Question 3:

1)

$$\mu = \int_{\text{all } x} P(x)\, x \, \partial x$$

$$\sigma^2 = \int_{\text{all } x} (x-\mu)^2 \, P(x) \, \partial x$$

2) unbiased : $\langle \bar{a} \rangle = a$

Show $\hat{\mu}$ is an unbiased estimator of $\langle x \rangle = \mu$

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\langle \hat{\mu} \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} x_i \right\rangle$$

sample mean

$$= \frac{1}{N} \sum_{i=1}^{N} \langle x_i \rangle$$

$$= \frac{1}{N} \left(N \times \langle x_i \rangle \right)$$

$$= \langle x \rangle = \mu \therefore \text{unbiased.}$$

population mean

3

$$\sigma^2_{True} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\sigma^2_{EST} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\hat{\mu})^2$$

show that σ^2_{EST} is $\overbrace{\text{not an unbiased}}^{\text{is biased}}$ estimator of σ^2_{True}

$$\langle\sigma^2_{EST}\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N}(x_i-\hat{\mu})^2\right\rangle$$

$$= \left\langle \frac{1}{N}\sum_{i=1}^{N}\left((x_i-\mu)-(\hat{\mu}-\mu)\right)^2\right\rangle$$

$$= \left\langle \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2 - 2(x_i-\mu)(\hat{\mu}-\mu)+(\hat{\mu}-\mu)^2\right\rangle$$

$(\times \frac{N}{N} = 1)$

$$= \left\langle \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2 \underbrace{- \frac{2}{N}(\hat{\mu}-\mu)\sum_{i=1}^{N}(x_i-\mu)} + (\hat{\mu}-\mu)^2\right\rangle$$

$$= -2(\hat{\mu}-\mu)\left[\frac{1}{N}\sum_{i=1}^{N}x_i - \frac{1}{N}\sum_{i=1}^{N}\mu\right]$$

$$= -2(\hat{\mu}-\mu)\left[\hat{\mu}-\mu\right]$$

$$\rightarrow = \left\langle \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2 - 2(\hat{\mu}-\mu)^2 + (\hat{\mu}-\mu)^2\right\rangle$$

$$\to \langle \sigma_{Est}^2 \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 - (\hat{\mu} - \mu)^2 \right\rangle$$

$$= \left\langle \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 \right\rangle - \langle (\hat{\mu} - \mu)^2 \rangle$$

$$= \langle \sigma_{True}^2 \rangle - \langle (\hat{\mu} - \mu)^2 \rangle < \sigma_{True}^2$$

$$= \left(1 - \frac{1}{N}\right) \sigma_{True}^2$$

$$\therefore \quad \frac{\langle \sigma_{Est}^2 \rangle}{1 - \frac{1}{N}} = \frac{N}{N-1} \langle \sigma_{Est}^2 \rangle$$

is an unbiased estimator.