Assignment #05

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Class: BSCS-2020
Section: D
Subject: Maths

Q1 Explain why each of the following integrals is Improper.

i $\int_1^2 \frac{x}{x-1}\,dx$

Sol

Since $y = \frac{x}{x-1}$ has an infinite discontinuity at $x = 1$, $\int_1^2 \frac{x}{x-1}\,dx$ is a Type-2 improper integral.

ii $\int_0^\infty \frac{1}{1+x^3}\,dx$

Sol

Since $\int_0^\infty \frac{1}{1+x^3}\,dx$ has an infinite interval of integration, it is an improper integral of Type 1

iii $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$

Sol

Since $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$ has an infinite interval of integeration, it is an improper integral of Type 1

ii

iv $\int_0^{\pi/4} \cot x \, dx$

Sol

Since $y = \cot x$ has an infinite discontinuity at $x = 0$, $\int_0^{\pi/4} \cot dx$ is a Type-2 improper integeral.

Q2 Determine whether each integral is convergent or divergent. Evalute those that are convergent.

i $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$

Sol

$$= \int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$$

$$= \lim_{t \to \infty} \int_3^t (x-2)^{-3/2} dx$$

$$= \lim_{t \to \infty} \left[-2(x-2)^{-1/2} \right]_3^t$$

$$= \lim_{t \to \infty} \left(\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}} \right)$$

$= 0 + 2$

$= 2$ (convergent)

ii $\int_{-\infty}^{0} \frac{x}{x^4+4}\,dx$

Sol

$= \int_{-\infty}^{0} \frac{x}{x^4+4}\,dx$

$= \lim_{t\to-\infty} \int_{t}^{0} \frac{x}{x^4+4}\,dx$

$= \lim_{t\to-\infty} \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)\right]_t^0$

$= \lim_{t\to-\infty} \left[0 - \frac{1}{4}\tan^{-1}\left(\frac{t^2}{2}\right)\right]$

$= -\frac{1}{4}\left(\frac{\pi}{2}\right)$

$= -\frac{\pi}{8}$ (convergent)

iv $\int_{1}^{\infty} \frac{dx}{\sqrt{x} + x\sqrt{x}}$

Sol

Let $x = u^2$

$dx = 2u\,dx$

$\int \frac{2u\,du}{\sqrt{u^2} + u^2}$

$\int \frac{2\,du}{1+u^2}$

$2\tan^{-1}u + C$

$2\tan^{-1}\sqrt{x} + C$

iii $\int_1^{\infty} \frac{1}{x(\ln x)^2} dx$

let $\ln x = u$

$\frac{dx}{x} = du$

$\int \frac{du}{u^2}$

$\frac{1}{t} + c$

Substitute

$u = \ln x$

$\frac{-1}{\sqrt{\ln x}} + C$

$\int_1^{\infty} \frac{1}{x(\ln x)^2} dx = \left[\frac{-1}{\ln x}\right]_1^{\infty}$

$\frac{-1}{\ln(\infty)} + \frac{1}{\ln(1)}$

$-\frac{1}{\infty} + \frac{1}{\infty} = 0$

v $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\lim_{x \to 1^-} \frac{1}{\sqrt{1-x^2}} = \lim_{x \to 1^+} \frac{1}{\sqrt{1-x^2}}$

$\frac{1}{\sqrt{0}^+} = \frac{1}{\sqrt{0}^+}$

integeral is discontinous at $x=1$

Impropertype 2

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$

$$\lim_{b\to 1^-} \left[\sin^{-1} x\right]_0^b$$

$$\lim_{x\to 1^-} \sin^{-1} b - \sin^{-1} 0$$

$$\sin^{-1} 1 - 0 = \frac{\pi}{2}$$

vi $\int_0^{\pi/2} \frac{\cos}{\sqrt{\sin\theta}} d\theta$

Sol $\sin\theta = x$

lim of integration will change form $\int_0^{\pi/2}$ to $\int_{\sin 0}^{\sin \pi/2} = \int_0^1$

$$= \int_0^1 \frac{1}{\sqrt{x}} dx$$

$$= [2\sqrt{x}]_0^1$$

$$= 2\sqrt{1} - 2\sqrt{0} - 0$$

$$= 2$$

vii $\int_0^1 \frac{e^{1/x}}{x^3} dx$

Sol let $\frac{1}{x} = t$

$$\frac{-dx}{x^2} = dt$$

$$\int -te^t dt$$

Now we will perform integeration by parts

$u = -t \quad dv = e^t dt$

$du = -dt$

$v = e^t$

$$\int -te^t dt = -te^t - \int e^t(-dt)$$

$-te^{t} + \int e^{t} dt$

$-te^{t} + e^{t} + C$

$t = \frac{1}{x}$

$-\frac{e^{-1x}}{x} + e^{1/x} + C$

$\int_0^1 \frac{e^{1/x}}{x^2} dx = \lim_{x \to 0^+} \left[\frac{-e^{1/x}}{x} + e^{1/x} \right]_t^1$

$\left[\frac{-e^{1/t}}{1} + e^{1/t} \right] - \lim_{t \to 0^+} \left[\frac{-e^{1/t}}{t} + e^{1/t} \right]$

$e^{\infty}[e_0 - 1]$

Diverge

viii $\int_0^1 x \ln x \, dx$

$u = \ln x$

$du = x dx$

$du = \frac{dx}{x}$

$v = \frac{x^2}{x}$

$\int x \ln x \, dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \frac{dw}{w}$

$\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$

$\int_0^1 x \ln x \, dx = \lim_{t \to 0^+} \left[\frac{x^2 \ln x}{2} - \frac{x^2}{4} \right]_t^1$

$0 - \frac{1}{4} - \lim_{t \to 0^+} \frac{t^2 \ln t}{2} + \lim_{t \to 0^+} \frac{t^2}{4}$

$-\frac{1}{4} + \lim_{t \to 0^+} - \frac{\ln t}{2/t^2}$

lim of form $\frac{\infty}{\infty}$, therefore appling l Hospital

$$\frac{1}{4} + \lim_{t\to 0^+} \frac{-1/t}{-4/t^3}$$

$$-\frac{1}{4} + \lim_{t\to 0^+} \frac{t^2}{4}$$

$$-\frac{1}{4} + 0 = -\frac{1}{4} \rightarrow \text{Converage}$$

Q3 Sketch the region enclosed by the given curves and find its area.

ii $y = \sec^2 x$, $y = 8\cos x$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

Sol

The curves intersect when $8\cos x = \sec^2 x$

$\Rightarrow 8\cos x^3 = 1$

$\cos x^3 = \frac{1}{8}$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ for $0 < x < \frac{\pi}{2}$

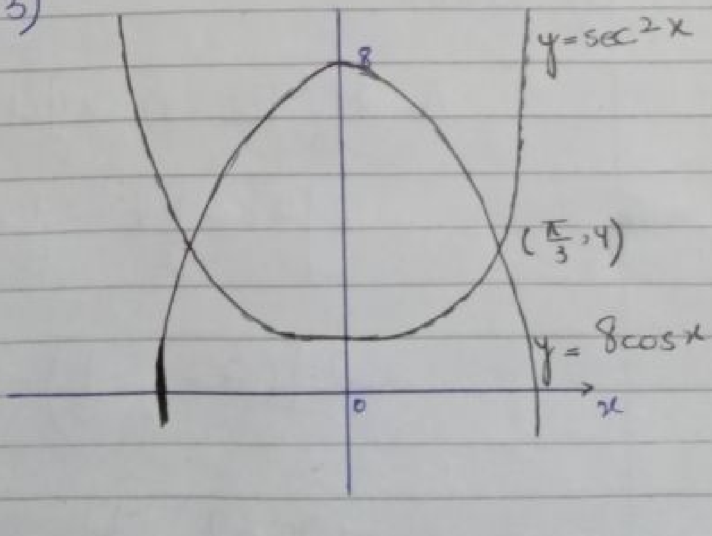
$A = 2\int_0^{\pi/3} (8\cos x - \sec^2 x)\,dx$

$= 2\left[8\sin x - \tan x\right]_0^{\pi/3}$

$= 2\left(8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right)$

$= 2(3 - \sqrt{3})$

$= 6\sqrt{3}$



iii $y = x^4$, $y = 2 - |x|$

Sd

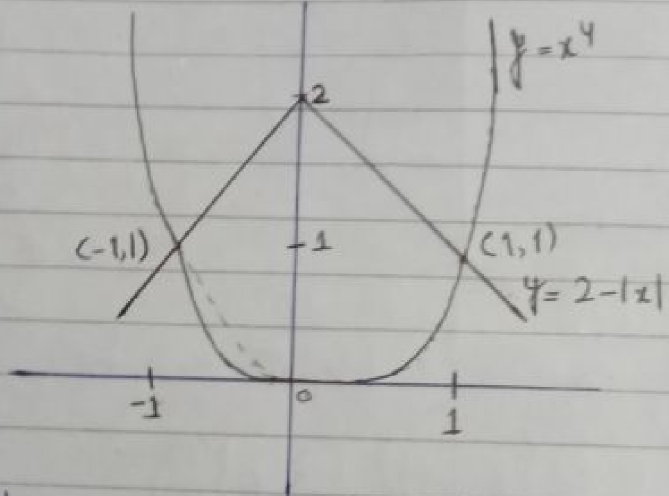
The curves intersect when

$$x^4 = 2 - |x|$$

$$x^4 - 2 + |x| = 0$$

$x^4 - 2 + x = 0$ $\quad$ $x^4 - 2 - x = 0$

$x = 1$ $\quad$ $x = -1$



By inspection we can see that the area of the region enclosed by the curves is twice the area enclosed in the first quadrant

$$A = 2\int_0^1 [(2-x) - x^4]\,dx$$

$$= 2\left[2x - \frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1$$

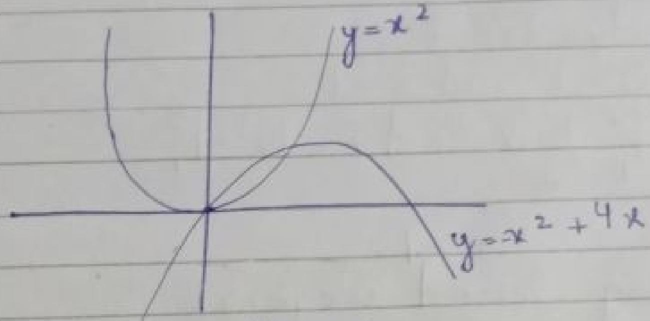
$$= 2\left[\left(2 - \frac{1}{2} - \frac{1}{5}\right) - 0\right]$$

$$\Rightarrow 2\left[\frac{13}{10}\right] = \frac{13}{5}$$

i $y=x^2, y=4x-x^2$

Sol

$$4-x^2=x^2$$
$$0=2x^2-4x$$
$$0=2x(x-2)$$
$$x=0,2$$



top is $y=4x-x^2$

bottom is $y=x^2$

$$A=\int_0^2 ((4x-x^2)-x^2)dx$$
$$\int_0^2 (4x-2x^2)dx$$
$$\left[2x-2\cdot\frac{1}{3}x^3\right]_0^2$$
$$2(2)^2-\frac{2}{3}(2)^3-(0-0)$$
$$8-\frac{16}{3}=\frac{24-16}{3}=\frac{8}{3}$$

Q4

i $y = \frac{x}{\sqrt{1+x^2}}$, $y = \frac{x}{\sqrt{9-x^2}}$ $x \geq 0$

Sol

$$A = \int_0^2 \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{9-x^2}} \, dx$$

$V = 1 + x^2$

$dv = 2x\,dx$

$0 - 1$ and $2 \to 5$

$$\int_1^5 \frac{1}{2} v^{-1/2} \, dv = |v^{1/2}|_1^5$$

$\sqrt{5} - 1$

Solving Second integeral

$z = 9 - x^2$

$dz = -2x\,dx$

$0 \to 9$ and $2 - 5$

$$\int^5 -\frac{1}{2} v^{-1/2} \, dz = \int_5^1 \frac{1}{2} v^{-1/2} \, dz$$

$[v^{1/2}]_5^1 = 3 - \sqrt{5}$

0.472

ii $y = \frac{\ln x}{x}$, $y = \frac{\ln y^2}{x}$

$$\int_1^e \frac{\ln x}{x} - \frac{(\ln x)^2}{x} \, dx$$

$$-\int_1^e \frac{\ln x - \ln x^2}{x} \, dx$$

$\ln x = v$ $\frac{dx}{x} = dv$

$$\left[\frac{v^2}{2} - \frac{v^3}{3}\right]_0^1$$

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

limit of integeration will change from

$$\int_1^e \longrightarrow \int_{\ln 1}^{\ln e} = \int_0^1$$

$$\int_0^1 v - v^2 \, dv$$

Q5 Find the limit, use L'Hospital Rules where appropriate.

i $\lim\limits_{x\to 4} \dfrac{x^2-2x-8}{x-4}$

Sol

This limit has the form $\frac{0}{0}$

Therefore using L'Hospital rule

$$= \lim_{x\to 4} \frac{x^2-2x-8}{x-4} \overset{H}{=}$$

$$= \lim_{x\to 4} \frac{2x-2}{+4}$$

$$= 2(4)-2$$

$$= 6$$

ii $\lim\limits_{x\to(\frac{\pi}{2})^+} \dfrac{\cos x}{1-\sin x}$

Sol

The limit has the form $\frac{0}{0}$

Therefore using L'Hospital rule

$$= \lim_{x\to(\frac{\pi}{2})^+} \frac{\cos x}{1-\sin x} \overset{H}{=}$$

$$= \lim_{x\to(\pi/2)^+} \frac{-\sin x}{-\cos x}$$

$$\Rightarrow \lim_{x\to(\pi/2)^+} \tan x = -\infty$$

v $\lim_{x \to 1} \frac{x^a - 1}{x^b - 1}, b \neq 0$

Sol

The limit has the form $\frac{0}{0}$

Therefore using L'Hospital Rule

$$= \lim_{x \to 1} \frac{x^a - 1}{x^b - 1}, [b \neq 0] \underline{\underline{H}}$$

$$= \lim_{x \to 1} \frac{a x^{a-1}}{b x^{b-1}}$$

$$= \frac{a x^{1-1}}{b x^{1-1}}$$

$$= \frac{a}{b}$$

vi $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

Sol

The limit has the form $\infty - \infty$

$$= \lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$$

$$= \lim_{x \to 1} \left(\frac{x \ln x - (x-1)}{(x-1)\ln x} \right) \underline{\underline{H}}$$

$$= \lim_{x \to 1} \frac{x(1/x) + \ln x - 1}{(x-1)(1/x) + \ln x}$$

$$= \lim_{x \to 1} \frac{\ln x}{1 - (1/x) + \ln x}$$

$$= \lim_{x \to 1} \frac{1/x}{1/x^2 + 1/x} \cdot \frac{x^2}{x^2} \quad \underline{\underline{H}}$$

$$= \lim_{x \to 1} \frac{x}{1 + x}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

iii. $\displaystyle \lim_{\theta \to \pi} \frac{1 + \cos\theta}{1 - \cos\theta}$

Sol

$$\lim_{\theta \to \pi} \frac{1 + \cos\theta}{1 - \cos\theta}$$

$$= \frac{1 + \cos(\pi)}{1 - \cos(\pi)}$$

$$= \frac{1 + (-1)}{1 - (-1)}$$

$$= \frac{1 - 1}{2}$$

$$= 0$$

v. $\displaystyle \lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$

Evaluate the numerator or denominator form $\frac{0}{0}$

$$\lim_{x \to 0} \left(\frac{\cos mx - \cos nx}{x^2} \right)$$

$$= \frac{\cos(m.0) - \cos(n.0)}{0^2}$$

$$\Rightarrow \frac{1-1}{0} = \frac{0}{0}$$

Because x^2 is nonzero near $x=0$ and so derivate $2x$. we can use L Hospitals rule

$$\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$$

$$\lim_{x \to 0} \frac{\frac{d(\cos mx - \cos nx)}{dx}}{\frac{d(x^2)}{dx}}$$

$$\lim_{x \to 0^-} \frac{m \sin nx + n \sin nx}{2x}$$

$\frac{0}{0} \rightarrow$ Using L'Hospital rule

$$\lim_{x \to 0} \frac{-m \sin mx + \sin nx}{2x}$$

$$= \lim_{x \to 0} \left(\frac{-m \sin mx + xS}{12x} \right)^1$$

vii $\lim_{x\to 0} (\text{cosec}\, x - \cot x)$

Sol $\cot x = \frac{\cos x}{\sin x}$, $\text{cosec}\, x = \frac{1}{\sin x}$

$\lim_{x\to 0} (\text{cosec}\, x - \cot x) = \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$

$\lim_{x\to 0} \left(\frac{1-\cos x}{\sin x}\right)$

$= \frac{0}{0}$

$\lim_{x\to 0} (1-\cos x) = 1-\cos 0$

$= 1-1$

$= 0$

$\lim_{x\to 0} \sin x = \sin 0$

$= 0$

$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{(1-\cos x)'}{(\sin x)'}$

$\lim_{x\to 0} \frac{\sin x}{\cos x}$

$\frac{\sin(0)}{\cos(0)} = \frac{0}{1}$

$= 0$