Assignment #04

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Subject : Maths

Q1 $f(x) = x^2 - 4x + 5 \quad 0 \le x \le 3$, find the Riemann sum with $n=6$, taking the simple points to be
i left end points (ii) Right endpoints (iii) Mid points

Sol

we know that

$$\Delta x = \frac{b-a}{n}$$

So

$$\Delta x = \frac{3-0}{6}$$

$$\Delta x = \frac{1}{2} \text{ or } 0.5$$

Now

$\Delta x_0 = a + \Delta x(0) = 0$

$x_1 = a + \Delta x(1) = 0.5$

$x_2 = a + \Delta x(2) = 1$

$x_3 = a + \Delta x(3) = 1.5$

$x_4 = a + \Delta x(4) = 2$

$x_5 = a + \Delta x(5) = 2.5$

$x_6 = a + \Delta x(6) = 3$

So,

$f(x_0) = 5$

$f(x_1) = 3.25$

$f(x_2) = 2$
$f(x_3) = 1.25$
$f(x_4) = 1$
$f(x_5) = 1.25$
$f(x_6) = 2$

And Now

$$\breve{x}_1 = \frac{x_0 + x_1}{2}$$
$$= 0.25$$

$$\breve{x}_2 = \frac{x_1 + x_2}{2}$$
$$= 0.75$$

$$\breve{x}_3 = \frac{x_2 + x_3}{2}$$
$$= 1.25$$

$$\breve{x}_4 = \frac{x_3 + x_4}{2}$$
$$= 1.75$$

$$\breve{x}_5 = \frac{x_4 + x_5}{2}$$
$$= 2.25$$

$$\breve{x}_6 = \frac{x_5 + x_6}{2}$$
$$= 2.75$$

So,

$f(x_1^*) = 4.0625$

$f(x_2^*) = 2.5625$

$f(x_3^*) = 1.5625$

$f(x_4^*) = 1.0625$

$f(x_5^*) = 1.0625$

$f(x_6^*) = 1.5625$

Now

- Left End Points

$$\sum_{i=0}^{5} f(x_i)\Delta x = \Delta x\,[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$

$$= 0.5\,(13.75)$$

$$= 6.875$$

- Right End Points:

$$\sum_{i=1}^{6} f(x_i)\Delta x = \Delta x\,(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)]$$

$$= 0.5\,(10.75)$$

$$= 5.375$$

- Mid Point:

$$\sum_{i=1}^{6} f(x_i^*)\Delta x = \Delta x\,[f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + f(x_5^*) + f(x_6^*)]$$

$$= 0.5\,(11.875)$$

$$= 5.9375$$

b Express the limit as a

b Use the midpoint Rule with $n=4$ to approximate the integral

$$\int_0^{\pi} x \sin^2 x \, dx$$

Sol

We know that

$$\Delta x = \frac{\pi - 0}{4}$$

$$\Delta x = \frac{\pi}{4}$$

Now,

$x_0 = a + \Delta x(0) = 0$

$x_1 = a + \Delta x(1) = \frac{\pi}{4}$

$x_2 = a + \Delta x(2) = \pi/2$

$x_3 = a + \Delta x(3) = \frac{3\pi}{4}$

$x_4 = a + \Delta x(4) = \pi$

So we have

$x_1^{*} = \frac{x_0 + x_1}{2} = \frac{\pi}{8}$, $x_2^{*} = \frac{x_1 + x_2}{2} = \frac{3\pi}{8}$

$x_3^{*} = \frac{x_2 + x_3}{2} = \frac{5\pi}{6}$, $x_4^{*} = \frac{x_3 + x_4}{2} = \frac{7\pi}{8}$

So,

$f(x_1^{*}) = 0.575$, $f(x_2^{*}) = 1.0056$

$f(x_3^{*}) = 1.6759$, $f(x_4^{*}) = 0.4026$

$$\sum_{i=1}^{4} f(x^{*}) \Delta x = \Delta x [f(x_1^{*}) + f(x_2^{*}) + f(x_3^{*}) + f(x_4^{*})]$$

$$= \frac{\pi}{4}(3.1416)$$

$$= 2.4674$$

Q2 a Express the integral as a limit of Reimann sum. Donot evaluate the limit

$$\int_0^5 \left(x^2+\frac{1}{x}\right)dx$$

Sol Here $a=2$ & $b=5$

So, $\Delta x = \frac{b-a}{n}$

$= \frac{5-2}{n}$

$= \frac{3}{n}$

So we have,

$x_i = a+(\Delta x)(i)$

$= a+\Delta x(i)$

$= 2+\frac{3i}{n}$

So

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{6i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\left(\frac{6i}{n}\right)^2+\frac{n}{6i}\right)$$

b Express the limit as a definite integral on the given interval.

i $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin x_i}{1+x_i} \Delta x \; ; [0,\pi]$

Sol

Here $a=0$, $b=\pi$

So

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin x_i}{1+x_i} = \int_0^{\pi} \left(\frac{\sin x}{1+x}\right) dx$$

ii $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4} \Delta x , [1,3]$

Sol

Here $a=1$, $b=3$

So

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4} \Delta x = \int_1^3 \frac{x}{x^2+4} dx$$

Q3 Find $\int_0^5 f(x)dx$ if $f(x) = \begin{cases} 3 & \text{for } x<3 \\ x+e^x & \text{for } x \geq 3 \end{cases}$

Sol

$$\int_0^5 f(x)dx = \int_0^3 f(x)dx + \int_3^5 f(x)dx$$

$$= \int_0^3 (3)dx + \int_3^5 (x+e^x)dx$$

So,

$$\int_0^3 3\,dx = 3\int_0^3 1\,dx$$

$$= 3 \cdot x\Big|_0^3$$

$$= 3(3-0)$$

$$= 9$$

And

$$\int_3^5 (x+e^x)\,dx = \int_3^5 x\,dx + \int_3^5 e^x\,dx$$

$$= \frac{x^2}{2}\Big|_3^5 + e^x\Big|_3^5$$

$$= \left(\frac{(5)^2-(3)^2}{2}\right) + (e^5 - e^3)$$

$$= 8 + 128.3$$
$$= 136.3$$

So,

$$\int_0^5 f(x) = 9 + 136.3$$
$$= 145.3$$

Q4 Evaluate the integral:

i $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx$

Sol

$$= 4\int_{1/2}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= 4\int_{1/2}^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$$

$$= 4(\sin^{-1} x)\Big|_{1/2}^{1/\sqrt{2}}$$

$$= 4\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$= 4\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= 4\left(\frac{\pi}{12}\right)$$

$$= \frac{\pi}{3}$$

ii $\int_0^{\pi} f(x)\,dx$, where $f(x)\begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le 2 \end{cases}$

Sol

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \cos x\,dx$$

$$= -\cos x\Big|_0^{\pi/2} + \sin x\Big|_{\pi/2}^{\pi}$$

$$= -[\cos(0) - \cos(\pi/2)] + [\sin(\pi/2) - \sin(\pi)]$$

$$= -(1-0) + (1-0)$$

$$= -1+1$$

$$= 0$$

iii $f(x) = \int_{2\pi}^{0} \sqrt{1+\sec t}\,dt$

Sol

$$-\int_0^{2\pi} \sqrt{1+\sec t}\,dt$$

Multiplying and dividing by $\sqrt{\sec t - 1}$

$$= -\int_0^{2\pi} \sqrt{\sec t + 1} - \frac{\sqrt{\sec t - 1}}{\sqrt{\sec t - 1}}$$

$$= -\int_0^{2\pi} \frac{\sqrt{\sec^2 t - 1}}{\sqrt{\sec t - 1}}$$

iv $\int_0^3 \frac{y^3 - 2y^2 - y}{y^2} dy$

Sol

$= \int_1^3 \left(\frac{y^3}{y^2} - \frac{2y^2}{y^2} - \frac{y}{y^2}\right) dy$

$= \int_1^3 y\, dy - 2\int_0^3 1\, dy - \int_0^3 \frac{1}{y}\, dy$

$= \frac{y^2}{2}\Big|_1^3 - 2y\Big|_1^3 - \ln|y|\Big|_1^3$

$= \frac{1}{2}((3)^2 - (1)^2) - 2(3-1) - (\ln|3| - \ln|1|)$

$= \frac{1}{2}(9-1) - 2(2) - 1.1$

$= 4 - 4 - 1.1$

$= -1.1$

v $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta} d\theta$

Sol

$= \int_0^{\pi/3} \frac{\sin\theta(1+\tan^2\theta)}{\sec^2\theta} d\theta$

$= \int_0^{\pi/3} \frac{\sin\theta \cancel{\sec^2\theta}}{\cancel{\sec^2\theta}} d\theta$

$= \int_0^{\pi/3} \sin\theta\, d\theta$

$= -\cos\theta\Big|_0^{\pi/3}$

$= -(\cos(\pi/3) - \cos(0))$

$= -\left(\frac{1}{2} - 1\right)$

$= -(0.5)$

-0.5

vi $\int_0^2 |2x-1|\,dx$

Sol

$$|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq 1/2 \\ -2x+1 & \text{if } x < 1/2 \end{cases}$$

So,

$$\int_0^2 |2x-1|\,dx = \int_0^{1/2} (-2x+1)\,dx + \int_{1/2}^{2} (2x-1)\,dx$$

$$\int_0^{1/2} (-2x+1)\,dx = -2\int_0^{1/2} x\,dx + \int_0^{1/2} dx$$

$$= -2\left(\frac{x^2}{2}\right)\Big|_0^{1/2} + x\Big|_0^{1/2}$$

$$= -\left((1/2)^2 - (0)^2\right) + (1/2 - 0)$$

$$= -\frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4}$$

And now

$$\int_{1/2}^{2} (2x+1)\,dx = 2\int_{1/2}^{2} x\,dx + \int_{1/2}^{2} dx$$

$$= \cancel{2}\left(\frac{x^2}{\cancel{2}}\right)\Big|_{1/2}^{2} + x\Big|_{1/2}^{2}$$

$$= \left[(2)^2 - (1/2)^2\right] + \left[2 - 1/2\right]$$

$$\Rightarrow \frac{15}{4} - \frac{3}{2} = \frac{9}{4}$$

So

$$\int_0^3 |2x-1| = \frac{1}{4} + \frac{9}{4}$$

$$= \frac{5}{2}$$

Q5 Use part 1 of fundamental theorem of calculus to find derivative of

i $g(x) = \int_x^0 \sqrt{t+t^2}\, dt$

Sol

We see this that this function is continuous everywhere also, $a \le x \le b$. So we can apply fundamental law so we get

$$g(x) = -\sqrt{x+x^2}$$

ii $h(x) = \int_0^{e^x} \ln t\, dt$

Sol

As this function is also continuous as logarithmic function is not negative so, also $a \le x \le b$

So,

$$h'(x) = \frac{d}{dx}\int_0^{e^x} \ln t\, dt$$

let $v = e^x$, so $\frac{dv}{dx} = e^x$ and $dv = e^x dx$ so

$$h'(x) = \frac{d}{dx} \cdot \frac{dv}{dt}\int_0^{v} \ln t\, dt$$

$$= \frac{d}{dx}\left[\int_0^v \ln t \, dt\right] \frac{dv}{dt}$$

$$= \ln v \cdot e^x$$

$$= \ln e^x \cdot e^x$$

$$= x \ln e \cdot e^x$$

$$= x(1) \cdot e^x$$

$$= xe^x$$

iii $h(x) = \int_2^{\sqrt{x}} \frac{z^2}{z^4+1} \, dz$

Sol

let $v = \sqrt{x}$, so $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$

Now

$$h(x) = \int_2^v \frac{z^2}{z^4+1} \, dz$$

$$h'(x) = \frac{d}{dx}\int_0^v \frac{z^2}{z^4+1} \, dz$$

$$= \frac{d}{dv}\left[\int_2^v \frac{z^2}{z^4+1} \, dz\right] \frac{dv}{dx}$$

$$= \frac{v^2}{v^4+1} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\sqrt{x}}{2(x^2+1)}$$

iv $h(x) = \int_{2x}^{3x} \frac{y^2-1}{y^2+1}\, dy$

Sol

$$\int_{2x}^{0} \frac{y^2-1}{y^2+1}\, dy + \int_{0}^{3x} \frac{y^2-1}{y^2+1}\, dy$$

Now

$$\int_{2x}^{0} \frac{y^2-1}{y^2+1} = -\int_{0}^{2x} \frac{y^2-1}{y^2+1}\, dy$$

let $v = 2x$ then $\frac{dv}{dx} = 2$

$$= -\frac{d}{dx}\int_{0}^{v} \frac{y^2-1}{y^2+1}\, dy$$

Using chain rule

$$= -\frac{d}{dv}\left[\int_{0}^{v} \frac{y^2-1}{y^2+1}\right]\frac{dv}{dx}$$

$$= -\left[\frac{v^2-1}{v^2+1}\right]2$$

$$= -2\left[\frac{(2x)^2-1}{(2x)^2+1}\right]$$

$$= -2\left(\frac{4x^2-1}{4x^2+1}\right)$$

$$\frac{-8x^2+2}{4x^2-1}$$

Now

$$\int_0^{3x} \frac{y^2-1}{y^2+1}\,dy$$

Let $v = 3x$ so $\frac{dv}{dx} = 3$

Taking $\frac{d}{dx}$

$$= \frac{d}{dx}\int_0^{v} \frac{y^2-1}{y^2+1}\,dy$$

Using Chain rule

$$= \frac{d}{dv}\left[\int_0^{v} \frac{y^2-1}{y^2+1}\right]\frac{dv}{dx}$$

$$= \frac{v^2-1}{v^2+1}\cdot 3$$

$$= \frac{9x^2-1}{9x^2+1}\cdot 3$$

$$= \frac{27x^2-3}{9x^2+1}$$

So,

$$h'(x) = \frac{27x^2-3}{9x^2+1} - \frac{8x^2-2}{4x^2+1}$$

b what is wrong with the equation?

i $\int_{1}^{2} \frac{4}{x^3} dx = -\frac{2}{x^2}\Big]_{-1}^{2} = \frac{2}{3}$

When $x=0$ this equation tends to go to infinity
So this function is discontinuous at $x=0$
therefore this solution is not possible

ii $\int_{0}^{\pi} \sec^2 x \, dx = \tan x \Big]_{0}^{\pi} = 0$

when $x = \frac{\pi}{2}$ this function tends towards infinity so this function is discontinuous at $x = \pi/2$, therefore this solution is not possible.

Q6 Evaluate the Integral

1 $\int \sin^5(2t) \cos^2(2t) \, dt$

Sol

Let $v = 2t$ then $\frac{dv}{dt} = 2$, so $dt = \frac{dv}{2}$

$$\int \sin^5(v) \cos^2(v) \frac{dv}{2}$$

$$\frac{1}{2}\int \sin^4(v) \cdot \sin(v) \cdot \cos^2 v \, dv$$

$$\frac{1}{2}\int (1-(\cos^2 v)^2, \sin(v) \cdot \cos^2 v \, dv$$

Let $v = \cos v$, so $\frac{dv}{dv} = -\sin v \Rightarrow \sin v = -\frac{dv}{dv}$

$$-\frac{1}{2}\int (1-v)^2 \cdot \frac{dv}{dv} \cdot v^2 \cdot dv$$

$$-\frac{1}{2}\int v^2(1+v^4-2v^2)\,dv$$

$$-\frac{1}{2}\left[\int v^2\,dv + \int v^6\,dv - 2\int v^4\,dv\right]$$

$$-\frac{1}{2}\left[\frac{v^3}{3} + \frac{v^7}{7} - \frac{2v^5}{5}\right]$$

$$-\frac{1}{2}\left[\frac{\cos^3(2t)}{3} + \frac{\cos^7(2t)}{7} + \frac{2\cos^5(2t)}{5}\right]$$

ii $\int \frac{\sin^2(1/t)}{t^2}\,dt$

Sol

Let $v=\frac{1}{t}$ then $\frac{dv}{dt}=\frac{-1}{t^2}$

$$dt=-t^2\,dv$$

So,

$$\int \frac{\sin v}{(1/v^2)}(-t^2\,dv)$$

$$-\int \cancel{v^2}\sin^2 v\left(\frac{1}{\cancel{v^2}}\,dv\right)$$

$$-\int \sin^2 v\,dv$$

$$-\int \frac{1-\cos^2 v}{2}\,dv$$

$$= -\frac{1}{2}\int dv + \frac{1}{2}\int \cos 2v \, dv$$

$$= -\frac{v}{2} + \frac{2\sin 2v}{2} + C$$

$$= -\frac{v}{2} + \sin 2v + C$$

$$= \frac{-1}{2t} + \sin(2/t) + C$$

iii $\int \tan^3 x \sec^6 x \, dx$

Sol

$$\int \tan^2 x \cdot \tan x \cdot \sec^6 x \, dx$$

$$\int (\sec^2 x - 1) \cdot \tan x \cdot \sec^6 x \, dx$$

let $v = \sec x$, so $dv = \tan x \sec x \, dx$

$$\int (v^2 - 1) \cdot v^5 \, dv$$

$$\int v^2 \, dv - \int v^5 \, dv$$

$$\frac{v^3}{3} - \frac{v^6}{6} + C$$

$$\frac{\sec^3 x}{3} - \frac{\sec^6 x}{6} + C$$

iv $\int_0^{\pi/4} \sqrt{1-\cos 4\theta}\, d\theta$

Let $v = 2\theta$ So $dv = 2d\theta$

$d\theta = \frac{dv}{2}$

So

$= \int_0^{\pi/4} \sqrt{1-\cos 2v} \cdot \frac{dv}{2}$

$= \frac{1}{2} \int_0^{\pi/4} \sqrt{2\sin^2 v}\, dv$

$= \frac{1}{2} \int_0^{\pi/4} \sqrt{2} \cdot \sqrt{\sin^2 v}\, dv$

$= \frac{\sqrt{2}}{2} \int_0^{\pi/4} \sin v\, dv$

$= -\frac{\sqrt{2}}{2} \cos v \Big|_0^{\pi/4}$

$= -\frac{\sqrt{2}}{2} (\cos(\pi/4) - \cos(0))$

$= -\frac{\sqrt{2}}{2} \left(\frac{\sqrt{2}}{2} - 1\right)$

$= 0.207$