepsilon)}{(1 - \rho_0^2)^2} L^2 \|\theta - \tilde{\theta}\|_2,$$

where  $\rho_0 = \tanh(1/2)$ .

*Proof.* Let  $\theta, \tilde{\theta} \in B_\star$ . By Lemma 8.6, for any point on the line segment between them, the Cayley transform of the Weyl-Titchmarsh function satisfies  $|v(z; \cdot)| \leq \rho_0 = \tanh(1/2)$  uniformly on  $\partial\Omega_{\eta_0}(L)$ . This provides the necessary strict Schur bound to apply the double-insertion estimate from Lemma 8.3. The proof then proceeds exactly as in Proposition 8.5: the difference of Jacobian entries is bounded by a double-insertion integral, which, after summing over all pairs of blocks  $(j_1, j_2)$ , yields the  $O(L^2)$  Lipschitz bound with the required denominator of  $(1 - \rho_0^2)^2$ .  $\square$

*Remark 9.4* (Optional length normalization for  $G_K$ ). With  $\tilde{\Pi} := |\partial\Omega_{\eta_0}(L)|^{-1/2} \Pi_K^{\text{mix}}$ , one has  $\sigma_{\min}(D(\tilde{\Pi} \circ G_K)) \asymp \sqrt{L}$  and  $\text{Lip}(D(\tilde{\Pi} \circ G_K)) \asymp (1 - \rho_0^2)^{-2} L^{3/2}$ .

**Definition 9.5** (Square map at bandwidth  $K$  (weighted)). Let  $\Theta^{(K)} \subset \mathbb{R}^{p_K}$  be the set of truncated DST parameter vectors  $\theta^{(K)}$  defined above, with  $p_K = 2K$ . Let  $\Pi_K^{\text{mix}}$  select exactly  $p_K$  coefficients from the bulk seam core, but applied to the *weighted, gauge-normalized, normalized* streams  $\tilde{A}_L := A_L / W_\eta^+$ . Define

$$F_K : \Theta^{(K)} \rightarrow \mathbb{R}^{p_K}, \quad F_K(\theta) := \Pi_K^{\text{mix}} (\text{weighted, normalized boundary streams of } A_L(\cdot; \theta)).$$

**Definition 9.6** (Length-normalized square map). Set  $\tilde{\Pi}_K^{\text{mix}} := |\partial\Omega_\eta(L)|^{-1/2} \Pi_K^{\text{mix}}$  and define

$$\begin{aligned} \tilde{F}_K(\theta) &:= \tilde{\Pi}_K^{\text{mix}} \left( \text{weighted, normalized, gauge-normalized boundary streams of } A_L(\cdot; \theta) \right) \\ &= |\partial\Omega_\eta(L)|^{-1/2} F_K(\theta). \end{aligned}$$

**Lemma 9.7** (Coefficient normalization). *For any  $f \in C^{1,1}(\partial\Omega_\eta(L))$ ,*

$$\|\tilde{\Pi}_K^{\text{mix}}(f)\|_2 \leq C(\eta) \left( \|f\|_{L^\infty(\partial\Omega_\eta(L))} + \|\partial_\tau f\|_{\text{Lip}(\partial\Omega_\eta(L))} \right),$$

with  $C(\eta)$  independent of  $L, K$ .

*Proof.* On each  $C^{1,1}$  boundary component, reparametrize by arclength; the coordinate map is biLipschitz with geometry constants depending only on  $\eta$  (Appendix D D). Jackson's inequality on the circle then yields uniform trigonometric coefficient bounds in  $K$  and in the geometry. For  $f \in C^{1,1}$ , Jackson's inequality on the circle gives

$$\|\widehat{f}_{[1:K]}\|_{\ell^2} \leq C \left( \|f \circ \gamma\|_{L^\infty(0,S)} + \|\partial_\tau f\|_{\text{Lip}(\partial\Omega_\eta(L))} \right),$$

uniformly in  $K$  and in the geometry. The extractor  $\tilde{\Pi}_K^{\text{mix}}$  is a fixed linear combination of orthonormal trigonometric/DST coefficients on each component (after our gauge normalization/normalization), and the normalizer acts as a bounded isomorphism on the mode space with operator norm depending only on  $\eta$  and the  $C^{1,1}$  constants (by Lemma D.7). Combining these bounds yields the claim with  $C(\eta)$  independent of  $K$  and  $L$ .  $\square$

*Remark 9.8.* Multiplying all rows of the extractor by the constant  $|\partial\Omega_\eta(L)|^{-1/2}$  scales both singular values and Jacobian norms by that factor. Since  $|\partial\Omega_\eta(L)| \asymp L$  and  $N \asymp L^2$ , we have

$$\sigma_{\min}(D\tilde{F}_K) = |\partial\Omega_\eta(L)|^{-1/2} \sigma_{\min}(DF_K) \asymp \frac{\sqrt{N}}{\sqrt{L}} \asymp \sqrt{L}.$$

All inverse-function constants used later are unchanged up to absolute constants (the uniform row rescaling affects both sides equally). Here  $|\partial\Omega_\eta(L)| \asymp L$  because the two horizontal segments each have length  $\approx 2\pi L$  (total  $\approx 4\pi L$ ), while the corner arcs have  $L$ -independent radii.

*Remark 9.9.* On each  $C^{1,1}$  boundary component the gauge normalization together with the intrinsic normalization acts as a bounded isomorphism on the trigonometric mode space; the operator norms are uniform in  $L$  by Appendix E. Hence  $L^\infty$  and coefficient norms are equivalent up to constants independent of  $L$ .

**Lemma 9.10** (Base Jacobian conditioning on the first  $K$  modes). *Fix  $\delta \in (0, \frac{1}{4})$  and  $\eta > 0$ . Using the DST-I parametrization  $\theta^{(K)}$  and the mixed selection  $F_K$  from Definition 9.5, there exists  $c_\delta \in (0, 1)$  such that for any  $K \leq c_\delta N$ ,*

$$c_1(\delta, \eta) \sqrt{N} \leq \sigma_{\min}(DF_K(\theta_{\text{base}})) \leq \sigma_{\max}(DF_K(\theta_{\text{base}})) \leq C_1(\delta, \eta) \sqrt{N}.$$

*Proof.* Use Lemma 4.3 (two-frequency structure for  $\partial_\theta A_L$  at the base) and the height-equivalence/intrinsic normalization of §5. On the bulk, the DST-I Gram bounds (§6) give  $\Theta(\sqrt{N})$  singular values; the mixed selection preserves these by Proposition 6.4.  $\square$

**Lemma 9.11** (Transfer-matrix relation from  $G_K$  to  $F_K$ ). *on  $\text{Im } z \geq \eta$  let  $\tilde{A} := A_L/W_\eta^+$  and  $\tilde{E} := E_L/W_\eta^+$ , so  $\tilde{A} = \frac{1}{2}(1+v)\tilde{E}$ . Then on any ball where  $\sup_{\partial\Omega_{\eta_0}(L)} |v| \leq \rho_0 < 1$ ,*

$$\|DF_K(\theta) - DF_K(\tilde{\theta})\| \leq \frac{C(\eta_0, \eta, \delta, \varepsilon)}{(1 - \rho_0^2)^2} L^2 \|\theta - \tilde{\theta}\|.$$

*Consequently all conditioning and inverse-function radii obtained for  $G_K$  transfer to  $F_K$  up to absolute constants.*

*Proof.* Write on  $\text{Im } z \geq \eta$ :  $\tilde{A} = \frac{1}{2}(1+v)\tilde{E}$ , so  $D\tilde{A} = \frac{1}{2}((Dv)\tilde{E} + (1+v)D\tilde{E})$ . The extractor  $\Pi_K^{\text{mix}}$  is linear and parameter-independent, and division by  $W_\eta^+$  is parameter-independent with  $|W_\eta^+| \geq 1$  (Lemma 8.11). Along a strict Schur bound path  $\sup |v| \leq \rho_0 < 1$ , we therefore have

$$\|DF_K(\theta) - DF_K(\tilde{\theta})\| \leq C(\eta, \delta) \left( \|DG_K(\theta) - DG_K(\tilde{\theta})\| + \|D\tilde{E}(\theta) - D\tilde{E}(\tilde{\theta})\| \right).$$

The  $DG_K$  term is controlled by Lemma 9.3, while the  $D\tilde{E}$  term is estimated exactly as in Theorem 8.8 (single/double insertions and the same oscillatory kernel), giving the stated  $(1 - \rho_0^2)^{-2}L^2$  bound.  $\square$

**Lemma 9.12** (Persistence of invertibility in a ball). *There is  $r = r(\delta, \eta, \varepsilon) > 0$  such that for all  $\theta$  with  $\|\theta - \theta_{\text{base}}\| \leq r$ ,*

$$\sigma_{\min}(DF_K(\theta)) \geq \frac{1}{2} \sigma_{\min}(DF_K(\theta_{\text{base}})) \gtrsim_{\delta, \eta} \sqrt{N}.$$

*Proof.* By Theorem 8.8,  $\|DF_K(\theta) - DF_K(\tilde{\theta})\| \leq CL^2\|\theta - \tilde{\theta}\|$  on the strict–Schur ball. Set

$$r := \frac{1}{2CL^2} \sigma_{\min}(DF_K(\theta_{\text{base}})).$$

Then for any  $\theta$  with  $\|\theta - \theta_{\text{base}}\| \leq r$  we have  $\|DF_K(\theta) - DF_K(\theta_{\text{base}})\| \leq \frac{1}{2} \sigma_{\min}(DF_K(\theta_{\text{base}}))$ . Weyl’s inequality gives

$$\sigma_{\min}(DF_K(\theta)) \geq \sigma_{\min}(DF_K(\theta_{\text{base}})) - \|DF_K(\theta) - DF_K(\theta_{\text{base}})\| \geq \frac{1}{2} \sigma_{\min}(DF_K(\theta_{\text{base}})),$$

as claimed.  $\square$

**Theorem 9.13** (Local exact solvability at bandwidth  $K$ ). *Let  $c \in \mathbb{R}^{p_K}$  satisfy  $\|c - F_K(\theta_{\text{base}})\| \leq \frac{1}{4} \sigma_{\min}(DF_K(\theta_{\text{base}})) r$ , with  $r$  as in Lemma 9.12. Then there exists a unique  $\theta_K$  in the ball  $\|\theta - \theta_{\text{base}}\| \leq r$  such that*

$$F_K(\theta_K) = c.$$

$$\text{Moreover } \|\theta_K - \theta_{\text{base}}\| \leq \frac{2}{\sigma_{\min}(DF_K(\theta_{\text{base}}))} \|c - F_K(\theta_{\text{base}})\|.$$

*Proof.* Let  $J(\theta) := DF_K(\theta)$  and set  $\sigma_0 := \sigma_{\min}(J(\theta_{\text{base}}))$ ,  $L_J := \sup_{\theta \neq \tilde{\theta}} \frac{\|J(\theta) - J(\tilde{\theta})\|}{\|\theta - \tilde{\theta}\|}$ .

By Lemma 9.12 we have  $\sigma_{\min}(J(\theta)) \geq \frac{1}{2} \sigma_0$  on the ball  $\|\theta - \theta_{\text{base}}\| \leq r$  once  $r \leq \sigma_0/(2L_J)$ . Take

$$r := \frac{\sigma_0}{2L_J}, \quad \delta := \frac{1}{4} \sigma_0 r = \frac{\sigma_0^2}{8L_J}.$$

If  $\|c - F_K(\theta_{\text{base}})\| \leq \delta$ , the Newton–Kantorovich theorem for square  $C^1$  systems (with  $J L_J$ -Lipschitz and  $\sigma_{\min} \geq \frac{1}{2} \sigma_0$  on the ball) yields a unique  $\theta_K$  with  $\|\theta_K - \theta_{\text{base}}\| \leq r$  and  $F_K(\theta_K) = c$ , as well as

$$\|\theta_K - \theta_{\text{base}}\| \leq \frac{2}{\sigma_0} \|c - F_K(\theta_{\text{base}})\|.$$

Using Theorem 8.8 gives  $L_J \lesssim L^2$ , while Lemma 9.10 gives  $\sigma_0 \asymp \sqrt{N}$ . Hence  $r \asymp \sqrt{N}/L^2$  and  $\delta = \frac{1}{4} \sigma_{\min}(\theta_{\text{base}}) r \asymp N/L^2$ , matching the stated quantitative radius.  $\square$

Quantitatively, combining  $\sigma_{\min}(DG_K) \asymp \sqrt{N} \asymp L$  with Lemma 9.3 (via the Newton–Kantorovich estimate) yields an admissible inverse-function radius  $r \gtrsim (1 - \rho_0^2)^2/L$ . By Lemma 9.11, the same bound applies to  $F_K$ .

Next we project the *target* boundary data through the same extractor, the following lemma provides the required boundary approximation.

**Lemma 9.14** (Trigonometric/DST approximation on  $C^{1,1}$  boundary). *Let  $\gamma$  be an arclength parametrization of any boundary component of  $\partial\Omega_\eta(L)$ . For  $f \in C^{1,1}(\partial\Omega_\eta(L))$  there exist trigonometric polynomials  $P_K$  of degree  $K$  such that*

$$\|f \circ \gamma - P_K\|_{L^\infty} \leq \frac{C}{K} \|\partial_\tau f\|_{\text{Lip}}, \quad \|\partial_\tau f - \partial_\tau P_K\|_{L^\infty} \leq C \|\partial_\tau f\|_{\text{Lip}}.$$

The constants depend only on  $\eta$  and the  $C^{1,1}$  geometry (hence are uniform in  $L$ ). Along the horizontal seams, after our gauge normalization/normalization the DST-I basis realizes  $P_K$  with the same rates.

*Proof.* Jackson's theorem on the circle applied to  $f \circ \gamma$  (and to  $\partial_\tau f$ ), plus the bilipschitz change of variables for  $C^{1,1}$  arcs; see Appendix E for the uniform  $C^{1,1}$  bounds. Along seams, the normalizer is a bounded isomorphism on the mode space (Lemma D.7), preserving  $L^\infty$  rates up to a constant, and the DST-I realization follows.  $\square$

**Lemma 9.15** (Boundary-to-coefficients  $H^\infty \rightarrow \ell^2$ ). *Let  $y_1 = \eta_0 + \varepsilon$ ,  $y_2 = 2\eta_0$  and let  $\tilde{\Pi}_K^{\text{mix}}$  be the length-normalized mixed trigonometric projector of Definition 9.6 acting on the two horizontal seams  $\Gamma_{y_1}, \Gamma_{y_2}$ . If  $f$  is holomorphic on a neighborhood of  $\{y_1 \leq \text{Im } z \leq y_2\}$ , then*

$$\|\tilde{\Pi}_K^{\text{mix}} f\|_{\ell^2} \leq C \|f\|_{L^\infty(\Gamma_{y_1} \cup \Gamma_{y_2})},$$

with a universal constant  $C$  independent of  $L$  and  $K$ .

*Proof.* On each seam  $\Gamma_{y_j} = \{x \in [-X(L), X(L)]\}$  endow arclength with the normalized measure  $d\mu_j := d\sigma/|\Gamma_{y_j}|$  so that  $\mu_j(\Gamma_{y_j}) = 1$ . Let  $\{\varphi_m^{(j)}\}_{m=1}^K$  be the first  $K$  trigonometric/DST modes orthonormal in  $L^2(\Gamma_{y_j}, \mu_j)$ . The coefficient vector on  $\Gamma_{y_j}$  is  $a^{(j)} = (\langle f, \varphi_m^{(j)} \rangle_{L^2(\mu_j)})_{m=1}^K$ , hence

$$\|a^{(j)}\|_{\ell^2} = \|P_K^{(j)} f\|_{L^2(\mu_j)} \leq \|f\|_{L^2(\mu_j)} \leq \|f\|_{L^\infty(\Gamma_{y_j})}.$$

Stacking the two seams and using  $\|u \oplus v\|_{\ell^2} \leq \sqrt{\|u\|_{\ell^2}^2 + \|v\|_{\ell^2}^2}$  gives  $\|\Pi_K^{\text{mix}} f\|_{\ell^2} \leq \sqrt{2} \|f\|_{L^\infty(\Gamma_{y_1} \cup \Gamma_{y_2})}$  when coefficients are computed with normalized seam measures. By Definition 9.6,  $\tilde{\Pi}_K^{\text{mix}} = |\partial\Omega_{\eta_0}(L)|^{-1/2} \Pi_K^{\text{mix}}$  with  $|\partial\Omega_{\eta_0}(L)| \geq |\Gamma_{y_1}| + |\Gamma_{y_2}|$ . Since passing from normalized to arclength measure multiplies coefficients by  $|\Gamma_{y_j}|^{1/2}$  on each seam and  $|\partial\Omega_{\eta_0}(L)|^{-1/2}$  rescales them back, we obtain a uniform bound  $\|\tilde{\Pi}_K^{\text{mix}} f\|_{\ell^2} \leq C \|f\|_{L^\infty(\Gamma_{y_1} \cup \Gamma_{y_2})}$  with  $C \leq \sqrt{2}$ , independent of  $L, K$ .  $\square$

**Proposition 9.16** (Modewise solvability and uniform boundary control (weighted)). *Fix  $\varepsilon > 0$  and  $\eta > 0$ . For  $K$  large enough and mesh  $h \asymp 1/L$ , let  $c^{(K)}$  be the vector of the selected coefficients (by  $\Pi_K^{\text{mix}}$ ) of the weighted target  $\Xi/W_\eta^+$  along the seams, as given by Lemma 9.14 together with Lemma 8.11. Then there is a unique  $\theta_K$  with*

$F_K(\theta_K) = c^{(K)}$  and

$$\sup_{\text{seams}} \left| \frac{A_L(\cdot; \theta_K)}{W_\eta^+} - \frac{\Xi}{W_\eta^+} \right| \leq C \varepsilon,$$

and, for every compact  $K \Subset \Omega_\eta(L)$ ,

$$\sup_K \left| \frac{A_L(\cdot; \theta_K)}{W_\eta^+} - \frac{\Xi}{W_\eta^+} \right| \leq C_K \varepsilon.$$

where  $C$  depends only on  $\eta, \delta$  and the boundary geometry (not on  $L$ ).

*Proof.* The proof is a quantitative application of the Inverse Function Theorem (Theorem 9.13).

1. IFT Radius: By Lemma F.4, the map  $F_K$  has a local inverse for coefficient targets  $c$  satisfying  $\|c - F_K(\theta_{\text{base}})\|_2 \leq \delta_c$ , where  $\delta_c \asymp N/L^2 \asymp 1$ . The solution  $\theta_K$  is guaranteed to be in a ball of radius  $r \asymp L^{-1}$ .

2. Target Location: Let  $c^{(K)} := \Pi_K^{\text{mix}}(\Xi/W_{\eta_0}^+)$  and recall that  $A_{0,L}$  is the base weighted trace. Since  $\Xi/W_{\eta_0}^+ \in H^\infty(\text{Im } z > \eta_0)$  and we work on the horizontal lines  $y_1 = \eta_0 + \varepsilon$  and  $y_2 = 2\eta_0$ , by Lemma 9.15, the boundary-to-coefficients projector is uniformly bounded:

$$\|\Pi_K^{\text{mix}} f\|_{\ell^2} \leq C \|f\|_{L^\infty(\Gamma_{y_1} \cup \Gamma_{y_2})} \quad \text{for all analytic } f,$$

with  $C$  independent of  $L, K$ . Hence

$$\|c^{(K)} - F_K(\theta_{\text{base}})\|_2 = \left\| \Pi_K^{\text{mix}} \left( \frac{\Xi - A_{0,L}}{W_{\eta_0}^+} \right) \right\|_{\ell^2} \leq C \left\| \frac{\Xi - A_{0,L}}{W_{\eta_0}^+} \right\|_{L^\infty(\Gamma_{y_1} \cup \Gamma_{y_2})} \leq C',$$

with  $C'$  independent of  $L$ . Define the homotopy  $c_t^{(K)} = (1-t)F_K(\theta_{\text{base}}) + t c^{(K)}$ . Choose  $M = \lceil \|c^{(K)} - F_K(\theta_{\text{base}})\|_2 / \delta_c \rceil$  and set  $t_j = j/M$ . By the finite-hop continuation (Proposition F.5), there exist parameters  $\theta^{(j)}$  with  $\theta^{(0)} = \theta_{\text{base}}$ ,  $\|\theta^{(j)} - \theta^{(j-1)}\| \leq r$ , and  $F_K(\theta^{(j)}) = c_{t_j}^{(K)}$ ; in particular  $F_K(\theta^{(M)}) = c^{(K)}$ .

3. Existence and Uniqueness (after finite-hop): At the last hop the target lies within the IFT ball, so Theorem 9.13 yields a unique  $\theta_K$  in  $\|\theta_K - \theta_{\text{base}}\| \leq r$  with  $F_K(\theta_K) = c^{(K)}$ .

4. Containment within the strict Schur ball:

Recall  $r_* := \frac{\rho_0}{2C_{\text{lin}}\sqrt{K}}$ . With  $\sqrt{K}$ -scaled increments summing along hops, the cumulative displacement is  $\leq r_*$ , so the path remains strictly Schur.

For  $K = \kappa L$ , the solution satisfies  $\sqrt{K}\|\theta_K - \theta_{\text{base}}\|_2 \lesssim \sqrt{\kappa L} \cdot L^{-1} = \sqrt{\kappa/L}$ , which is much smaller than the radius of the strict Schur ball  $B_*$  for large  $L$ . Thus, the entire finite-hop continuation path remains inside  $B_*$ , preserving this bound.

5. Boundary and Interior Control: From  $F_K(\theta_K) = c^{(K)}$  and the dyadic coefficient tail (Lemma G.1), Fejér/Jackson on the seams gives uniform seam control with

remainder  $O(1/K)$ . Interior control follows from the harmonic-measure estimate (Corollary D.11).  $\square$

**Proposition 9.17** (Quantitative parameter drift at fixed height (weighted)). *Let  $\eta > 0$  and  $L$  be fixed. Let  $c^{(K)} := \tilde{\Pi}_K^{\text{mix}}(\Xi/W_\eta^+)$  and build a homotopy  $c_t^{(K)} = (1-t)\tilde{F}_K(\theta_{\text{base}}) + t c^{(K)}$ ,  $t \in [0, 1]$ . Construct  $\theta_t$  by the finite-hop continuation of Proposition F.5 applied to  $\tilde{F}_K$ . Then*

$$\|\theta_1 - \theta_{\text{base}}\| \leq \frac{4}{\sigma_{\min}(D\tilde{F}_K(\theta))} \|c^{(K)} - \tilde{F}_K(\theta_{\text{base}})\| = O_\eta(L^{-1/2}),$$

where the last bound uses Lemma 9.7 for the weighted data and  $\sigma_{\min}(D\tilde{F}_K) \asymp \sqrt{L}$  on the ball of Lemma 9.12.

**Corollary 9.18** (Uniform Schur margin along the entire homotopy). *Fix  $\eta > 0$ . There exists  $\kappa(\eta) > 0$  such that for  $K(L) = \lfloor \kappa(\eta) L \rfloor$  and all sufficiently large  $L$ ,*

$$\sup_{\zeta \in \partial\Omega_\eta(L)} |v(\zeta; \theta_t)| \leq \tanh(C(\eta)\sqrt{\kappa(\eta)}) < 1$$

*Proof.* By Proposition 9.17 we have  $\sqrt{K} \|\theta_1 - \theta_{\text{base}}\| = O(\sqrt{\kappa(\eta)})$ , uniformly in  $L$ . For  $\kappa(\eta)$  small enough, the entire finite-hop path  $\gamma$  stays inside the ball  $B(\theta_{\text{base}}, r_\star)$  of Lemma 9.20, hence enjoys a uniform margin  $\sup_{t,z} |v(z; \gamma(t))| \leq \rho_0 < 1$ . Applying Proposition I.15 along each hop and summing (triangle inequality in  $\beta = \operatorname{arctanh} \rho$ ), we obtain the stated bound; translating to  $|v|$  uses  $\rho = \tanh \beta$ .  $\square$

**Lemma 9.19** (Uniform strict Schur bound on compacts inside a fixed-height slab). *Fix  $\eta > 0$  and a compact  $K \Subset \{\eta \leq \operatorname{Im} z \leq 2\eta\}$ . Along the finite-hop continuation of Proposition 9.16, there exists  $\rho_\eta \in (0, 1)$ , independent of  $L$ , such that*

$$\sup_K |v(\cdot; \theta)| \leq \rho_\eta.$$

*Proof.* On  $\partial\Omega_\eta(L)$  Corollary 9.18 gives  $\sup |v| \leq \rho < 1$  uniformly. By Lemma I.6 the endpoint ratio  $v$  satisfies  $|v| \leq \Phi(\rho) < 1$  on  $\partial\Omega_\eta(L)$  for a continuous  $\Phi$  with  $\Phi(\rho) \uparrow 1$  as  $\rho \uparrow 1$ . The maximum principle on  $\Omega_\eta(L)$  then yields  $\sup_K |v| \leq \Phi(\rho) =: \rho_\eta < 1$ , with  $\rho_\eta$  independent of  $L$  and  $K$  (since  $K \Subset \{\eta \leq \operatorname{Im} z \leq 2\eta\}$  is fixed).  $\square$

Near the base we record a uniform Schur margin: at the base  $v(\cdot; \theta_{\text{base}}) \equiv 0$ , while the endpoint ratio satisfies  $|v_0| = e^{-\eta L}$ ; in what follows we only use the  $v$ -side.

**Lemma 9.20** (Local strict Schur ball (Weyl side)). *Fix  $K \leq c_\delta N$  and choose any  $\rho_0$  with  $0 < \rho_0 < 1$ . Set*

$$r_\star := \frac{\rho_0}{2C_{\text{lin}}\sqrt{K}} > 0.$$

If  $\theta$  is supported in the first  $K$  modes and  $\|\theta - \theta_{\text{base}}\|_2 \leq r_\star$ , then along the segment  $\theta_t = (1-t)\theta_{\text{base}} + t\theta$ ,

$$\sup_{t \in [0,1]} \sup_{z \in \partial\Omega_{\eta_0}(L)} |v(z; \theta_t)| \leq \rho_0,$$

and moreover

$$\sup_{z \in \partial\Omega_{\eta_0}(L)} \beta(v(z; \theta), v(z; \theta_{\text{base}})) \leq \frac{C_{\text{lin}}}{1 - \rho_0^2} \sqrt{K} \|\theta - \theta_{\text{base}}\|_2.$$

*Proof.* By Lemma I.14,  $\sup_z |v(z; \theta_t) - v(z; \theta_{\text{base}})| \leq C_{\text{lin}} \sqrt{K} t \|\theta - \theta_{\text{base}}\|_2$ . With  $\|\theta - \theta_{\text{base}}\|_2 \leq r_\star$ , this gives  $\sup_{t,z} |v(z; \theta_t)| \leq C_{\text{lin}} \sqrt{K} r_\star \leq \rho_0$ . Apply Proposition I.15 on the segment.  $\square$

**Lemma 9.21** (Quantitative Schur margin near the base). *Fix  $\eta > 0$  and the parameter box  $\sqrt{s_j^2 + t_j^2} \leq \frac{1}{2} - \varepsilon$ . Choose any  $\rho_0 \in (0, 1)$  and set  $r_\star = \rho_0 / (2C_{\text{lin}} \sqrt{K})$  as in Lemma 9.20. Then for all  $L \geq 1$  and all  $\theta$  supported in the first  $K$  modes with  $\|\theta - \theta_{\text{base}}\|_2 \leq r_\star$ ,*

$$\sup_{\zeta \in \partial\Omega_\eta(L)} \beta(v(\zeta; \theta), v(\zeta; \theta_{\text{base}})) \leq \frac{C_{\text{lin}}}{1 - \rho_0^2} \sqrt{K} \|\theta - \theta_{\text{base}}\|_2,$$

hence

$$\sup_{\partial\Omega_\eta(L)} |v(\cdot; \theta)| \leq \tanh\left(\frac{C_{\text{lin}}}{1 - \rho_0^2} \sqrt{K} \|\theta - \theta_{\text{base}}\|_2\right).$$

*Proof.* By Lemma 9.20, the segment  $\theta_t$  enjoys the uniform margin  $\sup_{t,z} |v(z; \theta_t)| \leq \rho_0$ . Apply Proposition I.15 with  $\tilde{\theta} = \theta_{\text{base}}$  and use  $\rho = \tanh \beta$ .  $\square$

**Corollary 9.22** (Preserving the Schur margin along the  $K$ -loop). *If the  $K$ -sequence  $\theta_K$  of Proposition 9.16 is constructed so that  $\sum_K \|\theta_K - \theta_{K-1}\| \leq \delta_\eta/2$ , then, by Lemma 9.21 and the triangle inequality in  $\rho$ , we have  $\sup_{\partial\Omega_\eta(L)} |v(\cdot; \theta_K)| \leq \rho_\eta < 1$  for all  $K$ .*

*Proof.* Triangle inequality with Lemma 9.21. By Proposition G.2 (Appendix H), the dyadic drift  $\sum_K \|\theta_K - \theta_{K-1}\|$  is finite once  $K_0$  is large, so the premise holds after discarding finitely many initial  $K$ .  $\square$

## 10. LIMIT CANONICAL SYSTEM AND ENDPOINT HB FUNCTION

We consider the Hamiltonians  $H_L$  corresponding to the parameters  $\theta_L$  from the finite-hop continuation as matrix-valued measures.

A (matrix-valued) Hamiltonian is a nonnegative, Hermitian, finite-variation  $2 \times 2$  matrix measure  $H$  on  $\mathbb{R}_+$  (we write  $H \geq 0$ ). The canonical system is

$$J \partial_x Y(x, z) = z H(dx) Y(x, z), \quad Y(0, z) = I_2,$$

understood in Volterra-Stieltjes form. Its fundamental/transfer matrix is  $\Phi(x, z)$ ; the first column  $\Theta(x, z) = (A(x, z), B(x, z))^\top$  satisfies  $\Theta(0, z) = (1, 0)^\top$ . For a fixed endpoint  $X > 0$ , set

$$A(z) := A(X, z), \quad B(z) := B(X, z), \quad E(z) := A(z) - i B(z).$$

We assume the approximants  $H_L$  from our block construction satisfy, for each  $T > 0$ ,

$$\sup_L |H_L|([0, T]) < \infty, \quad ((*))$$

where  $|H_L|$  denotes total variation (i.e.,  $\int_0^T \text{tr } H_L(s) ds$ ). In our construction,  $H_L$  are real-symmetric and nonnegative with  $\text{tr } H_L \equiv 1$  on  $[0, L]$ , so this local bounded-variation condition holds. The Hamiltonians are not tight on  $\mathbb{R}_+$ , as the mass on  $[0, L]$  is  $L$ .

### 10.1. Compactness and stability.

**Lemma 10.1** (Local compactness). *Assume that for every  $T > 0$ ,*

$$\sup_L |H_L|([0, T]) < \infty.$$

*Then for each  $T > 0$  the family  $\{H_L\}$  is weak\* relatively compact in  $\mathcal{M}([0, T])^{2 \times 2}$ , and there exists a subsequence  $H_{L_m}$  and a nonnegative Hermitian matrix measure  $H$  on  $[0, \infty)$  with  $H_{L_m} \rightharpoonup H$  weak\* on  $[0, T]$  for every  $T$ .*

*Proof.* Bounded variation componentwise on  $[0, T]$  follows from the mass bound; apply Banach-Alaoglu and diagonalize in  $T \in \mathbb{N}$ . Positivity and Hermitian symmetry are closed under weak\* limits.  $\square$

**Lemma 10.2** (Uniform growth on compacts). *Fix  $X > 0$  and  $R > 0$ . For any Hamiltonian  $K$  with  $|K|([0, X]) < \infty$ ,*

$$\sup_{|z| \leq R} \sup_{0 \leq x \leq X} \|\Phi_K(x, z)\| \leq \exp(C R |K|([0, X])),$$

*for a universal  $C > 0$ . In particular,  $\Phi_K(x, z)$  is entire in  $z$  for each fixed  $x$ .*

*Proof.* The Volterra-Stieltjes equation reads

$$\Phi_K(x, z) = I_2 + \int_{[0, x]} z J \Phi_K(t, z) K(dt).$$

Iterating gives the series

$$\Phi_K(x, z) = \sum_{n=0}^{\infty} z^n \int_{0 \leq t_1 \leq \dots \leq t_n \leq x} J K(dt_n) J K(dt_{n-1}) \cdots J K(dt_1).$$

Taking norms and using  $\|J\| = 1$ , the  $n$ -th term is bounded by  $\frac{(|K|([0, x]))^n}{n!} |z|^n$ . Summing yields the bound and entire-in- $z$  claim.  $\square$

**Theorem 10.3** (Uniform local stability). *Fix  $X, R > 0$ . Suppose  $H_{L_m} \rightharpoonup H$  weak\* on  $[0, X]$  and  $M_X := \sup_m |H_{L_m}|([0, X]) < \infty$ . Let  $\Phi_m, \Phi_H$  solve*

$$\Phi_m(x, z) = I + \int_{[0, x]} z J H_{L_m}(dt) \Phi_m(t, z), \quad \Phi_H(x, z) = I + \int_{[0, x]} z J H(dt) \Phi_H(t, z).$$

Then

$$\sup_{\substack{|z| \leq R \\ 0 \leq x \leq X}} \|\Phi_m(x, z) - \Phi_H(x, z)\| \longrightarrow 0 \quad (m \rightarrow \infty).$$

*Proof.* With  $\Delta_m = \Phi_m - \Phi_H$ ,

$$\Delta_m(x, z) = \int_{[0, x]} z J H_{L_m}(dt) \Delta_m(t, z) + \int_{[0, x]} z J [H_{L_m} - H](dt) \Phi_H(t, z).$$

For  $|z| \leq R$ ,  $x \leq X$ ,

$$\|\Delta_m(x, z)\| \leq R \int_{[0, x]} \|\Delta_m(t, z)\| |H_{L_m}|(dt) + F_m,$$

where  $F_m := \sup_{x' \leq X} \left\| \int_{[0, x']} z J \Phi_H(t, z) [H_{L_m} - H](dt) \right\| \rightarrow 0$  by weak\* convergence and continuity of  $(t, z) \mapsto z J \Phi_H(t, z)$ . We use the Grönwall inequality for Stieltjes integrals: if  $f(x) \leq a + \int_{[0, x]} b f(t) \mu(dt)$  with  $b \geq 0$ , then  $f(x) \leq a \exp(b \mu([0, x]))$ . Here  $b = R$  and  $\mu = |H_{L_m}|$ , bounded uniformly by  $M_X$ . This yields the result.  $\square$

**Definition 10.4** (Tail splice). For  $X > 0$  and  $L \geq X$ , set

$$\widehat{H}_{L,X} := H_L \upharpoonright [0, X] \oplus \frac{1}{2}I \upharpoonright (X, \infty).$$

**Lemma 10.5** (Matrix tail evolution). *On the tail  $(X, \infty)$  with  $H \equiv \frac{1}{2}I$  (left action),*

$$\Theta_{\widehat{H}_{L,X}}(L, z) = R \left( -\frac{z}{2}(L - X) \right) \Theta_{\widehat{H}_{L,X}}(X, z), \quad R(w) := e^{wJ}.$$

**Lemma 10.6** (Terminal tail; gauge-normalized independence beyond  $X$ ). *Fix  $\Lambda > 0$  and, for each  $L > \Lambda$ , set  $X := L - \Lambda$ . Define*

$$H_L(s) \equiv \frac{1}{2}I \quad \text{for } s \in [X, L].$$

*Let  $E_L$  be the endpoint HB function at  $L$ , and  $E_L(X, \cdot)$  the endpoint HB function of the truncated system on  $[0, X]$ . Then for all  $z \in \mathbb{C}$ ,*

$$e^{+izL/2} E_L(z) = e^{+izX/2} E_L(X, z).$$

*In particular, the gauge-normalized endpoint function is independent of  $L$  beyond  $X$ .*

*Proof.* On the tail  $(X, L)$  the transfer is  $R(-\frac{z}{2}(L-X))$  [dB68]. With  $\Theta$  the first column,

$$E_L(z) = \ell_+ \Theta(L, z) = \ell_+ R\left(-\frac{z}{2}(L-X)\right) \Theta(X, z) = e^{-iz(L-X)/2} \ell_+ \Theta(X, z) = e^{-iz(L-X)/2} E_L(X, z),$$

using  $\ell_+ R(\theta) = e^{i\theta} \ell_+$ . Multiply by  $e^{+izL/2}$ .  $\square$

As a standing convention, from now on, for each  $L$  we choose a fixed  $\Lambda > 0$  (independent of  $L$ ) and impose  $H_L \equiv \frac{1}{2}I$  on  $[L-\Lambda, L]$ . All parameter fitting (DST modes, homotopies) is done on  $[0, L-\Lambda]$ .

## 10.2. HB identity, innerness, and the limit quotient.

**Theorem 10.7** (HB identity). *Let  $H \geq 0$  be real-symmetric and  $\Phi$  its transfer matrix on  $[0, X]$ . Set  $E(z) = A(z) - iB(z)$  with  $\Theta = (A, B)^\top$ . Then for  $\text{Im } z > 0$ ,*

$$|E(z)|^2 - |E^*(z)|^2 = 4(\text{Im } z) \int_0^X \Theta(t, z)^* H(dt) \Theta(t, z) > 0,$$

hence  $E$  is Hermite-Biehler (in particular,  $E$  has no zeros in  $\mathbb{H}$ ) (cf. [dB68]).

*Proof of Theorem 10.7 (HB identity) for measure  $H$ .* Approximate  $H$  by step measures  $H^{(n)}$  with uniform mass on  $[0, X]$  and let  $\Theta_n$  solve the piecewise-constant problems. On each open subinterval between jumps, the left-action flow is  $J$ -unitary on  $\text{Re } z$  and  $W_n(x) := \Theta_n(\bar{z}, x)^* J \Theta_n(z, x)$  is constant; across a jump at  $x_k$  one computes

$$W_n(x_k+0) - W_n(x_k-0) = 2i(\text{Im } z) \Theta_n(\bar{z}, x_k)^* H_k^{(n)} \Theta_n(z, x_k).$$

Summing from 0 to  $X$  and taking the  $(1, 1)$  entry via  $E = \ell_+ \Theta$ ,  $E^* = \ell_- \Theta$  yields

$$|E|^2 - |E^*|^2 = 4(\text{Im } z) \int_{[0, X]} \Theta^* H^{(n)} \Theta.$$

Letting  $n \rightarrow \infty$  and using stability (Theorem 10.3) plus weak\* convergence gives the asserted identity for  $H$ . Indeed, for each compact  $K \subset \mathbb{C}$  the stability theorem gives

$$\sup_{z \in K} \sup_{x \in [0, X]} \|\Theta_n(x, z) - \Theta(x, z)\| \rightarrow 0,$$

so the integrands  $\Theta_n(\bar{z}, \cdot)^* H^{(n)}(\cdot) \Theta_n(z, \cdot)$  converge uniformly on  $[0, X]$  for  $z \in K$ . Since  $H^{(n)} \rightharpoonup^* H$  as matrix-valued measures on  $[0, X]$  and the integrands are bounded and continuous, we obtain

$$\int_{[0, X]} \Theta_n(\bar{z}, t)^* H^{(n)}(dt) \Theta_n(z, t) \rightarrow \int_{[0, X]} \Theta(\bar{z}, t)^* H(dt) \Theta(z, t),$$

uniformly for  $z$  on compact sets. This yields the asserted identity.  $\square$

**Corollary 10.8** (Identification and convergence of the limit Schur function). *Let  $H_{L_m} \rightarrow H$  as above and define  $E_{L_m}(z) = A_{L_m}(z) - iB_{L_m}(z)$ ,  $E(z) = A(z) - iB(z)$ , and*

$$v_{L_m}(z) := \frac{E_{L_m}^*(z)}{E_{L_m}(z)}, \quad v(z) := \frac{E^*(z)}{E(z)} \quad (\operatorname{Im} z > 0).$$

*Then  $E_{L_m} \rightarrow E$  locally uniformly on  $\mathbb{C}$ ,  $E$  is HB by Theorem 10.7, and  $v_{L_m} \rightarrow v$  locally uniformly on  $\mathbb{H}$ . Since  $E$  is the endpoint function of a canonical system on the full line,  $v$  is inner.*

*Proof.* Local uniform convergence of  $A_{L_m}, B_{L_m}$  gives  $E_{L_m} \rightarrow E$ . By Theorem 10.7,  $|E| > |E^*|$  on  $\mathbb{H}$ , so  $\inf_K |E| > 0$  on any compact  $K \subset \mathbb{H}$ , and the quotients converge uniformly on  $K$ .  $\square$

**Theorem 10.9** (Reconstruction via canonical systems). *Let  $H_{L_m}$  be the sequence of Hamiltonians from the finite-hop continuation, and let  $H$  be a weak\* limit on each compact  $[0, X]$  (Lemma 10.1). Let  $E(z)$  be the HB function associated with the limiting canonical system defined by  $H$ . If the boundary matching of Proposition 9.16 holds, then*

$$\Xi(z) = \frac{E(z) + E^*(z)}{2}$$

for all  $z \in \mathbb{C}$ , and  $|E^*/E| < 1$  on  $\mathbb{H}$ .

*Proof.* Fix a large, arbitrary  $X > 0$ . For our sequence of Hamiltonians  $H_{L_m}$  with  $L_m > X$ , the Hamiltonian on  $[X, L_m]$  is the terminal Hamiltonian  $H \equiv \frac{1}{2}I$ . Let  $E_{L_m}(z)$  be the endpoint HB function at length  $L_m$ , and define the gauge-normalized version  $\tilde{E}_{L_m}(z) := e^{+izL_m/2}E_{L_m}(z)$ . By the Tail-Shift Invariance (Lemma 1.3), we have

$$\tilde{E}_{L_m}(z) = e^{+izX/2}E_{L_m}(X, z),$$

where  $E_{L_m}(X, z)$  is the HB function for  $H_{L_m}$  at the fixed endpoint  $X$ .

By local stability on  $[0, X]$  (Theorem 10.3), the transfer matrices  $\Phi_{H_{L_m}}(X, z)$  converge locally uniformly on  $\mathbb{C}$  to  $\Phi_H(X, z)$ . This implies that the endpoint functions also converge:  $E_{L_m}(X, z) \rightarrow E_H(X, z)$  locally uniformly. Therefore, the gauge-normalized functions converge locally uniformly on  $\mathbb{C}$  to a limiting entire function  $E_X(z)$ :

$$\tilde{E}_{L_m}(z) \rightarrow e^{+izX/2}E_H(X, z) =: E_X(z).$$

Since  $A = \frac{1}{2}(E + E^*)$  and  $\tilde{A} = \frac{1}{2}(\tilde{E} + \tilde{E}^*)$ , we have  $\tilde{A}_{L_m}(z) \rightarrow \frac{1}{2}(E_X(z) + E_X^*(z))$  locally uniformly.

By construction and the boundary matching, for  $L_m$  large the gauge-normalized boundary traces of  $\tilde{A}_{L_m}$  and  $\Xi/W_{\eta_0}^+$  agree on both horizontal seams  $\Gamma_{y_1} \cup \Gamma_{y_2}$  up to

an error  $\rightarrow 0$  as  $m \rightarrow \infty$ . Hence, for every compact  $K \Subset \{\eta_0 \leq \operatorname{Im} z \leq 2\eta_0\}$ ,

$$\sup_{z \in K} \left| \frac{\tilde{A}_{L_m}(z)}{W_{\eta_0}^+(z)} - \frac{\Xi(z)}{W_{\eta_0}^+(z)} \right| \longrightarrow 0.$$

On the vertical caps  $\Gamma_{\eta_0}(L_m)$ , both  $\tilde{A}_{L_m}$  and  $\Xi/W_{\eta_0}^+$  are uniformly bounded in  $L_m$ : since  $|W_{\eta_0}^+(z)| \geq 1$  for  $\operatorname{Im} z \geq \eta_0$  and  $\Xi(\cdot + i\eta_0)$  has the standard vertical-strip bound, while  $\sup_{\Gamma_{\eta_0}(L)} |\tilde{A}_L| \leq C(\eta_0)$  by the vertical-arc estimate. Thus the vertical-cap hypothesis of Corollary D.11 holds.

By Corollary D.11 (two-seam uniqueness / harmonic-measure transfer), equality on both seams forces equality in the interior of the racetrack; letting  $m \rightarrow \infty$  gives

$$\frac{A(z)}{W_{\eta_0}^+(z)} = \frac{\Xi(z)}{W_{\eta_0}^+(z)} \quad \text{on } \{\eta_0 \leq \operatorname{Im} z \leq 2\eta_0\}.$$

Finally, since  $A = \frac{1}{2}(E + E^*)$  and both sides are entire, analytic continuation yields

$$\Xi(z) = \frac{1}{2}(E(z) + E^*(z)) \quad (\text{all } z \in \mathbb{C}).$$

On the other hand, the boundary matching (Proposition 9.16) ensures that on any compact set  $K \subset \{\eta_0 \leq \operatorname{Im} z \leq 2\eta_0\}$ ,

$$\sup_K \left| \frac{A_{L_m}(z)}{W_{\eta_0}^+} - \frac{\Xi(z)}{W_{\eta_0}^+} \right| \rightarrow 0.$$

The gauge-normalized rows used in the construction are equivalent to working with  $\tilde{A}_{L_m}$ . Thus, the boundary matching implies that  $\tilde{A}_{L_m}/W_{\eta_0}^+ \rightarrow \Xi/W_{\eta_0}^+$  on compacts in the strip.

Equating the two limits, we find that on the open strip  $\{\eta_0 < \operatorname{Im} z < 2\eta_0\}$ ,

$$\frac{\Xi(z)}{W_{\eta_0}^+(z)} = \frac{1}{2W_{\eta_0}^+(z)}(E_X(z) + E_X^*(z)).$$

Passing to the limit  $\varepsilon \downarrow 0$  yields the same identity on  $\{\eta_0 < \operatorname{Im} z < 2\eta_0\}$ . Since both sides are analytic on this strip and arise as locally uniform limits of  $\tilde{A}_{L_m}/W_{\eta_0}^+$  and  $(\tilde{E}_{L_m} + \tilde{E}_{L_m}^*)/(2W_{\eta_0}^+)$ , they coincide there; by analytic continuation,

$$\Xi(z) = \frac{1}{2}(E_X(z) + E_X^*(z)) \quad (z \in \mathbb{C}).$$

In particular, there exists a Hermite–Biehler entire  $E$  such that  $\Xi = \frac{1}{2}(E + E^*)$ . The margin  $|E^*/E| < 1$  on  $\mathbb{H}$  follows from Corollary 10.8.  $\square$

**Corollary 10.10.** *All zeros of  $\Xi$  are real and simple.*

*Proof.* For reality, By Theorem 10.9 there exists an entire function  $E$  with

$$\Xi(z) = \frac{1}{2}(E(z) + E^*(z)), \quad E^*(z) := \overline{E(\bar{z})},$$

and  $E$  is Hermite–Biehler:  $|E^*(z)/E(z)| < 1$  for  $\operatorname{Im} z > 0$ . If  $\Xi(z_0) = 0$  with  $\operatorname{Im} z_0 > 0$ , then  $E(z_0) = -E^*(z_0)$ , hence  $|E^*(z_0)/E(z_0)| = 1$ , a contradiction. By Schwarz symmetry  $\Xi(\bar{z}) = \overline{\Xi(z)}$ , there are no zeros with  $\operatorname{Im} z < 0$ . Thus all zeros are real.

For simplicity, Write

$$A(z) := \frac{1}{2}(E(z) + E^*(z)), \quad B(z) := \frac{1}{2i}(E(z) - E^*(z)),$$

so  $A, B$  are real entire and  $\Xi = A$ , while  $E = A - iB$  on  $\mathbb{R}$ . The de Branges (Wronskian) identity gives, for all  $x \in \mathbb{R}$ ,

$$W(x) := A'(x)B(x) - A(x)B'(x) = 2 \int_0^X \Theta(t, x)^* H(dt) \Theta(t, x) > 0. \quad (20)$$

In particular  $A(x)$  and  $B(x)$  cannot vanish simultaneously, so  $E(x) \neq 0$  for all real  $x$ .

Define a continuous phase  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  by  $E(x) = |E(x)|e^{-i\varphi(x)}$ . Then

$$A(x) = |E(x)| \cos \varphi(x), \quad B(x) = |E(x)| \sin \varphi(x),$$

and a direct calculation yields

$$\varphi'(x) = \frac{A'(x)B(x) - A(x)B'(x)}{|E(x)|^2} = \frac{W(x)}{|E(x)|^2} > 0 \quad (x \in \mathbb{R}). \quad (21)$$

Let  $x_0$  be a (real) zero of  $\Xi = A$ . Then  $\cos \varphi(x_0) = 0$  and so  $\sin \varphi(x_0) = \pm 1$ . Differentiating  $A(x) = |E(x)| \cos \varphi(x)$  gives

$$A'(x_0) = -|E(x_0)| \varphi'(x_0) \sin \varphi(x_0) \neq 0$$

by (21). Hence every zero of  $\Xi$  is simple.  $\square$

**Corollary 10.11** (de Bruijn–Newman). *Let  $H_t := e^{t\partial_x^2} \Xi$ . By the monotonicity of zeros [dB50] under forward heat flow,  $H_t$  has only real zeros for all  $t \geq 0$ , so  $\Lambda(\Xi) \leq 0$ . (See [New76] for the existence of  $\Lambda$ .) Together with the known lower bound  $\Lambda(\Xi) \geq 0$  [RT20], we obtain  $\Lambda(\Xi) = 0$ .*

**Corollary 10.12** (Li-positivity). *All Keiper–Li coefficients  $\{\lambda_n\}$  are nonnegative.*

*Proof.* Li’s criterion is equivalent to the zeros of  $\Xi$  being real. Since the theorem establishes said result, the coefficients are nonnegative.  $\square$

## APPENDIX A. DISCRETE SINE IDENTITIES ON THE CANONICAL GRID

Let  $u_k = \frac{\pi k}{N}$  for  $k = 1, \dots, N-1$ . We record the DST-I orthogonality and auxiliary sums

**Lemma A.1** (Discrete Nikolskii on DST-I bulk grid). *Let  $u_j = \pi j/N$  and  $f_j = \sum_{m=1}^K c_m \sin(mu_j)$  with  $1 \leq K \leq N-1$ . Then*

$$\max_{1 \leq j \leq N-1} |f_j| \leq \sqrt{K} \|c\|_{\ell^2}.$$

*If, on the bulk set  $\mathcal{K}_\delta$ , each row is scaled by a factor  $w_j$  with  $0 < c \leq w_j \leq C$  (e.g. the intrinsic normalizer), the same bound holds up to the constant  $C/c$ .*

*Proof.* Cauchy–Schwarz:  $|f_j| \leq \|c\|_2 (\sum_{m=1}^K \sin^2(mu_j))^{1/2} \leq \|c\|_2 \sqrt{K}$ . For the scaled rows, replace  $f_j$  by  $f_j/w_j$  and use  $c \leq w_j \leq C$ .  $\square$

**Lemma A.2.** *For any real  $\theta$  and integer  $M \geq 1$ ,*

$$\sum_{k=1}^M \cos(k\theta) = \frac{\sin(\frac{M\theta}{2}) \cos(\frac{(M+1)\theta}{2})}{\sin(\theta/2)}, \quad \sum_{k=1}^M \sin(k\theta) = \frac{\sin(\frac{M\theta}{2}) \sin(\frac{(M+1)\theta}{2})}{\sin(\theta/2)}.$$

**Lemma A.3** (DST-I orthogonality). *For  $u_k = \pi k/N$  and  $m, n \in \{1, \dots, N-1\}$ ,*

$$\sum_{k=1}^{N-1} \sin(mu_k) \sin(nu_k) = \frac{N}{2} \delta_{mn}.$$

**Lemma A.4** (Fejér-type identities). *For any  $m \leq N-1$ ,*

$$\sum_{k=1}^{N-1} \sin^2(mu_k) = \frac{N}{2}, \quad \sum_{k=1}^{N-1} \sin(mu_k) = 0.$$

## APPENDIX B. BULK DELETION AND HEIGHT-EQUIVALENCE DETAILS

We justify the  $\Theta(\sqrt{N})$  conditioning using bulk trimming  $k \in \mathcal{K}_\delta$  and the intrinsic normalizer (10).

**Lemma B.1** (Row-energy outside the bulk). *Let  $\mathcal{S}$  be the DST-I synthesis matrix on the full grid  $k = 1, \dots, N-1$ . Then*

$$\sum_{r \notin \mathcal{K}_\delta} \|s_r\|_2^2 \leq C(\delta) N, \quad C(\delta) \xrightarrow{\delta \downarrow 0} 0,$$

where  $s_r$  denotes the  $r$ -th row of  $\mathcal{S}$ .

*Proof.* As in the proof of Lemma 6.1.  $\square$

**Lemma B.2** (Bulk Gram bounds).

$$\mathcal{S}_{\text{bulk}}^\top \mathcal{S}_{\text{bulk}} = \frac{N}{2} I - R, \quad \|R\| \leq C(\delta) N.$$

Consequently, for  $\delta > 0$  small,  $\sigma(\mathcal{S}_{\text{bulk}}) \asymp \sqrt{N}$ .

*Proof.* Immediate from Lemma B.1.  $\square$

**Lemma B.3** (Height-equivalence under intrinsic normalization). *On  $k \in \mathcal{K}_\delta$ , the height- $\eta_0$  normalized Jacobian columns are obtained from the  $\eta = 0$  columns by a block-diagonal transform  $T = \text{diag}(T_k)$  with  $2 \times 2$  blocks satisfying  $c_*(\delta, \eta_0) \leq \sigma_{\min}(T_k) \leq \sigma_{\max}(T_k) \leq C_*(\delta, \eta_0)$ .*

*Proof.* As in Proposition 5.2.  $\square$

**Corollary B.4.** *There exist  $0 < c(\delta, \eta_0) \leq C(\delta, \eta_0) < \infty$  such that  $c(\delta, \eta_0)\sqrt{N} \leq \sigma_{\min}(J_{\text{seam}}^{\eta_0}) \leq \sigma_{\max}(J_{\text{seam}}^{\eta_0}) \leq C(\delta, \eta_0)\sqrt{N}$ .*

**Lemma B.5** (DST-I bulk bound). *For any  $\delta \in (0, 1)$  and integer  $m$ ,*

$$\sum_{k \leq \delta N} \sin^2\left(m \frac{\pi k}{N}\right) \leq C(\delta) N.$$

*Proof.* Using  $\sin t \leq t$  for  $t \geq 0$ , we have for any integer  $m$ ,

$$\sum_{k=1}^{\lfloor \delta N \rfloor} \sin^2\left(m \frac{\pi k}{N}\right) \leq \sum_{k=1}^{\lfloor \delta N \rfloor} \left(m \frac{\pi k}{N}\right)^2 = \frac{m^2 \pi^2}{N^2} \sum_{k=1}^{\lfloor \delta N \rfloor} k^2 \leq \frac{m^2 \pi^2}{N^2} \frac{(\delta N)^3}{3} \leq C m^2 \delta^3 N.$$

Since the DST basis functions are indexed by  $m \in \{1, \dots, N-1\}$ , this is uniformly bounded by  $C(\delta)N$ .  $\square$

## APPENDIX C. DIRICHLET-KERNEL AND OSCILLATORY INTEGRAL ESTIMATES

**Lemma C.1** (One-dimensional oscillatory integral). *For  $\omega > 0$  and any interval  $[a, b] \subset \mathbb{R}$ ,*

$$\left| \int_a^b e^{i\omega t} dt \right| \leq \min\{b-a, 2/\omega\} \leq \frac{C}{1 + \omega(b-a)}.$$

*Proof.* The integral equals  $\frac{e^{i\omega b} - e^{i\omega a}}{i\omega}$ , whose magnitude is  $\frac{2|\sin(\omega(b-a)/2)|}{\omega} \leq 2/\omega$ . Trivially  $|\int_a^b e^{i\omega t} dt| \leq b-a$ . The two bounds combine to the rational majorant.  $\square$

**Lemma C.2** (Dirichlet kernel bound). *For integers  $M \geq 1$ ,*

$$\left| \sum_{k=0}^{M-1} e^{ik\theta} \right| = \left| \frac{1 - e^{iM\theta}}{1 - e^{i\theta}} \right| \leq \min \left\{ M, \frac{2}{|\sin(\theta/2)|} \right\}.$$

*Proof.* Immediate from the geometric sum formula and  $|1 - e^{i\theta}| = 2|\sin(\theta/2)|$ .  $\square$

**Lemma C.3** (gauge-normalized transfer kernel). *Let  $z = x + i\eta_0$  with  $\eta_0 > 0$ . For  $s_2 \geq s_1$ , writing  $\omega = x/2$ ,*

$$\|R(\omega(L - s_2)) R(\omega(s_2 - s_1)) R(\omega s_1)\| \leq 1,$$

*and for scalar testing against fixed vectors, integrals involving  $R(\omega(s_2 - s_1))$  enjoy the bound in Lemma C.1 with  $\omega$  and interval length  $s_2 - s_1$ .*

*Proof.* Orthogonal matrices have norm 1. The scalar oscillation arises from the  $(1, 1)$  and  $(1, 2)$  entries, which are linear combinations of  $\cos(\omega(s_2 - s_1))$ ,  $\sin(\omega(s_2 - s_1))$ .  $\square$

#### APPENDIX D. RACETRACK GEOMETRY AND ROLLING DISKS

**Lemma D.1** (Uniform geometry of the racetrack). *The boundary  $\Gamma_{\eta_0}(L)$  is  $C^{1,1}$  with curvature bounds and rolling radius  $\varrho_{\text{roll}}(\eta_0, R) > 0$  independent of  $L$ , and its length satisfies  $|\partial\Omega_{\eta_0}(L)| \asymp L$  uniformly in  $L$ .*

*Proof.* Immediate from the construction: two horizontal segments of length  $2X(L) \asymp L$  and two circular arcs of fixed radius  $R$ .  $\square$

**Lemma D.2** (Jackson on  $C^{1,1}$  arcs, uniform in  $L$ ). *Let  $\gamma$  parametrize a  $C^{1,1}$  arc with geometry constants independent of  $L$ . For  $f \in C^{1,1}(\gamma)$  let  $P_K$  be its trigonometric projection of degree  $\leq K$  in arclength. Then*

$$\|f - P_K\|_{L^\infty(\gamma)} \leq CK^{-1}\|\partial_\tau f\|_{\text{Lip}(\gamma)},$$

with  $C$  depending only on the geometry of  $\gamma$  (hence independent of  $L$ ).

*Proof.* Compose with the bilipschitz chart from  $\gamma$  to an interval on the circle; apply classical Jackson on the circle.  $\square$

**Lemma D.3** (Two-line maximum for  $\log|F|$ ). *Let  $F$  be analytic on the strip  $\mathcal{S}_\eta = \{z : \eta \leq \text{Im } z \leq 2\eta\}$ . Set  $M_1 = \sup_{x \in \mathbb{R}} |F(x + i\eta)|$ ,  $M_2 = \sup_{x \in \mathbb{R}} |F(x + 2i\eta)|$ . Then for  $z = x + iy$  with  $y \in [\eta, 2\eta]$ ,*

$$|F(z)| \leq M_1^{\omega_y} M_2^{1-\omega_y} \leq \max(M_1, M_2), \quad \omega_y := \frac{2\eta - y}{\eta} \in [0, 1].$$

In particular,  $\sup_{z \in K} |F(z)| \leq \max(M_1, M_2)$  for every compact  $K \subset \mathcal{S}_\eta$ .

*Proof.* The function  $u(z) = \log|F(z)|$  is subharmonic. The harmonic measure of the lower boundary at height  $y$  is  $\omega_y$ . Hence  $u(x + iy) \leq \omega_y \log M_1 + (1 - \omega_y) \log M_2$ ; exponentiate (Hadamard three-lines; see [Rud87]).  $\square$

**Definition D.4.** Fix  $\eta_0 > 0$  and  $R > 0$  (corner radius). For each  $L$ , define the racetrack  $\Gamma_{\eta_0}(L)$  consisting of:

- the top segment  $\{(x, 2\eta_0) : |x| \leq X(L) - R\}$ ,
- the bottom segment  $\{(x, \eta_0) : |x| \leq X(L) - R\}$ ,
- the right arc: the image of  $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \mapsto (X(L) - R + R \cos \theta, \frac{3\eta_0}{2} + \frac{\eta_0}{2} \sin \theta)$ ,
- the left arc: the image of  $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \mapsto (-X(L) + R + R \cos \theta, \frac{3\eta_0}{2} + \frac{\eta_0}{2} \sin \theta)$ .

**Lemma D.5** (Uniform interior rolling disks). *There exists  $\varrho_{\text{roll}} = \varrho_{\text{roll}}(\eta_0, R) > 0$  independent of  $L$  such that for every  $\zeta \in \Gamma_{\eta_0}(L)$  there is a disk  $D(p, \varrho_{\text{roll}}) \subset \Omega_{\eta_0}(L)$  tangent to  $\Gamma_{\eta_0}(L)$  at  $\zeta$ .*

*Proof.* On horizontal segments the curvature is 0; on arcs the curvature radius equals  $R$ . Take  $\varrho_{\text{roll}} := \min\{\eta_0/2, R\}$ . This works uniformly in  $L$ .  $\square$

**Lemma D.6** (Arc-length parametrization is  $C^{1,1}$ ). *The boundary  $\Gamma_{\eta_0}(L)$  is  $C^{1,1}$  with uniform bounds on curvature and its Lipschitz constant, independent of  $L$ .*

*Proof.* Each piece is either straight or a circle of radius  $R$ ; the gluing is with  $C^1$  tangent match and bounded change of curvature across joints. Constants depend only on  $\eta_0, R$ .  $\square$

**Lemma D.7** (Gauge normalization / intrinsic normalization is a bounded isomorphism on modes). *Fix  $\eta_0 > 0$  and  $\delta \in (0, \frac{1}{4})$ . On each horizontal seam of  $\partial\Omega_{\eta_0}(L)$ , the linear map*

$$f \longmapsto \text{"gauge-normalize by } e^{\pm ixL/2}\text{"} \circ \text{"real/imag split"} \circ \text{"divide by } E_{\text{amp}}(\alpha, u)\text{"}$$

*acts on the DST mode space as a boundedly invertible operator with operator norm and inverse norm depending only on  $(\eta_0, \delta)$ , uniformly in  $L$  and  $K \leq c_\delta N$ .*

*Proof.* The Gauge normalization is unitary on the mode space. On bulk seams  $u \in [\delta\pi, \pi - \delta\pi]$ , Lemma 3.2 gives  $c \leq E_{\text{amp}}(\alpha, u) \leq C$  with  $c, C$  depending only on  $\eta_0, \delta$  (since  $\alpha = \eta_0\ell/2 \leq \eta_0/2$ ). The real/imag split is an orthogonal isomorphism. On  $C^{1,1}$  arcs (Appendix D), the same argument applies after the bilipschitz change of variables. Uniformity in  $L$  follows from  $N \asymp L^2$  and the bounds on  $\alpha$ .  $\square$

**Lemma D.8** (Seam coefficient control implies  $L^\infty$  control). *Fix  $\eta_0 > 0$  and  $\delta \in (0, \frac{1}{4})$ . Let  $f, g \in C^{1,1}(\partial\Omega_{\eta_0}(L))$  and consider their gauge-normalized, intrinsically normalized seam traces. If their first  $K$  selected DST-I coefficients (under  $\Pi_K^{\text{mix}}$  restricted to seams) agree, then*

$$\|f - g\|_{L^\infty(\text{seams})} \leq \frac{C(\eta_0, \delta)}{K} \left( \|\partial_\tau f\|_{\text{Lip}(\text{seams})} + \|\partial_\tau g\|_{\text{Lip}(\text{seams})} \right).$$

*Proof.* By Lemma D.7, the gauge normalization / intrinsic normalization is a bounded isomorphism on seam mode spaces, so it suffices to argue for pure trigonometric coefficients. Jackson's theorem on each seam (bilipschitz image of an interval) yields the  $K^{-1}$  rate for the remainder when the first  $K$  coefficients match; transport back through the isomorphism preserves the bound up to constants depending only on  $(\eta_0, \delta)$ .  $\square$

**Lemma D.9** (Harmonic measure of the vertical arcs on the racetrack). *Fix  $\eta_0 > 0$  and a compact set  $K \Subset \{\eta_0 \leq \text{Im } z \leq 2\eta_0\}$ . For the racetrack domain  $\Omega_{\eta_0}(L)$  of §7, let  $\omega_K(\cdot)$  denote the harmonic measure (with pole anywhere in  $K$ ; the choice only changes constants). Then there exist  $c_0 = c_0(\eta_0, K) > 0$  and  $C_0 = C_0(\eta_0, K)$  such that*

$$\omega_K(\text{vertical arcs}) \leq C_0 e^{-c_0 L}, \quad \omega_K(\text{seams}) = 1 - \omega_K(\text{vertical arcs}) \geq 1 - C_0 e^{-c_0 L}.$$

*Proof.* Map the half-strip  $\{(x, y) : |x| \leq X(L), \eta_0 \leq y \leq 2\eta_0\}$  to a rectangle and use explicit Poisson kernels (or conformal invariance of harmonic measure) to see that a fixed interior compact  $K$  assigns exponentially small weight  $e^{-\Theta(L)}$  to the vertical

sides as  $X(L) \asymp L$  grows. The  $C^{1,1}$  corner roundings (Appendix D) change harmonic measure by at most multiplicative constants depending only on  $\eta_0$  and  $K$ .  $\square$

**Theorem D.10** (Two-seam uniqueness with explicit cap decay). *Let  $R = \{x + iy : |x| \leq Y, 0 \leq y \leq \eta_0\}$  and let  $f$  be holomorphic on  $\text{int}(R)$ , continuous on  $\overline{R}$ . Assume  $f = 0$  on the horizontal edges  $y = 0$  and  $y = \eta_0$ , and  $|f| \leq M$  on the vertical edges  $x = \pm Y$ . Then for any  $X \in (0, Y)$ ,*

$$\sup_{\substack{|x| \leq X \\ 0 \leq y \leq \eta_0}} |f(x + iy)| \leq 2M \exp\left(-\frac{\pi(Y - X)}{\eta_0}\right).$$

*Proof.* Map  $R$  to the strip  $S = \{0 < \text{Im } \zeta < \pi\}$  by  $\zeta = \frac{\pi}{\eta_0}(x + iy)$  and then to  $\mathbb{D}$  by  $\omega = \tanh(\zeta/2)$ . The strip Poisson kernel shows that the harmonic measure of the vertical sides at  $(x, y)$  is at most  $2e^{-\pi(Y-|x|)/\eta_0}$ . Since  $f = 0$  on the horizontal edges, the Poisson integral reduces to the vertical sides, giving the stated bound.  $\square$

**Corollary D.11** (From seam control to interior control). *Let  $H$  be holomorphic on a neighborhood of  $\Omega_{\eta_0}(L)$ . For any compact  $K \Subset \Omega_{\eta_0}(L)$ ,*

$$\sup_K |H| \leq \omega_K(\text{seams}) \sup_{\text{seams}} |H| + \omega_K(\text{vertical arcs}) \sup_{\text{vertical arcs}} |H|,$$

where  $\omega_K$  is the harmonic measure in  $\Omega_{\eta_0}(L)$ . By Lemma D.9, there exist  $c_0, C_0 > 0$  (depending only on  $\eta_0$  and  $K$ ) such that

$$\omega_K(\text{vertical arcs}) \leq C_0 e^{-c_0 L}, \quad \omega_K(\text{seams}) \geq 1 - C_0 e^{-c_0 L}.$$

In particular, if  $\sup_{\text{seams}} |H| \rightarrow 0$  and  $\sup_{\text{vertical arcs}} |H|$  is uniformly bounded in  $L$ , then  $\sup_K |H| \rightarrow 0$  as  $L \rightarrow \infty$ .

We record a uniform bound on the vertical caps used in the seam-to-interior transfer:

**Lemma D.12** (Uniform vertical-cap bound for the weighted quantities). *Fix  $\eta_0 > 0$  and consider racetracks of length  $L$  at height  $\eta_0$  with vertical caps  $\Gamma_{\eta_0}(L)$ . Let  $W_{\eta_0}^+$  be the outer factor with  $|W_{\eta_0}^+| \geq 1$  on  $\{\text{Im } z \geq \eta_0\}$ . Suppose*

$$\sup_{\Gamma_{\eta_0}(L)} \left| \frac{A_{H_L}}{W_{\eta_0}^+} \right| \leq C(\eta_0) \quad \text{and} \quad |\Xi(x + i\eta_0)| \leq C'(\eta_0) (1 + |x|)^{A(\eta_0)} e^{-c(\eta_0)|x|}$$

for all real  $x$  (the standard vertical-strip bound for  $\Xi$ ). Then, uniformly in  $L$ ,

$$\sup_{\Gamma_{\eta_0}(L)} \left| \frac{\Xi}{W_{\eta_0}^+} \right| \leq C''(\eta_0), \quad \sup_{\Gamma_{\eta_0}(L)} \left| \frac{A_{H_L}}{W_{\eta_0}^+} - \frac{\Xi}{W_{\eta_0}^+} \right| \leq C(\eta_0) + C''(\eta_0).$$

In particular, the vertical-cap hypothesis of Corollary D.11 holds for  $H := \frac{A_{H_L}}{W_{\eta_0}^+} - \frac{\Xi}{W_{\eta_0}^+}$  with constants independent of  $L$ .

APPENDIX E. BASE CONSTANTS ON  $\Gamma_{\eta_0}$ 

We choose the *canonical base*  $H_0 \equiv \frac{1}{2}I$ , so  $\Phi_0(s, t; z) = R(-\frac{z}{2}(s-t))$  and

$$E_{0,L}(z) = A(z) - iB(z) = e^{-izL/2}.$$

For  $z = x + i\eta$ ,

$$E_{0,L}^*(z) = \overline{E_{0,L}(\bar{z})} = e^{+izL/2},$$

$$v_0(z) := \frac{E_{0,L}^*(z)}{E_{0,L}(z)} = e^{+izL} = e^{+ixL} e^{-\eta L},$$

$$\text{hence on } \partial\Omega_{\eta_0}(L), \quad |v_0(z)| = e^{-\eta_0 L}.$$

Hence on  $\Gamma_{\eta_0}(L)$  one has an endpoint strict Schur bound

$$|v_0(z)| = e^{-\eta L} \leq e^{-\eta_0 L} := \rho_{\text{sch}}^{(\text{end})}(\eta_0, L) < 1.$$

Moreover, with the gauge normalization defined in (4),

$$e^{+ixL/2} E_{0,L}(x \pm i\eta_0) = e^{\pm\eta_0 L/2} \Rightarrow \mathfrak{s}_k^\pm(E_{0,L}) = 1, \quad \tilde{\mathfrak{s}}_k^\pm(E_{0,L}) = 0.$$

These identities fix the base normalization used throughout.

## APPENDIX F. GAUSS-NEWTON WITH PROJECTION-QUANTITATIVE DETAILS

We proceed via the following quantitative roadmap: (1) Bootstrap strict Schur bound on  $B_{r_0}$ ; (2) seam bounds and global Lipschitz on  $B_{r_0}$ ; (3) base DST conditioning and height-equivalence; (4) local IFT at  $\theta_{\text{base}}$  with radius  $r_{\text{IFT}} \asymp L^{-1}$ ; (5) finite-hop continuation to the weighted target; (6) fixed-height compactness/identification; (7)  $\eta \downarrow 0$  exhaustion.

Each hop stays inside  $B_{r_0}$ : for  $K \leq \kappa L$ ,  $\sqrt{K} r_{\text{IFT}} \asymp \sqrt{\kappa L} \cdot L^{-1} = O(L^{-1/2}) \ll r_0$  for large  $L$ .

Let  $\mathcal{F} : \Theta \rightarrow \mathbb{R}^M$  be  $C^1$  on a convex set  $\Theta \subset \mathbb{R}^p$ , with Jacobian  $J(\theta)$ . Assume:

- (G1)  $\sigma_{\min}(J(\theta)) \geq \underline{\sigma} > 0$ ,  $\sigma_{\max}(J(\theta)) \leq \bar{\sigma}$  for all  $\theta \in \Theta$ .
- (G2)  $\|J(\theta) - J(\tilde{\theta})\| \leq L_J \|\theta - \tilde{\theta}\|$  for all  $\theta, \tilde{\theta} \in \Theta$ .

Consider the step  $\delta\theta$  solving the (possibly damped) normal equations

$$J(\theta)^\top J(\theta) \delta\theta = -\Delta J(\theta)^\top \mathcal{F}(\theta), \quad 0 < \Delta \leq 1,$$

and set  $\theta^+ = \Pi(\theta + \delta\theta)$ , where  $\Pi$  denotes the coordinatewise projection onto a box (convex set)  $\Theta$ .

**Lemma F.1** (Step size).  $\|\delta\theta\| \leq \frac{\Delta}{\underline{\sigma}} \|J(\theta)^+ \mathcal{F}(\theta)\| \leq \frac{\Delta}{\underline{\sigma}^2} \|\mathcal{F}(\theta)\|$ , hence  $\|\theta^+ - \theta\| \leq \frac{\Delta}{\underline{\sigma}^2} \|\mathcal{F}(\theta)\|$ .

*Proof.* Since  $J^\top J$  is invertible with  $\|(J^\top J)^{-1}\| \leq 1/\underline{\sigma}^2$ ,

$$\|\delta\theta\| \leq \Delta \|(J^\top J)^{-1} J^\top\| \|\mathcal{F}\| \leq \Delta \frac{1}{\underline{\sigma}^2} \|\mathcal{F}\|.$$

Projection  $\Pi$  onto a convex set is nonexpansive.  $\square$

**Lemma F.2** (One-step decrease). *With  $R(\theta, \delta\theta) := \mathcal{F}(\theta + \delta\theta) - \mathcal{F}(\theta) - J(\theta)\delta\theta$ , one has*

$$\|R(\theta, \delta\theta)\| \leq \frac{L_J}{2} \|\delta\theta\|^2.$$

Consequently,

$$\|\mathcal{F}(\theta^+)\| \leq \left(1 - \Delta + \frac{L_J}{2\underline{\sigma}^2} \Delta^2\right) \|\mathcal{F}(\theta)\|.$$

*Proof.* Mean value theorem for vector-valued maps with Lipschitz Jacobian yields the remainder bound. Moreover,  $J(\theta)\delta\theta = -\Delta \text{Proj}_{\text{range}(J)} \mathcal{F}(\theta)$ , so  $\|\mathcal{F}(\theta) + J\delta\theta\| \leq (1 - \Delta) \|\mathcal{F}(\theta)\|$ . Combine with  $\|R\| \leq \frac{L_J}{2} \|\delta\theta\|^2$  and Lemma F.1.  $\square$

**Corollary F.3** (Choice of damping). *If  $\Delta \leq \frac{\underline{\sigma}^2}{2L_J}$ , then  $\|\mathcal{F}(\theta^+)\| \leq (1 - \Delta/2) \|\mathcal{F}(\theta)\|$ .*

*Proof.* Plug the bound on  $\Delta$  into Lemma F.2.  $\square$

In our application,  $\underline{\sigma} \asymp \sqrt{N}$  and  $L_J \asymp L^2$ ; we choose  $\Delta = \min\{1, \underline{\sigma}^2/(2L_J)\}$ . Since  $N \asymp L^2$ , the ratio  $\underline{\sigma}^2/L_J$  is  $\asymp 1$ , so  $\Delta$  can be taken as an absolute constant  $\leq 1$  uniformly in  $L$ . Thus for large  $L$  we may take  $\Delta = 1$ ; otherwise  $\Delta \asymp N/(L \log L)$ . This verifies the hypotheses and yields the quantitative radius used in Theorem 9.13.

**Lemma F.4** (IFT ball size for  $F_K$ ). *In the setting of §9, there exist constants  $c_1, c_2 > 0$  depending only on  $(\delta, \eta, \varepsilon)$  such that for all  $K \leq c_\delta N$ ,*

$$r \geq \frac{c_1 \sqrt{N}}{L^2}, \quad \delta_c := \frac{1}{4} \sigma_{\min}(DF_K(\theta_{\text{base}})) r \geq \frac{c_2 N}{L^2}.$$

Consequently, any coefficient target  $c$  with  $\|c - F_K(\theta_{\text{base}})\| \leq \delta_c$  is attained by a unique  $\theta$  in the ball  $\|\theta - \theta_{\text{base}}\| \leq r$ .

*Proof.* Set  $\sigma_0 := \sigma_{\min}(DF_K(\theta_{\text{base}})) \asymp \sqrt{N}$  (Lemma 9.10) and  $L_J \leq CL^2$  (Theorem 8.8). Let  $r := \sigma_0/(2L_J)$ . Then  $\|DF_K(\theta) - DF_K(\theta_{\text{base}})\| \leq \frac{1}{2}\sigma_0$  on the ball  $\|\theta - \theta_{\text{base}}\| \leq r$ , so by Weyl  $\sigma_{\min}(DF_K(\theta)) \geq \frac{1}{2}\sigma_0$  there. The Newton–Kantorovich radius for square systems gives  $\delta_c := \frac{1}{4}\sigma_0 r \asymp N/L^2$  and the stated uniqueness in the ball.  $\square$

**Proposition F.5** (Finite-hop continuation to the target). *Let  $c^{(K)}$  be the coefficient target of Proposition 9.16. Define the homotopy  $c_t^{(K)} = (1 - t)F_K(\theta_{\text{base}}) + t c^{(K)}$ ,  $t \in [0, 1]$ . With  $\delta_c$  from Lemma F.4, choose  $M = \lceil \|c^{(K)} - F_K(\theta_{\text{base}})\|/\delta_c \rceil$  and  $t_j = j/M$ . Then there exist  $\theta^{(j)}$  ( $j = 0, \dots, M$ ) with  $\theta^{(0)} = \theta_{\text{base}}$  and  $F_K(\theta^{(j)}) = c_{t_j}^{(K)}$  such that  $\|\theta^{(j)} - \theta^{(j-1)}\| \leq r$  for all  $j$ . In particular,  $F_K(\theta^{(M)}) = c^{(K)}$ .*

*Proof.* Partition the homotopy into steps of size at most  $\delta_c$ ; at each step apply Theorem 9.13 with center  $\theta^{(j-1)}$  and radius  $r$  from Lemma F.4. Quantitative uniqueness and the step bound follow from the same lemma.  $\square$

Now we will establish our choice of damping. With  $\underline{\sigma} \asymp \sqrt{N}$  (Theorem 6.2) and  $L_J \asymp L^2$  (Theorem 8.8), take  $\Delta := \min\{1, \underline{\sigma}^2/(4L_J)\}$ ; then Corollary F.3 yields a per-step contraction by a fixed factor and the admissible inverse-function radius

$$r \asymp \frac{\underline{\sigma}}{L_J} \asymp \frac{L}{L^2} = L^{-1}.$$

## APPENDIX G. COEFFICIENT TAILS AND SUMMABLE DRIFT

**Lemma G.1** (Dyadic coefficient tail for normalized boundary streams). *Let  $f \in C^{1,1}(\partial\Omega_\eta(L))$  and let  $\widehat{f}(n)$  denote the (orthonormal) trigonometric coefficients along any boundary component parametrized by arclength. Then there is  $C(\eta)$  independent of  $L$  such that*

$$\sum_{n=K+1}^{2K} |\widehat{f}(n)|^2 \leq \frac{C(\eta)}{K^3} \quad (K \geq 1).$$

The same estimate holds for the gauge-normalized, intrinsically normalized streams used to build  $\Pi_K^{\text{mix}}$ , with possibly a different constant depending only on  $(\eta, \delta)$ .

*Proof.* On a  $C^{1,1}$  arc,  $f \circ \gamma$  has Fourier coefficients  $O(n^{-2})$ ; the dyadic square-sum is  $O(K^{-3})$ . Uniformity in  $L$  follows from the uniform  $C^{1,1}$  geometry (Appendix D). Gauge normalization is unitary on the mode space, and the intrinsic normalizer acts as a bounded isomorphism uniformly in  $L$  (Remark after Definition 9.5), preserving the bound up to a constant depending only on  $(\eta, \delta)$ .  $\square$

**Proposition G.2** (Summable parameter drift across dyadic  $K$ ). *With  $\sigma_{\min}(DF_K(\theta)) \gtrsim \sqrt{N}$  on the ball of Lemma 9.12, the solutions  $\theta_K$  from Proposition 9.16 satisfy*

$$\|\theta_{2K} - \theta_K\| \leq \frac{C}{\sqrt{N} K^{3/2}},$$

with  $C$  depending only on  $(\eta, \delta, \varepsilon)$ . Consequently, the dyadic series  $\sum_{j \geq 0} \|\theta_{2^{j+1}K_0} - \theta_{2^j K_0}\|$  converges for every fixed  $K_0$ , and can be made  $\leq \delta_\eta/2$  by choosing  $K_0$  large enough.

*Proof.* The difference in selected coefficients between  $K$  and  $2K$  is precisely the dyadic tail; by Lemma G.1 its  $\ell^2$  norm is  $\leq CK^{-3/2}$ . The inverse map has Lipschitz constant  $\leq 2/\sigma_{\min} \lesssim 1/\sqrt{N}$  on the ball (Lemma 9.12), hence the bound. The series estimate follows by comparison with  $\sum K^{-3/2}$ .  $\square$

## APPENDIX H. THE STAR OPERATION AND LOCALLY UNIFORM LIMITS

**Lemma H.1** (Star commutes with locally uniform limits (entire case only)). *Let  $E_n$  be entire and suppose  $E_n \rightarrow E$  locally uniformly on  $\mathbb{C}$ . Then  $E_n^*(z) := \overline{E_n(\bar{z})}$*

converges locally uniformly to  $E^*(z) := \overline{E(\bar{z})}$  on  $\mathbb{C}$ . In particular, if  $w_n \rightarrow w$  locally uniformly on  $\mathbb{H}$  and  $w_n E_n \equiv E_n^*$  for all  $n$ , then  $w E \equiv E^*$ .

Note. We use this lemma only when we have global (entire) convergence. Our main argument avoids this issue by constructing the limiting entire function  $E$  directly from the weak\* limit of the Hamiltonians (see Theorem 10.9), which guarantees local uniform convergence on all of  $\mathbb{C}$ .

*Proof.* On any compact disc, expand  $E_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$ ; local uniform convergence implies  $a_{n,k} \rightarrow a_k$  for each  $k$ . Thus  $E_n^*(z) = \sum_{k \geq 0} \overline{a_{n,k}} z^k \rightarrow \sum_{k \geq 0} \overline{a_k} z^k = E^*(z)$  locally uniformly. The stated consequence follows by taking limits in  $w_n E_n - E_n^* \equiv 0$ .  $\square$

## APPENDIX I. $SU(1,1)$ ACTION AND PSEUDOHYPERBOLIC CONTROL

Throughout Appendix I we write  $v := \frac{m-i}{m+i}$  for the Cayley transform of the Weyl–Titchmarsh function and  $v := E^*/E$  for the endpoint ratio. We use  $\rho(a, b) = \left| \frac{a-b}{1-\bar{a}b} \right|$  for the pseudohyperbolic metric on  $\mathbb{D}$ .

We will repeatedly use the intertwinement  $v(z) = F_z(v(z))$  (Lemma I.6) and the strict Schur transfer (Corollary I.8).

**I.1. Off-real  $J$ -contractivity and quantitative transfer.** Fix  $z \in \mathbb{H}$  and write the disk Möbius transfer

$$F_z(w) := \frac{\alpha(z)w + \beta(z)}{\overline{\beta(z)}w + \overline{\alpha(z)}}, \quad w \in \mathbb{D},$$

so that  $v(z) = F_z(v(z))$ . (Here  $U(z) = C^{-1}\Phi(L; z)C = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ , with  $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$  the fixed change of basis.)

For standard background on  $J$ -contractive matrix functions and the  $SU(1,1)$  disk model, see [AD08, Ch. 0] and [Pot60].

**Lemma I.1** (Lagrange identity and  $J$ -contractivity). *For real  $x$ ,  $|\alpha(x)|^2 - |\beta(x)|^2 = 1$  ( $J_{1,1}$ -unitarity). For  $z \in \mathbb{H}$ ,  $U(z)$  is  $J_{1,1}$ -contractive (see, e.g., [AD08, §0.5]):*

$$J_{1,1} - U(z)^* J_{1,1} U(z) = 2 \operatorname{Im} z \int_0^L U(z, t)^* \tilde{H}(t) U(z, t) dt \succeq 0,$$

with  $\tilde{H} = C^* H C \succeq 0$ . Strictness holds unless  $H \equiv 0$  on  $[0, L]$ .

*Proof.* Let  $M$  solve  $J\partial_t M = zHM$  with  $M(\cdot, 0) = I$ , and set  $Q(t) = M^*JM$ . Then  $\partial_t Q = (\bar{z} - z)M^*HM = 2i \operatorname{Im} z M^*HM$ . Integrate on  $[0, L]$ . Conjugate by  $C$  using  $C^* J_{1,1} C = -iJ$  and  $U = C^{-1}MC$ . For  $z \in \mathbb{R}$  the integral vanishes, so  $U$  is  $J_{1,1}$ -unitary.  $\square$

**Theorem I.2** (Disk self-map off  $\mathbb{R}$ ). *For  $z \in \mathbb{H}$ ,  $F_z : \mathbb{D} \rightarrow \mathbb{D}$  is a strict holomorphic self-map. For real  $x$ ,  $F_x \in SU(1,1)$  is a disk automorphism.*

*Proof.* For  $T = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$  and  $w \in \mathbb{D}$ ,

$$1 - \left| \frac{\alpha w + \beta}{\bar{\beta}w + \bar{\alpha}} \right|^2 = \frac{{w \choose 1}^*(J_{1,1} - T^*J_{1,1}T) {w \choose 1}}{|\bar{\beta}w + \bar{\alpha}|^2} (1 - |w|^2).$$

Apply with  $T = U(z)$  and use Lemma I.1. For  $x \in \mathbb{R}$ ,  $U(x)^*J_{1,1}U(x) = J_{1,1}$ .  $\square$

**Lemma I.3** (Quantitative transfer via Schwarz–Pick). *Let  $a(z) := F_z(0) = \beta(z)/\overline{\alpha(z)} \in \mathbb{D}$ . If  $|v(z)| \leq \rho < 1$ , then*

$$|v(z)| = |F_z(v(z))| \leq \frac{|a(z)| + \rho}{1 + |a(z)|\rho} < 1.$$

*Proof.* Set  $G = \phi_{-a} \circ F_z$ , where  $\phi_{-a}(w) = (w - a)/(1 - \bar{a}w)$ . Then  $G : \mathbb{D} \rightarrow \mathbb{D}$  and  $G(0) = 0$ , so  $|G(w)| \leq |w|$ . Since  $F_z = \phi_a \circ G$ , the displayed bound follows by maximizing at  $|w| = \rho$ .  $\square$

**Lemma I.4** (Nonvanishing of the denominator). *If  $|w| < 1$  then  $\bar{\beta}(z)w + \bar{\alpha}(z) \neq 0$ . Hence  $F_z$  is holomorphic on  $\mathbb{D}$ .*

*Proof.* Otherwise  $(\alpha w + \beta, \bar{\beta}w + \bar{\alpha}) = (*, 0)$ . With  $u = (w, 1)^\top$  we would have  $u^*U^*J_{1,1}Uu \geq 0$  while  $u^*J_{1,1}u = |w|^2 - 1 < 0$ , contradicting  $J_{1,1}$ -contractivity (Lemma I.1).  $\square$

*Remark I.5* (SU(1,1) identities). For  $\phi(v) = \frac{\alpha v + \beta}{\bar{\beta}v + \bar{\alpha}}$  with  $|\alpha|^2 - |\beta|^2 = 1$  and  $\gamma = -\beta/\alpha$ ,

$$1 - |\phi(v)|^2 = \frac{(1 - |\gamma|^2)(1 - |v|^2)}{|1 - \bar{\gamma}v|^2}, \quad |\bar{\beta}v + \bar{\alpha}|^2 = \frac{1 - |v|^2}{1 - |\phi(v)|^2} = \frac{|1 - \bar{\gamma}v|^2}{1 - |\gamma|^2}.$$

**Lemma I.6** (Endpoint–Weyl intertwinement). *Let  $\Phi(L; z) = \begin{pmatrix} A^\Phi & B^\Phi \\ C^\Phi & D^\Phi \end{pmatrix}$ ,  $E = A^\Phi - iB^\Phi$ ,  $v = (m - i)/(m + i)$ , and define  $U = C^{-1}\Phi C = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ . Then for all  $z \in \mathbb{H}$ ,*

$$v(z) = \frac{E^*(z)}{E(z)} = \frac{\alpha(z)v(z) + \beta(z)}{\bar{\beta}(z)v(z) + \bar{\alpha}(z)} = F_z(v(z)).$$

*Proof.* On the Weyl side,  $m \mapsto \frac{A^\Phi m + C^\Phi}{B^\Phi m + D^\Phi}$ ; conjugating by the Cayley map  $m \mapsto v = \frac{m - i}{m + i}$  yields  $v \mapsto \frac{\alpha v + \beta}{\bar{\beta}v + \bar{\alpha}}$  (the entries  $\alpha, \beta$  are the Cayley conjugates of  $A^\Phi, \dots, D^\Phi$ ). On the endpoint side,  $E = A^\Phi - iB^\Phi$  and  $E^* = \overline{A^\Phi(\bar{z})} + i\overline{B^\Phi(\bar{z})}$ ; the same conjugation applied to the first column gives the displayed fractional-linear relation.  $\square$

*Remark I.7* (Tail-independence of  $a(z)$ ). If  $\Phi(L; z) = \Phi_{\text{left}}(z)\Phi_{\text{tail}}(z)$  and the tail is normalized to fix  $v = 0$  on the seams, then  $a(z) = F_z(0)$  depends only on  $\Phi_{\text{left}}(z)$  and is independent of  $L$ .

**Corollary I.8** (Strict Schur bound transfer). *If  $|v(z)| \leq \rho < 1$  on a set  $S \subset \mathbb{H}$ , then with  $v$  from Lemma I.6,*

$$|v(z)| \leq \sup_{|a| \leq \rho} \sup_{|w| \leq \rho} \left| \frac{\alpha w + \beta}{\bar{\beta} w + \bar{\alpha}} \right| < 1 \quad \text{uniformly on } S,$$

and the RHS is continuous and  $\uparrow 1$  as  $\rho \uparrow 1$  along real  $z$ .

## I.2. Block maps, smallness choice, and composition control.

**Lemma I.9** (HB increment;  $J$ -contractivity at height). *Let  $H \geq 0$  be Hermitian and  $\Theta$  solve the left-action system. For  $z = x + iy$  with  $y > 0$ ,*

$$\Theta(\bar{z}, x)^* J \Theta(z, x) - J = 2y \int_{[0, x]} \Theta(\bar{z}, t)^* H(dt) \Theta(z, t) \succeq 0.$$

*Proof.* Repeat the proof of Lemma I.1 on the left-action fundamental matrix  $\Theta$ .  $\square$

**Lemma I.10** (Each block map is a Schur self-map). *If  $H$  is constant on  $I_j = [a_j, b_j]$  with  $\|H - \frac{1}{2}I\| \leq \frac{1}{2} - \varepsilon$  and length  $\ell$ , then for  $z$  with  $\operatorname{Im} z = \eta_0 > 0$  the Weyl map across  $I_j$ ,*

$$v \mapsto \phi_j(v; z) = \frac{A_j(z)v + B_j(z)}{C_j(z)v + D_j(z)},$$

is a holomorphic self-map of  $\mathbb{D}$  with Schwarz–Pick contraction, and base drift  $\delta_{\mathbb{D}}(0, \phi_j(0; z)) \leq C(\varepsilon)|z|\ell$ .

*Proof.*  $J$ -contractivity at  $\operatorname{Im} z = \eta_0$  follows from Lemma I.9; hence  $\phi_j$  is Schur. A Duhamel expansion gives  $M_j(z, \ell) = I - zJH\ell + O(\ell^2)$ ; conjugating to the disk basis yields  $|\phi_j(0; z)| \leq C(\varepsilon)|z|\ell + O(\ell^2)$ . Schwarz–Pick implies the contraction.  $\square$

**Remark I.11** (One-time smallness choice of block length). Fix  $\eta_0 > 0$  and  $Y > 0$ . Choose a uniform subdivision with  $\ell \leq \ell_{\max}$  so that

$$r := \sup_{|x| \leq Y} C(\varepsilon) |x + i\eta_0| \ell_{\max} < 1.$$

This linearizes hyperbolic accumulation via  $\operatorname{arctanh} u \leq u/(1-u^2)$  and is independent of  $L$ .

**Lemma I.12** (Composition and pseudohyperbolic control). *Let  $A_\nu, \tilde{A}_\nu \in \operatorname{SU}(1, 1)$  be the block automorphisms ( $\nu = 0, \dots, N-1$ ). Then for any  $W_0 \in \mathbb{D}$ ,*

$$\rho\left(T_{A_{N-1}} \circ \dots \circ T_{A_0}(W_0), T_{\tilde{A}_{N-1}} \circ \dots \circ T_{\tilde{A}_0}(W_0)\right) \leq C \sum_{\nu=0}^{N-1} \|A_\nu - \tilde{A}_\nu\|.$$

If  $A_\nu, \tilde{A}_\nu$  arise from blocks at  $\operatorname{Im} z = \eta_0$ , then

$$\rho(\dots) \leq C(\eta_0) \sum_{\nu=0}^{N-1} |z| \ell \|H_\nu - \tilde{H}_\nu\|.$$

*Proof.* Triangle inequality in the Poincaré metric and invariance under automorphisms give

$$\rho(\Phi_N, \tilde{\Phi}_N) \leq \sum_{\nu=0}^{N-1} \sup_{|w|<1} \rho(T_{A_\nu}(w), T_{\tilde{A}_\nu}(w)).$$

The single-block bound  $\sup_w \rho(T_A(w), T_B(w)) \leq C \|A - B\|$  holds by smoothness of the  $A \mapsto T_A$  map on a compact parameter set (ensured by Lemma I.10 and Remark I.11). The second estimate follows from the Duhamel expansion controlling  $A_\nu - \tilde{A}_\nu$  by  $|z|\ell\|H_\nu - \tilde{H}_\nu\|$ .  $\square$

### I.3. Linear sensitivity and nonlinear control with margin.

**Lemma I.13** (Base linear control for  $v$ ). *At the base tail  $\theta_{\text{base}}$  one has  $v(\cdot; \theta_{\text{base}}) \equiv 0$ . Fix  $\eta_0 > 0$ . There exists  $C = C(\eta_0, \delta, \varepsilon)$  such that for every  $\delta\theta$  supported in the first  $K$  DST modes ( $K \leq c_\delta N$ ),*

$$\sup_{z \in \partial\Omega_{\eta_0}(L)} |\partial_{\delta\theta} v(z; \theta_{\text{base}})| \leq C \sqrt{K} \|\delta\theta\|_2.$$

*Proof.* Differentiate the finite-length Weyl formula  $m = (iA^\Phi - C^\Phi)(D^\Phi - iB^\Phi)^{-1}$  at the base  $(A^\Phi, D^\Phi) = \cos(zL/2)$ ,  $(B^\Phi, -C^\Phi) = \sin(zL/2)$ . Each entry variation is a DST packet with two odd frequencies; intrinsic normalization fixes height mixing. Apply the Nikolskii bound on the seam grid and compose with the Cayley derivative at  $i$ .  $\square$

**Lemma I.14** (Uniform linear bound on the parameter box). *Let  $K \leq c_\delta N$  and let  $\Theta$  denote the admissible compact parameter box (PSD/trace constraints,  $K$ -bandlimited support). Then there exists  $C_{\text{lin}} = C_{\text{lin}}(\eta_0, \delta, \varepsilon, K)$  such that for every  $\theta \in \Theta$ ,*

$$\sup_{z \in \partial\Omega_{\eta_0}(L)} \|\partial_\theta v(z; \theta)\|_{\ell^2 \rightarrow \mathbb{C}} \leq C_{\text{lin}} \sqrt{K}.$$

*Proof.*  $(z, \theta) \mapsto \partial_j v(z; \theta)$  is continuous on the compact set  $\partial\Omega_{\eta_0}(L) \times \Theta$ . Set  $M_j := \sup |\partial_j v| < \infty$ . For  $h = \sum_{j \leq K} h_j e_j$ ,  $|\partial_\theta v(z; \theta)[h]| \leq (\sum_{j \leq K} |\partial_j v|^2)^{1/2} \|h\|_2 \leq \sqrt{K} (\max_{j \leq K} M_j) \|h\|_2$ . Take  $\sup_z$  and set  $C_{\text{lin}} = \max_{j \leq K} M_j$ .  $\square$

**Proposition I.15** (Nonlinear pseudohyperbolic control with margin). *Let  $\theta, \tilde{\theta}$  be supported in the first  $K$  DST modes ( $K \leq c_\delta N$ ) and set  $\theta_t = (1-t)\tilde{\theta} + t\theta$ . Assume there exists  $\rho_0 \in (0, 1)$  such that*

$$\sup_{t \in [0, 1]} \sup_{z \in \partial\Omega_{\eta_0}(L)} |v(z; \theta_t)| \leq \rho_0.$$

*Then*

$$\sup_{z \in \partial\Omega_{\eta_0}(L)} \beta(v(z; \theta), v(z; \tilde{\theta})) \leq \frac{C_{\text{lin}}}{1 - \rho_0^2} \sqrt{K} \|\theta - \tilde{\theta}\|_2,$$

hence

$$\sup_{z \in \partial\Omega_{\eta_0}(L)} \rho(v(z; \theta), v(z; \tilde{\theta})) \leq \tanh\left(\frac{C_{\text{lin}}}{1 - \rho_0^2} \sqrt{K} \|\theta - \tilde{\theta}\|_2\right).$$

*Proof.* For fixed  $z$ , write  $w(t) := v(z; \theta_t) \in \mathbb{D}$ . By Lemma I.14,  $|w'(t)| = |\partial_\theta v(z; \theta_t)[\dot{\theta}_t]| \leq C_{\text{lin}} \sqrt{K} \|\theta - \tilde{\theta}\|_2$ . Since  $ds = |dw|/(1 - |w|^2)$  and  $1 - |w(t)|^2 \geq 1 - \rho_0^2$ , integrate to bound  $\beta(w(1), w(0))$ ; use  $\rho = \tanh \beta$ .  $\square$

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