Numerical Solutions for Viscous Burgers Equation

Course project for the course titled MATH F422

Sidhant Singh Karthik A. Shardool Kulkarni

Birla Institute of Technology and Science

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Outline

Introduction

Burgers Equation

- Burgers Equation $u_t + uu_x = \alpha u_{xx}$ is a non-linear, parabolic, one dimensional PDE.
- For a given field u(x,t) and the viscosity α
  this is the general form of the Viscous Burgers' Equation.
- An acoustic wave of a finite amplitude while travelling in a viscous medium steepens in it's waveform. Such steepening is governed by the Burgers Equation.

BTCS Method

- The domain for the PDE is defined as $0<x<1$, $t>0$
- The initial condition for the problem is taken as

$$u(x,0) = sin(\pi x)$$

- The Boundary conditions are

$$u(0,t) = u(1,t) = 0$$

- Burger's equation can be written as

$$u_t + f[u]_x = \alpha u_{xx}$$

Where $f[u]$ is $\frac{u^2}{2}$ this is done so as to write in a conserved form ;

- The discretization is as follows:

$$\frac{u_j^n - u_j^{n-1}}{\delta t} + \frac{(u_{j-1}^n)^2 - (u_{j+1}^n)^2}{4(\delta x)} = \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2(\delta x)^2}$$

- The scheme is of the order of

$$O(\delta x^2 + \delta t)$$

Crank Nicholson Scheme

- The domain for the PDE is defined as 0<x<1 , t>0
- The initial condition for the problem is taken as

$$u(x,0) = sin(\pi x)$$

- The Boundary conditions are

$$u(0,t) = u(1,t) = 0$$

- The discretization is as follows:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{u_j^n(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + u_j^{n+1}(u_{j+1}^n - u_{j-1}^n)}{4\delta x}$$

$$= \alpha \frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} + u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{2(\delta x)^2}$$

- The scheme is of the order of

$$O(\delta x^2 + \delta t^2)$$

DuFort Frankel Scheme

- The domain for the PDE is defined as 0<x<1 , t>0
- The initial condition for the problem is taken as

$$u(x,0) = sin(\pi x)$$

- The Boundary conditions are

$$u(0,t) = u(1,t) = 0$$

- Burger's equation can be written as

$$u_t + f[u]_x = \alpha u_{xx}$$

  Where $f[u]$ is $\frac{u^2}{2}$ this is done so as to write in a conserved form ;
- The discretization is as follows:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\delta t} + \frac{(u_{j+1}^n)^2 - (u_{j-1}^n)^2}{4(\delta x)} = \alpha \frac{(u_{j+1}^n + u_{j-1}^n - u_j^{n+1} - u_j^{n-1})}{(\delta x)^2}$$

- The scheme is of the order of

$$O(\delta x^2 + \delta t^2)$$

Numerical results and Analysis

- For N=80, dt=0.001,α=0.1 .

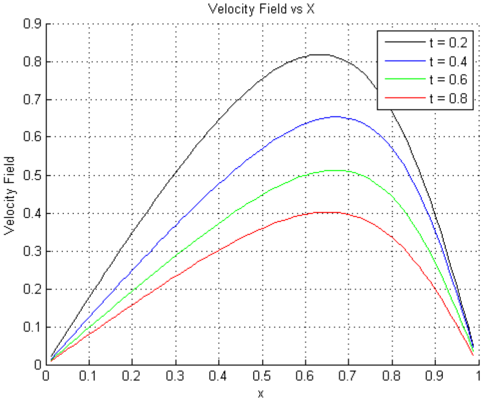

Figure: BTCS Scheme

Numerical results and Analysis

- For N=80, dt=0.001,α=0.1 .

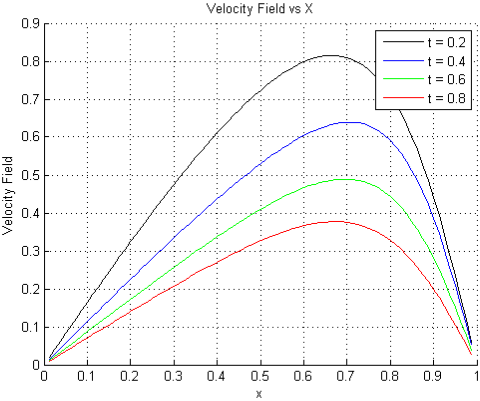

Figure: Crank Nicholson Scheme

Numerical results and Analysis

- For N=80, dt=0.001,α=0.1 .

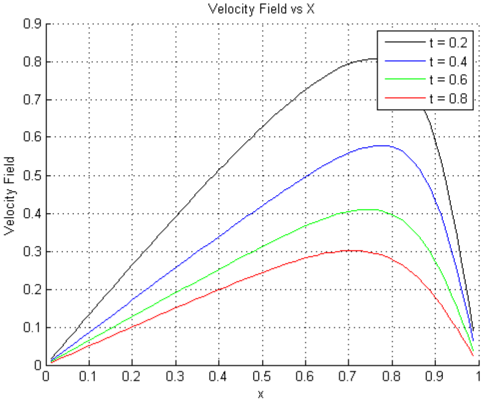

Figure: Dufort Frankel Scheme

- The exact solution was found by taking a very fine mesh which was used to find l_∞ norm.
- The order of convergence for all three schemes was found using l_∞ norm and it was found to be '2' for spatial convergence.

Summary

- This study was aimed at exploring different Finite Difference Schemes to numerically solve the Viscous Burgers Equation.
- The schemes used were BTCS , Crank Nicholson and Dufort-Frankel
- BTCS was solved using the Newtons Method.
- Dufort-Frankel was solved in an Explicit manner.
- Outlook
  - Other Boundary conditions can be studied.
  - Crank Nicolson and Dufort-Frankel can be done implicitly by using the Newton's Method.
  - L-2 norm and methods of higher order of accuracy can be studied.

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Burgers Equation
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