Elements of Network Science

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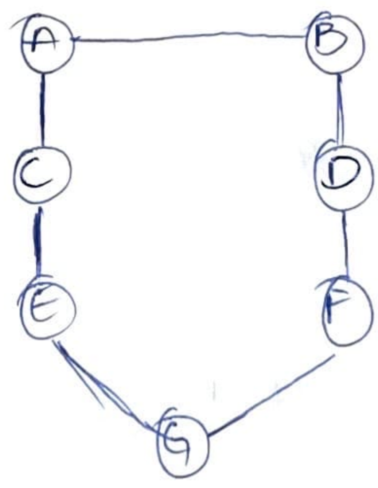
EXERCISE - II

Question ①:-

a) Example of graph in which every node is pivotal for at least one pair of nodes.

GRAPH:-



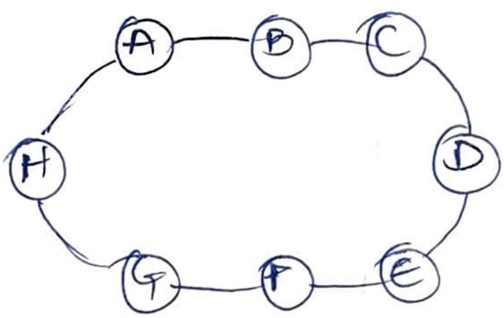
Here, all the nodes are pivotal for atleast one pair of nodes.

* Node Ⓐ is pivotal for Ⓑ & Ⓒ
* Node Ⓑ is pivotal for Ⓐ & Ⓓ
* Node Ⓒ is pivotal for Ⓐ & Ⓔ
* Node Ⓓ is pivotal for Ⓑ & Ⓕ

* Node E is pivotal for C & G
* Node F is pivotal for D & G
* Node G is pivotal for E & F

(b) Example of graph in which every node is pivotal for at least two different pairs of nodes.

GRAPH :-



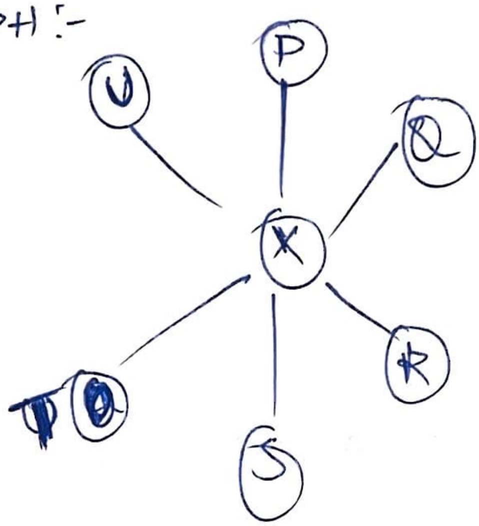
Here,

* A is pivotal to (B, H) & (C, G)
* B is pivotal to (A, C) & (D, H)
* C is pivotal to (B, D) & (A, E)
* D is pivotal to (C, E) & (B, F)
* E is pivotal to (D, F) & (C, G)
* F is pivotal to (E, G) & (D, H)
* G is pivotal to (H, F) & (A, E)

* (H) is pivotal to (A, G) & (B, F)

(c) Example of Graph with at least four nodes in which there is a single node x that is pivotal for every pair of nodes.

GRAPH :-

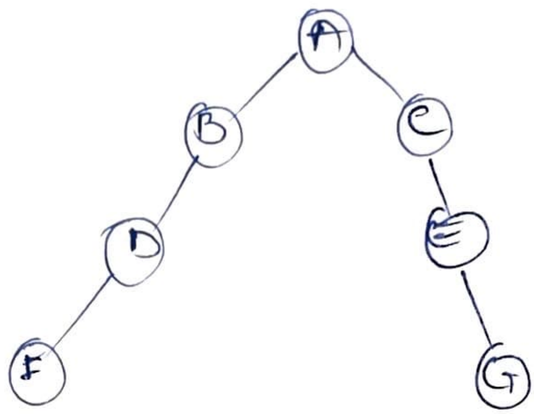


Here, x is pivotal to every possible pair in the above graph.

* (X) is pivotal to Pairs :- (P, Q), (Q, R), (R, S), (S, T), (T, U), (P, U) and the all remaining combinations.

Question (2) :-

(a) Example of a graph in which more than half of all nodes are gate keepers.



Here,

* All the nodes B, C, D, E, A, ~~are~~ are gate keepers.

* For F to reach node B, A, C, E & G is has to go through D.

* For node D, for pair, FB, FA, FC, FE, FG is a gate keeper

* Node B is a gate keeper for DA, DC, DE, DG.

* Similarly, it will be true for the nodes D, B, A, C, E

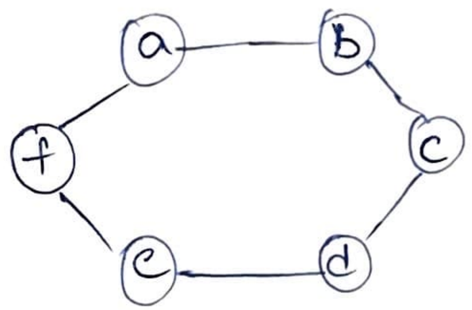
∴ Therefore, total number of gate keepers = 5

total number of nodes = 7

∴ More than half of all nodes are gate keepers.

(b) Example of a Graph in which there are no gate keepers, but in which every node is a local gate keeper.

GRAPH :-



Here, all the nodes are local gatekeepers.

Here, ~~for~~ node (a), node (b) & (f) are adjacent nodes but they are not connected directly.

But, there is a route from (f) — (e) — (d) — (c) — (b)

∴ Therefore, even without crossing node (a) there is another path to reach from node (b) to node (f)

Similarly, we have paths,

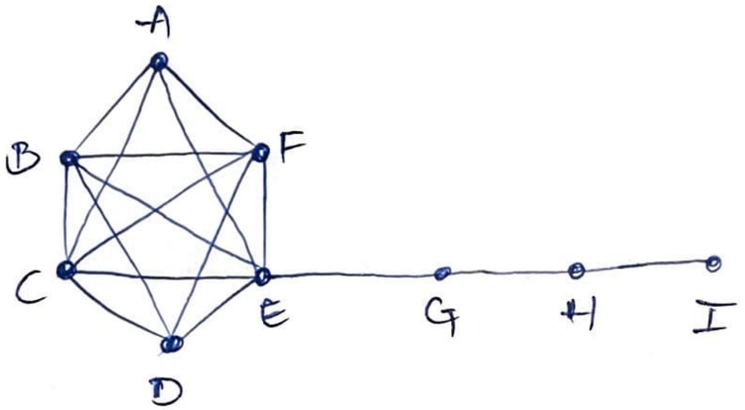
- to reach a from c, without crossing b
- to reach b from d, without crossing c
- to reach c from e, without crossing d.
- to reach d from f, without crossing e

to reach a from e, without touching f //

Question ③:-

ⓐ Example of a graph where the diameter is more than three times as large as the average distance.

GRAPH:-



Average length $d = \frac{1}{l_{max}} \sum_{i,j>i} d_{ij}$

for value $n = 50$ nodes.

$$d = \frac{\sum_{ij} d_{ij}}{\frac{(n+3)(n+2)}{2}} = \frac{\frac{n(n-1)}{2} + an + 1}{\frac{(n+3)(n+2)}{2}}$$

for $n=50$, $d = \frac{1676}{1378}$

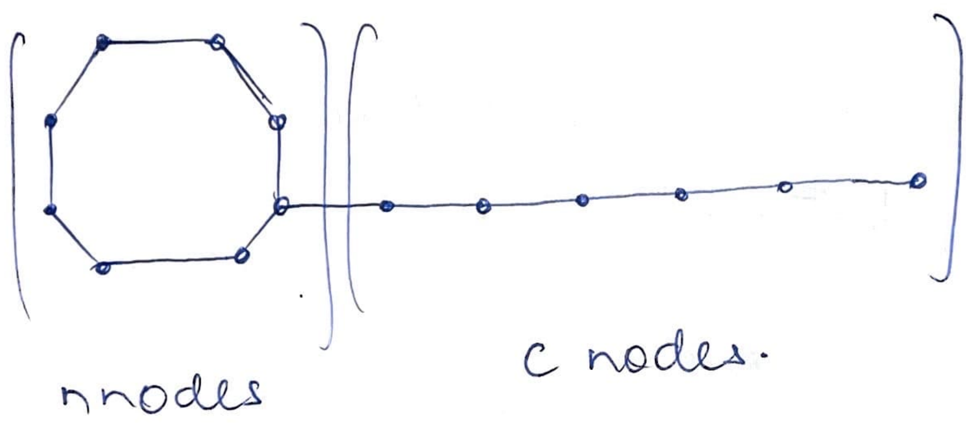
$$\boxed{d \approx 1.21}$$

which is almost $4/3$

$\therefore \frac{\text{diameter}}{3}$

$\therefore$ Therefore, the diameter is more than three times as large as the average length.

(b)



for a graph with, $n+c$ nodes

diameter $= c+1$

$$\text{average path} = \frac{\sum_{i<j} d_{ij}}{\frac{(n+c)(n+c-1)}{2}}$$

$$= \frac{\frac{n(n-1)}{2} + \sum_{m=2}^{i+1} (n-1)m + \sum_{k=1}^{p-1} (1-k)k}{\frac{(n+c)(n+c-1)}{2}}$$

$\therefore$ when the number of nodes goes to infinity, the average path tends to become 1.

$$\frac{\text{diameter}}{\text{average path length}} = c+1$$

$\therefore$ as $n \to \infty$

$\Rightarrow$ Diameter can exceed the average path length by a factor.