Notes of Unit I (Sequences and Series)_Mathematics I

For

B.E. I year O.U.

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Unit - I

Sequences and Series

Sequence :-

An ordered set of real numbers $a_1, a_2, a_3 \ldots a_n$ is called a sequence and is denoted by $\{a_n\}$.

Note :- (1) If the no. of terms are unlimited, then $\{a_n\}$ is said to be infinite sequence.

(2) The general term is denoted by a_n.

Ex :- (i) $1, 3, 5, 7, \ldots, 2n-1 \ldots$

here $\{2n-1\}$, $a_n = 2n-1$.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

here $\{\frac{1}{n}\}$, $a_n = \frac{1}{n}$

(iii) $1, -1, 1, -1, \ldots, (-1)^n, \ldots$

here $\{(-1)^n\}$ and $a_n = (-1)^n$

(iv) Let 'c' be a constant

$c, c, c, \ldots c, \ldots$

here $\{c\}$, $a_n = c$. Constant sequence.

Limit of a Sequence :-

A sequence $\{a_n\}$ is said to have limit 'l', if for every $\epsilon > 0$, $\exists\ N > 0 \ni |a_n - l| < \epsilon \ \forall\ n \geq N$.

It is denoted by $\lim_{n\to\infty} a_n = l$.

Ex :- Let $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

here $a_n = \frac{1}{n}$, $l = 0$

Let $\epsilon = \frac{1}{2}$ then $\exists\ N = 3$

then $\left|\frac{1}{n} - 0\right| < \frac{1}{2} \ \forall\ n \geq 3$

Convergent Sequence :-

A sequence $\{a_n\}$ is said to be convergent if $\lim_{n\to\infty} a_n$ is finite.

Ex :- $\{a_n\} = \{\frac{1}{2^n}\}$

terms are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$

here $a_n = \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$ finite.

$\therefore \{\frac{1}{2^n}\}$ is convergent.

Divergent Sequent :-

A sequence $\{a_n\}$ is said to be divergent if $\lim_{n\to\infty} a_n$ is not finite i.e if

$$\lim_{n\to\infty} a_n = +\infty \quad \text{or} \quad -\infty.$$

Ex :- (1) Let $\{n^2\}$.

$$\lim_{n\to\infty} n^2 = \infty \Rightarrow \{n^2\} \text{ is divergent.}$$

Oscillatory Sequence :-

If a sequence $\{a_n\}$ niether converges nor diverges to $+\infty$ or $-\infty$, then it is called oscillatory sequence.

Ex :- $\{(-1)^n\}$.

$$\left.\begin{aligned} \lim_{n\to\infty} (-1)^n &= 1 \quad \text{if } n \text{ is even.} \\ \lim_{n\to\infty} (-1)^n &= -1 \quad \text{if } n \text{ is odd.} \end{aligned}\right\} \text{limit does not exist.}$$

$\therefore \{a_n\}$ is oscillatory.

<u>Problems</u>:- Examine the following sequences for convergence.

(1) $a_n = \frac{n^2-2n}{3n^2+n}$.

(3)

<u>Sol</u>:- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-2n}{3n^2+n}$

$$= \lim_{n\to\infty} \frac{n^2\left[1-\frac{2}{n}\right]}{n^2\left[3+\frac{1}{n}\right]} = \lim_{n\to\infty} \frac{1-\frac{2}{n}}{3+1/n} = \frac{1}{3} \text{ (finite) and unique}$$

$\therefore \{a_n\}$ is convergent.

(2) $a_n = 2^n$.

<u>Sol</u>:- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^n = \infty$

$\therefore \{a_n\}$ is divergent.

(3) $a_n = 3+(-1)^n$

<u>Sol</u>:- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3+(-1)^n = \begin{cases} 4 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$

limit is finite but not unique.

$\therefore \{a_n\}$ oscillates.

(4) $a_n = 1+\frac{2}{n}$.

<u>Sol</u>:- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1+\frac{2}{n} = 1$ finite & unique.

$\therefore \{a_n\}$ convergent.

(5) $a_n = [n+(-1)^n]^{-1}$

<u>Sol</u>:- $a_n = \frac{1}{[n+(-1)^n]}$

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$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+(-1)^n} = 0 \quad \text{unique \& finite.}$$

$\therefore \{a_n\}$ is convergent.

(6) $a_n = \dfrac{3n-1}{1+2n}$.

Sol :- $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n-1}{1+2n} = \lim_{n\to\infty} \frac{n\left(3-\frac{1}{n}\right)}{n\left(\frac{1}{n}+2\right)} = \frac{3}{2}$ finite & unique

$\therefore \{a_n\}$ is convergent.

Series :-

If $u_1, u_2, u_3, \ldots, u_n, \ldots$ be an infinite sequence of real numbers, then

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \infty$$

is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} u_n$.

Ex :- (i) $\sum_{n=1}^{\infty} (-1)^n$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n}$ (iii) $\sum_{n=1}^{\infty} 2^n$.

Partial Sums :-

Let $\sum_{n=1}^{\infty} u_n$ be an infinite series. Then

$S_n = u_1 + u_2 + \cdots + u_n$ is called n^{th} partial sum of $\sum_{n=1}^{\infty} u_n$.

Here, $S_1 = u_1$ first partial sum

$S_2 = u_1 + u_2$ second partial sum.

$S_3 = u_1 + u_2 + u_3$

$\vdots$

$S_n = u_1 + u_2 + \cdots + u_n$.

$\vdots$

$\therefore \{S_n\}$ is called as sequence of partial sums of $\sum_{n=1}^{\infty} u_n$.

Convergence, divergent and Oscillations of a series:-

Consider $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots \infty$.

Let $S_n = u_1 + u_2 + \cdots + u_n \Rightarrow \{S_n\}$ is a sequence of partial sums.

(i) $\sum u_n$ converges if $\{S_n\}$ converges. i.e $\lim_{n\to\infty} S_n =$ finite & unique

(ii) $\sum u_n$ diverges if $\{S_n\}$ diverges i.e $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$

(iii) $\sum u_n$ oscillates if $\{S_n\}$ oscillates i.e. $\lim_{n\to\infty} S_n =$ finite but not unique.

Problems:- Examine the convergence of following serieses.

(1) $1+2+3+\cdots+n+\cdots\infty$

Sol:- Given $\sum_{n=1}^{\infty} n$.

here $S_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \frac{1}{2}\infty = \infty$

$\therefore \sum_{n=1}^{\infty} n$ diverges to ∞.

(2) $1^2+2^2+3^2+\cdots+n^2+\cdots$

Sol:- Given $\sum_{n=1}^{\infty} n^2$.

here $S_n = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} = \infty$

$\therefore \sum_{n=1}^{\infty} n^2$ diverges to ∞.

(3) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$

Sol:- Given $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

here $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

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$$u_1 = \frac{1}{1} - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

$$u_4 = \frac{1}{4} - \frac{1}{5}$$

$$\vdots$$

$$u_{n-1} = \frac{1}{n-1} - \frac{1}{n}$$

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

$\therefore S_n = u_1 + u_2 + \cdots + u_n = 1 - \frac{1}{n+1}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$. finite & unique

$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent

(4) Discuss the convergence of geometric series $1 + r + r^2 + r^3 + \cdots \infty$ and Show that

(i) it converges if $|r| < 1$

(ii) it diverges if $r \geq 1$

(iii) Oscillates if $r \leq -1$.

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Sol:- Given $1 + r + r^2 + r^3 + \cdots \infty = \sum_{n=1}^{\infty} r^{n-1}$

$\therefore S_n = 1 + r + r^2 + r^3 + \cdots + r^{n-1}$.

We know $S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$

(i) $|r| < 1 \Rightarrow -1 < r < 1 \Rightarrow \lim_{n\to\infty} r^n = 0$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{1-r} - \frac{r^n}{1-r}\right) = \frac{1}{1-r}$ finite

$\therefore$ given series is convergent if $|r| < 1$.

(ii) (a) $r = 1$ then

$$S_n = 1+1+1+\cdots+1 = n$$

$$\lim_{n\to\infty} S_n = \infty$$

$\therefore$ given series diverges if $r = 1$

(b) $r > 1 \Rightarrow \lim_{n\to\infty} r^n = \infty$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{1-r} - \frac{r^n}{1-r}\right) = -\infty$$

$\therefore$ given series diverges if $r > 1$

(iii) (a) $r = -1$ then given series is

$1-1+1-1+\cdots$ which is oscillatory series.

(b) $r < -1 \Rightarrow -r > 1$

Let $x = -r \Rightarrow x > 1$

and $r^n = (-1)^n x^n$

Now

$$S_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n x^n}{1+x} = \begin{cases} \dfrac{1-x^n}{1+x} & \text{if } n \text{ is even} \\ \dfrac{1+x^n}{1+x} & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore \lim_{n\to\infty} S_n = \begin{cases} -\infty \\ +\infty \end{cases}$$

$\therefore$ given series is oscillatory.

General properties of series :-

(i) The convergence or divergence of an infinite series remains uneffected by the addition or removal of a finite number of terms.

(ii) The convergence or divergence of an infinite series remains uneffected by multiplying each term with a finite number.

Series of positive terms :-

An infinite series in which all terms after some particular term are positive, is called as a positive term series.

Ex :- $-7-5-3-1+2+7+13+20+\cdots\infty$

Note :- A positive term series either converges or diverges to $+\infty$. but never oscillates.

Pf :- Let Σu_n be a positive term series $\Rightarrow u_{n+1} > 0$.

Let $S_n = u_1 + u_2 + \cdots + u_n$.

$\Rightarrow S_{n+1} = u_1 + u_2 + \cdots + u_n + u_{n+1}$

$\Rightarrow S_{n+1} - S_n = u_{n+1} > 0$.

$\Rightarrow S_{n+1} > S_n$.

$\Rightarrow \{S_n\}$ is increasing.

Two cases:
- $\{S_n\}$ is bdd above $\Rightarrow \{S_n\}$ convergent $\Rightarrow \Sigma u_n$ conv.
- $\{S_n\}$ is unbounded $\Rightarrow \{S_n\} \to +\infty \Rightarrow \Sigma u_n \to +\infty$

Necessary condition for convergence :-

Th:- If a series $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

Pf:- Given $\sum u_n$ is a series

Let $S_n = u_1 + u_2 + u_3 + \cdots + u_n$

Given $\sum u_n$ is convergent $\Rightarrow \lim_{n\to\infty} S_n = k$.

also $\lim_{n\to\infty} S_{n-1} = k$.

Then, $u_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = k - k = 0.$$

$$\boxed{\therefore \lim_{n\to\infty} u_n = 0}$$

Note:- Converse of above is not true.

i.e if $\lim_{n\to\infty} u_n = 0$ then $\sum u_n$ need not always be convergent.

Consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \sum \frac{1}{n}.$$

here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

But $\sum \frac{1}{n}$ is divergent from p-series.

Note:-

(i) If $\sum u_n$ is convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$

(ii) If $\lim_{n\to\infty} u_n = 0 \Rightarrow \sum u_n$ may converge or diverge

* (iii) If $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \sum u_n$ diverges.

Integral Test :-

A positive term series $f(1) + f(2) + \cdots + f(n) + \cdots$ where $f(n)$ decreases as n increases, converges or diverges according as the integral

$$\int_1^\infty f(x)\,dx \quad \text{is finite or infinite.}$$

Problem :-

(1) Show that p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \infty$$

(i) converges for $p > 1$ (ii) diverges for $p \le 1$.

Sol:- Let

$$f(n) = \frac{1}{n^p}$$

then from above test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges or diverges according as $\int_1^\infty \frac{1}{x^p}\,dx$ is finite or infinite.

If $p \neq 1$, then

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{m\to\infty} \int_1^m \frac{1}{x^p}\,dx$$

$$= \lim_{m\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^m$$

$$= \lim_{m\to\infty} \frac{m^{1-p} - 1}{1-p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$

If $p = 1$, then

$$\int_1^\infty \frac{1}{x}\,dx = \left[\log x\right]_1^\infty = \infty$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$

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Comparison Test (limit form) :–

If two positive term series $\sum u_n$ and $\sum V_n$ be such that $\lim_{n\to\infty} \frac{u_n}{V_n}$ = finite quantity $(\neq 0)$,

then $\sum u_n$ and $\sum V_n$ converge or diverge together.

Problems :–

(1) Test for convergence of the series

(i) $\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\cdots$

Sol :–

$$\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\cdots=\sum_{n=1}^{\infty}\frac{2n-1}{n(n+1)(n+2)}=\sum_{n=1}^{\infty}u_n$$

$$\therefore\quad u_n=\frac{2n-1}{n(n+1)(n+2)}=\frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$=\frac{1}{n^2}\,\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let $\sum V_n=\sum \frac{1}{n^2}$

$$\therefore\ \lim_{n\to\infty}\frac{u_n}{V_n}=\frac{(2-1/n)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=2\ (\neq 0)\ \text{finite}$$

Also $\sum\frac{1}{n^2}$ is a p-series with $p>1$

$\therefore \sum V_n$ convergent

$\therefore$ From Comparison Test $\sum u_n$ is convergent.

(ii) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots\infty$

Sol :– Here $u_n=\frac{n^n}{(1+n)^{1+n}}=\frac{1}{1+n}\left(\frac{n}{1+n}\right)^n$ ignoring first term

Let $V_n = \frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}\left(\frac{n}{n+1}\right)^n}{n}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n \qquad \left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \text{ for any } x.\right.$$

$$= 1 \cdot \frac{1}{e} \neq 0 \text{ \& finite.}$$

Also $\sum V_n = \sum \frac{1}{n}$ is p series with $p=1$

$\therefore \sum V_n$ diverges

$\therefore \sum u_n$ is divergent

(iii) $\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$

Sol:- Here $u_n = \sqrt{\frac{3^n-1}{2^n+1}} = \left(\frac{3}{2}\right)^{n/2} \sqrt{\frac{1-1/3^n}{1+1/2^n}}$

Let $V_n = \left(\frac{3}{2}\right)^{n/2} = \left(\sqrt{\frac{3}{2}}\right)^n$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \sqrt{\frac{1-1/3^n}{1+1/2^n}} = 1 \neq 0 \text{ \& finite.}$$

We know that

$\sum V_n = \sum\left(\sqrt{\frac{3}{2}}\right)^n$ is in Geometric series $\sum r^n$

with $r = \sqrt{\frac{3}{2}} > 1$

$\therefore \sum V_n$ is divergent

$\therefore \sum u_n$ is divergent.

(iv) $\sum \frac{1}{n} \sin \frac{1}{n}$

Sol:- We know $\sin x = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$

$$\therefore u_n = \frac{1}{n} \sin \frac{1}{n} = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \frac{1}{7!\,n^7} + \cdots\right]$$

$$= \frac{1}{n^2}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} - \frac{1}{7!\,n^6} + \cdots\right]$$

Take $V_n = \frac{1}{n^2}$,

$$\therefore \lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty}\left(1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} \cdots\right) = 1 \neq 0 \text{ finite.}$$

Also

$\sum V_n = \sum \frac{1}{n^2}$ is a p-series with $p > 1$.

$\therefore \sum V_n$ is convergent

$\therefore \sum u_n$ is convergent

(v) $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots \infty.$

Sol:- Let

$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left[\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right]}{n^3\left[\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right]} = \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{n^{5/2}\left[\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right]}$$

Let $V_n = \frac{1}{n^{5/2}}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}} = 1 \neq 0 \text{ finite.}$$

$\therefore \sum V_n = \sum \frac{1}{n^{5/2}}$ is p-series with $p = 5/2 > 1$

$\therefore \sum V_n$ convergent

$\therefore \sum u_n$ is convergent

June-19 SAQ

(VI) $\sum \frac{\sqrt{n}}{n^2+1}$

Sol :- Let $u_n = \frac{\sqrt{n}}{n^2+1} = \frac{n^{1/2}}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{1}{n^{3/2}\left(1+\frac{1}{n^2}\right)}$

Let $v_n = \frac{1}{n^{3/2}}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n^2}\right)} = 1 \neq 0$ finite

& $\sum v_n = \sum \frac{1}{n^{3/2}}$ is p series with $p = \frac{3}{2} > 1$

$\therefore \sum v_n$ converges

$\therefore \sum u_n$ converges

Dec-18

(VII) $\sum_{n=1}^{\infty} \frac{2+5n}{7n-3}$

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D'Alembert's Ratio Test :-

In a positive term series $\sum u_n$, if

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$, then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

Practical form :-

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = K$, then the series converges for $K > 1$ and diverges for $K < 1$.

Note :- This test fails if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = K = 1$.

Proof :- Let $\sum u_n = \sum \frac{1}{n^p}$.

Now $u_n = \frac{1}{n^p}$ & $u_{n+1} = \frac{1}{(n+1)^p}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1/n^p}{1/(n+1)^p} = \lim_{n\to\infty} \frac{(n+1)^p}{n^p} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p = 1 = K$

But $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Note:- This ratio Test is usefull, when in the given series, increasing powers of x are there or factorials are there.

Problems:- Test for the convergence of following series.

(1) $x+2x^2+3x^3+\cdots$

Sol:- Given $\sum u_n = \sum n x^n$

$$\therefore u_n = n x^n$$

$$\Rightarrow \quad u_{n+1} = (n+1)\, x^{n+1}$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n x^n}{(n+1)\, x^{n+1}} = \frac{n}{n(1+\frac{1}{n})\, x}$$

Now,
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ From D'Alemberts ratio test, given series

$\rightarrow$ Converges if $\frac{1}{x} > 1 \Rightarrow x<1$

$\rightarrow$ diverges if $\frac{1}{x} < 1 \Rightarrow x>1$.

and when $x=1 \Rightarrow u_n = n$

$$\therefore \lim_{n\to\infty} u_n = \infty \neq 0.$$

$\therefore$ Given series diverges

$\therefore \sum u_n$ converges if $x<1$, diverges if $x \geq 1$

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(2) Test the convergence of $\sum \frac{x^n}{n}$.

Sol:- Given $u_n = \frac{x^n}{n}$

$$\Rightarrow \quad u_{n+1} = \frac{x^{n+1}}{n+1}$$

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Now, $\frac{u_n}{u_{n+1}} = \frac{x^n/n}{x^{n+1}/n+1} = \frac{n+1}{n} \cdot \frac{1}{x}$

$= \left(1+\frac{1}{n}\right) \cdot \frac{1}{x}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ from Ratio test givens series,

→ converges if $\frac{1}{x} > 1 \Rightarrow x < 1$

→ diverges if $\frac{1}{x} < 1 \Rightarrow x > 1$

Now when $x = 1 \Rightarrow u_n = \frac{1}{n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$, we cant say anything

$\therefore \sum u_n = \sum \frac{1}{n}$ diverges since $p=1$.

$\therefore$ given series converges for $x<1$, diverges for $x \geq 1$

(3) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{10}{13}x^3 + \cdots$

Sol :- For $2, 6, 10, \cdots$ $\quad a_n = a+(n-1)d = 2+(n-1)4 = 4n-2$

For $5, 9, 13, \cdots$ $\quad a_n = a+(n-1)d = 5+(n-1)4 = 4n+1$

$$\therefore u_n = \frac{4n-2}{4n+1} x^n$$

Now $u_{n+1} = \frac{4(n+1)-2}{4(n+1)+1} x^{n+1} = \frac{4n+2}{4n+5} x^{n+1}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{4n-2}{4n+1} x^n \cdot \frac{4n+5}{4n+2\ (x^{n+1})}$$

$$= \frac{n\left(4-\frac{2}{n}\right)}{n\left(4+\frac{1}{n}\right)} \cdot \frac{n\left(4+\frac{5}{n}\right)}{n\left(4+\frac{2}{n}\right)} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ From Ratio test given series converges if $\frac{1}{x} > 1$ i.e $x < 1$
diverges if $\frac{1}{x} < 1$ i.e $x > 1$

and test fails for $\frac{1}{x} = 1 \Rightarrow x = 1$

$$\therefore \; u_n = \frac{4n-2}{4n+1} = \frac{n\left(4 - \frac{2}{n}\right)}{n\left(4 + \frac{1}{n}\right)}$$

$$\lim_{n\to\infty} u_n = 1 \neq 0$$

$\therefore$ Series diverges.

$\therefore$ Given series converges if $x < 1$ & diverges if $x \geq 1$

(4) $\sum_{n=0}^{\infty} \frac{(10+5i)^n}{n!}$

Sol:- Let $u_n = \frac{(10+5i)^n}{n!} \Rightarrow u_{n+1} = \frac{(10+5i)^{n+1}}{(n+1)!} = \frac{(10+5i)^{n+1}}{(n+1)\,n!}$

Now

$$\frac{u_n}{u_{n+1}} = \frac{(10+5i)^n}{n!} \cdot \frac{(n+1)\,n!}{(10+5i)^{n+1}}$$

$$= \frac{(n+1)}{(10+5i)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty > 1$$

$\therefore$ From Ratio test given series converges.

(5) $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$

Sol:- Let $u_n = \frac{n!\,2^n}{n^n} \Rightarrow u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}} = \frac{n!\,2^{n+1}}{(n+1)^n}$

$$\frac{u_n}{u_{n+1}} = \frac{n!\,2^n}{n^n} \cdot \frac{(n+1)^n}{n!\,2^{n+1}} = \frac{(n+1)^n}{2\,n^n} = \frac{(1+1/n)^n}{2}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(1+1/n)^n}{2} = \frac{e}{2} > 1$$

$\therefore$ the given series converges.

(6) $\sum \frac{(x-1)^n}{n^2}$

Sol:- Let $u_n = \frac{(x-1)^n}{n^2} \Rightarrow u_{n+1} = \frac{(x-1)^{n+1}}{(n+1)^2} = \frac{(x-1)^{n+1}}{n^2(1+\frac{1}{n})^2}$

Now,

$$\frac{u_n}{u_{n+1}} = \frac{(x-1)^n}{n^2} \cdot \frac{n^2(1+\frac{1}{n})^2}{(x-1)^{n+1}} = \frac{(1+\frac{1}{n})^2}{(x-1)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x-1}$$

$\therefore$ Given series converges if $\frac{1}{x-1} > 1 \Rightarrow x-1 < 1 \Rightarrow x < 2$

diverges if $\frac{1}{x-1} < 1 \Rightarrow x-1 > 1 \Rightarrow x > 2$

when $x = 2 \Rightarrow u_n = \frac{1}{n^2}$

$\sum u_n$ converges since $p = 2 > 1$.

$\therefore$ Given series converges for $x \leq 2$ and diverges for $x > 2$

---

(7) $\sum_{n=1}^{\infty} \frac{x^{2n}}{(n+1)\sqrt{n}}$

Sol:- Let $u_n = \frac{x^{2n}}{(n+1)\sqrt{n}} \Rightarrow u_{n+1} = \frac{x^{2n+2}}{(n+2)\sqrt{n+1}}$

Now,

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n+2}}$$

$$= \frac{n(1+\frac{2}{n})\sqrt{n}\sqrt{1+\frac{1}{n}}}{n(1+\frac{1}{n})\sqrt{n}} \frac{1}{x^2} = \frac{(1+\frac{2}{n})\sqrt{1+\frac{1}{n}}}{(1+\frac{1}{n})} \cdot \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$ from Ratio test series converges if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$

diverges if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$

when $x^2 = 1$ then

$$u_n = \frac{1}{(n+1)\sqrt{n}}$$

$\left\{ \lim_{n\to\infty} u_n = 0 \right.$ but of no use

$$u_n = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

Let $$V_n = \frac{1}{n^{3/2}}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{V_n} = 1$ finite & non zero.

$\therefore$ Also $\sum \frac{1}{n^{3/2}}$ converges bcz $p = \frac{3}{2} > 1$ from p-series

$\therefore$ from comparison test $\sum u_n$ converge if $x^2 = 1$

$\therefore$ Given series converge if $x^2 \le 1$

diverge if $x^2 > 1$.

---

(8):- $\sum \frac{x^{2n+1}}{2n+2}$

Sol:- Let $u_n = \frac{x^{2n+1}}{2n+2} \Rightarrow u_{n+1} = \frac{x^{2n+3}}{2n+4}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{x^{2n+1}}{2n+2} \cdot \frac{2n+4}{x^{2n+3}} = \frac{2n\left(1+\frac{4}{2n}\right)}{2n\left(1+\frac{2}{2n}\right)} \frac{1}{x^2}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$ Given series converge if $\frac{1}{x^2} > 1$ i.e. $x^2 < 1$

diverge if $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$

when $x^2 = 1 \Rightarrow x = 1$

Then $$u_n = \frac{1}{2n+2} = \frac{1}{n\left(2+\frac{2}{n}\right)}$$

Let $$V_n = \frac{1}{n}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1/n(2+\frac{2}{n})}{1/n} = \frac{1}{2} (\neq 0)$ & finite.

also $\sum v_n = \sum \frac{1}{n}$ diverge bcz $p=1$.

$\therefore \sum u_n$ diverge from comparison test.

$\therefore$ Given series converge for $x^2<1$ & div for $x^2 \geq 1$

---

(9) $\sum \frac{n^2-1}{n^2+1} x^n$

Sol :- Let $u_n = \frac{n^2-1}{n^2+1} x^n \Rightarrow u_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2-1}{n^2+1} x^n \cdot \frac{(n+1)^2+1}{(n+1)^2-1} \cdot \frac{1}{x^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{n^2}\left(1-\frac{1}{n^2}\right) \cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}{\cancel{n^2}\left(1+\frac{1}{n^2}\right) \cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}\right]} \cdot \frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ Given series converge if $\frac{1}{x}>1$ i.e. $x<1$

diverge if $\frac{1}{x}<1$ i.e. $x>1$

when $x=1$, $u_n = \frac{n^2-1}{n^2+1} = \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}}$

$$\lim_{n\to\infty} u_n = 1 \neq 0$$

$\therefore \sum u_n$ diverge.

$\therefore \sum u_n$ converge if $x<1$ & div if $x \geq 1$

(10) Determine the nature of the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$ (20)(a)

Sol:- Here $u_n = \frac{x^{n-1}}{n \cdot 3^n}$

Since the series involves increasing powers of x, so we use D'Alembert's ratio test.

$$u_{n+1} = \frac{x^{n+1-1}}{(n+1)\, 3^{n+1}} = \frac{x^n}{(n+1)\, 3^{n+1}}$$

Now,

$$\frac{u_n}{u_{n+1}} = \frac{x^{n-1}}{n \cdot \cancel{3^n}} \quad \frac{(n+1)\, \cancel{3^n} \cdot 3}{x^n} = \frac{3(n+1)}{n} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}} \cdot \frac{1}{x} = \frac{3}{x}$$

$\therefore$ From D'Alembert's ratio test the given series

$\to$ converges if $\frac{3}{x} > 1$

$\to$ diverges if $\frac{3}{x} < 1$

and the test fails if $\frac{3}{x} = 1 \Rightarrow x = 3$

Now if $x = 3$ then $u_n = \frac{3^{n-1}}{n \cdot 3^n} = \frac{1}{3n}$

Let $v_n = \frac{1}{n}$ then

$$\frac{u_n}{v_n} = \frac{1/3n}{1/n} = \frac{1}{3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0 \text{ and finite.}$$

Also $\sum v_n = \sum \frac{1}{n}$ is a p-series with $p=1$, which is divergent

$\therefore$ From Comparison test $\sum u_n$ also divergent if $x = 3$

Raabe's Test:-

Let Σu_n be a series of positive terms and let

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$ then, the series

(i) converges if $k>1$

(ii) diverges if $k<1$

(iii) test fails if $k=1$.

Note :- When D'Alembert's ratio test fails i.e when $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$.

and if $\frac{u_n}{u_{n+1}}$ does not involve the number 'e'. Then Raabe's Test is suggested to test the convergence.

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1) Test the convergence of $\sum \frac{4.7\cdots(3n+1)}{1.2.3\cdots n} x^n$.

Sol:- Since given series involves increasing powers of x
so we can use D'Alemberts ratio test.

here $u_n = \frac{4.7\cdots(3n+1)}{1.2.3\cdots n} x^n$

$\Rightarrow u_{n+1} = \frac{4.7\cdots(3n+1)(3n+4)}{1.2.3\cdots n(n+1)} x^{n+1}$

Now

$$\frac{u_n}{u_{n+1}} = \frac{4.7\cdots(3n+1)x^n}{1.2.3\cdots n} \cdot \frac{1.2.3\cdots n(n+1)}{4.7\cdots(3n+1)(3n+4)\, x^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{(3n+4)x} = \frac{1+1/n}{(3+4/n)x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{3x}$$

$\therefore$ Given series converges if $\frac{1}{3x} > 1 \Rightarrow x < 1/3$

diverges if $\frac{1}{3x} < 1 \Rightarrow x > 1/3$

when $x = 1/3$ Ratio test fails.

Here $\frac{u_n}{u_{n+1}} = \frac{n+1}{(3n+4)\cdot 1/3} = \frac{3n+3}{3n+4}$

$\therefore \frac{u_n}{u_{n+1}}$ does not contain 'e'.

$\therefore$ We can use Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+3}{3n+4} - 1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{3n}+3-\cancel{3n}-4}{3n+4}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{-1}{n(3+\frac{4}{n})}\right) = -1/3 < 1$$

$\therefore$ From Raabe's test given series diverge.

$\therefore$ Given series converge if $x < 1/3$ and diverge if $x \geq 1/3$

2) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\,\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^7}{7}+\cdots$

Sol :– Neglecting the first term, we can write :

$$\sum \frac{1\cdot3\cdot5\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n-2)}\,\frac{x^{2n-1}}{2n-1}$$

$$\therefore u_n=\frac{1\cdot3\cdot5\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n-2)}\,\frac{x^{2n-1}}{2n-1}$$

$$\Rightarrow u_{n+1}=\frac{1\cdot3\cdot5\cdots(2n-3)(2n-1)}{2\cdot4\cdot6\cdots(2n-2)(2n)}\,\frac{x^{2n+1}}{2n+1}$$

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$$\therefore \frac{u_n}{u_{n+1}}=\frac{2n}{(2n-1)}\,\frac{2n+1}{2n-1}\cdot\frac{1}{x^2}$$

$$\Rightarrow \frac{u_n}{u_{n+1}}=\frac{4n^2\left(1+\frac{1}{2n}\right)}{4n^2\left(1-\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x^2}$$

$\therefore$ From Ratio test given series converge if $\frac{1}{x^2}>1$ i.e. $x^2<1$

diverge if $\frac{1}{x^2}<1$ i.e $x^2>1$

when $x^2=1$

$$\frac{u_n}{u_{n+1}}=\frac{4n^2+2n}{(2n-1)^2}$$

Now,

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left(\frac{4n^2+2n}{(2n-1)^2}-1\right)=n\left(\frac{\cancel{4n^2}+2n-\cancel{4n^2}-1+4n}{(2n-1)^2}\right)$$

$$=\frac{n(6n-1)}{(2n-1)^2}=\frac{n^2(6-1/n)}{n^2\left(2-\frac{1}{n}\right)^2}=\frac{6-1/n}{(2-1/n)^2}$$

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=\frac{6}{4}=\frac{3}{2}>1$$

$\therefore$ From Raabe's test given series ~~diverges~~ converge.

$\therefore$ Given series converge if $x^2\le1$

diverge if $x^2>1$

Logarithmic Test :-

Let $\sum u_n$ be a series of positive terms such that $\lim_{n\to\infty}\left[n\log\frac{u_n}{u_{n+1}}\right] = k$, then the series

(i) converges for $k>1$ & (ii) diverges for $k<1$.

and this test fails for $k=1$.

Note:- When D'Alemberts ratio test fails i.e. when $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$ and if $\frac{u_n}{u_{n+1}}$ ~~does~~ ~~not~~ involve the number 'e'. Then Logarithmic test is suggested to test the convergence.

Problems :- Discuss the convergence of following series.

(1) $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \dots \infty$.

Sol:- Given series is

$$\sum u_n = \sum \frac{n^n x^n}{n!} \Rightarrow u_n = \frac{n^n x^n}{n!}$$

$$\Rightarrow u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!} = \frac{(n+1)^n x^{n+1}}{n!}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \cdot \frac{n!}{(n+1)^n x^{n+1}}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{e\cdot x}$$

$\therefore$ from Ratio test, $\sum u_n$ converges if $\frac{1}{e\cdot x} > 1$ i.e., $x < \frac{1}{e}$

diverges if $\frac{1}{ex} < 1$ i.e., $x > \frac{1}{e}$.

and when $x = 1/e$ Ratio test fails.

and
$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{(1/e)}$$

$$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$$

Logarithmic Series
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\Rightarrow \log \frac{u_n}{u_{n+1}} = \log e - \log\left(1+\frac{1}{n}\right)^n$$
$$= 1 - n\log\left(1+\frac{1}{n}\right)$$
$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right]$$
$$= 1 - \left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots\right]$$

$$\log \frac{u_n}{u_{n+1}} = \frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} - \cdots$$

$$\Rightarrow n\log \frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots$$

$$\therefore \lim_{n\to\infty}\left[n\log\frac{u_n}{u_{n+1}}\right] = \frac{1}{2} < 1$$

$\therefore$ From Log-test Σu_n diverges when $x = \frac{1}{e}$

$\therefore \Sigma u_n$ converges when $x < \frac{1}{e}$

diverges when $x \geq \frac{1}{e}$

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(2) $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots$

Sol:- Here
$$\Sigma u_n = \Sigma \frac{n!\,x^n}{(n+1)^n} \Rightarrow u_n = \frac{n!\,x^n}{(n+1)^n}$$

$$\Rightarrow u_{n+1} = \frac{(n+1)!\,x^{n+1}}{(n+2)^{n+1}} = \frac{(n+1)\,n!\,x^{n+1}}{(n+2)^{n+1}}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n!\,x^n}{(n+1)^n} \cdot \frac{(n+2)^{n+1}}{(n+1)\,n!\,x^{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{1}{x} \quad \text{——— (1)}$$

$$= \left(\frac{n+2}{n+1}\right)^n \left(\frac{n+2}{n+1}\right) \cdot \frac{1}{x}$$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)} \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{e^2}{e} \cdot \frac{1}{x} = \frac{e}{x}.$$

$\therefore$ From Ratio Test $\sum u_n$ converges if $\frac{e}{x} > 1$ i.e. $x < e$

and diverges if $\frac{e}{x} < 1$ i.e. $x > e$

when $x = e$ test fails.

and from (1),

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n+1}\right)^{n+1} \cdot \frac{1}{e}$$

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$$\frac{u_n}{u_{n+1}} = \left[1 + \frac{1}{n+1}\right]^{n+1} \Big/ e$$

$$\Rightarrow \log \frac{u_n}{u_{n+1}} = (n+1) \log\left[1 + \frac{1}{n+1}\right] - \log e$$

$$= (n+1)\left[\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} - \frac{1}{4(n+1)^4} + \cdots\right] - 1$$

$$\log \frac{u_n}{u_{n+1}} = -\frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} - \frac{1}{4(n+1)^3} + \cdots$$

$$\Rightarrow n \log \frac{u_n}{u_{n+1}} = -\frac{n}{2n\left(1+\frac{1}{n}\right)} + \frac{n}{3n^2\left(1+\frac{1}{n}\right)^2} - \frac{n}{4n^3\left(1+\frac{1}{n}\right)^3} + \cdots$$

$$\Rightarrow \lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = -\frac{1}{2} < 1$$

$\therefore$ From Log-Test $\sum u_n$ diverges when $x = e$.

$\therefore$ $\sum u_n$ converges if $x < e$ & diverges if $x \geq e$

Cauchy's n^{th} root test :-

Let Σu_n be an infinite series of positive terms and

$\lim_{n\to\infty} (u_n)^{1/n} = \lambda$, Then

(i) Σu_n converges if $\lambda < 1$

(ii) Σu_n diverges if $\lambda > 1$

(iii) No conclusion can be drawn if $\lambda = 1$.

Note:- Cauchy's n^{th} root test is applied when u_n involves the n^{th} power of itself as a whole.

Problems:-

Test the convergence of following series.

(1) $\Sigma (\log n)^{-2n}$

Sol:- Here $u_n = \dfrac{1}{(\log n)^{2n}}$

$$\Rightarrow (u_n)^{1/n} = \frac{1}{(\log n)^2}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{(\log n)^2} = 0 < 1$$

$\therefore$ From Cachy's n^{th} root test Σu_n converges.

(2) $\Sigma \dfrac{1}{n^n}$

Sol:- Here $u_n = \left(\dfrac{1}{n}\right)^n$

$$\Rightarrow (u_n)^{1/n} = \frac{1}{n}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

$\therefore$ Σu_n converges.

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(3) $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Sol :- Here $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$= \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$$

$$\Rightarrow (u_n)^{1/n} = \frac{1}{\left[\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}\right]^{1/n}}$$

$$= \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2-1}}}$$

$$(u_n)^{1/n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\Rightarrow \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

$\therefore \sum u_n$ converges.

(4) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots\infty$

Sol :- $\sum u_n = \sum\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{(n+1)}{n}\right]^{-n}$

$$\therefore u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-n}$$

$$= \left(\frac{n+1}{n}\right)^{-n}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{-n}$$

$$= \left(1+\frac{1}{n}\right)^{-n}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^{n}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{n}}$$

$$u_n^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^n - 1\right]}$$

$$\lim_{n\to\infty} u_n^{1/n} = \frac{1}{(e-1)} < 1$$

$\therefore \Sigma u_n$ converges.

(5) $\Sigma (\sqrt{n-1})^n$

Sol :- here $u_n = (\sqrt{n-1})^n$

$$\Rightarrow u_n^{1/n} = \sqrt{n-1}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \sqrt{n-1} = \infty > 1$$

$\therefore \Sigma u_n$ diverges

June-15

(6) :- $\Sigma \frac{(n+1)^n x^n}{n^{n+1}}$

Sol :- Here $u_n = \frac{(n+1)^n x^n}{n^{n+1}}$

$$\Rightarrow (u_n)^{1/n} = \frac{(n+1)x}{n^{1+\frac{1}{n}}} = \frac{n\left(1+\frac{1}{n}\right)x}{n \cdot n^{1/n}}$$

$$(u_n)^{1/n} = \left(1+\frac{1}{n}\right)\cdot\frac{1}{n^{1/n}}\cdot x$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\cdot\frac{1}{n^{1/n}}\cdot x\right]$$

$$\left\{\because \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1\right\}$$

$$= x$$

$\therefore \Sigma u_n$ converges for $x<1$

diverges for $x>1$

and when $x=1$ Cauchy's test fails.

$\therefore u_n = \frac{(n+1)^n}{n^{n+1}}$

June-19-S

(7) Determine the nature of $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

Sol:- here $u_n = \left(\frac{n}{3n+1}\right)^n$

$$u_n^{1/n} = \frac{n}{3n+1}$$

Now

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n}{n\left(3+\frac{1}{n}\right)} = \frac{1}{3} < 1$$

$$\therefore \lim_{n\to\infty} u_n^{1/n} < 1$$

$\therefore$ From Cauchy's n^{th} root test given series converges.

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$$u_n = \frac{(n+1)^n}{n^{n+1}}$$

$$u_n = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n \cdot n} = \frac{1}{n}\cdot\left(1+\frac{1}{n}\right)^n$$

Let $V_n = 1/n$

$\Rightarrow \lim\limits_{n\to\infty} \frac{u_n}{V_n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \,(\neq 0)$ & finite

Also $\Sigma V_n = \Sigma \frac{1}{n}$ is p-series with $p=1$

$\therefore \Sigma V_n$ diverges

$\therefore$ from Comp. Test Σu_n diverges when $x=1$.

$\therefore \Sigma u_n$ converges for $x<1$

diverges for $x \geq 1$.

Alternating Series :-

A series in which the terms are alternately positive and negative is called an alternating series.

Leibnitz's Test :- (For Alternating Series)

An alternating series

$u_1 - u_2 + u_3 - u_4 + \cdots$ converges if

(i) each term is numerically less than its preceding term and

(ii) $\lim\limits_{n\to\infty} u_n = 0$.

Problems :- Test the convergence of following series.

(1) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

Sol :- The terms of the given series are alternately positive and negative. and

$1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \cdots$

i.e. the terms are numerically decreasing.

Also $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\, \frac{1}{\sqrt{n}} = 0$

$\therefore$ From Leibnitz's test given series converges.

(2) :- $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} n}{2n-1}$

Sol :- The term are alternately positive and negative.

and $u_n - u_{n-1} = \frac{n}{2n-1} - \frac{n-1}{2n-3}$

$$= \frac{2n^2 - 3n - 2n^2 + n + 2n - 1}{(2n-1)(2n-3)}$$

$$= \frac{-1}{(2n-1)(2n-3)} < 0 \quad \forall \quad n \geq 2$$

$$\Rightarrow u_n < u_{n-1}$$

$\therefore$ terms are numerically decreasing.

Now $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\, \frac{n}{2n-1} = \underset{n\to\infty}{Lt}\, \frac{1}{2-1/n} = \frac{1}{2} (\neq 0)$

$\therefore$ The series oscillates

(3) $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$

Sol :- The terms are alternately positive and negative.

and $1 > \frac{1}{2!} > \frac{1}{4!} > \frac{1}{6!} > \cdots$

i.e. terms are numerically decreasing

Also $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\, \frac{1}{(2n-2)!} = 0$.

$\therefore$ From Leibnitz's test given series converges.

(4) :- $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$

Sol :-

$$\sum u_n = \sum \frac{\cos n\pi}{n^2+1} = \sum \frac{(-1)^n}{n^2+1}$$

$$= -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots$$

$\therefore$ terms are alternately +ve & −ve and

$\frac{1}{2} > \frac{1}{5} > \frac{1}{10} > \cdots$ i.e. terms are numerically increasing.

Also

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$$

$\therefore$ from Leibnitz's test given series converges.

(5) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots$

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Sol :- Terms are alternately +ve and −ve.

here $u_n = \frac{1}{\log(n+1)}$

We know

$$n+2 > n+1$$

$$\log(n+2) > \log(n+1)$$

$$\frac{1}{\log(n+2)} < \frac{1}{\log(n+1)} \Rightarrow u_{n+1} < u_n$$

$\therefore$ terms are numerically decreasing

Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\log(n+1)} = 0.$$

$\therefore$ From Leibnitz's test given series converges.

Dec-18

(32)(a)

(6):- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Sol:-

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Terms are alternately +ve and -ve

∴ Given series is an alternating series,

∴ We use Leibnitz's test.

Clearly $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} > \cdots$

also $\lim_{n \to \infty} \frac{1}{n} = 0$

∴ From Leibnitz's test given series converges.

June-18

(7):- Determine the nature of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Sol:-

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \cdots$$

Given is an alternating series, so we use Leibnitz's test.

Clearly $1 > \frac{1}{4} > \frac{1}{9} > \frac{1}{16} > \frac{1}{25} > \cdots$

also $\lim_{n \to \infty} \frac{1}{n^2} = 0$

∴ From Leibnitz's test given series converges.

Series of positive and negative terms :-

A series $\sum u_n$ in which some terms are positive and some terms are negative without any order i.e. terms are with arbitrary signs is called as positive and negtive term series.

Absolute Convergence :-

Let $\sum u_n$ be a series of positive and negative terms. If $\sum |u_n|$ is convergent then $\sum u_n$ is said to be absolutely convergent.

Theorem :- An absolutely convergent series is necessarily convergent.

Pf :- Let $\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$

be an absolutely convergent series.

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \leq |u_1| + |u_2| + |u_3| + \cdots + |u_3| + \cdots$$

which is convergent

$\therefore \sum u_n$ converges.

Example :-

(1) $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \cdots \infty$.

Sol :- Given series is of positive and negative terms series.

Series of absolute values is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots = \sum \frac{1}{n^2}$$

which is a p-series with $p = 2 > 1$

$\therefore \sum \frac{1}{n^2}$ converges.

$\therefore$ Given series absolutely converges.

Conditional Convergence :-

Let Σu_n be a series of positive and negative terms such that Σu_n is convergent and $\Sigma |u_n|$ is divergent then Σu_n is said to be conditionally convergent.

Ex :- (1) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ is convergent

by Leibnitz's test bcz terms are alternately +ve & -ve and numerically decreasing also $\lim_{n\to\infty} \frac{1}{n} = 0$.

But the absolute series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \Sigma \frac{1}{n} \text{ is a p-series with } p = 1$$

$\therefore$ it diverges.

$\therefore$ Given series conditionally converges.

Problems :-

june-17-saq

(1) Prove that the series is conditionally Convergent.

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

Sol :- Absolute series is

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots = \Sigma \frac{1}{\sqrt{n}} = \Sigma \frac{1}{n^{1/2}} \text{ is p-series}$$

with $p = 1/2 < 1$, $\therefore$ ~~abs~~ $\Sigma |u_n|$ diverges.

But $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is alternating series.

and

$$1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \cdots$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$\therefore$ By Leibnitz's rule it converges

$\therefore$ Given series is conditionally convergent. //

(2) Test the series for absolute/conditional convergence.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$$

Sol :- $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n(\log n)^2}\right| = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is a positive term series and $u_n = \frac{1}{n(\log n)^2}$ decreases as n increases,

$\therefore$ From Integral test.

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\log x)^2}\,dx$$

$$= \int_2^{\infty} (\log x)^{-2}\left(\frac{1}{x}\right)dx \qquad \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \left[\frac{(\log x)^{-2+1}}{-2+1}\right]_2^{\infty}$$

$$= \left[\frac{-1}{\log x}\right]_2^{\infty} = \left[\frac{-1}{\log\infty - \log 2}\right] = 0 \text{ finite}.$$

$\therefore$ $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n(\log n)^2}\right|$ is convergent.

$\therefore$ $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$ converges absolutely.

---

(3) Test whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ absolutely/conditionally converges.

Sol :- Given $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{2n-1}\right| = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$\therefore u_n = \frac{1}{2n-1} = \frac{1}{n(2-1/n)}$$

Let $v_n = 1/n$

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$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0 \text{ \& finite}$$

and $\Sigma v_n = \Sigma \frac{1}{n}$ diverges as $p=1$

From Comp. Test $\Sigma|u_n|$ ~~converges~~ diverges.

Now given series is

$$\Sigma u_n = \Sigma \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} + \cdots$$

is alternating series and terms are numerically decreasing

also $\underset{n\to\infty}{Lt}\frac{1}{2n-1} = 0$

From Liebnitz's test given series converges.

$\therefore$ Given series conditionally converges.

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Dec-17, LAQ.

(4) Test $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ for conditional convergence.

Sol:- $\sum u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

$$\sum |u_n| = \sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$|u_n| = \frac{n}{n^2+1} = \frac{n}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{1}{n} \cdot \frac{1}{\left(1+\frac{1}{n^2}\right)}$$

Let $V_n = \frac{1}{n}$

$\Rightarrow \frac{|u_n|}{V_n} = \frac{1}{\left(1+\frac{1}{n^2}\right)} \Rightarrow \lim_{n\to\infty} \frac{|u_n|}{V_n} = 1 (\neq 0)$ finite.

Also $\sum V_n = \sum \frac{1}{n}$ is p-series with $p=1$

$\therefore \sum V_n$ diverge.

$\therefore$ From Comp test $\sum |u_n|$ also diverge.

Now $\sum (-1)^{n-1} \frac{n}{n^2+1} = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \cdots$

which is an alternating series.

Also, $\frac{1}{2} > \frac{2}{5} > \frac{3}{10} > \cdots$

$\therefore$ terms are numerically decreasing.

also $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0$

$\therefore \sum (-1)^{n-1} \frac{n}{n^2+1}$ converges from Leibnitz's test.

$\therefore$ Given series converges conditionally.

June 17, LAQ

(5) Prove that the series $\sum (-1)^n \frac{\sin nx}{n^2}$ converges absolutely.

Sol:- $\sum u_n = \sum (-1)^n \frac{\sin nx}{n^2}$

Absolute series, $\sum |u_n| = \sum \left| (-1)^n \frac{\sin nx}{n^2} \right| = \sum \left| \frac{\sin nx}{n^2} \right| \leq \sum \frac{1}{n^2}$

which is p-series with $p=2>1$

$\therefore \sum \frac{1}{n^2}$ converges $\Rightarrow \sum \left| \frac{\sin nx}{n^2} \right|$ also converges

$\therefore \sum u_n$ converges absolutely.