

# Algorithm Analysis

# Algorithms

## Algorithm.

- [webster.com] A procedure for solving a mathematical problem (as of finding the greatest common divisor) in a finite number of steps that frequently involves repetition of an operation.
- [Knuth, TAOCP] An algorithm is a finite, definite, effective procedure, with some input and some output.

Great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing. - *Francis Sullivan*

# Etymology

Etymology. [Knuth, TAOCP]

- *Algorism* = process of doing arithmetic using Arabic numerals.
- A misperception: *algiros* [painful] + *arithmos* [number].
- True origin: Abu 'Abd Allah Muhammad ibn Musa al-Khwarizm was a famous 9th century Persian textbook author who wrote *Kitab al-jabr wa 'l-muqabala*, which evolved into today's high school algebra text.

# Algorithmic Paradigms

Design and analysis of computer algorithms.

- Greedy.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomized algorithms.
- Intractability.
- Coping with intractability.

Critical thinking and problem-solving.

# Applications

## Wide range of applications.

- Caching.
- Compilers.
- Databases.
- Scheduling.
- Networking.
- Data analysis.
- Signal processing.
- Computer graphics.
- Scientific computing.
- Operations research.
- Artificial intelligence.
- Computational biology.
- . . .

# Algorithm Analysis

**Analysis** refers to mathematical techniques for establishing both the correctness and efficiency of algorithms.

**Efficiency:** Given an algorithm  $A$ , we want to know how efficient it is. This includes several possible criteria:

- What is the asymptotic complexity of algorithm  $A$ ?
- How does the average-case complexity of  $A$  compare to the worst-case complexity?
- Is  $A$  the most efficient algorithm to solve the given problem? (For example, can we find a lower bound on the complexity of any algorithm to solve the given problem?)

# Polynomial-Time

**Brute force.** For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.

- Typically takes  $2^N$  time or worse for inputs of size  $N$ .
- Unacceptable in practice.

**Def.** An algorithm is **poly-time** if the below property holds.

There exists constants  $c > 0$  and  $d > 0$  such that on every input of size  $N$ , its running time is bounded by  $c N^d$  steps.

# Worst-Case Analysis

**Worst case running time.** Obtain bound on **largest possible** running time of algorithm on input of a given size  $N$ .

- Generally captures efficiency in practice.

**Average case running time.** Obtain bound on running time of algorithm on **random** input as a function of input size  $N$ .

- Hard (or impossible) to accurately model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.



# Worst-Case Polynomial-Time

**Def.** An algorithm is **practical** if its running time is polynomial.

**Justification:** **It really works in practice!**

- Although  $6.02 \times 10^{23} \times N^{20}$  is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.

**Exceptions.**

- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.

# Why It Matters

**Table 2.1** The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds  $10^{25}$  years, we simply record the algorithm as taking a very long time.

	$n$	$n \log_2 n$	$n^2$	$n^3$	$1.5^n$	$2^n$	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	$10^{25}$ years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	$10^{17}$ years	very long
$n = 1,000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

# Asymptotic Order of Growth

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**Upper bounds.**  $T(n)$  is  $O(f(n))$  if there exist constants  $c > 0$  and  $n_0 \geq 0$  such that for all  $n \geq n_0$  we have  $T(n) \leq c \cdot f(n)$ .

**Lower bounds.**  $T(n)$  is  $\Omega(f(n))$  if there exist constants  $c > 0$  and  $n_0 \geq 0$  such that for all  $n \geq n_0$  we have  $T(n) \geq c \cdot f(n)$ .

**Tight bounds.**  $T(n)$  is  $\Theta(f(n))$  if  $T(n)$  is both  $O(f(n))$  and  $\Omega(f(n))$ .

**Ex:**  $T(n) = 32n^2 + 17n + 32$ .

- $T(n)$  is  $O(n^2)$ ,  $O(n^3)$ ,  $\Omega(n^2)$ ,  $\Omega(n)$ , and  $\Theta(n^2)$ .
- $T(n)$  is not  $O(n)$ ,  $\Omega(n^3)$ ,  $\Theta(n)$ , or  $\Theta(n^3)$ .

# Notation

**Slight abuse of notation.**  $T(n) = O(f(n))$ .

- Not transitive:
  - $f(n) = 5n^3$ ;  $g(n) = 3n^2$
  - $f(n) = O(n^3) = g(n)$
  - but  $f(n) \neq g(n)$ .
- Better notation:  $T(n) \in O(f(n))$ .

**Meaningless statement.** Any comparison-based sorting algorithm requires at least  $O(n \log n)$  comparisons.

- Use  $\Omega$  for lower bounds.

# Properties

## Transitivity.

- If  $f = O(g)$  and  $g = O(h)$  then  $f = O(h)$ .
- If  $f = \Omega(g)$  and  $g = \Omega(h)$  then  $f = \Omega(h)$ .
- If  $f = \Theta(g)$  and  $g = \Theta(h)$  then  $f = \Theta(h)$ .

## Additivity.

- If  $f = O(h)$  and  $g = O(h)$  then  $f + g = O(h)$ .
- If  $f = \Omega(h)$  and  $g = \Omega(h)$  then  $f + g = \Omega(h)$ .
- If  $f = \Theta(h)$  and  $g = O(h)$  then  $f + g = \Theta(h)$ .

# Asymptotic Bounds for Some Common Functions

**Polynomials.**  $a_0 + a_1n + \dots + a_dn^d$  is  $\Theta(n^d)$  if  $a_d > 0$ .

**Polynomial time.** Running time is  $O(n^d)$  for some constant  $d$  independent of the input size  $n$ .

**Logarithms.**  $O(\log_a n) = O(\log_b n)$  for any constants  $a, b > 0$ .



can avoid specifying the  
base

**Logarithms.** For every  $x > 0$ ,  $\log n = O(n^x)$ .



log grows slower than every polynomial

**Exponentials.** For every  $r > 1$  and every  $d > 0$ ,  $n^d = O(r^n)$ .



every exponential grows faster than every polynomial

# A Survey of Common Running Times

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## Linear Time: $O(n)$

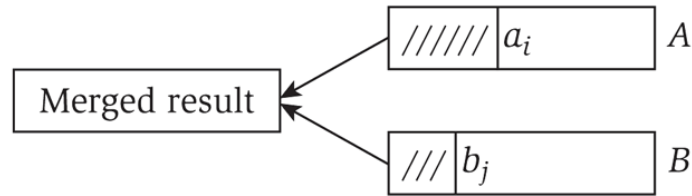
**Linear time.** Running time is proportional to input size.

**Computing the maximum.** Compute maximum of  $n$  numbers  $a_1, \dots, a_n$ .

```
max ← a1
for i = 2 to n {
    if (ai > max)
        max ← ai
}
```

## Linear Time: $O(n)$

**Merge.** Combine two sorted lists  $A = a_1, a_2, \dots, a_n$  with  $B = b_1, b_2, \dots, b_n$  into sorted whole.



```
i = 1, j = 1
while (both lists are nonempty) {
    if (ai ≤ bj) append ai to output list and increment i
    else          append bj to output list and increment j
}
append remainder of nonempty list to output list
```

**Claim.** Merging two lists of size  $n$  takes  $O(n)$  time.

**Pf.** After each comparison, the length of output list increases by 1.

## $O(n \log n)$ Time

$O(n \log n)$  time. Arises in divide-and-conquer algorithms.



also referred to as linearithmic time

**Sorting.** Mergesort and heapsort are sorting algorithms that perform  $O(n \log n)$  comparisons.

**Largest empty interval.** Given  $n$  time-stamps  $x_1, \dots, x_n$  on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

**$O(n \log n)$  solution.** Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

## Quadratic Time: $O(n^2)$

**Quadratic time.** Enumerate all pairs of elements.

**Closest pair of points.** Given a list of  $n$  points in the plane  $(x_1, y_1), \dots, (x_n, y_n)$ , find the pair that is closest.

**$O(n^2)$  solution.** Try all pairs of points.

```
min ←  $(x_1 - x_2)^2 + (y_1 - y_2)^2$ 
for i = 1 to n {
  for j = i+1 to n {
    d ←  $(x_i - x_j)^2 + (y_i - y_j)^2$ 
    if (d < min)
      min ← d
  }
}
```

← don't need to  
take square roots

**Remark.**  $\Omega(n^2)$  seems inevitable, but this is just an illusion.

## Cubic Time: $O(n^3)$

**Cubic time.** Enumerate all triples of elements.

**Set disjointness.** Given  $n$  sets  $S_1, \dots, S_n$  each of which is a subset of  $1, 2, \dots, n$ , is there some pair of these which are disjoint?

**$O(n^3)$  solution.** For each pairs of sets, determine if they are disjoint.

```
foreach set  $S_i$  {  
    foreach other set  $S_j$  {  
        foreach element  $p$  of  $S_i$  {  
            determine whether  $p$  also belongs to  $S_j$   
        }  
        if (no element of  $S_i$  belongs to  $S_j$ )  
            report that  $S_i$  and  $S_j$  are disjoint  
    }  
}
```

## Polynomial Time: $O(n^k)$ Time

**Independent set of size  $k$ .** Given a graph, are there  $k$  nodes such that no two are joined by an edge?

↖  
 $k$  is a constant

**$O(n^k)$  solution.** Enumerate all subsets of  $k$  nodes.

```
foreach subset S of k nodes {  
    check whether S is an independent set  
    if (S is an independent set)  
        report S is an independent set  
    }  
}
```

- Check whether  $S$  is an independent set =  $O(k^2)$ .
- Number of  $k$  element subsets =  $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)} \leq \frac{n^k}{k!}$
- $O(k^2 n^k / k!) = O(n^k)$ .

↖  
poly-time for  $k=17$ ,  
but not practical

# Exponential Time

**Independent set.** Given a graph, what is maximum size of an independent set?

**$O(n^2 2^n)$  solution.** Enumerate all subsets.

```
S* ← ∅  
foreach subset S of nodes {  
    check whether S is an independent set  
    if (S is largest independent set seen so far)  
        update S* ← S  
}  
}
```

# References

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## References

- Sections 2.1, 2.2, and 2.4 of the text book "algorithm design" by Jon Kleinberg and Eva Tardos
- The original slides were prepared by Kevin Wayne. The slides are distributed by Pearson Addison-Wesley.