Unit wise question bank with solutions

Mathematics I

For

B.E. I year (AICTE) O.U.

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Unit wise
Question Bank

Question Bank for Mathematics - I

Prepared by :- Dr. Mohd Ahmed

Unit - I (Sequences & Series)

SAQs :-

June-19
(1) Determine the nature of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

June-17
(2) Discuss the convergence of the series $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\cdots$

Dec-18
(3) Test the convergence of $\sum_{n=1}^{\infty} \frac{2+5n}{7n^2-3}$.

June-18
(4) Determine the nature of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

Dec-17
(5) Test the convergence of $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$

June-19
(6) Determine the nature of $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$.

Dec-17.
(7) Test $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ for conditional convergence.

June-17.
(8) Prove that the series $\sum (-1)^n \frac{\sin nx}{n^2}$ converges absolutely.

LAQs :-

Dec-18 (1) Determine the nature of the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$

Dec-18.
June-19, June-17.
(2) Test the convergence of $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\cdots$

June-15
(3) Discuss the convergence of $\sum \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \frac{x^{2n}}{2n}\right]$

June-15
(4) Test the convergence of $\sum \frac{(n+1)^n x^n}{n^{n+1}}$.

June-15
(5) Test the convergence of $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots$

Unit-II : Calculus of one Variable

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SAQs :-

June-17 (1) Verify Rolle's theorem for the function $f(x) = x^3 - 4x$ in $[-2, 2]$.

Dec-17 (2) Verify Lagrange's mean value theorem for $f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$.

June-11, June-14 (3) Using Lagrange's mean value theorem, show that $|\sin b - \sin a| \le |b - a|$.

Dec-15 (4) Find Taylor's series expansion of $f(x) = x^3 + 3x^2 + 2x + 1$ about $x = -1$.

June-15 (5) Obtain the curvature of the curve $x^4 + y^4 = 2$ at $(1, 1)$.

Dec-13 (6) Find the radius of curvature of the curve $r = a\sin\theta + b\cos\theta$ at $\theta = \pi/2$.

Apr-16 (7) Find the radius of curvature at $(0, 0)$ on the curve $y^2 = 4x$.

Dec-8 (8) Find the envelope of the family of lines $x\cos\alpha + y\sin\alpha = p$, where α is the parameter.

LAQs :-

Apr-7, May-8, June-13, Dec-13 (1) Find the coordinates of centre of curvature at any point of the parabola $y^2 = 4ax$. Hence show that its evolute is $27a\bar{y}^2 = 4(\bar{x} - 2a)^3$

Dec-18 (2) Find the evolute of the curve $x = a\cos^3 t$, $y = a\sin^3 t$.

Dec-8, June-11, June-12 (3) Find the envelope of family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where the parameters a & b are connected by $a + b = c$

Dec-17, Apr-7, June 13, June 19, June-15 (4) State and prove Lagrange's mean value theorem.

May 09, May 18, Dec 8, Dec-18 (5) State and prove Cauchy's mean value theorem.

July-17, Apr-7 (6) If $a<b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

May-09, Jan-12 (7) ~~8k~~ Find ρ at the origin for the curve.

$y - x = x^2 + 2xy + y^2$.

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SAQs :-

Dec-16 June-19 (1) Show that $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y}{x^6 + y^2}$ does not exist.

Dec-18 (2) Discuss the continuity of the function :

$$f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \text{ at } (0,0).$$

Dec-16 (3) Show that the function

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$ is not diff at $(0,0)$.

(4) If $u = x^2+y^2$ and $x = at^2$, $y = 2at$ then find $\frac{du}{dt}$.

June-19 (5) If $xy + y^2 - 3x - 3 = 0$, then evaluate $\frac{dy}{dx}$ at $(-1,1)$.

Dec-16 June-17 (6) If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$ then find $\frac{\partial(u,v)}{\partial(r,\theta)}$

June-15 (7) Find $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2}$

June-15 (8) If $u = \log(x^2+xy+y^2)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

LAQs :-

Apr-16 (1) If $z = f(x,y)$ where $x = e^u + e^{-v}$ and $y = e^{-u} + e^v$ then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

June-19 (2) Find the first three terms of Taylor's series of the function $f(x,y) = e^x \cos y$ around $(0,0)$.

June-19
May-17 (3) Investigate the function x^3+y^3-3axy for maxima and minima.

Dec-18 (4) Find the maximum values of $x+y+z$, subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$.

July-17 (5) If $f(x,y)=\begin{cases} \frac{x^2y(x-y)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$, then show that

$$\frac{\partial^2 f}{\partial y\,\partial x}\neq\frac{\partial^2 f}{\partial x\,\partial y} \text{ at } (0,0).$$

Unit IV : Multivariable Calculus (Integration)

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SAQs :-

June-18 (1) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$.

June-11 Dec-15 (2) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$.

Dec-18 (3) Evaluate $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$.

Dec-18 (4) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz$.

June-19 (5) Evaluate $\int_0^{\pi/6} \int_0^1 \int_{-1}^3 y \sin z\, dx\,dy\,dz$.

(6) Evaluate $\iiint_V (xy+yz+zx)\, dx\,dy\,dz$ where V is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$.

(7) Evaluate $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

June-19 (8) Evaluate $\int_0^\pi \int_0^1 x \cos xy\, dy\, dx$.

LAQs :-

Dec-17 Dec-18 (1) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$ by changing to polar coordinates.

June-19 (2) Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dy\, dx$ by changing to polar coordinates.

Dec-18 (3) Evaluate $\int_0^2 \int_x^2 2y^2 \sin xy$ by changing the order of integration.

(4) Evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration.

Dec-18 (5) Find the volume of the unit sphere $x^2+y^2+z^2=1$.

(6) Evaluate $\iiint z(x^2+y^2)\,dx\,dy\,dz$ over $x^2+y^2 \le 1$, $2 \le z \le 3$

★ (7) Evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$, by transforming to spherical coordinates.

Unit V : Vector Calculus

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SAQs:-

Dec17 (1) Find ∇f at $(1,2,-1)$ if $f(x,y,z) = \log_e(x+y+z)$.

June-15
(2) Find $\operatorname{div}\bar{F}$, where $\bar{F} = \operatorname{grad}(x^3+y^3+z^3+3xyz)$.

Dec-15
(3) Evaluate $\nabla\left(\frac{1}{r}\right)$ where $r = |\bar{r}|$, $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$.

Dec18
Dec17
June-19 (4) If $\bar{a}$ is a constant vector and $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ then evaluate $\operatorname{div}(\bar{a}\times\bar{r})$. and $\operatorname{Curl}(\bar{a}\times\bar{r})$. and prove that $\nabla(\bar{a}\times\bar{r}) = 0$.

Dec17 (5) Show that $\bar{V} = 12x\bar{i} - 15y^2\bar{j} + \bar{k}$ is irrotational.

July17 (6) Show that the vector $e^{x+y-2z}(\bar{i}+\bar{j}+\bar{k})$ is solenoidal.

Dec-16
Reg,
Supply. (7) Prove that $\operatorname{curl}(\operatorname{grad} f) = \bar{0}$ where f is differentiable scalar field.

Dec18 (8) Find the unit normal vector to the surface $f(x,y,z) = x^2y - y^2z - xyz$ at $P(1,-1,0)$.

Dec16 (9) In what direction from $(3,1,-2)$ is the directional derivative of $f(x,y,z) = xy^2z^3$ maximum?

June-19 (10) Evaluate $\int_C \bar{V}\cdot d\bar{r}$ where $\bar{V} = x\bar{i}+y\bar{j}+z\bar{k}$ and C is the line segment from $A(1,2,2)$ to $B(3,6,6)$.

Dec15
Dec17 (11) State Greens theorem and Stoke's theorem.

LAQs :-

Dec17 (1) Prove that $\nabla \cdot (\nabla \times \bar{A}) = 0$ where $\bar{A}$ is a vector point function.

Jul-17 (2) Prove that $\nabla(\log r) = \frac{\bar{r}}{r^2}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$.

Jul-17 (3) Find the constants a, b, c such that $\bar{F} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$ is irrotational and find a scalar function f such that $\bar{F} = \nabla f$.

Dec17 (4) Find the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of normal to the surface $x \log_e z - y^2 = -4$ at $(-1,2,1)$.

June19 (5) Find the angle between the surfaces $x^2+y^2+z^2=9$, $z+3=x^2+y^2$ at $(-2,1,2)$.

Dec15 (6) Find the values of a & b such that the surface $5x^2-2yz-9x=0$ intersects the surface $ax^2+by^3=4$ orthogonally at $(1,-2,1)$.

Dec17 (7) If $\bar{F} = (5xy-6x^2)\bar{i} + (2y-4x)\bar{j}$, evaluate $\int_C \bar{F} \cdot d\bar{R}$ along the curve C in xy-plane given by $y=x^3$ from $(1,1)$ to $(2,8)$.

Dec18 (8) Using Gauss divergence theorem evaluate $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ where S is the surface of the sphere $(x-2)^2+(y-2)^2+(z-2)^2=16$.

Dec18 (9) Verify Green's theorem for $\oint_C (xy^2+2xy)\,dx + x^2\,dy$ where C is the boundary of the region enclosing $y^2=4x$, $x=3$

Dec16 (10) Verify Stokes theorem for the vector field $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on xy-plane.

Solutions to
Unit wise
Question Bank

UNIT-1 (Sequences & Series)

SAQs:

J-19
1. Determine the nature of series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Sol Given: $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

$$u_n = \frac{\sqrt{n}}{n^2+1}$$

$$= \frac{n^{1/2}}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \frac{1}{n^{3/2}} \cdot \frac{1}{\left(1+\frac{1}{n^2}\right)}$$

Let $v_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \ (\neq 0) \text{ finite}$$

$\sum v_n = \sum \frac{1}{n^{3/2}}$ is a p-series where $p = \frac{3}{2} > 1$

$\therefore \sum v_n$ is convergent.

Hence, from Comparison test, $\sum u_n$ is also convergent.

J-17
2. Discuss the convergence of series $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots$

Sol Given series is $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots$

Let $\sum u_n = 2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots \infty = \sum \frac{n+1}{n}$

$$u_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 \neq 0$$

$\therefore \sum u_n$ diverges.

D-18
3. Test the convergence of $\sum_{n=1}^{\infty} \frac{2+5n}{7n^2-3}$

Sol. Given: $\sum u_n = \sum \frac{2+5n}{7n^2-3}$

$$u_n = \frac{2+5n}{7n^2-3} = \left[\frac{5+2/n}{7-3/n^2}\right] \times \frac{1}{n}$$

Let $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{5+2/n}{7-3/n^2} = \frac{5}{7} \ (\neq 0) \text{ Finite}$$

$\therefore \sum v_n = \sum \frac{1}{n}$ is a p-series with $p=1$

$\therefore \sum v_n$ is divergent

From Comparison test, $\sum u_n$ is also divergent.

4. J-18 Determine the nature of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Given: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\cdots$ is an alternating series

So, we use Leibnitz's test.

i) clearly $1>\frac{1}{4}>\frac{1}{9}>\frac{1}{16}\cdots$ terms are numerically decreasing

ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore$ From Leibnitz's test, $\sum u_n$ converges.

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5. D-17 Test the convergence of $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$ $\{\because \cos n\pi = (-1)^n$

Given: $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = -\frac{1}{2}+\frac{1}{5}-\frac{1}{10}+\frac{1}{17}\cdots$ is an alternating series.

So we use Leibnitz's test.

i) $\frac{1}{2}>\frac{1}{5}>\frac{1}{10}>\frac{1}{17}\cdots$ terms are numerically decreasing.

ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$

$\therefore$ From Leibnitz's test, $\sum u_n$ converges.

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6. J-19 Determine the nature of $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

Given: $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n = \sum u_n$

$$u_n = \left(\frac{n}{3n+1}\right)^n, \quad u_n^{1/n} = \frac{n}{3n+1} = \frac{1}{3+\frac{1}{n}}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{3+1/n} = \frac{1}{3} < 1$$

$\therefore$ From Cauchy's n^{th} root test, $\sum u_n$ converges.

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7. D-17 Test $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ for conditional convergence.

Given: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

Absolute term series $\sum |u_n| = \sum \frac{n}{n^2+1}$

$|u_n| = \frac{1}{n}\left(\frac{1}{1+1/n^2}\right)$

Let $v_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 (\neq 0)$ finite

$\therefore \sum v_n = \sum \frac{1}{n}$ is a p-series with $p=1$

$\sum v_n$ is divergent.

From Comparison test, $\sum |u_n|$ is also divergent.

But $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1} = -\frac{1}{2}+\frac{2}{5}-\frac{3}{10}+\cdots$ is an alternating series

So, we use Leibnitz's test

i) Clearly, $\frac{1}{2} > \frac{2}{5} > \frac{3}{10} \cdots$ terms are numerically decreasing

ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \frac{1}{n(1+\frac{1}{n^2})} = 0$

From Leibnitz's test, $\sum u_n$ converges.

Hence, given series converges conditionally.

J-18

8. Prove that the series $\sum (-1)^n \frac{\sin nx}{n^2}$ converges absolutely.

Given: $\sum (-1)^n \frac{\sin nx}{n^2}$

Absolute terms series: $\sum |u_n| = \sum \frac{|\sin nx|}{n^2} \le \sum \frac{1}{n^2}$

$\sum |u_n| \le \sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$

$\therefore \sum \frac{1}{n^2}$ is convergent.

Then $\sum |u_n|$ also converges & given series converges absolut-
-ely.

LAQs:

D-18

1. Determine the nature of series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$

Given: $\sum u_n = \sum \frac{x^{n-1}}{n \cdot 3^n}$

$$u_n = \frac{x^{n-1}}{n \cdot 3^n}, \quad u_{n+1} = \frac{x^n}{(n+1)3^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{x \cdot n \cdot 3^n} \times \frac{(n+1)3^n \cdot 3}{x^n} = 3\left(\frac{n+1}{n}\right) \times \frac{1}{x} = \frac{3}{x}\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{3}{x}$$

From ratio's test, $\sum u_n$ converges if $\frac{3}{x} > 1 \Rightarrow x < 3$ & diverges if $\frac{3}{x} < 1 \Rightarrow x > 3$ & test fails if $x = 3$

If $x = 3$, $u_n = \frac{3^n}{3n3^n} = \frac{1}{3n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3n} = 0 \rightarrow$ no conclusion can be drawn.

$\sum u_n = \frac{1}{3} \sum \frac{1}{n}$ is a p-series with $p = 1$

$\therefore \sum u_n$ diverges if $x = 3$

J-17 D-18 J-19

2. Test the convergence of $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots$

Given: $\sum u_n = \sum_{n=0}^{\infty} \frac{n! x^n}{(n+1)^n}$,

$u_n = \frac{n!x^n}{(n+1)^n}$, $u_{n+1} = \frac{(n+1)!x^{n+1}}{(n+2)^{n+1}}$

$\frac{u_n}{u_{n+1}} = \frac{n!x^n}{(n+1)^n} \times \frac{(n+2)^n(n+2)}{(n+1)n!x^n x} = \frac{(n+2)^n(n+2)}{(n+1)^n(n+1)} \times \frac{1}{x}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(1+2/n)^n(1+2/n)}{(1+1/n)^n(1+1/n)} \times \frac{1}{x} = \frac{e^2}{e} \times \frac{1}{x} = \frac{e}{x}$

From ratio's test, Σu_n converges if $\frac{e}{x} > 1 \Rightarrow x < e$, diverges if $x > e$ & test fails if $x = e$.

If $x = e$, then $\frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n+1}\right)^{n+1} \times \frac{1}{e}$

$\frac{u_n}{u_{n+1}}$ involves the no. 'e' so, we use logarithmic test.

$n \log \frac{u_n}{u_{n+1}} = n\left[\log\left(\frac{n+2}{n+1}\right)^{n+1} + \log\frac{1}{e}\right]$

$= n\left[(n+1)\log\left(1+\frac{1}{n+1}\right) - \log e\right]$

$\boxed{\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots}$

$= n\left[(n+1)\left(\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} - \cdots\right) - 1\right]$

$= n\left[1 - \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} - \cdots - 1\right]$

$= -\frac{1}{2(1+\frac{1}{n})} + \frac{1}{3(1+\frac{1}{n})^2 n} - \frac{1}{4(1+\frac{1}{n})^3 n^2} + \cdots$

$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = -\frac{1}{2} < 1$

From logarithmic test, Σu_n is divergent if $x = e$.

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3. J-15 Discuss the convergence of $\Sigma \left[\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \frac{x^{2n}}{2n}\right]$

Given: $\Sigma u_n = \Sigma\left[\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \frac{x^{2n}}{2n}\right]$

$u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \times \frac{x^{2n}}{2n}$, $u_{n+1} = \frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots(2n)(2n+2)} \times \frac{x^{2n+2}}{2n+2}$

$\frac{u_n}{u_{n+1}} = \frac{x^{2n}}{2n} \times \frac{2n+2}{2n+1} \times \frac{2n+2}{x^{2n} x^2} = \frac{(2n+2)}{2n}\frac{(2n+2)}{2n+1} \times \frac{1}{x^2}$

$= \left(1+\frac{1}{n}\right)\left(\frac{2+2/n}{2+1/n}\right) \times \frac{1}{x^2}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(\frac{2+2/n}{2+1/n}\right) \times \frac{1}{x^2} = \frac{1}{x^2}$

From ratio's test, Σu_n converges if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$, diverges if $x^2 > 1$ & test fails if $x^2 = 1 \Rightarrow x = 1$

If $x = 1$ then $\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{2n(2n+1)}$ doesnot involve the no. 'e'. So, we use Raabe's test

$$\frac{u_n}{u_{n+1}} = \frac{4n^2+4+8n}{4n^2+2n}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+4+8n-4n^2-2n}{4n^2+2n} = \frac{6n+4}{4n^2+2n}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{6n^2+4n}{4n^2+2n} = \frac{n^2(6+\frac{4}{n})}{n^2(4+\frac{2}{n})} = \frac{6+4/n}{4+2/n}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{6+4/n}{4+2/n} = \frac{3}{2} > 1$$

$\therefore$ $\sum u_n$ converges from Raabe's test if $x=1$.

---

4. Test the convergence of $\sum \frac{(n+1)^n x^n}{n^{n+1}}$

J-15

Given: $u_n = \frac{(n+1)^n x^n}{n^{n+1}}$

$$u_n^{1/n} = \frac{(n+1)x}{n \cdot n^{1/n}}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\frac{x}{n^{1/n}} = x \qquad \{\because \lim_{n\to\infty} n^{1/n} = 1$$

From Cauchy's n^{th} root test, $\sum u_n$ converges if $x<1$, diverges if $x>1$ & test fails if $x=1$

If $x=1$, $u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

Let $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \ (\neq 0) \text{ finite}$$

$\therefore$ $\sum v_n = \sum 1/n$ is a p-series with $p=1$

$\sum v_n$ diverges & from Comparison test, $\sum u_n$ also diverges if $x=1$.

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5. Test the convergence of $x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots$

J-15

Given: $u_n = \frac{1.3.5\ldots(2n-1-2)x^{2n-1}}{2.4.6\ldots(2n-1-1)(2n-1)} = \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)} \times \frac{x^{2n-1}}{2n-1}$

$$u_{n+1} = \frac{1.3.5\ldots(2n-3)(2n-1)}{2.4.6\ldots(2n-2)(2n)} \times \frac{x^{2n+1}}{2n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-1}}{(2n-1)^2} \times \frac{(2n+1)2n}{x^{2n+1}} = \frac{(2n+1)2n}{(2n-1)^2} \times \frac{1}{x^2} = \frac{(2n)^2(1+1/2n)}{(2n)^2(1-1/2n)^2} \times \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

From Ratio's test, $\sum u_n$ converges if $\frac{1}{x^2}>1 \Rightarrow x^2<1$, diverges if $x^2>1$ & test fails if $x^2=1 \Rightarrow x=1$.

If $x=1$, $\frac{u_n}{u_{n+1}} = \frac{4n^2+2n}{(2n-1)^2}$ doesn't involve the no. 'e' so, we use Raabe's test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+2n}{4n^2+1-4n} - 1 = \frac{4n^2+2n-4n^2-1+4n}{4n^2-4n+1} = \frac{6n-1}{4n^2-4n+1}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{6n^2-n}{4n^2-4n+1} = \frac{n^2(6-1/n)}{n^2(4-4/n+1/n^2)}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{6-1/n}{4-4/n+1/n^2} = \frac{3}{2} > 1$$

$\therefore$ From Rabee's test, Σu_n converges if $x^2=1$.

UNIT-2 (Calculus in one variable)

SAQ's:

1\. Verify Rolle's theorem for the function $f(x)=x^3-4x$ in $[-2,2]$

J-17

Given: $f(x)=x^3-4x$ in $[-2,2]$

i) $f(x)$ is continuous on $[-2,2]$

ii) $f'(x)=3x^2-4$ exist on $(-2,2)$

$f(x)$ is differentiable in $(-2,2)$

iii) $f(-2)=(-2)^3-4(-2)=0$, $f(2)=2^3-4(2)=0$

$f(-2)=f(2)$

$\therefore$ $f(x)$ satisfies all conditions of Rolle's theorem then,

from Rolle's theorem $f'(c)=0$

$3c^2-4=0$

$c=\frac{2}{\sqrt{3}}\in(-2,2)$

$\therefore$ Rolle's theorem is verified.

2\. Verify Lagrange's mean value theorem for $f(x)=x(x-1)(x-2)$ in $[0,1/2]$.

D-17

Given: $f(x)=x(x-1)(x-2)$ on $[0,1/2]$

$f(x)=x^3-3x^2+2x$

i) $f(x)$ is continuous on $[0,1/2]$

ii) $f'(x)=3x^2-6x+2$ exist in $(0,1/2)$

$f(x)$ is differentiable on $(0,1/2)$

$\therefore$ $f(x)$ satisfies all conditions of LMVT.

From LMVT, $f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{f(1/2)-f(0)}{1/2-0}$

$3c^2-6c+2=\frac{3}{4} \Rightarrow 12c^2-24c+5=0$

$c=1.764$ or $0.236\in(0,1/2)$

$\therefore$ LMVT is verified.

3\. Using LMVT, show that $|\sin b-\sin a|\le|b-a|$

J-11

J-14

Let $f(x)=\sin x$ defined on $[a,b]$

$f'(x)=\cos x$ on (a,b)

From LMVT, $\exists\, c\in(a,b) \ni f'(c)=\frac{f(b)-f(a)}{b-a}$

$\cos c=\frac{\sin b-\sin a}{b-a}$

$|\cos c|=\frac{|\sin b-\sin a|}{|b-a|}$

$(\because |\cos c| \le 1) \Rightarrow \frac{|\sin b - \sin a|}{|b-a|} = |\cos c| \le 1$

$\therefore |\sin b - \sin a| \le |b-a|$

4. Find Taylor's series expansion of $f(x) = x^3+3x^2+2x+1$ about $x=-1$.
D-15
Taylor's series expansion about $x=-1$ is –

$$f(x) = f(-1) + \frac{(x+1)}{1!}f'(-1) + \frac{(x+1)^2}{2!}f''(-1) + \frac{(x+1)^3}{3!}f'''(-1) + \cdots \quad \text{—} ①$$

Given $f(x) = x^3+3x^2+2x+1 \Rightarrow f(-1) = -1+3-2+1 = 1$

$f'(x) = 3x^2+6x+2 \Rightarrow f'(-1) = 3-6+2 = -1$

$f''(x) = 6x+6 \Rightarrow f''(-1) = 0$

$f'''(x) = 6 \Rightarrow f'''(-1) = 6$

$f^{IV}(x) = 0 \Rightarrow f^{IV}(-1) = 0$

From ①, $x^3+3x^2+2x+1 = 1+(x+1)(-1) + \frac{(x+1)^2}{2}(0) + \frac{(x+1)^3}{6}(6) + 0$

$\therefore x^3+3x^2+2x+1 = 1-(x+1)+(x+1)^3$

5. Obtain the curvature of the curve $x^4+y^4=2$ at $(1,1)$.
J-15
Given: $x^4+y^4=2$

$\boxed{\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}}$

Differentiate w.r.t x, $4x^3+4y^3y'=0 \Rightarrow y' = -\frac{x^3}{y^3}$

$y'|_{(1,1)} = -1$, $y'' = \frac{-3x^2y^3+3x^3y^2y'}{y^6} \Rightarrow y''|_{(1,1)} = \frac{-3(1)(1)-3(1)(1)}{1} = -6$

$\rho_{(1,1)} = \frac{(1+y'^2)^{3/2}}{y''} = \frac{(1+1)^{3/2}}{-6} = \frac{2\sqrt{2}}{-\sqrt{2}\sqrt{2}.3} = \frac{\sqrt{2}}{3}$

Curvature, $k = \frac{1}{\rho} = \frac{3}{\sqrt{2}}$.

6. Find the radius of curvature of the curve $r = a\sin\theta + b\cos\theta$ at $\theta = \frac{\pi}{2}$.
D-13

Given $r = a\sin\theta + b\cos\theta \Rightarrow r_{(\theta=\pi/2)} = a\sin\frac{\pi}{2} + b\cos\frac{\pi}{2} = a$

$r' = a\cos\theta - b\sin\theta \Rightarrow r'_{(\theta=\pi/2)} = a\cos\frac{\pi}{2} - b\sin\frac{\pi}{2} = -b$

$r'' = -a\sin\theta - b\cos\theta \Rightarrow r''_{(\theta=\pi/2)} = -a\sin\frac{\pi}{2} - b\cos\frac{\pi}{2} = -a$

$$\rho = \frac{(r^2+r'^2)^{3/2}}{r^2+2r'^2-rr''} = \frac{(a^2+b^2)^{3/2}}{a^2+2b^2+a^2} = \frac{1}{2}\frac{(a^2+b^2)^{3/2}}{(a^2+b^2)} = \frac{1}{2}(a^2+b^2)^{1/2}$$

7. Find the radius of curvature at $(0,0)$ on curve $y^2=4x$
A-16
Given curve is $y^2=4x$ —①
Clearly, it passes through origin
Equating lowest degree term to zero $\Rightarrow x=0$
$\therefore$ Y-axis is tangent to given curve.
Divide (1) by $2x$, $y^2/2x = 2$

Take $\lim_{\substack{x\to 0\\ y\to 0}}$ so that $\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x} = \rho$

$\therefore \rho = 2.$

8. D-18 Find envelope of family of lines $x\cos\alpha + y\sin\alpha = p$, where α is the parameter.

Given: Family of lines is $x\cos\alpha + y\sin\alpha = p$

Differentiate partially w.r.t 'α', $-x\sin\alpha + y\cos\alpha = 0$

$$x\sin\alpha - y\cos\alpha = 0$$

$(1)^2 + (2)^2 \Rightarrow x^2\cos^2\alpha + y^2\sin^2\alpha + x^2\sin^2\alpha + y^2\cos^2\alpha = p^2$

$$x^2(\cos^2\alpha + \sin^2\alpha) + y^2(\cos^2\alpha + \sin^2\alpha) = p^2$$

$$\therefore x^2 + y^2 = p^2.$$

LAQs:

1. A-7 M-8 J-13 D-13 Find the coordinates of centre of curvature at any point of parabola $y^2 = 4ax$. Hence show that its evolute is $27a\bar{y}^2 = 4(\bar{x}-2a)^3$

Given parabola is $y^2 = 4ax$

Differentiate w.r.t 'x', $2yy' = 4a \Rightarrow y' = \frac{2a}{y}$

Diff w.r.t 'x', $y'' = -\frac{2a}{y^2}y' = -\frac{4a^2}{y^3}$

Now, $\bar{x} = x - \frac{y'(1+y'^2)}{y''} = x + \frac{2a}{y}\left(\frac{y^3}{4a^2}\right)\left(1+\frac{4a^2}{y^2}\right) = x + \frac{2}{4a}(y^2+4a^2)$

$$\bar{x} = x + 2x + 2a = 3x + 2a \quad \text{—} (1)$$

$\bar{y} = y + \frac{(1+y'^2)}{y''} = y + \left(1+\frac{4a^2}{y^2}\right)\left(-\frac{y^3}{4a^2}\right) = y - y - \frac{y^3}{4a^2} = \frac{-y^3}{4a^2} = -\frac{8(ax)^{3/2}}{4a^2}$

$$\bar{y} = -\frac{2}{\sqrt{a}}x^{3/2} \quad \text{—} (2)$$

From (1), $x = \frac{\bar{x}-2a}{3}$ sub// in (2)

$$\bar{y} = -\frac{2}{\sqrt{a}}\left(\frac{\bar{x}-2a}{3}\right)^{3/2}$$

$$\bar{y}^2 = \frac{4}{27a}(\bar{x}-2a)^3$$

$$\therefore 27a\bar{y}^2 = 4(\bar{x}-2a)^3$$

2. D-18 Find the evolute of the curve $x = a\cos^3 t$, $y = a\sin^3 t$.

Given: $x = a\cos^3 t$, $y = a\sin^3 t$

$\frac{dx}{dt} = -3a\sin t\cos^2 t$, $\frac{dy}{dt} = 3a\sin^2 t\cos t$

$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = y' = \frac{3a\sin^2 t\cos t}{-3a\sin t\cos^2 t} = -\tan t$

$y'' = -\sec^2 t$

$\bar{x} = x - y'\frac{(1+y'^2)}{y''} = a\cos^3 t + \tan t\frac{(1+\tan^2 t)}{-\sec^2 t} = a\cos^3 t - \tan t\frac{(\cancel{\sec^2 t})}{\cancel{\sec^2 t}}$
$= a\cos^3 t - \tan t.$

$\bar{y} = y + \frac{(1+y'^2)}{y''} = a\sin^3 t + \frac{(\sec^2 t)}{-\sec^2 t} = a\sin^3 t - 1$

$\therefore$ Required evolute is

$\bar{x} = a\cos^3 t - \tan t \;,\; y = a\sin^3 t - 1.$

3\. Find the envelope of family of straight lines $\frac{x}{a}+\frac{y}{b}=1$ where parameters a & b are connected by relation $a+b=c$.

D-8
J-11
J-12

Given family of straight lines is $\frac{x}{a}+\frac{y}{b}=1$ — (1)

$a+b=c$ — (2)

Diff (1) w.r.t a partially, treating 'b' as a function of 'a'.

$\frac{\partial}{\partial a}\left(\frac{x}{a}\right)+\frac{\partial}{\partial b}\left(\frac{y}{b}\right)\frac{db}{da}=0 \Rightarrow -\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$ — (3)

Diff (2), $\frac{\partial}{\partial a}(a)+\frac{\partial}{\partial b}(b)\frac{db}{da}=0 \Rightarrow \frac{db}{da}=-1$ — (4)

Sub^{ll} (4) in (3), $\frac{x}{a^2}=\frac{y}{b^2}$

$\frac{x}{a^2}=\frac{y}{b^2} \Rightarrow \frac{\frac{x}{a}}{a}=\frac{\frac{y}{b}}{b} \Rightarrow \frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{c}$ (from (1) & (2)).

$\frac{x}{a^2}=\frac{1}{c}$ & $\frac{y}{b^2}=\frac{1}{c}$

$a=\sqrt{cx}$, $b=\sqrt{cy}$ sub^{ll} in (1) or (2)

$\sqrt{cx}+\sqrt{cy}=c$

$\therefore \sqrt{x}+\sqrt{y}=c^{1/2}.$

4\. State and prove Lagranges mean value theorem.

D-17
A-7
J-13
J-19
J-15

Statement: Let $f(x)$ be a function defined on $[a,b] \ni$

i) $f(x)$ is continuous on $[a,b]$

ii) $f(x)$ is differentiable in (a,b)

then $\exists$ atleast one point $c\in(a,b) \ni f'(c)=\frac{f(b)-f(a)}{b-a}$

Proof: Define $g(x)=f(x)+Ax$ on $[a,b]$ where, — (1)

A is to be determined in such a way that $g(a)=g(b)$

$f(a)+Aa = f(b)+Ab \quad \Rightarrow A=-\frac{f(b)-f(a)}{b-a}$ — (2)

Given, $f(x)$ is continuous on $[a,b] \Rightarrow g(x)$ is continuous on $[a,b]$

$f(x)$ is differentiable on $(a,b) \Rightarrow g(x)$ is differentiable on (a,b)

sub^{ll} (2) in (1) $\Rightarrow g(a)=g(b)$

$\therefore g(x)$ is satisfying all conditions of Rolle's theorem, then $\exists$

$c \in (a,b) \ni g'(c) = 0$

$f'(c) + A = 0 \Rightarrow f'(c) = -A$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Hence proved.

5. State and prove Cauchy's Mean Value theorem:

M-9
M-18
D-8
D-18

Statement: Let $f(x)$ & $g(x)$ be two functions defined on $[a,b] \ni$

i) $f(x)$ & $g(x)$ are continuous on $[a,b]$

ii) $f(x)$ & $g(x)$ are differentiable on (a,b)

iii) $g'(x) \neq 0 \; \forall x \in (a,b)$ then $\exists$ atleast one point $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Proof: Define $h(x) = f(x) + kg(x)$ (1) on $[a,b]$ where k is to be defined in such a way that $h(a) = h(b)$.

$$f(a) + kg(a) = f(b) + kg(b) \Rightarrow k = \frac{f(b)-f(a)}{g(a)-g(b)} \quad (2)$$

Given, $f(x)$ & $g(x)$ are continuous on $[a,b] \Rightarrow h(x)$ is continuous on $[a,b]$.

$f(x)$ & $g(x)$ are differentiable on $(a,b) \Rightarrow h(x)$ ~~is~~ differentiable on (a,b).

Sub// (2) in (1) $\Rightarrow h(a) = h(b)$

$\therefore h(x)$ satisfies all conditions of Rolles theorem then $\exists c \in (a,b)$ $\ni h'(c) = 0$.

$$f'(c) + kg'(c) = 0$$

$$k = -\frac{f'(c)}{g'(c)}$$

$$\therefore \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Hence proved.

6. If $a<b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

J-17

Let $f(x) = \tan^{-1}x$ to be defined on $[a,b]$

$f'(c) = \frac{1}{1+x^2}$ on (a,b)

From LMVT, $\exists \; c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a} \quad (1)$$

Now, $c \in (a,b) \Rightarrow a < c < b$

$$1+a^2 < 1+c^2 < 1+b^2.$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

From (1), $\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$

$$\therefore \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

7. Find ρ at the origin for the curve $y-x=x^2+2xy+y^2$.

A-7
J-17

Given Curve is $y-x=x^2+2xy+y^2$ —— ①

Clearly, above curve passes through origin.

Equating lowest degree term to zero $\Rightarrow x=y$

$\therefore$ Neither x-axis nor Y-axis is tangent to ①.

Put $y=px+\frac{qx^2}{2}$ in ①

$$px+\frac{qx^2}{2}-x=x^2+2x\left(px+\frac{qx^2}{2}\right)+\left(px+\frac{qx^2}{2}\right)^2$$

$$px+\frac{qx^2}{2}-x=x^2+2px^2+qx^3+p^2x^2+q^2\frac{x^4}{4}+\cancel{2}pq\frac{x^3}{\cancel{2}} \quad —— ②$$

From ②, $p^2+2p+1=q/2$ $\quad$ | $p-1=0$ $\quad$ {Equating Coefficients of x and x^2.

$(1+2+1)2=q$ $\quad$ | $p=1$

$q=8$

$$\rho=\frac{(1+p^2)^{3/2}}{q}=\frac{2^{3/2}}{8}=\frac{1}{2\sqrt{2}}$$

$$\therefore \rho=\frac{1}{2\sqrt{2}}$$

UNIT-3 (Calculus in multi Variable) - Differentiation

SAQs:

J-19 D-16

1. Show that $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y}{x^6 + y^2}$ doesnot exist.

Sol $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$

Let $y = mx^3$ such that $y \to 0 \Rightarrow x \to 0$

$\lim_{x\to 0} \frac{mx^6}{x^6 + m^2x^6} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$ which is different for different values of 'm'.

$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y}{x^6 + y^2}$ doesnot exist.

D-18

2. Discuss the continuity of function -

$$f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases} \quad \text{at } (0,0)$$

Sol Consider: $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(x-y)^2}{x^2+y^2}$

Let $y = mx$ so that $y \to 0 \Rightarrow x \to 0$

$\lim_{x\to 0} \frac{x^2(1-m)^2}{x^2(1+m^2)} = \frac{(1-m)^2}{1+m^2}$ which is different for different values of m.

$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y)$ doesnot exist.

$\therefore f(x,y)$ is discontinuous at $(0,0)$.

D-16

3. Show that the function $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is not differentiable at $(0,0)$.

Sol We know, $\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2-0}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{1}{h}$ doesn't exist

$\therefore f$ is not differentiable at $(0,0)$.

4. If $u=x^2+y^2$ & $x=at^2$, $y=2at$ then find $\frac{du}{dt}$

Sol: We know, $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$= 2x(2at)+2y(2a)$$
$$= 4at(at^2)+4a(2at)$$
$$= 4a^2t^3 + 8a^2t$$
$$= 4a^2t(t^2+2).$$

J-19

5. If $xy+y^2-3x-3=0$, then evaluate $\frac{dy}{dx}$ at $(-1,1)$.

Sol Given: $xy+y^2-3x-3=0$.

We know, $\frac{dy}{dx} = -\frac{f_x}{f_y}$ —①

$f_x = y-3$, $f_y = x+2y$

From ①, $\frac{dy}{dx} = \frac{3-y}{x+2y}$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{3-1}{-1+2} = 2$$

J-17
D-16

6. If $u=x^2-y^2$, $v=2xy$ & $x=r\cos\theta$, $y=r\sin\theta$ then find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Sol Given: $u=x^2-y^2$, $v=2xy$

$x=r\cos\theta$, $y=r\sin\theta$

We know, $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(r,\theta)}$ —①

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2+4y^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta+\sin^2\theta) = r$$

From (1), $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r(x^2+y^2) = 4r(r^2\cos^2\theta+r^2\sin^2\theta) = 4r^3$

J-15

7. Find $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$

Sol Given: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$

Let $y=mx$ so that $y\to 0 \Rightarrow x\to 0$

$\lim\limits_{x\to 0} \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}$ which is different for different values of m.

J-15 $\therefore$ It doesnot exist.

8. If $u=\log(x^2+xy+y^2)$, then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Sol Given: $u=\log(x^2+xy+y^2)$

$$\frac{\partial u}{\partial x}=\frac{2x+y}{x^2+xy+y^2}\quad,\quad \frac{\partial u}{\partial y}=\frac{x+2y}{x^2+xy+y^2}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2x^2+xy+xy+2y^2}{x^2+xy+y^2}=\frac{2(x^2+xy+y^2)}{x^2+xy+y^2}=2.$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2$$

A-16 LAQs:

1. If $z=f(x,y)$ where $x=e^u+e^{-v}$ & $y=e^{-u}+e^v$ then show that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$.

Sol Given: $z=f(x,y)$, $x=e^u+e^{-v}$, $y=e^{-u}+e^v$ —— (1)

$$\Rightarrow \frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}(e^u)+\frac{\partial z}{\partial y}(-e^{-u})$$

$$\frac{\partial z}{\partial u}=e^u\frac{\partial z}{\partial x}-e^{-u}\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(e^v)$$

$$\frac{\partial z}{\partial v}=-\frac{\partial z}{\partial x}e^{-v}+\frac{\partial z}{\partial y}e^v$$

Now, $$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}e^u-\frac{\partial z}{\partial y}e^{-u}+\frac{\partial z}{\partial x}e^{-v}-\frac{\partial z}{\partial y}e^v$$

$$=\frac{\partial z}{\partial x}[e^u+e^{-v}]-\frac{\partial z}{\partial y}[e^{-u}+e^v]$$

$$=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}\qquad \{\because \text{From (1)}$$

$$\therefore\ \frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

J-19

2. Find the first three terms of Taylor's series of the function $f(x,y)=e^x\cos y$ around $(0,0)$.

Sol Given: $f(x,y)=e^x\cos y$.

$$f(x,y)=f(0,0)+[xf_x(0,0)+yf_y(0,0)]+\frac{1}{2!}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0),\ y^2f_{yy}(0,0)]+\cdots \text{—— (1)}$$

Given $\Rightarrow f(x,y)=e^x\cos y \Rightarrow f(0,0)=1$

$f_x(x,y) = e^x \cos y \Rightarrow f_x(0,0) = 1$

$f_y(x,y) = -e^x \sin y \Rightarrow f_y(0,0) = 0$

$f_{xx}(x,y) = e^x \cos y \Rightarrow f_{xx}(0,0) = 1$

$f_{yy}(x,y) = -e^x \cos y \Rightarrow f_{yy}(0,0) = -1$

$f_{xy}(x,y) = -e^x \sin y \Rightarrow f_y(0,0) = 0$

From ①, $e^x \cos y = 1 + [x+0] + \frac{1}{2!}[x^2 + 0 - y^2] + \cdots$

$\therefore e^x \cos y = 1 + x + \frac{1}{2}[x^2 - y^2] + \cdots$

F-10 M-12

3. If $f(x,y) = \begin{cases} \frac{x^2 y(x-y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ then show that $\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$ at $(0,0)$.

Sol

$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^3 y - x^2 y^2}{x^2+y^2}\right] = \frac{(x^2+y^2)(3x^2 y - 2xy^2) - (x^3 y - x^2 y^2)(2x)}{(x^2+y^2)^2}$

$= \frac{3x^4 y - 2x^3 y^2 + 3x^2 y^3 - 2xy^4 - 2x^4 y + 2x^3 y^2}{(x^2+y^2)^2}$

$= \frac{x^4 y + 3x^2 y^3 - 2xy^4}{(x^2+y^2)^2}$

Now, $f_x(x,y) = \begin{cases} \frac{x^4 y + 3x^2 y^3 - 2xy^4}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$f_{xy} = \frac{\partial}{\partial y} f_x = \lim_{k \to 0} \frac{f_x(x, y+k) - f_x(x,y)}{k} = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$ at $(0,0)$

$= \lim_{k \to 0} \frac{0-0}{k} = 0.$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^3 y - x^2 y^2}{x^2+y^2}\right] = \frac{(x^2+y^2)(x^3 - 2x^2 y) - (2y)(x^3 y - x^2 y^2)}{(x^2+y^2)^2}$

$= \frac{x^5 - 2x^4 y + x^3 y^2 - 2x^2 y^3 - 2x^3 y^2 + 2x^2 y^3}{(x^2+y^2)^2}$

$= \frac{x^5 - x^3 y^2 - 2x^4 y}{(x^2+y^2)^2}$

Now, $f_y(x,y) = \begin{cases} \frac{x^5 - x^3 y^2 - 2x^4 y}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$f_{yx} = \frac{\partial}{\partial x} f_y = \lim_{h \to 0} \frac{f_y(x+h, y) - f(x,y)}{h} = \lim_{h \to 0} \frac{f_y(h,0) - f(0,0)}{h}$ at $(0,0)$

$= \lim_{k \to 0} \frac{h^5/h^4}{h} = 1$

$$\frac{\partial^2 f}{\partial x \partial y} = 1 \quad \& \quad \frac{\partial^2 f}{\partial y \partial x} = 0 \quad \text{at } (0,0)$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x} \text{ at } (0,0).$$

D-18

4. Investigate the function x^3+y^3-3axy for maxima and minima.

Sol Let $u(x,y) = x^3+y^3-3axy$

$p = u_x = 3x^2-3ay$, $q = u_y = 3y^2-3ax$, $r = u_{xx} = 6x$,
$s = u_{xy} = 3a$, $t = u_{yy} = 6y$

Now, $p=0 \Rightarrow 3x^2-3ay=0 \Rightarrow x^2 = ay$ — (1)
$q=0 \Rightarrow 3y^2-3ax=0 \Rightarrow y^2 = ax$ — (2)

Sq^n (1) $\Rightarrow x^4 = a^2y^2 = a^2(ax)$
$\Rightarrow x^4 - a^3x = 0 \Rightarrow x(x^3-a^3) = 0$
$\Rightarrow x=0,\ x=a$

If $x=0$ then from (1), $y=0$ & if $x=a$ then $y=a$
$\therefore$ Stationary points are $(0,0)$ & (a,a).

At $(0,0) \Rightarrow r=0,\ s=-3a,\ t=0.$
$rt-s^2 = -9a^2 < 0$
$\therefore (0,0)$ is a saddle point.

At $(a,a) \Rightarrow r=6a,\ s=-3a,\ t=6a$
$rt-s^2 = 36a^2-9a^2 = 27a^2>0$ & $r=6a$
$\therefore f(x,y)$ is minimum at (a,a) if $a>0$
$f(x,y)$ is maximum at (a,a) if $a<0$.

J-17

5. Find the minimum value of $x+y+z$, subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$.

Sol Let $f(x,y,z) = x+y+z$ subject to $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ — (1)

From Lagrange's method,
$F = \text{function} + \lambda(\text{Condition}) = (x+y+z) + \lambda\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right)$

$\frac{\partial F}{\partial x} = 1 - \frac{a\lambda}{x^2} = 0 \Rightarrow a\lambda = x^2 \Rightarrow \frac{x^2}{a} = \lambda$ — (2)

$\frac{\partial F}{\partial y} = 1 - \frac{b\lambda}{y^2} = 0 \Rightarrow b\lambda = y^2 \Rightarrow \frac{y^2}{b} = \lambda$ — (3)

$\frac{\partial F}{\partial z} = 1 - \frac{c\lambda}{z^2} = 0 \Rightarrow c\lambda = z^2 \Rightarrow \frac{z^2}{c} = \lambda$ — (4)

Solve (1), (2), (3) & (4) to get x, y, z & λ.

Now, from (2), (3) & (4),

$$\frac{x^2}{a} = \frac{y^2}{b} = \frac{z^2}{c}$$

$$\frac{x}{\sqrt{a}} = \frac{y}{\sqrt{b}} = \frac{z}{\sqrt{c}} = k \text{ (say)} \quad \text{—} (5)$$

$$x = \sqrt{a}k, \quad y = \sqrt{b}k, \quad z = \sqrt{c}k$$

Sub^n these in (1), we get

$$\frac{a}{\sqrt{a}k} + \frac{b}{\sqrt{b}k} + \frac{c}{\sqrt{c}k} = 1$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = k \quad \text{—} (6)$$

Now, from (5), $x = \sqrt{a}k = \sqrt{a}(\sqrt{a} + \sqrt{b} + \sqrt{c})$

$y = \sqrt{b}k = \sqrt{b}(\sqrt{a} + \sqrt{b} + \sqrt{c})$

$z = \sqrt{c}k = \sqrt{c}(\sqrt{a} + \sqrt{b} + \sqrt{c})$

$\therefore$ These are the values of x, y, z for which $f(x,y,z)$ attains min.

Min value is

$$f(x,y,z) = (x + y + z)$$

$$= \sqrt{a}(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sqrt{b}(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sqrt{c}(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$$= (\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$= (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$ is the minimum value.

UNIT-4. Multi-Variable Calculus (Integration).

SAQ's -

1. Evaluate $\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dx\,dy$

J-18

$$\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dx\,dy=\int_{x=0}^1\left[\int_{y=x}^{\sqrt{x}}(x^2+y^2)dy\right]dx$$

$$=\int_{x=0}^1\left[x^2y+\frac{y^3}{3}\right]_{y=x}^{\sqrt{x}}dx=\int_{x=0}^1\left[x^{5/2}+\frac{x^{3/2}}{3}-x^3-\frac{x^3}{3}\right]dx$$

$$=\int_{x=0}^1\left[x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right]dx$$

$$=\left[\frac{x^{5/2+1}}{\frac{5}{2}+1}+\frac{1}{3}\frac{x^{3/2+1}}{\frac{3}{2}+1}-\frac{4}{3}\frac{x^4}{4}\right]_0^1=\frac{2}{7}+\frac{2}{15}-\frac{1}{3}=\frac{3}{35}.$$

2. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

J-11, D-15

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy=\int_0^\infty e^{-y^2}\left[\int_0^\infty e^{-x^2}dx\right]dy=\int_0^\infty e^{-y^2}\left[\frac{\sqrt{\pi}}{2}\right]dy\qquad\left[\because\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\right]$$

$$=\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-y^2}dy=\frac{\sqrt{\pi}}{2}\frac{\sqrt{\pi}}{2}=\frac{\pi}{4}.$$

3. Evaluate $\int_0^{\pi/4}\int_{\sin x}^{\cos x}dy\,dx$.

D-18

$$\int_0^{\pi/4}\int_{\sin x}^{\cos x}dy\,dx=\int_0^{\pi/4}[y]_{y=\sin x}^{\cos x}dx=\int_0^{\pi/4}[\cos x-\sin x]dx$$

$$=[\sin x+\cos x]_{x=0}^{\pi/4}=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}-\sin 0-\cos 0$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1=\frac{2}{\sqrt{2}}-1$$

$$=\sqrt{2}-1$$

4. $\int_0^1\int_0^{1-x}\int_0^{1-x-y}dx\,dy\,dz.$

D-18

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y}dx\,dy\,dz=\int_{x=0}^1\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}dz\,dy\,dx=\int_{x=0}^1\int_{y=0}^{1-x}[1-x-y]\,dy\,dx$$

$$=\int_{x=0}^1\left[y-xy-\frac{y^2}{2}\right]_{y=0}^{1-x}dx=\int_{x=0}^1\left[(1-x)-x(1-x)-\frac{(1-x)^2}{2}\right]dx$$

$$=\int_{x=0}^1\left[(1-x)^2-\frac{1}{2}(1-x)^2\right]dx=\frac{1}{2}\int_{x=0}^1(1-x)^2dx$$

$$=\frac{1}{2}\left[\frac{-(1-x)^3}{3}\right]_0^1=-\frac{1}{2}\left[\frac{(1-1)^3}{3}-\frac{1}{3}\right]=\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)=\frac{1}{6}$$

5. J-19 Evaluate $\int_0^{\pi/6}\int_0^1\int_{-1}^3 y\sin z \, dx\,dy\,dz$

$$\int_0^{\pi/6}\int_0^1\int_{-1}^3 y\sin z \, dx\,dy\,dz = \int_0^{\pi/3} \sin z\,dz \int_0^1 y\,dy \int_{-1}^3 dx$$

$$= [-\cos z]_0^{\pi/6} \left[\frac{y^2}{2}\right]_0^1 [x]_{-1}^3$$

$$= \left[\cos 0 - \cos\frac{\pi}{6}\right]\left[\frac{1}{2}\right]\left[3-(-1)\right]$$

$$= 2\left[1-\frac{\sqrt{3}}{2}\right] = 2-\sqrt{3}.$$

6. Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$ where V is region of space bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$

$$\iiint_V (xy+yz+zx)\,dx\,dy\,dz = \int_{z=0}^3 \int_{y=0}^2 \int_{x=0}^1 (xy+yz+zx)\,dx\,dy\,dz$$

$$= \int_{z=0}^3 \int_{y=0}^2 \left[\frac{x^2y}{2} + xyz + \frac{zx^2}{2}\right]_0^1 dy\,dz$$

$$= \int_{z=0}^3 \int_{y=0}^2 \left[\frac{y}{2} + yz + \frac{z}{2}\right] dy\,dz$$

$$= \int_{z=0}^3 \left[\frac{y^2}{4} + \frac{zy^2}{2} + \frac{zy}{2}\right]_{y=0}^2 dz$$

$$= \int_{z=0}^3 [1+2z+z]\,dz = \left[z + 2\frac{z^2}{2} + \frac{z^2}{2}\right]_{z=0}^3$$

$$= 3+9+\frac{9}{2} = \frac{33}{2}.$$

7. Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of circle $x^2+y^2=a^2$.

Given: $\iint xy\,dx\,dy$

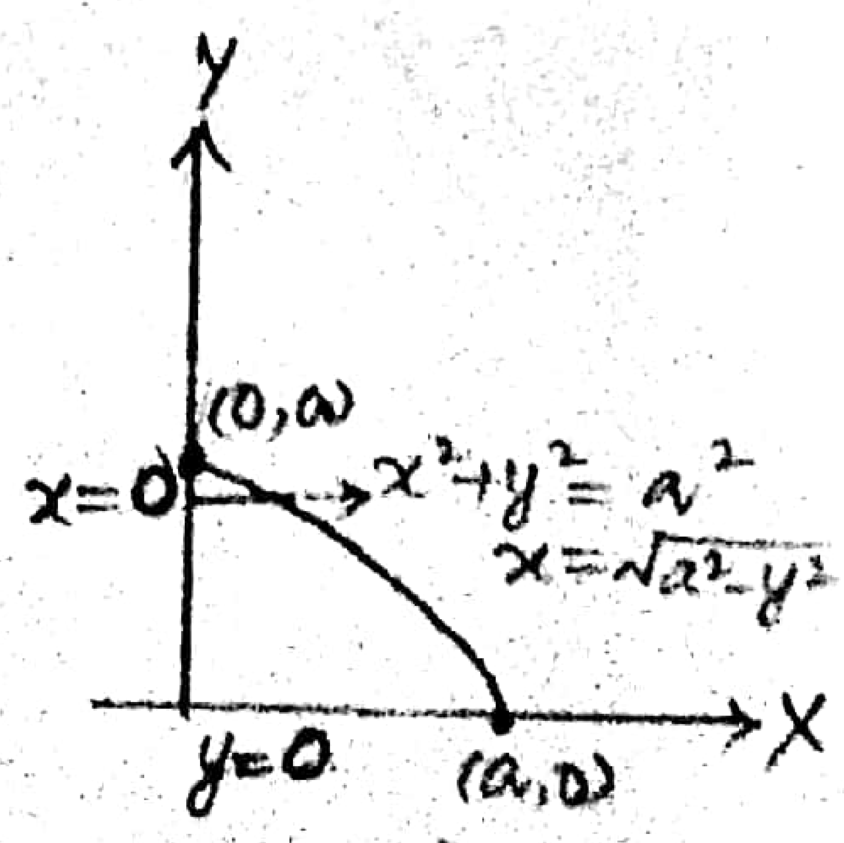
Limits for given region are $0 \le x \le \sqrt{a^2-y^2}$ & $0 \le y \le a$

$$\therefore \iint xy\,dx\,dy = \int_{y=0}^a \left[\int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\right] dy = \int_{y=0}^a \left[y\frac{x^2}{2}\right]_{x=0}^{\sqrt{a^2-y^2}} dy$$

$$= \int_{y=0}^a y\frac{(a^2-y^2)}{2}\,dy = \int_{y=0}^a \left[\frac{a^2y-y^3}{2}\right] dy$$

$$= \left[\frac{a^2}{2}\frac{y^2}{2} - \frac{1}{2}\frac{y^4}{4}\right]_{y=0}^a = \frac{a^2}{2}\frac{a^2}{2} - \frac{1}{2}\frac{a^4}{4}$$

$$= \frac{a^4}{4} - \frac{a^4}{8} = \frac{2a^4-a^4}{8} = \frac{a^4}{8}.$$



8. Evaluate $\int_0^{\pi}\int_0^{1} x\cos xy\,dy\,dx$.

$\int_0^{\pi}\int_0^{1} x\cos xy\,dy\,dx = \int_0^{\pi}\left[x\,\frac{\sin xy}{x}\right]_{y=0}^{1} dx = \int_0^{\pi}[\sin x - \sin 0]\,dx$

$= [-\cos x]_0^{\pi} = \cos 0 - \cos\pi = 1-(-1) = 2.$

LAQ's:

1. Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar co-ordinates.

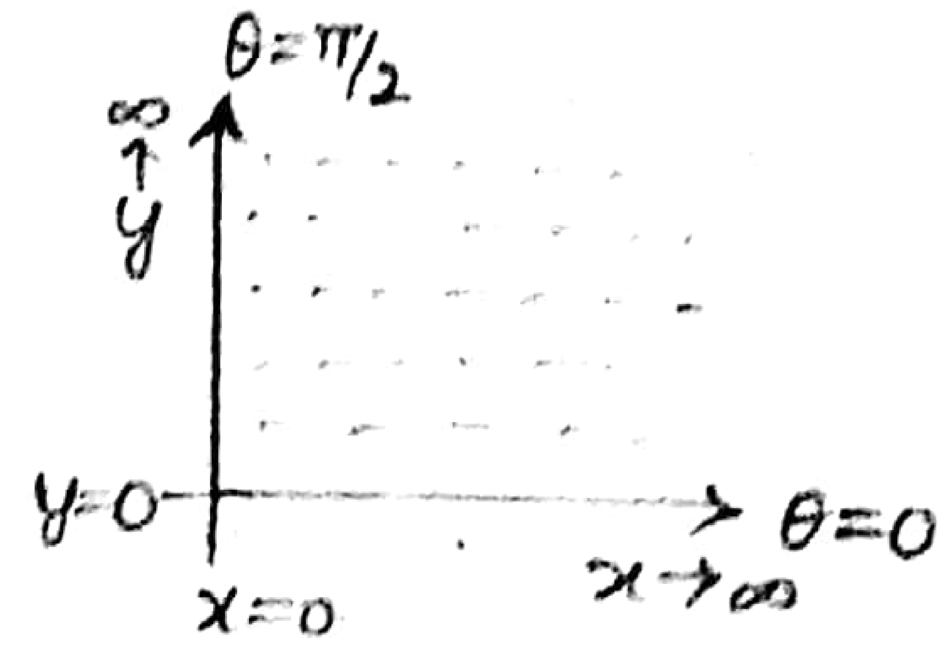
D-18,17

Given limits are $0 \le x \le \infty$, $0 \le y < \infty$

For changing to polar coordinates,

put $x = r\cos\theta$, $y = r\sin\theta$ & $dx\,dy = r\,dr\,d\theta$

Polar limits will be $0 \le \theta \le \pi/2$, $0 \le r < \infty$



$\therefore \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$

Put $r^2 = t$

$2r\,dr = dt$

limits,

$r = 0 \Rightarrow t = 0$

$r \to \infty \Rightarrow t \to \infty$

$= \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_{t=0}^{\infty} d\theta$

$= \frac{1}{2}\int_{\theta=0}^{\pi/2}[-e^{-\infty}+e^{0}]\,d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}[0+1]\,d\theta$

$= \frac{1}{2}[\theta]_0^{\pi/2} = \pi/4.$

2. Evaluate $\int_0^{1}\int_0^{\sqrt{1-y^2}}(x^2+y^2)\,dy\,dx$ by changing to polar coordinates.

J-19

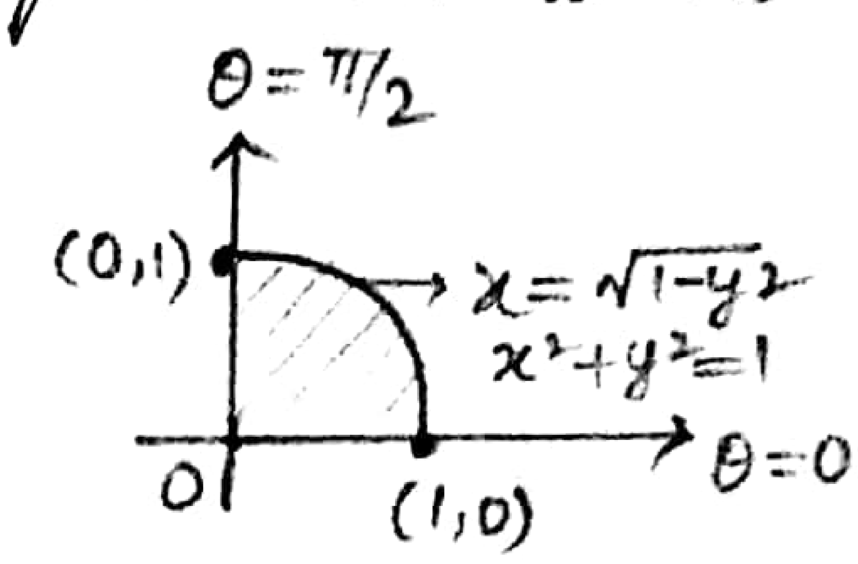
Given limits are $0 \le y \le 1$ & $0 \le x \le \sqrt{1-y^2}$

$\Rightarrow y = 0$ to $y = 1$ & $x = 0$ to $x = \sqrt{1-y^2}$

For changing polar coordinates,

Put $x = r\cos\theta$, $y = r\sin\theta$ & $dx\,dy = r\,dr\,d\theta$.

Polar limits will be $0 \le \theta \le \pi/2$, $0 \le r \le 1$



$\therefore \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(r^2)\,r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 d\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2} d\theta$

$\frac{1}{4}[\theta]_{\theta=0}^{\pi/2} = \frac{\pi}{2}\left(\frac{1}{4}\right) = \frac{\pi}{8}.$

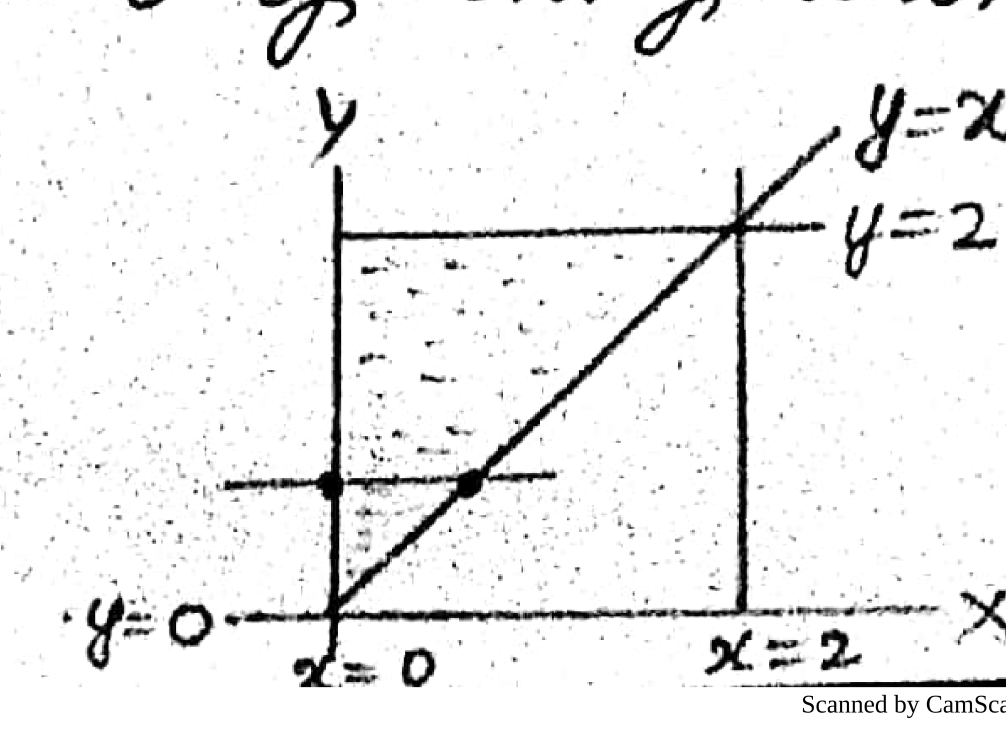
3. Evaluate $\int_0^{2}\int_x^{2} 2y^2\sin xy$ by changing the order of integration.

D-18

Given limits are $0 \le x \le 2$, $x \le y \le 2$

New limits will be, $0 \le x \le y$, $0 \le y \le 2$

After changing order, we get



$$\int_{y=0}^{2}\int_{x=0}^{y} 2y^2 \sin xy \, dx\, dy = 2\int_{y=0}^{2} y^2\left[\frac{-\cos xy}{y}\right]_{x=0}^{y} dy$$

$$= -2\int_{y=0}^{2} y\left[\cos y^2 - 1\right] dy = 2\int_{y=0}^{2} y\left[1-\cos y^2\right] dy$$

$$= \cancel{2}\int_{\cancel{t}=0}^{4} (1-\cos t)\frac{dt}{\cancel{2}} = \left[t - \sin t\right]_{t=0}^{4}$$

$$= 4 - \sin 4 - 0 + 0 = 4 - \sin 4$$

Put $y^2 = t$
$\Rightarrow 2y\,dy = dt$
Limits:
$y \to 0 \Rightarrow t \to 0$
$y \to 2 \Rightarrow t \to 4$.

---

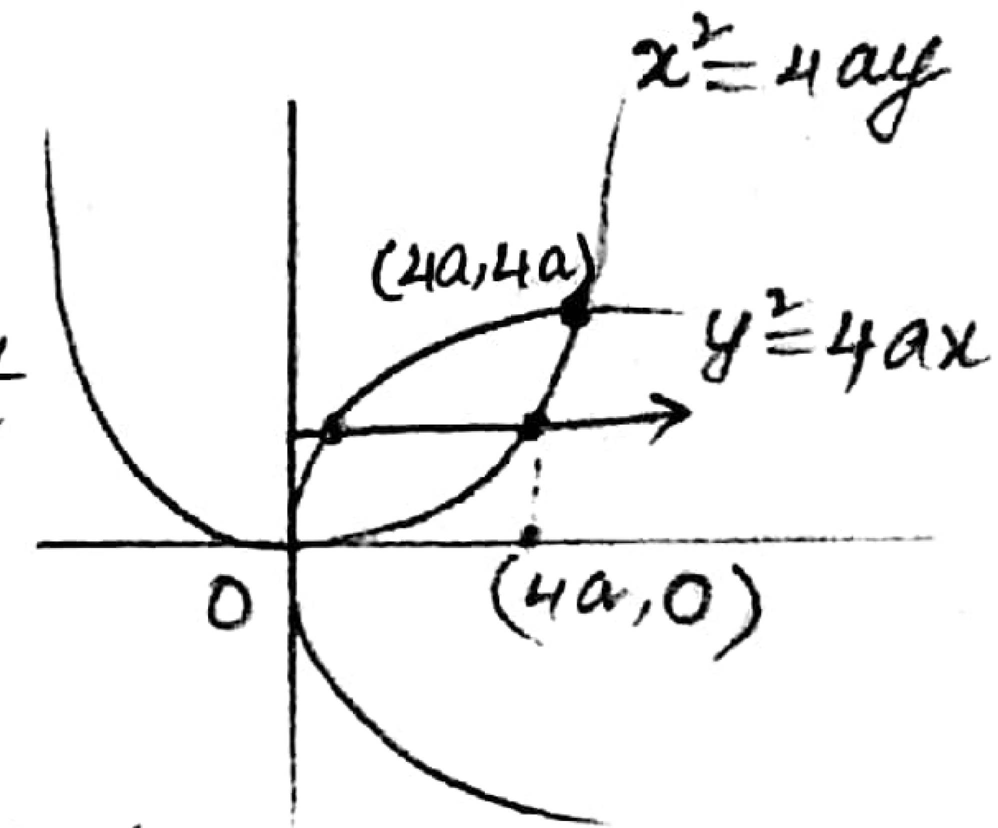
4. Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration

Given limits are, $0 \le x \le 4a$, $\frac{x^2}{4a} \le y \le 2\sqrt{ax}$

$\Rightarrow x^2 = 4ay$, $y^2 = 4ax$ & $x=0$ to $x=4a$

By changing order of integration, we get

$\frac{y^2}{4a} \le x \le 2\sqrt{ay}$ & $0 \le y \le 4a$



$$\therefore \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}\left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy$$

$(\int dx = x)$

$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3}\right]_{y=0}^{4a}$$

$$= \frac{4}{3}a^{1/2}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

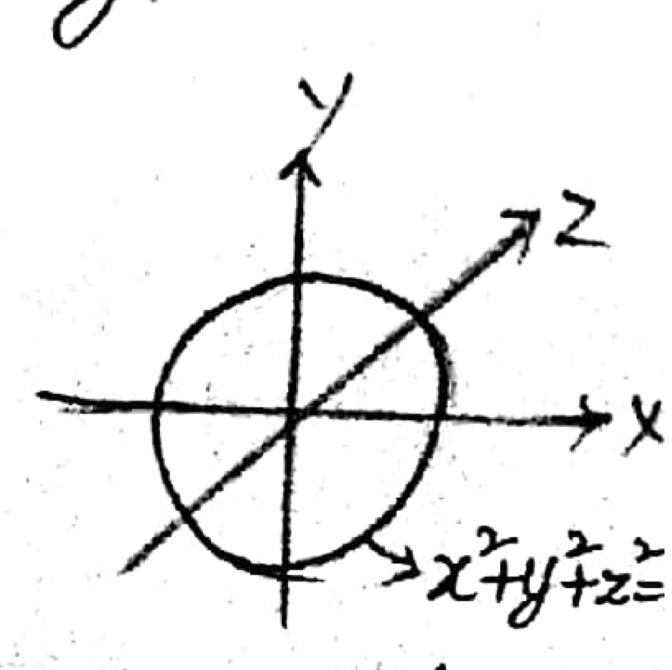
$$= \frac{32}{3}a^2 - \frac{64a^2}{12} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16a^2}{3}.$$

---

5. Find the volume of unit sphere $x^2+y^2+z^2=1$.
D-18

Given region is a sphere & its volume is given by

$v = \iiint_S dx\,dy\,dz$ — ① where $S: x^2+y^2+z^2=1$



Since, given is a sphere, we transform ① to spherical coordinates. Put

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ & $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

New limits will be $0 \le r \le 1$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$.

$$\therefore V = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2\sin\theta\,dr\,d\theta\,d\phi = \int_{\phi=0}^{2\pi} d\phi\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{r=0}^{1} r^2\,dr$$

$$= [\phi]_0^{2\pi}[-\cos\theta]_0^{\pi}\left[\frac{r^3}{3}\right]_0^1 = 2\pi(-(-1)+1)\frac{1}{3} = 4\frac{\pi}{3}.$$

6. Evaluate $\iiint z(x^2+y^2)\,dx\,dy\,dz$ over $x^2+y^2\le 1,\ 2\le z\le 3$.

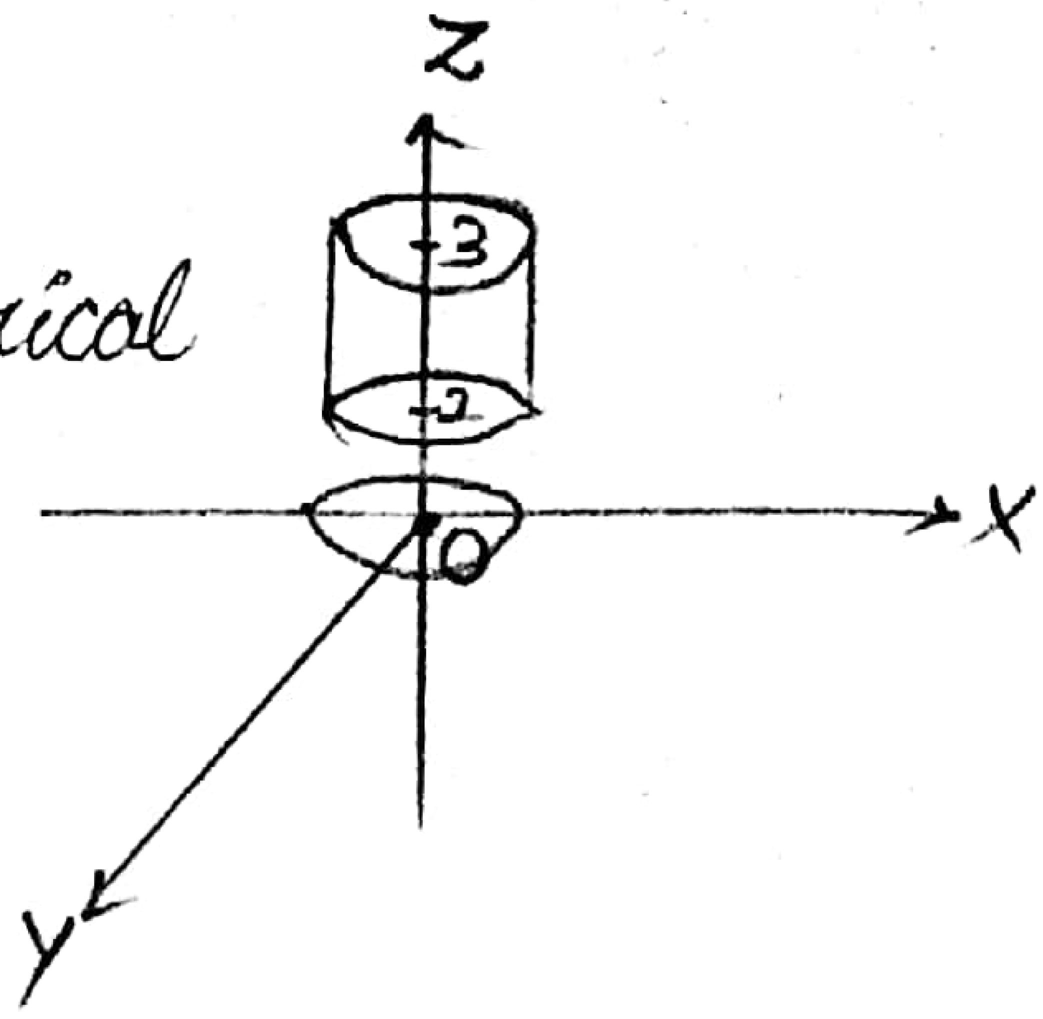
Given: $\iiint z(x^2+y^2)\,dx\,dy\,dz$ — ①

Region is formed by $x^2+y^2\le 1,\ 2\le z\le 3$

Changing this triple integral to cylindrical co-ordinates,

put $x=r\cos\theta,\ y=r\sin\theta$ & $z=z$.

$\therefore dx\,dy\,dz = r\,dr\,d\theta\,dz$.



New limits are: $0\le\theta\le 2\pi,\ 0\le r\le 1,\ 2\le z\le 3$

From (1), $\int_{z=2}^{3}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} z(r^2\cos^2\theta+r^2\sin^2\theta)\,r\,dr\,d\theta\,dz$

$$=\int_{z=2}^{3} z\,dz\int_{\theta=0}^{2\pi}d\theta\int_{r=0}^{1} r^3\,dr$$

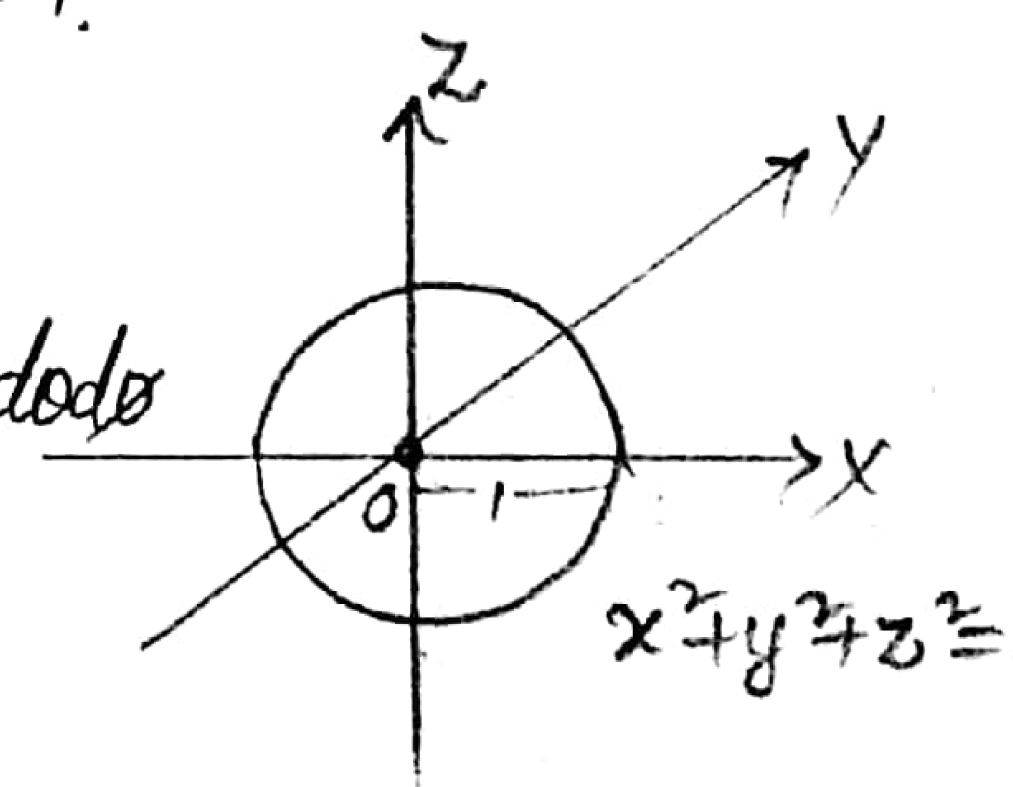
$$=\left[\frac{z^2}{2}\right]_2^3\left[\theta\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 = \frac{5}{2}\cdot 2\pi\cdot\frac{1}{4} = \frac{5\pi}{4}$$

★ 7. Evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$, by transforming co-ordinates.

Given: $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ — ①, $x^2+y^2+z^2\le 1$.

Put $x=r\sin\theta\cos\phi$
$y=r\sin\theta\sin\phi$
$z=r\cos\theta$

$x^2+y^2+z^2=r^2$
$J=r^2\sin\theta$
i.e., $dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\phi$



From ①, $\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2(r^2\sin\theta\,dr\,d\theta\,d\phi)$

$$=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\frac{r^5}{5}\right]_{r=0}^{1}\sin\theta\,d\theta\,d\phi$$

Limits are:
$0\le\phi\le 2\pi$
$0\le\theta\le\pi$
$0\le r\le 1$

$$=\frac{1}{5}\int_{\phi=0}^{2\pi}\left[-\cos\theta\right]_{\theta=0}^{\pi}d\phi$$

$$=\frac{1}{5}\int_{\phi=0}^{2\pi}(-(-1)+1)\,d\phi=\frac{2}{5}(\phi)_{\phi=0}^{2\pi}$$

$$=\frac{4\pi}{5}.$$

UNIT-5 (Vector Calculus).

SAQs:

D-14
1. Find ∇f at $(1,2,-1)$ if $f(x,y,z)=\log_e(x+y+z)$

Given $f(x,y,z)=\log(x+y+z)$

$$\frac{\partial f}{\partial x}=\frac{1}{x+y+z},\ \frac{\partial f}{\partial y}=\frac{1}{x+y+z},\ \frac{\partial f}{\partial z}=\frac{1}{x+y+z}$$

$$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k=\frac{1}{x+y+z}i+\frac{1}{x+y+z}j+\frac{1}{x+y+z}k$$

$$\nabla f=\frac{1}{x+y+z}(i+j+k)$$

At $(1,2,-1)$, $\nabla f=\frac{1}{2}(i+j+k)$

J-15
2. Find $\operatorname{div}\bar{F}$, where $\bar{F}=\operatorname{grad}(x^3+y^3+z^3-3xyz)$

Given: $\bar{F}=\operatorname{grad}(x^3+y^3+z^3-3xyz)$

$$=i(3x^2-3yz)+j(3y^2-3xz)+\bar{k}(3z^2-3xy)$$

Now, $\operatorname{div}F=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial F_3}{\partial z}$

$$=6x+6y+6z=6(x+y+z)$$

D-15
3. Evaluate $\nabla\left(\frac{1}{r}\right)$ where $r=|\bar{r}|$, $\bar{r}=xi+yj+zk$.

Let $\bar{r}=xi+yj+zk \Rightarrow |\bar{r}|=\sqrt{x^2+y^2+z^2}$

$$r^2=x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r}\ \&\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\therefore \nabla\frac{1}{r}=\frac{\partial}{\partial x}\left(\frac{1}{r}\right)i+\frac{\partial}{\partial y}\left(\frac{1}{r}\right)j+\frac{\partial}{\partial z}\left(\frac{1}{r}\right)k$$

$$=\frac{\partial}{\partial x}\left(\frac{i}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial y}\left(\frac{j}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial z}\left(\frac{k}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}i+\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}j+\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}k$$

$$=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2xi)-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2yj)-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2zk)$$

$$=-\frac{1}{(x^2+y^2+z^2)^{3/2}}[xi+yj+zk]=-\frac{1}{(r^2)^{3/2}}[\bar{r}]$$

$$=-\frac{\bar{r}}{r^3}$$

$$\therefore \nabla\left(\frac{1}{r}\right)=-\frac{\bar{r}}{r^3}.$$

4. If $\bar{a}$ is a constant vector & $\bar{r} = xi+yj+zk$ then evaluate $\operatorname{div}(\bar{a}\times\bar{r})$ & $\operatorname{curl}(\bar{a}\times\bar{r})$ & prove that $\nabla(\bar{a}\times\bar{r})=0$.

D-18, J-19

$\bar{a} = a_1 i + a_2 j + a_3 k$, $\bar{r} = xi+yj+zk$

$$\bar{a}\times\bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = i(a_2 z - a_3 y) - j(a_1 z - a_3 x) + k(a_1 y - a_2 x)$$

$\therefore \operatorname{div}(\bar{a}\times\bar{r}) = \sum \frac{\partial}{\partial x}(a_2 z - a_3 y) = 0$

$$\operatorname{Curl}(\bar{a}\times\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & a_1 z - a_3 x & a_1 y - a_2 x \end{vmatrix}$$

$= \bar{i}(a_1 - a_1) - \bar{j}(-a_2 - a_2) + \bar{k}(a_3 + a_3) = 2a_2 j$

$\therefore \operatorname{curl}(\bar{a}\times\bar{r}) = 2a_2\bar{j}$

$\nabla(\bar{a}\times\bar{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y)\bar{i} + \frac{\partial}{\partial y}(a_1 z - a_3 x)\bar{j} + \frac{\partial}{\partial z}(a_1 y - a_2 x)\bar{k} = 0$

$\therefore \nabla(\bar{a}\times\bar{r}) = 0.$

5. Show that $\bar{v} = 12xi - 15y^2 j + k$ is irrotational.

D-17

Given $\bar{v} = 12x\bar{i} - 15y^2\bar{j} + k$

$$\operatorname{curl} f = \nabla\times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 12x & -15y^2 & 1 \end{vmatrix} = i(0-0) - j(0-0) + k(0-0) = 0$$

$\therefore \operatorname{curl} f = 0$, $\bar{v}$ is irrotational.

6. Show that vector $e^{x+y-2z}(\bar{i}+\bar{j}+\bar{k})$ is solenoidal

JL-17

Given $\bar{f} = e^{x+y-2z}(i+j+k)$

$f_1 = f_2 = f_3 = e^{x+y-2z}$

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$= e^{x+y-2z} + e^{x+y-2z} - 2e^{x+y-2z}$

$= 2e^{x+y-2z} - 2e^{x+y-2z}$

$= 0.$

7. Prove that $\operatorname{curl}(\operatorname{grad} f) = \bar{0}$ where f is differentiable scalar field.

D-16

Let f be a scalar function.

$$\operatorname{grad}\bar{f} = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

$$\operatorname{Curl}(\operatorname{grad}\bar{f}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$= \sum i\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right]$

$= \sum i(0) = 0$

$\therefore$ curl(grad $\bar{f}$) = 0.

---

8. Find unit normal vector to the surface $f(x,y,z) = x^2y - y^2z - xyz$ at $P(1,-1,0)$

D-18

Let $\phi(x,y,z) = x^2y - y^2z - xyz$

Normal to ϕ is $\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\nabla\phi = (2xy - yz)i + (x^2 - 2yz - xz)j + (-y^2 - xy)k$

Normal to ϕ at $P(1,-1,0)$ $= \nabla\phi|_P = [2(-1)-(-1)0]i + (1-0)j + 0k$

$= -2i + j$

$|\nabla\phi| = \sqrt{4+1} = \sqrt{5}$

$\therefore$ Unit normal to ϕ is $\frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{5}}(-2i+j)$

---

9. In what direction from $(3,1,-2)$ is directional derivative of $f(x,y,z) = xy^2z^3$ maximum?

D-16

Given: $\phi = xy^2z^3$

$\frac{\partial\phi}{\partial x} = y^2z^3, \quad \frac{\partial\phi}{\partial y} = 2xyz^3, \quad \frac{\partial\phi}{\partial z} = 3xy^2z^2$

Given point is $P(3,1,-2)$

Directional derivative will be max in direction of

$\text{grad}\,\phi|_{P(3,1,-2)} = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = y^2z^3i + 2xyz^3j + 3xy^2z^2k$

$= (1)(-2)^3i + 2(3)(-2)^3j + 3(3)(-2)^2k$

$= -8i - 48j + 36k$

$= -4[2i + 12j - 9k]$

---

10. Evaluate $\int_C \bar{v}\cdot d\bar{r}$ where $v = xi + yj + zk$ & C is line segment from $A(1,2,2)$ to $B(3,6,6)$.

J-19

We know, $d\bar{r} = dxi + dyj + dzk$

$\int_C \bar{v}\cdot d\bar{r} = \int_C xdx + ydy + zdz$

Given C is a L.S joining $A(1,2,2)$ & $B(3,6,6)$.

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \Rightarrow \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-2}{4}$

$x-1 = \frac{y-2}{2} = \frac{z-2}{2}$

$x-1 = \frac{y-2}{2}$ $\qquad$ $x-1 = \frac{z-2}{2}$

$2x-2 = y-2$ $\qquad$ $2x-2 = z-2$

$y = 2x \Rightarrow dy = 2dx$ $\qquad$ $z = 2x$

$$\int_C \bar{v}\cdot d\bar{r} = \int_{x=1}^{3} x\,dx + 2x\cdot 2dx + 2x\cdot 2dx = \int_{x=1}^{3} 9x\,dx = 9\left[\frac{x^2}{2}\right]_1^3 = \frac{9}{2}[9-1] = 36.$$

$$\therefore \int_C \bar{v}\cdot d\bar{r} = 36.$$

LAQs:

1\. Prove that $\nabla\cdot(\nabla\times\bar{A}) = 0$ where $\bar{A}$ is a vector point function.

D-17

Let $\bar{A} = A_1 i + A_2 j + A_3 k$

$$\nabla\times\bar{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = i\left(\frac{\partial}{\partial y}A_3 - \frac{\partial}{\partial z}A_2\right) - j\left(\frac{\partial}{\partial x}A_3 - \frac{\partial}{\partial z}A_1\right) + k\left(\frac{\partial}{\partial x}A_2 - \frac{\partial}{\partial y}A_1\right)$$

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

$$\Rightarrow \nabla\cdot(\nabla\times A) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}A_3 - \frac{\partial}{\partial z}A_2\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}A_3 - \frac{\partial}{\partial z}A_1\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}A_2 - \frac{\partial}{\partial y}A_1\right)$$

$$= \frac{\partial^2}{\partial x\partial y}A_3 - \frac{\partial^2}{\partial x\partial z}A_2 - \frac{\partial^2}{\partial y\partial x}A_3 + \frac{\partial^2}{\partial y\partial z}A_1 + \frac{\partial^2}{\partial z\partial x}A_2 - \frac{\partial^2}{\partial z\partial y}A_1$$

$$= 0$$

$$\therefore \nabla\cdot(\nabla\times\bar{A}) = 0.$$

2\. Prove that $\nabla(\log r) = \frac{\bar{r}}{r^2}$ where $\bar{r} = xi + yj + zk$ & $r = |\bar{r}|$

JL-17

Let $\bar{r} = xi + yj + zk$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$$

$$r^2 = x^2+y^2+z^2$$

$$\therefore \log r = \log\sqrt{x^2+y^2+z^2}$$

$$\nabla(\log r) = \frac{\partial}{\partial x}(\log r)i + \frac{\partial}{\partial y}(\log r)j + \frac{\partial}{\partial z}(\log r)k$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1\,(2xi)}{2\sqrt{x^2+y^2+z^2}} + \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1}{2}\frac{2yj}{\sqrt{x^2+y^2+z^2}}$$

$$+ \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1\,2zk}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{xi}{x^2+y^2+z^2} + \frac{yj}{x^2+y^2+z^2} + \frac{zk}{x^2+y^2+z^2}$$

$$= \frac{xi+yj+zk}{x^2+y^2+z^2} = \frac{\bar{r}}{r^2}.$$

3. Find the constants a, b, c such that $F = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$ is irrotational & find a scalar function f such that $\bar{F} = \nabla f$.

J-17

Given: $\bar{F} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$ is irrotational.

$$\nabla \times F = 0 \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = 0$$

$= i(c-3) - j(2-a) + k(b+3) = 0$

$c = 3, \quad a = 2, \quad b = -3.$

Now, $F = (2x+3y+2z)i + (-3x+2y+3z)j + (2x+3y+3z)k$

Given, $\bar{F} = \nabla f$

$(2x+3y+2z)i + (-3x+2y+3z)j + (2x+3y+3z)k = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

Comparing, $\frac{\partial f}{\partial x} = 2x+3y+2z \Rightarrow f = \frac{2x^2}{2} + 3xy + 2xz + f(y,z)$

$\frac{\partial f}{\partial y} = -3x+2y+3z \Rightarrow f = -3xy + \frac{2y^2}{2} + 3zy + f(x,z)$

$\frac{\partial f}{\partial z} = 2x+3y+3z \Rightarrow f = 2xz + 3yz + \frac{3z^2}{2} + f(x,y)$

$\therefore f = x^2 + y^2 + \frac{3z^2}{2} + 2xz + 3yz \pm 3xy + C.$

---

4. Find directional derivative of $\phi = xy^2 + yz^3$ at point $(2,-1,1)$ in direction of normal to surface $x\log z - y^2 = -4$ at $(-1,2,1)$.

D-17

Given: $\phi_1 = xy^2 + yz^3$

$P(x,y,z) = (2,-1,1)$

$\text{Grad}\, \phi_1 = \frac{\partial \phi_1}{\partial x} i + \frac{\partial \phi_1}{\partial y} j + \frac{\partial \phi_1}{\partial z} k = y^2 i + (2xy + z^3) j + 3yz^2 k$

$\text{Grad}\, \phi|_P = i + (-4+1)j + 3(-1)k = i - 3j - 3k$

Let $\phi_2 = x\log z - y^2 + 4$

$Q(x,y,z) = (-1,2,1)$

Normal to ϕ_2 at Q is $\bar{n} = \nabla \phi_2|_{Q(-1,2,1)}$

$\nabla \phi_2 = \frac{\partial \phi_2}{\partial x} i + \frac{\partial \phi_2}{\partial y} j + \frac{\partial \phi_2}{\partial z} k$

$= \log z\, i - 2y j + \frac{x}{z} k$

Unit vector $\bar{e} = \frac{n}{|n|} = \frac{\log z\, i - 2y j + \frac{x}{z} k}{\sqrt{(\log z)^2 + 4y^2 + \frac{x^2}{z^2}}}$

at $Q(-1,2,1)$, $\bar{e} = \frac{-4j - k}{\sqrt{16+1}} = \frac{-4j - k}{\sqrt{17}}$

Directional derivative $= \bar{e}.\text{grad}\,\phi_1 \Big|_{P(2,-1,1)} = -\dfrac{4j-k}{\sqrt{17}}\,[i-3j-3k]$

$$= \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}.$$

5. Find angle b/w surfaces $x^2+y^2+z^2=9$, $z+3=x^2+y^2$ at $(-2,1,2)$.

J-19 Given: $\phi_1 = x^2+y^2+z^2-9$, $\phi_2 = x^2+y^2-z-3$

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}i + \frac{\partial\phi_1}{\partial y}j + \frac{\partial\phi_1}{\partial z}k = 2xi+2yj+2zk$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}i + \frac{\partial\phi_2}{\partial y}j + \frac{\partial\phi_2}{\partial z}k = 2xi+2yj-k$$

Angle b/w surfaces ϕ_1 & ϕ_2 is angle b/w normals $\bar{n}_1$ & $\bar{n}_2$ to surfaces ϕ_1 & ϕ_2 at $(-2,1,2)$.

$\bar{n}_1 = \nabla\phi_1$ at $(-2,1,2) = -4i+2j+4k$

$\bar{n}_2 = \nabla\phi_2$ at $(-2,1,2) = -4i+2j-k$

$$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|n_1||n_2|} = \frac{16+4-4}{\sqrt{16+4+16}\cdot\sqrt{16+4+1}} = \frac{16}{3\sqrt{21}}$$

$\therefore \theta = \cos^{-1}\left(\dfrac{16}{3\sqrt{21}}\right).$

6. Find the values of a & b such that surface $5x^2-2yz-9z=0$ intersects the surface $ax^2+by^3=4$ orthogonally at $(1,-2,1)$.

D-15 Given: $f_1 = 5x^2-2yz-9z$ & $f_2 = ax^2+by^3-4$

$$\nabla f_1 = \frac{\partial f_1}{\partial x}i + \frac{\partial f_1}{\partial y}j + \frac{\partial f_1}{\partial z}k \qquad \nabla f_2 = 2axi+3by^2j$$

$$= 10i - 2zj - (2y+9)k$$

Normals to surfaces at $(1,-2,1)$ are–

$\bar{n}_1 = 10i-2j-5k$ & $\bar{n}_2 = 2ai+12bj$

Given that f_1 & f_2 intersects orthognally i.e, $\theta = \pi/2$

$\bar{n}_1\cdot\bar{n}_2 = 0$

$20a-24b=0 \Rightarrow a = \dfrac{6b}{5}$ where 'b' is arbitrary.

7. If $\bar{F} = (5xy-6x^2)i+(2y-4x)j$, evaluate $\int_C F.d\bar{r}$ along curve C in

D-17 XY plane given by $y=x^3$ from $(1,1)$ to $(2,8)$.

We know, $dxi+dyj = dr$

$F.dr = [(5xy-6x^2)i+(2y-4x)j][dxi+dyj]$

$\quad = (5xy-6x^2)dx+(2y-4x)dy$

Now, $\int_C F.d\bar{r} = \int_C (5xy-6x^2)dx+(2y-4x)dy$

Given : C is a curve $y=x^3$ from $(1,1)$ to $(2,8)$

$y=x^3 \Rightarrow dy=3x^2dx$

$$\int_{x=1}^{2}(5x^4-6x^2)dx+(2x^3-4x)3x^2dx$$

$$\int_{x=1}^{2}(5x^4-6x^2)dx+(6x^5-12x^3)dx$$

$$\left.\frac{5x^5}{5}-\frac{6x^3}{3}+\frac{6x^6}{6}-\frac{12x^4}{4}\right]_1^2 = 32-16+64-48+3=35$$

---

8. D-18 Using Gauss divergence theorem, evaluate $\iint_S x\,dydz+y\,dzdx+z\,dxdy$ where S is surface of sphere $(x-2)^2+(y-2)^2+(z-2)^2=16$.

We know from Gauss divergence theorem, $\int_V \operatorname{div}\bar{F}\,dv=\int_S \bar{F}\cdot\bar{n}\,ds$

$$\iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz=\iint_S F_1dydz+F_2dzdx+F_3dxdy$$

$$\therefore \iint_S x\,dydz+y\,dzdx+z\,dxdy=\iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz$$

$$=\iiint_V(1+1+1)dxdydz$$

$$=3\iiint_V dxdydz$$

V is given by $(x-2)^2+(y-2)^2+(z-2)^2=4^2$

i.e., sphere at $(2,2,2)$ with radius 4

$\therefore$ Changing into spherical coordinates

Put $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, $dxdydz=r^2\sin\theta\,dr\,d\theta\,d\phi$

Limits will be $0\le r\le 4$, $0\le\theta\le\pi$, $0\le\phi\le 2\pi$

$$=3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{4}r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=3\int_{\phi=0}^{2\pi}d\phi\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{r=0}^{4}r^2dr$$

$$=3[2\pi][-\cos\theta]_{\theta=0}^{\pi}\left[\frac{r^3}{3}\right]_{r=0}^{4}$$

$$=[2\pi][-(-1)+1][64]$$

$$=256\pi.$$

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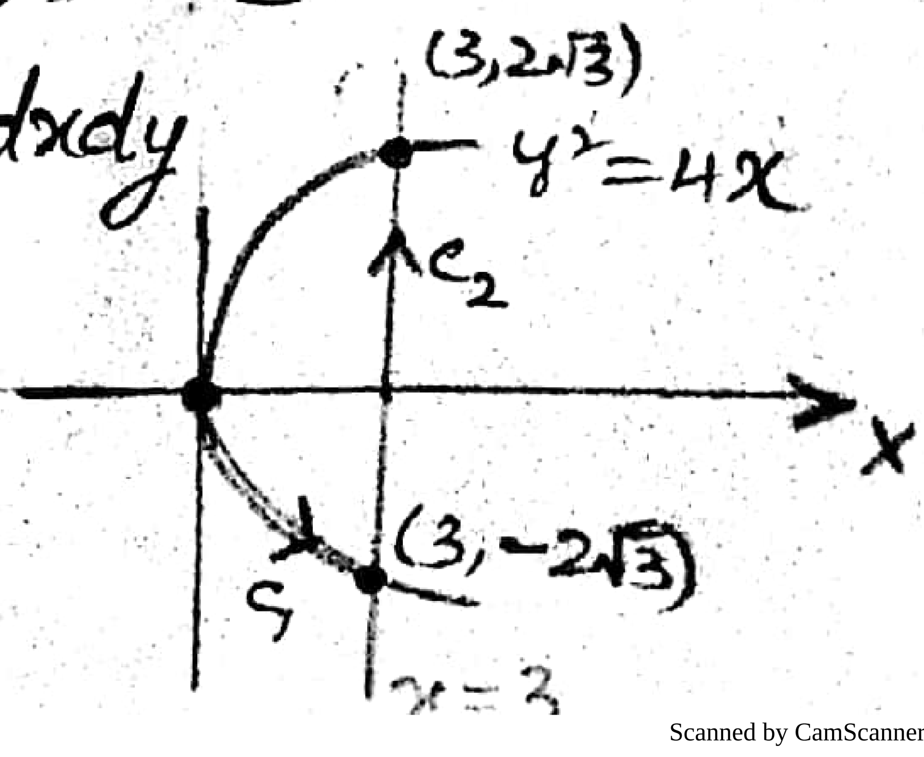
9. D-18 Verify Green's theorem for $\oint_C(xy^2+2xy)dx+x^2dy$ where C is the boundary of the region enclosing $y^2=4x$, $x=3$.

From Green's theorem, $\oint Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$



$LHS=\oint Mdx+Ndy=\int_{C_1}Mdx+Ndy+\int_{C_2}Mdx+Ndy$

$\Rightarrow \int_{C_1}Mdx+Ndy=\int_{C_1}(xy^2+2xy)dx+x^2dy$

Given: $C_1 : y^2 = 4x$, $x=3$

$y^2 = 4(3) \Rightarrow y = \pm\sqrt{12} = \pm 2\sqrt{3}$

Points of intersection are $(3, 2\sqrt{3}), (3, -2\sqrt{3})$

Now on C_1, $y^2 = 4x \Rightarrow x = \frac{y^2}{4} \Rightarrow dx = \frac{y\,dy}{2}$

y is varying from $2\sqrt{3}$ to $-2\sqrt{3}$

$$\int_{C_1} (xy^2+2xy)\,dx + x^2 dy = \int_{y=2\sqrt{3}}^{-2\sqrt{3}} \left(\frac{y^4}{4} + \frac{y^3}{2}\right)\frac{y\,dy}{2} + \frac{y^4}{16}dy$$

$$= \frac{1y^6}{8.6}\Big]_{2\sqrt{3}}^{-2\sqrt{3}} + \frac{y^5}{20}\Big]_{2\sqrt{3}}^{-2\sqrt{3}} + \frac{y^5}{80}\Big]_{2\sqrt{3}}^{-2\sqrt{3}}$$

$$= \left[\frac{(-2\sqrt{3})^6 - (2\sqrt{3})^6}{48}\right] + \frac{[-2\sqrt{3}]^5 - [2\sqrt{3}]^5}{20} + \frac{[-2\sqrt{3}]^5 - [2\sqrt{3})^5}{80}$$

$$= \frac{-32\,3^{5/2} - 32\,3^{5/2}}{20} - \frac{32\,3^{5/2} - 32\,3^{5/2}}{80} = \frac{-64\,3^{5/2}}{20} - \frac{64\,3^{5/2}}{80}$$

$$= \frac{-16(3^{5/2}) - 4(3^{5/2})}{5} = -(4)3^{5/2} = -4 \cdot 3^{1/2+2} = -36\sqrt{3}$$

$$\Rightarrow \int_{C_2} M\,dx + N\,dy = \int_{C_2} (xy^2+2xy)\,dx + x^2 dy$$

Now on C_2, $x = 3 \Rightarrow dx = 0$ & y is from $-2\sqrt{3}$ to $2\sqrt{3}$.

$$\int_{C_2} (xy^2+2xy)\,dx + x^2 dy = \int_{y=-2\sqrt{3}}^{2\sqrt{3}} 9\,dy = 9y\Big]_{-2\sqrt{3}}^{2\sqrt{3}} = 9[2\sqrt{3} + 2\sqrt{3}]$$

$$= 36\sqrt{3}$$

$$\therefore \oint_C = \oint_{C_1} + \oint_{C_2} = -36\sqrt{3} + 36\sqrt{3} = 0$$

LHS $= 0$

$$\text{RHS} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\frac{\partial N}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = 2xy + 2x$$

$$\int_{y=-2\sqrt{3}}^{2\sqrt{3}} \int_{x=0}^{3} (2x - 2xy - 2x)\,dx\,dy = -2\int_{y=-2\sqrt{3}}^{2\sqrt{3}} \int_{x=0}^{3} yx\,dx\,dy$$

$$= -2\int_{y=-2\sqrt{3}}^{2\sqrt{3}} y \cdot \left[\frac{x^2}{2}\right]_0^3 dy = -2 \times \frac{9}{2} \int_{y=-2\sqrt{3}}^{2\sqrt{3}} y\,dy = -9\left[\frac{y^2}{2}\right]_{-2\sqrt{3}}^{2\sqrt{3}}$$

$= -\frac{9}{2}\left[(2\sqrt{x})^2 - (-2\sqrt{x})^2\right] = 0$

RHS $= 0$

LHS $=$ RHS $= 0$

$\therefore$ Green's theorem is verified.

10. Verify Stokes theorem for vector field $F = (2x-y)i - yz^2 j - y^2 z k$ over the upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on XY-plane.

D-16

Stokes theorem states that $\oint_C F \cdot d\bar{r} = \int_S \text{curl}\, F \cdot \bar{n}\, ds$

Given sphere $x^2+y^2+z^2=1$ over $\overline{XY}$ plane.

$\therefore$ The boundary C of S is a circle in $\overline{XY}$ plane i.e, $x^2+y^2=1, z=0$

Now $\int_C \bar{F} \cdot d\bar{r} = \int_C [(2x-y)i - yz^2 j - y^2 z k] \cdot [i\,dx + j\,dy + k\,dz]$

$$= \int_C (2x-y)\,dx \quad \{\because z=0 \;\&\; dz=0\}$$

Put $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta\, d\theta$ & $\theta: 0 \to 2\pi$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\, d\theta$$

$$= -\int_{\theta=0}^{2\pi} [\sin 2\theta - \sin^2\theta]\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1-\cos 2\theta}{2} - \sin 2\theta\right) d\theta$$

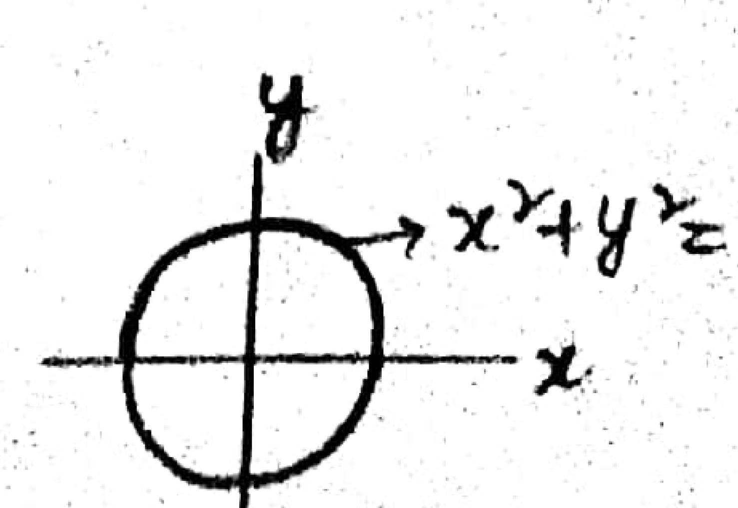
$$= \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4} + \frac{\cos 2\theta}{2}\right]_{\theta=0}^{2\pi} = \left[\pi - 0 + \tfrac{1}{2} - \tfrac{1}{2}\right] = \pi \quad \text{--- (1)}$$

Now, $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = i(-2yz + 2yz) - j(0-0) + \bar{k}(0+1) = k$

$\int_S (\nabla \times F) \cdot \bar{n}\, ds = \iint_R \frac{(\nabla \times F) \cdot \bar{n}}{\bar{n} \cdot \bar{k}}\, dx\,dy$, where R is projection of S on $\overline{XY}$ plane.

$$= \iint_R \frac{\bar{k} \cdot \bar{n}}{\bar{n} \cdot k}\, dx\,dy = \iint_R dx\,dy$$

$$= 4\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} dy\,dx$$



$$= 4\int_{x=0}^{1} \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 4\left[\frac{1}{2}\sin^{-1}x\right] = 2\sin^{-1}\left(\sin\frac{\pi}{2}\right) = \pi$$

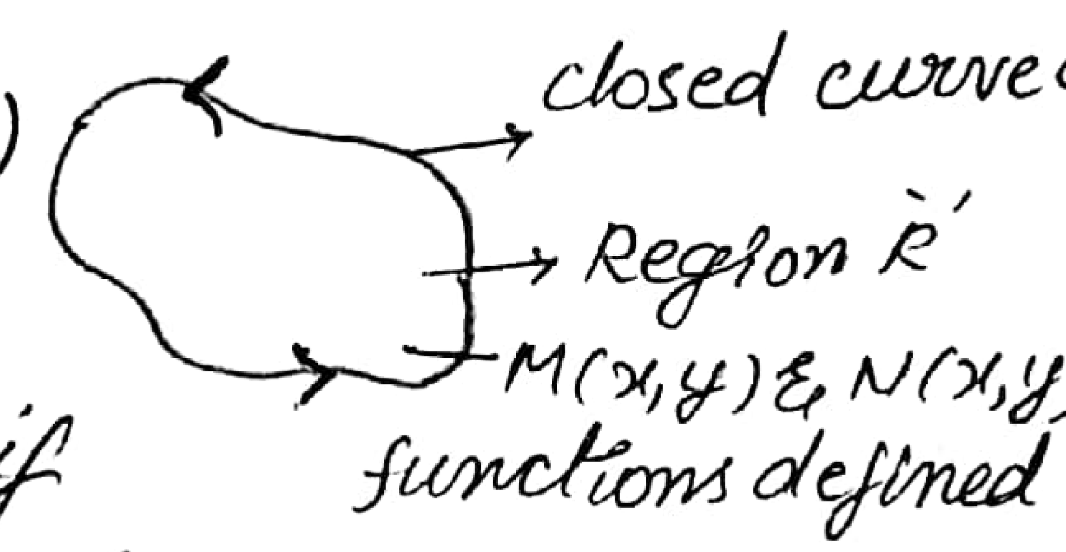
$$\therefore \oint \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds.$$

Stokes theorem is verified.

D-15,17 SAQ 11. State Green's theorem & Stokes theorem

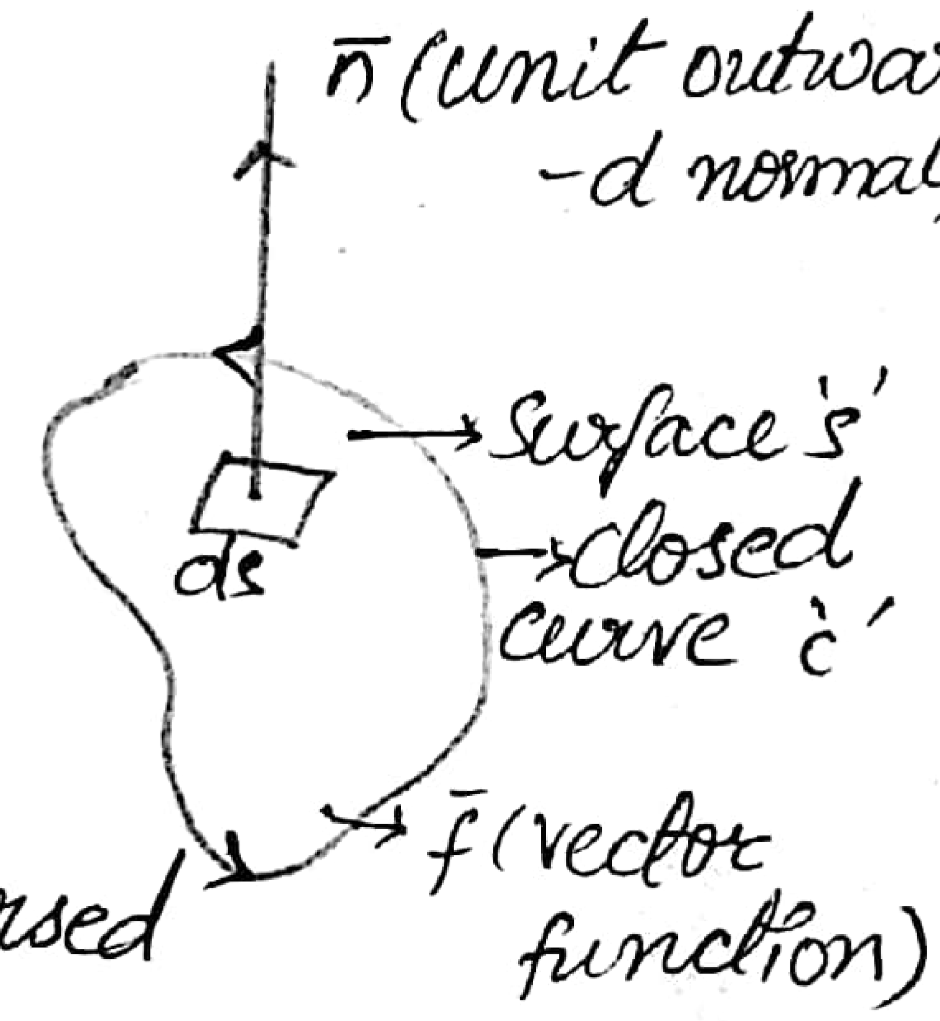
GREEN'S THEOREM:

(Transformation b/w line & double integral)



If 'R' is a closed region in XY plane bou-
-nded by a simple closed curve 'c' & if
$M(x,y)$ & $N(x,y)$ are continuous functions in R
then, $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$, where 'c' is traversed anti-clockwise direction.

STOKES THEOREM:



Let 's' be an open surface bounded by a closed non-intersecting curve 'c'. If $\bar{F}$ is any differe-
-ntiable vector point function then -

$$\oint \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds,$$ where 'c' is transversed in +ve direction & $\bar{n}$ is unit otward normal at any point of surface.