Single Source Shortest Path Problem

Outline for Today

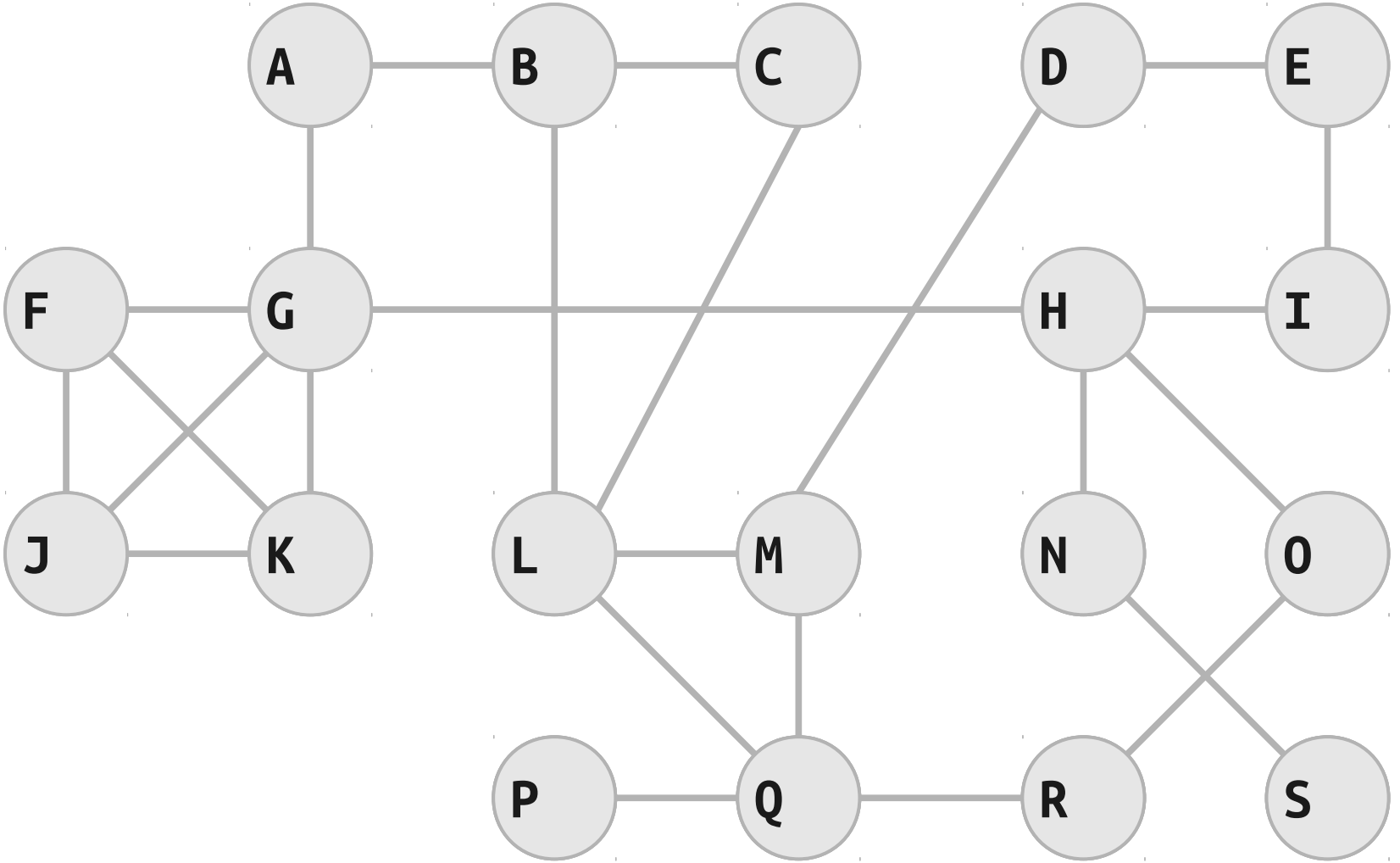
- **BFS/DFS**
- **Dijkstra's Algorithm**
  - An algorithm for finding shortest paths in more realistic settings
- **Bellman-Ford**
- **Practical Concerns**

Breadth-First Search

- Given an arbitrary graph $G = (V, E)$ and a starting node $s \in V$, **breadth-first search** finds shortest paths from s to each reachable node v.
- When implemented using an adjacency list, runs in $O(m + n)$ time, which we defined to be linear time on a graph.
- One correctness proof worked in terms of "layers:" the algorithm finds all nodes at distance 0, 1, 2, ... in order.

A Second Intuition for BFS

Breadth-First Search



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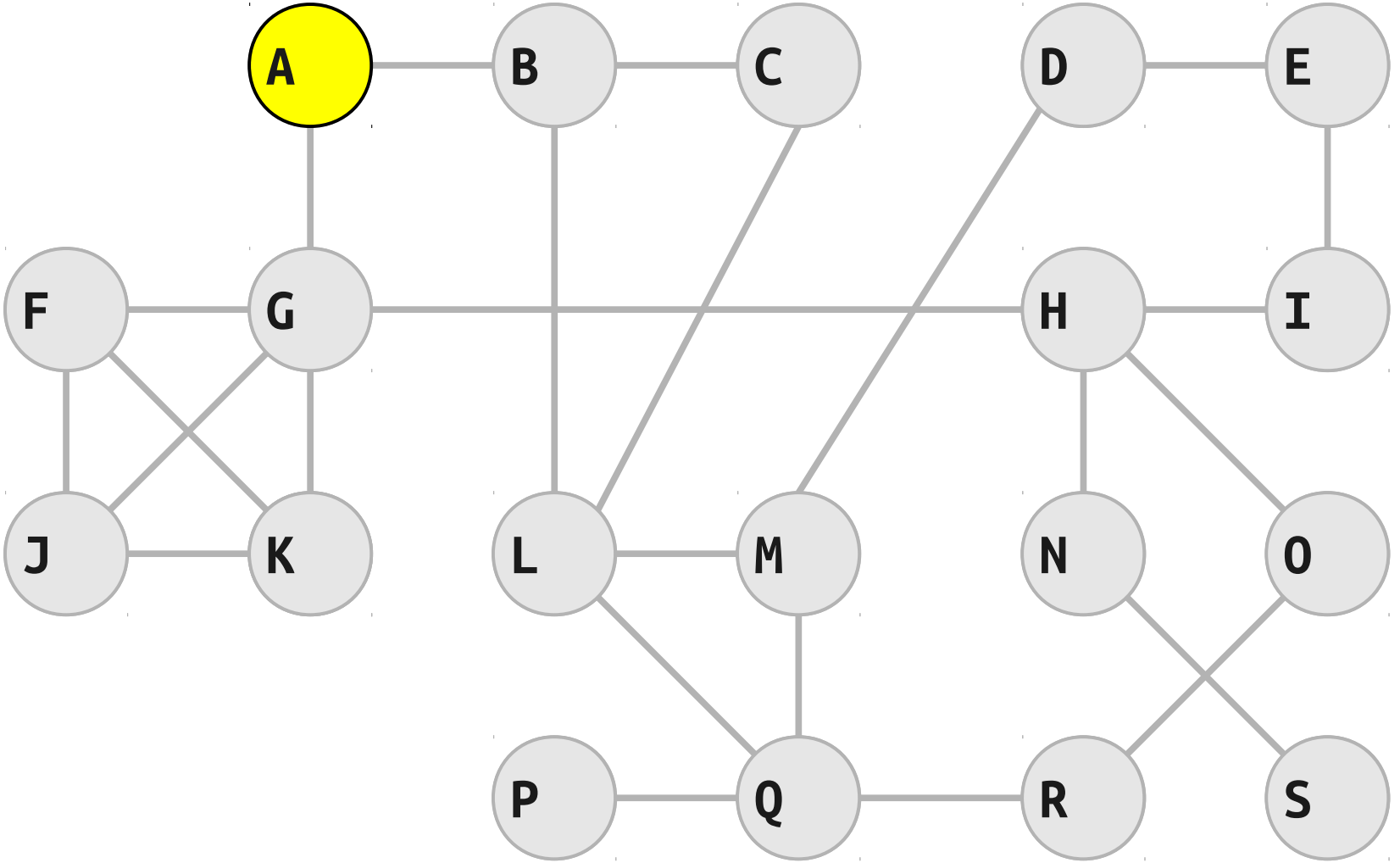

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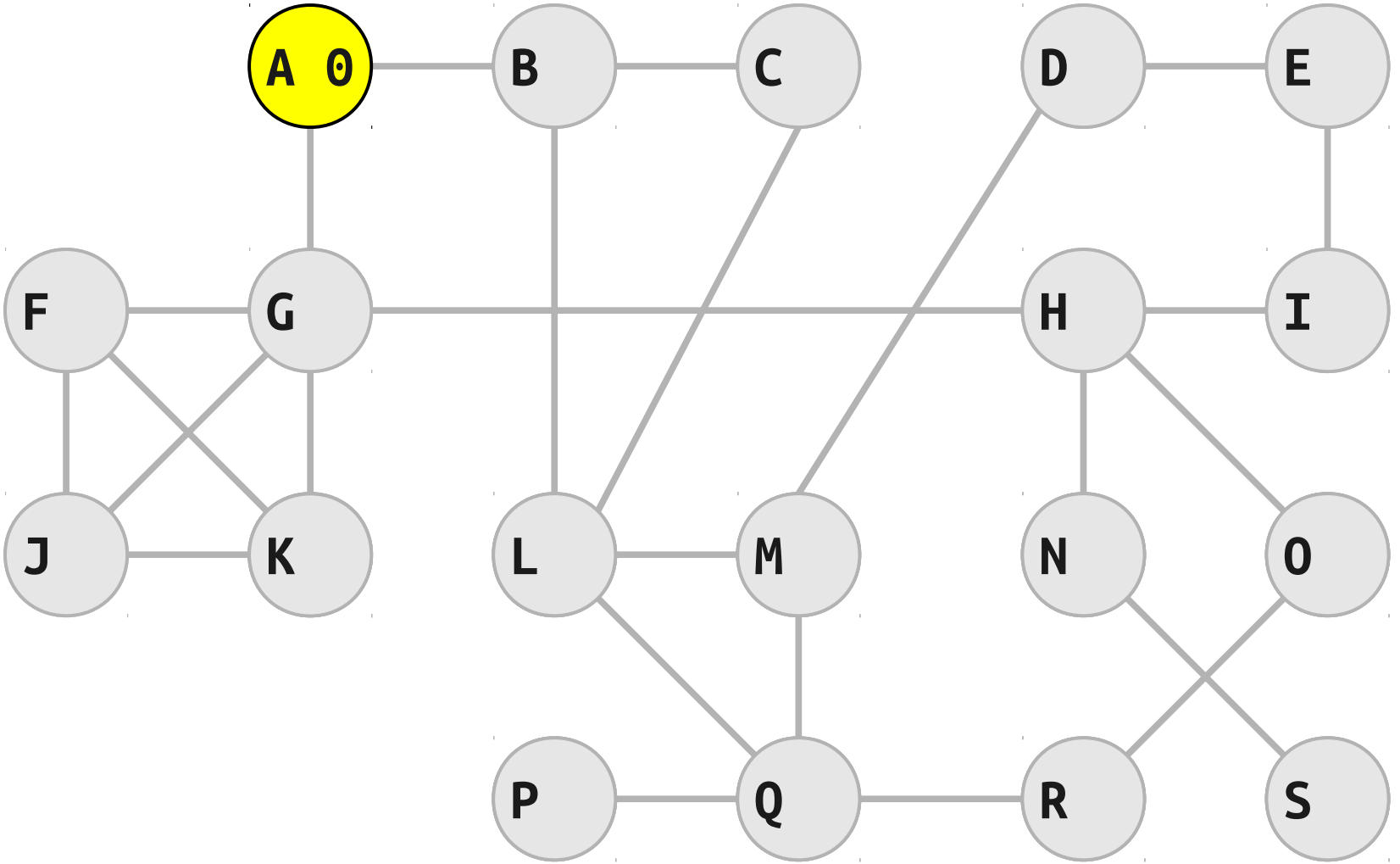

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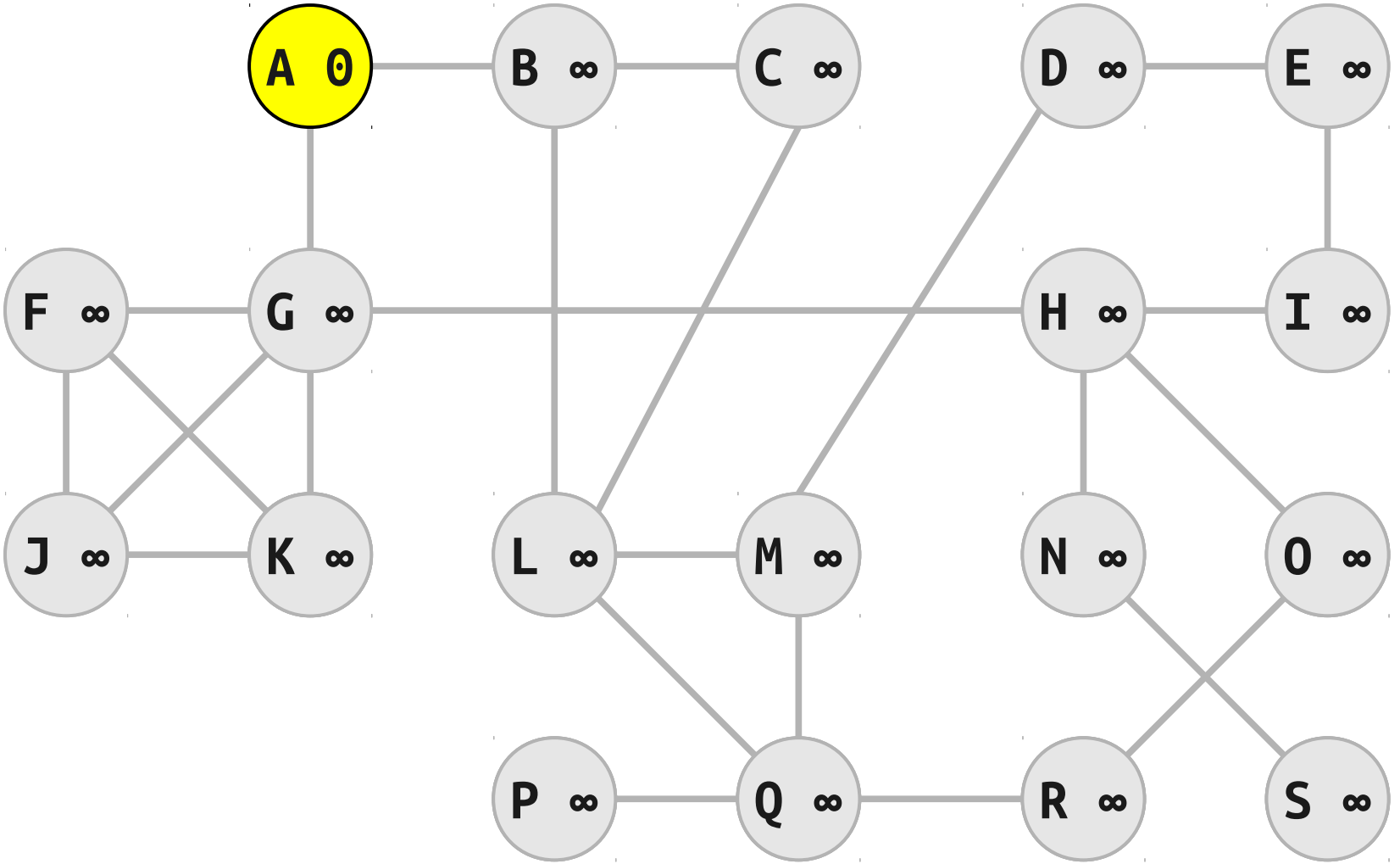

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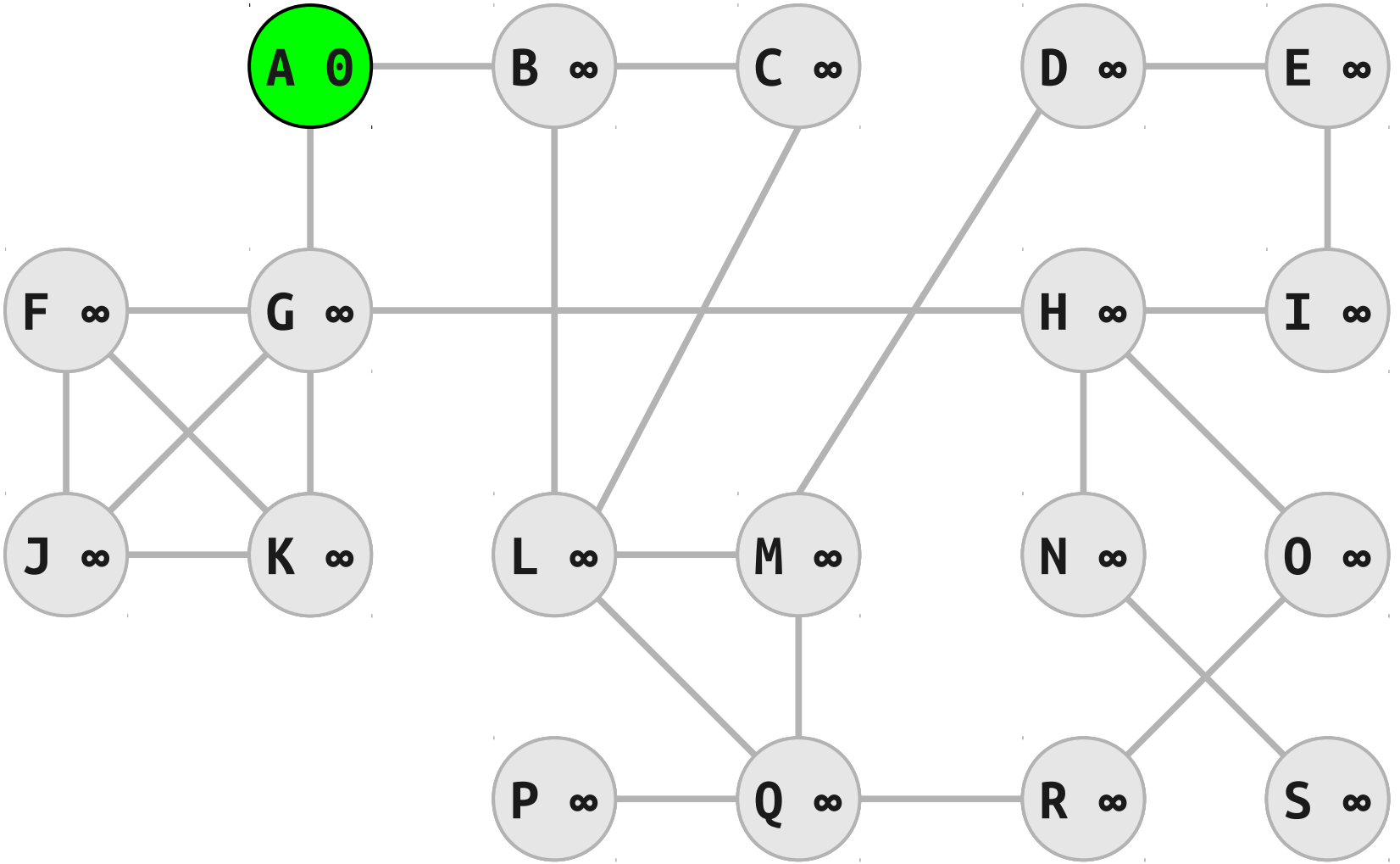

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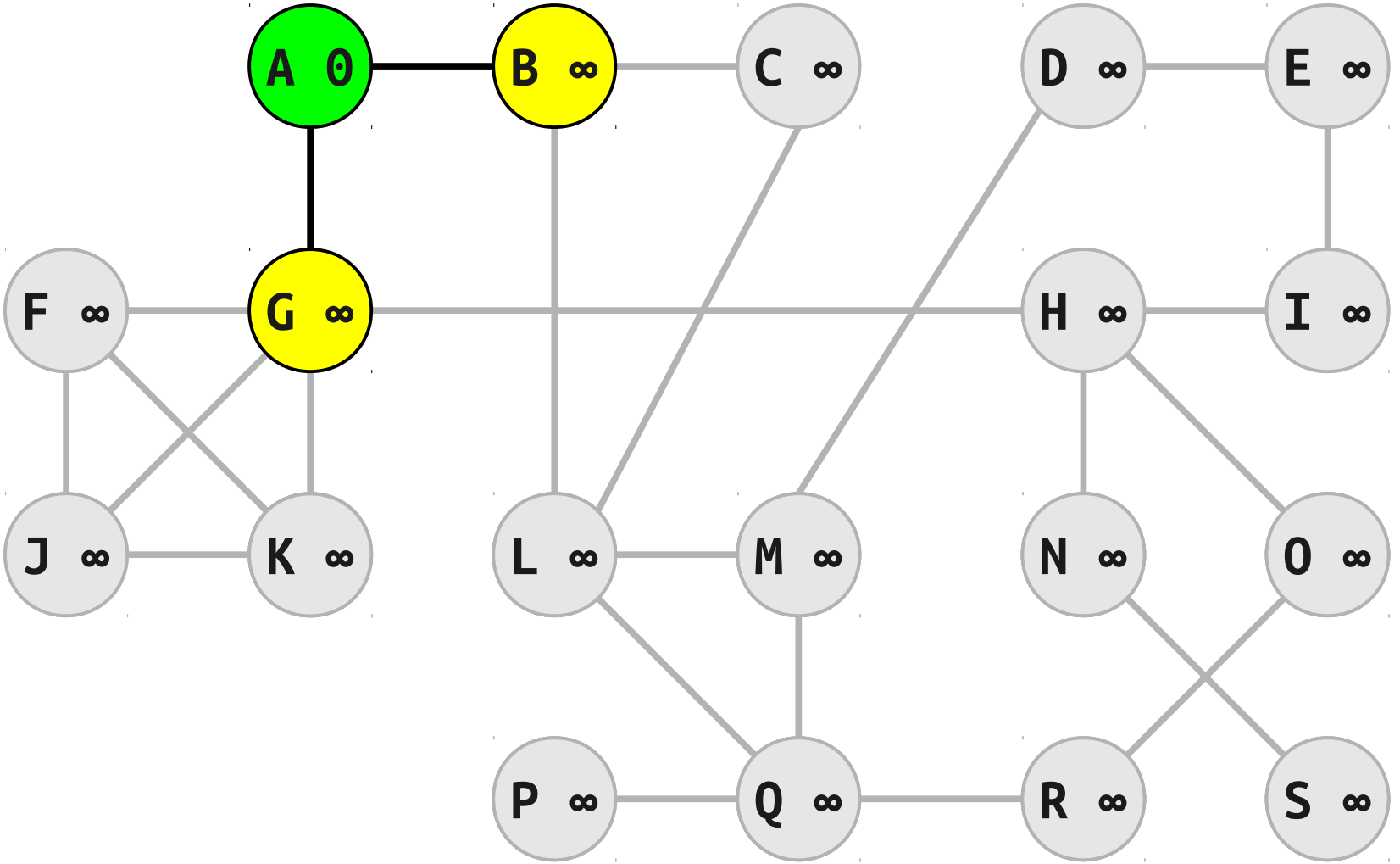

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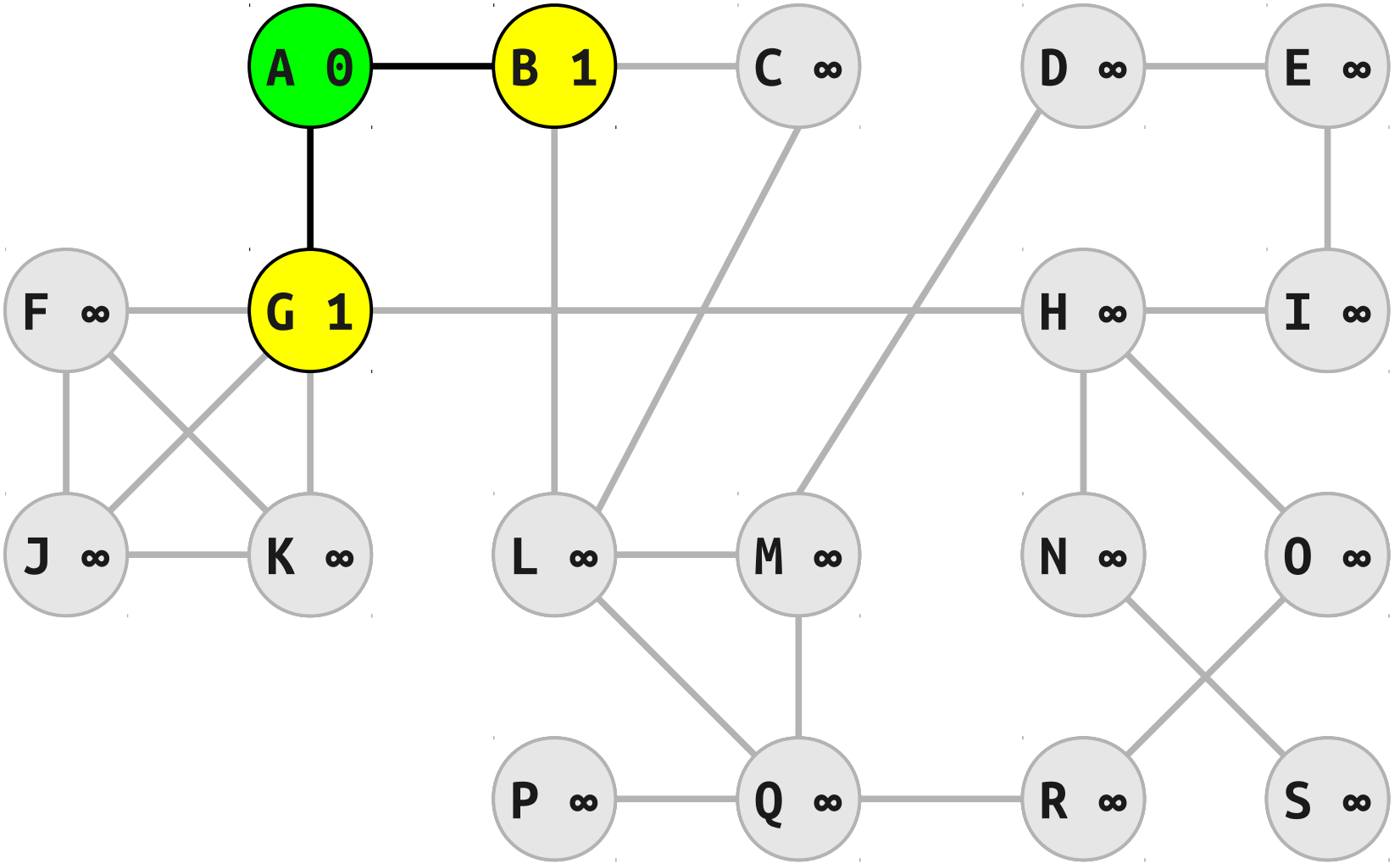

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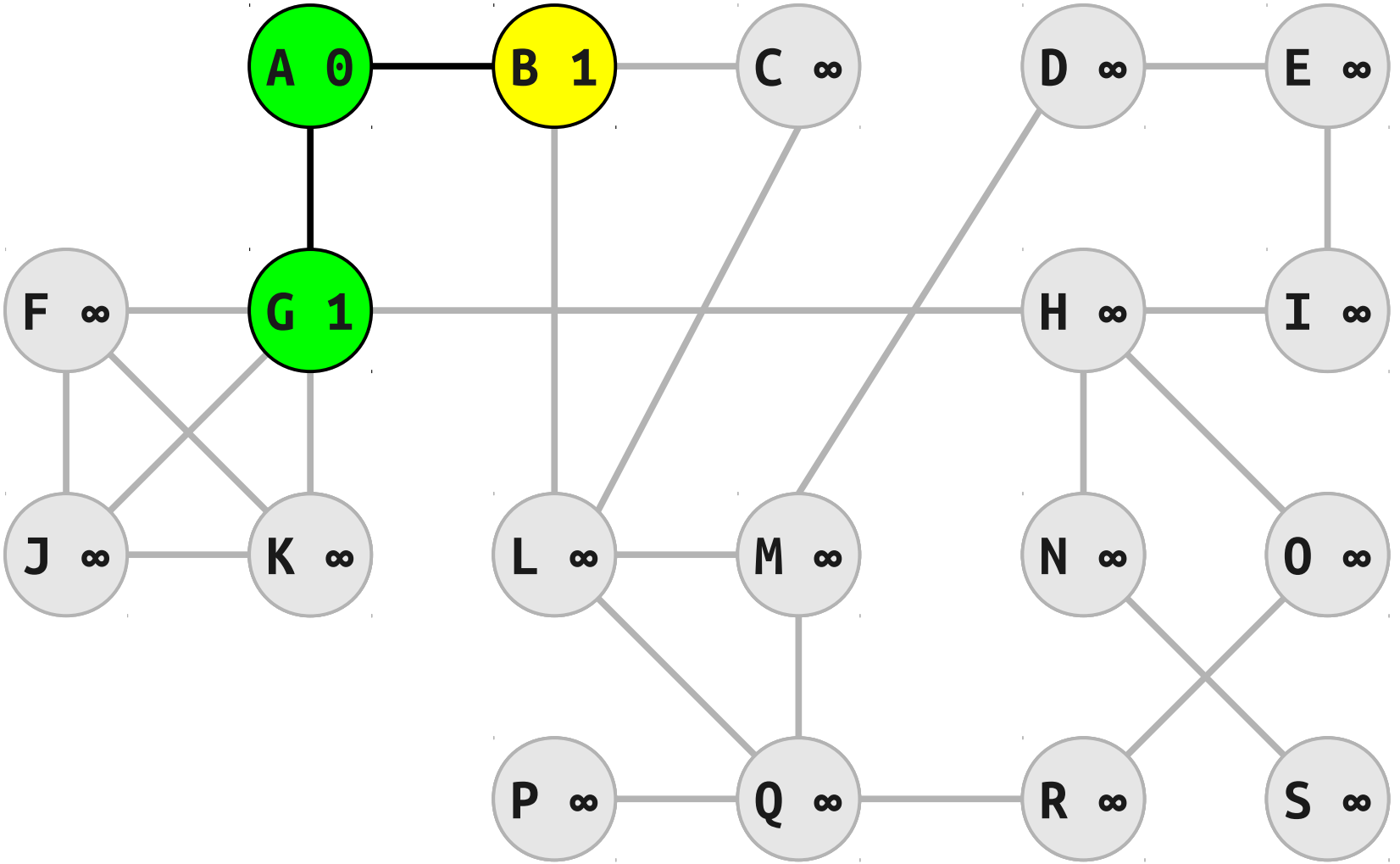

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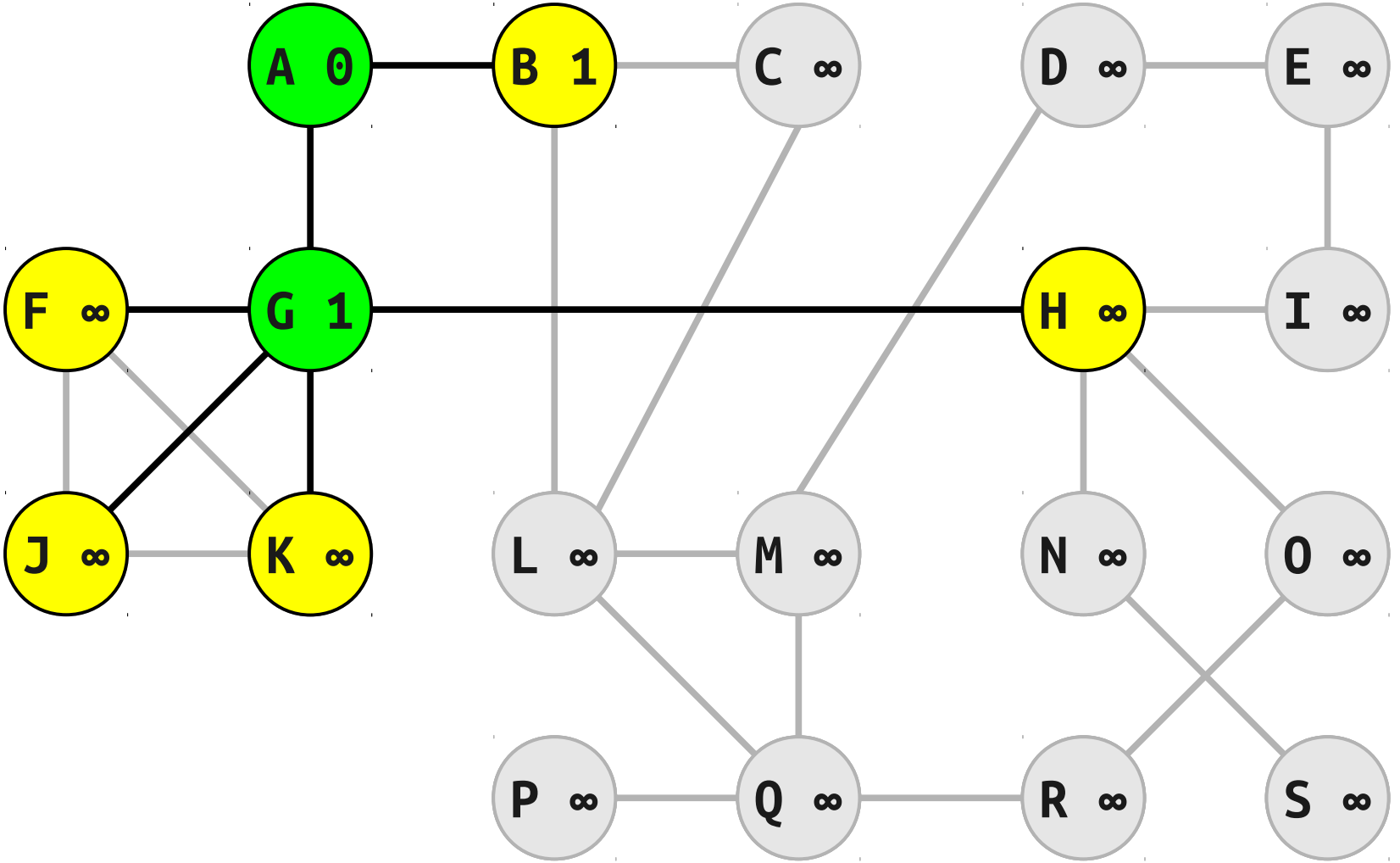

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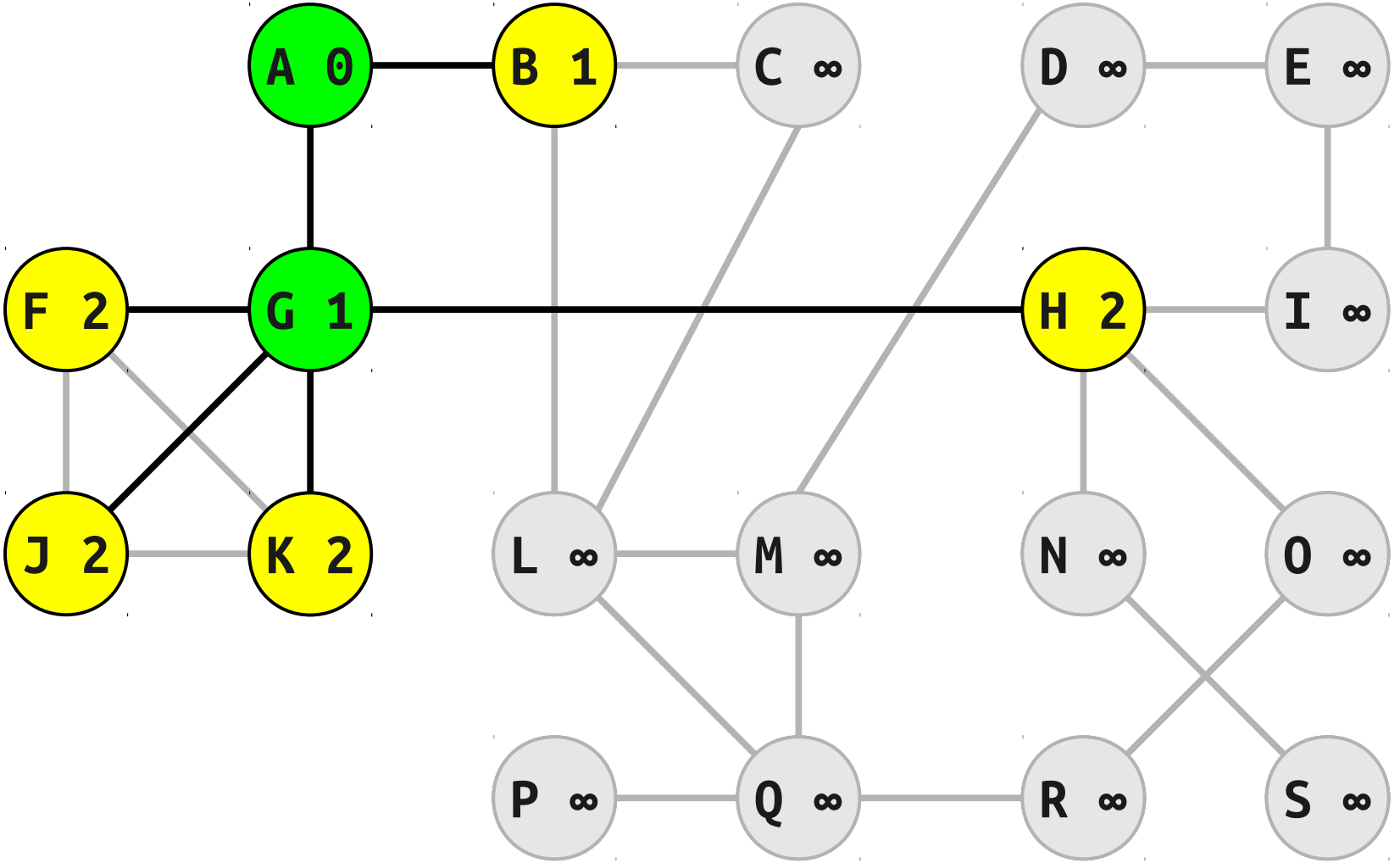

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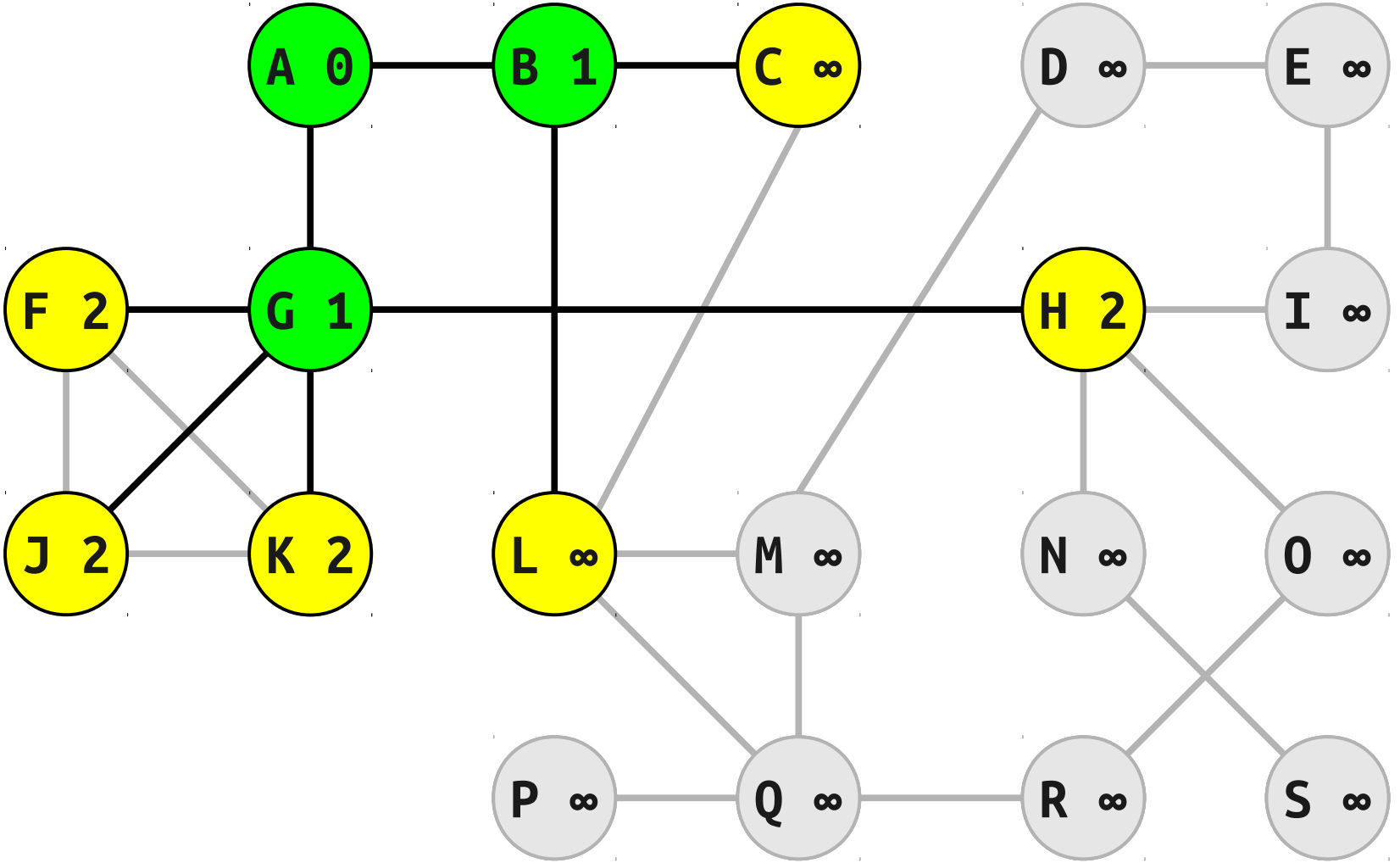

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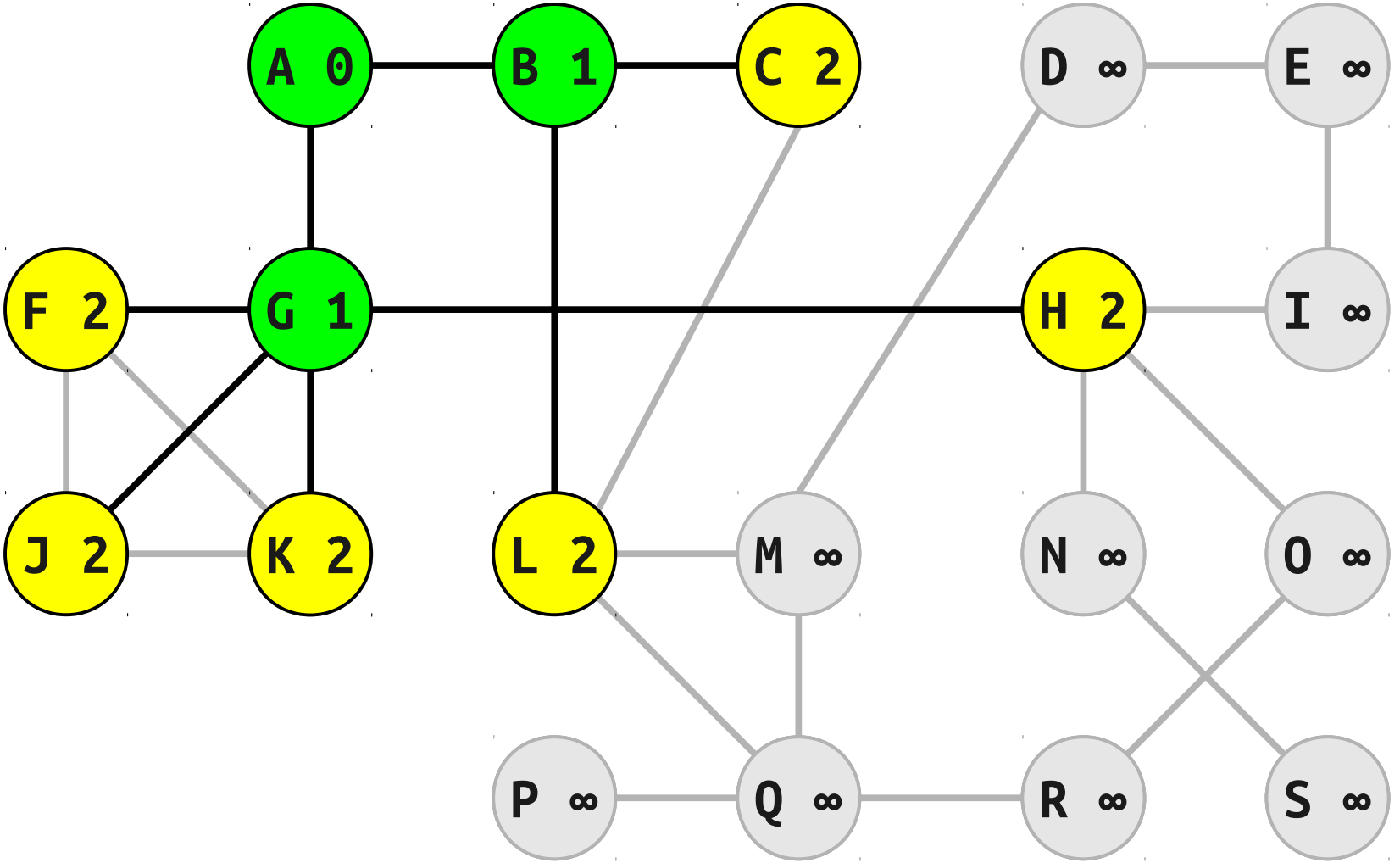

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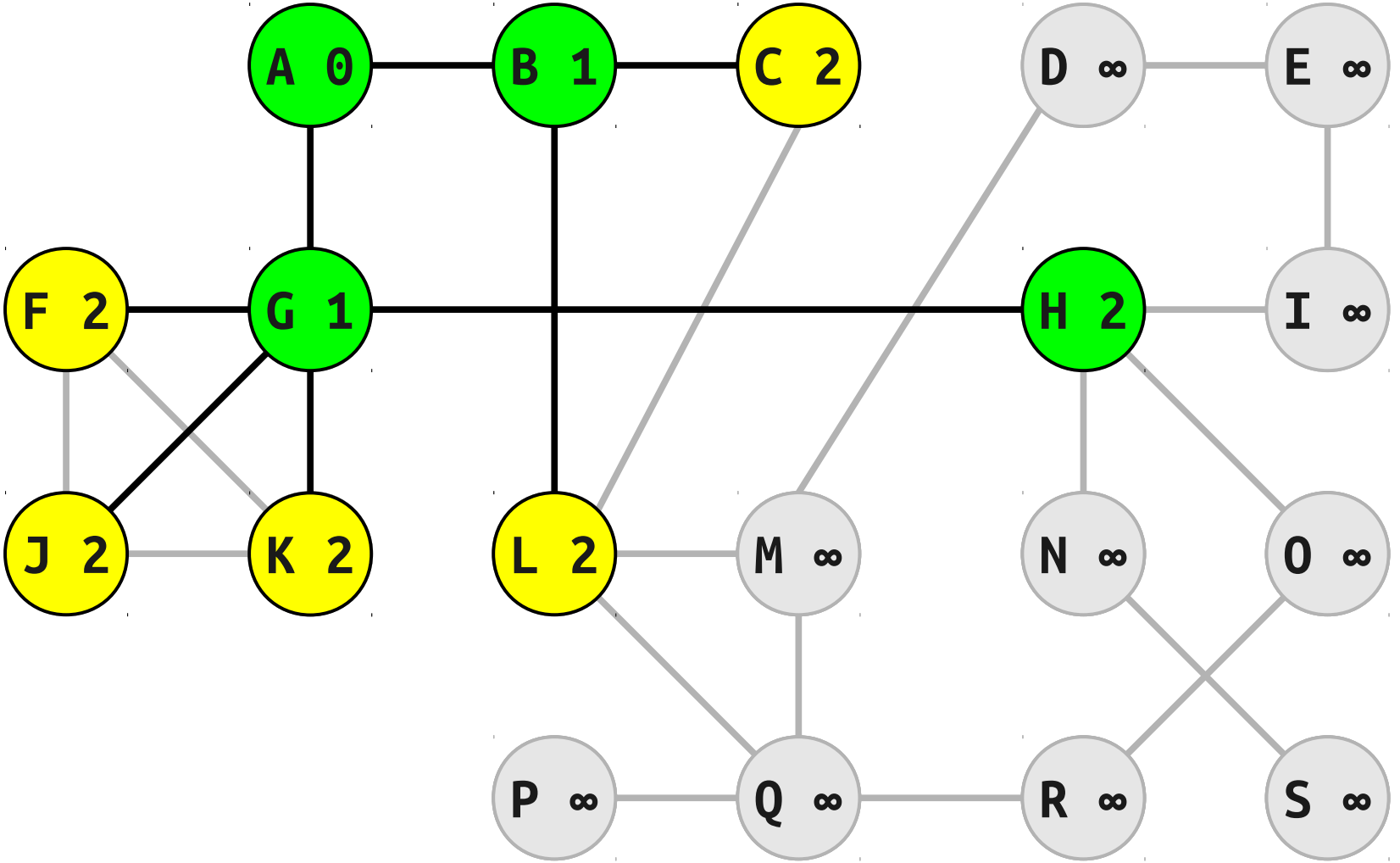

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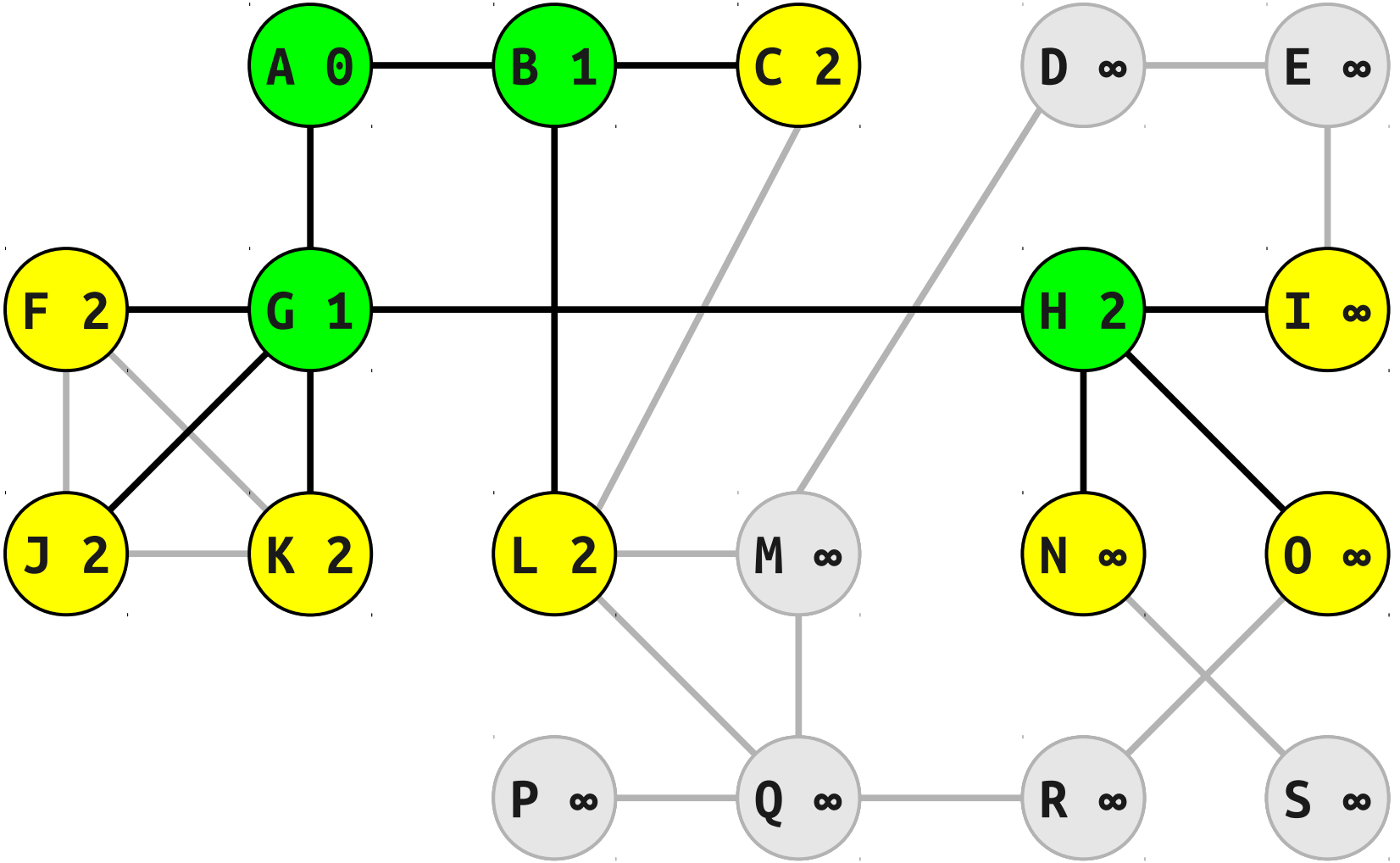

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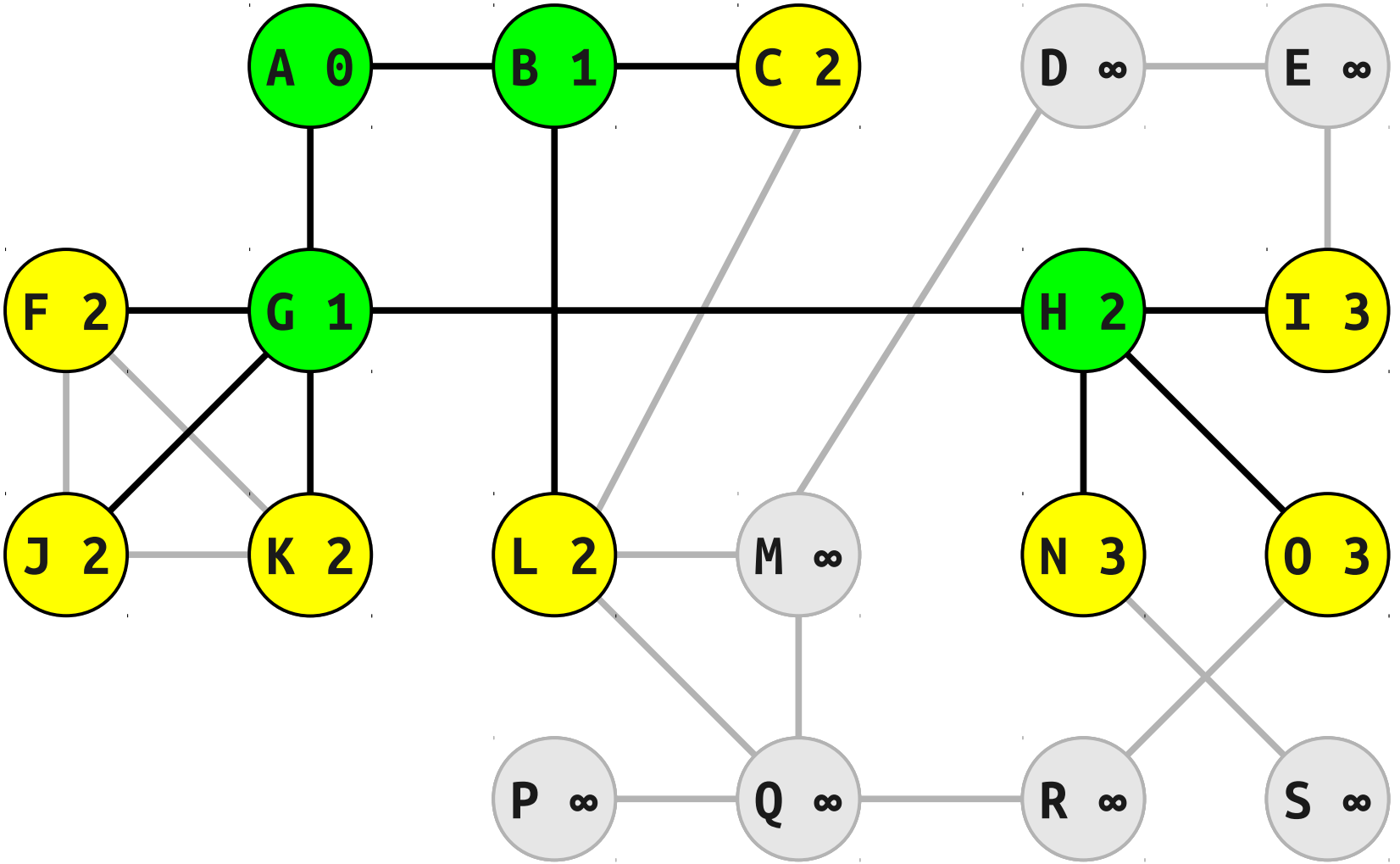

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procedure breadthFirstSearch(s, G):
    let q be a new queue.
    for each node v in G:
      dist[v] = ∞

    dist[s] = 0
    enqueue(s, q)

    while q is not empty:
       let v = dequeue(q)
       for each neighbor u of v:
          if dist[u] = ∞:
             dist[u] = dist[v] + 1
             enqueue(u, q)
```

O(1)

O(*n*)

O(1)

O(*n*)

O(*m* + *n*)

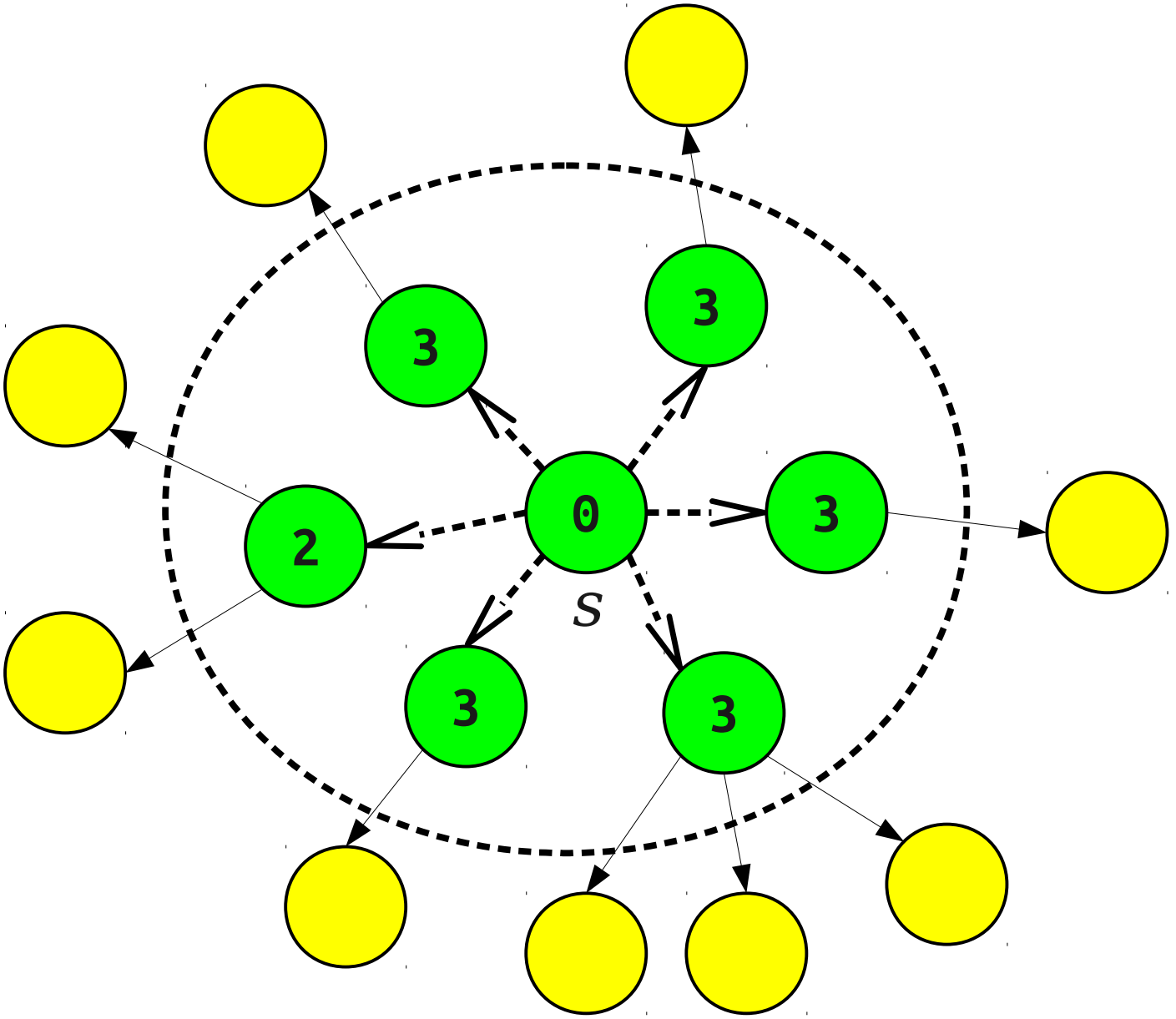
A Second Intuition

- At each point in the execution of BFS, a node v is either
  - green, and we have the shortest path to v;
  - yellow, and it is connected to some green node; or
  - gray, and v is undiscovered.
- Each iteration, we pick a yellow node with minimal distance from the start node and color it green. So what is the cost of the lowest-cost yellow node?
- If v is yellow, it is connected to a green node u by an edge.
- The cost of getting from s to v is then $d(s, u) + 1$.
- BFS works by picking the yellow node v minimizing

$$\mathbf{d(s, u) + 1}$$

where (u, v) is an edge and u is green.

Pick yellow node v minimizing d(s, u) + 1, where (u, v) is an edge and u is green.



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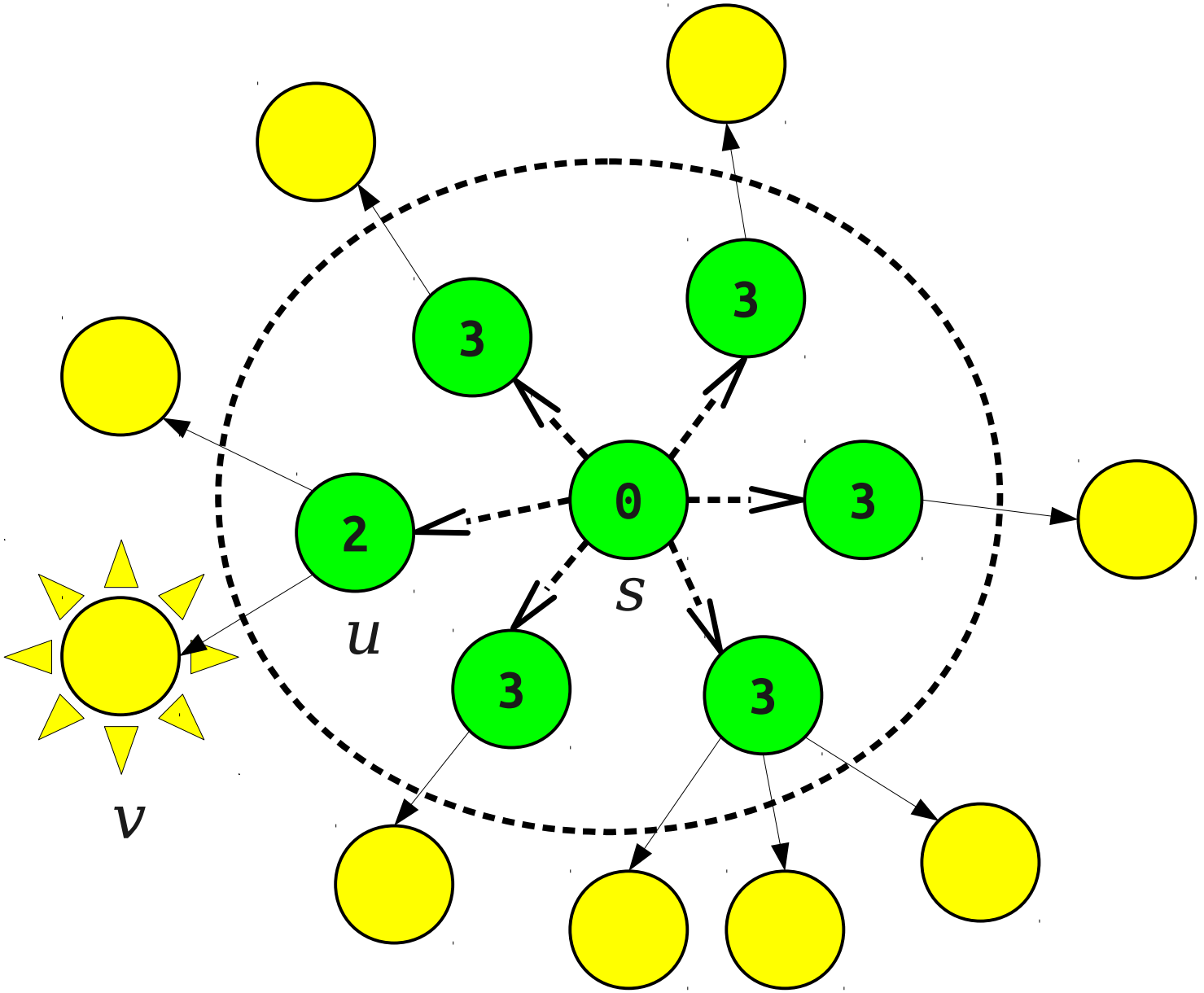

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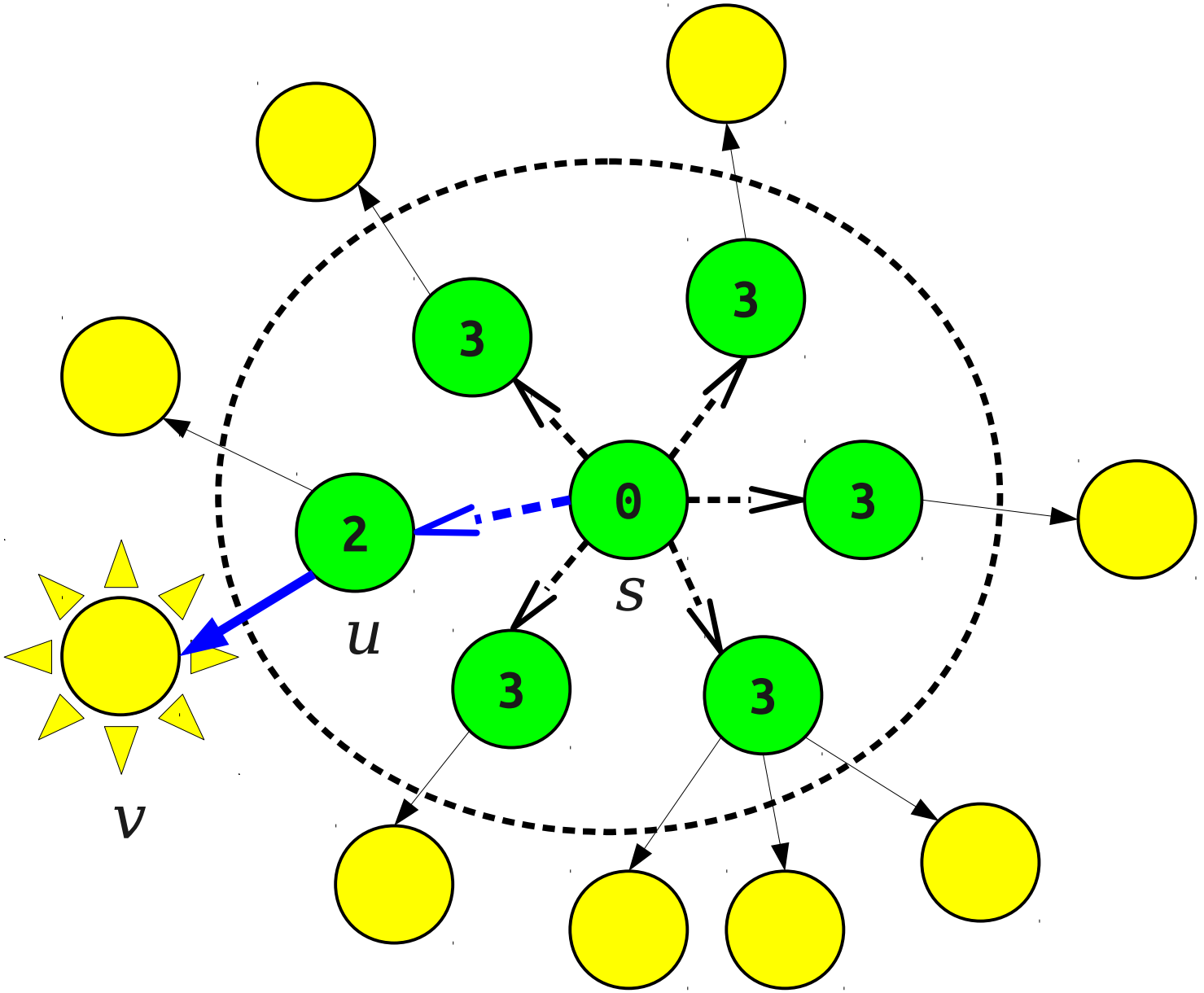

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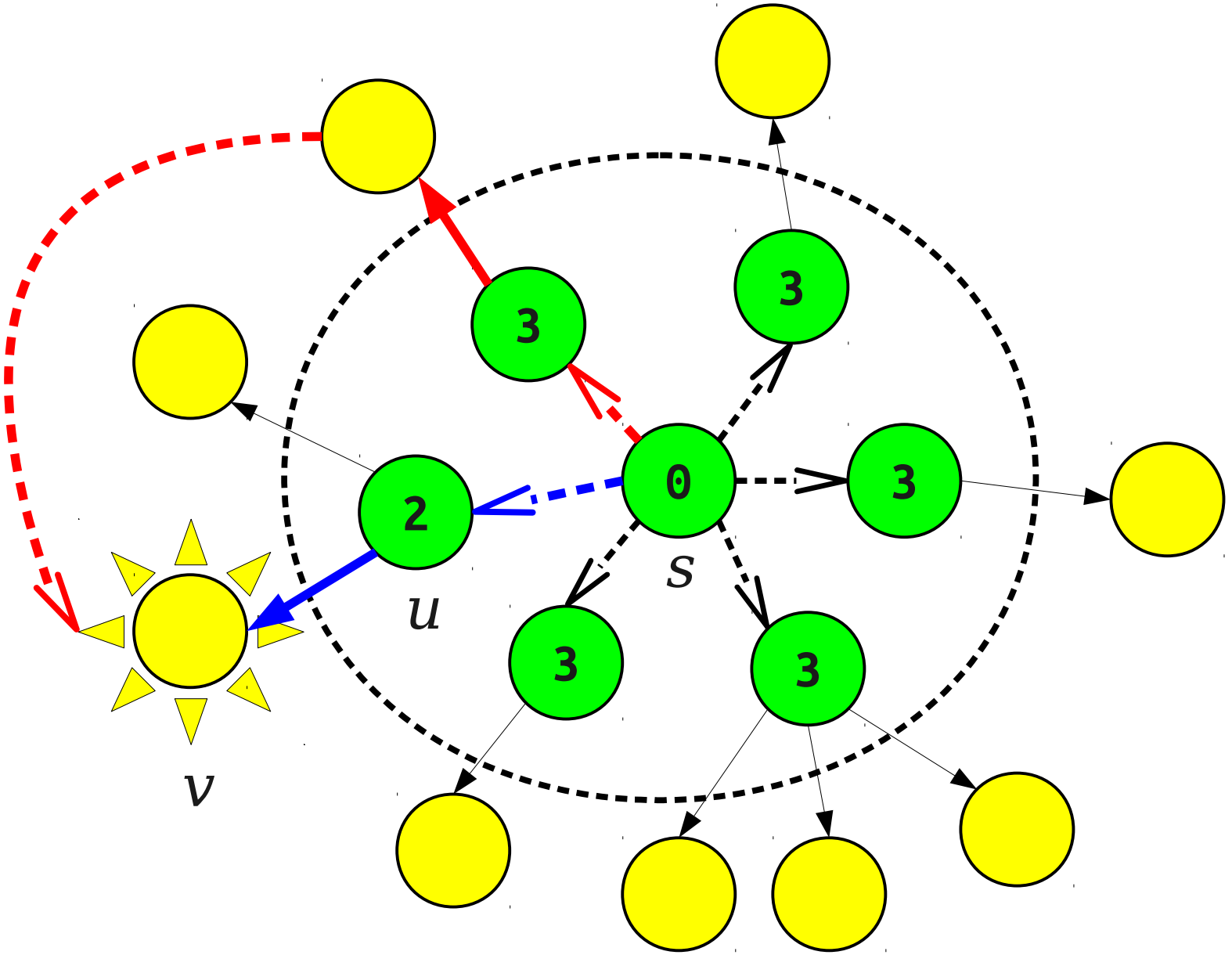

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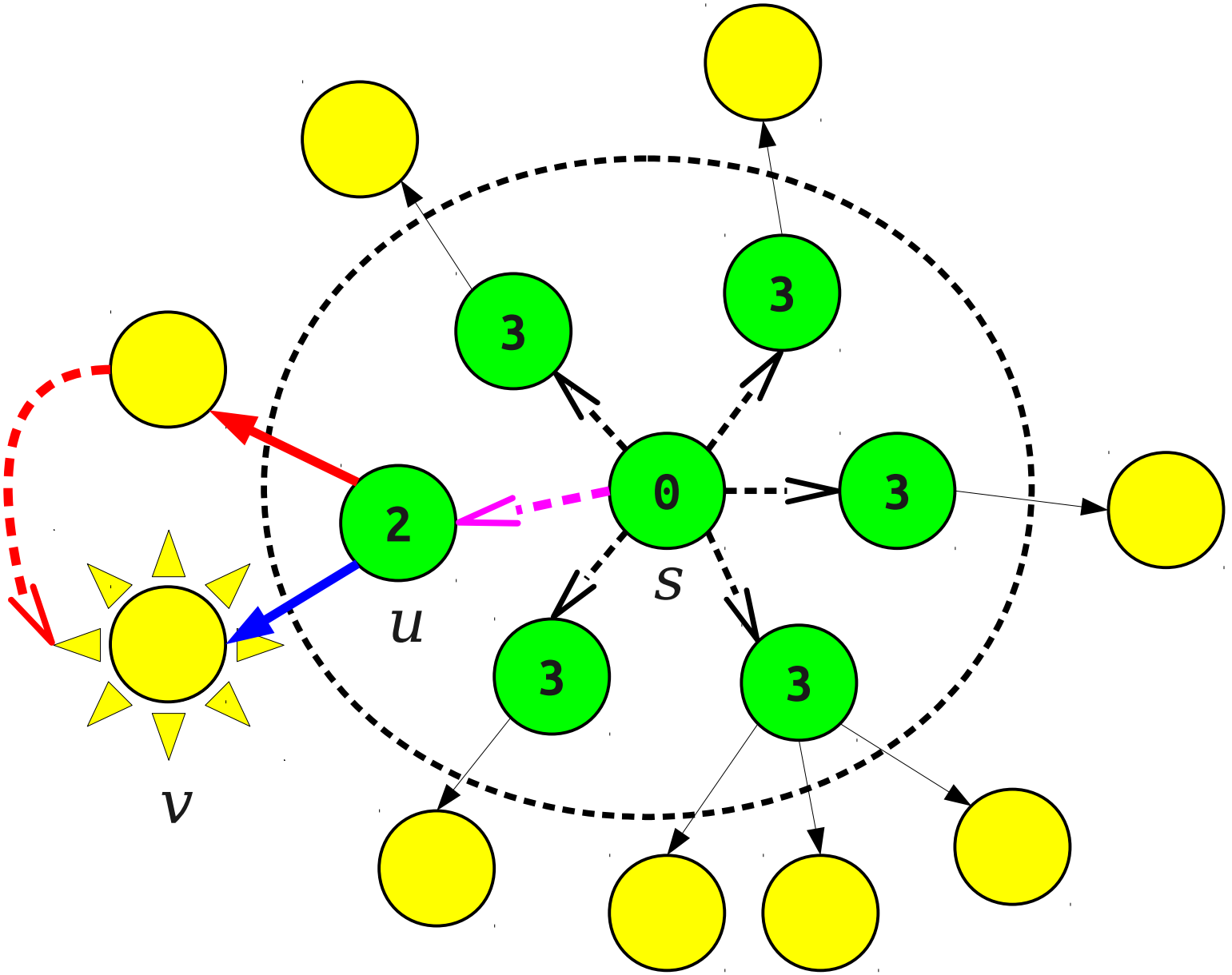

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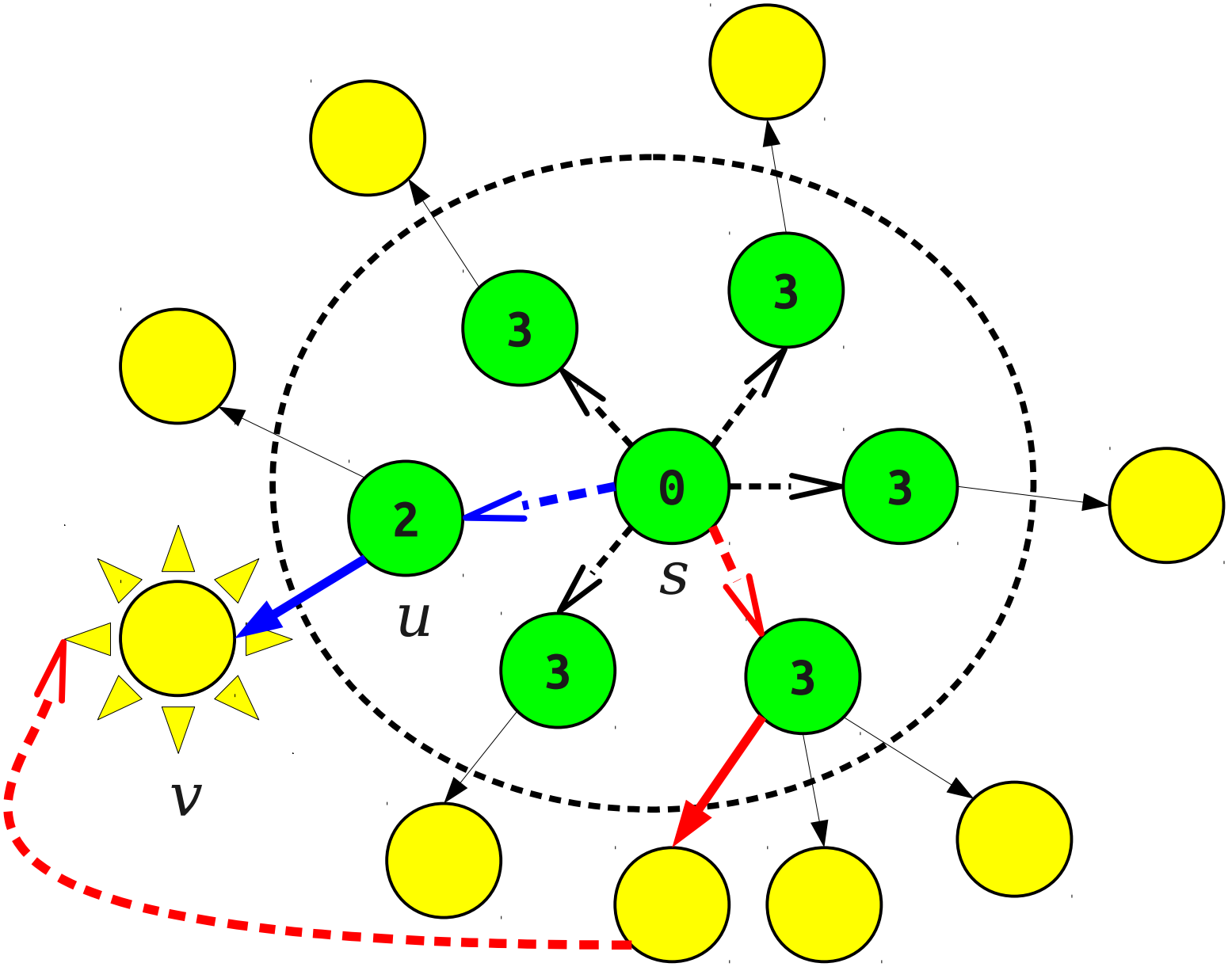

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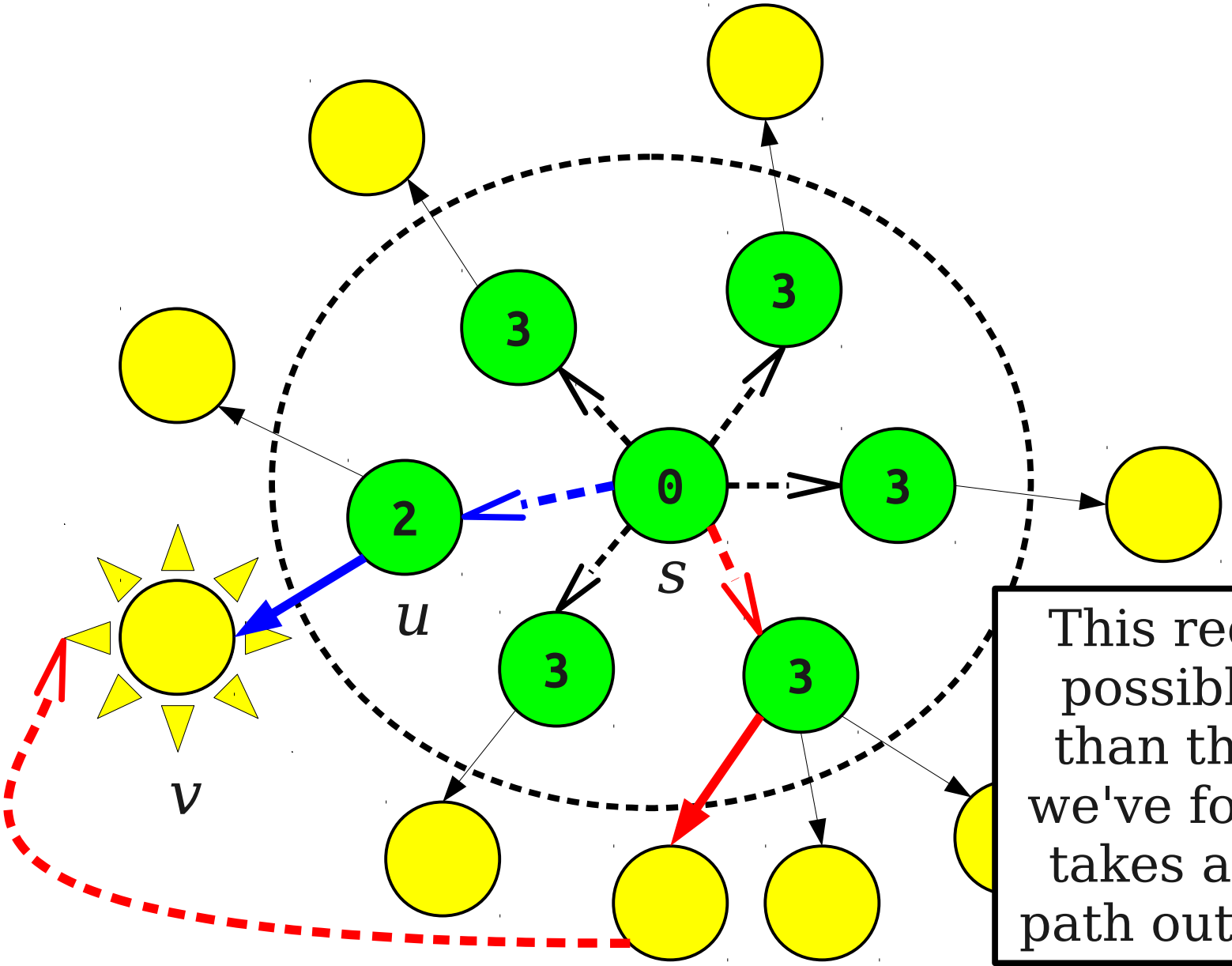

This red path can't possibly be better than the blue path we've found, since it takes a suboptimal path out of the circle!

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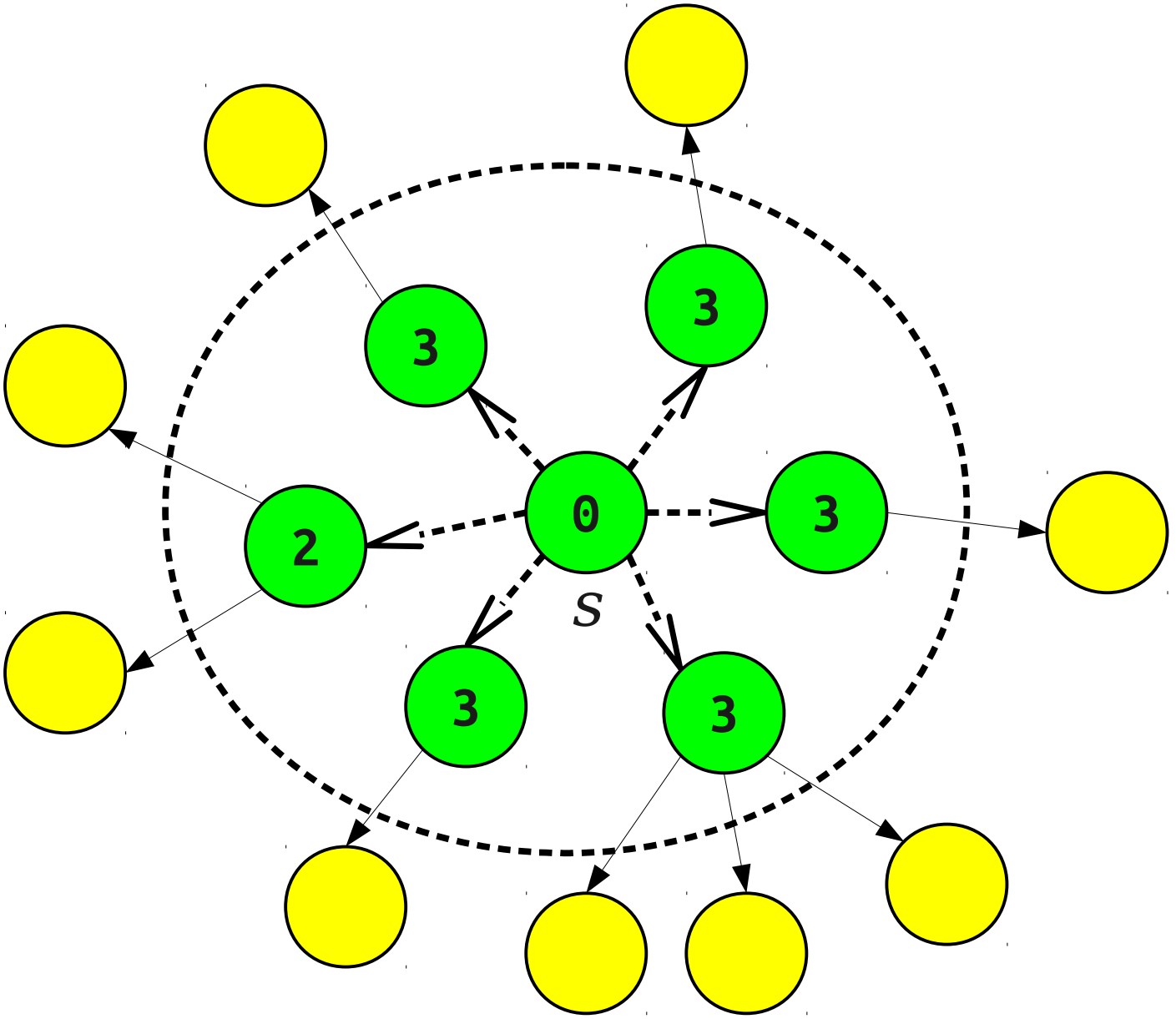

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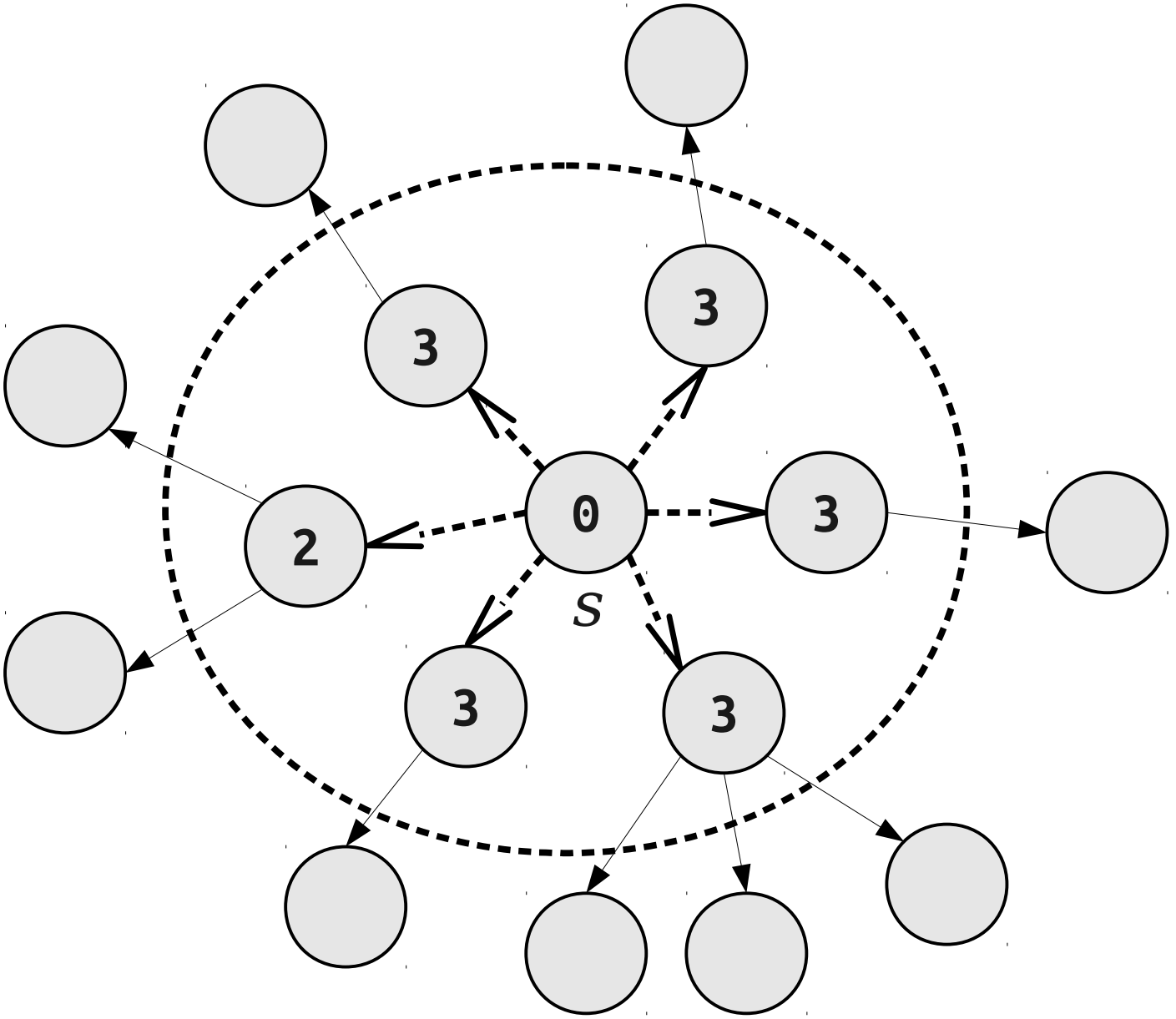

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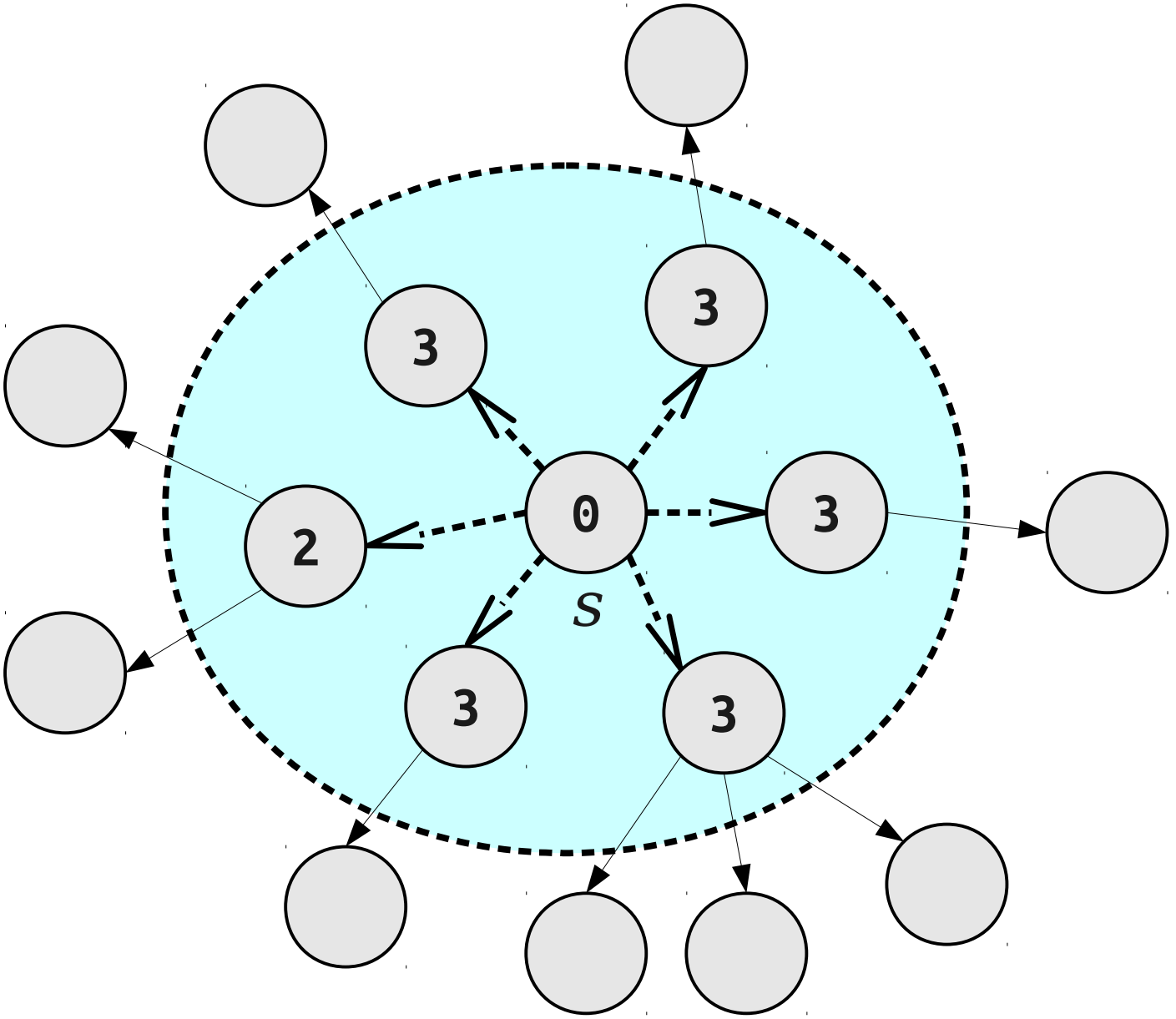

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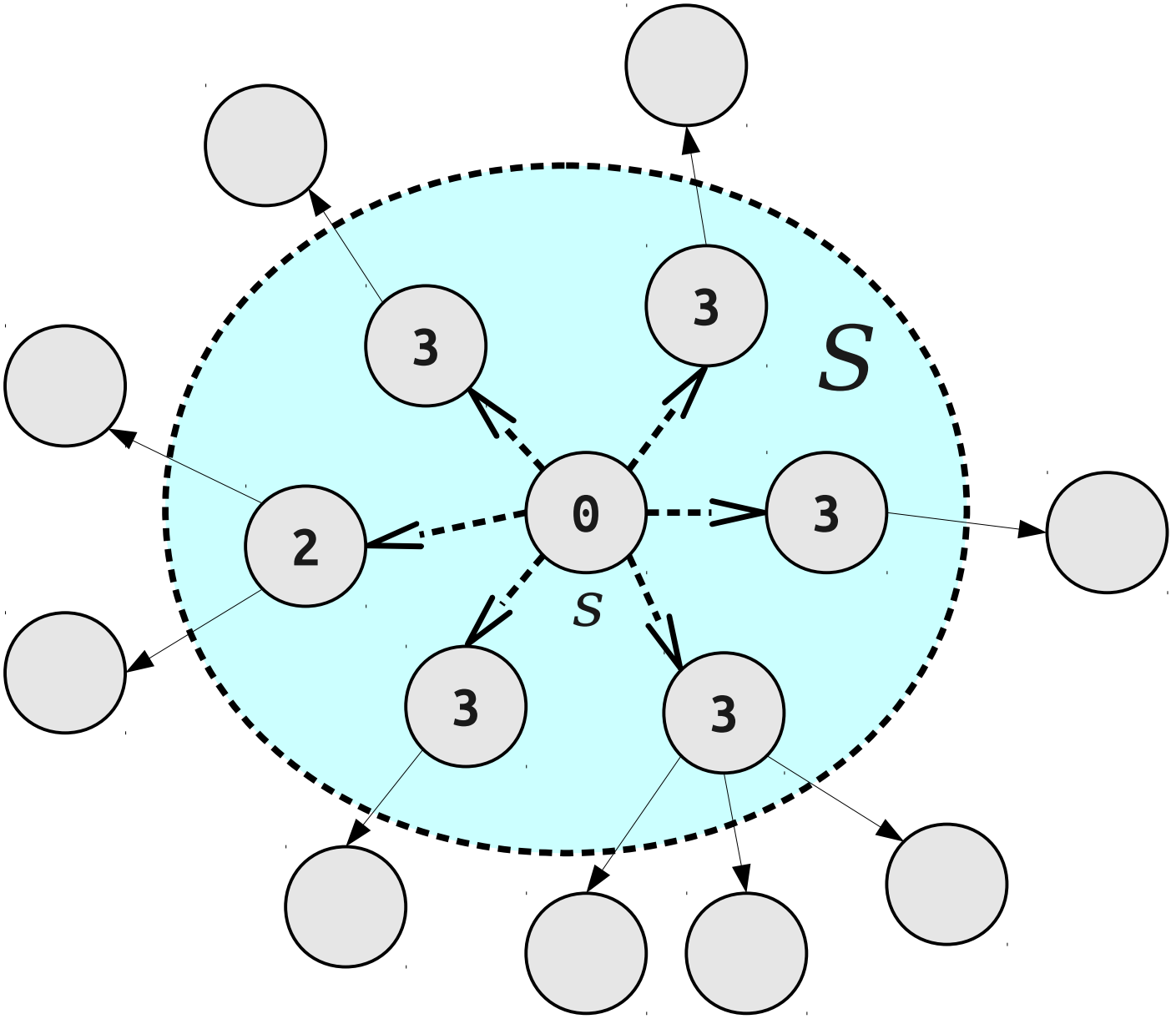
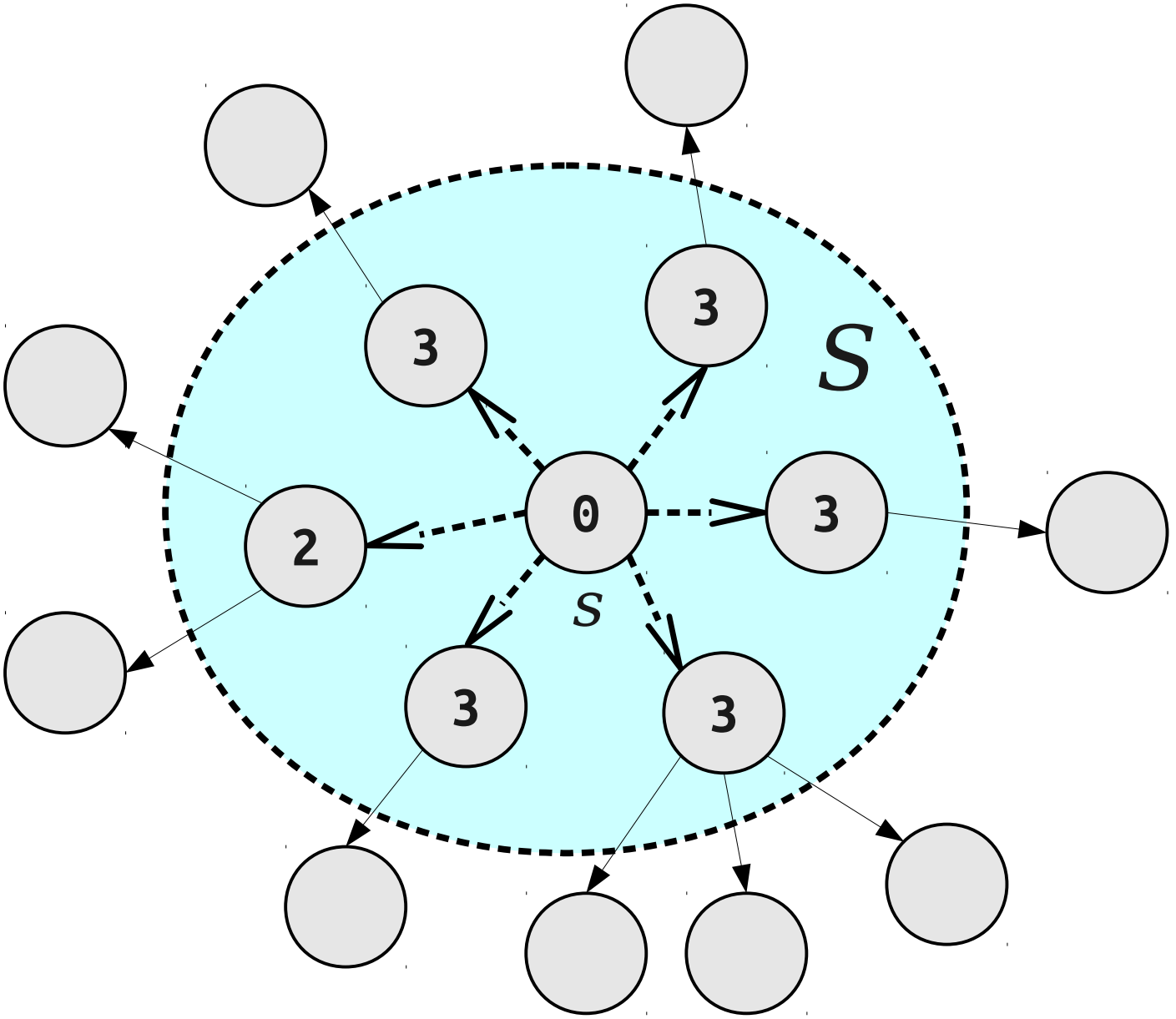


Pick node $v \notin S$ minimizing d(s, u) + 1,
where (u, v) is an edge and $u \in S$



Lemma: Suppose we have shortest paths computed for nodes $S \subseteq V$, where $s \in S$. Consider a node v where $(u, v) \in E$, $u \in S$, and the quantity $\mathrm{d}(s, u) + 1$ is minimized. Then $d(s, v) = d(s, u) + 1$.

Proof: There is a path to v of cost $\mathrm{d}(s, u) + 1$: follow the shortest path to u (which has cost $\mathrm{d}(s, u)$), then follow one more edge to v for total cost $\mathrm{d}(s, u) + 1$.

Now suppose for the sake of contradiction that there is a shorter path P to v. This path must start in S (since $s \in S$) and leave S (since $v \notin S$). So consider when P leaves S. When this happens, P must go from s to some node $x \in S$, cross an edge (x, y) to some node y, then continue from y to v. This means that $|P|$ is at least $\mathrm{d}(s, x) + 1$, since the path goes from s to x and then follows at least one more edge.

Since v was picked to minimize $\mathrm{d}(s, u) + 1$ for any choice of $u \in S$ adjacent to an edge (u, v), we know

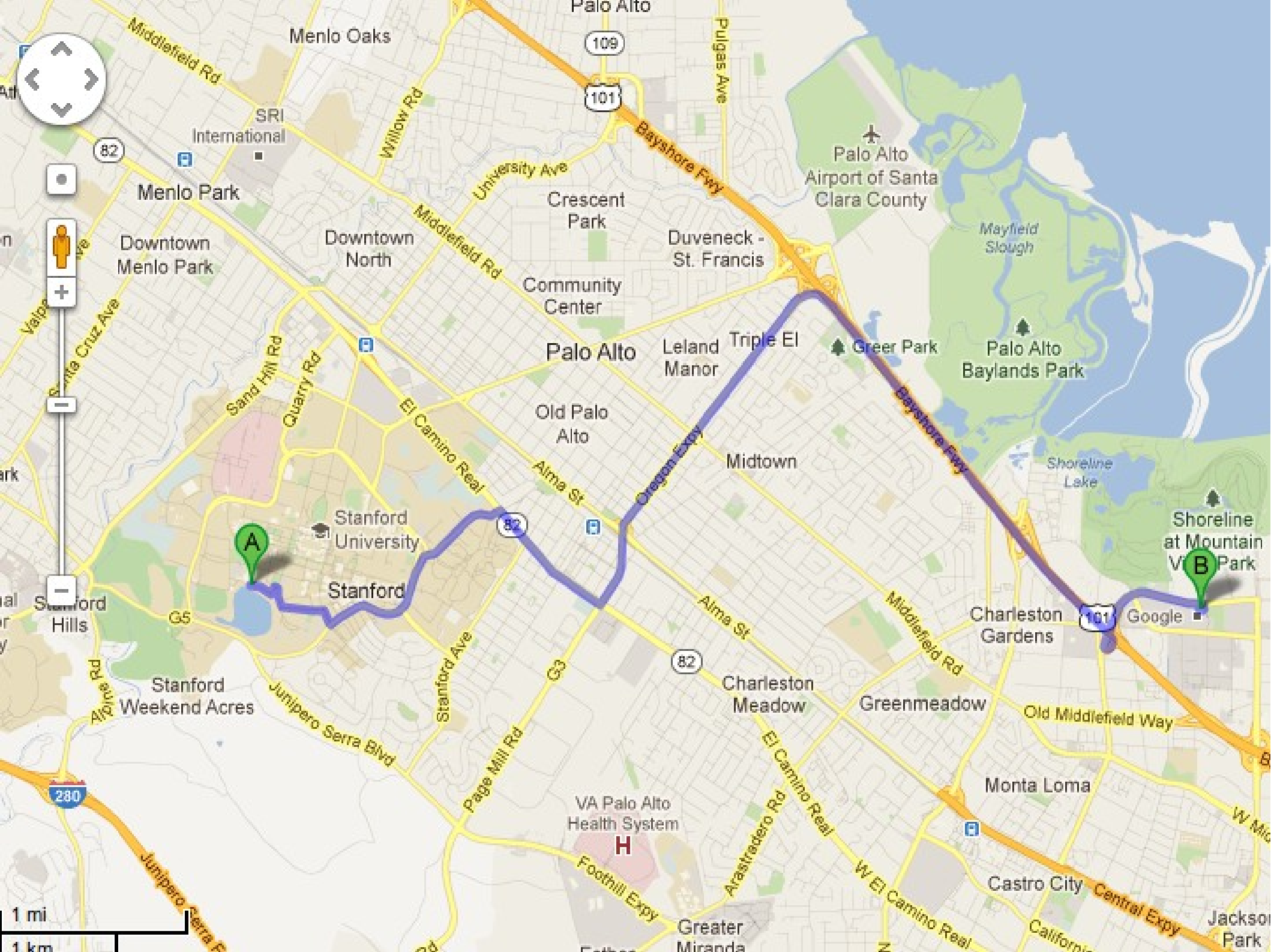
$$\mathrm{d}(s, u) + 1 \leq d(s, x) + 1 \leq |P|$$

contradicting the fact that $|P| < \mathrm{d}(s, u) + 1$. We have reached a contradiction, so our assumption was wrong and no shorter path exists.

Since there is a path of length $\mathrm{d}(s, u) + 1$ from s to v and no shorter path, this means that $\mathrm{d}(s, v) = \mathrm{d}(s, u) + 1$. ■

Why These Two Proofs Matter

- The first proof of correctness (based on layers) is based on our first observation: the nodes visited in BFS radiate outward from the start node in ascending order of distance.
- The second proof of correctness (based on picking the lowest yellow node) is based on our second observation: picking the lowest-cost yellow node correctly computes a shortest path.
- Interestingly, this second correctness proof can be generalized to a larger setting...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Palo Alto Baylands Park
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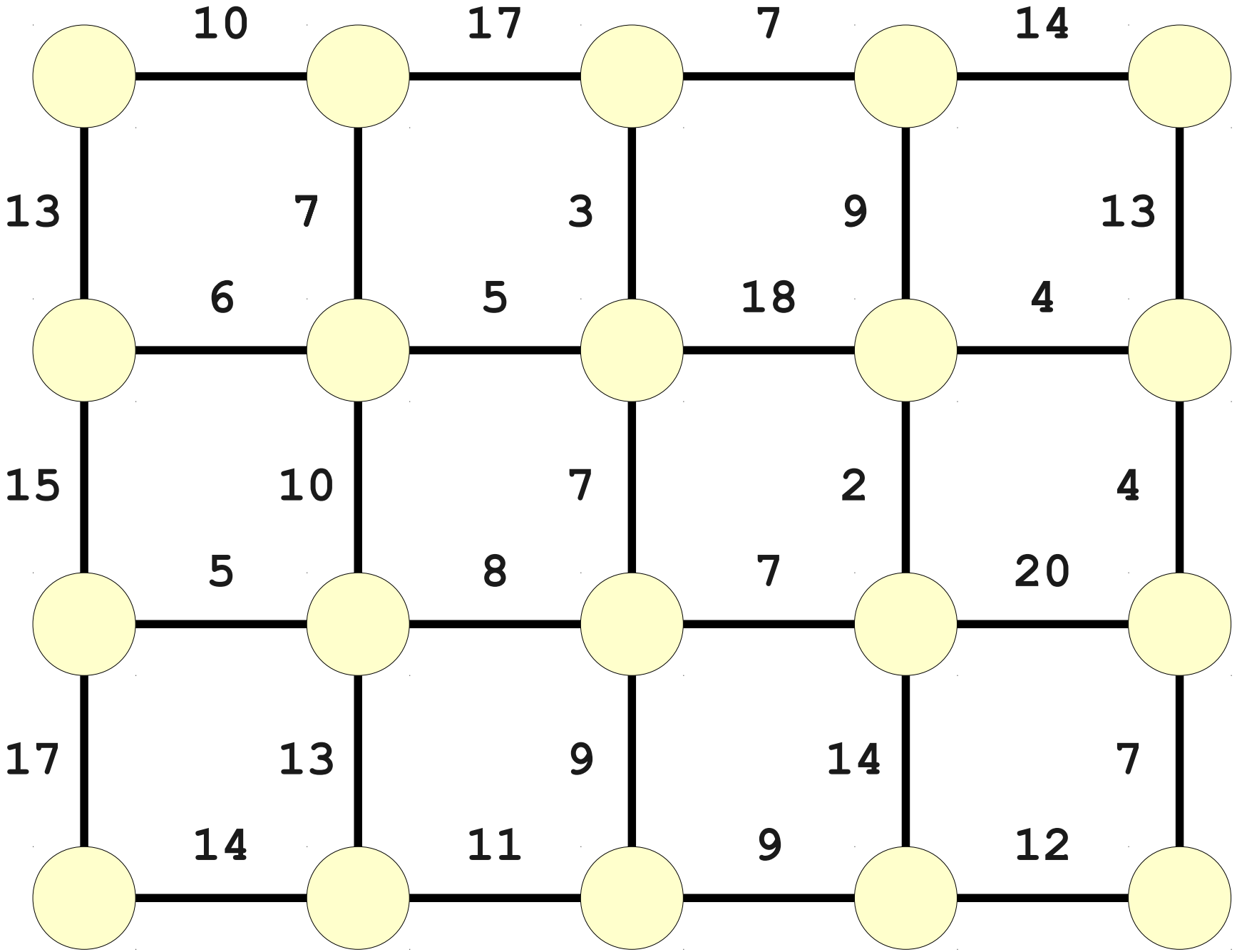
Edges with Costs

- In many applications, edges have an associated length (or cost, weight, etc.), denoted ***l*(*u*, *v*)**.
- **Assumption**: Lengths are nonnegative. (We'll revisit this later in the quarter.)
- Let's say that the length of a path P (denoted ***l*(*P*)**) is the sum of all the edge lengths in the path P.
- Goal: find the shortest path from s to every node in V, taking costs into account.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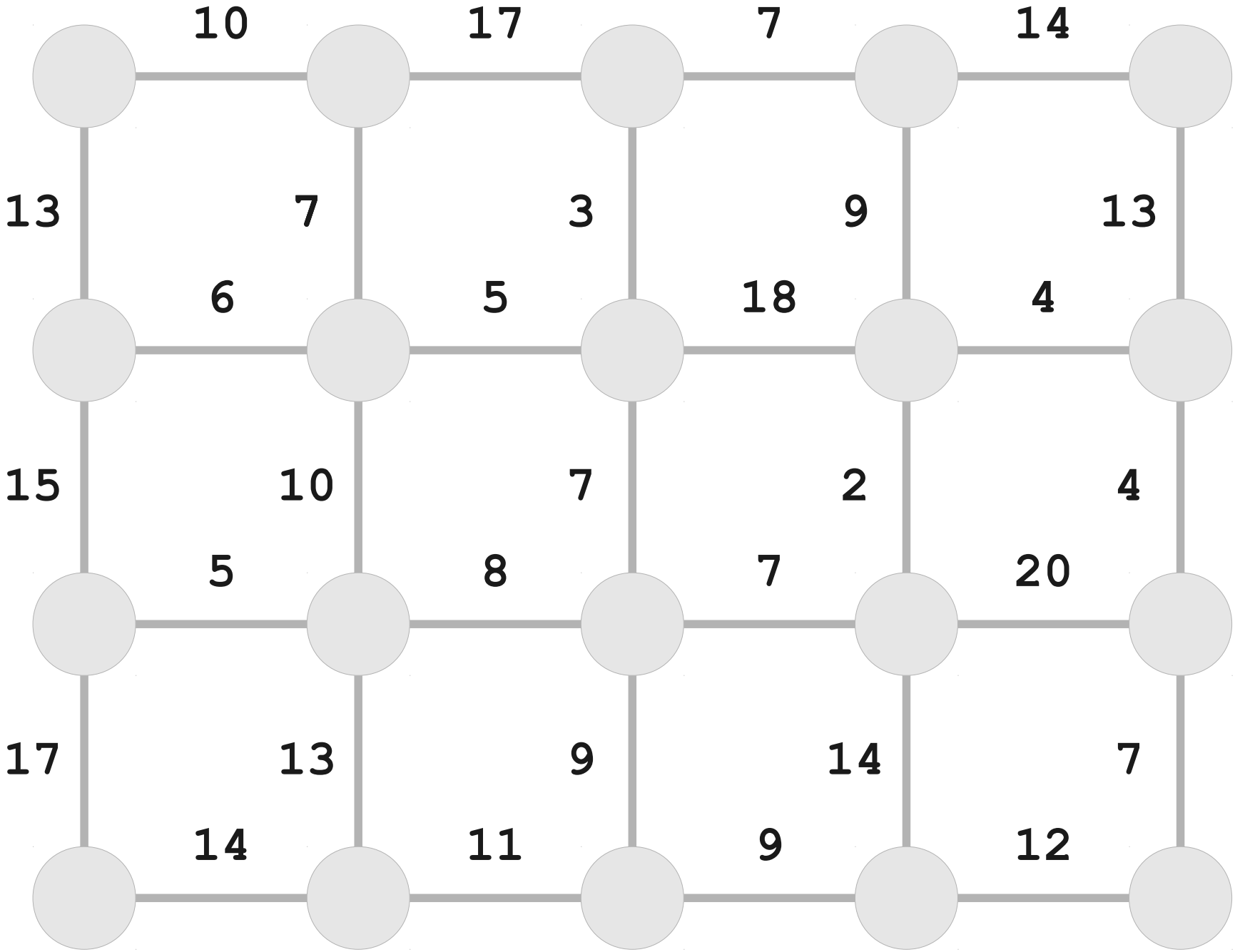
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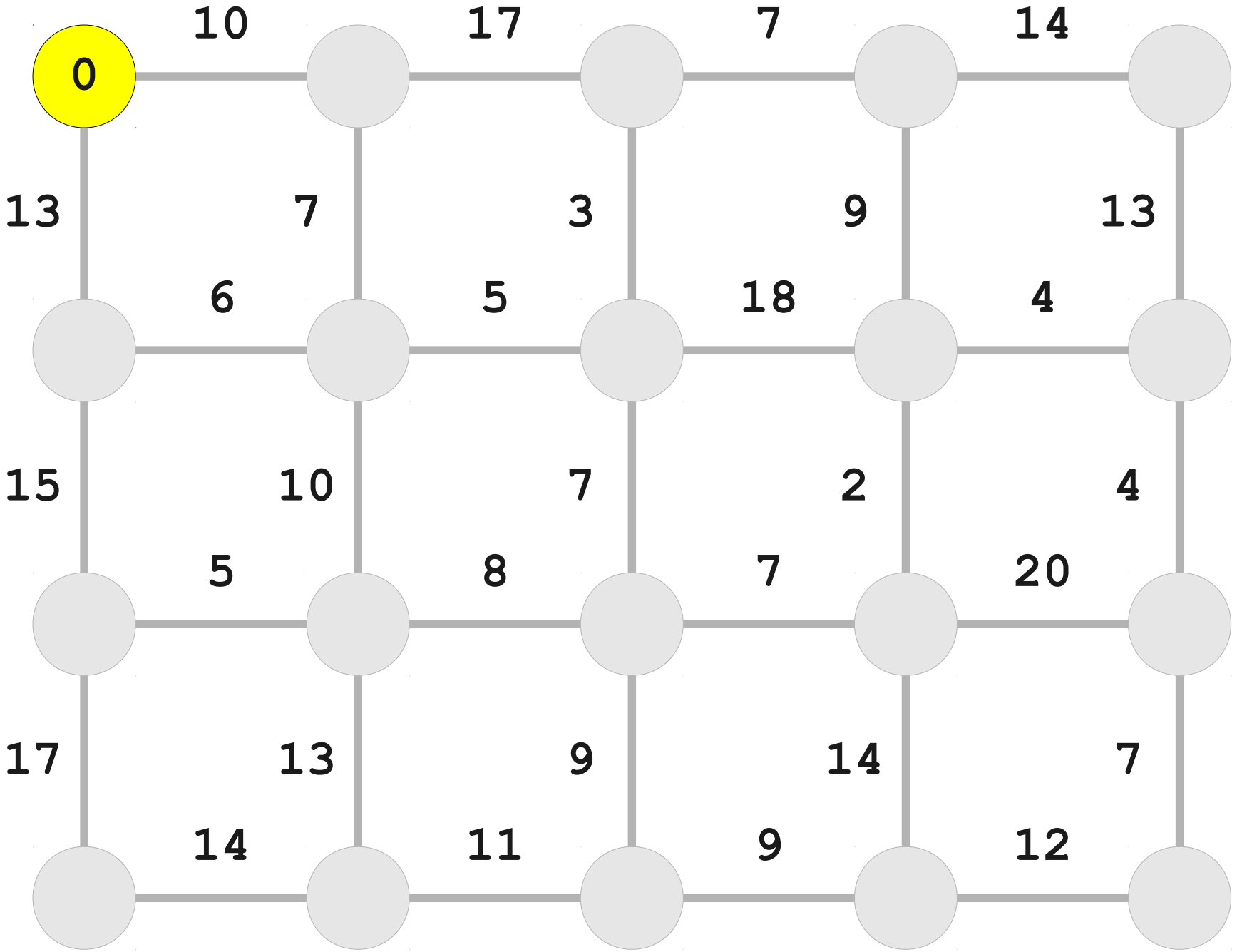
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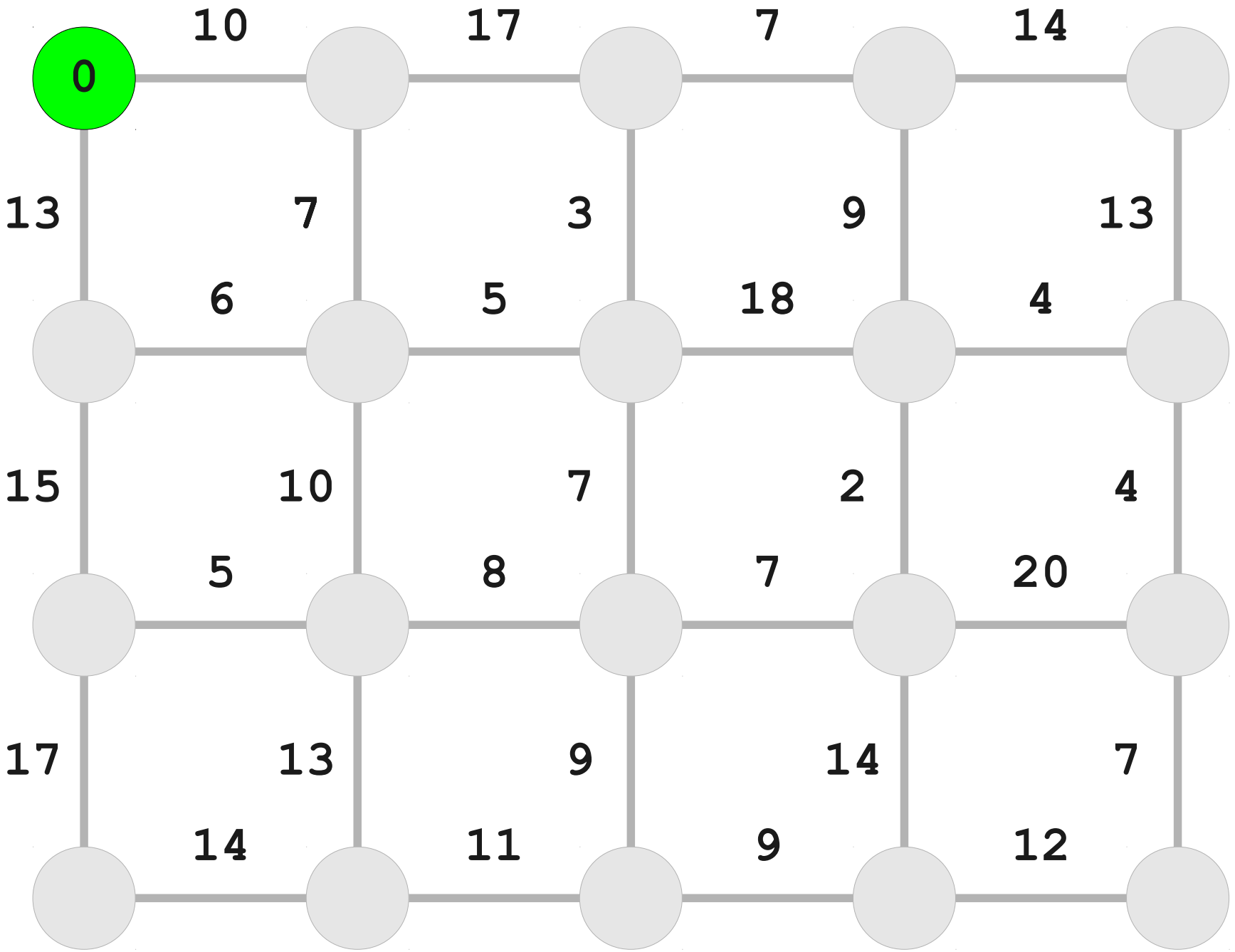
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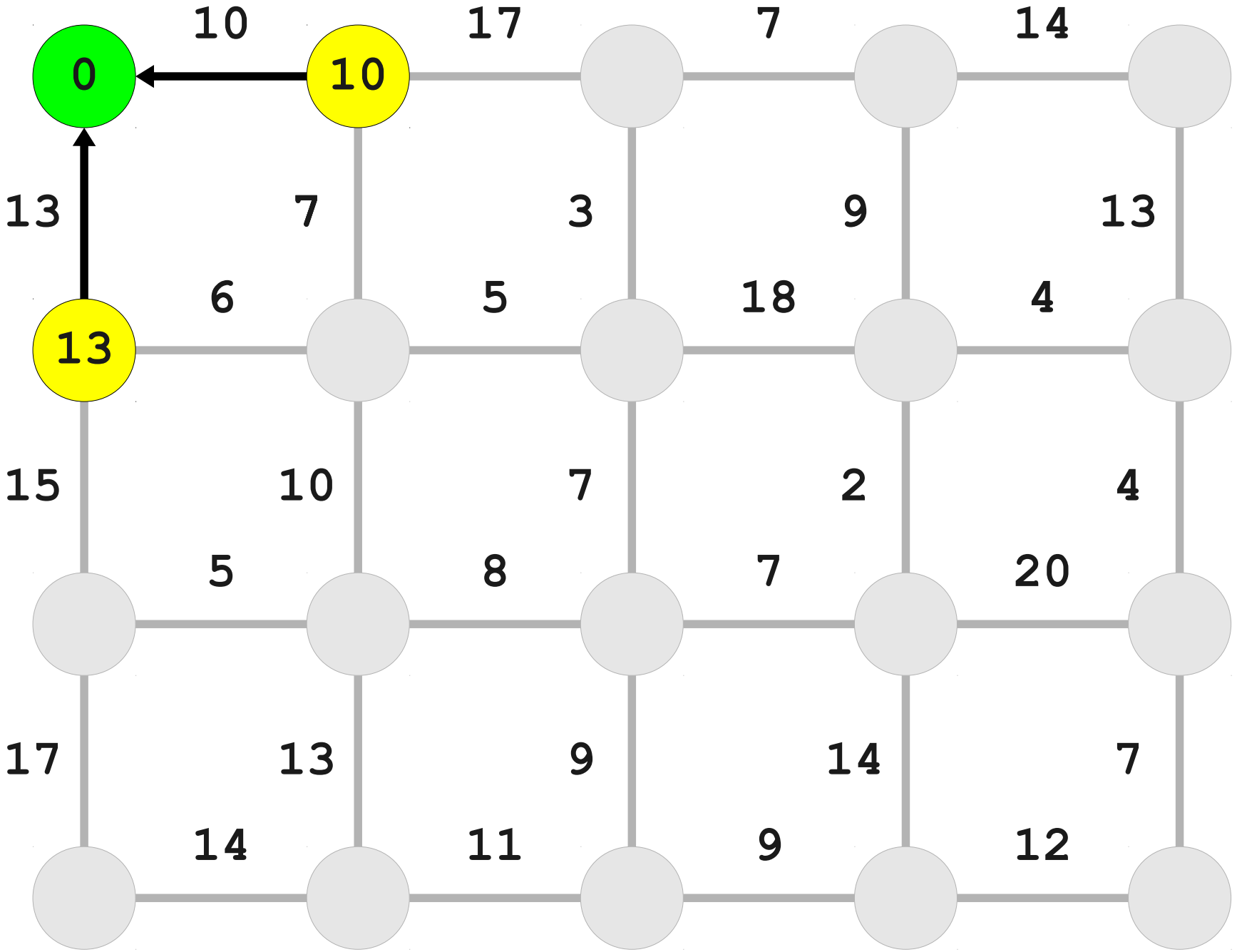
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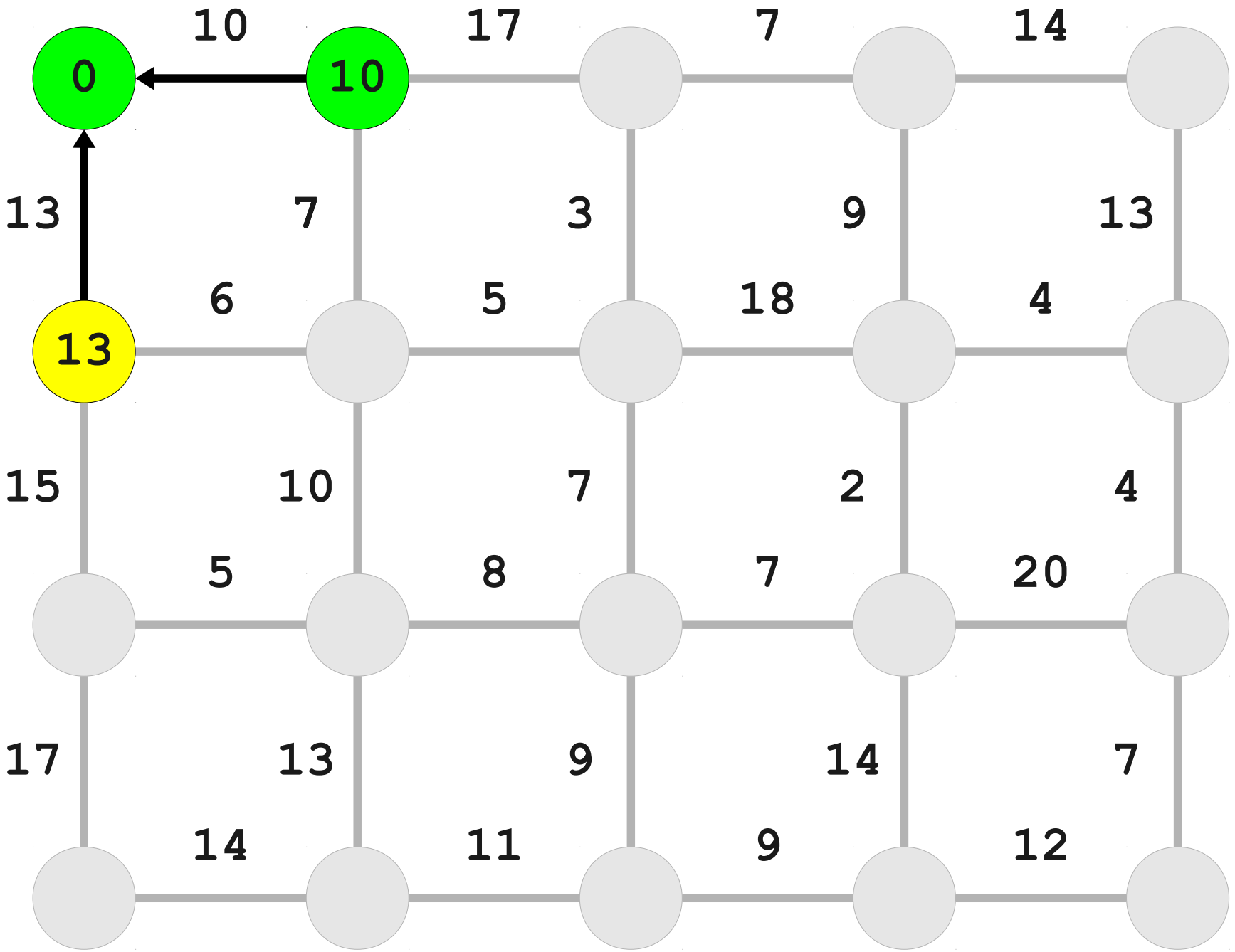
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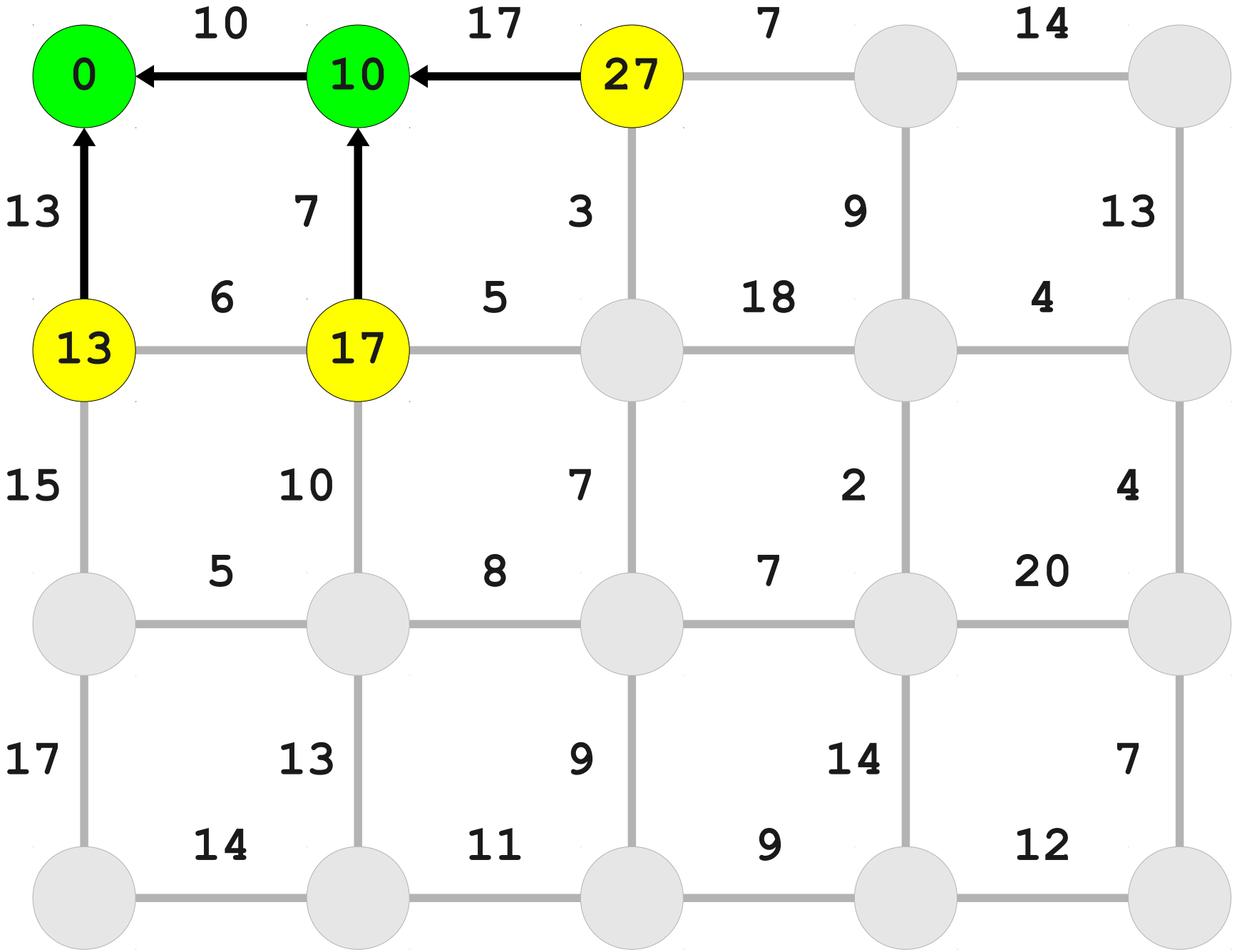
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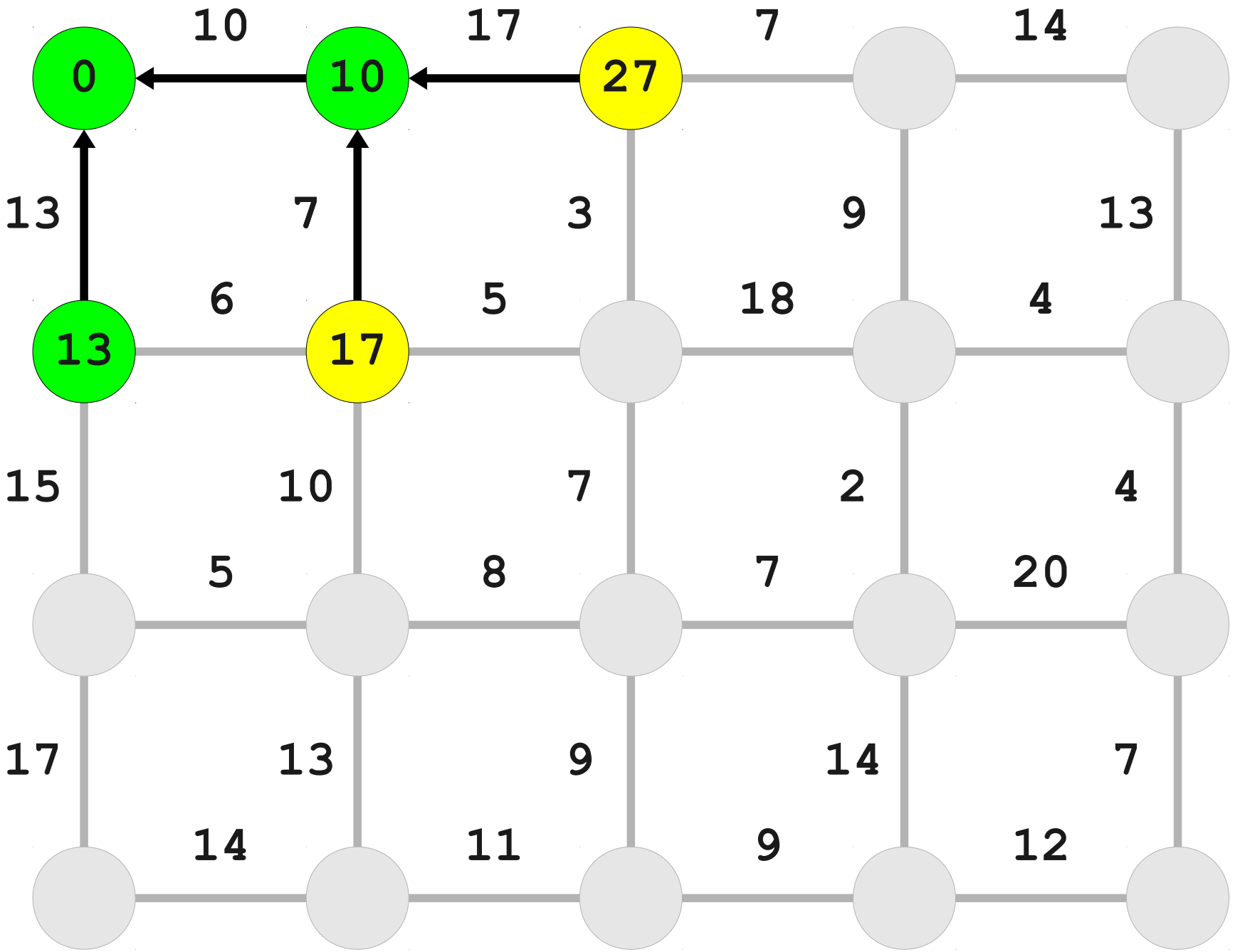
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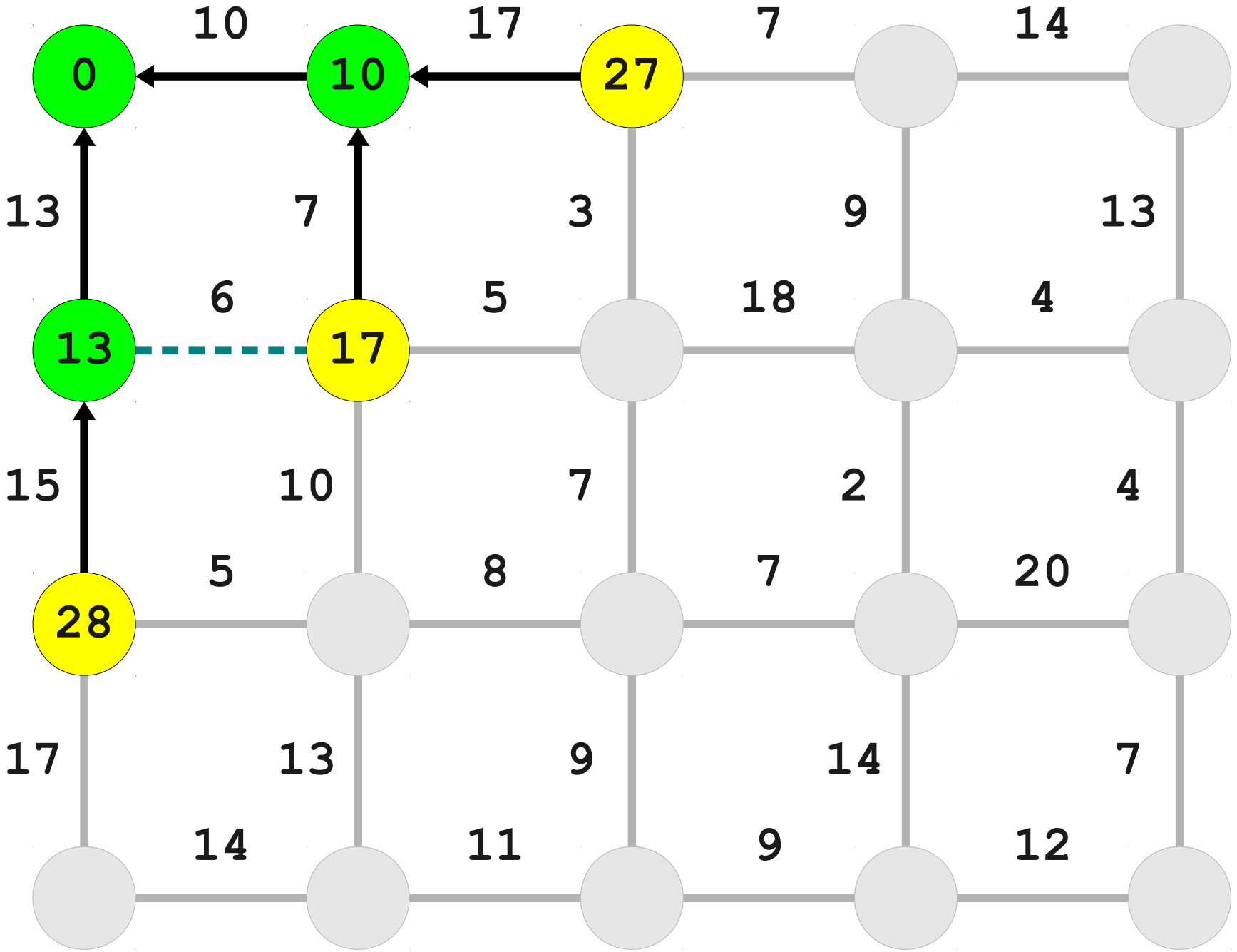
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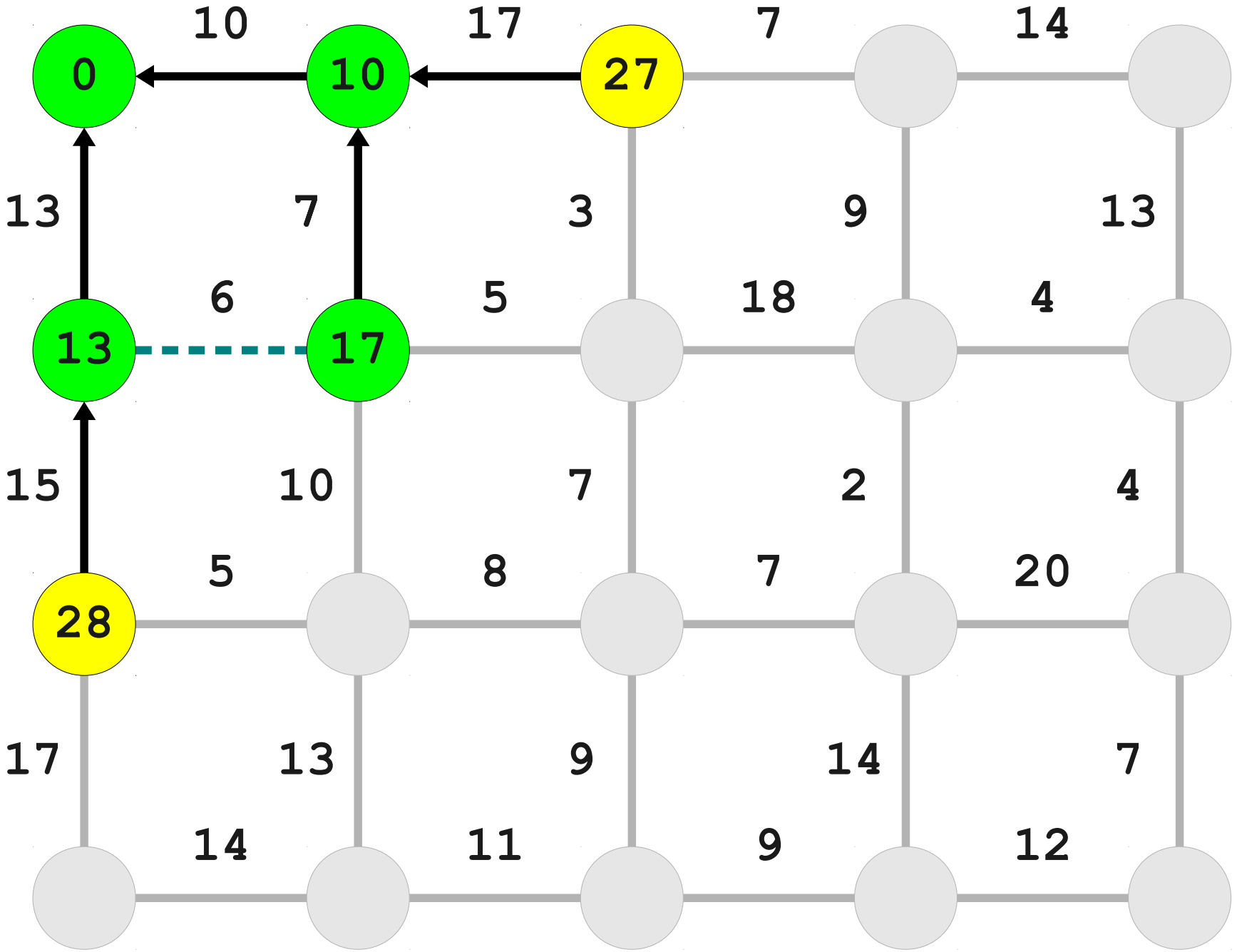
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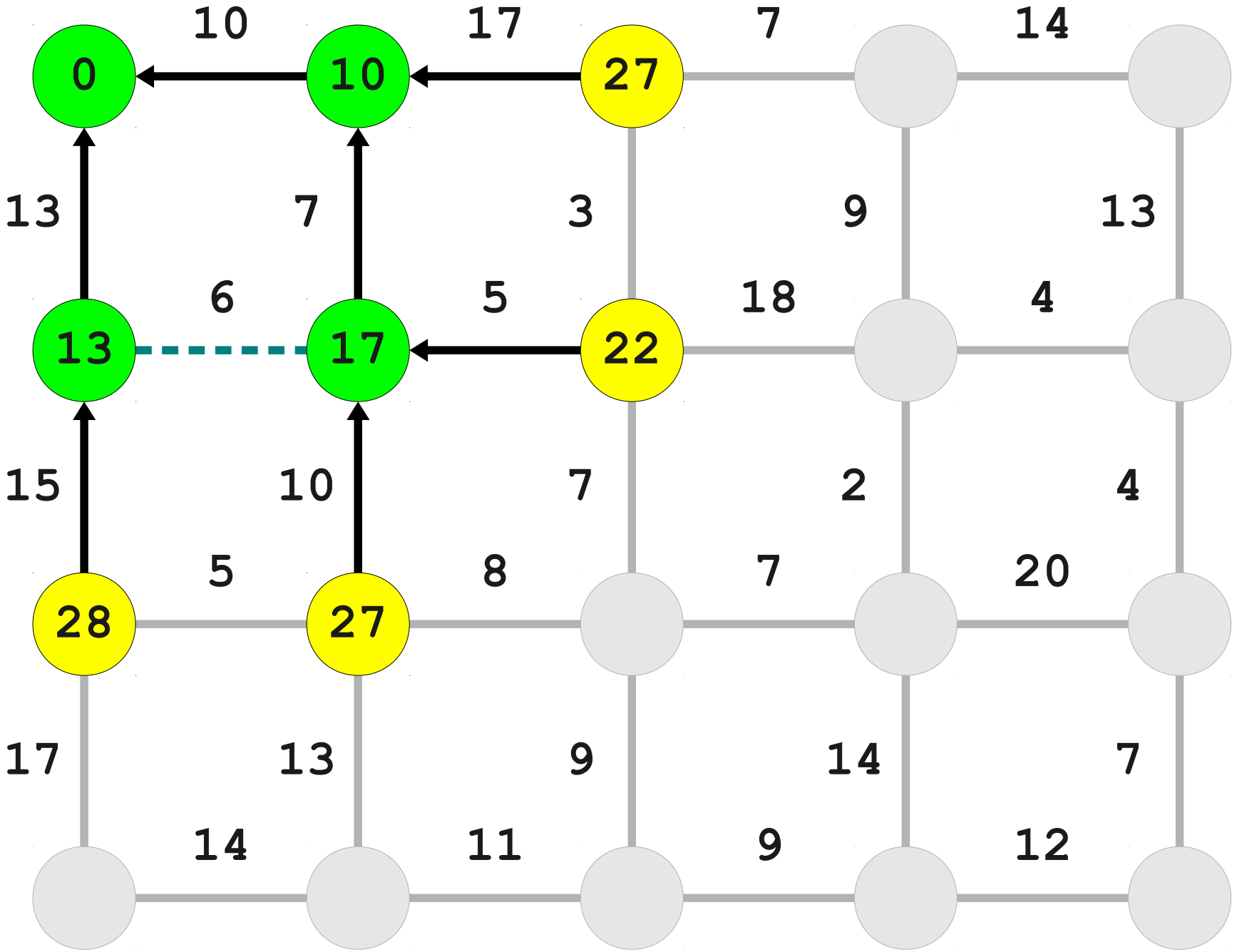
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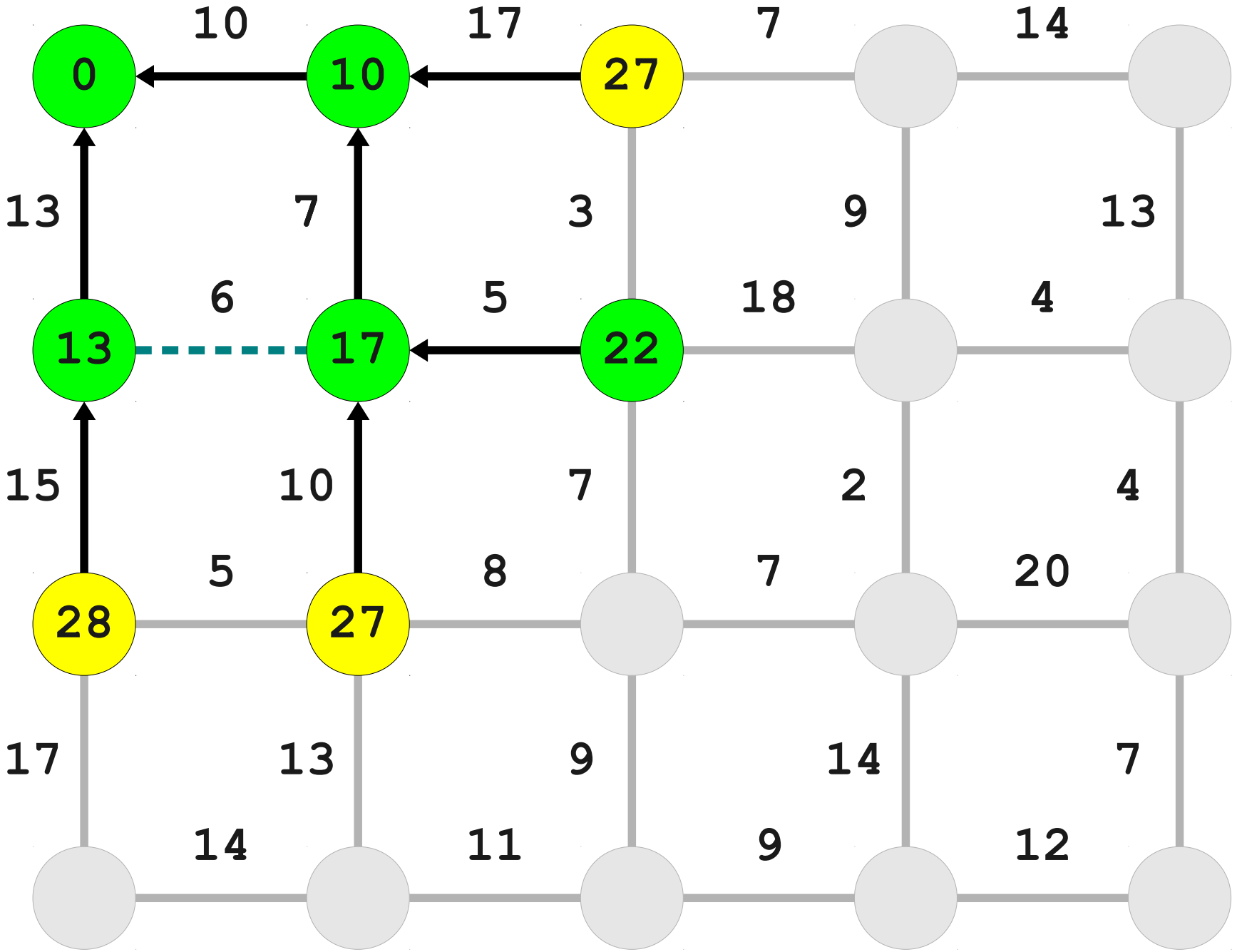
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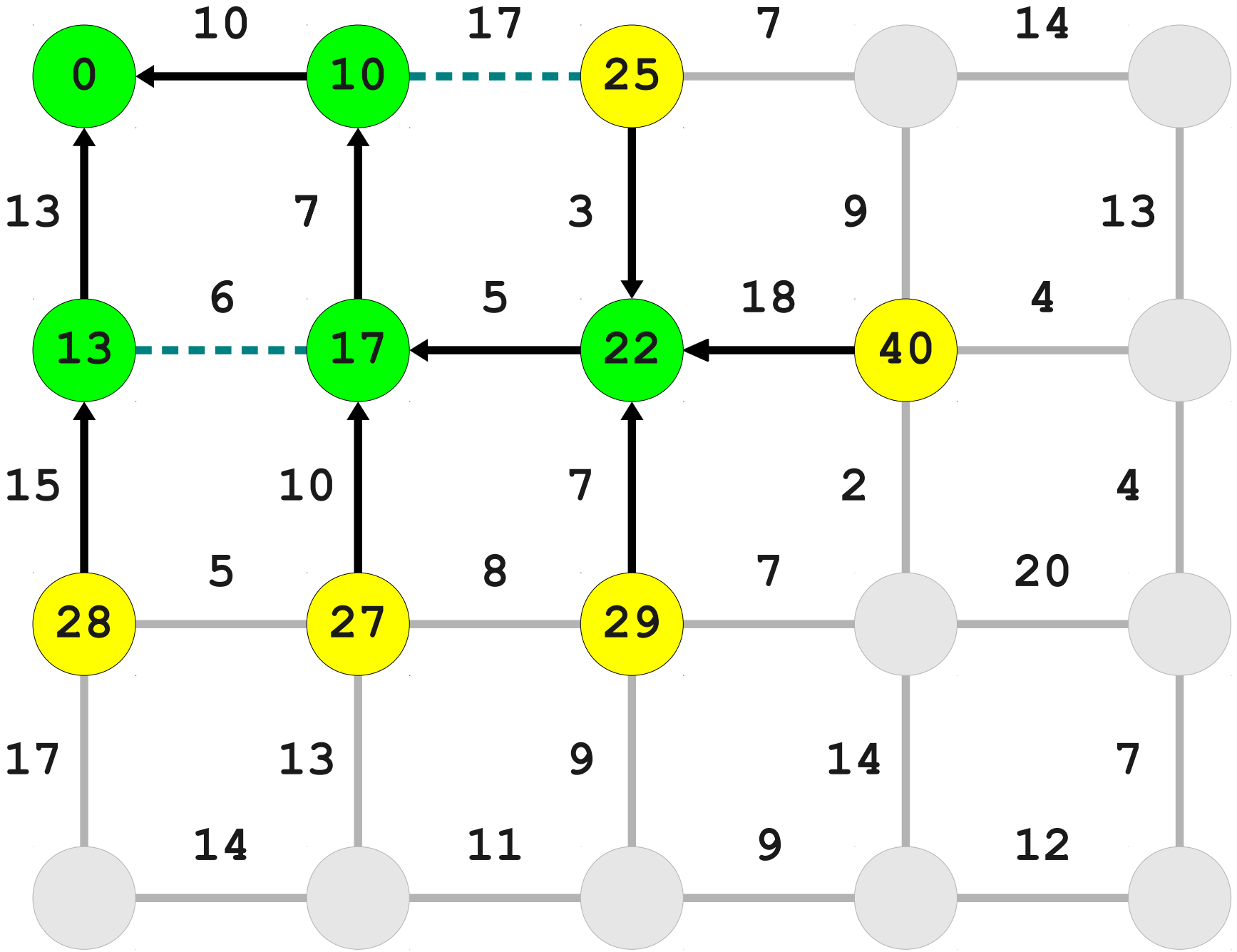
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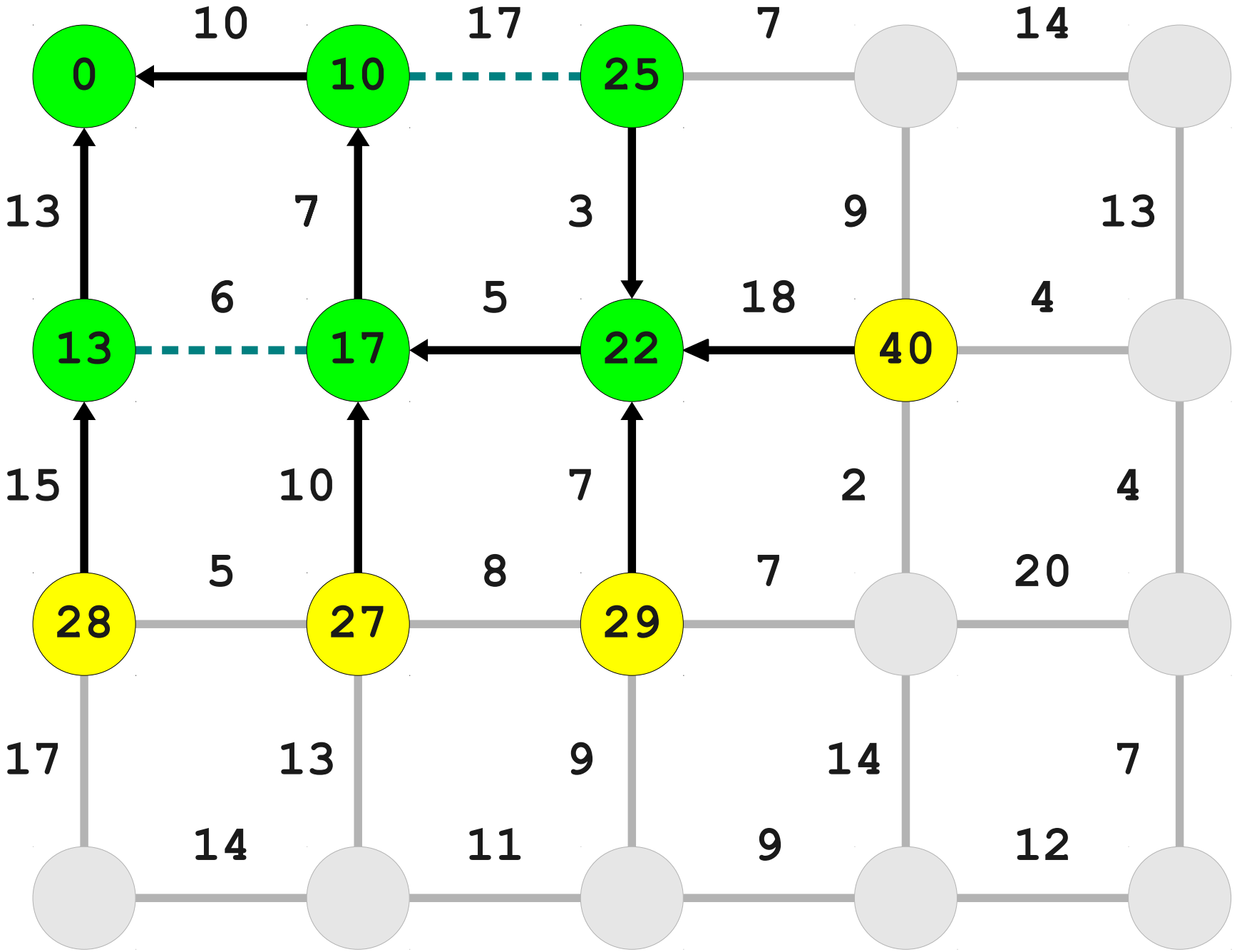
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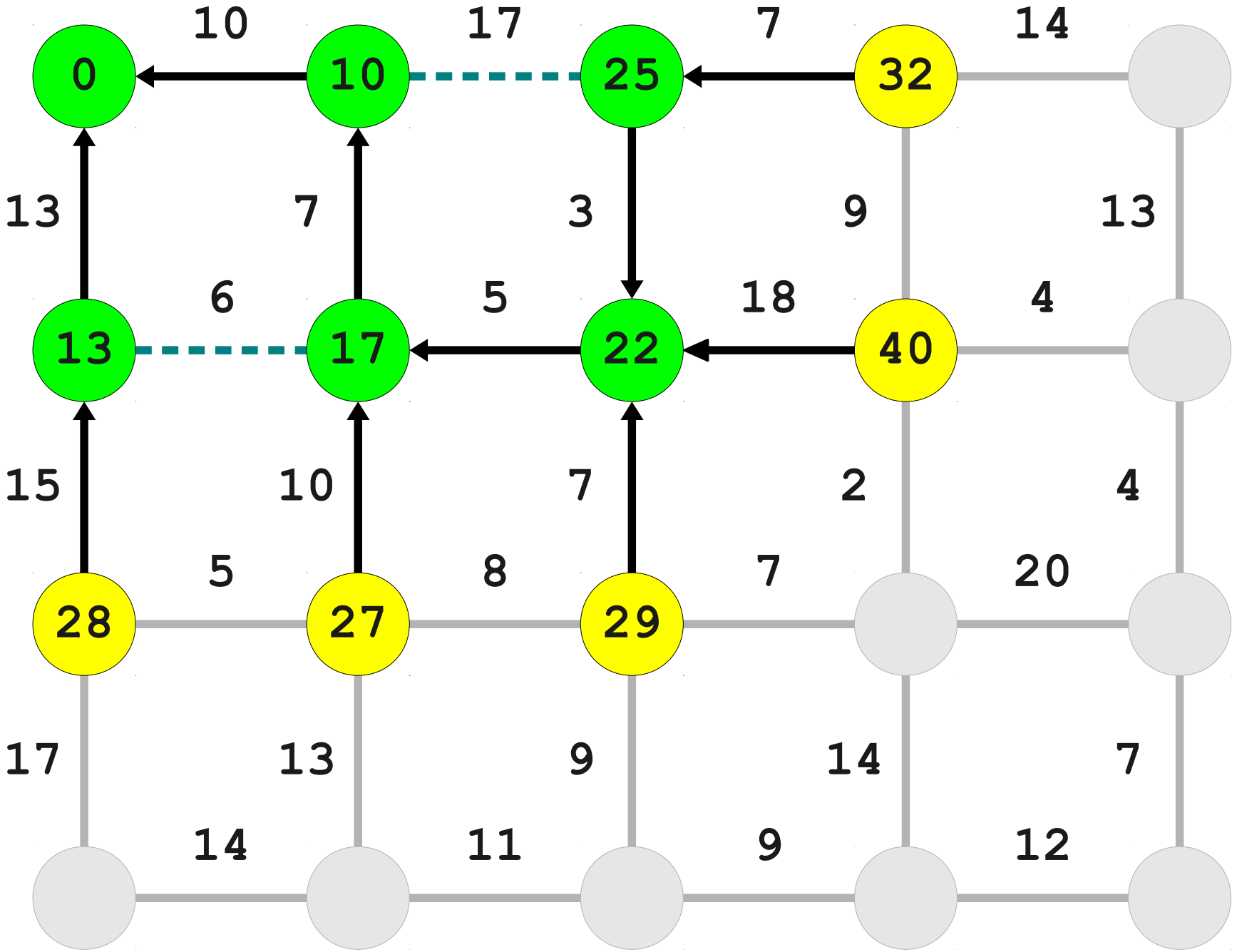
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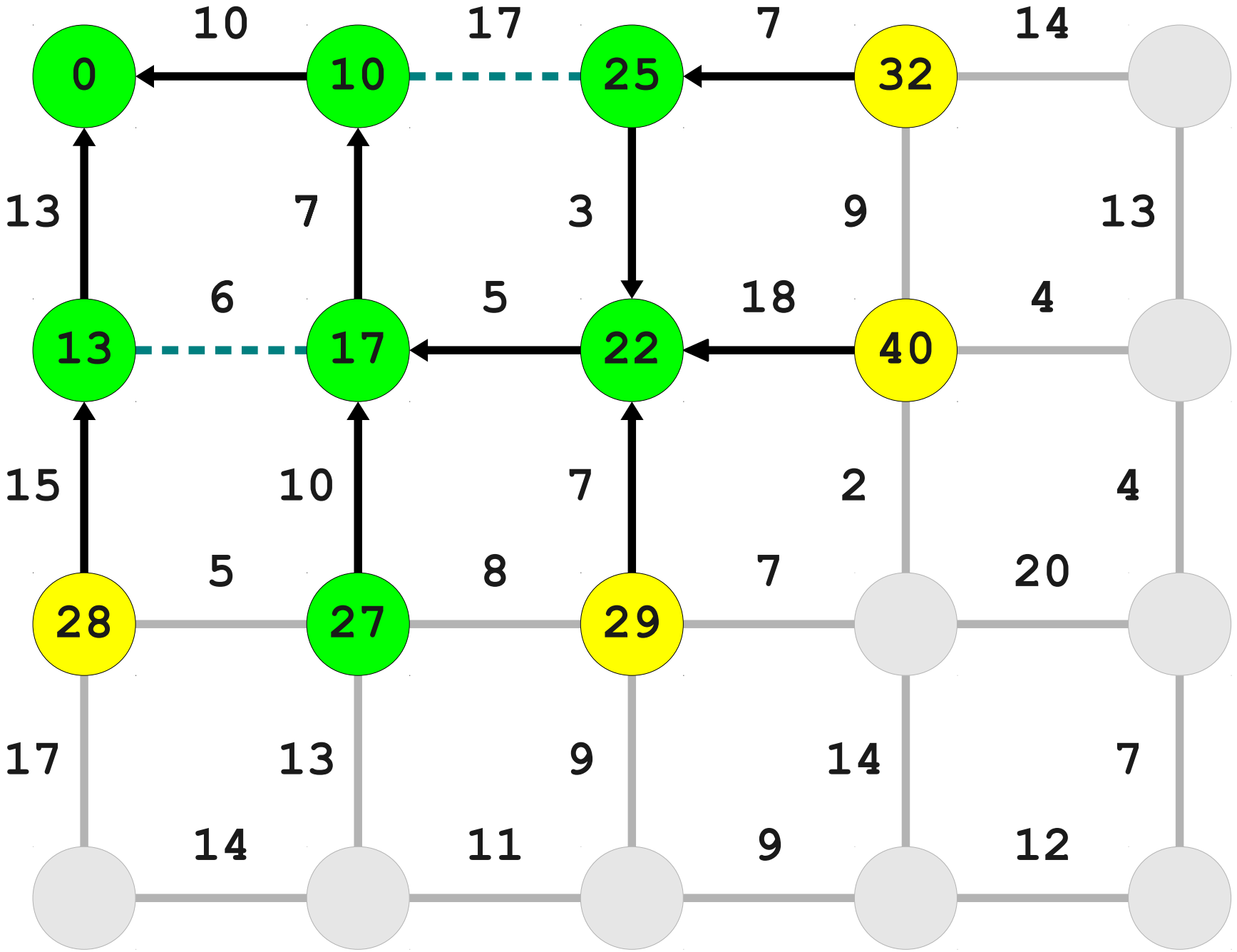
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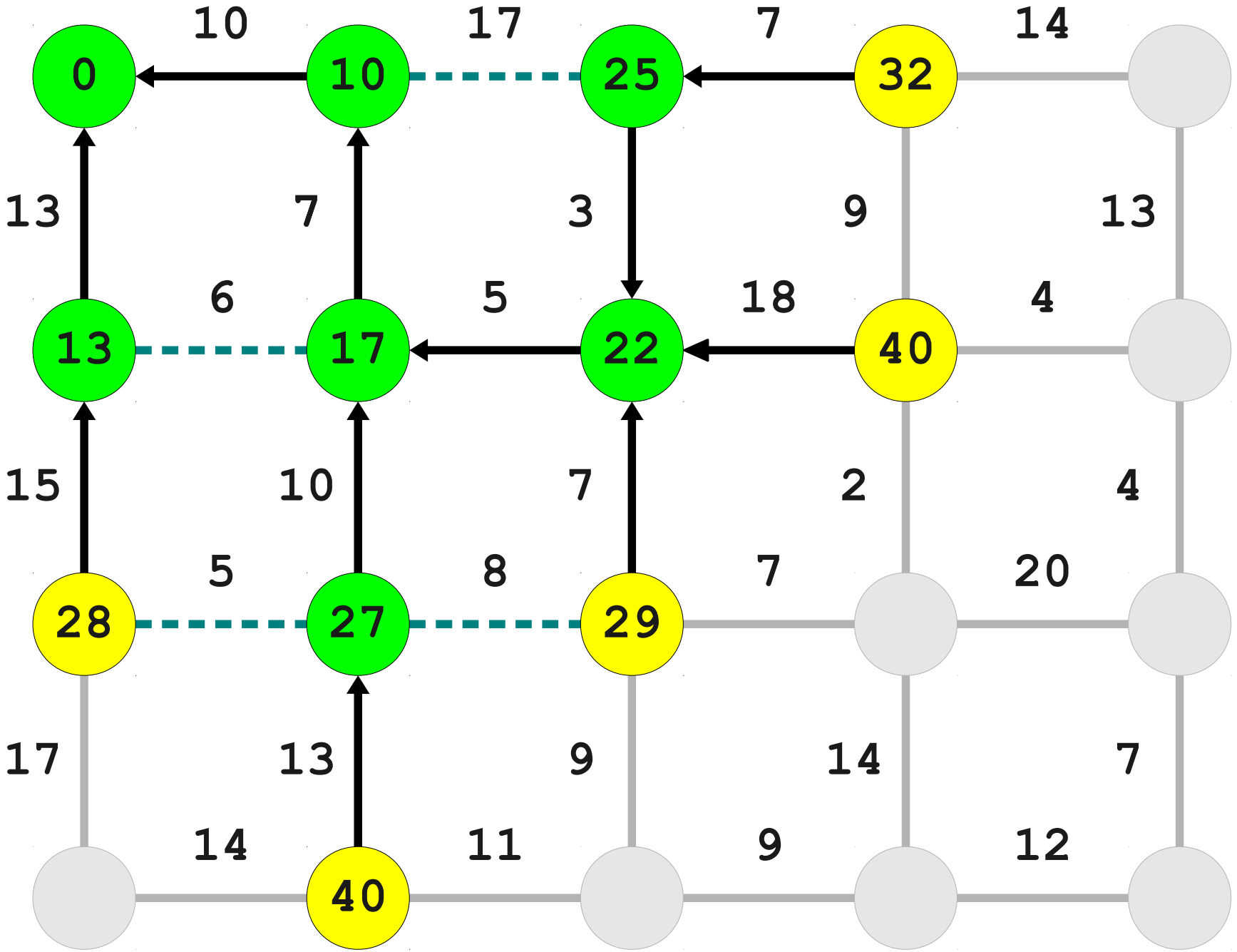
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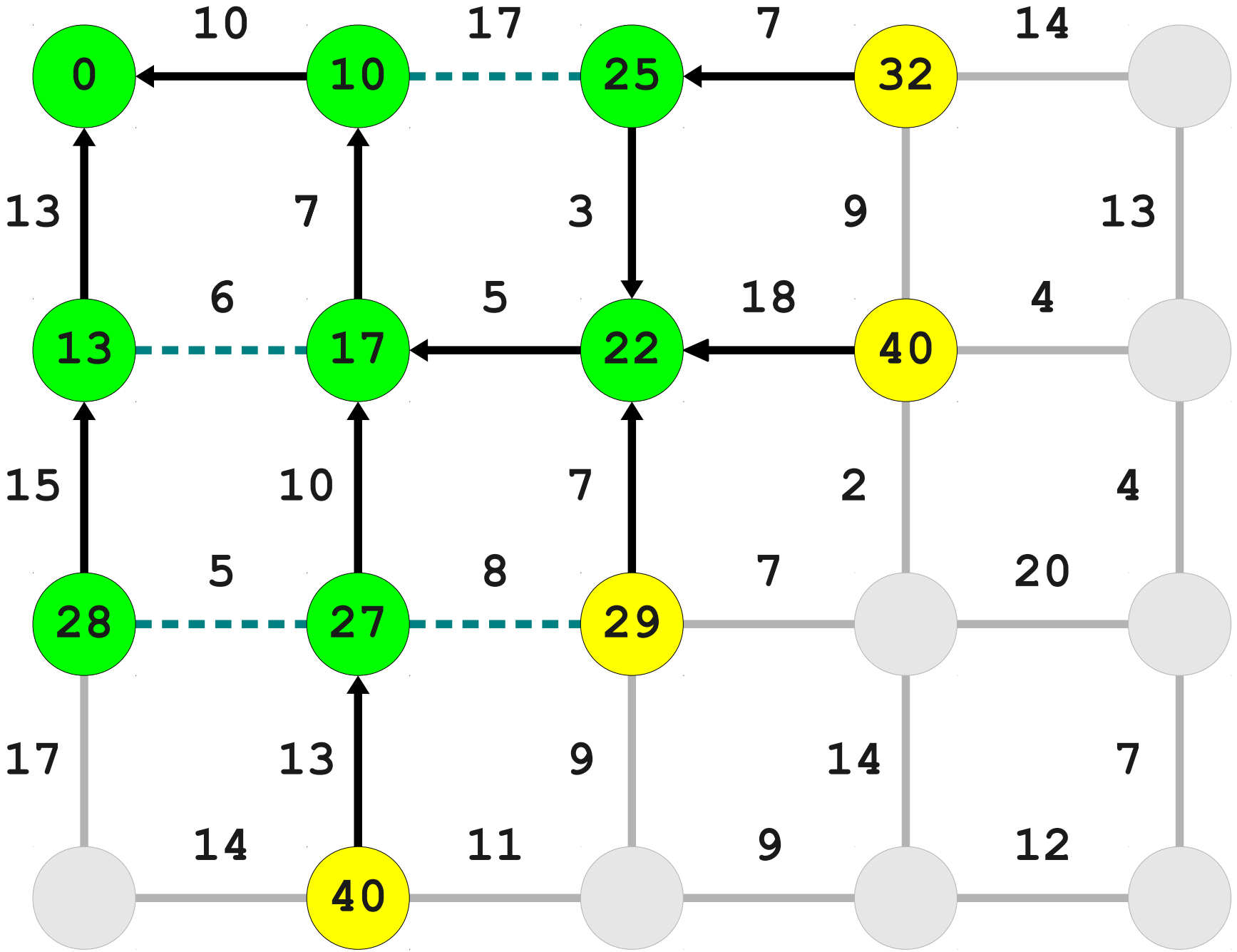
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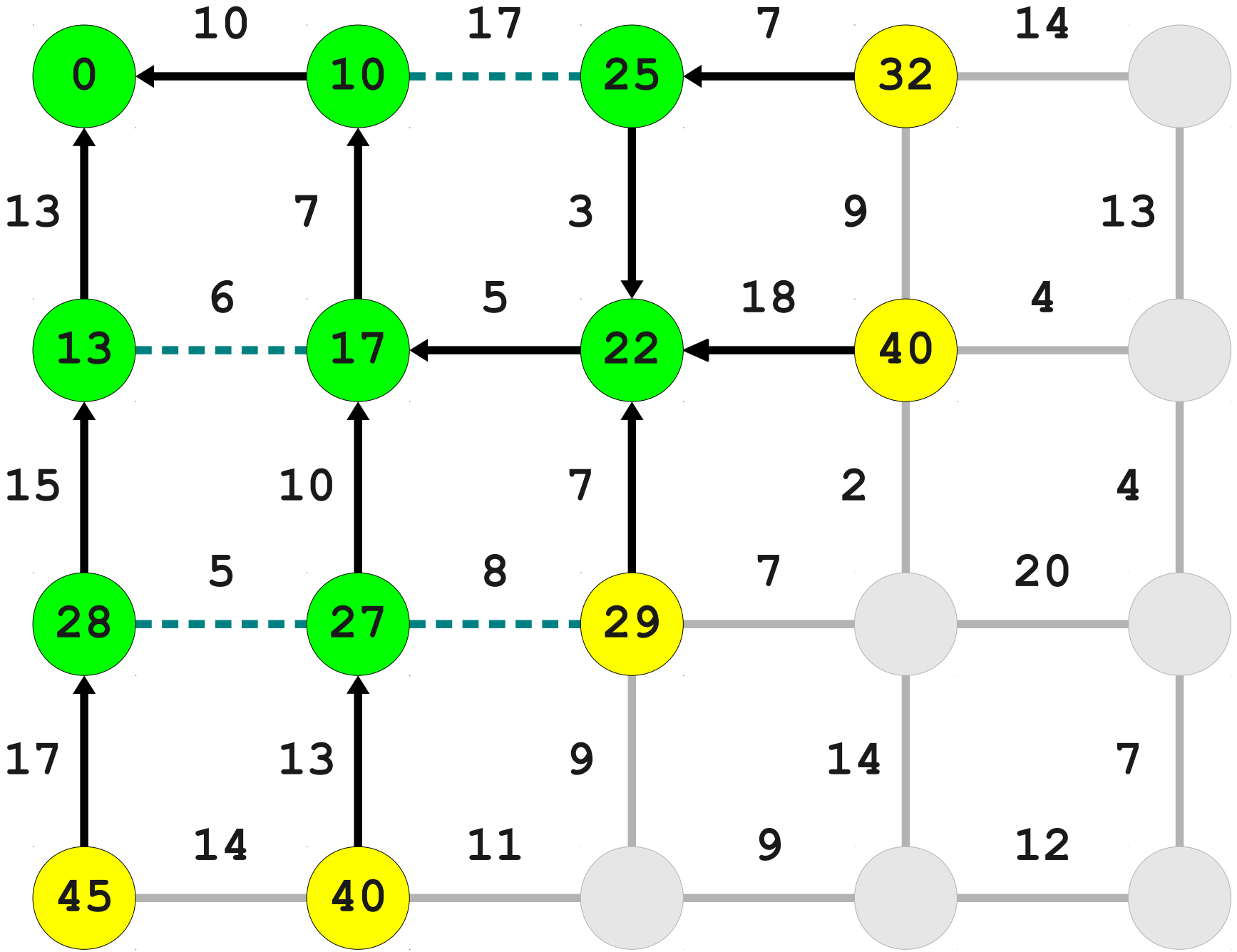
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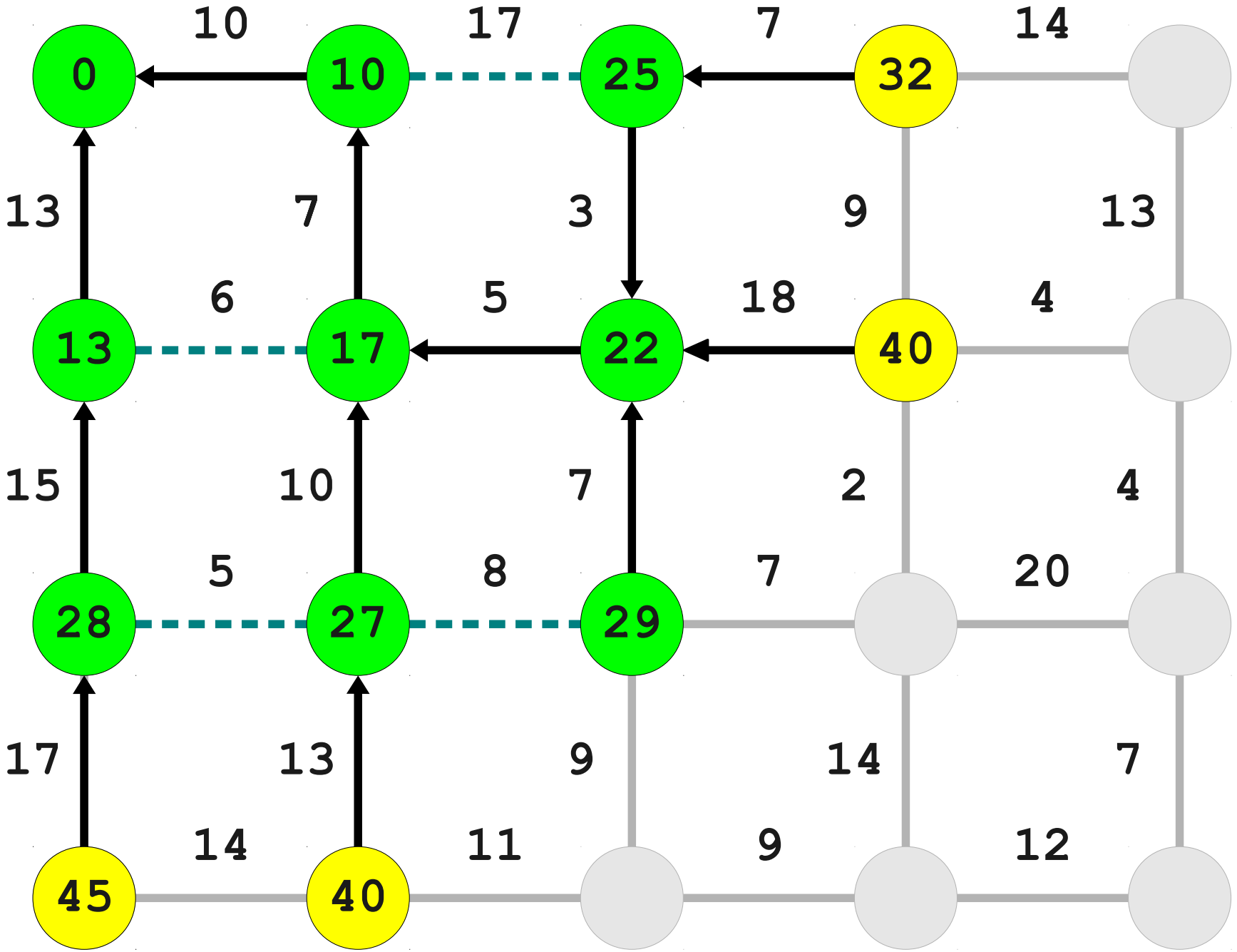
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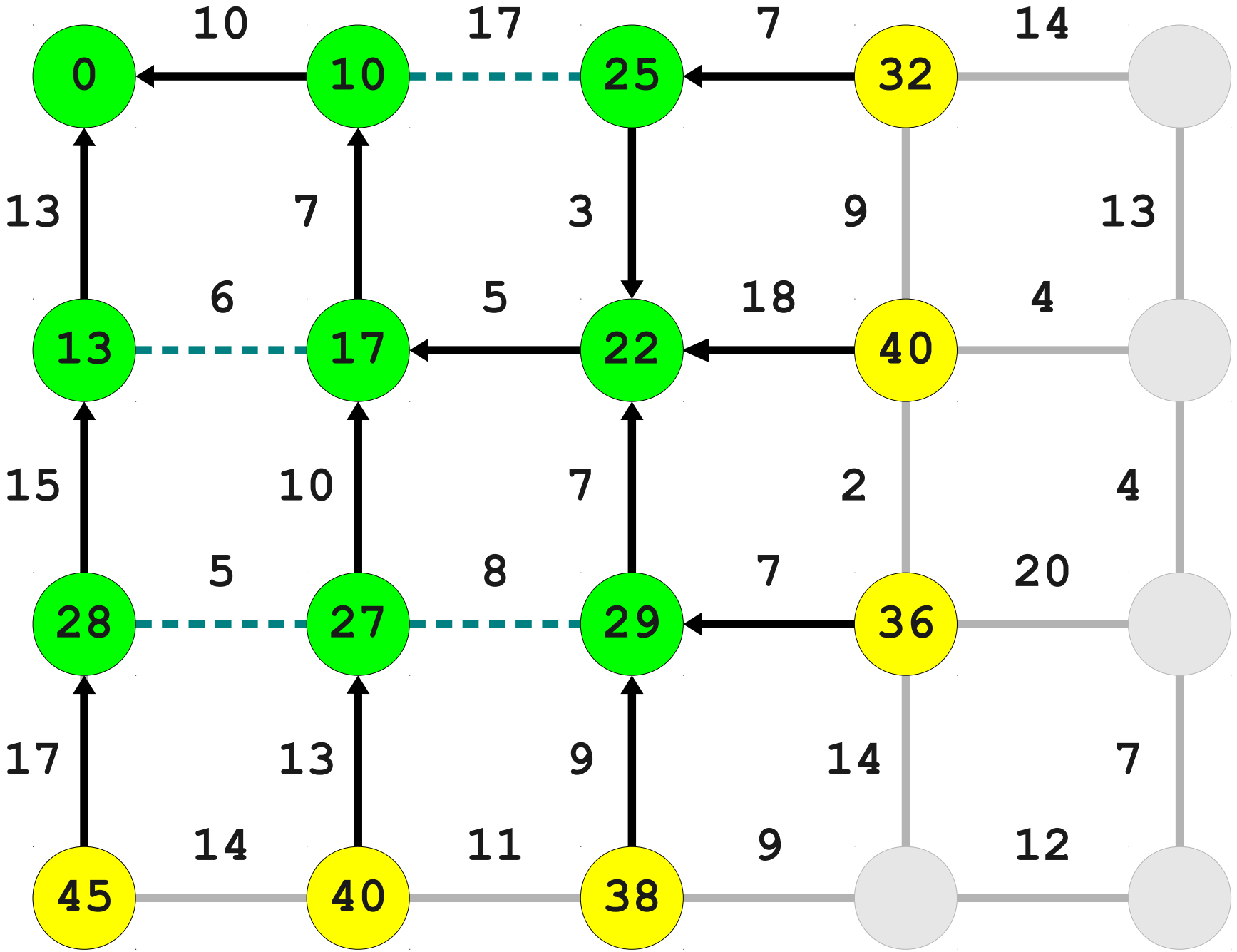
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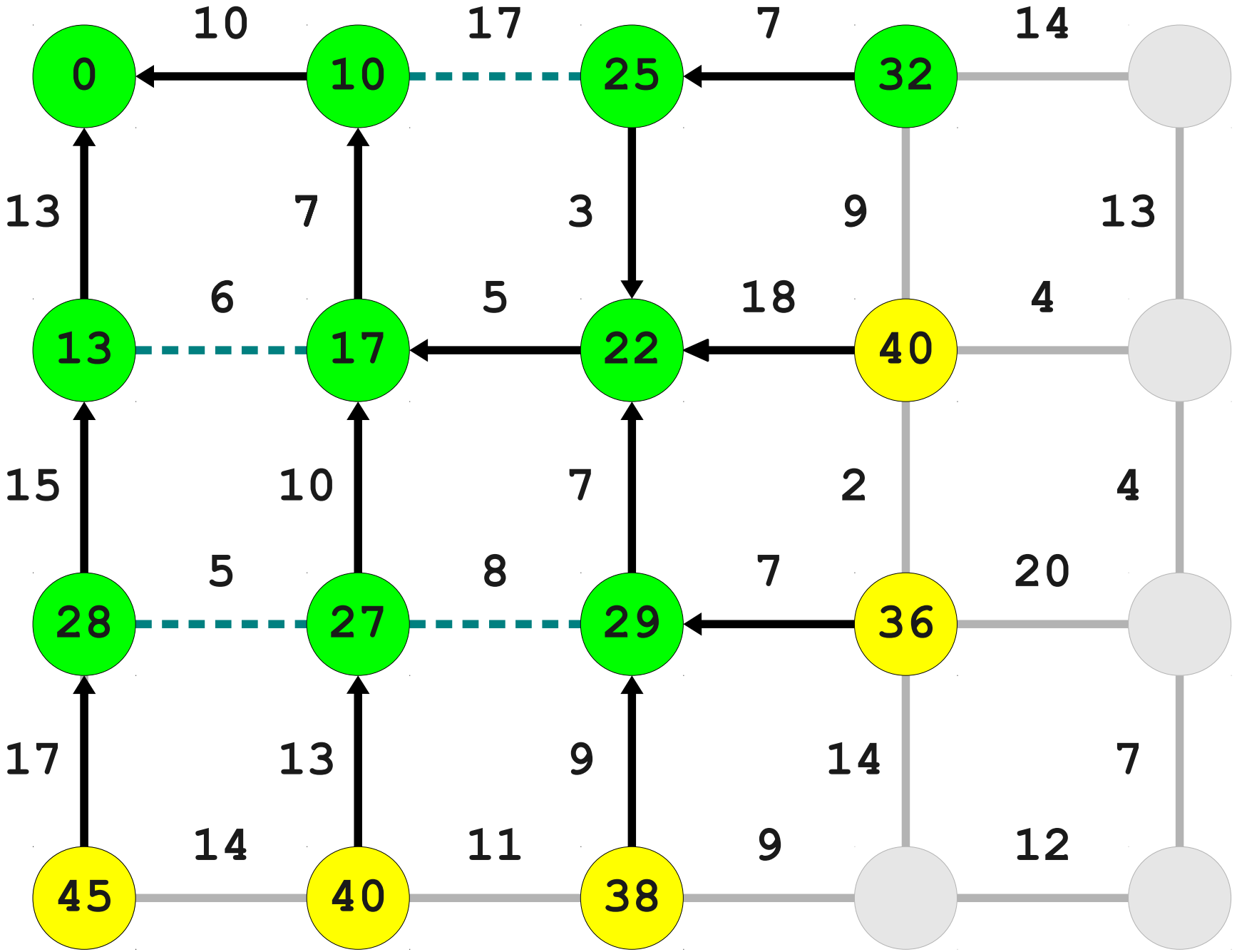
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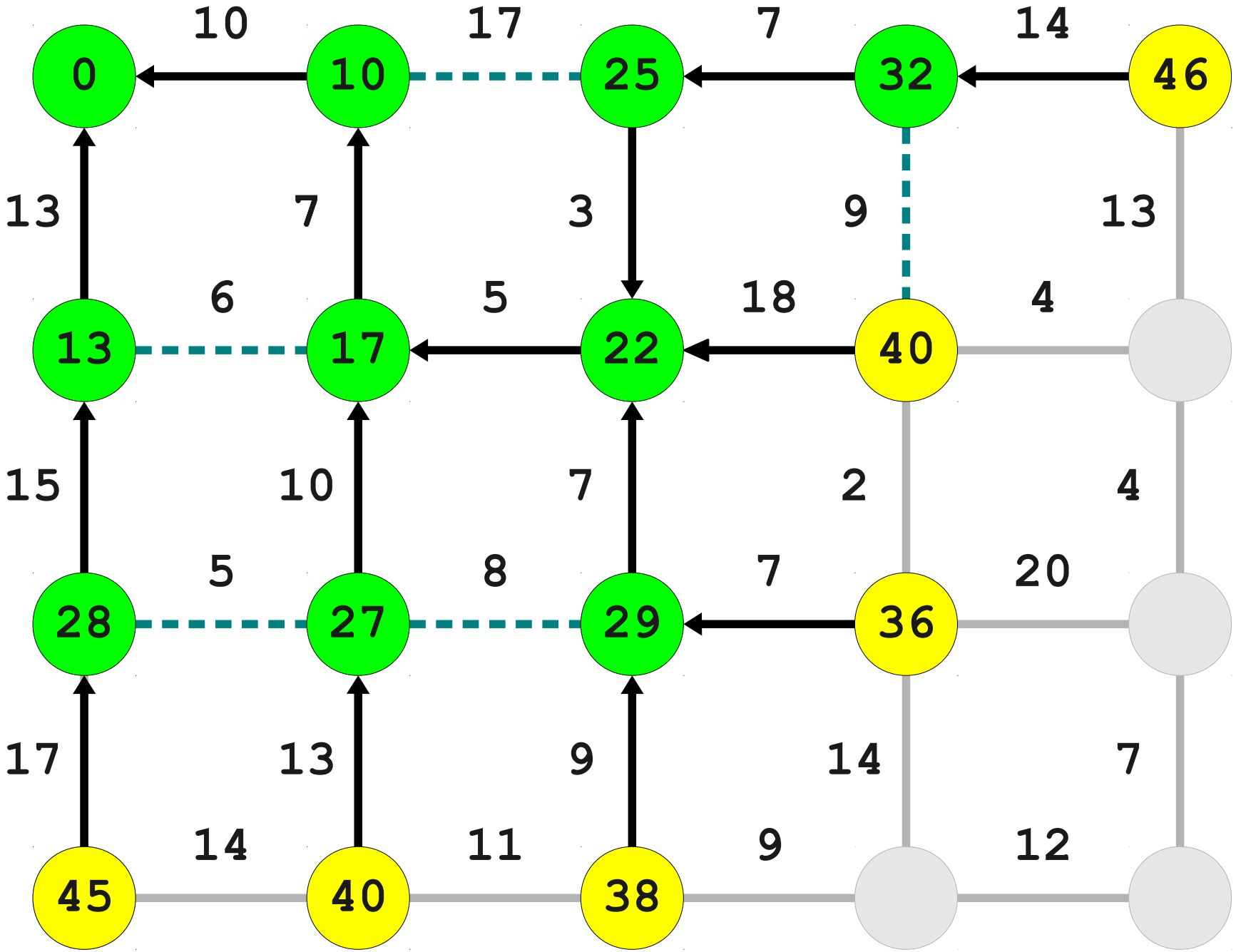
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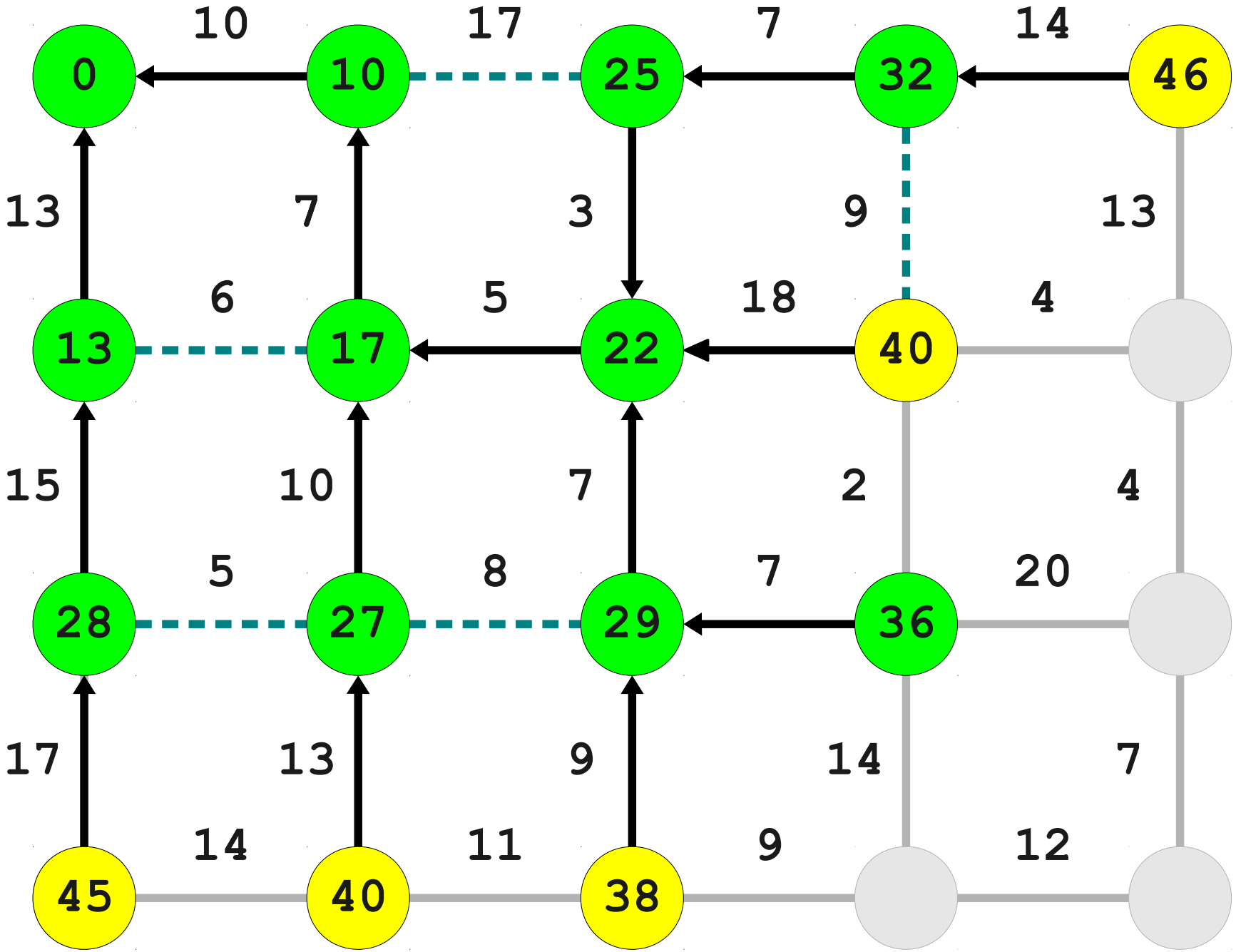
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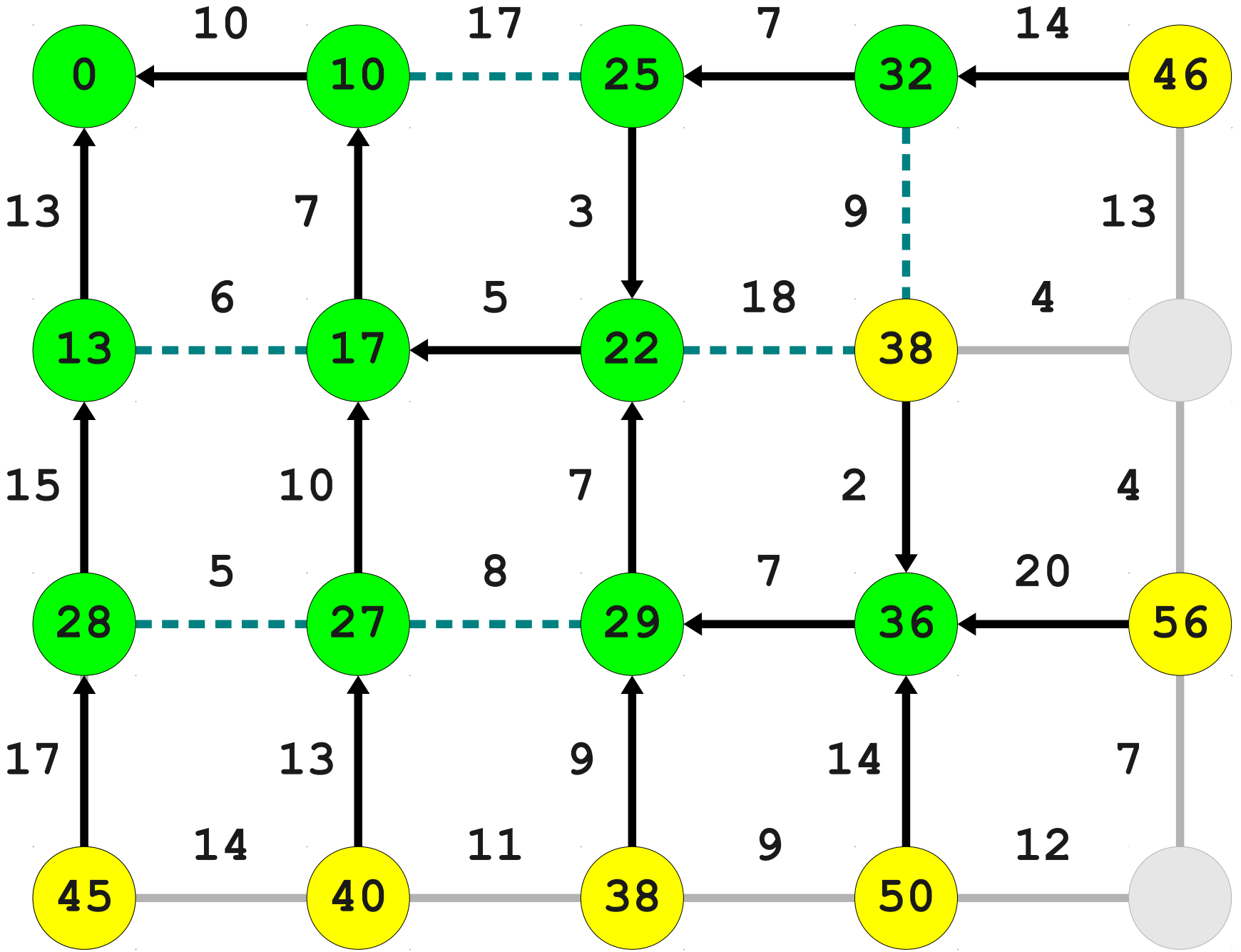
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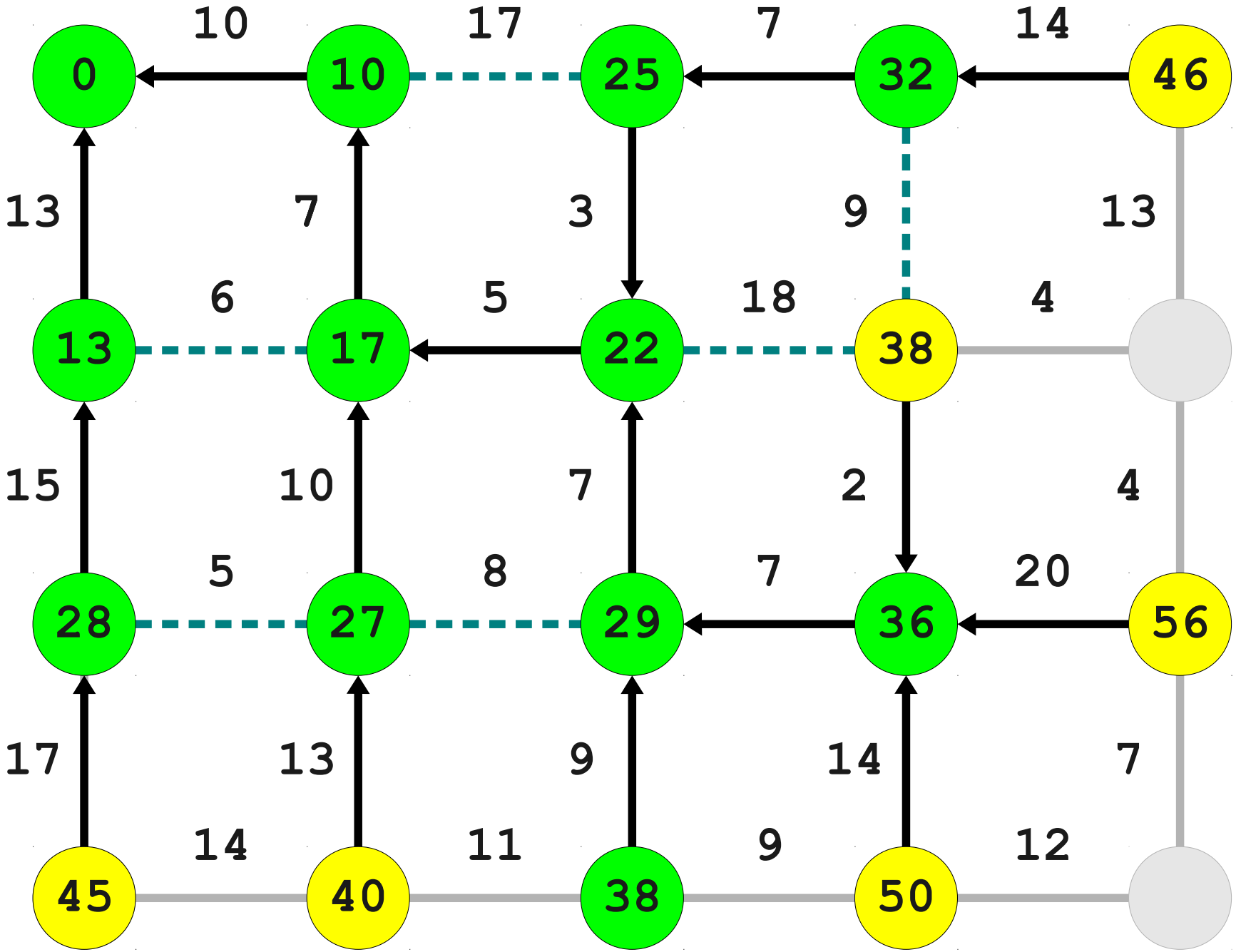
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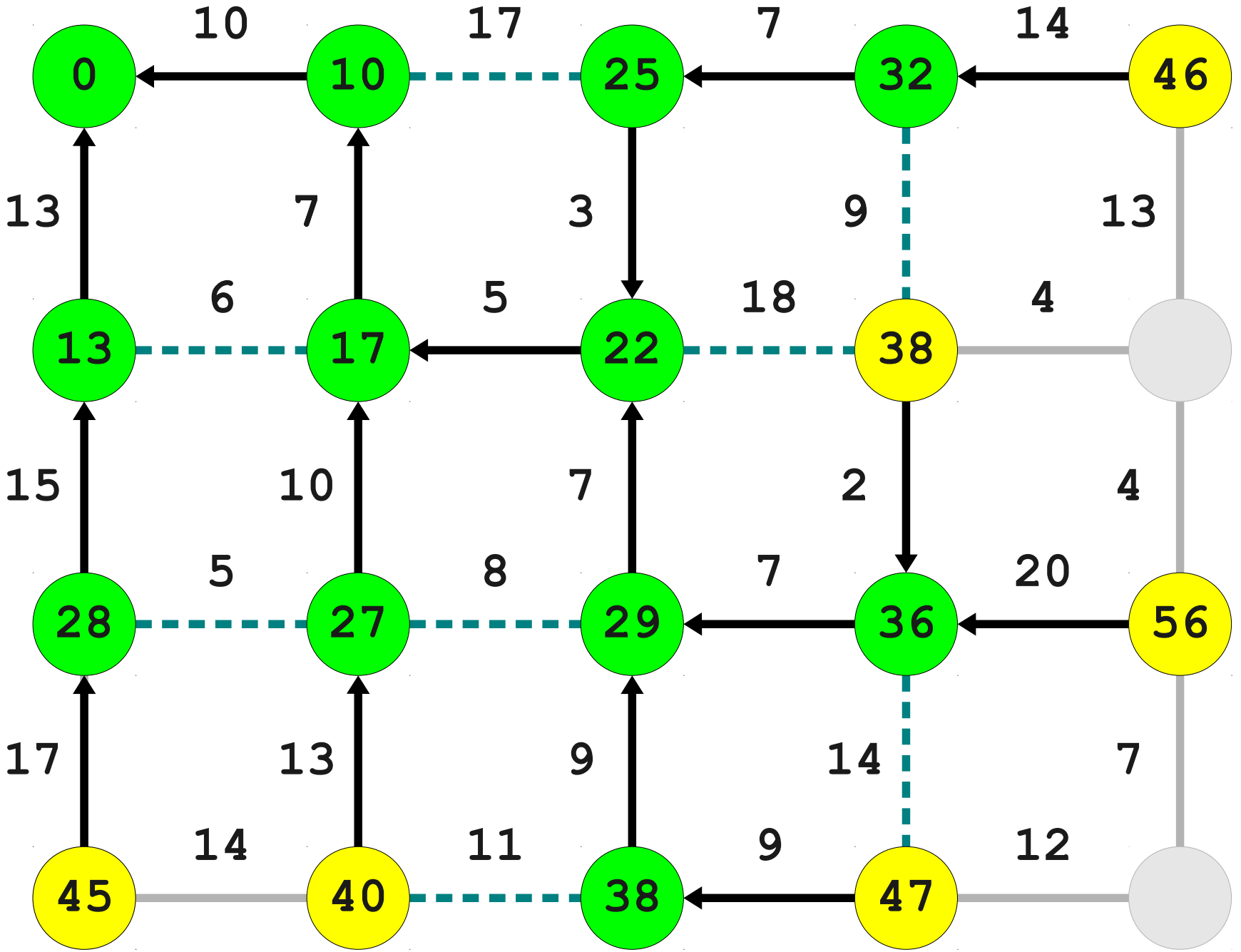
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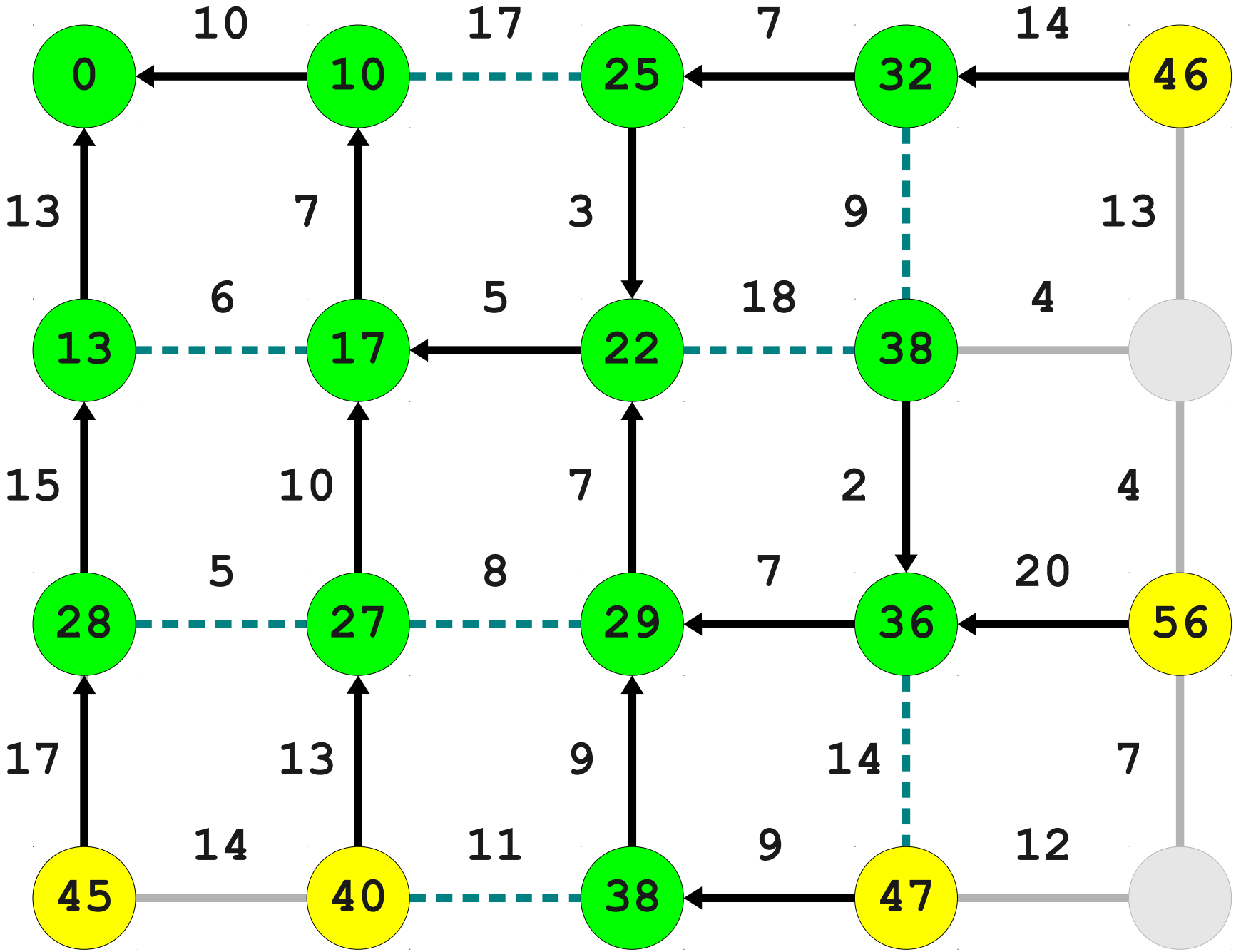
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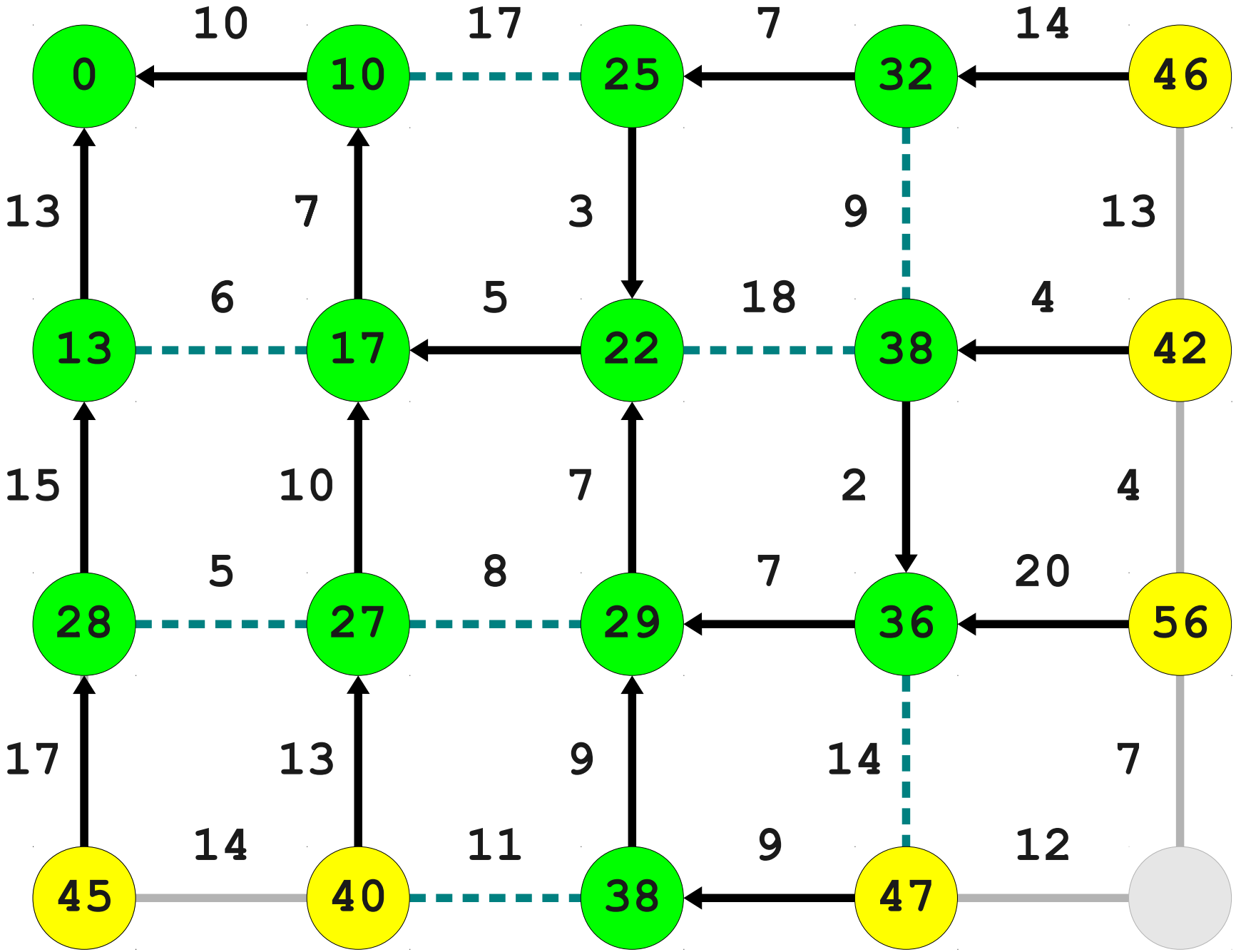
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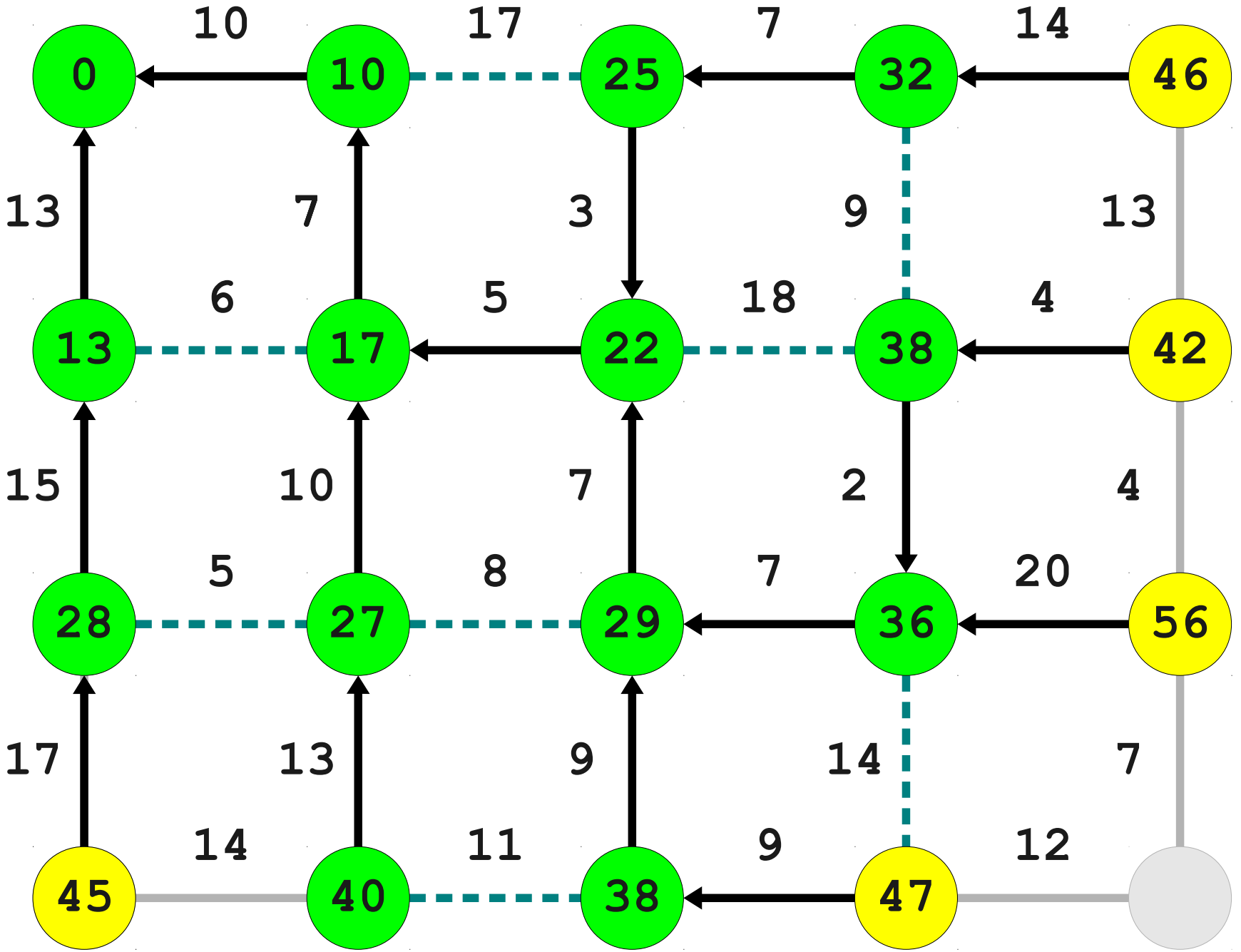
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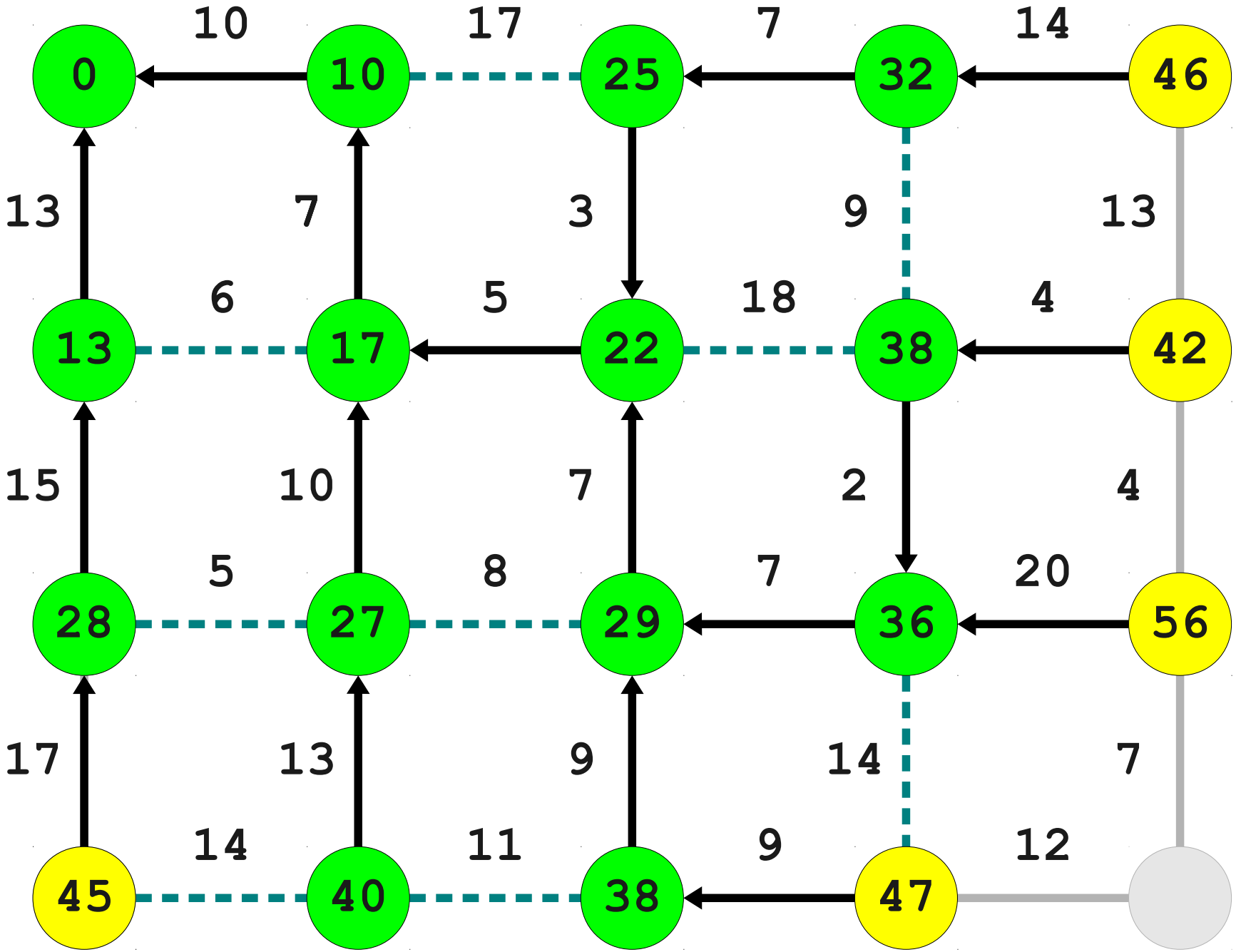
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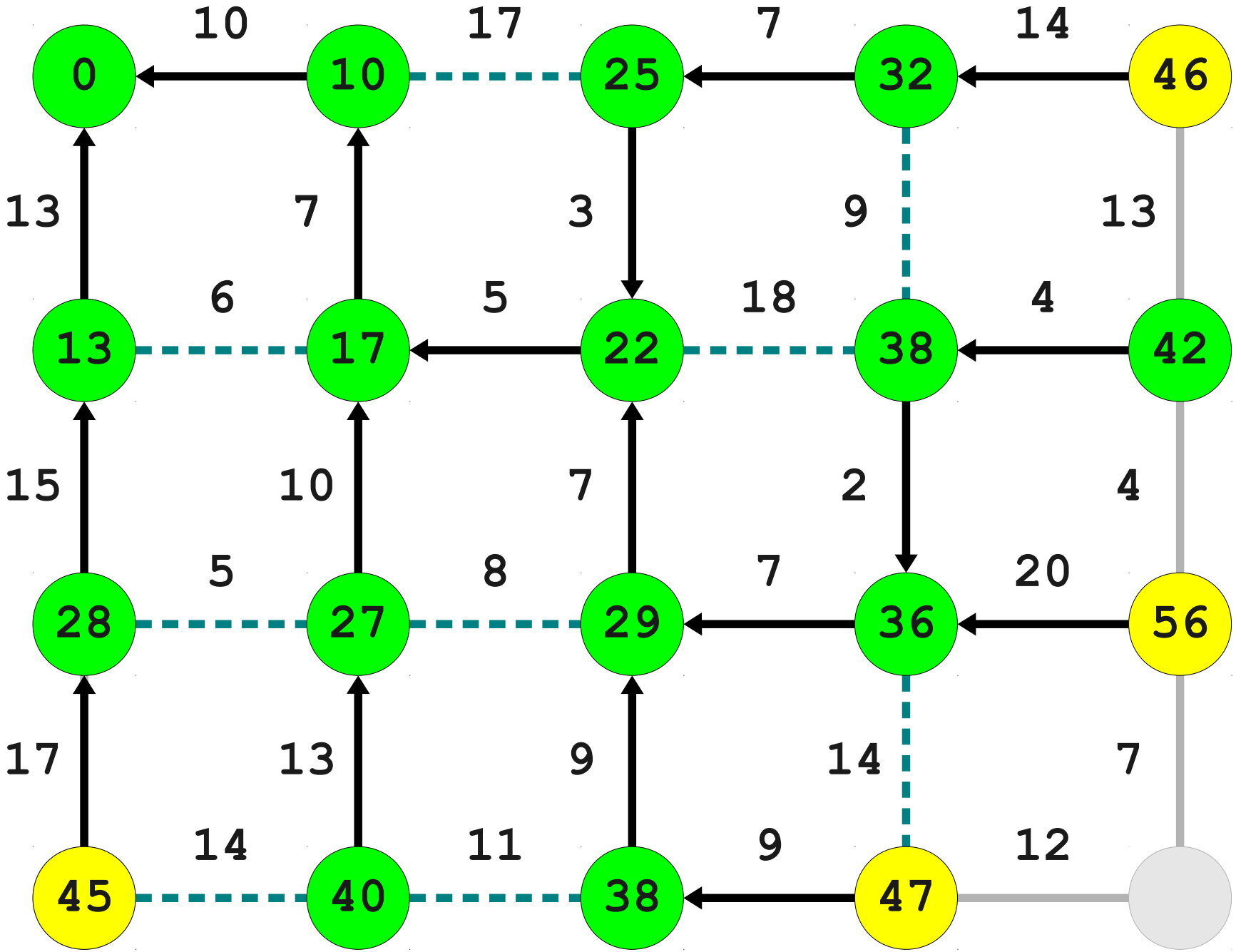
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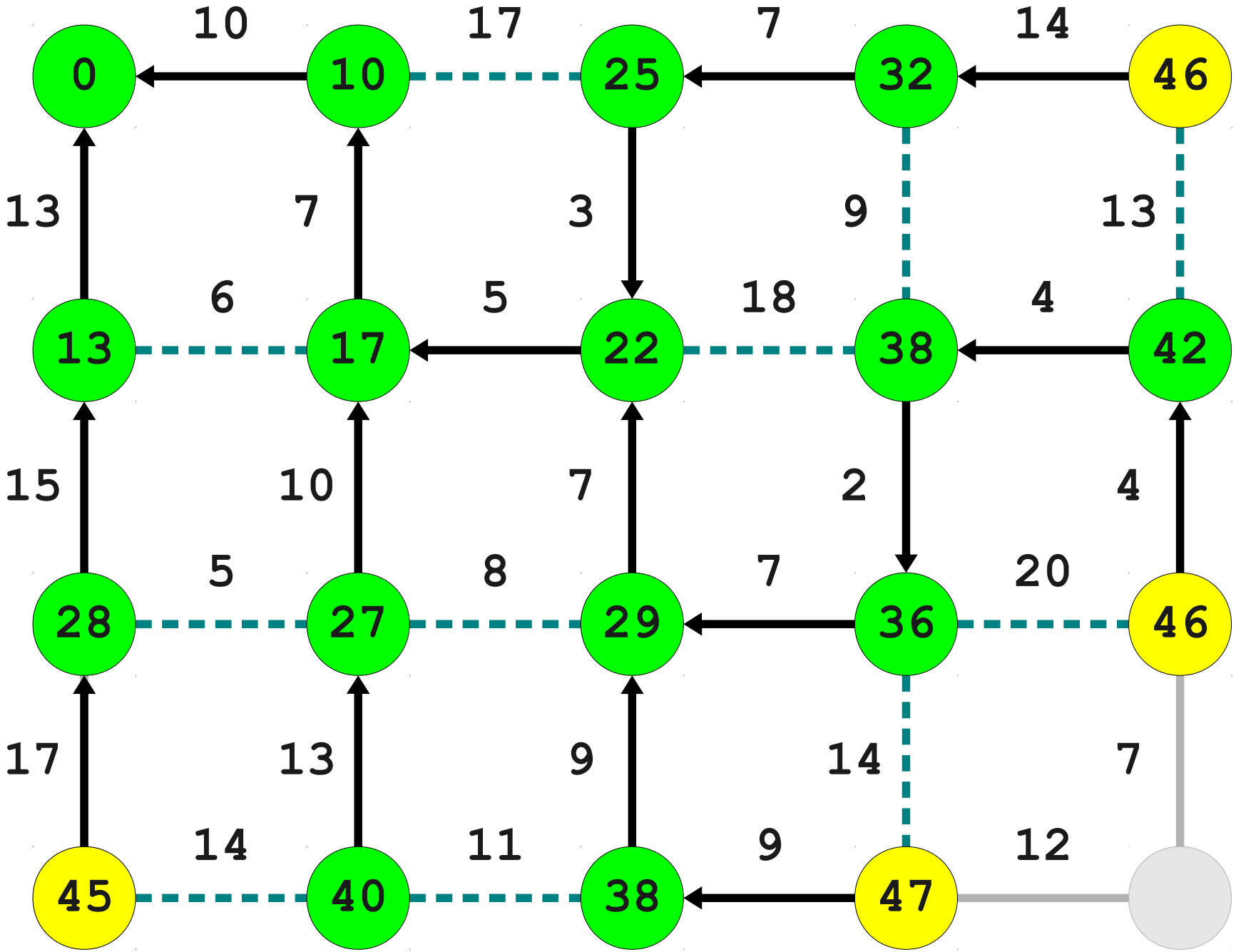
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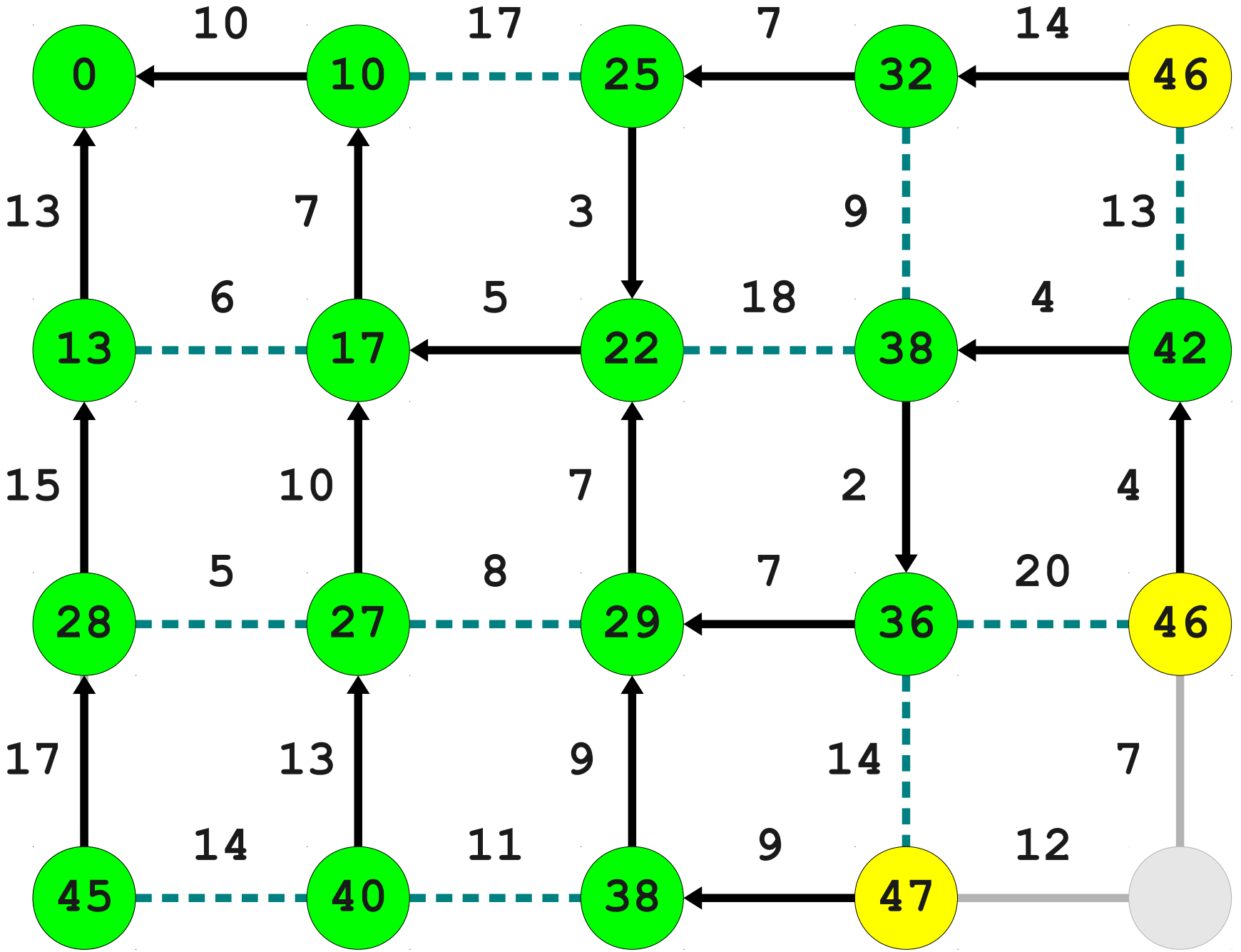
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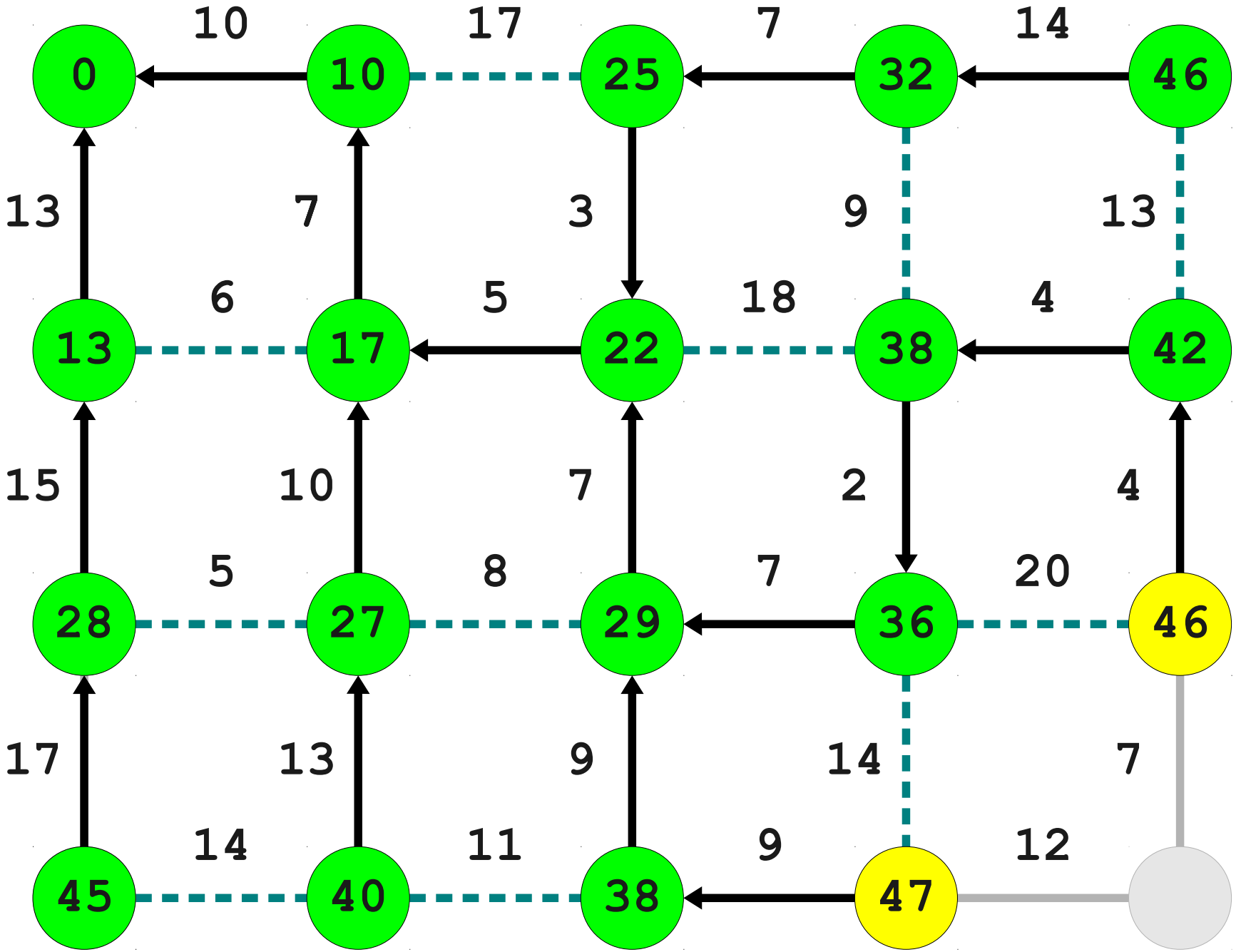
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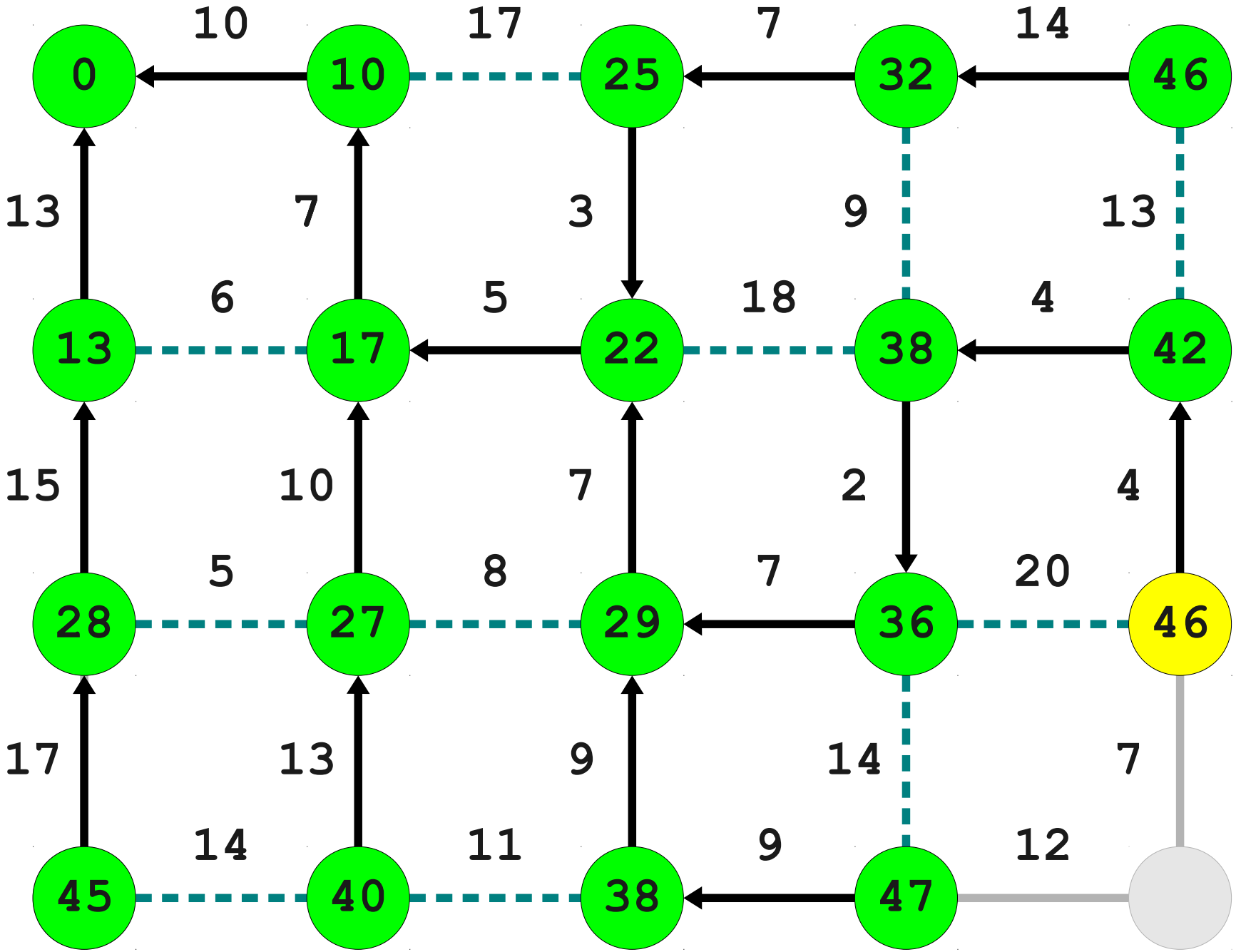
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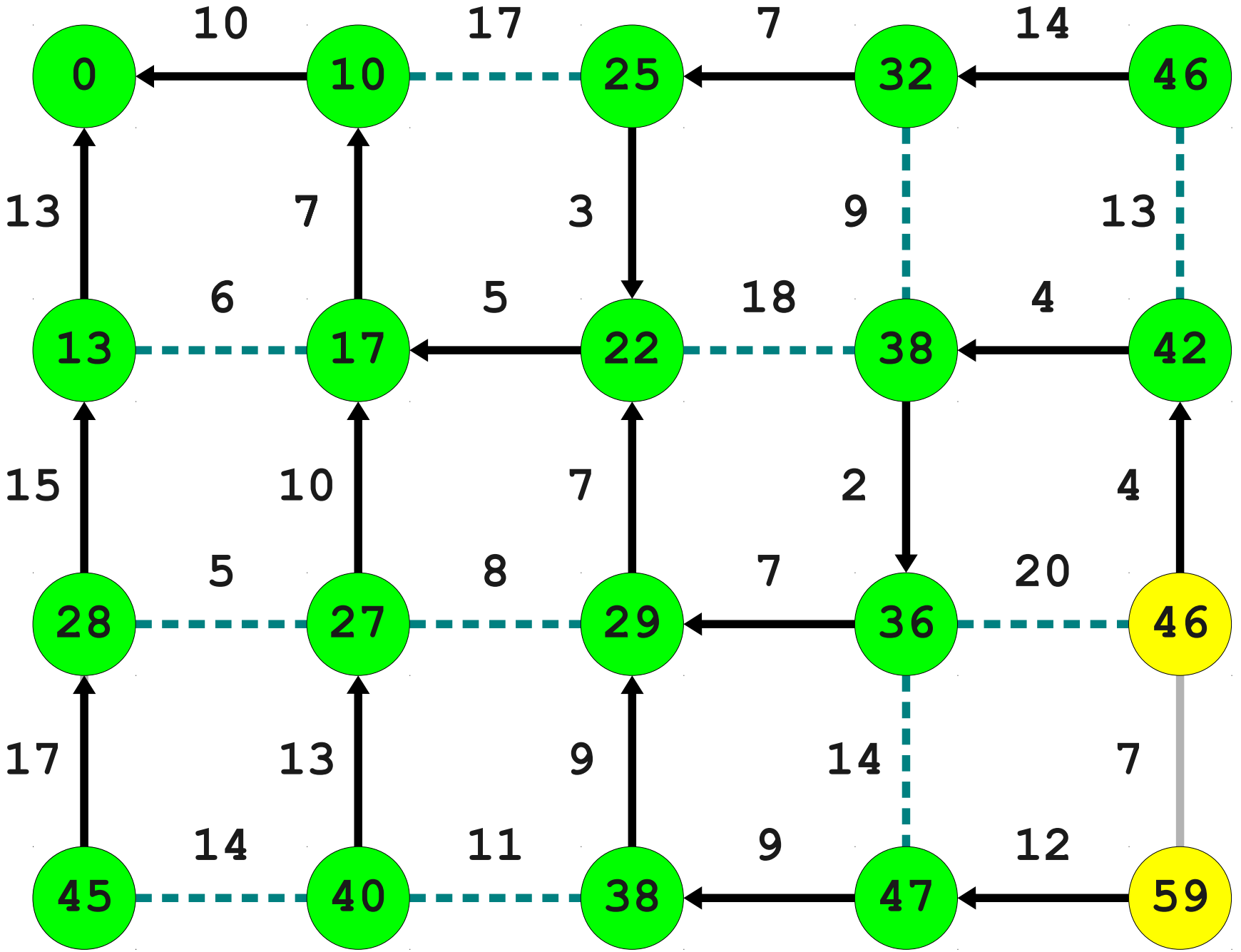
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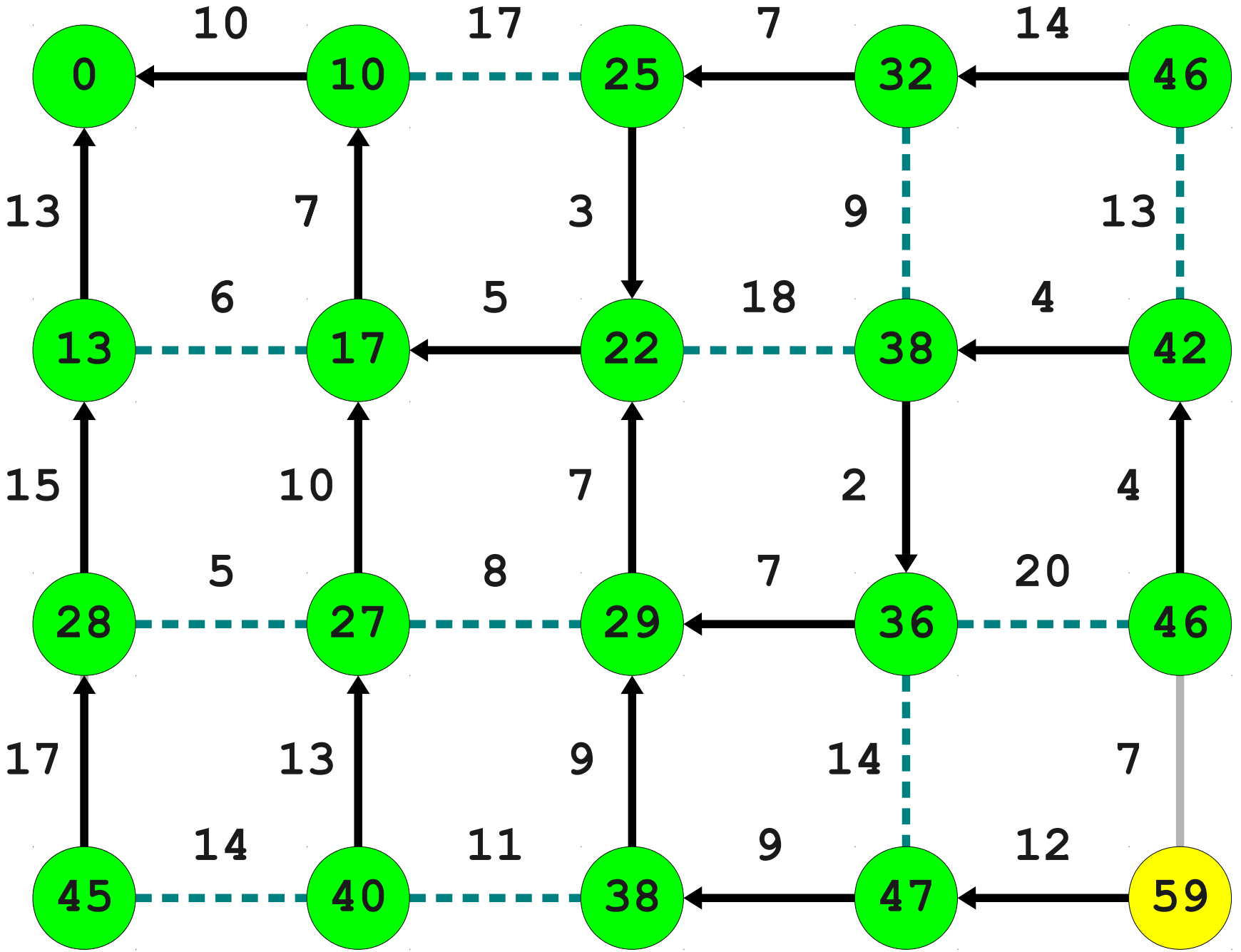
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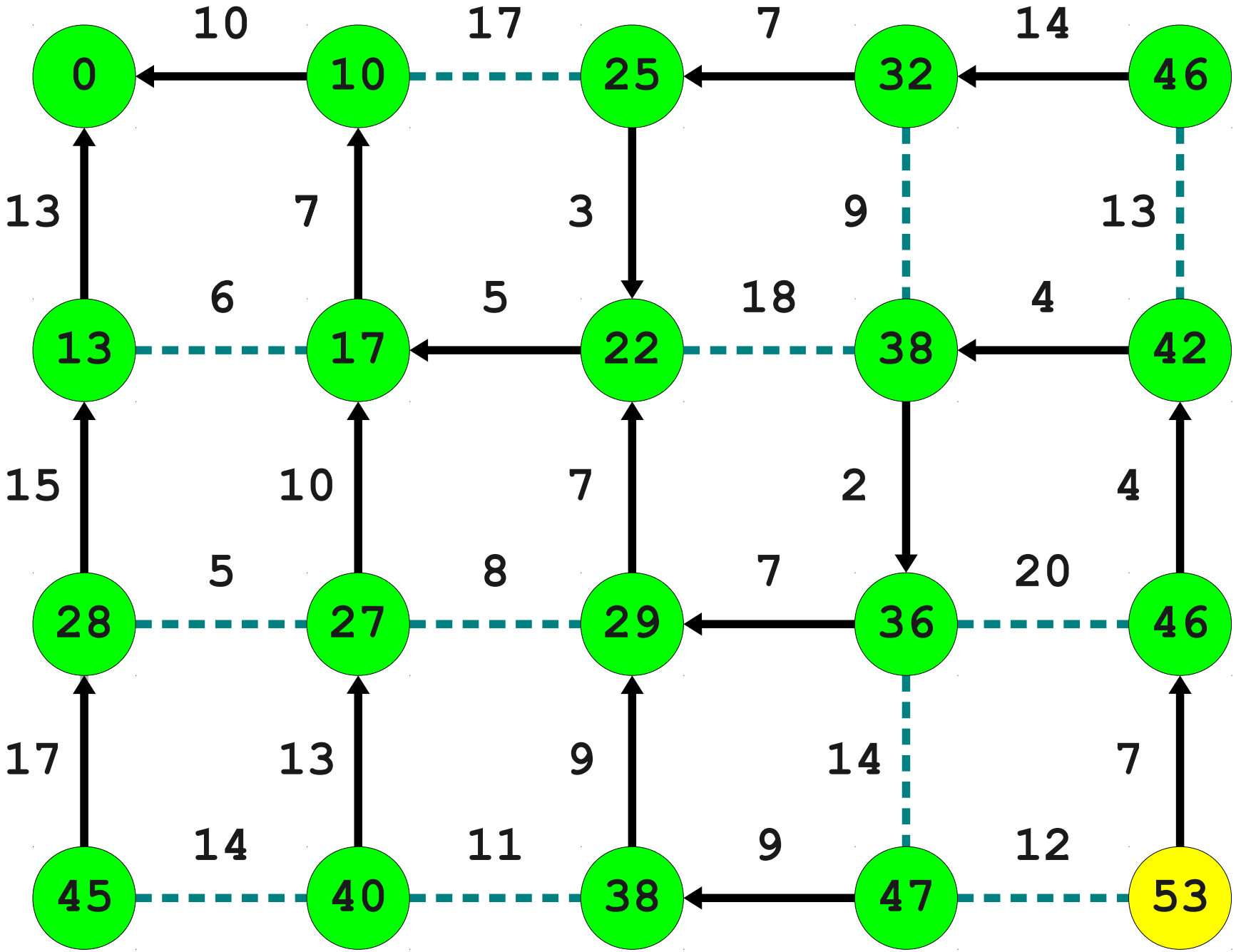
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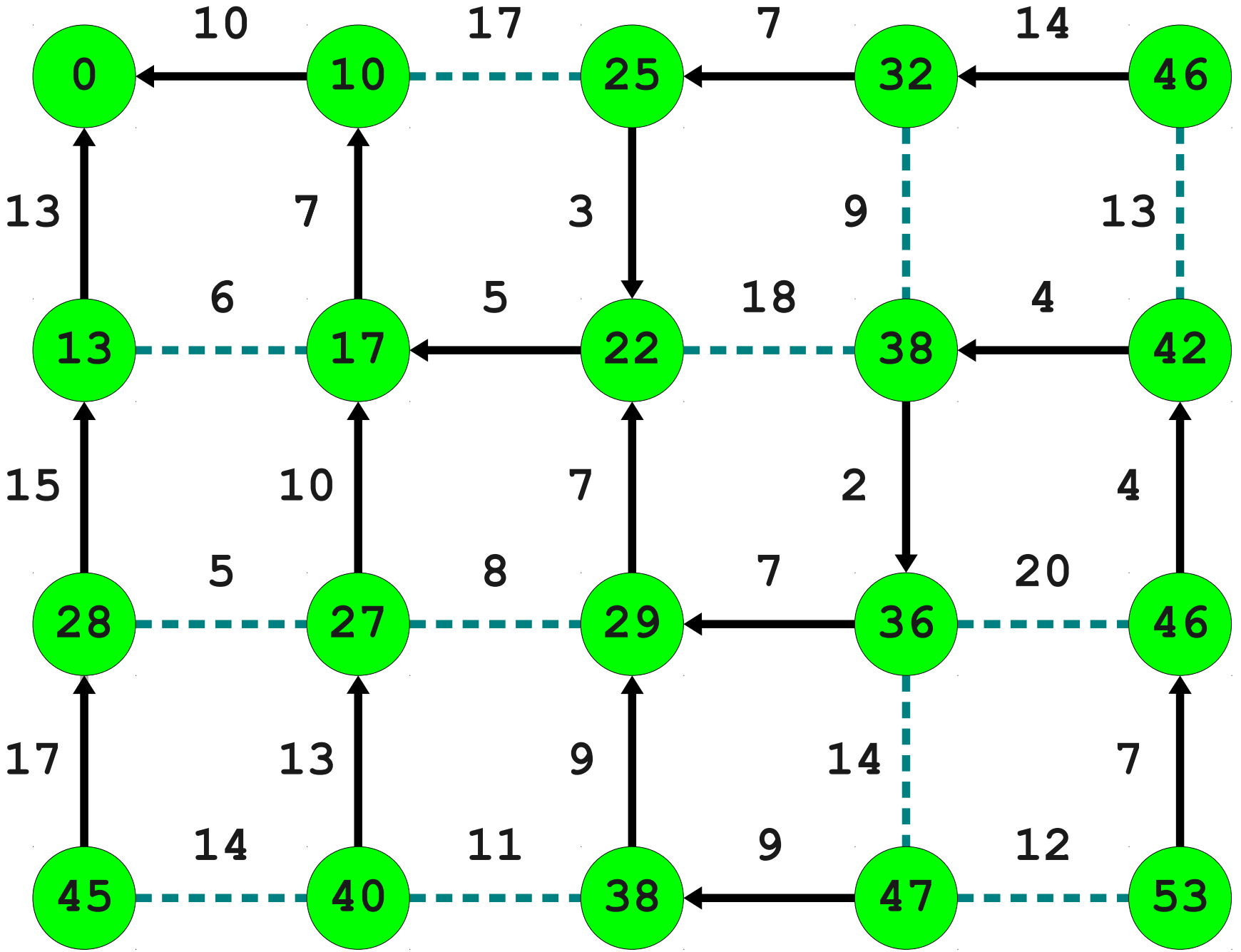
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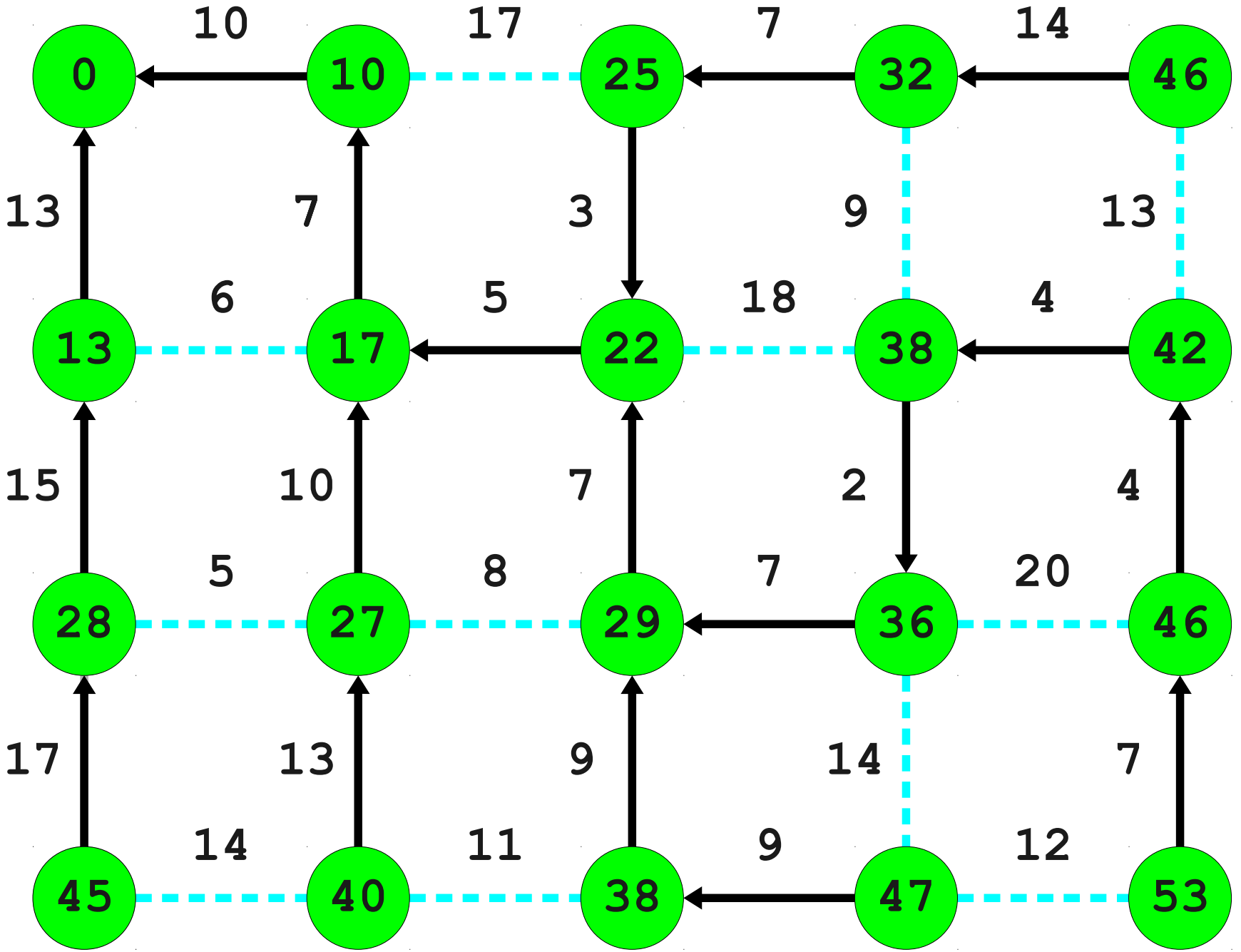
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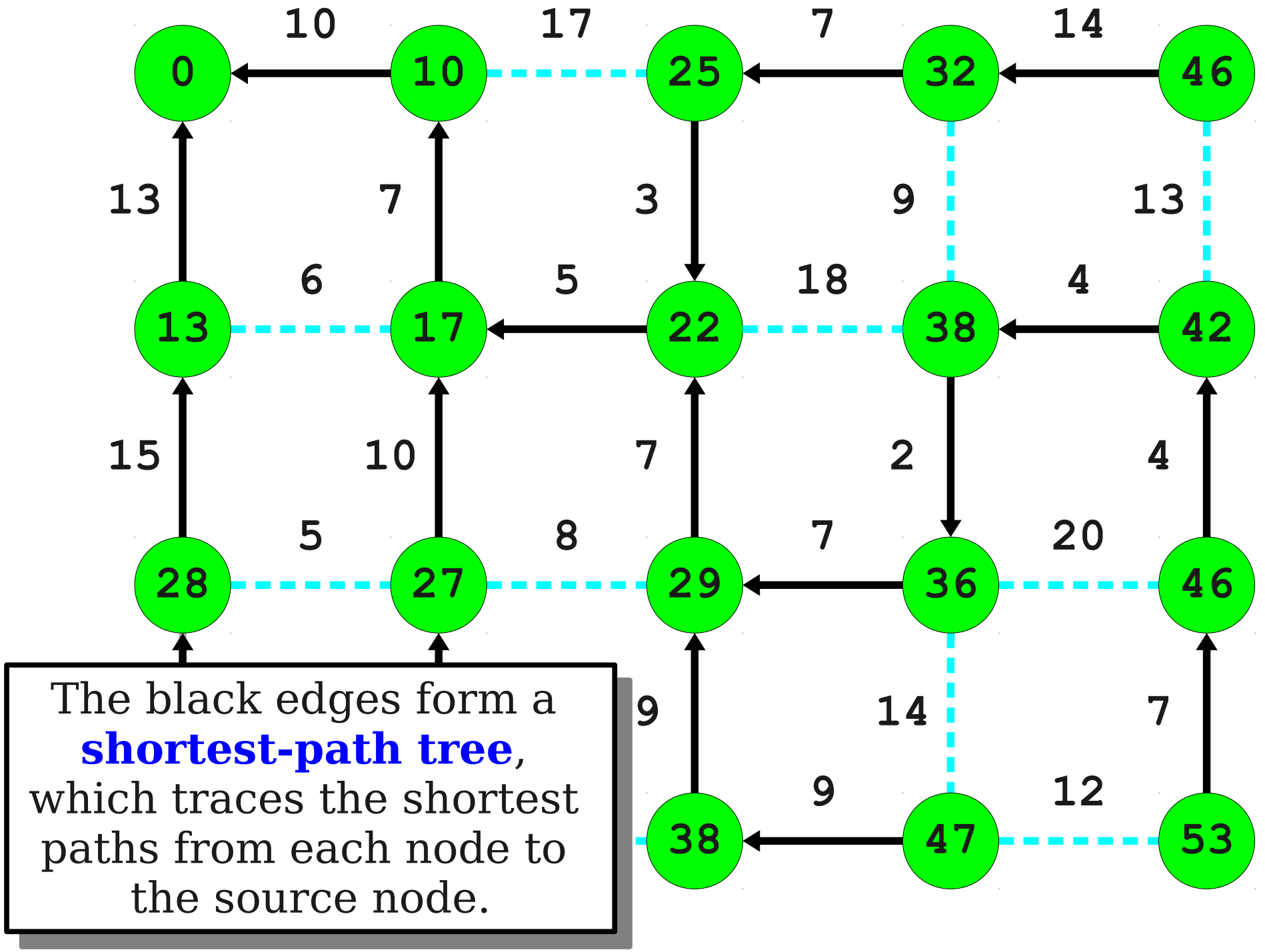
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The black edges form a shortest-path tree, which traces the shortest paths from each node to the source node.

```
procedure dijkstrasAlgorithm(s, G):
 let q be a new queue
 for each v in V:
    dist[v] = ∞

 dist[s] = 0
 enqueue(s, q)

 while q is not empty:
    let v be a node in q minimizing dist[v]
    remove(v, q)

    for each node u connected to v:
       if dist[u] > dist[v] + l(u, v):
          dist[u] = dist[v] + l(u, v)
          if u is not enqueued into q:
             enqueue(u, q)
```

Dijkstra's Algorithm

- Assuming nonnegative edge lengths, finds the shortest path from s to each node in G.
- Correctness proof sketch is based on the second argument for breadth-first search:
  - Always picks the node v minimizing $\mathrm{d}(s, u) + l(u, v)$ for yellow v and green u.
  - If a shorter path P exists to v, it must leave the set of green nodes through some edge (x, y).
  - But then $l(P)$ is at least $\mathrm{d}(s, x) + l(x, y)$, which is at least $\mathrm{d}(s, u) + l(u, v)$.
  - So the “shorter” path costs at least as much as the path we found.

```
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$O(m + n)$

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```

$O(n^2)$

Dijkstra Runtime

- Using a standard implementation of a queue, Dijkstra's algorithm runs in time **$O(n^2)$**.
  - $O(n + m)$ time processing nodes and edges, plus $O(n^2)$ time finding the lowest-cost node.
  - Since $m = O(n^2)$, $O(n + m + n^2) = O(n^2)$.
- Using a slightly fancier data structure (a binary heap), can be made to run in time **$O(m \log n)$**.
  - Is this necessarily more efficient?
  - More on how to do this later this quarter.
- Using a *much* fancier data structure (the *Fibonacci heap*), can be made to run in time **$O(m + n \log n)$**.
  - Take CE354 for details!

Shortest Path Algorithms

- If all edges have the same weight, can use breadth-first search to find shortest paths.
  - Takes time $O(m + n)$.
- If edges have nonnegative weight, can use Dijkstra's algorithm.
  - Takes time $O(n^2)$, or less using more complex data structures.
- What about the case where edges can have negative weight?
  - More on that later in the quarter...

```
INITIALIZE-SINGLE-SOURCE(G, s)
1   for each vertex v ∈ G.V
2       v.d = ∞
3       v.π = NIL
4   s.d = 0
```

```
DIJKSTRA(G, w, s)
 1  INITIALIZE-SINGLE-SOURCE(G, s)
 2  S = ∅
 3  Q = ∅
 4  for each vertex u ∈ G.V
 5      INSERT(Q, u)
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      S = S ∪ {u}
 9      for each vertex v in G.Adj[u]
10          RELAX(u, v, w)
11          if the call of RELAX decreased v.d
12              DECREASE-KEY(Q, v, v.d)
```

```
RELAX(u, v, w)
1   if v.d > u.d + w(u, v)
2       v.d = u.d + w(u, v)
3       v.π = u
```

Dijkstra's algorithm: which priority queue?

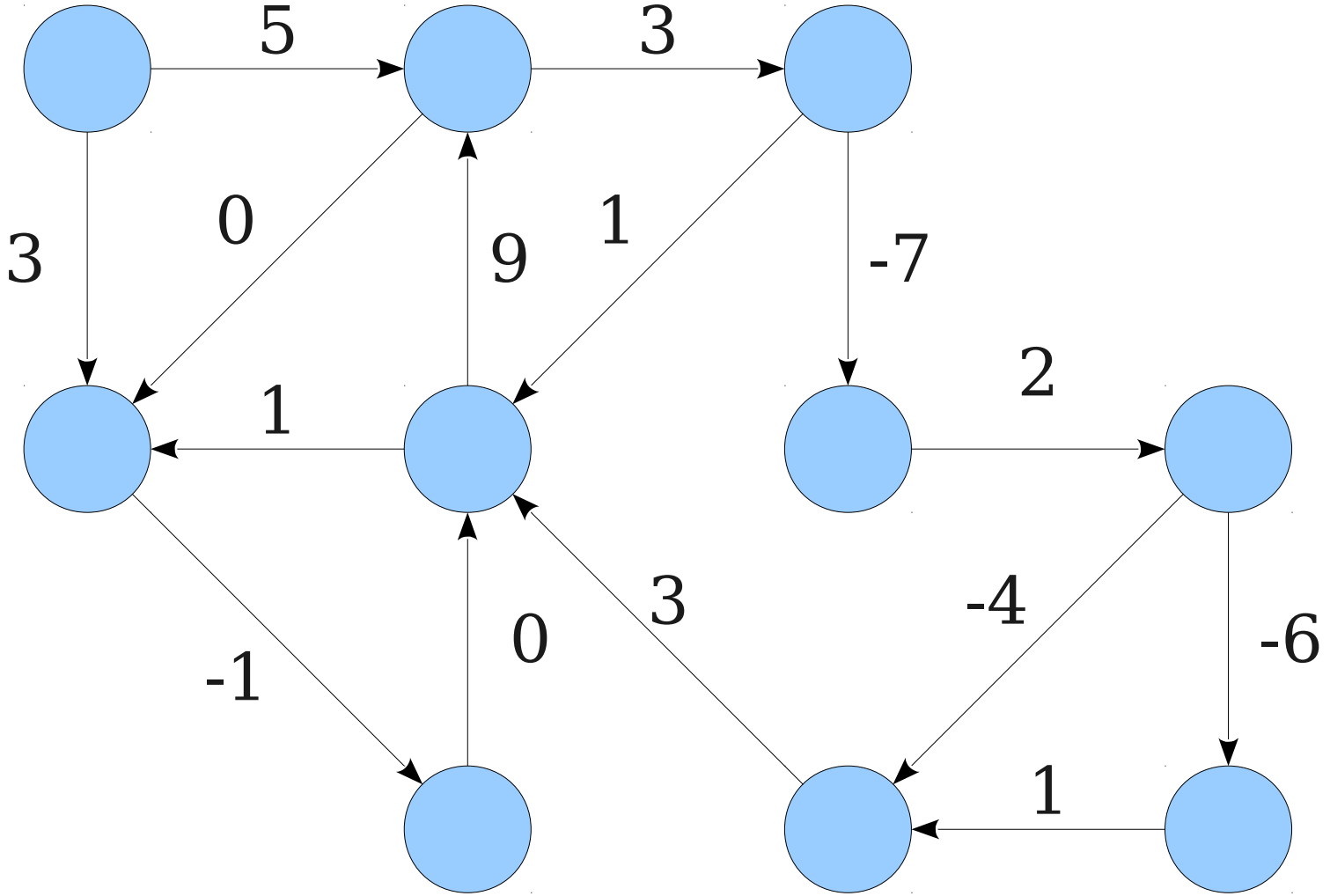
Performance. Depends on PQ: n INSERT, n DELETE-MIN, $\leq m$ DECREASE-KEY.

- Array implementation optimal for dense graphs. ⟵ $\Theta(n^2)$ edges
- Binary heap much faster for sparse graphs. ⟵ $\Theta(n)$ edges
- 4-way heap worth the trouble in performance-critical situations.

priority queue	INSERT	DELETE-MIN	DECREASE-KEY	total
node-indexed array (A[i] = priority of i)	$O(1)$	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(m \log n)$
d-way heap (Johnson 1975)	$O(d \log_d n)$	$O(d \log_d n)$	$O(\log_d n)$	$O(m \log_{m/n} n)$
Fibonacci heap (Fredman–Tarjan 1984)	$O(1)$	$O(\log n)$ †	$O(1)$ †	$O(m + n \log n)$
integer priority queue (Thorup 2004)	$O(1)$	$O(\log \log n)$	$O(1)$	$O(m + n \log \log n)$

assumes $m \geq n$ † amortized

Negative Edge Weights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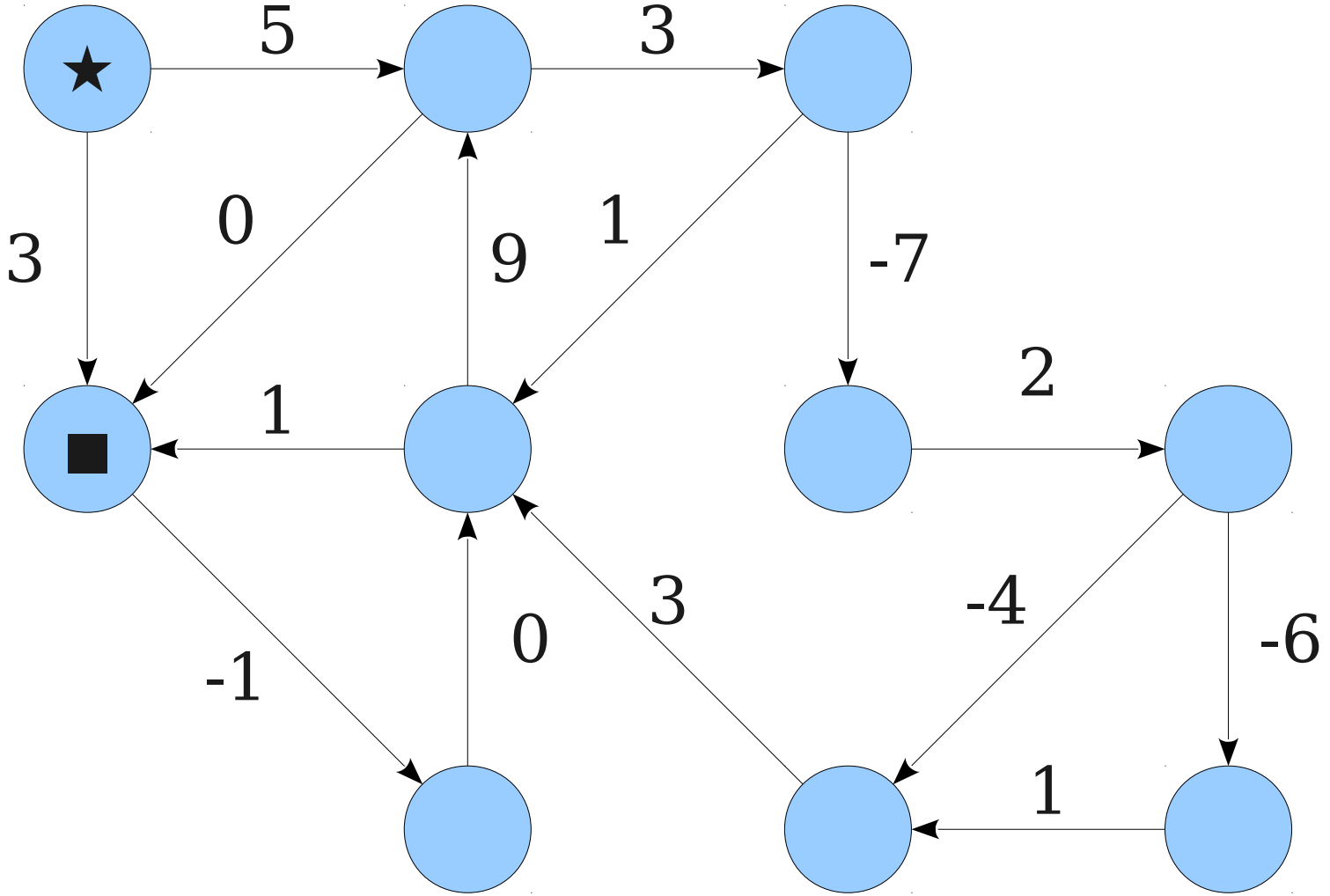
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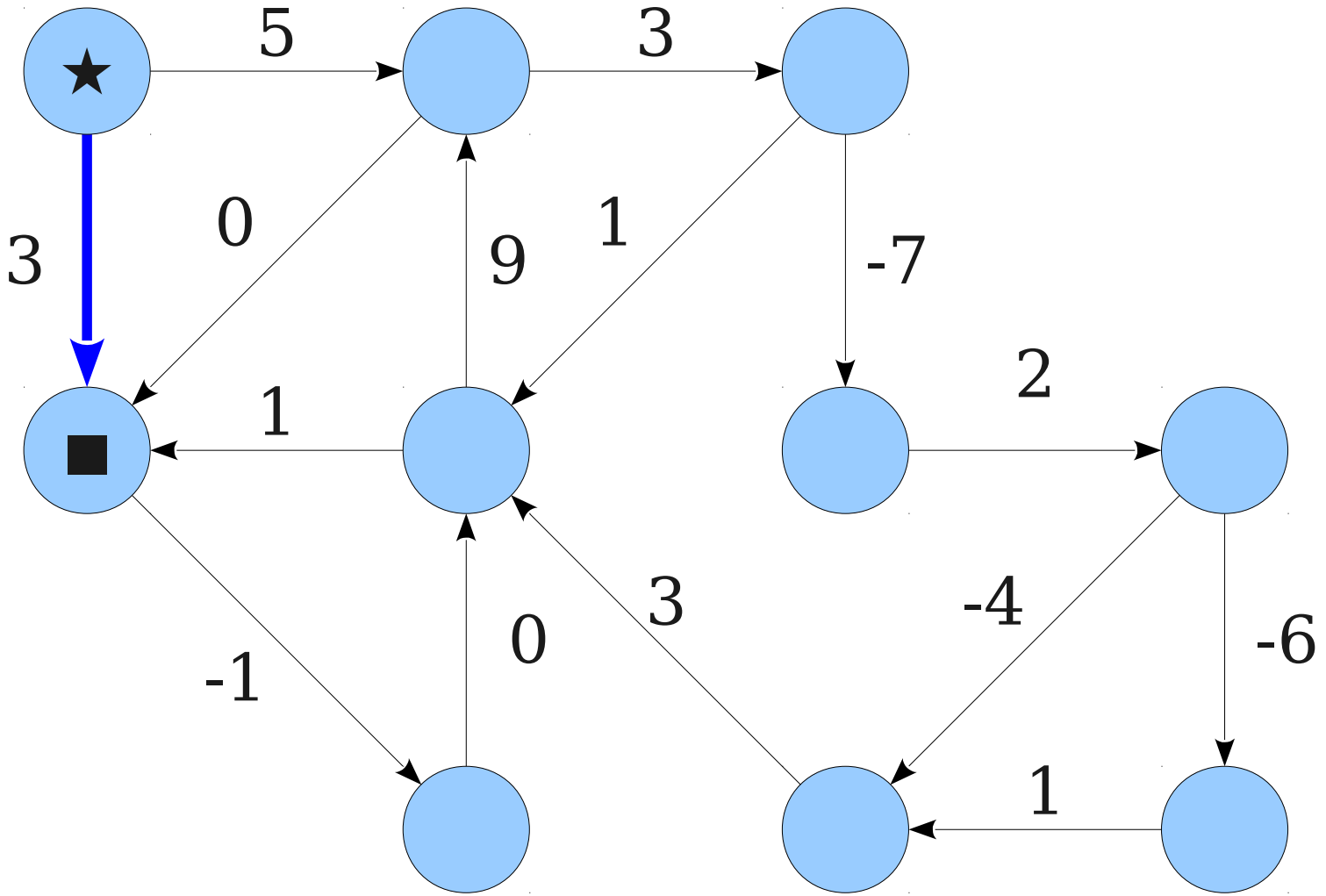
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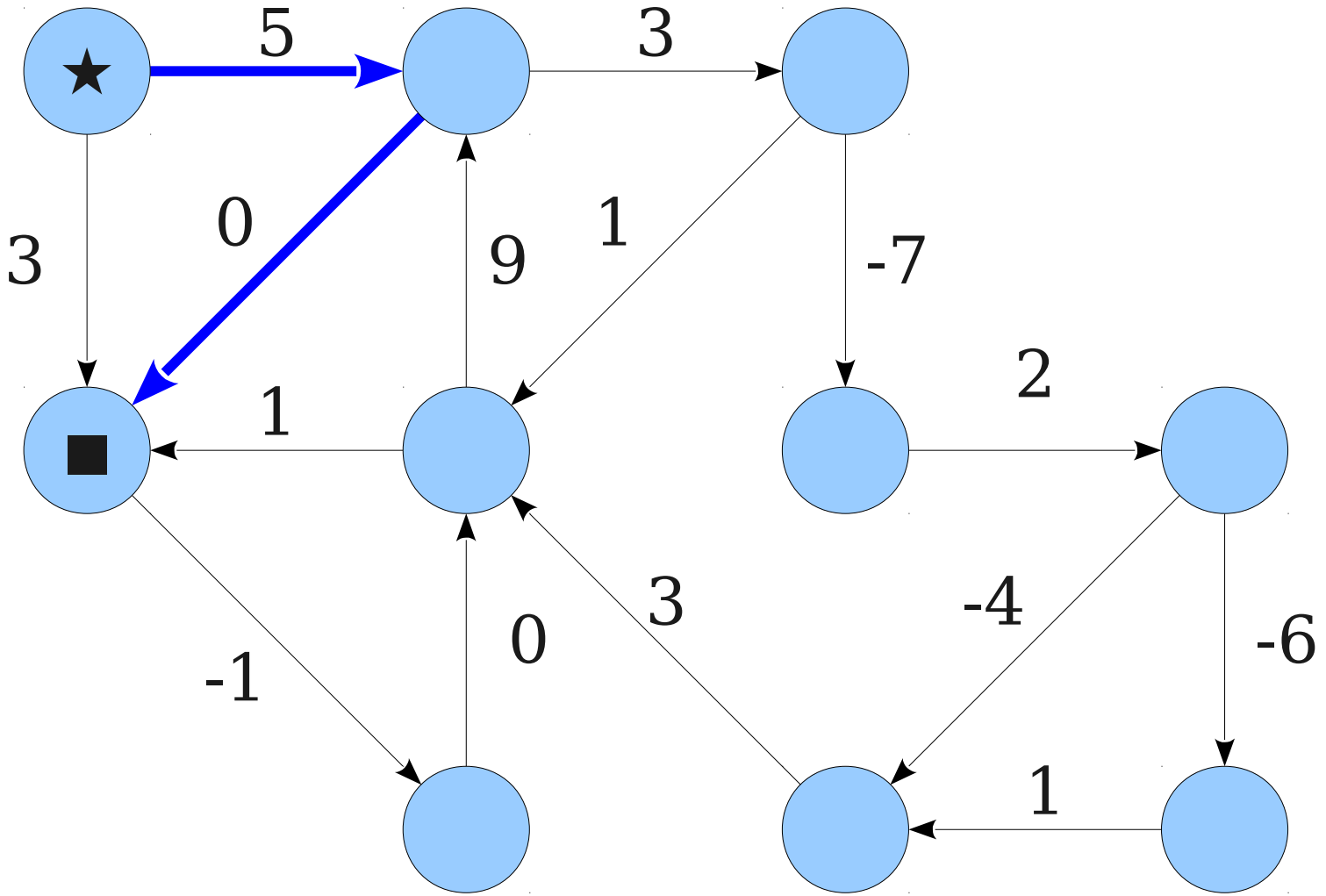
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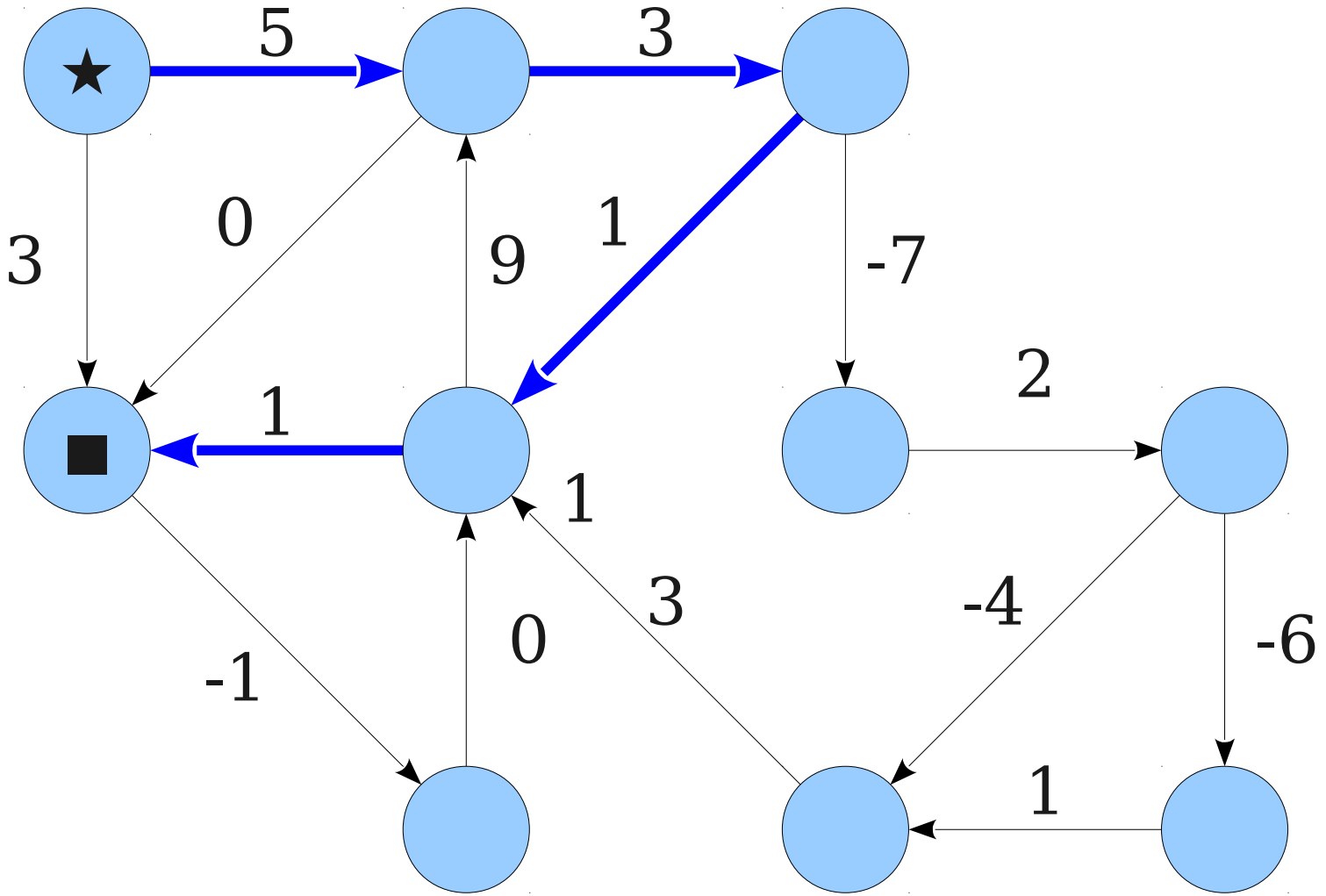
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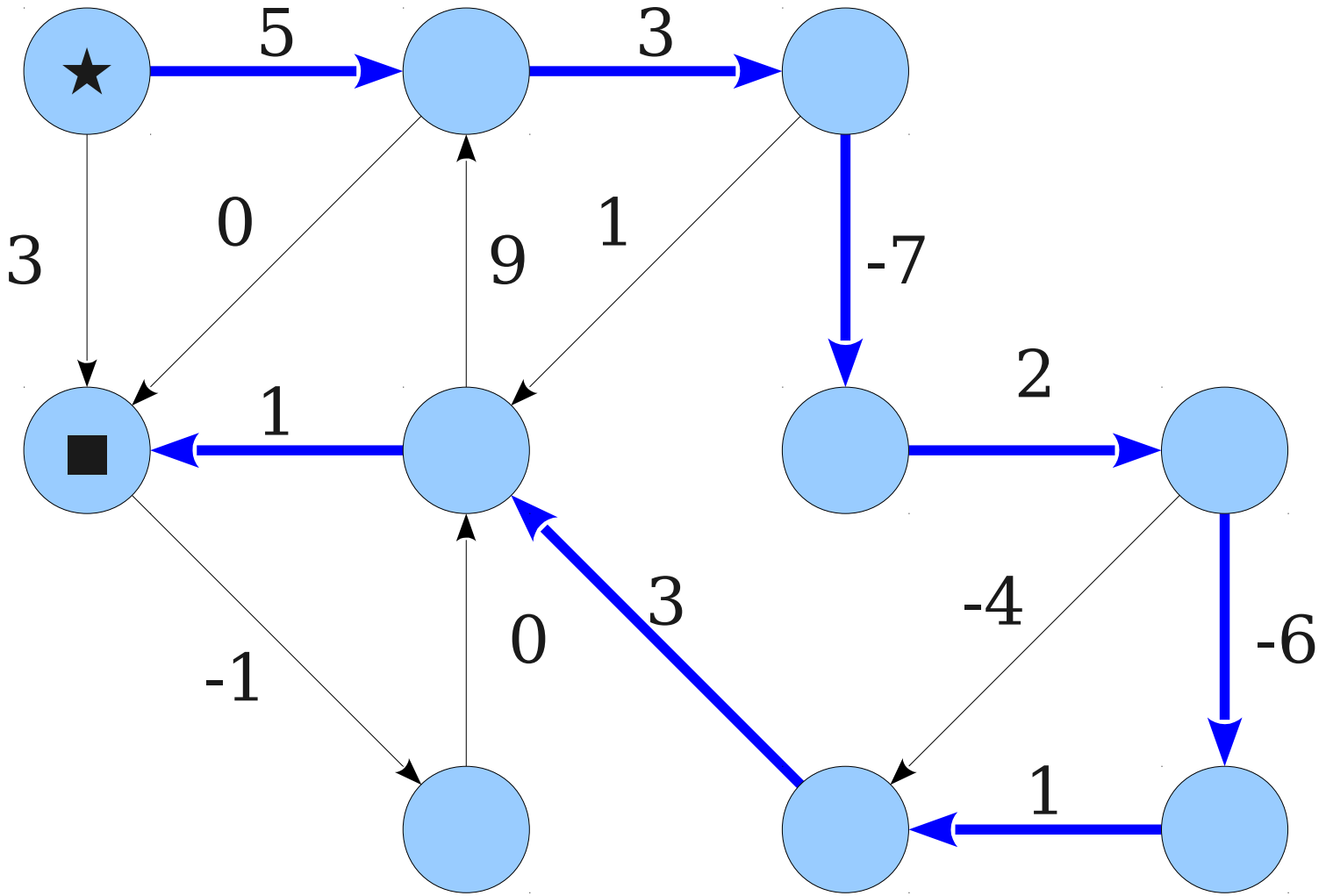
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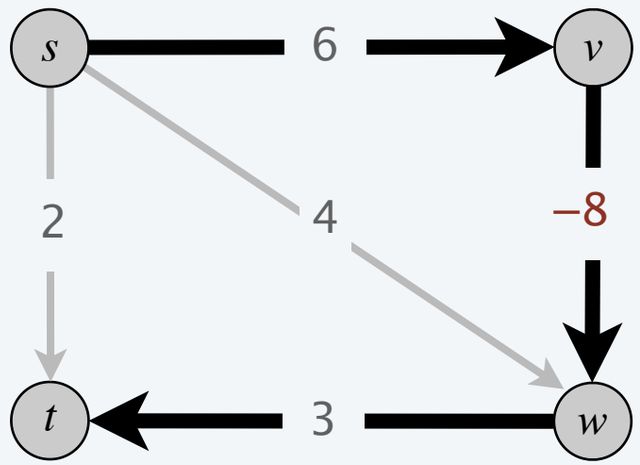
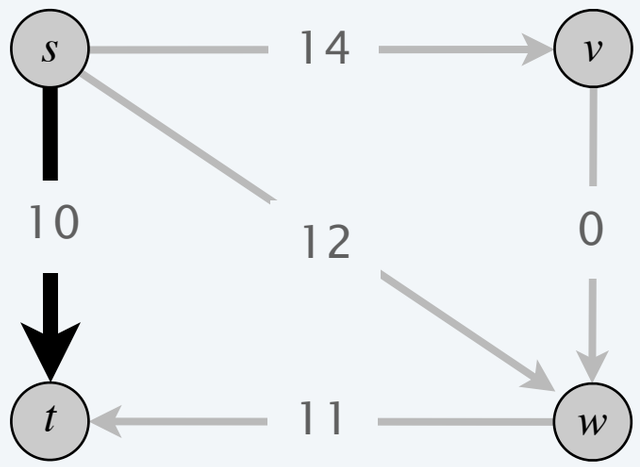
Shortest paths with negative weights: failed attempts

Dijkstra. May not produce shortest paths when edge lengths are negative.



Dijkstra selects the vertices in the order s, t, w, v
But shortest path from s to t is $s \rightarrow v \rightarrow w \rightarrow t$.

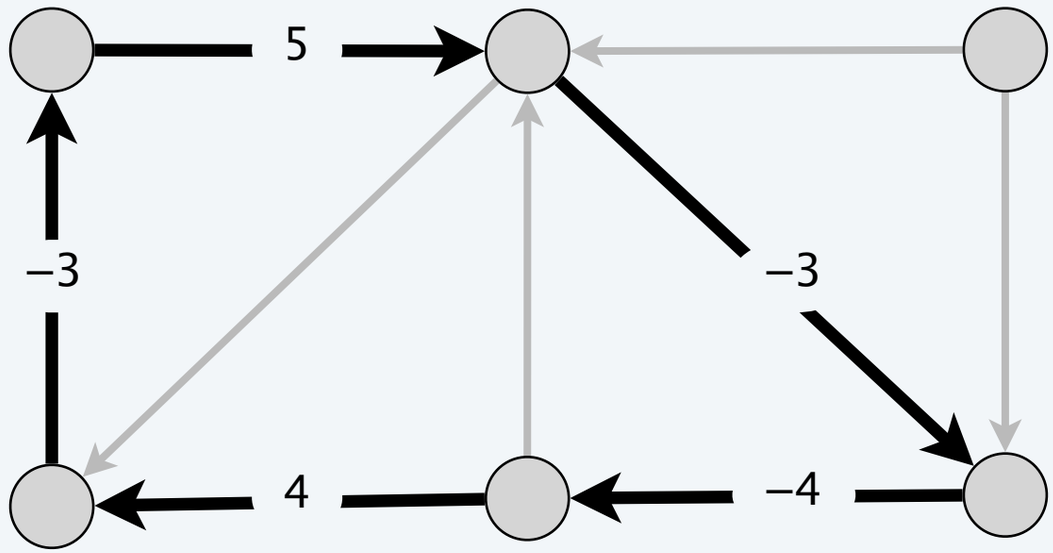
Reweighting. Adding a constant to every edge length does not necessarily make Dijkstra's algorithm produce shortest paths.



Adding 8 to each edge weight changes the shortest path from $s \rightarrow v \rightarrow w \rightarrow t$ to $s \rightarrow t$.

Negative cycles

Def. A negative cycle is a directed cycle for which the sum of its edge lengths is negative.

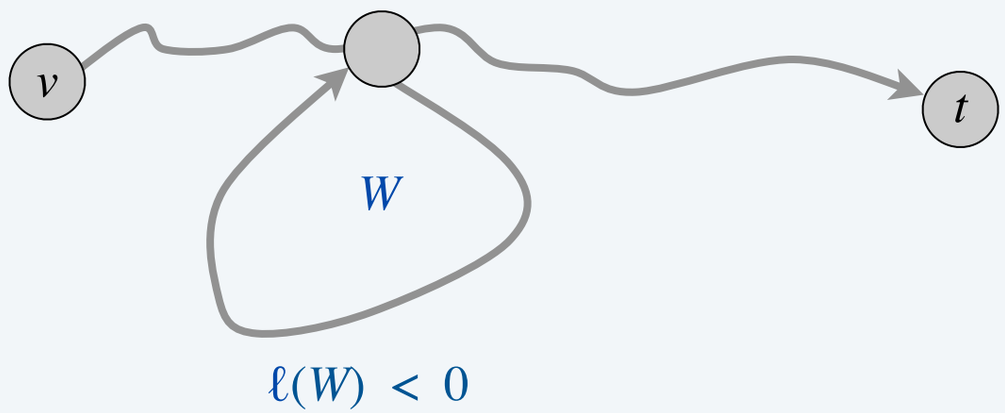


a negative cycle W : $\ell(W) = \sum_{e \in W} \ell_e < 0$

Shortest paths and negative cycles

Lemma 1. If some $v \rightsquigarrow t$ path contains a negative cycle, then there does not exist a shortest $v \rightsquigarrow t$ path.

Pf. If there exists such a cycle W, then can build a $v \rightsquigarrow t$ path of arbitrarily negative length by detouring around W as many times as desired. ▪

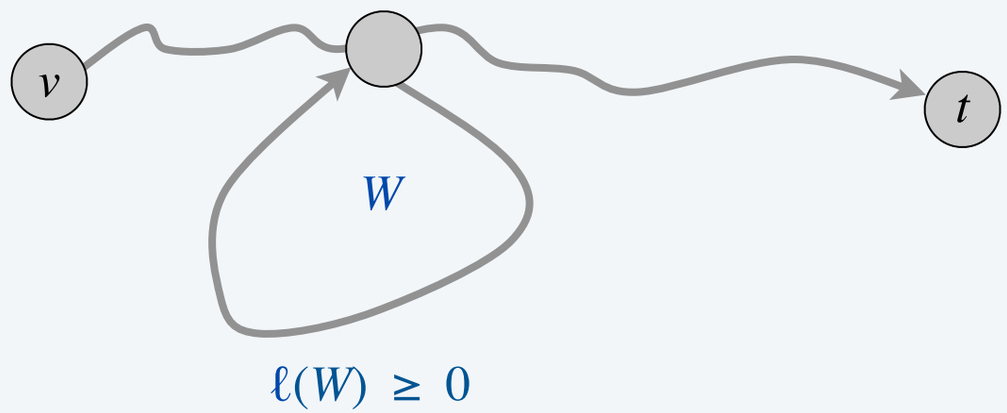


Shortest paths and negative cycles

Lemma 2. If G has no negative cycles, then there exists a shortest $v \rightsquigarrow t$ path that is simple (and has $\leq n - 1$ edges).

Pf.

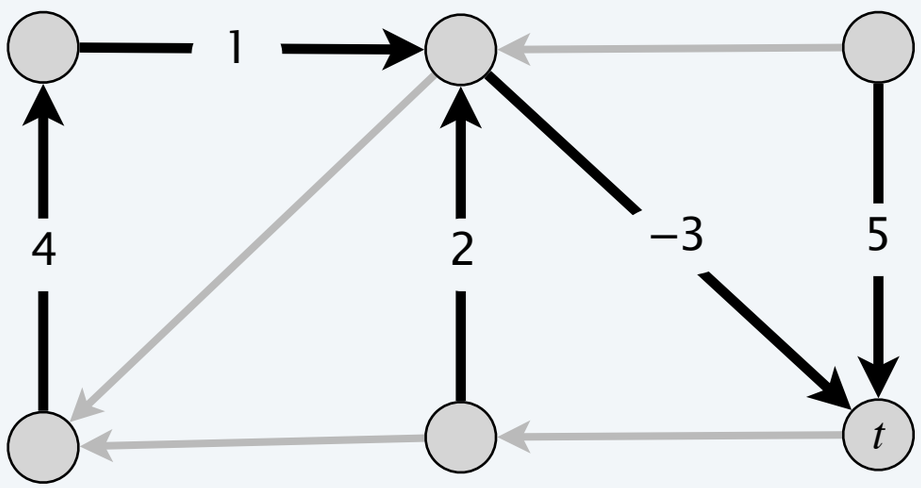
- Among all shortest $v \rightsquigarrow t$ paths, consider one that uses the fewest edges.
- If that path P contains a directed cycle W, can remove the portion of P corresponding to W without increasing its length. ▪



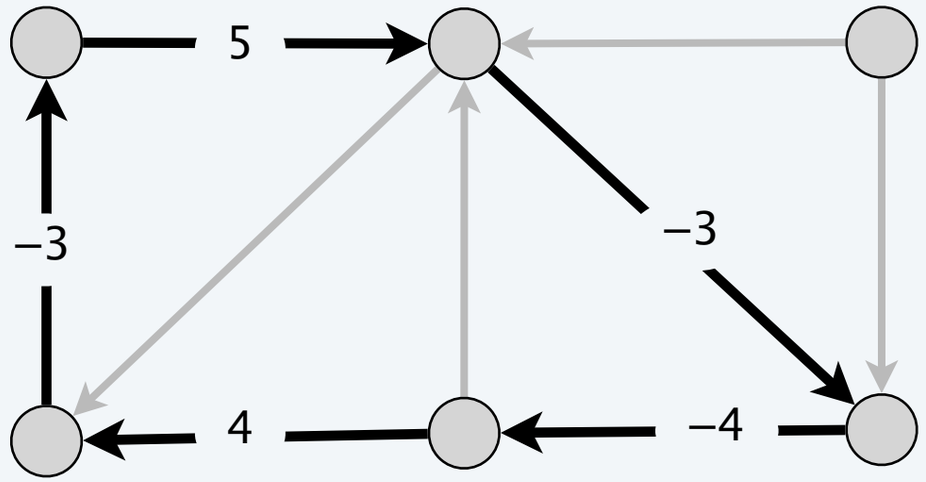
Shortest-paths and negative-cycle problems

Single-destination shortest-paths problem. Given a digraph $G = (V, E)$ with edge lengths ℓ_{vw} (but no negative cycles) and a distinguished node t, find a shortest $v \rightsquigarrow t$ path for every node v.

Negative-cycle problem. Given a digraph $G = (V, E)$ with edge lengths ℓ_{vw}, find a negative cycle (if one exists).

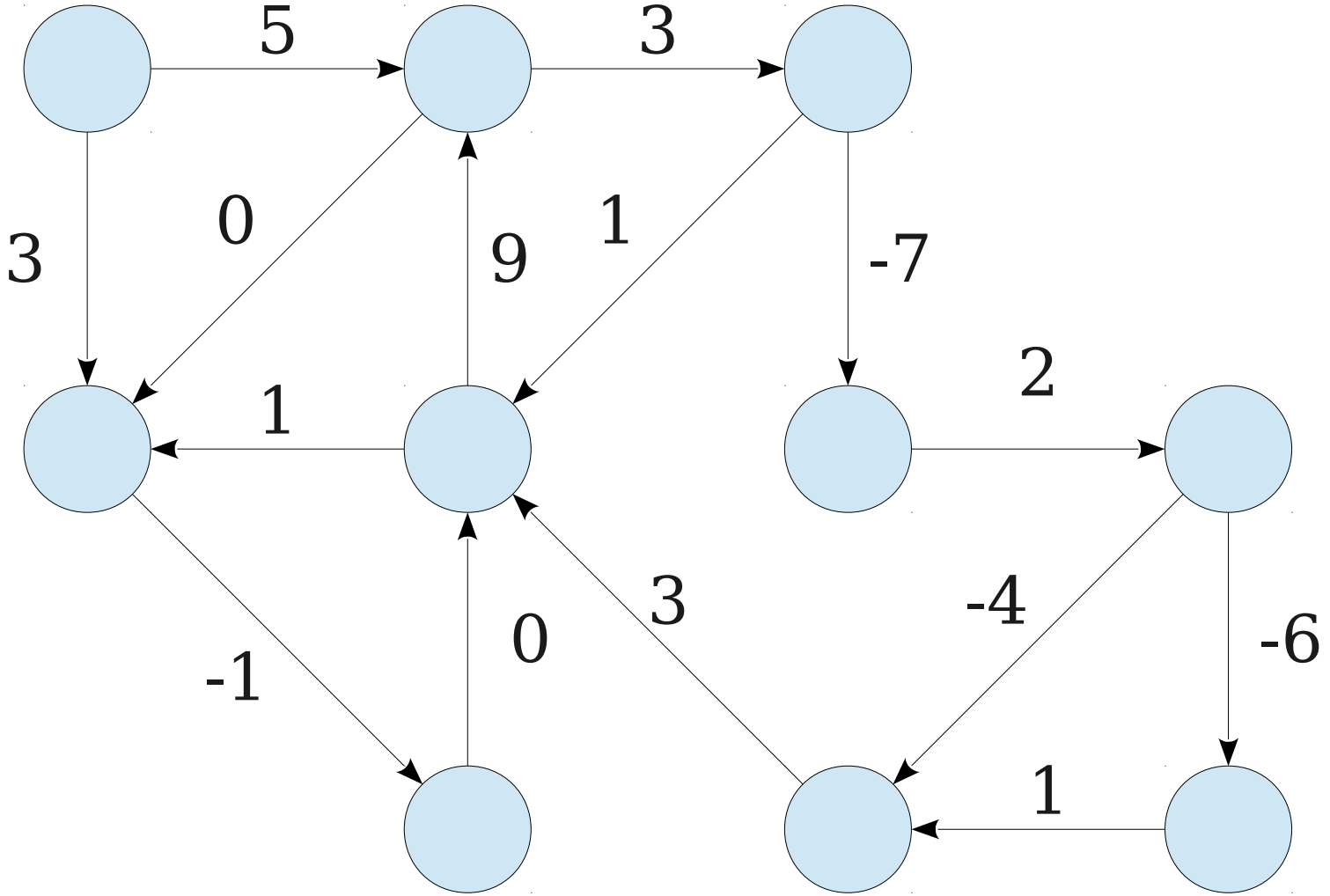


shortest-paths tree



negative cycle

A Different Intuition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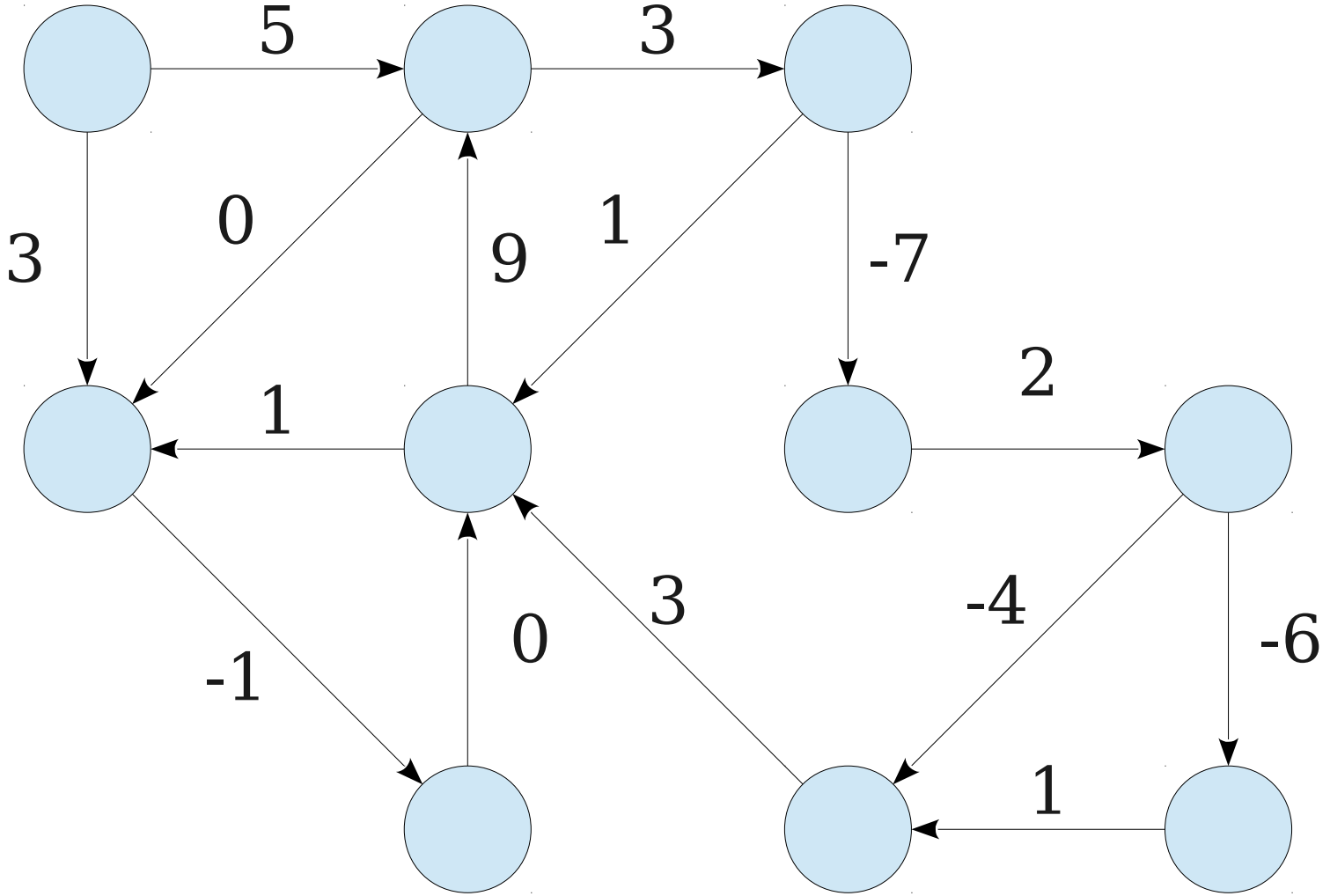
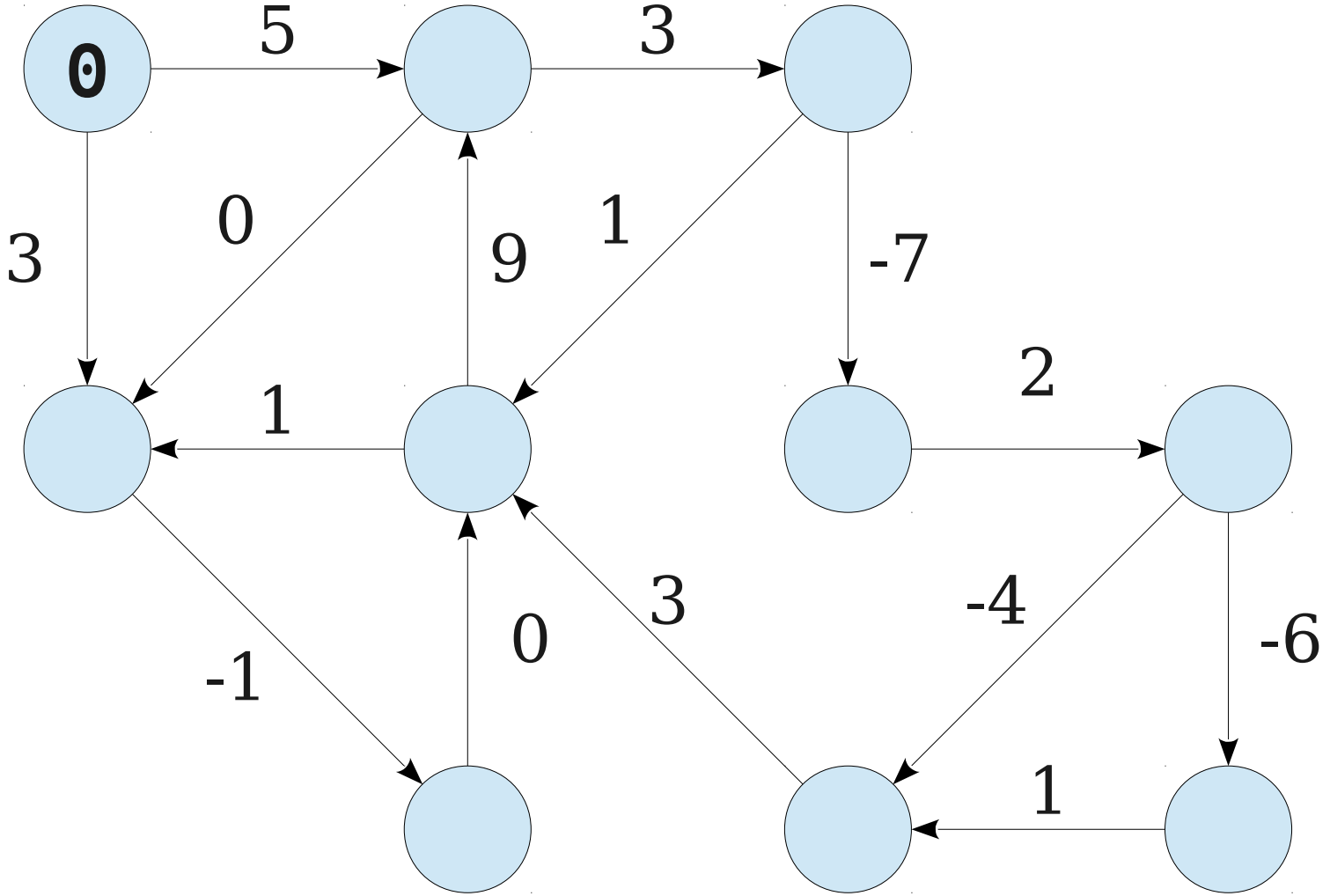


Consider paths of length **0**.



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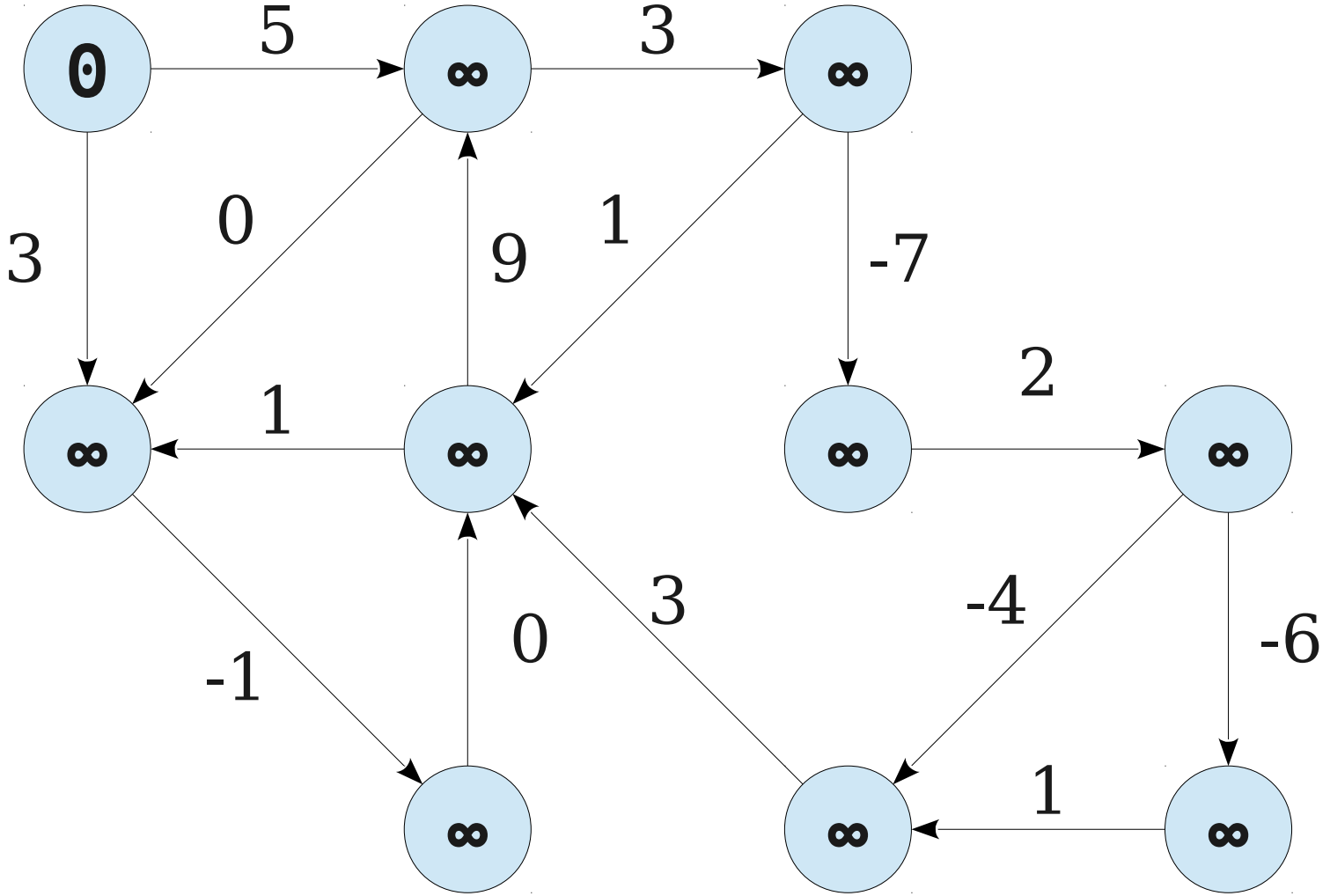

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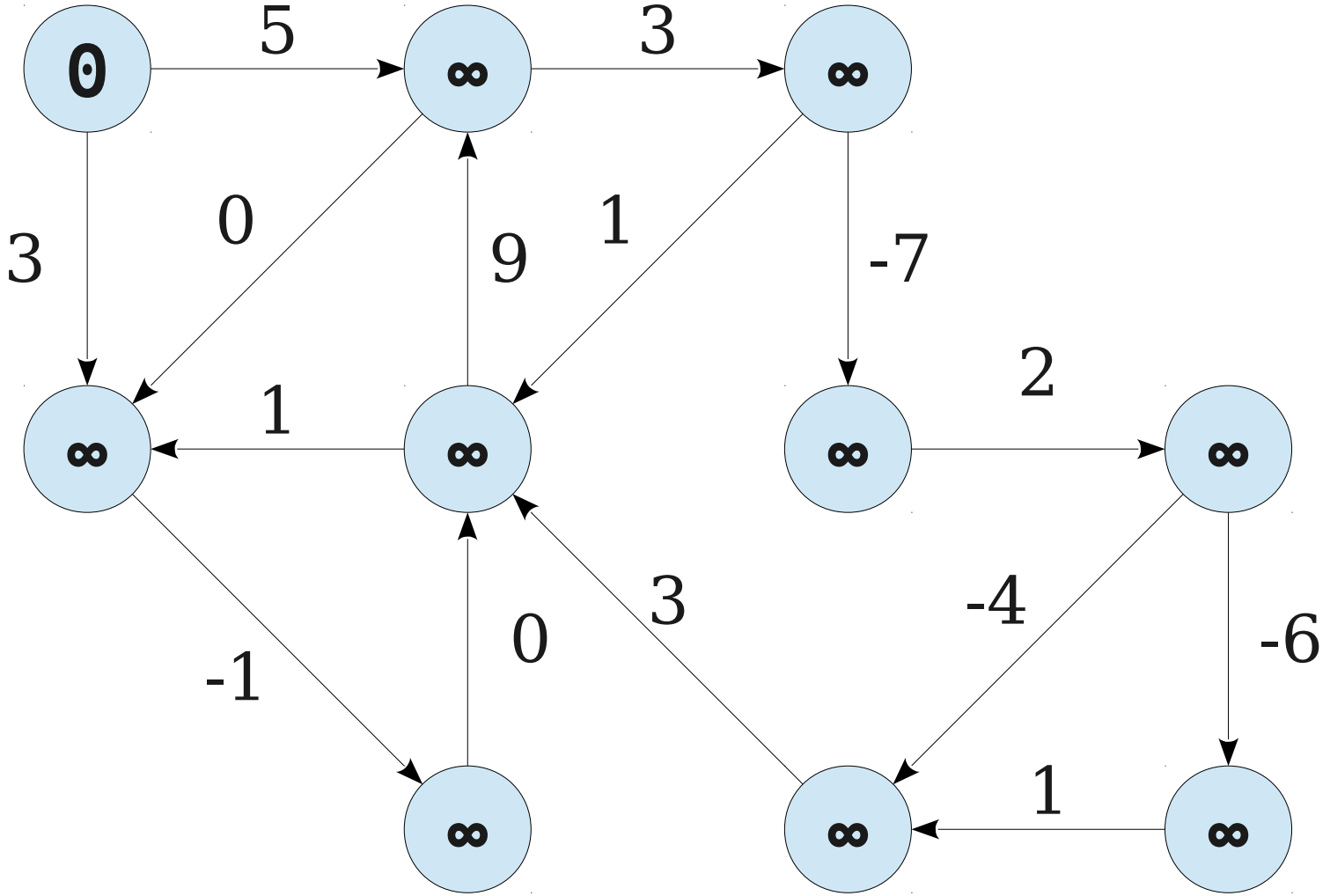
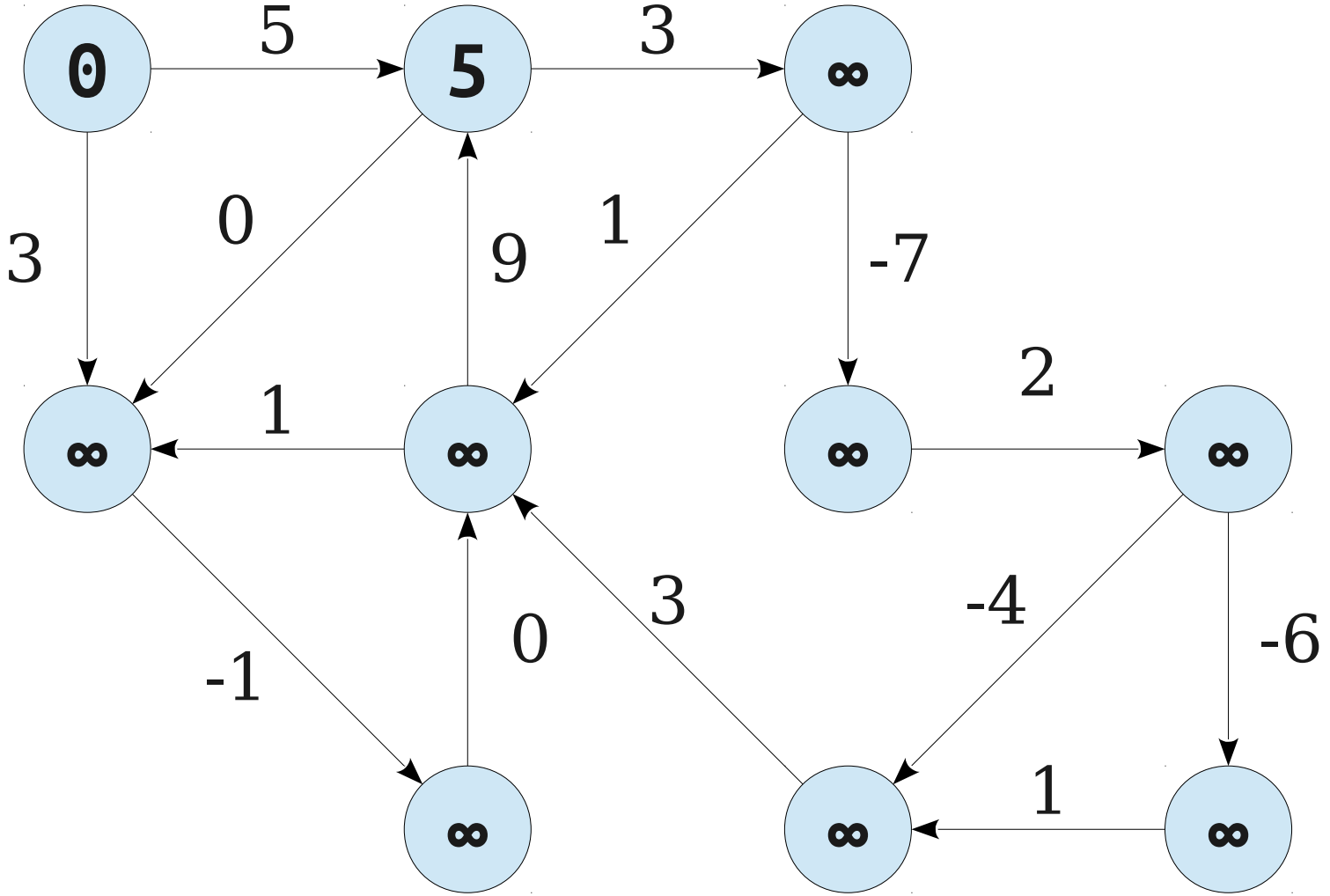


Consider paths of length **1**.



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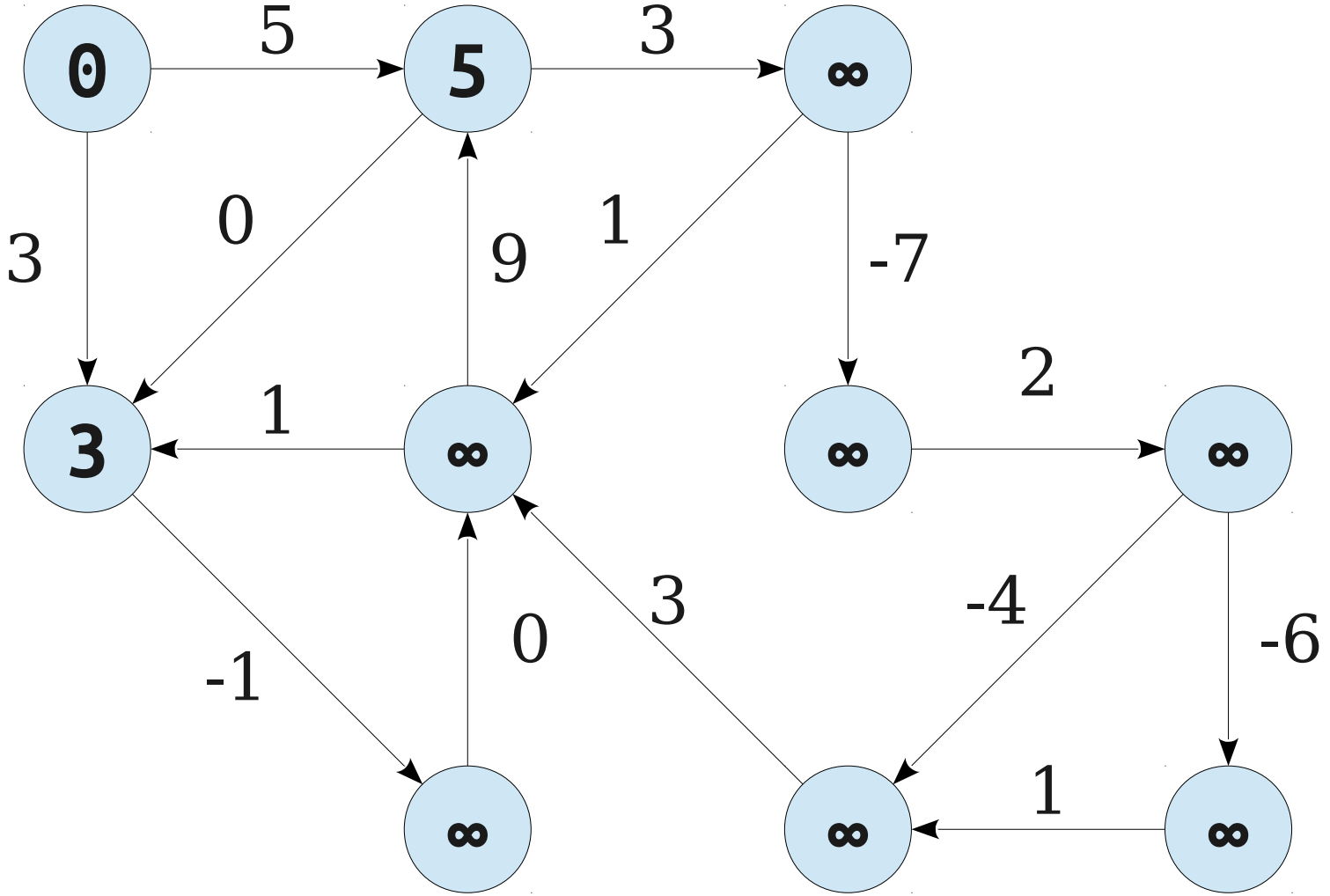

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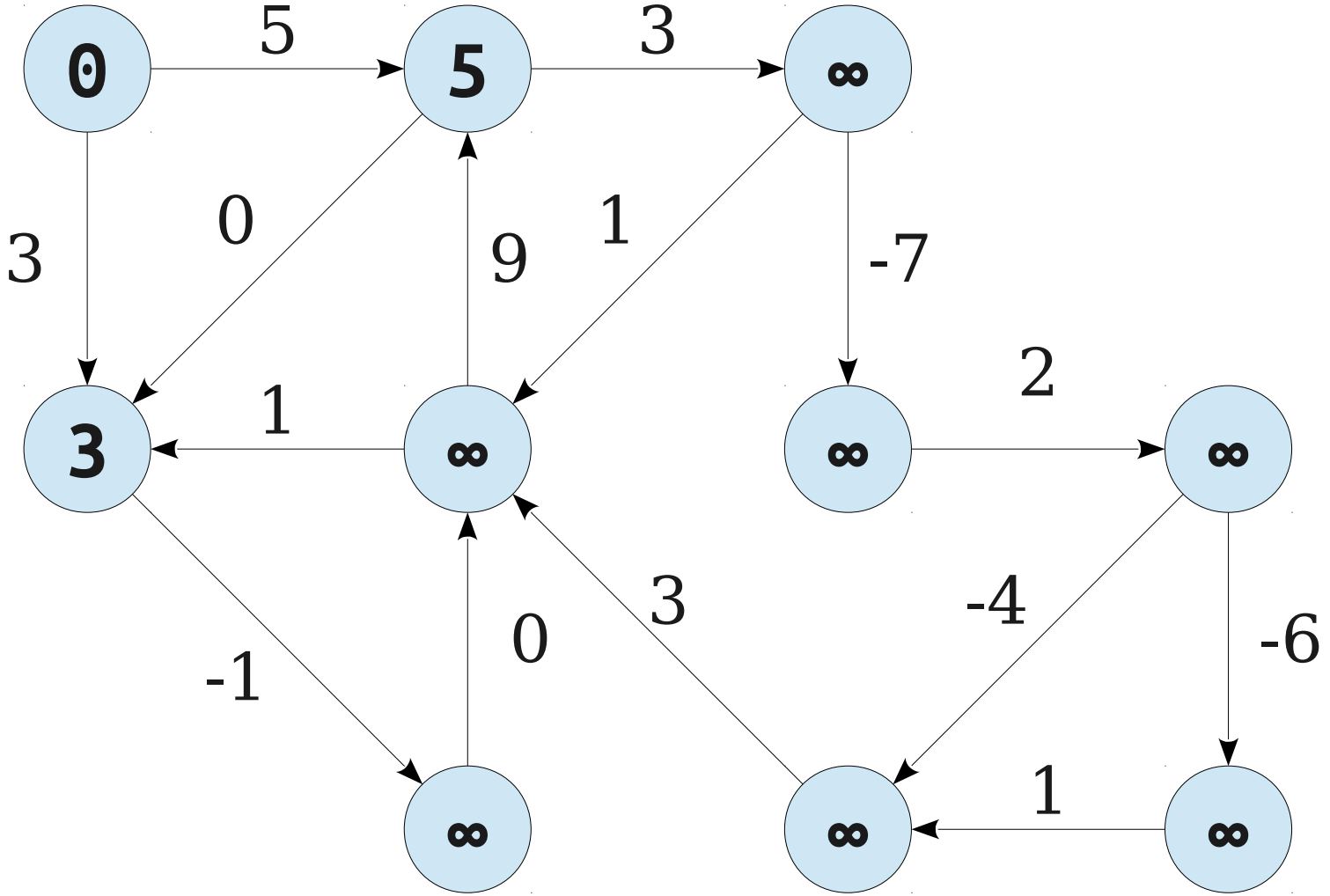
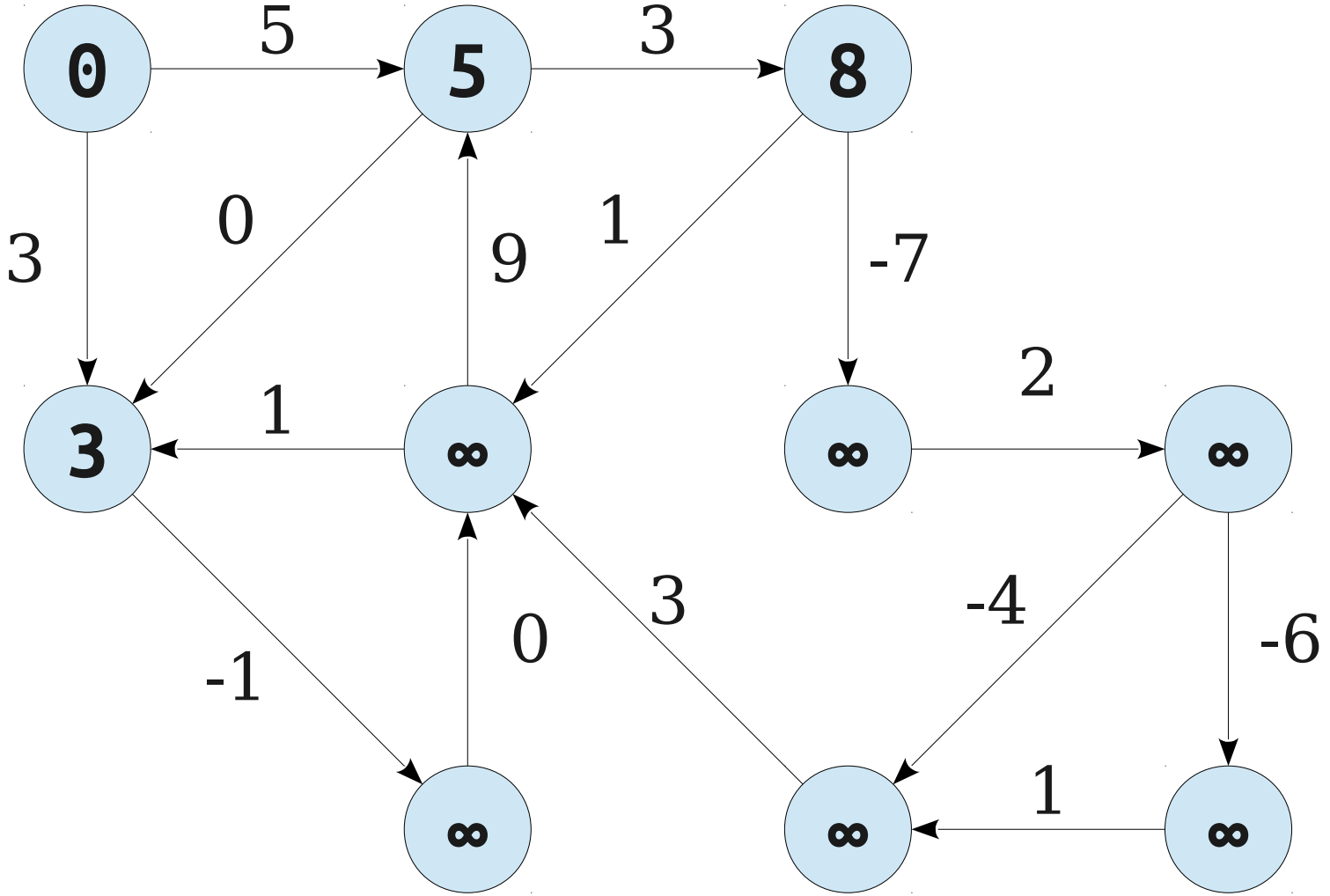


Consider paths of length **2**.



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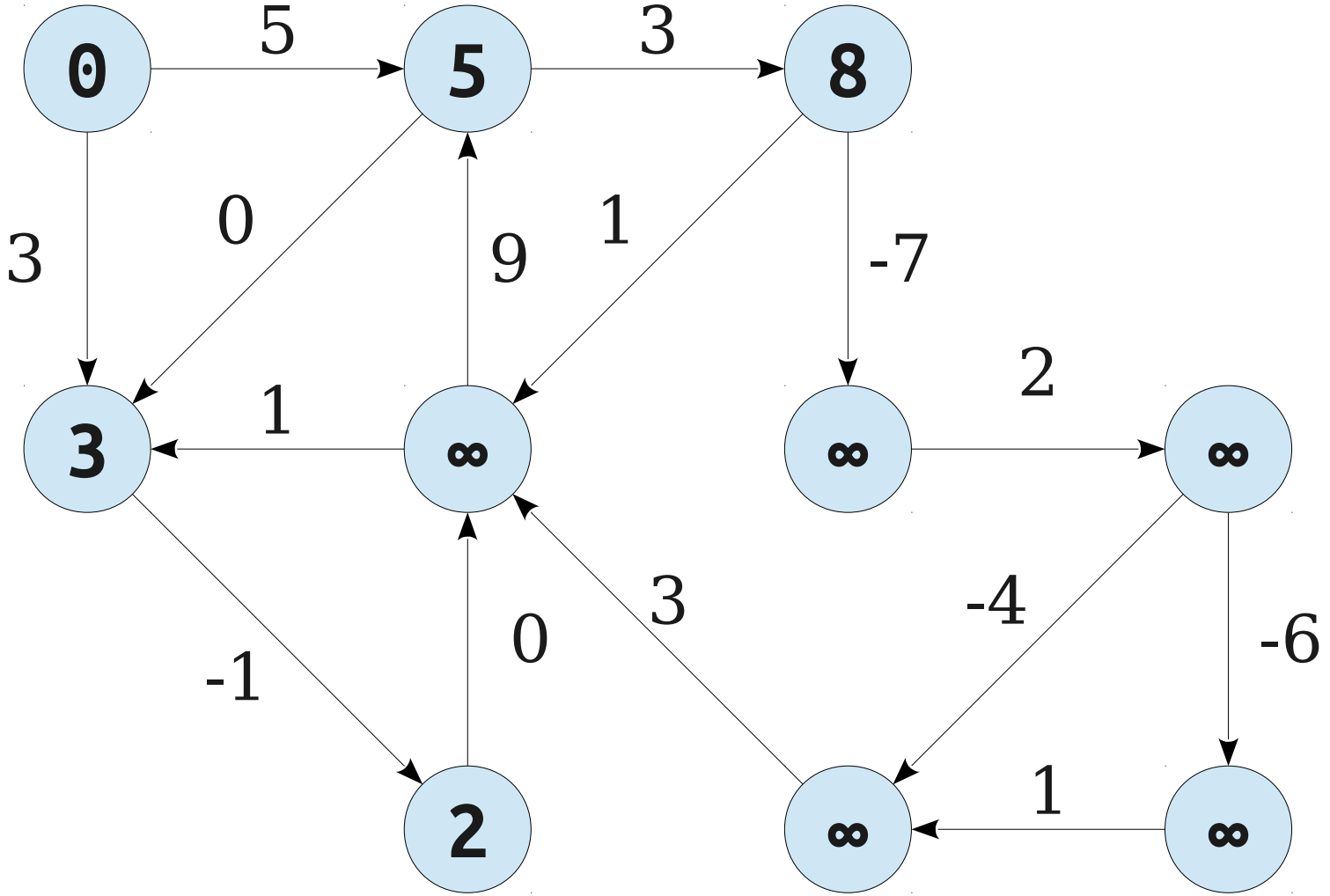

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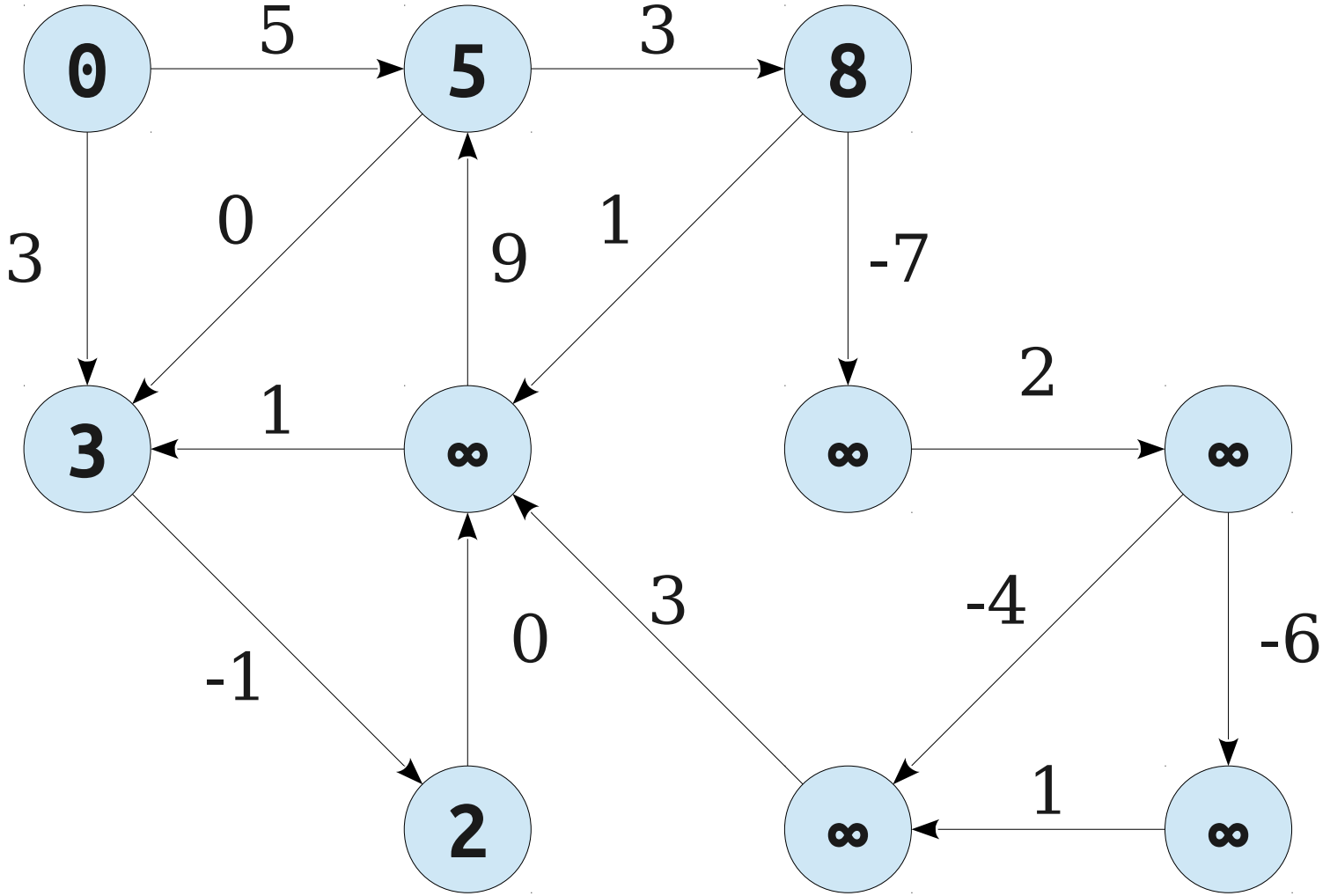
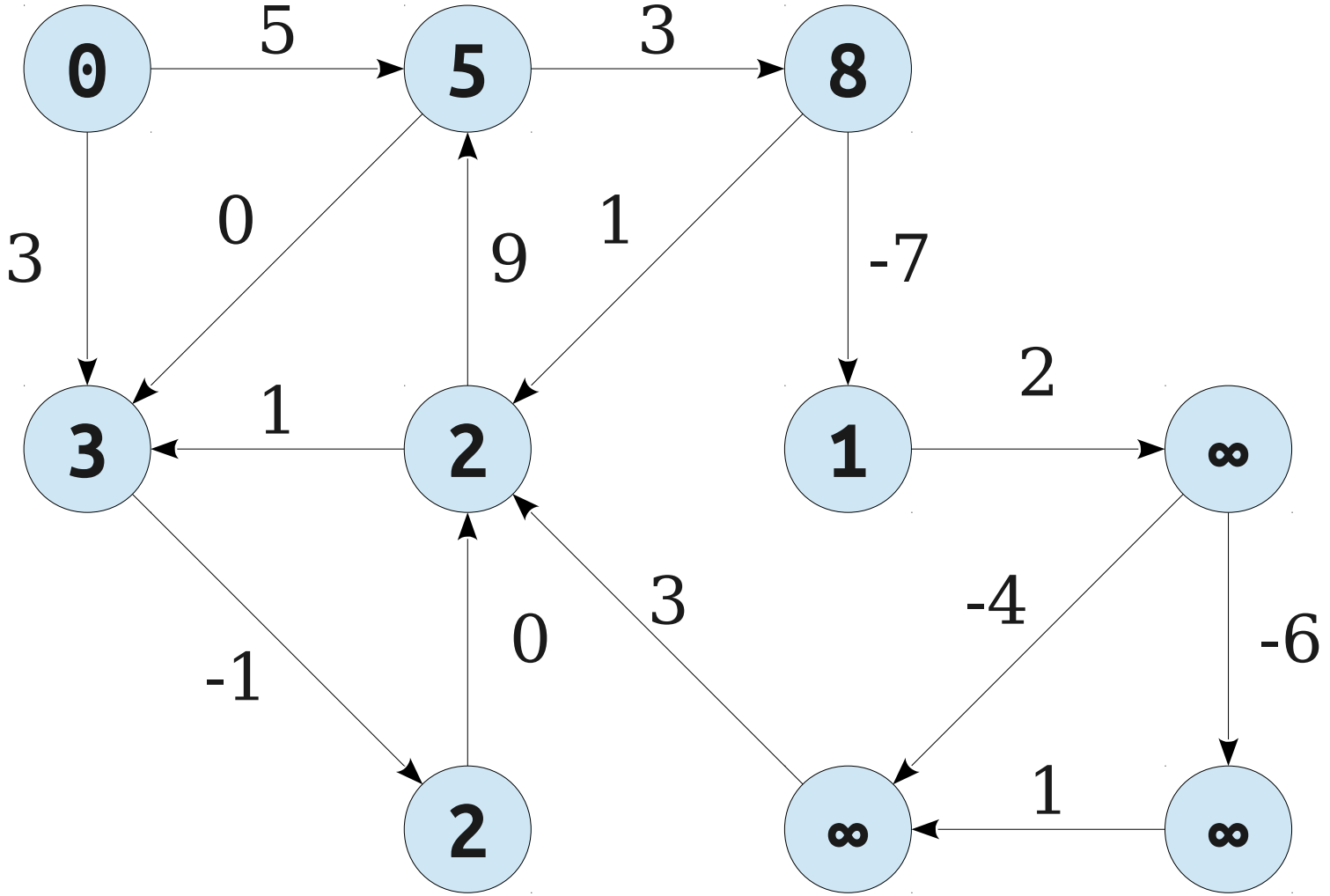
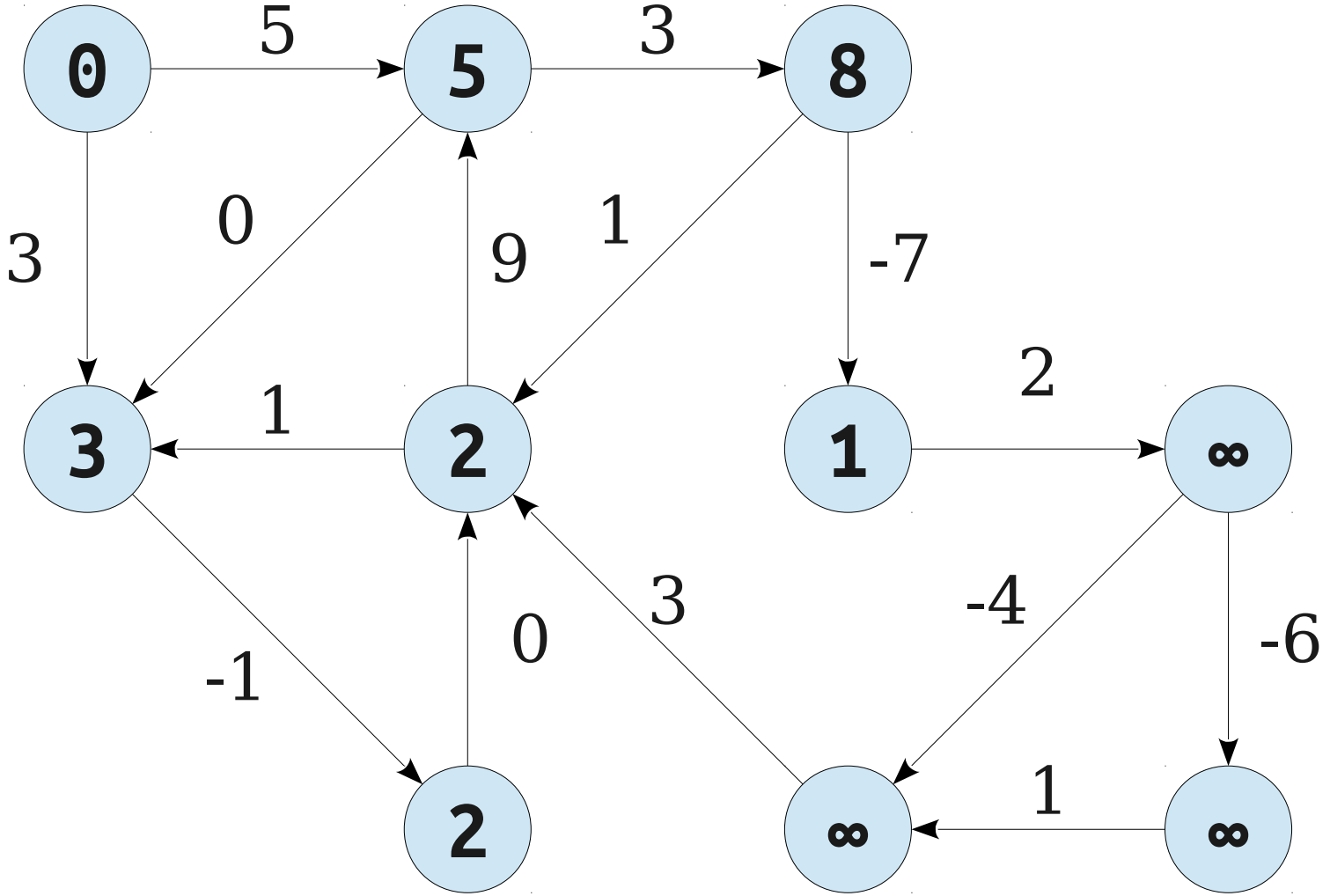


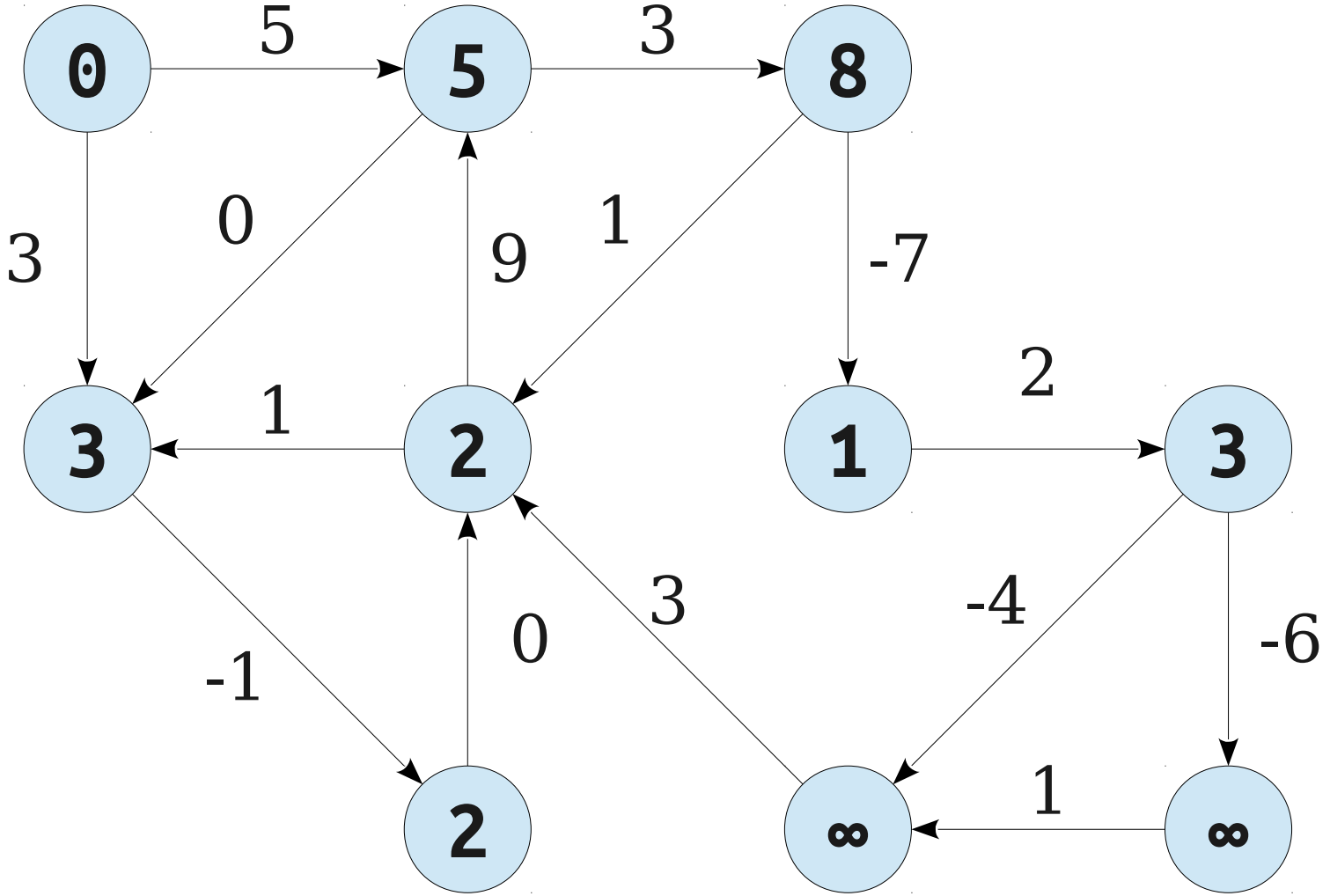
Consider paths of length **3**.



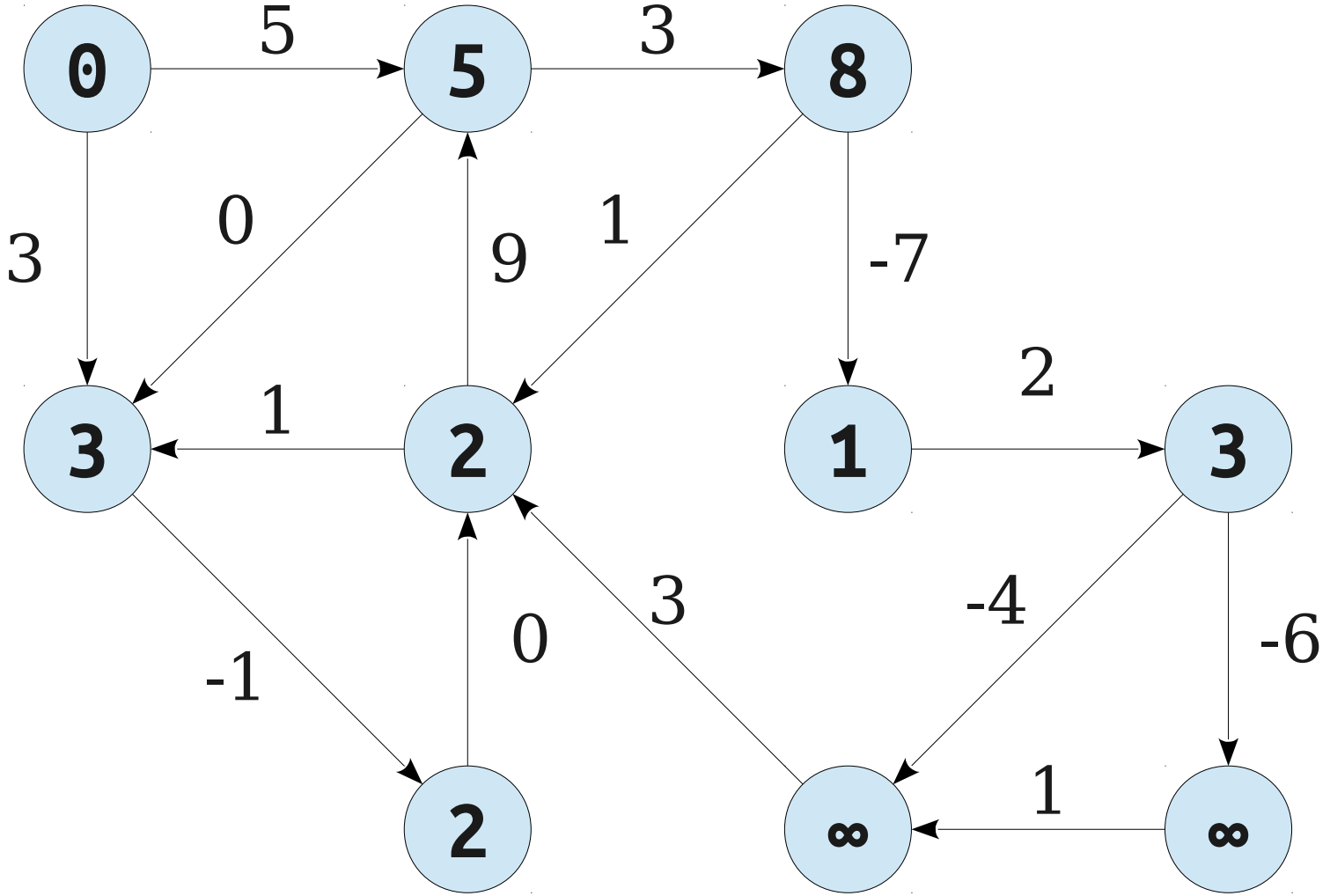
Consider paths of length **3**.



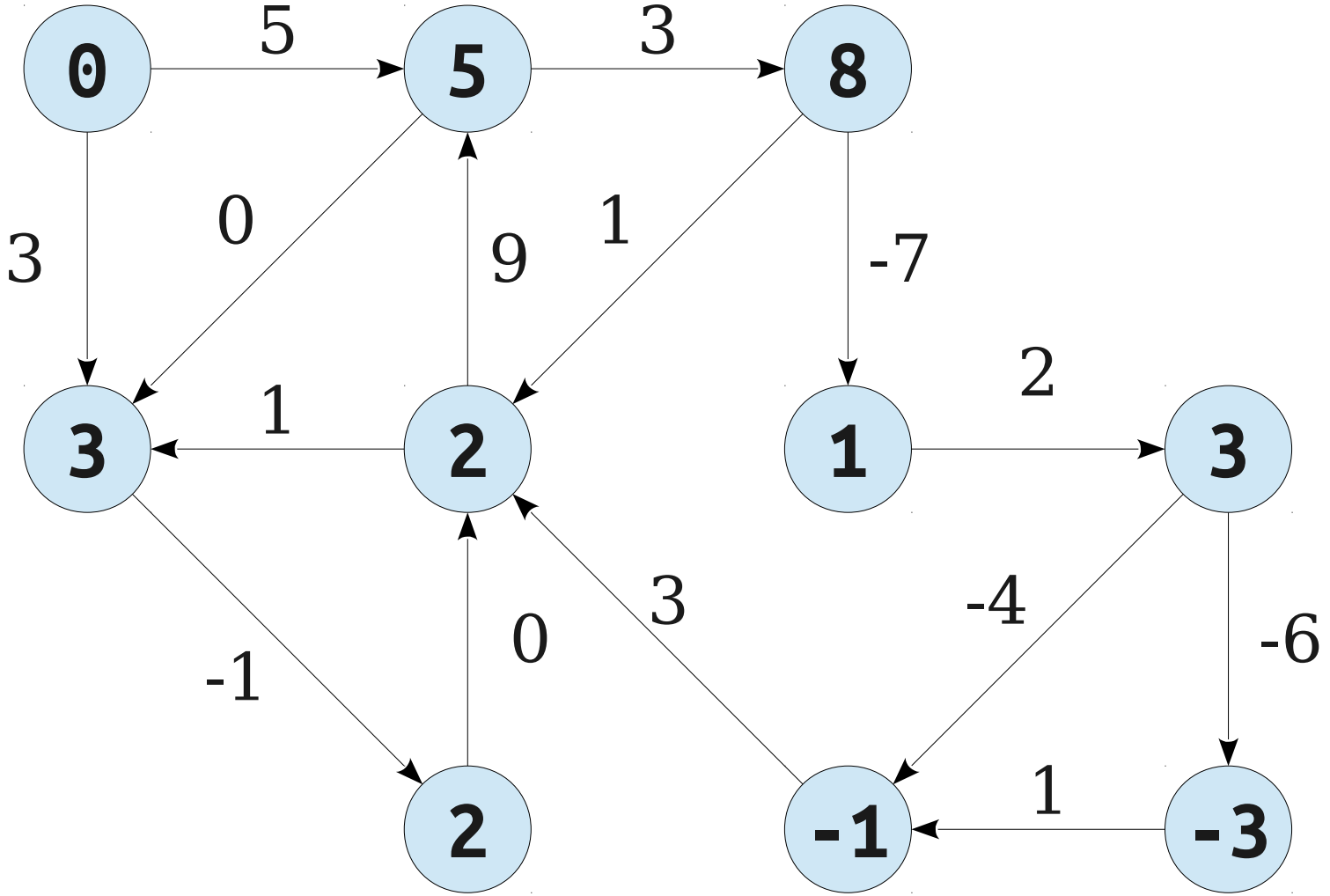
Consider paths of length **4**.



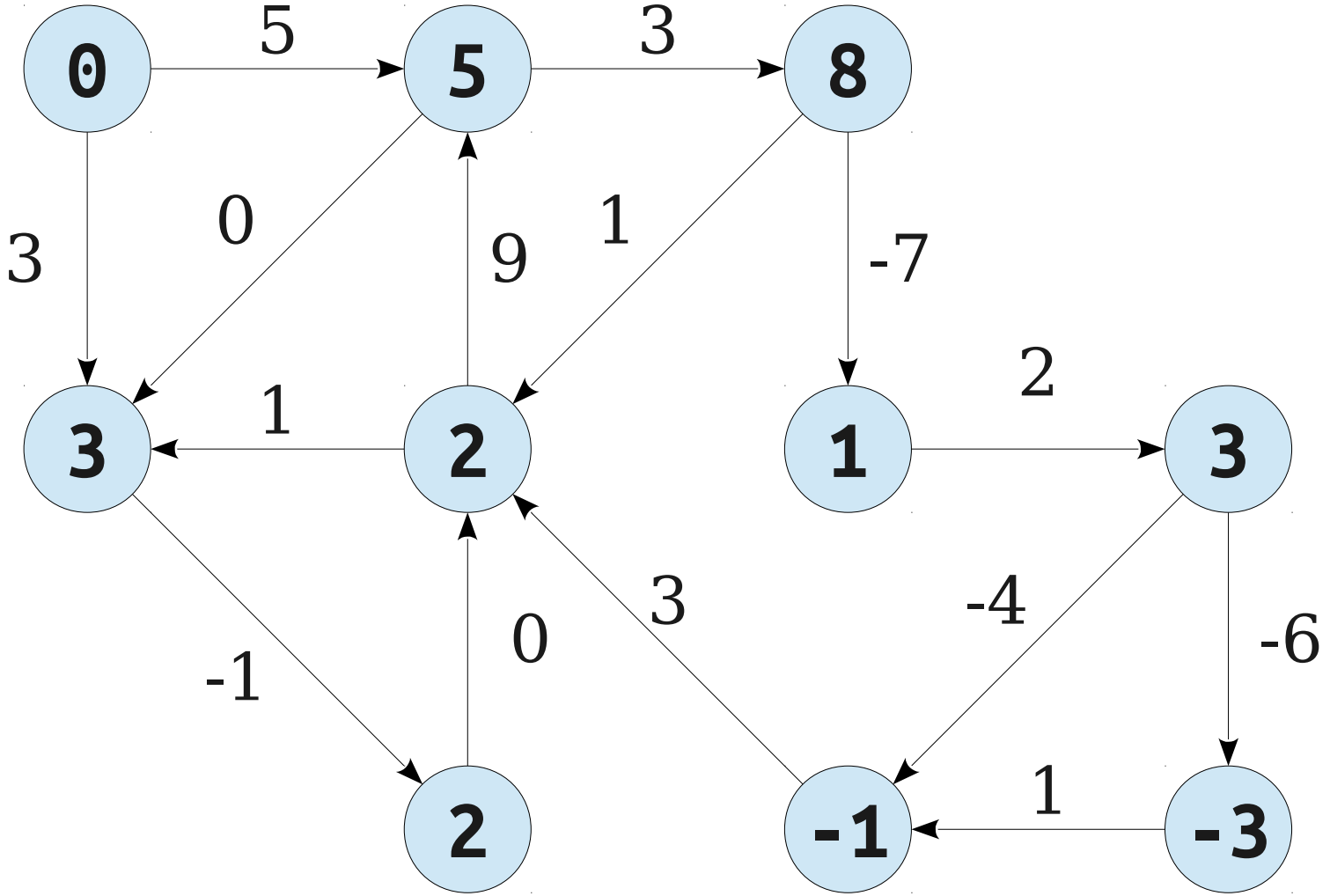
Consider paths of length **4**.



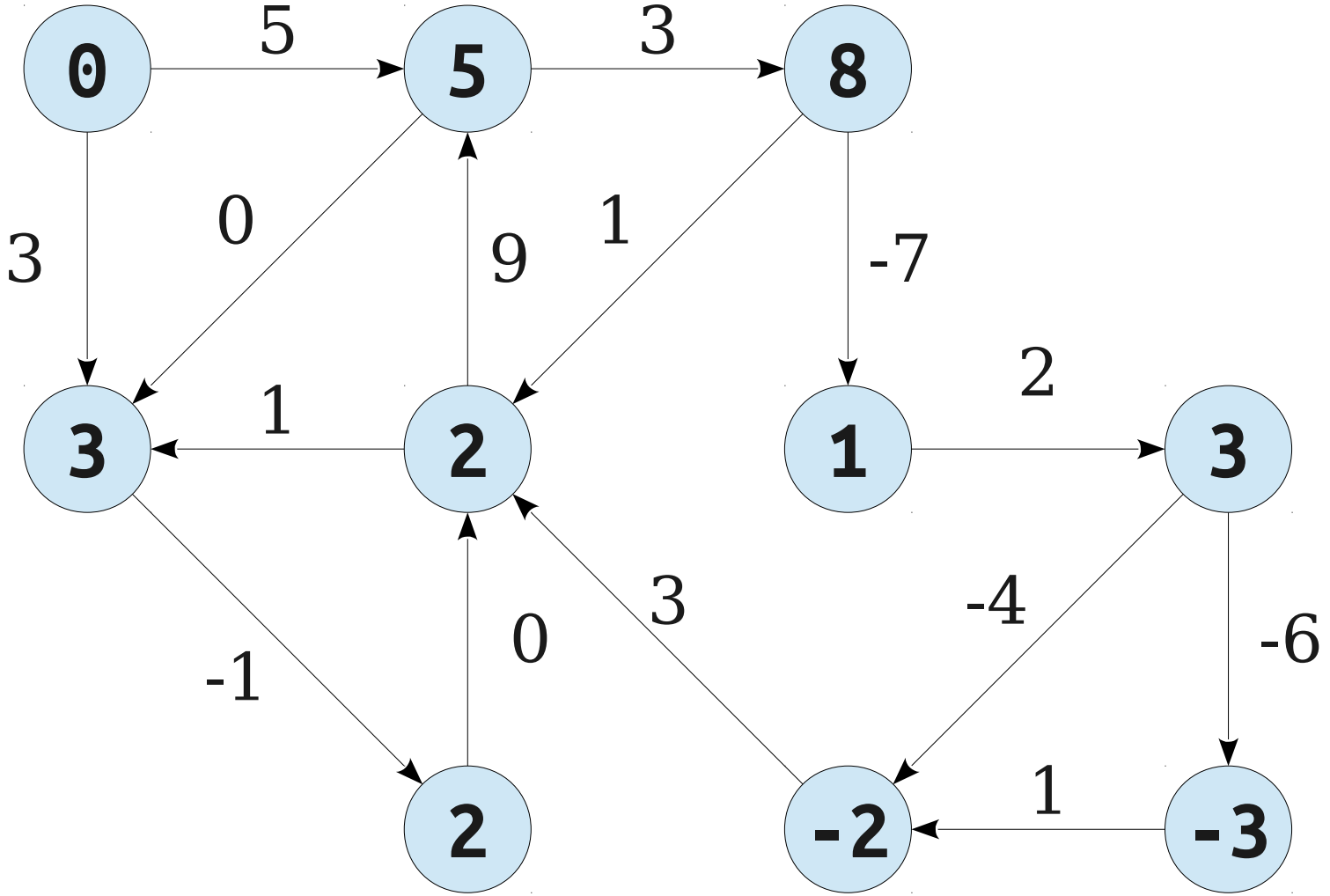
Consider paths of length **5**.



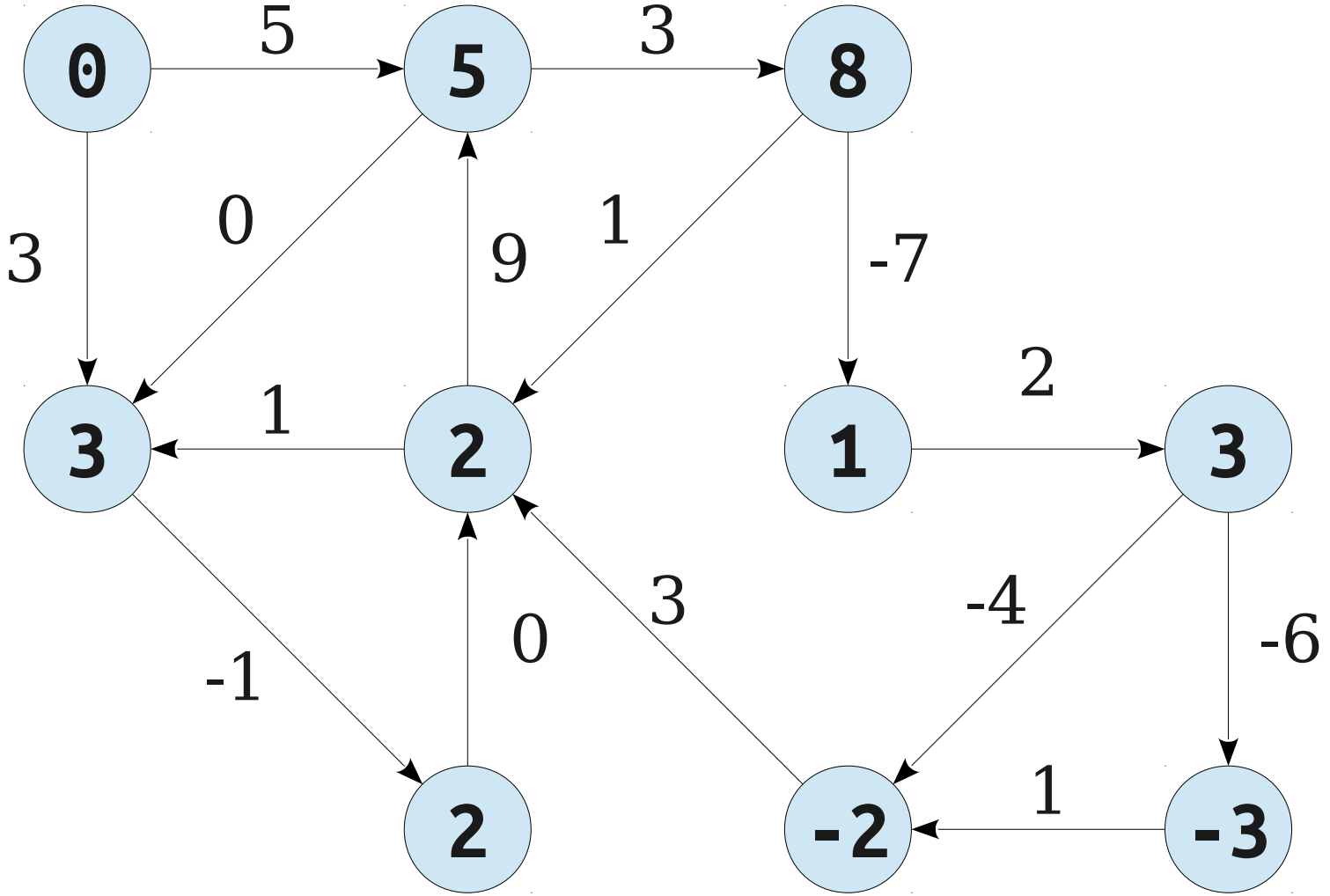
Consider paths of length **5**.



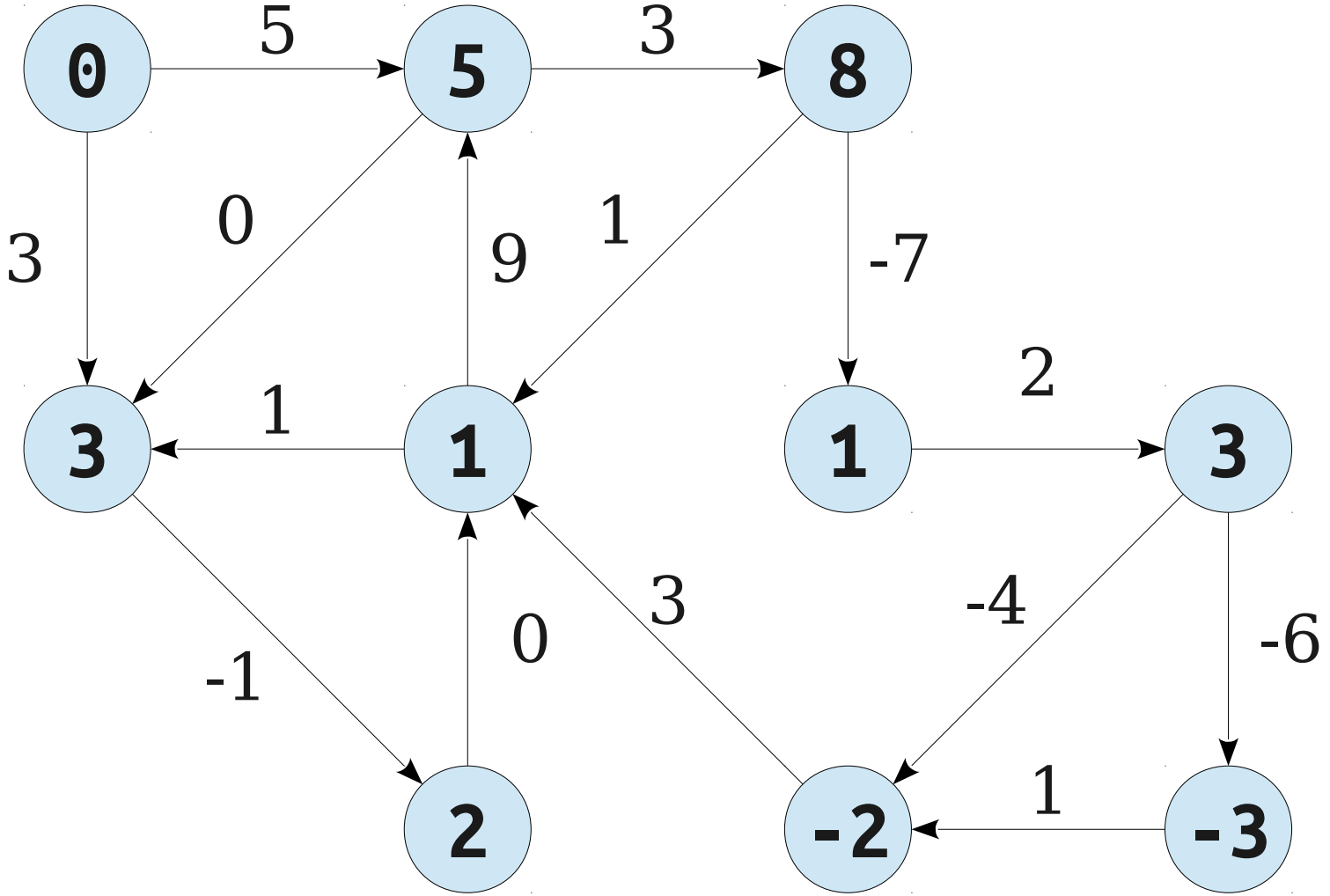
Consider paths of length **6**.



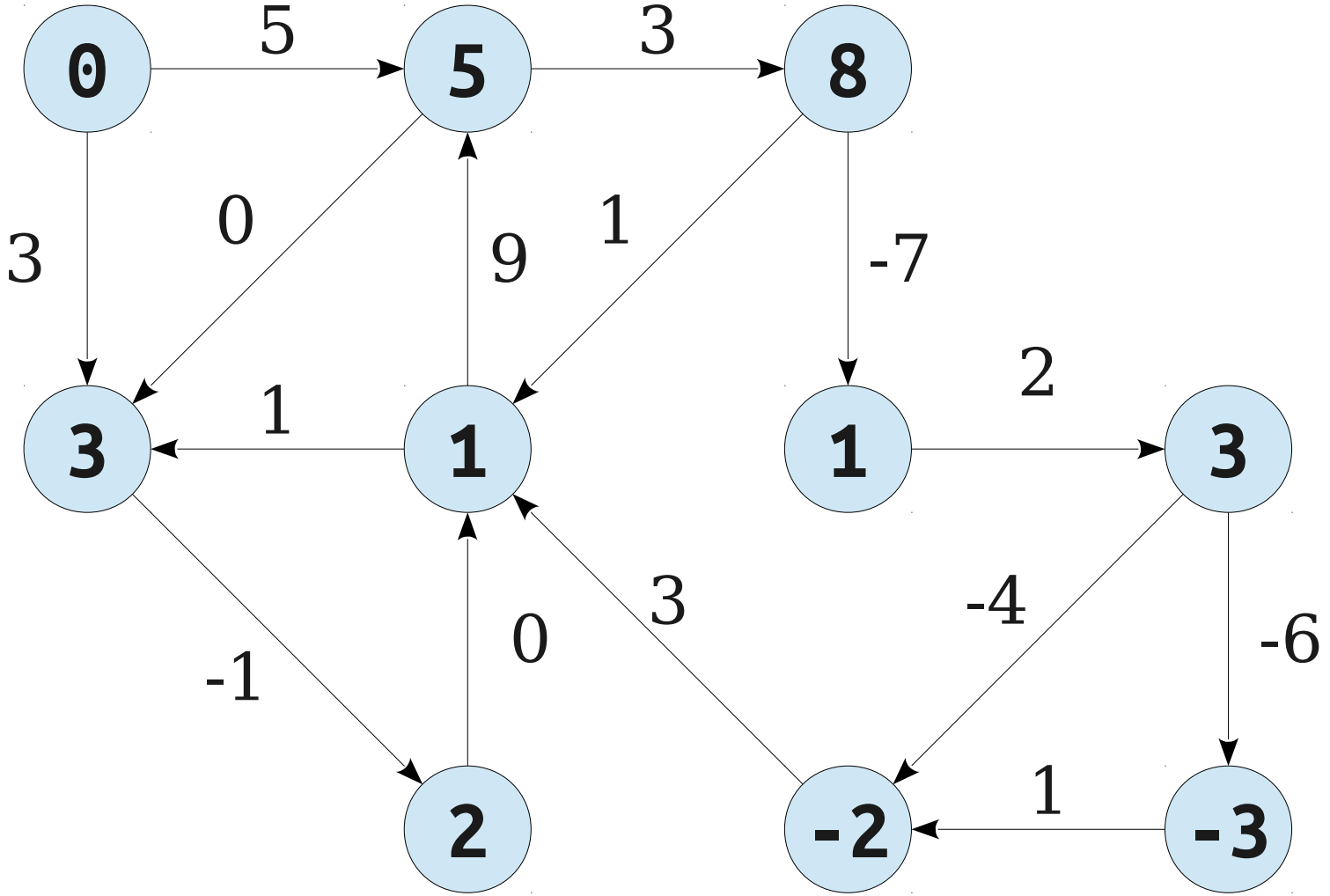
Consider paths of length **6**.



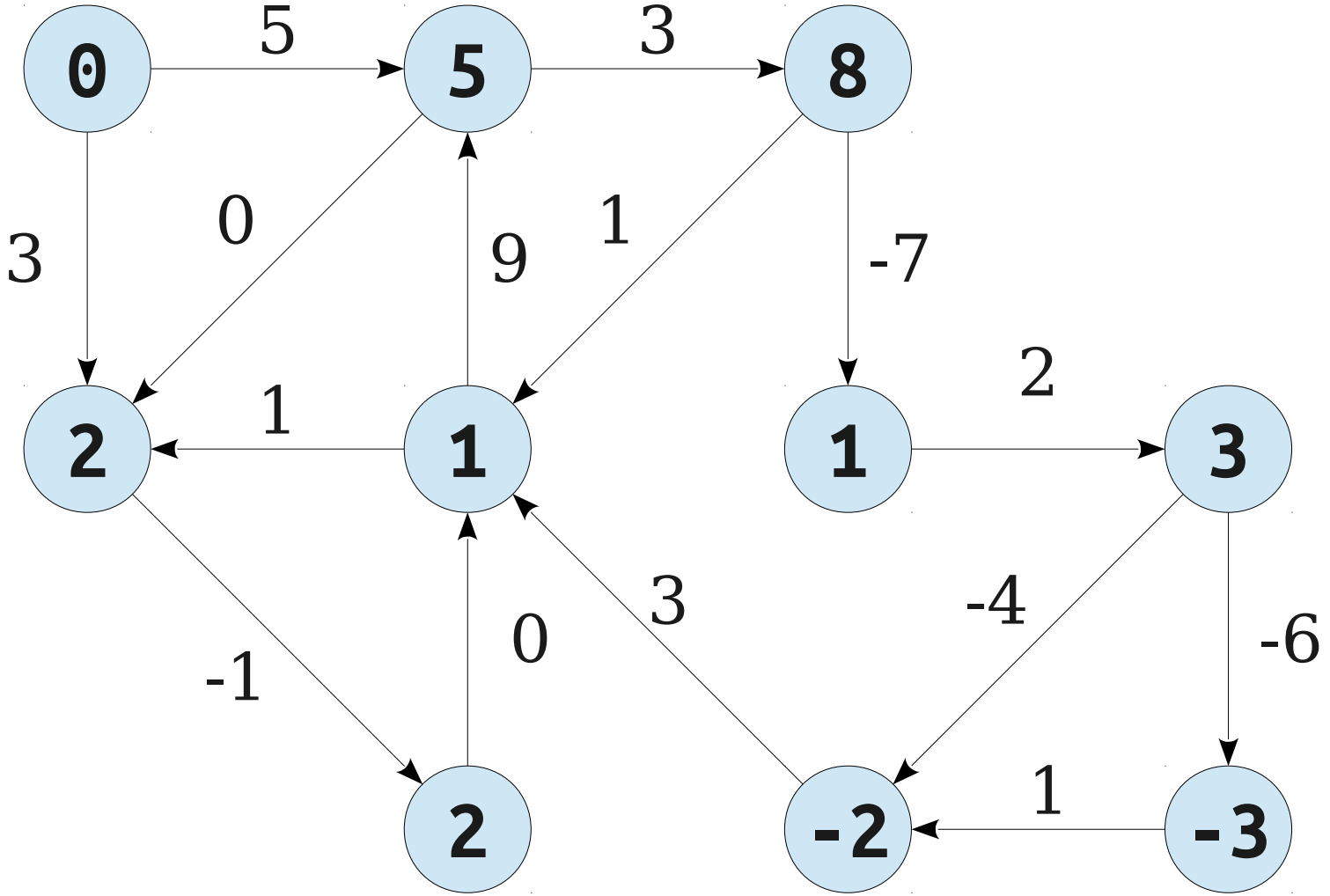
Consider paths of length 7.



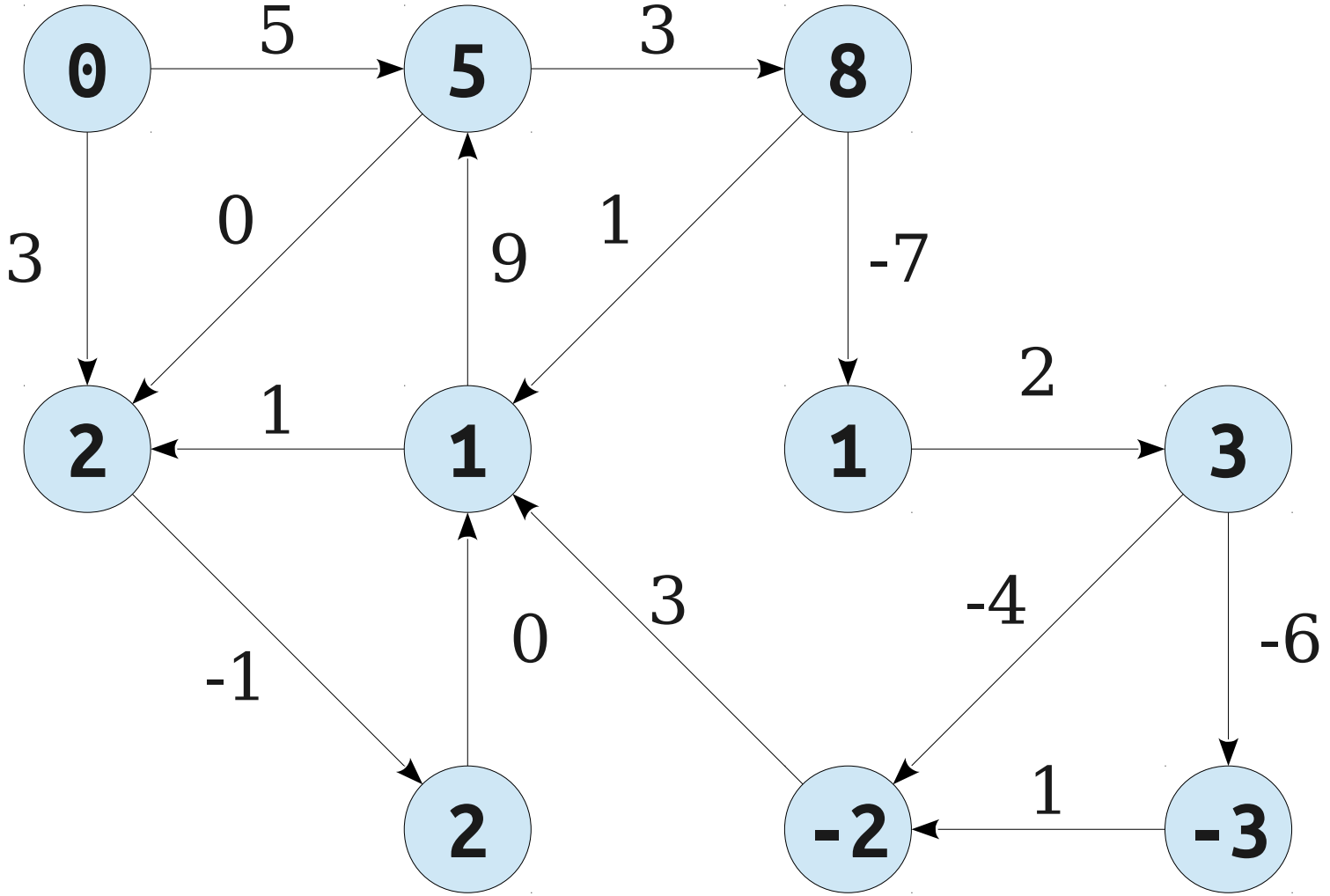
Consider paths of length 7.



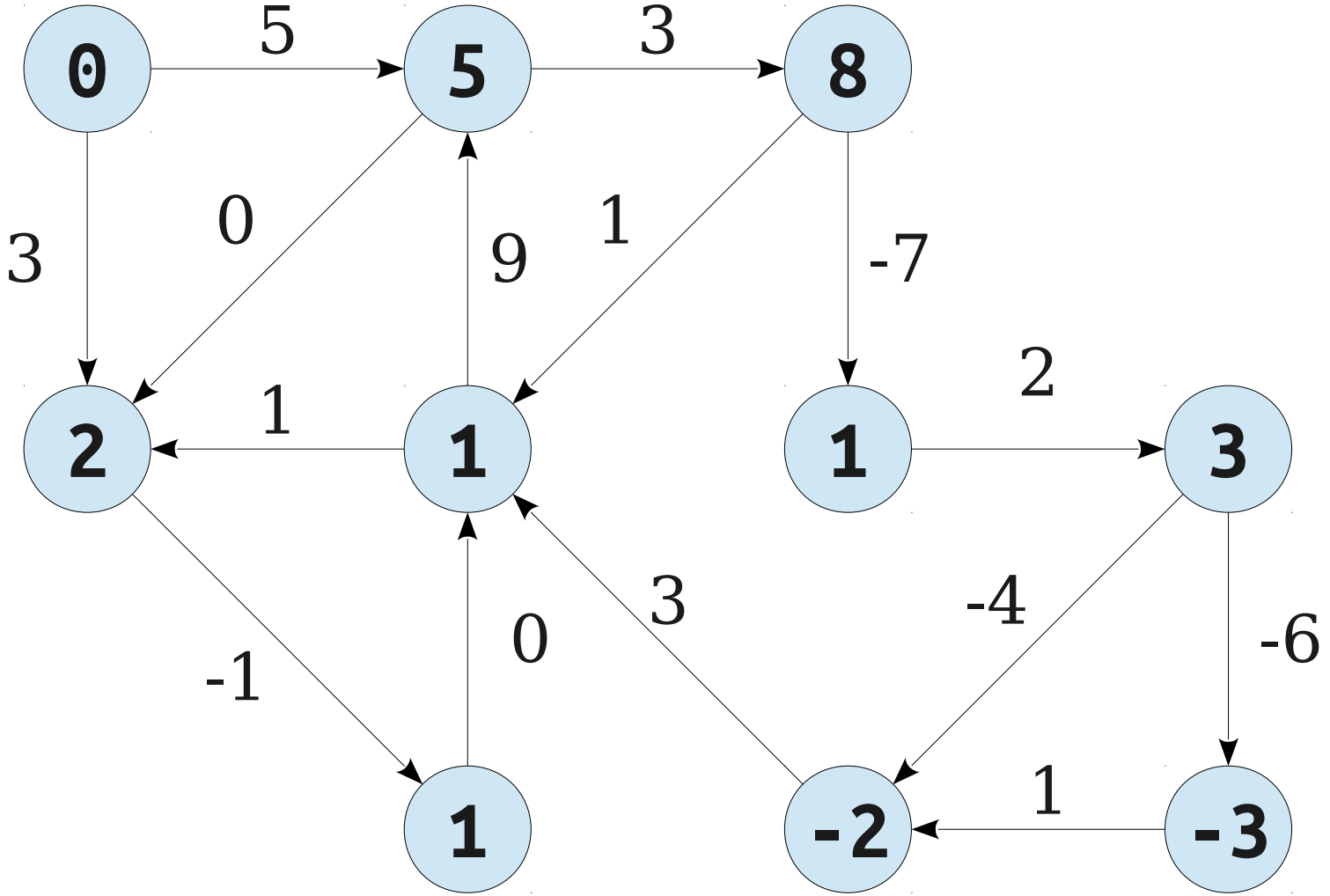
Consider paths of length **8**.



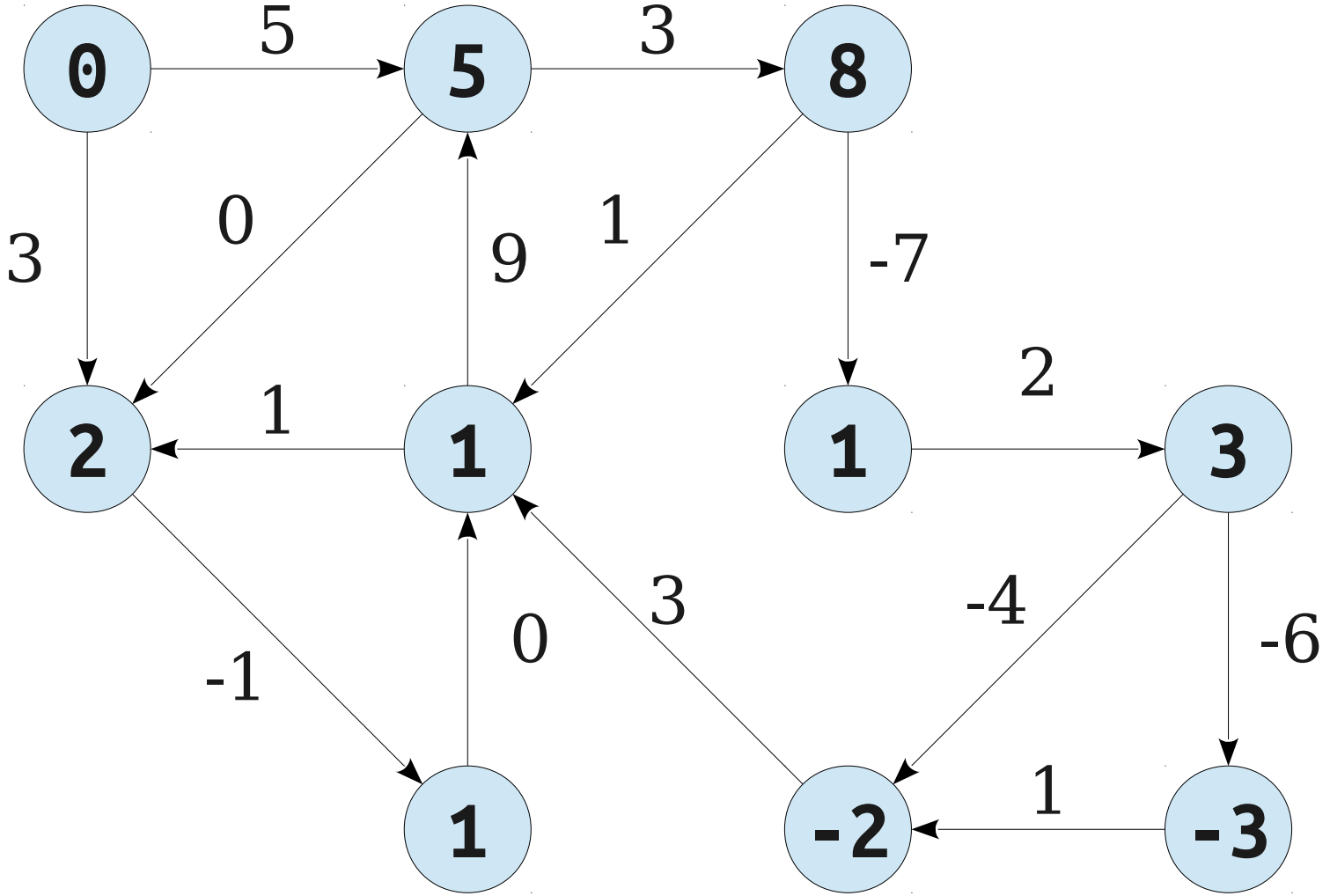
Consider paths of length **8**.



Consider paths of length **9**.



Consider paths of length **9**.



Consider paths of length **10+**.

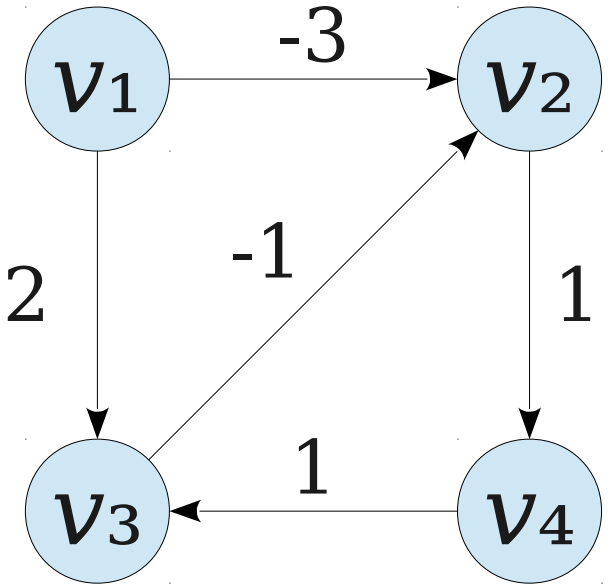
The Recurrence

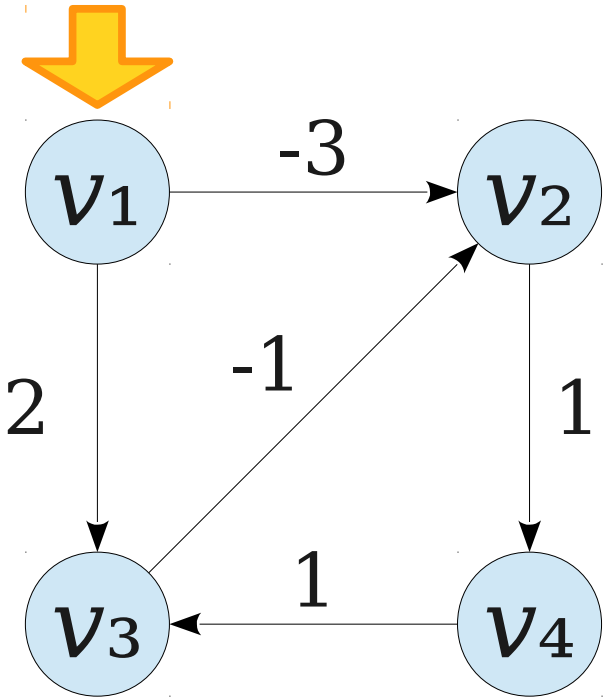
- **Idea:** Find paths of lengths at most 0, 1, 2, …, n.
- Let $w(u, v)$ denote the weight of edge (u, v).
- Let s be our start node. Let OPT(v, i) be the length of the shortest s – v path whose length is at most i, or ∞ if no path exists.
- **Claim:** OPT(v, i) satisfies the following recurrence:

$$\text{OPT}(v,i)=\begin{cases} 0 & \text{if } i=0 \text{ and } v=s \\ \infty & \text{if } i=0 \text{ and } v\neq s \\ \min\left\{ \begin{array}{l} \text{OPT}(v,i-1), \\ \min\limits_{(u,v)\in E}\{\text{OPT}(u,i-1)+w(u,v)\} \end{array} \right\} & \text{otherwise} \end{cases}$$

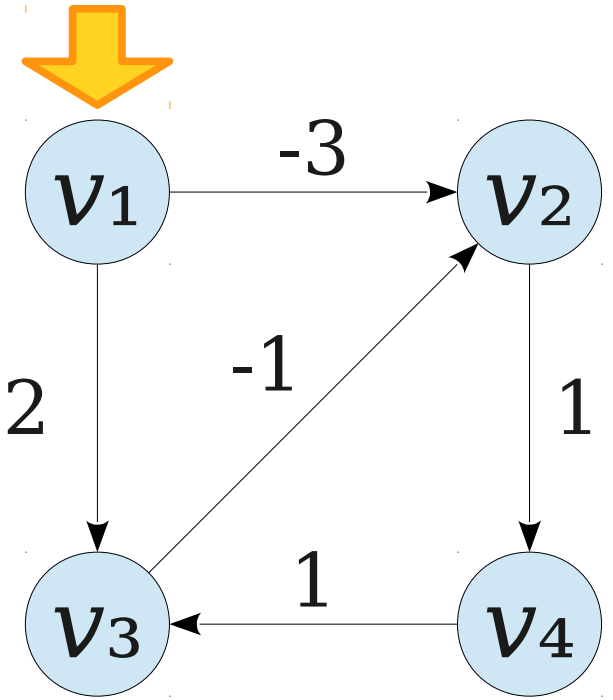
The Bellman-Ford Algorithm

- The **Bellman-Ford algorithm** evaluates this recurrence bottom-up:
  - Create a table DP of size $n \times n$.
  - Set DP[v][0] = ∞ for all $v \neq s$.
  - Set DP[s][0] = 0
  - For $i = 1$ to $n - 1$, for all $v \in V$:
    - Set DP[v][i] =
      min {
        DP[v][$i - 1$],
        min { DP[u][$i - 1$] + $w(u, v)$ } (where $(u, v) \in E$)
      }
  - Return row n of DP.

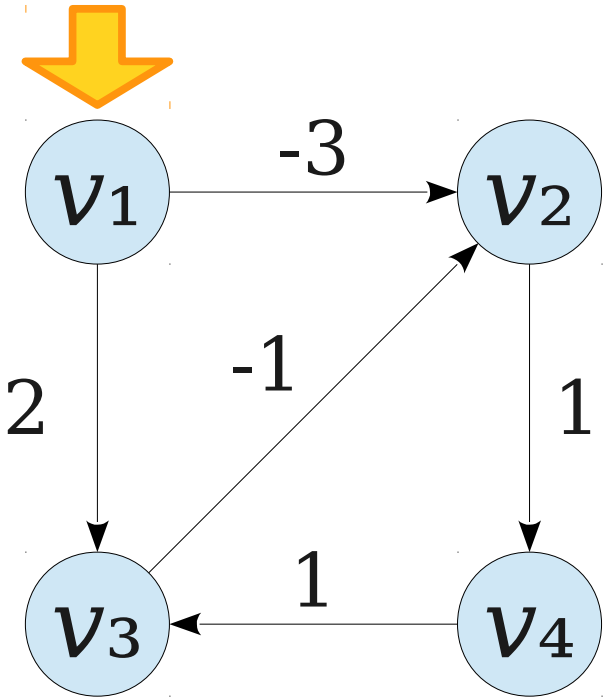

v_1
v_2
v_3
v_4
-3
2
-1
1
1


v_1
v_2
v_3
v_4
-3
2
-1
1
1

	v_1	v_2	v_3	v_4
3				
2				
1				
0				

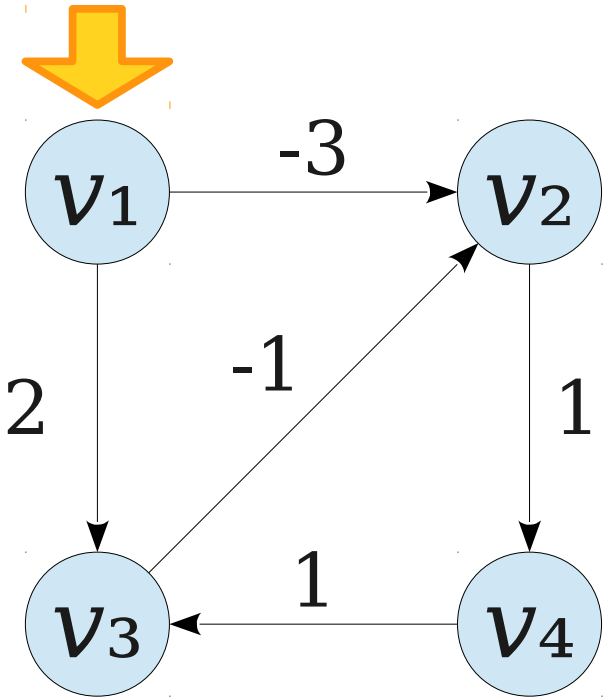


	v_1	v_2	v_3	v_4
3				
2				
1				
0				



$$\mathrm{OPT}(v,i)=\begin{cases} 0 & \text{if } i=0 \text{ and } v=s \\ \infty & \text{if } i=0 \text{ and } v\neq s \\ min\left\{ \begin{array}{l} \mathrm{OPT}(v,i-1), \\ \underset{(u,v)\in E}{min}\left\{\mathrm{OPT}(u,i-1)+w(u,v)\right\} \end{array} \right. & \text{otherwise} \end{cases}$$

	v_1	v_2	v_3	v_4
3				
2				
1				
0	**0**	∞	∞	∞



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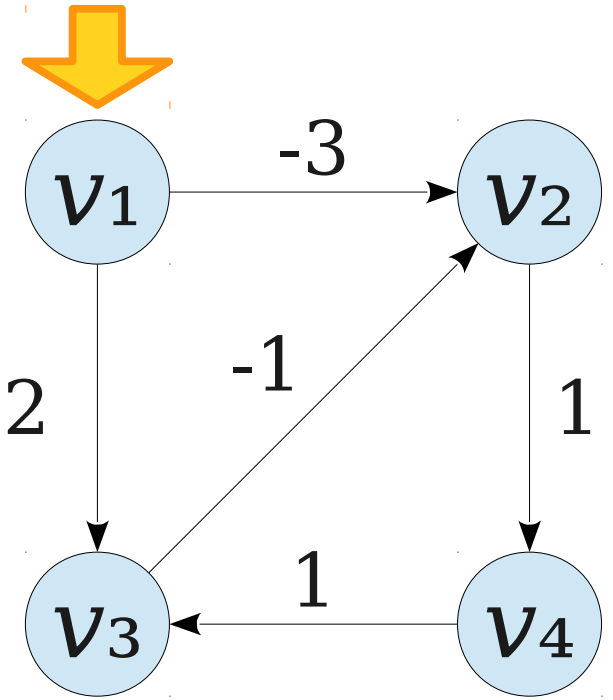
	v_1	v_2	v_3	v_4
3				
2				
1				
0	**0**	∞	∞	∞



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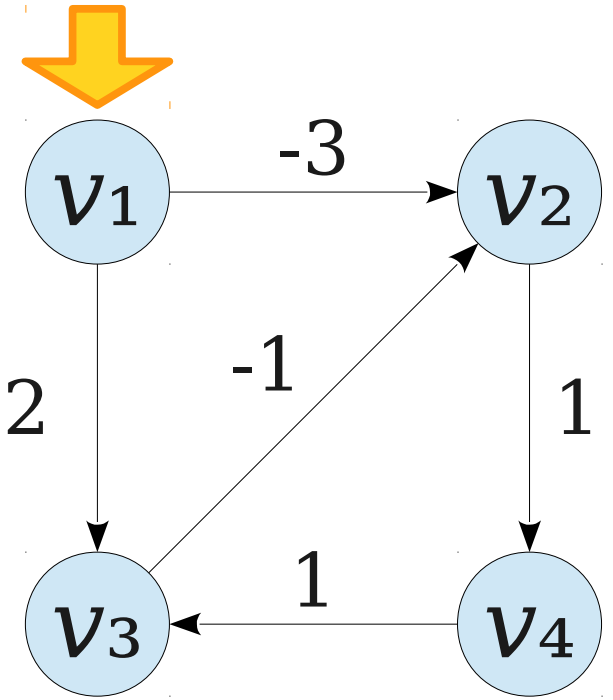
	v_1	v_2	v_3	v_4
3				
2				
1	**0**			
0	**0**	∞	∞	∞



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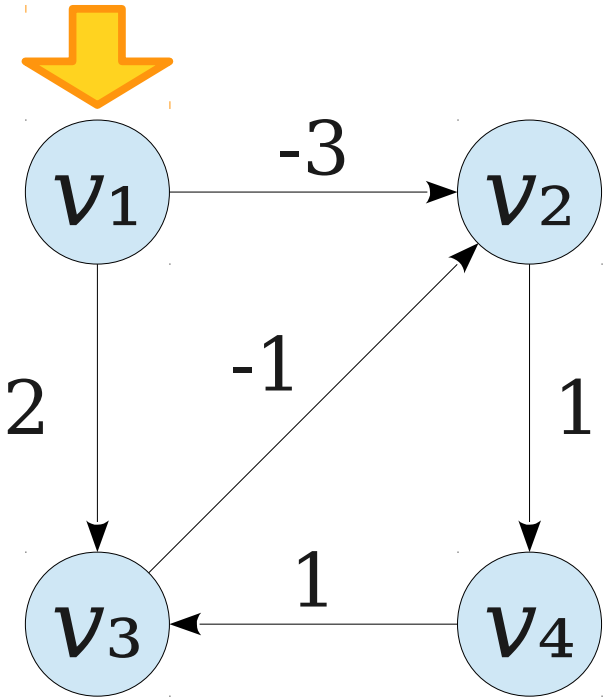
	v_1	v_2	v_3	v_4
3				
2				
1	**0**	**-3**		
0	**0**	∞	∞	∞



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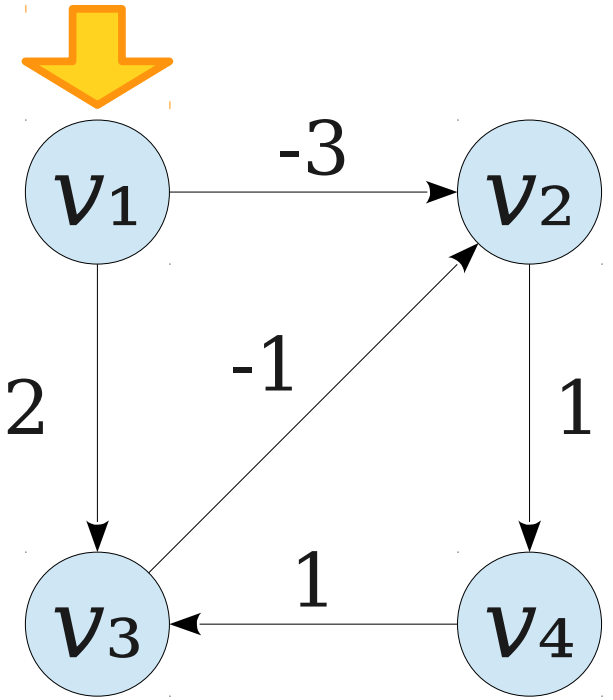
	v_1	v_2	v_3	v_4
3				
2				
1	**0**	**-3**	**2**	
0	**0**	∞	∞	∞



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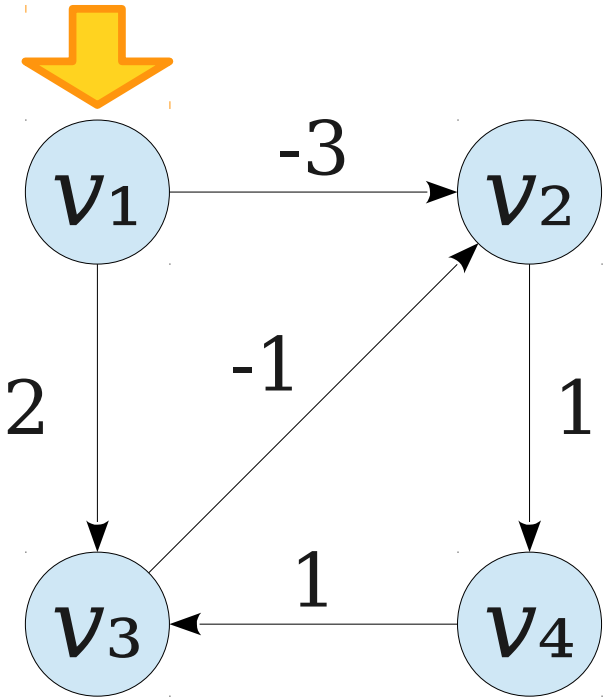
	v_1	v_2	v_3	v_4
3				
2				
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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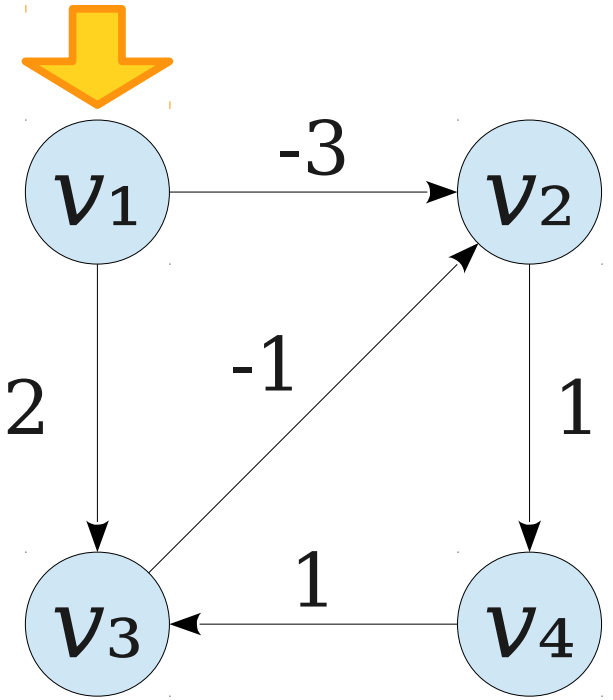
	v_1	v_2	v_3	v_4
3				
2				
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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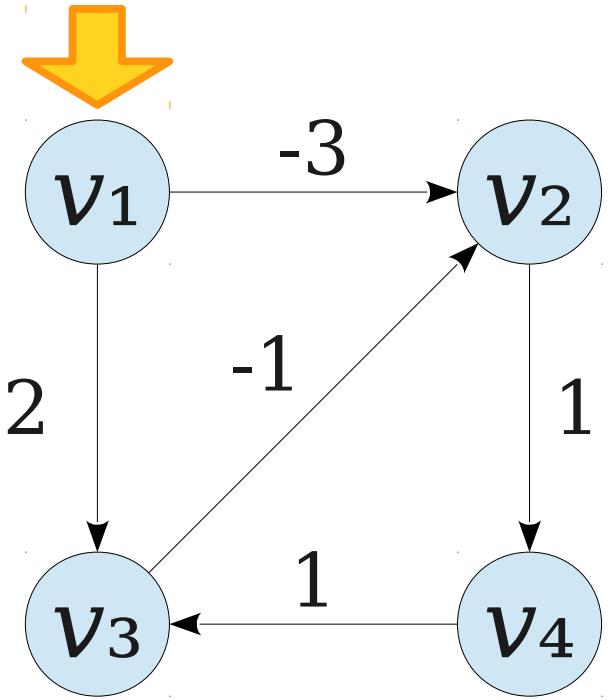
	v_1	v_2	v_3	v_4
3				
2	**0**			
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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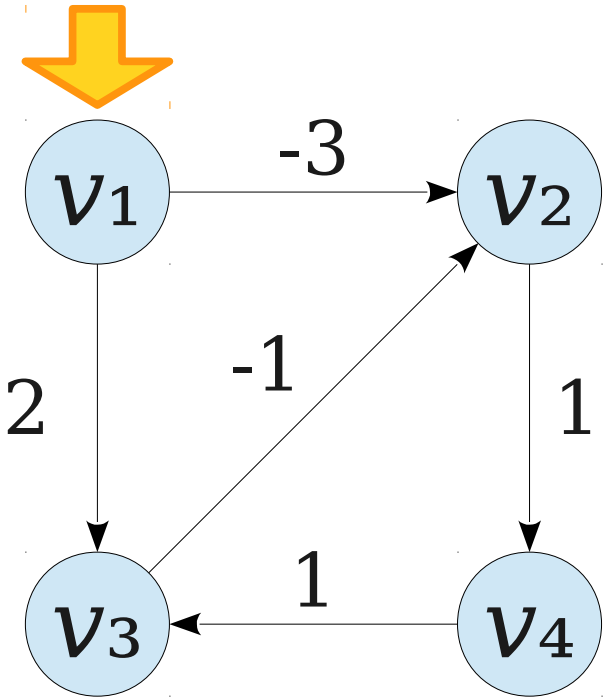
	v_1	v_2	v_3	v_4
3				
2	**0**	**-3**		
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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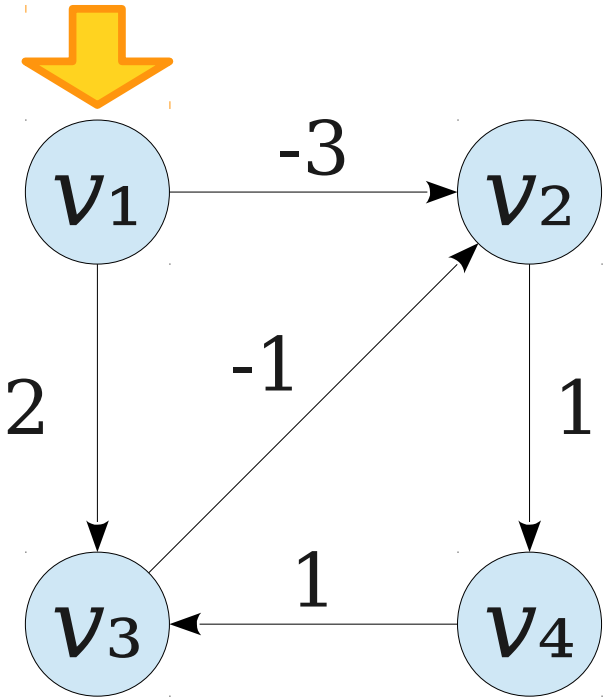
	v_1	v_2	v_3	v_4
3				
2	**0**	**-3**	**2**	
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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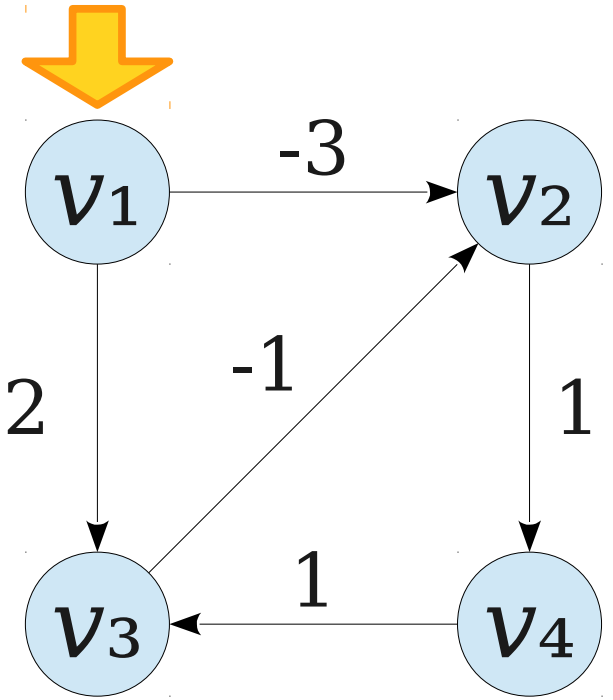
	v_1	v_2	v_3	v_4
3				
2	**0**	**-3**	**2**	**-2**
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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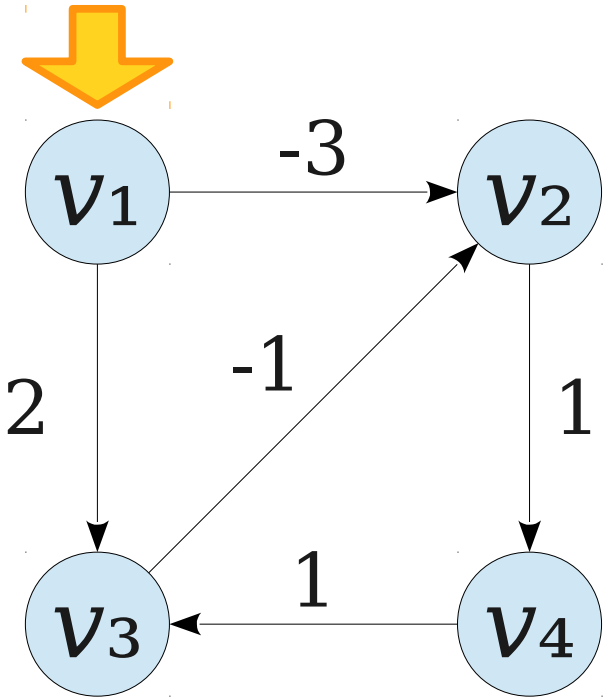
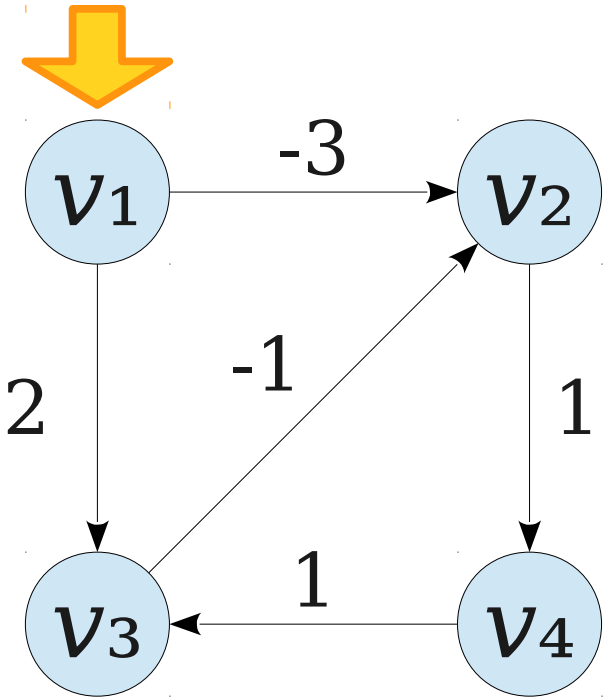
	v_1	v_2	v_3	v_4
3				
2	**0**	**-3**	**2**	**-2**
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



$$\mathrm{OPT}(v,i)=\begin{cases} 0 & \text{if } i=0 \text{ and } v=s \\ \infty & \text{if } i=0 \text{ and } v\neq s \\ min\left\{ \begin{array}{l} \mathrm{OPT}(v,i-1), \\ \underset{(u,v)\in E}{min}\left\{\mathrm{OPT}(u,i-1)+w(u,v)\right\} \end{array} \right. & \text{otherwise} \end{cases}$$

	v_1	v_2	v_3	v_4
3	**0**			
2	**0**	**-3**	**2**	**-2**
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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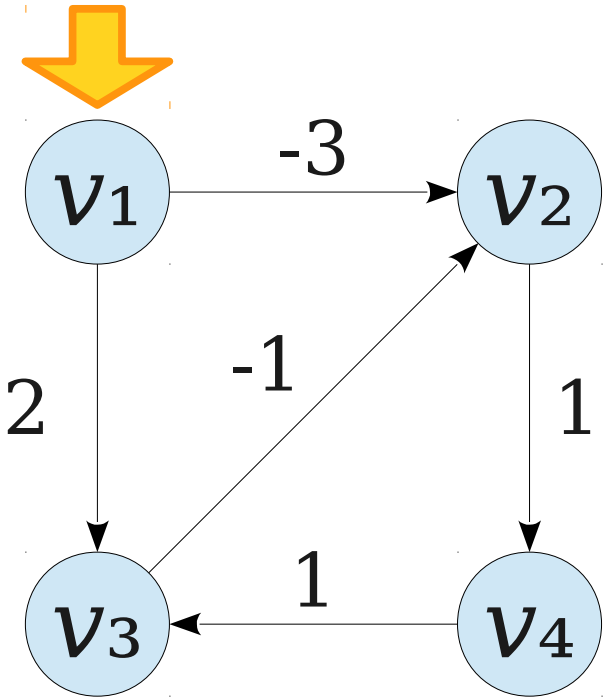
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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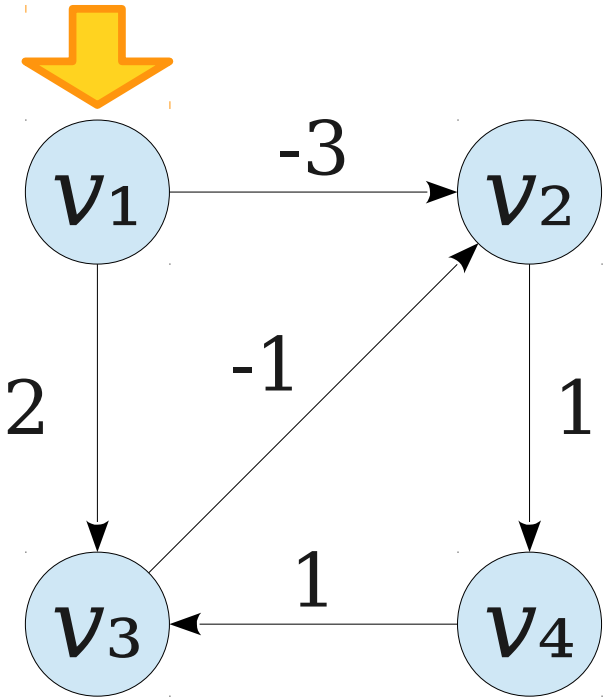
	v_1	v_2	v_3	v_4
3	**0**	**-3**	**-1**	
2	**0**	**-3**	**2**	**-2**
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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3	**0**	**-3**	**-1**	**-2**
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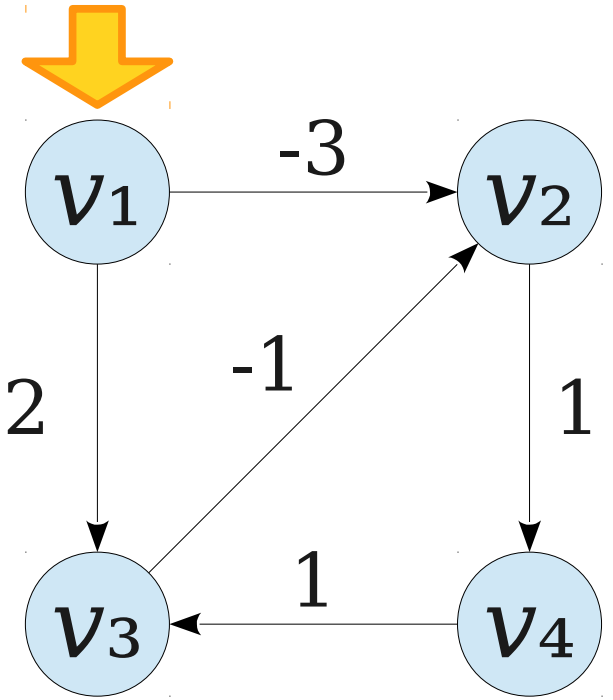
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



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Analyzing Time Complexity

- What is the time complexity of this algorithm?
  - Create a table DP of size $n \times n$.
  - Set DP[v][0] = ∞ for all $v \neq s$.
  - Set DP[s][0] = 0
  - For $i = 1$ to $n - 1$, for all $v \in V$:
    - Set DP[v][i] =
      min {
        DP[v][$i - 1$],
        min { DP[u][$i - 1$] + $w(u, v)$ } *(where $(u, v) \in E$)*
      }
  - Return row n of DP.
- Answer: **O(*mn*)**, i you reverse G prior to running the algorithm.

Analyzing Space Complexity

- What is the *space* complexity of this algorithm?
  - Create a table DP of size $n \times n$.
  - Set DP[v][0] = ∞ for all $v \neq s$.
  - Set DP[s][0] = 0
  - For $i = 1$ to $n - 1$, for all $v \in V$:
    - Set DP[v][i] =
      min {
        DP[v][$i - 1$],
        min { DP[u][$i - 1$] + $w(u, v)$ } *(where $(u, v) \in E$)*
      }
  - Return row n of DP.
- Answer: **$O(n^2)$**. *(Can we reduce this?)*

Shortest paths with negative weights: practical improvements

Space optimization. Maintain two 1D arrays (instead of 2D array).

- $d[v]$ = length of a shortest $v \rightsquigarrow t$ path that we have found so far.
- $successor[v]$ = next node on a $v \rightsquigarrow t$ path.

Performance optimization. If $d[w]$ was not updated in iteration $i - 1$, then no reason to consider edges entering w in iteration i.

Bellman–Ford–Moore: efficient implementation

BELLMAN–FORD–MOORE(V, E, c, t)

FOREACH node $v \in V$:

  $d[v] \leftarrow \infty$.

  $successor[v] \leftarrow null$.

$d[t] \leftarrow 0$.

FOR $i = 1$ TO $n - 1$

  FOREACH node $w \in V$:

    IF ($d[w]$ was updated in previous pass)

      FOREACH edge $(v, w) \in E$:

        IF ($d[v] > d[w] + \ell_{vw}$)

          $d[v] \leftarrow d[w] + \ell_{vw}$.

          $successor[v] \leftarrow w$.

  IF (no $d[\cdot]$ value changed in pass i) STOP.

pass i
$O(m)$ time

Bellman–Ford–Moore: analysis

Lemma 3. For each node v : $d[v]$ is the length of some $v \rightsquigarrow t$ path.

Lemma 4. For each node v : $d[v]$ is monotone non-increasing.

Lemma 5. After pass i, $d[v] \leq$ length of a shortest $v \rightsquigarrow t$ path using $\leq i$ edges.

Pf. [by induction on i]

- Base case: $i = 0$.
- Assume true after pass i.
- Let P be any $v \rightsquigarrow t$ path with $\leq i + 1$ edges.
- Let (v, w) be first edge in P and let P' be subpath from w to t.
- By inductive hypothesis, at the end of pass i, $d[w] \leq \ell(P')$ because P' is a $w \rightsquigarrow t$ path with $\leq i$ edges.
- After considering edge (v, w) in pass $i + 1$:

and by Lemma 4, $d[w]$ does not increase

$$\begin{aligned} d[v] &\leq \ell_{vw} + d[w] \\ &\leq \ell_{vw} + \ell(P') \\ &= \ell(P) \quad \blacksquare \end{aligned}$$

and by Lemma 4, $d[v]$ does not increase

Bellman–Ford–Moore: analysis

Theorem 2. Assuming no negative cycles, Bellman–Ford–Moore computes the lengths of the shortest $v \rightsquigarrow t$ paths in $O(mn)$ time and $\Theta(n)$ extra space.

Pf. Lemma 2 + Lemma 5. ▪

shortest path exists and has at most $n-1$ edges

after i passes, $d[v] \leq$ length of shortest path that uses $\leq i$ edges

Remark. Bellman–Ford–Moore is typically faster in practice.

- Edge (v, w) considered in pass $i+1$ only if $d[w]$ updated in pass i.
- If shortest path has k edges, then algorithm finds it after $\leq k$ passes.

```
BELLMAN-FORD(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| − 1
3      for each edge (u, v) ∈ G.E
4          RELAX(u, v, w)
5  for each edge (u, v) ∈ G.E
6      if v.d > u.d + w(u, v)
7          return FALSE
8  return TRUE
```

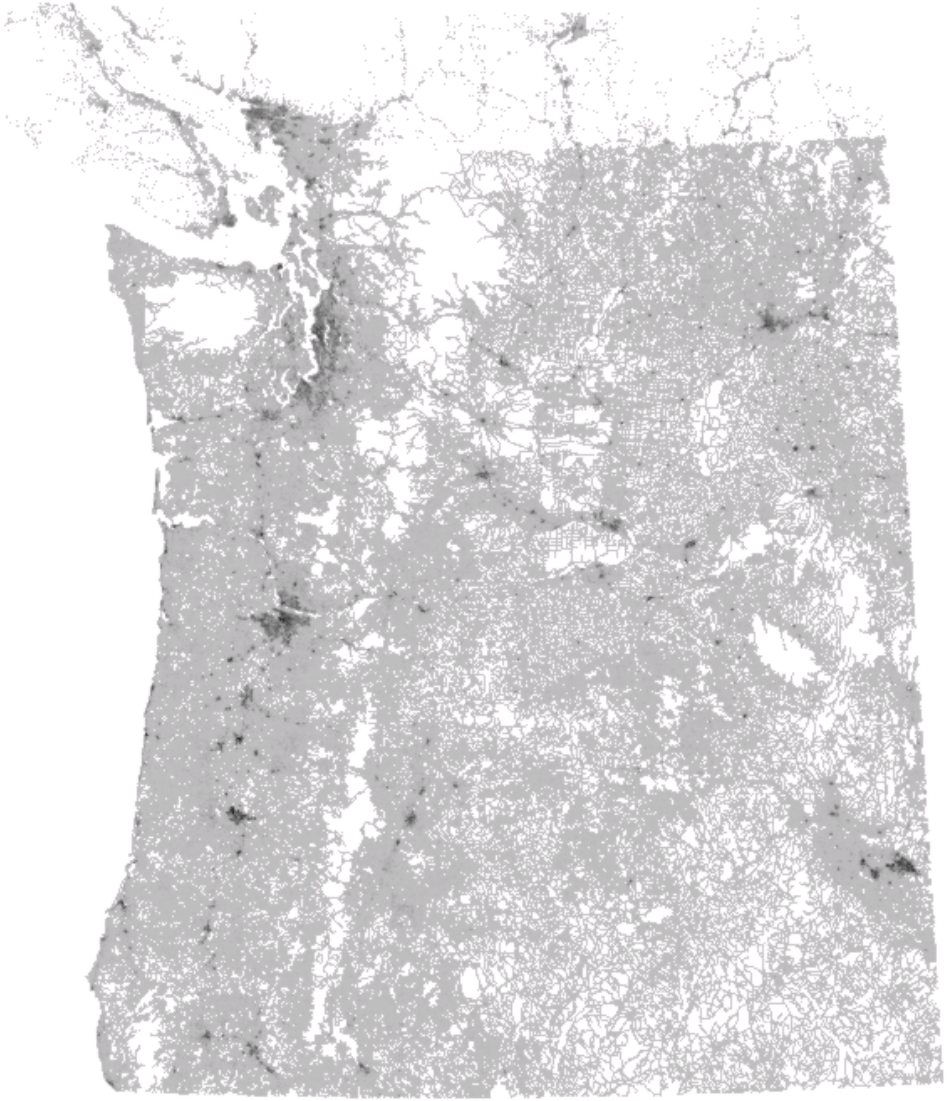
Single-source shortest paths with negative weights

year	worst case	discovered by
1955	$O(n^4)$	Shimbel
1956	$O(m\, n^2\, W)$	Ford
1958	$O(m\, n)$	Bellman, Moore
1983	$O(n^{3/4}\, m \log W)$	Gabow
1989	$O(m\, n^{1/2} \log(nW))$	Gabow–Tarjan
1993	$O(m\, n^{1/2} \log W)$	Goldberg
2005	$O(n^{2.38}\, W)$	Sankowsi, Yuster–Zwick
2016	$\tilde{O}(n^{10/7} \log W)$	Cohen–Mądry–Sankowski–Vladu
20xx		

single-source shortest paths with weights between –W and W

Practical Concerns

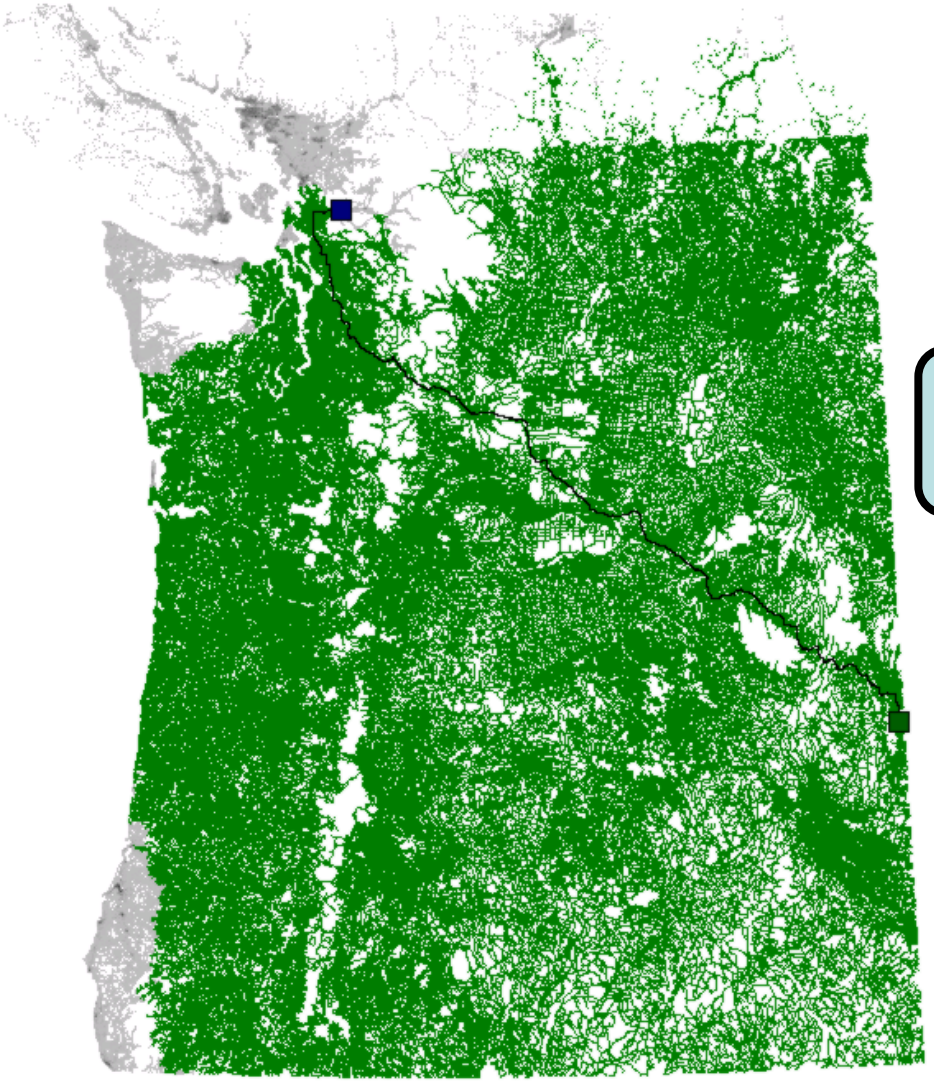
Dijkstra Example



1.6 million vertices
3.8 million edges
Distance = travel time.

Images comes from A.V. Goldberg

Dijkstra Example

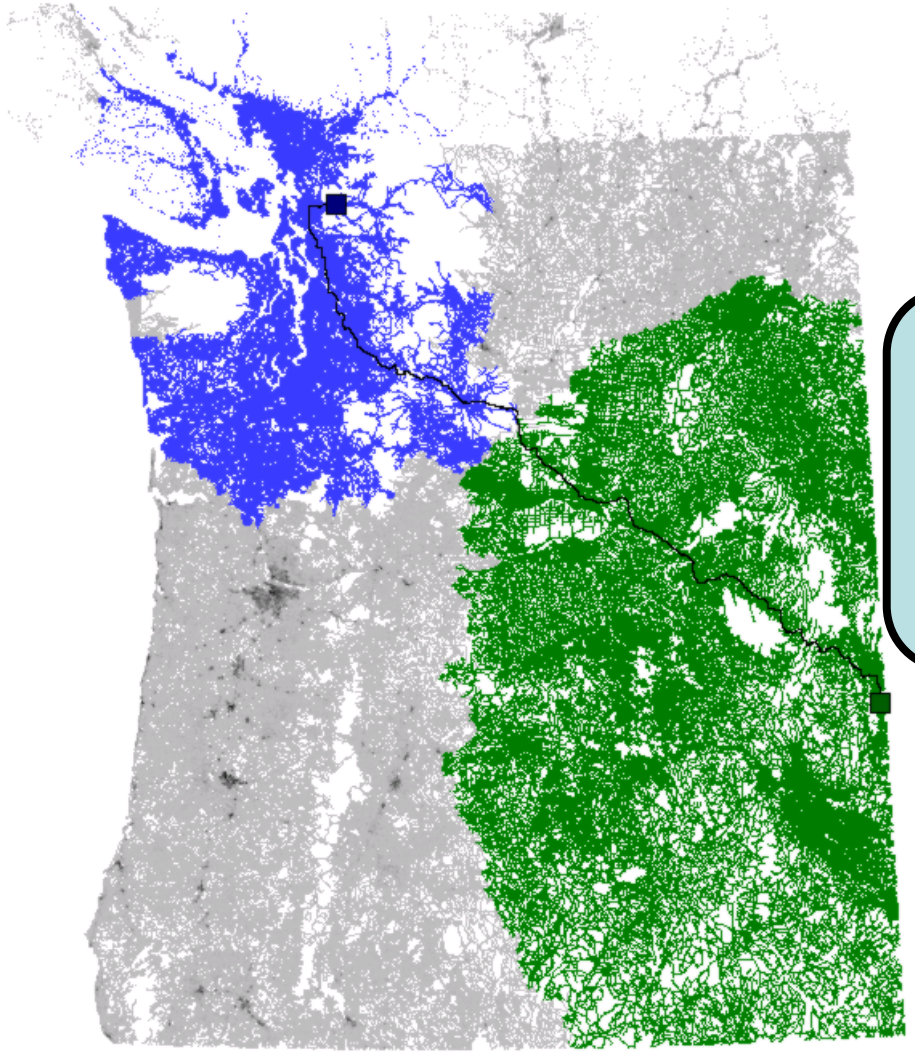


Searched Area
(starting from green point)

Problem of Dijkstra:
Didn't take account of where is t

340ms

Bidirectional Dijkstra



Forward search
Backward search

Problem of Bidirectional Dijkstra:
Forward search did not take account of t
Backward search did not take account of s.

A^* Search

```
AStar(G, c, s, t) {
    Initialize set of explored nodes S ← {s}

    // Maintain distance from s to each vertices in S
    d[s] ← 0

    while (S ≠ V)
    {
        Pick an edge (u, v) such that u ∈ S and v ∉ S and
                d[u] + c(u,v) + h(v) is as small as possible.

        Add v to S and define d[v] = d[u] + c(u,v) .
        Parent(v) ← u .
    }
```

BFS

Dijkstra

A^*

- $h(v)$ is the estimate of distance from v to t
- If $h(v)$ is exactly the shortest distance from v to t, then the algorithm would go directly to t.

Routing Problem in Large Graphs

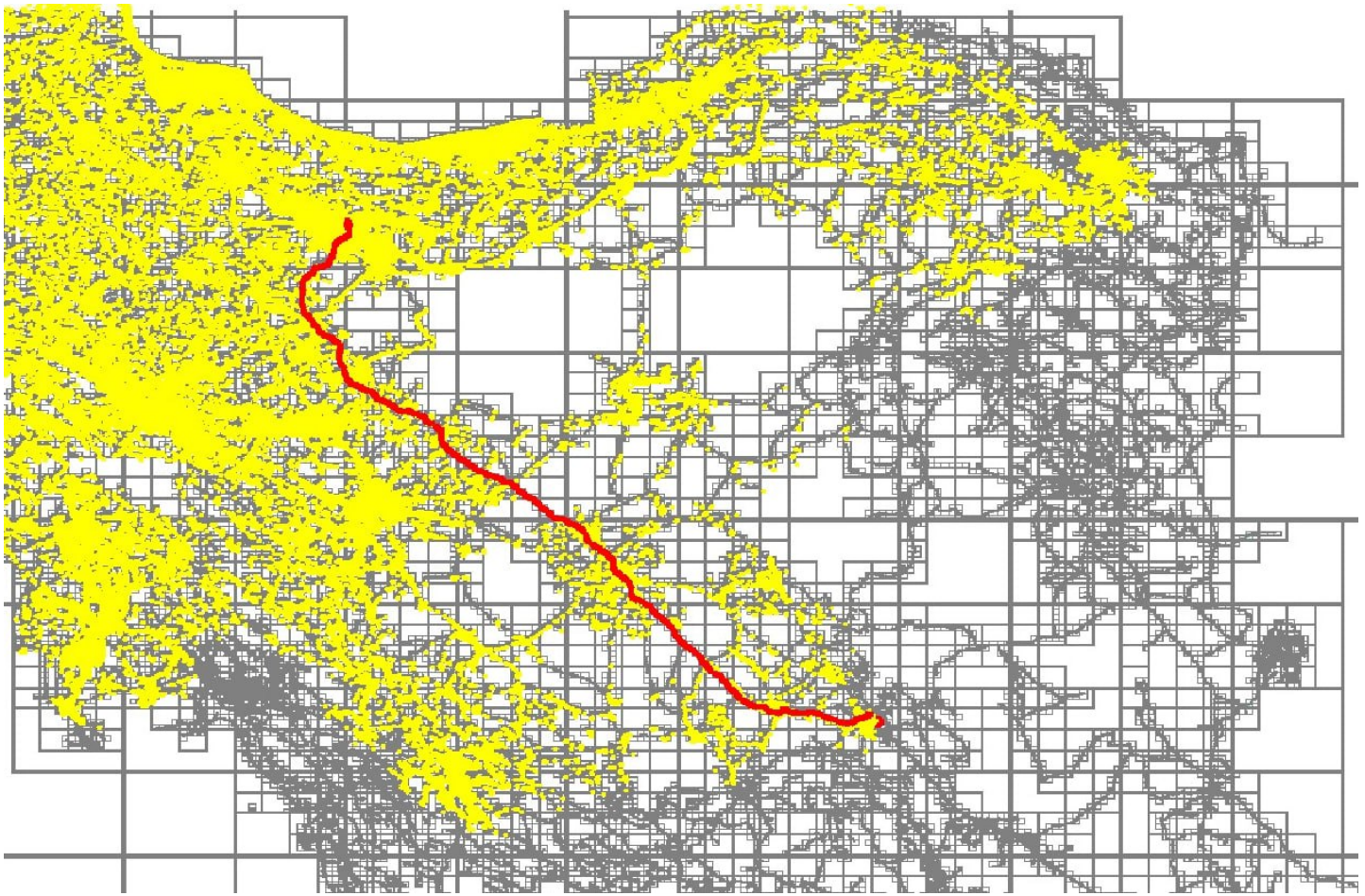
- Iran's Graph has 2,052,067 vertices and 2707945 edges.

- Consider Routing From Azadi Gate to Kerman
- The shortest path contains 3437 many edges.
- Dijkstra algorithm visits 1722445 many nodes!

Algorithms & Photo By: A. R. Tofighi Mohammadi

Dijkstra Performance

#Visited Nodes: 1,722,445



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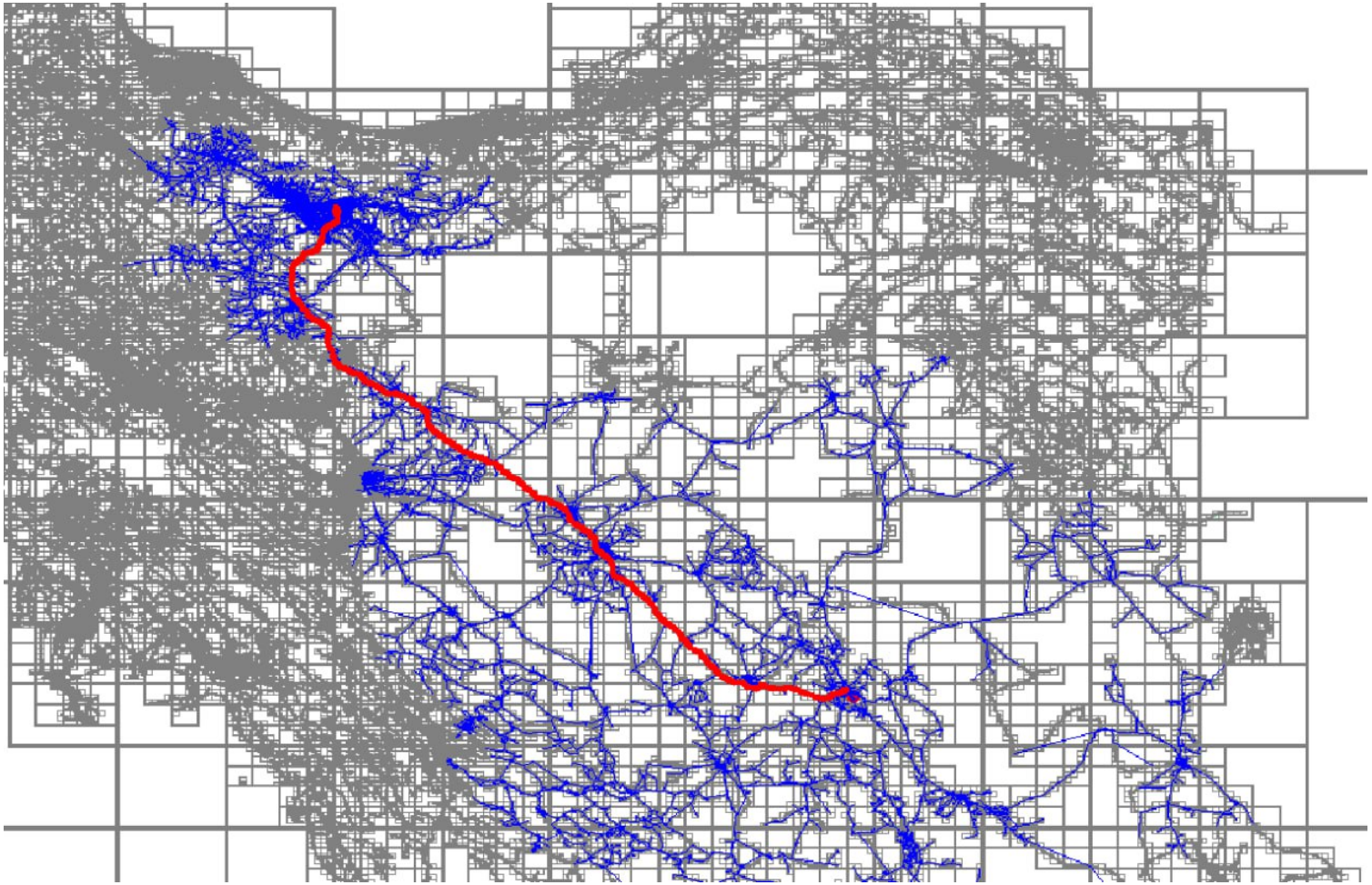
Bi-Directional Dijkstra

#Visited Nodes: 660,564



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Landmark Selection

Preprocessing

- Random selection is fast.
- Many heuristics find better landmarks.
- Local search can find a good subset of candidate landmarks.
- We use a heuristic with local search.

Preprocessing/query trade-off.

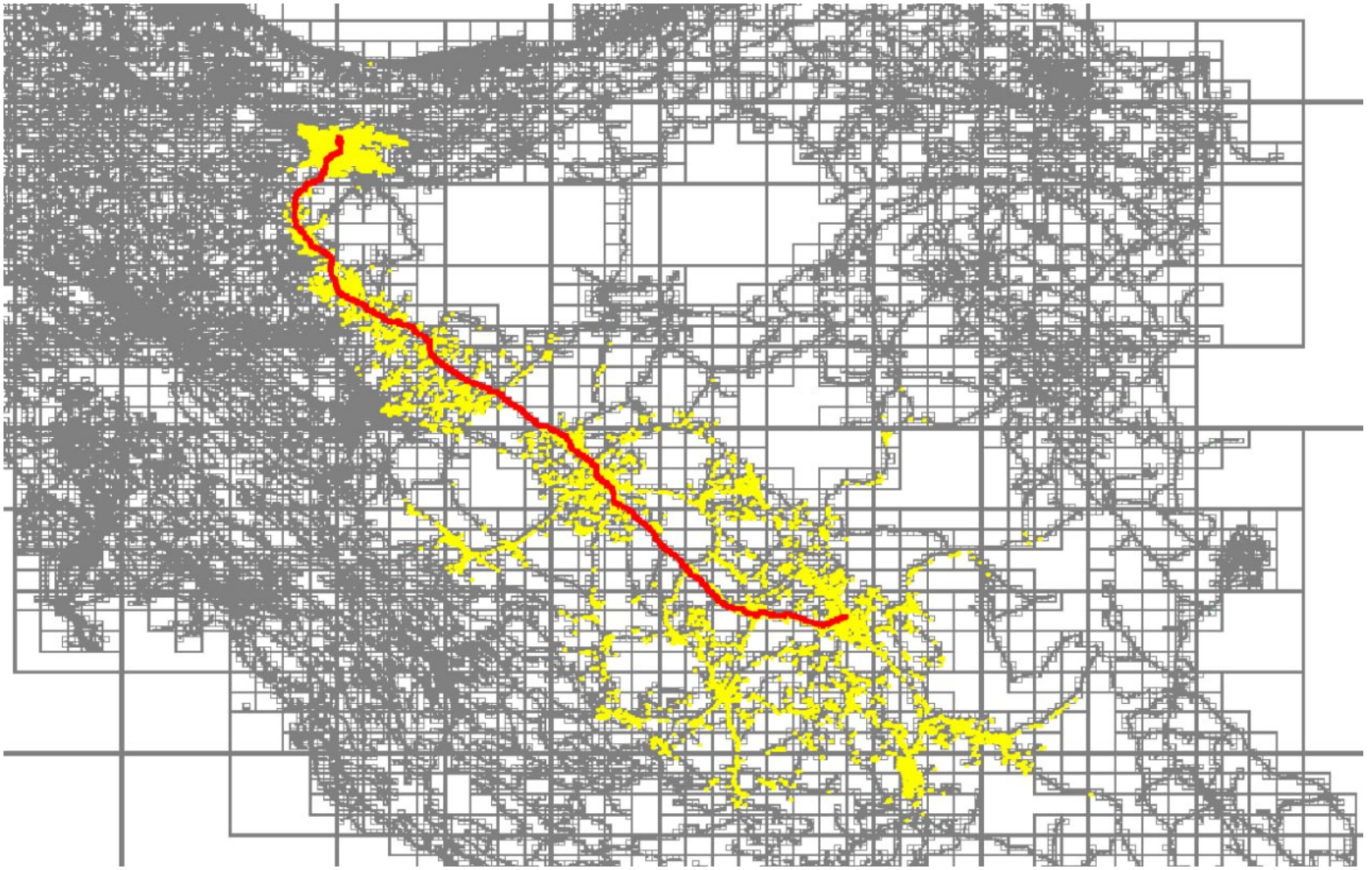
Query

- For a specific s, t pair, only some landmarks are useful.
- Use only active landmarks that give best bounds on dist(s, t).
- If needed, dynamically add active landmarks (good for the search frontier).
- Only three active landmarks on the average.

Allows using many landmarks with small time overhead.

Dijkstra + Landmark

#Visited Nodes: 405,784



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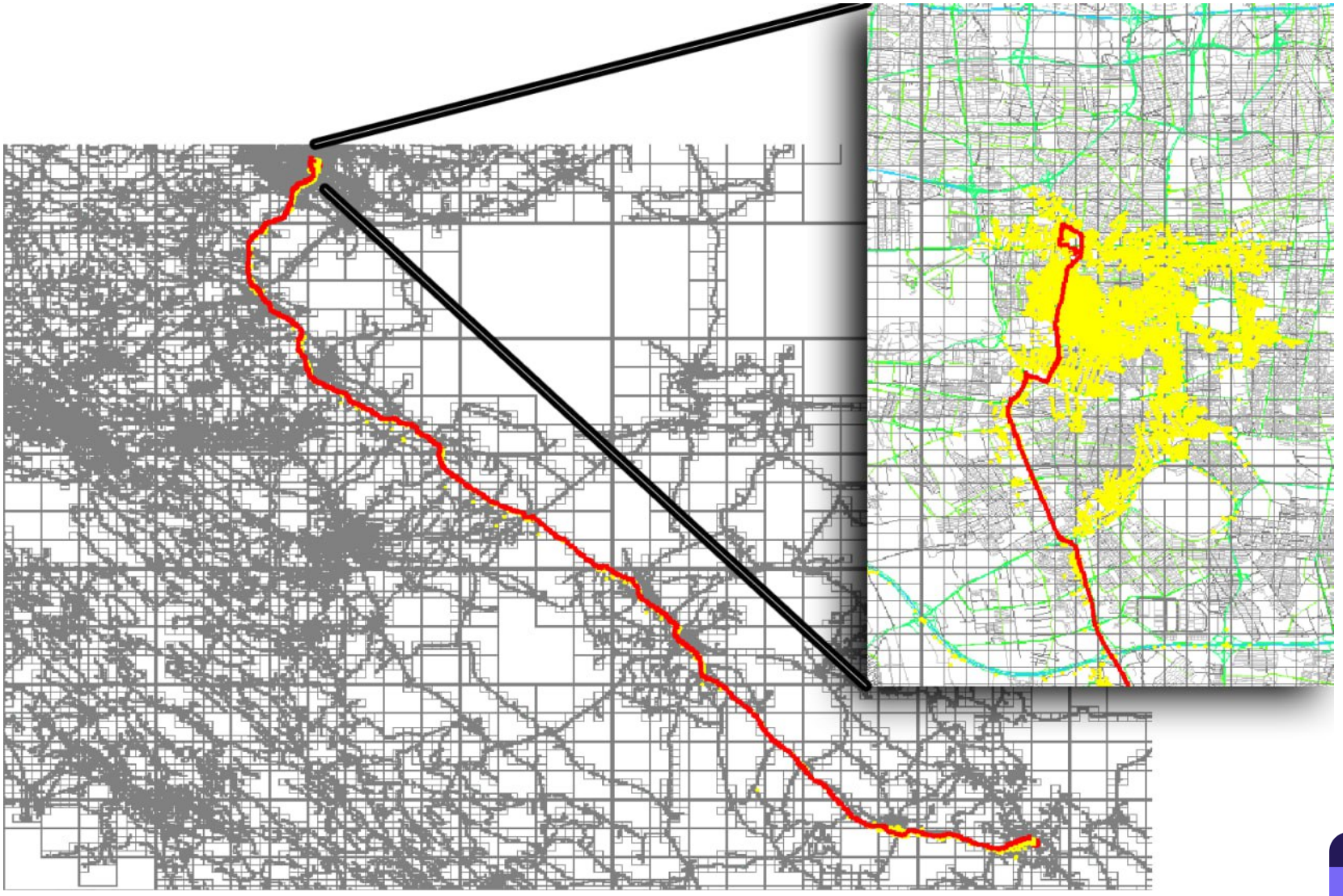
Bi-directional Dijkstra + Landmark

#Visited Nodes: 19,056



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