Nutcracker 1

September 2, 2016

Rules

- Points for each questions will be awarded only to the complete and clear proof.
- Pools will be ranked according to number of points obtained.
- Ties will be resolved in favour of a pool having an earlier submission.
- No submissions will be considered after the deadline.
- Have fun!

Problems

1. Evaluate $\int_0^1 \frac{1}{x^x} dx$. The answer need not be in a closed form i.e. you can give me a series summation of constants. (5 points)

2. It is quite trivial to see that the power of a prime p in $n!$ is

$$\sum_1^n \left\lfloor \frac{n}{p^k} \right\rfloor$$

   Using this result get a lower bound on $\pi(n)$ (Number of primes $\leq n$) and thus deduce that there are infinite primes. (5 points)
   Note - Solution which do not use this result get 0 marks.

3. Find the distinct number of ways to colour the cube using the colours fuschia, blue and sarcoline such that fuschia must be used exactly twice. (5 points)

4. Prove that there exists no 2 polynomials f, g with positive degrees and integer coefficients such that

$$f(x)g(x) = x^7 + x^6 + 4x^5 + 8x^4 + 5x^3 + 5x^2 + 5x + 9$$

   (5 points)

5. If every second positive integer except 2 is remaining, then every third remaining integer except 3, then every fourth remaining integer etc., an infinite number of the remaining integers are prime.
   Prove this. ($\frac{1}{\pi^e}$ points)