Asset Allocation
Simulation Project Report

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Introduction

Our objective is to generate synthetic datasets and test the empirical performances of various asset allocation strategies. We will simulate the dataset and test these strategies to see which strategy works best for a given situation or a condition.

In this report, we will be covering the data generation process, and create two kinds of datasets: a heavy tailed distribution and a light tailed distribution. A heavy tailed distribution has thicker tails than a lighter tailed distribution and has more extreme values than a lighter tailed distribution. Afterwards we will implement twelve different asset allocation strategies and test their performances to determine the best strategy under a given situation.

Data Generation Process

Throughout the project, we **fix** the risk-free rate r to be 0.002.

We first simulate the data that we need in order to apply the strategies. In the market, there is one risk-free asset with risk-free rate r and N risky assets. In particular, we adopt the K-factor model on the **excess return** R_t at time t of $N-K$ risky assets:

$$R_t = \alpha + \mathbf{B}R_{factor,t} + \epsilon_t,$$

where we denote $\alpha \in \mathbb{R}^{N-K}$ as the mispricing vector, $\mathbf{B} \in \mathbb{R}^{(N-K)\times K}$ as the factor value matrix, $R_{factor,t} \in \mathbb{R}^K$ as the factor coefficient vector at time t, and $\epsilon_t \in \mathbb{R}^{N-K}$ as the idiosyncratic noise at time t. Please note that in this factor model, K factors stand for K different assets. And the returns of the rest $N-K$ risky assets are linear combination of these K factors plus some noise.

(Light tailed distribution) Suppose $\alpha = \mathbf{0}$, $R_{factor,t} \sim \mathcal{N}(0.007, 0.002)$, $\epsilon_t \sim \mathcal{N}(0, \Sigma_\epsilon)$. We further assume Σ_ϵ is diagonal, and each diagonal element $\sqrt{\Sigma_{\epsilon ii}} \sim \mathcal{U}_{[\frac{0.10}{\sqrt{12}}, \frac{0.30}{\sqrt{12}}]}$. Apply this 1-factor model to generate the excess return of $N = 10$ risky assets for $T = 24000$ time steps.

1. Simulate the loading vector $\mathbf{B}$ for all $N-1$ risky assets. For each i, $\mathbf{B}_i \sim \mathcal{U}_{[0.50,1.50]}$.
2. Simulate the noise covariance matrix Σ_ϵ for all $N-1$ risky assets. For each i, $\sqrt{\Sigma_{\epsilon ii}} \sim \mathcal{U}_{[\frac{0.10}{\sqrt{12}}, \frac{0.30}{\sqrt{12}}]}$.
3. For $t = 1$ to T:
   Simulate $R_{factor,t} \sim \mathcal{N}(0.007, 0.002)$, $\hat{\epsilon_t} \sim \mathcal{N}(0, \Sigma_\epsilon)$. Then $R_t = \alpha + \mathbf{B}R_{factor,t} + \hat{\epsilon_t}$.

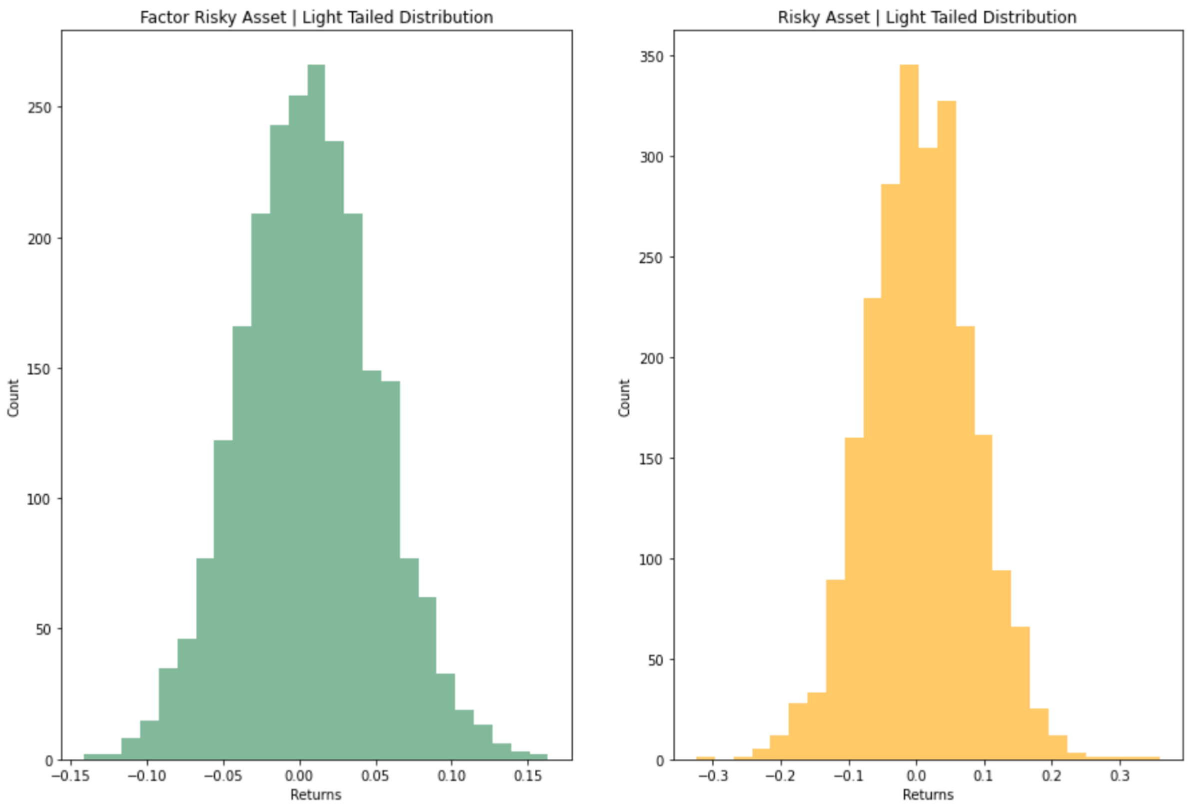

Figure 1: Histogram of Factor Risky Asset and Factor Asset for Light Tailed Distribution

(Heavy tailed distribution) Again suppose $\alpha = \mathbf{0}$, but $R_{factor,t} \sim \sqrt{0.002} \cdot t_{df=5} + 0.007$, where $t_{df=5}$ denotes a T-distribution with degrees of freedom 5. Then we achieve a heavy tail distribution for the factor risky asset. In order to achieve a heavy tail for the excess return of other risky assets, we need to change the distribution of ϵ_t as well. Consider ϵ_t follows a multivariate T distribution with degree of freedom ν:

$$\epsilon_t \sim \mu + \sqrt{\frac{\nu}{U}} Z,$$

where $Z \sim \mathcal{N}(0, \Sigma)$, $U \sim \chi^2_\nu$, Σ is the covariance matrix and ν is the degree of freedom. Then ϵ_t follows a $t_{df=\nu}(\mu, \Sigma)$ multivariate distribution. To create a similar setting to the light tailed case above, we let $\mu = 0$, $\nu = 5$ and $\Sigma = \Sigma_\epsilon$ (which is already generated above).

Adopt following procedure to simulate the heavy-tailed noise and apply to the 1-factor model to generate the excess return of $N = 10$ assets for $T = 24000$ time steps.

1. Use the loading vector $\mathbf{B}$ sampled in the light tailed distribution case.
2. Use the noise $\hat{\epsilon_t}$ sampled in the light tailed distribution case in Sec. 1.
3. For $t = 1$ to T:
   Simulate $r_t \sim t_{df=5}$. Then $R'_{factor,t} = \sqrt{0.002} \times r_t + 0.007$.
4. For $t = 1$ to T:
   Simulate $U_t \sim \chi^2_\nu$. Heavy tail noise $\epsilon'_t = \sqrt{\frac{\nu}{U_t}} \hat{\epsilon_t}$ and $R'_t = \alpha + \mathbf{B} R'_{factor,t} + \epsilon'_t$.

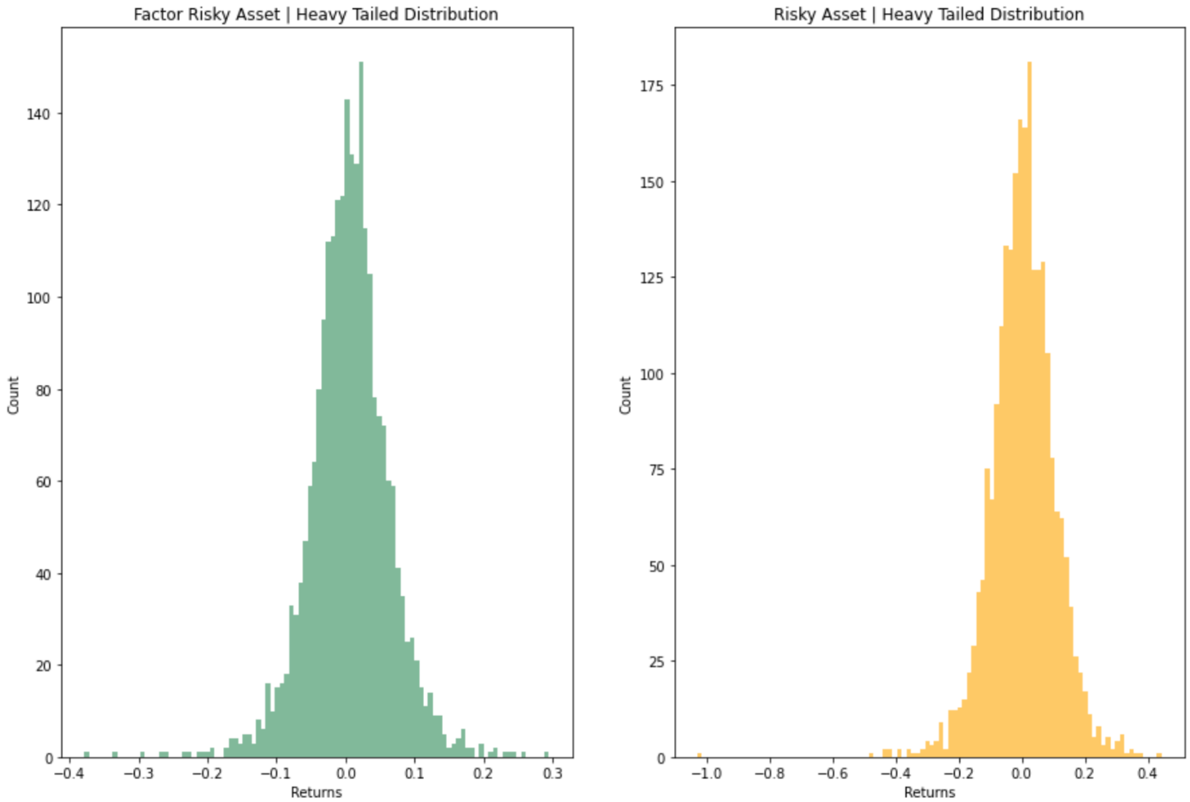

Figure 2: Histogram of Factor Risky Asset and Factor Asset for Heavy Tailed Distribution

Methods Implemented

In this project, we will implement and compare 12 asset allocation methods:

1. $1/N$ with rebalancing (ew)

The portfolio is divided equally among the assets. Weight of each asset for every period t is as follows:

$$w_{i,t} = \frac{1}{N};\ where\ N\ is\ the\ number\ of\ risky\ assets\ and\ t\ is\ the\ period$$

The weights is $1/N$ for all assets at all time period t.

2. Market/ Factor Portfolio (mkt)

In this allocation strategy the entire weight is given to the factor portfolio, and hence the name market portfolio.

$$w_{i,t} = \begin{cases} 1 & \text{for } i = factor \\ 0 & \text{otherwise} \end{cases}$$

3. Sample Based Mean Variance (mv)

The aim to minimize the variance of the portfolio. We can formulate the problem as the following optimization problem:

$$\max_{a \in \mathbb{R}^d} -var(X^a) = \max_{a \in \mathbb{R}^d} \vec{a}^T \Sigma \vec{a}$$

$$\text{such that } E[X^a] = r + (\vec{\mu} - r\vec{e})^T \vec{a} \geq \kappa$$

We have the following closed form solution for weight vector:

$$a^* = \frac{\kappa - r}{(\vec{\mu} - r\vec{e})^T \Sigma^{-1} (\vec{\mu} - r\vec{e})} \Sigma^{-1} (\vec{\mu} - r\vec{e}) = \gamma \Sigma^{-1} (\vec{\mu} - r\vec{e})$$

4. Sample Based Unbiased Mean Variance (u-mv)

The estimation in method 3: Sample Based Mean Variance (mv) of the optimal portfolio allocation is biased. The bias can be eliminated as follows:

$$\max_{a \in \mathbb{R}^d} -var(X^a) = \max_{a \in \mathbb{R}^d} \vec{a}^T \Sigma \vec{a}$$

$$\text{such that } E[X^a] = r + (\vec{\mu} - r\vec{e})^T \vec{a} \geq \kappa$$

$$\hat{\Sigma}_{sample} = \frac{\mathrm{M}}{\mathrm{M} - 1} \hat{\Sigma}$$

$$E[\hat{\Sigma}^{-1}] = \frac{\mathrm{M}}{\mathrm{M-N-2}}\Sigma^{*-1}$$

$$\Sigma^{*-1}(\vec{\mu}^{*} - r\vec{e}) = \frac{\mathrm{M-N-2}}{\mathrm{M}}\hat{\Sigma}^{-1}(\hat{\mu} - r\vec{e}) \;= \frac{\mathrm{M-N-2}}{\mathrm{M-1}}\hat{\Sigma}^{-1}{}_{sample}(\hat{\mu} - r\vec{e})$$

We have the following closed form solution for weight vector at the end of each rebalancing period:

$$w = \left(\frac{M-N-2}{M-1}\right)\frac{(\kappa - r)}{\hat{\mu}'^{T}\hat{\Sigma}^{-1}\hat{\mu}'}\hat{\Sigma}^{-1}\hat{\mu}'$$

Where excess returns $\hat{\mu}' = (\hat{\mu} - re)$

5. Bayes-Stein (bs)

James-Stein estimator allows us to shrink the data towards a point specified at μ0 with a pre-defined weight as follows:

$$\mu_0 = \frac{\mu^T\Sigma^{-1}e}{e^T\Sigma^{-1}e}e$$

The point is corresponding to the minimum variance portfolio.

The biased estimate is as follows:

$$\vec{\mu}_{JS} = \mu_0 + (1-\alpha)(\mu - \mu_0)$$

Weight can be estimated with the help of the Bayesian criteria as follows:

where $\mu_0 = \frac{\mu^T\Sigma^{-1}e}{e^T\Sigma^{-1}e}e$ and $\alpha = \frac{N+2}{N+2+(M-N-2)(\mu-\mu_0)^T\Sigma^{-1}(\mu-\mu_0)}$

We have the following closed form solution for weight vector at the end of each rebalancing period:

$$w = \frac{(\kappa - r)}{(\hat{\mu}^{JS} - re)^T\hat{\Sigma}^{-1}(\hat{\mu}^{JS} - re)}\hat{\Sigma}^{-1}(\hat{\mu}^{JS} - re)$$

6. Minimum Variance (min)

In this allocation strategy there is no requirement of the required rate of return and it choses a portfolio that minimizes the variance. We also impose self-financing condition here otherwise all our allocation would go into the risk free rate. We can translate this condition into the following optimization problem:

$$\max_{w\in\mathbb{R}^d} -w^T\hat{\Sigma}w$$
$$\text{subject to } \; e^Tw = 1$$

The optimal weights can be found as:

$$w^* = \frac{\Sigma^{-1}\vec{e}}{\vec{e}^{\mathrm{T}}\Sigma^{-1}\vec{e}}$$

7. Sample Bases Mean-Variance with no short sale (mv-c)

The optimization problem is as follows:

$$\begin{aligned} &\max_{w \in \mathbb{R}^d} \; -w^T \hat{\Sigma} w \\ s.t. \quad & r + \hat{\mu}'^T w \geq \kappa \\ & w \geq 0 \\ & 1 - e^T w \geq 0 \end{aligned}$$

In this strategy short selling is not allowed.

8. Bayes-Stein with no short sale (bs-c)

James-Stein estimator allows us to shrink the data towards a point specified at μ0 with a pre-defined weight as follows:

$$\mu_0 = \frac{\mu^T \Sigma^{-1} e}{e^T \Sigma^{-1} e} e$$

The biased estimate is as follows:

$$\hat{\mu}^{JS} = \alpha \mu_0 + (1 - \alpha)\hat{\mu}$$

Weight can be estimated with the help of the Bayesian criteria as follows:

$$\alpha = \frac{N + 2}{N + 2 + (M - N - 2)(\mu - \mu_0)^T \Sigma^{-1} (\mu - \mu_0)}$$

The weights can thus be computed as follows:

$$w = \frac{(\kappa - r)}{(\hat{\mu}^{JS} - re)^T \hat{\Sigma}^{-1} (\hat{\mu}^{JS} - re)} \hat{\Sigma}^{-1} (\hat{\mu}^{JS} - re)$$

The optimization problem is as follows:

$$\max_{w\in\mathbb{R}^d} \; -w^T\hat{\Sigma}w$$

$$s.t. \quad r + (\hat{\mu}^{JS} - re)^T w \geq \kappa$$

$$w \geq 0$$

$$1 - e^T w \geq 0$$

9. Minimum Variance with no short sale (min-c)

For this scenario, we shall assume the self-financing condition wherein the entirety of the allocated weights would contribute towards the risk-free rate. The aim of this strategy is to minimize the variance. We can translate this condition into the following optimization problem:

$$\max_{a\in\mathbb{R}^d} \vec{\mathrm{w}}^T\Sigma\vec{\mathrm{w}}$$

$$\text{such that } \vec{e}^T\vec{\mathrm{w}} = 1$$

$$\text{and } \mathrm{w_i} \geq 0 \text{ for } i \in [1; N]$$

10. Uncertainty in mean with box uncertainty set (r-m-1)

For this scenario, we shall consider parameter uncertainty in terms of μ. This is because of the complexity of estimation of this parameter. This can be defined as:

$$U = \prod_{k=1}^{d} [\hat{\mu}_k - \delta_k, \hat{\mu}_k + \delta_k] \text{ x } \{\Sigma\}$$

Our optimization problem will get transformed to the below:

$$\max_{w\in\mathbb{R}} -w^T\hat{\Sigma}w$$

$$s.t. \quad r + \hat{\mu}'w - \delta^T|w| \geq \kappa$$

Here, $\delta = 0.004$ and $\gamma = (\hat{\mu} - r\vec{e})^T\hat{\Sigma}^{-1}(\hat{\mu} - r\vec{e})$ herein, but is in general an exogenous tuning parameter.

11. Uncertainty in mean with ellipsoid uncertainty set (r-m-2)

For this scenario, we shall consider parameter uncertainty in terms of μ. This is because of the complexity of estimation of this parameter. This can be defined as:

$$U = \{\mu : (\mu - \hat{\mu})^T \Sigma^{-1} (\mu - \hat{\mu}) \leq \delta^2\} \text{ x } \{\Sigma\}$$

The optimization problem can be edited for ellipsoid uncertainty as follows:

$$\max_{w \in \mathbb{R}} -w^T \hat{\Sigma} w$$

$$s.t. \quad r + \hat{\mu}' w - \delta^T \sqrt{w^T \hat{\Sigma} w} \geq \kappa$$

Or Equivalently

$$\max_{a} \hat{\mu}^T \vec{\mathrm{w}} - \frac{\gamma}{2} \mathrm{w}^T \Sigma \vec{\mathrm{w}} - \delta \sqrt{\mathrm{w} \Sigma \vec{\mathrm{w}}}$$

$$\text{such that } \hat{\mu}^T \vec{\mathrm{w}} - \delta \sqrt{\vec{\mathrm{w}} \Sigma \mathrm{w}} \geq \kappa$$

where $\gamma = (\hat{\mu} - r\vec{e})^T \hat{\Sigma}^{-1} (\hat{\mu} - r\vec{e})$ and $\delta = 0.004$ herein, but is in general an exogenous tuning parameter.

12. Distributional Robust (dr-mv)

In this case we assume distributional uncertainity. We solve the following optimization problem

$$\min_{a} \left(\sqrt{\vec{\mathrm{w}}^T \hat{\Sigma} \vec{\mathrm{w}}} + \delta \left\| \mathrm{w} \right\|_p \right)^2$$

such that $\frac{1}{N} \sum_{i=1}^{N} \vec{R}_i^T \vec{\mathrm{w}} - \delta \left\| \vec{\mathrm{w}} \right\|_p \geq \kappa$ and $\delta = 0.004$ herein, but is in general an exogenous tuning parameter.

Where p=2

Performance Comparison of Strategies with Estimation

We will analyze and compare our strategies using the rolling window technique, which consists of the following three metrics: Out-of-Sample Sharpe Ratio (OSR), In-Sample Sharpe Ratio (ISR) and Turnover.

Out of Sample Sharpe Ratio (OSR)

$$OSR = \frac{\mu_{OSR} - r}{\hat{\sigma}_{OSR}},$$

where $\hat{\mu}_{OSR}$ denotes the sample average of the out-of-sample return μ_t achieved by the portfolio over all time steps from $t = M+1$ to T, and $\hat{\sigma}_{OSR}$ denotes the corresponding sample standard deviation. Specifically, out-of-sample μ_t is measured in the following way: when we apply different strategies using estimation with rolling window technique at step t, we use the excess returns of all N risky assets in steps $t-M$ to $t-1$ to estimate and solve for the optimal weight $w^{(t)}$. Then we calculate the return at time t:

$$\mu_t = \sum_{\text{asset } i \text{ in all assets}} w^{(t)}_{\text{asset } i} \times \text{return of asset } i \text{ at time } t$$

In-Sample Sharpe Ratio (ISR)

$$ISR = \frac{\hat{\mu}_{ISR} - r}{\hat{\sigma}_{ISR}},$$

where $\hat{\mu}_{ISR}$ denotes the sample average of the return in-sample μ_t achieved by the portfolio over all time steps from $t = M$ to T, and $\hat{\sigma}_{ISR}$ denotes the corresponding sample standard deviation. Specifically, in-sample μ_t is measured in the following way: when we apply different strategies using estimation with rolling window technique at step t, we use the excess returns of all N risky assets in steps $t-M-1$ to t to estimate and solve for the optimal weight $w^{(t)}$. Then we calculate the return at time t:

$$\mu_t = \sum_{\text{asset } i \text{ in all assets}} w^{(t)}_{\text{asset } i} \times \text{return of asset } i \text{ at time } t$$

Turnover

$$Turnover = \frac{\sum_{t=M+1}^{T} \sum_{j=1}^{N} |w_j^{(t+1)} - \tilde{w}_j^{(t+1)}|}{T-M},$$

where $\tilde{w}^{(t)}$, $w^{(t)}$ denotes the weight vector before and after rebalancing at time t respectively.

Here is an example: for the $1/N$ strategy, suppose there are only two assets, and the asset prices are \$1 and \$1 at time $t=1$. The $1/N$ strategy will place $w_1^{(1)} = 0.5$ and $w_2^{(1)} = 0.5$ of our wealth on them. At the next time step $t = 2$, the prices of the two assets become \$2 and \$0.5. Then, the our strategy at time $t = 2$ before rebalancing will have $\tilde{w}_1^{(2)} = \frac{1}{1+0.25} = 0.8$ while $\tilde{w}_2^{(2)} = \frac{0.25}{1+0.25} = 0.2$. To maintain our $1/N$ strategy at time $t = 2$, we need to rebalance the weight to $w_1^{(2)} = 0.5$ and $w_2^{(2)} = 0.5$. Then the turnover at time $t = 2$ is:

$$|\tilde{w}_1^{(2)} - w_1^{(2)}| + |\tilde{w}_2^{(2)} - w_2^{(2)}| = |0.8 - 0.5| + |0.2 - 0.5| = 0.6$$

Simulations Result
Part 2.1
N=10, M=120, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

	ISR	OSR	Turnover
ew	0.129738141	0.130084284	0.041384091
mkt	0.150069162	0.150238868	0
mv	0.324577518	0.094232994	0.255305329
u-mv	0.324577518	0.094232994	0.231313039
bs	0.267486304	0.106220272	0.346382958
min	0.149501626	0.136728661	0.108451298
mv-c	0.184680265	0.12315133	0.179836887
bs-c	0.171944096	0.116542632	0.222780988
min-c	0.139486745	0.130793553	0.149832736
r-m-1	0.30383073	0.087517244	0.273828142
r-m-2	0.31013637	0.080070899	0.289843827
dr-mv	0.297412418	0.090914312	0.280578521

Table 1: N=10, M=120, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

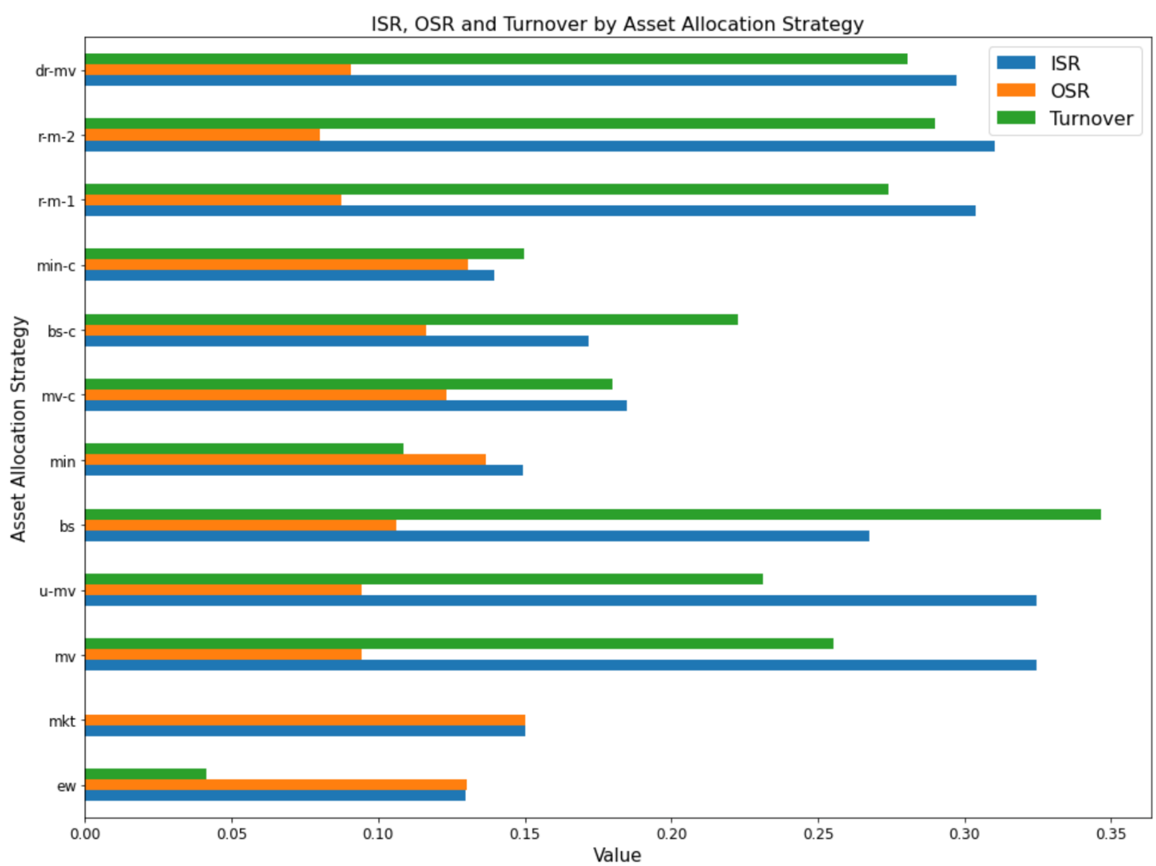

Figure 3: Bar Chart | N=10, M=120, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

N=10, M=120, T=2400 | Normal Distribution | α=0 | Heavy Tailed Distribution

	ISR	OSR	Turnover
ew	0.089495167	0.089172769	0.050886383
mkt	0.10511784	0.104831957	0
mv	0.303746175	0.042160448	0.261026426
u-mv	0.303746175	0.042160448	0.236560571
bs	0.251877915	0.089918356	0.385718635
min	0.104504563	0.090299872	0.129581355
mv-c	0.144265192	0.070156119	0.17577503
bs-c	0.133944973	0.057997041	0.199302681
min-c	0.102348691	0.09565604	0.09615336
r-m-1	0.28717074	0.037831077	0.283720619
r-m-2	0.288907801	0.037271452	0.347996209
dr-mv	0.279876828	0.036865836	0.290241402

Table 2: N=10, M=120, T=2400 | Normal Distribution | α=0 |Heavy Tailed Distribution

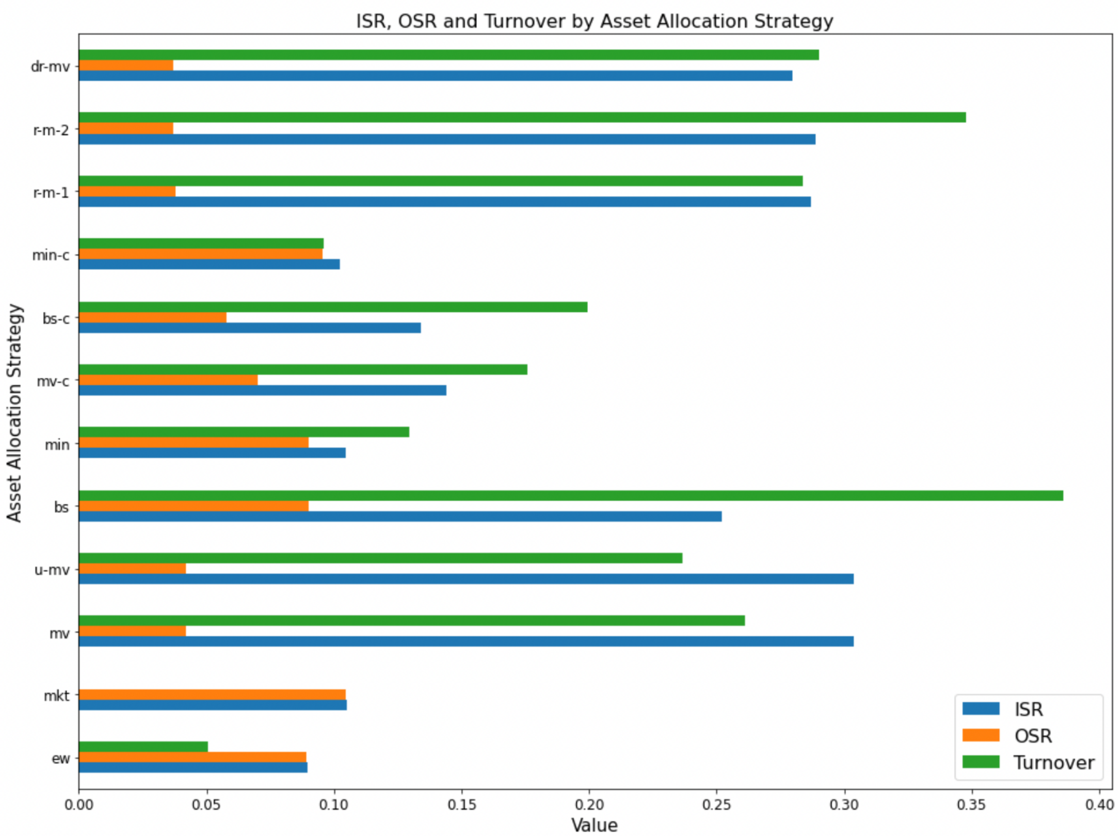

Figure 4: Bar Chart | N=10, M=120, T=2400 | Normal Distribution | α=0 |Heavy Tailed Distribution

2.2 Performance of Optimal Mean-variance

We still use the simulated data generated in Sec. 1. Since we know the true mean vector and covariance matrix in our simulation study, apply them to the mean variance problem, and compare with the methods 1-12 in terms of the metrics mentioned in Sec. 2.1.

Recall that in theory, any mean-variance efficient allocation achieves the same and the highest Sharpe ratio.

$$R_t' = B' R_{factor_t,t} + \hat{\epsilon}_t'$$

$$\mathbb{E}[R_t'] = B' \mathbb{E}[R_{factor,t}] = 0.007 B'$$

Therefore,

$Var[R_t'] = Var(R_{factor,t}) B' B'^T \ + \ \Sigma_\epsilon$

And,

$Var(R_{factor,t}) = 0.002$

Solving these equations gives us the optimal weights for Optimal Mean Variance Portfolio.

Light Tailed Distribution:
ISR: 0.15006916247039506
OSR: 0.15023886773056128
Turnover: 0.026393626428313194

Heavy Tailed Distribution:
ISR: 0.10511784014383459
OSR: 0.10483195671269742
Turnover: 0.03260832542015525

This is in line with our results in Part 2.1

3 Tuning-parameter Ablation Study

3.1 The Effect of Window Length

N=10, M=500, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

	ISR	OSR	Turnover
ew	0.131474967	0.132015369	0.041430226
mkt	0.152985926	0.153344292	0
mv	0.211663035	0.134517823	0.174237282
u-mv	0.211663035	0.134517823	0.169703754
bs	0.197939113	0.156589584	0.124735539
min	0.153668391	0.150550418	0.065998091
mv-c	0.135330126	0.11458617	0.107076929
bs-c	0.137546775	0.116833964	0.147626544
min-c	0.145180968	0.142990655	0.096225312
r-m-1	0.184845382	0.117872505	0.205051421
r-m-2	0.20046445	0.117139987	0.257067345
dr-mv	0.179459517	0.123046649	0.194353916

Table 3: N=10, M=500, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

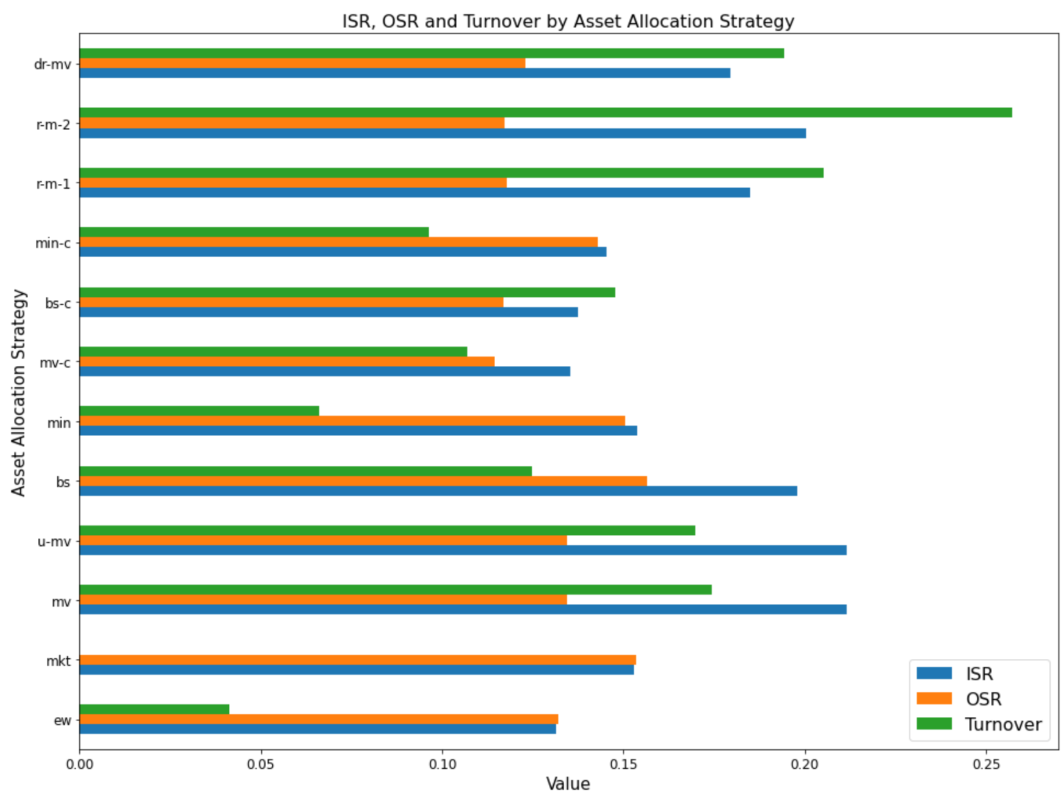

Figure 5: Bar Chart | N=10, M=500, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

N=10, M=500, T=2400 | Normal Distribution | α=0 | Heavy Tailed Distribution

	ISR	OSR	Turnover
ew	0.078725638	0.078260164	0.050728021
mkt	0.089275215	0.088708706	0
mv	0.149304785	0.046939194	0.218056941
u-mv	0.149304785	0.046939194	0.212671574
bs	0.124269567	0.076769884	0.174086927
min	0.080962851	0.075256138	0.079149185
mv-c	0.099999019	0.031824042	0.15793051
bs-c	0.094542813	0.028028323	0.142789353
min-c	0.088229443	0.08383218	0.033738172
r-m-1	0.136947732	0.050550349	0.234097927
r-m-2	0.137847814	0.04138514	0.290668107
dr-mv	0.124584645	0.04849397	0.232287097

Table 4: N=10, M=500, T=2400 | Normal Distribution | α=0 |Heavy Tailed Distribution

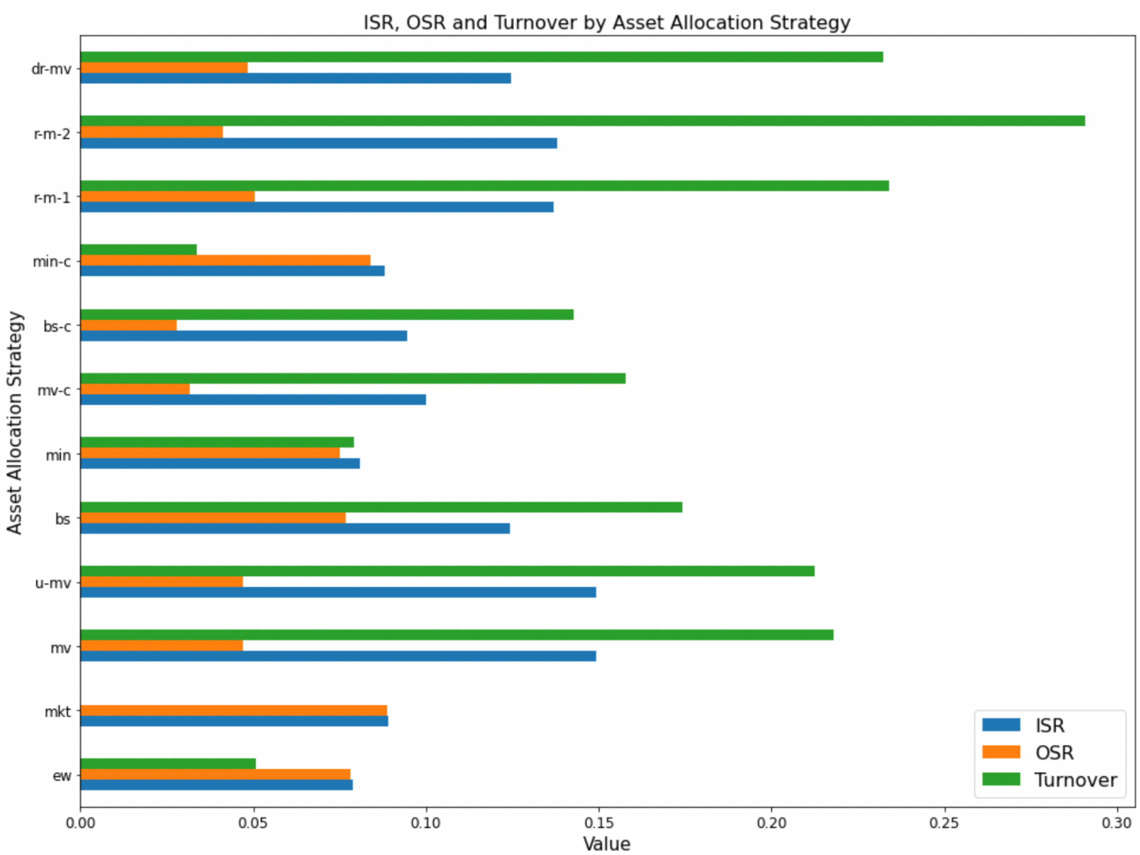

Figure 6: Bar Chart | N=10, M=500, T=2400 | Normal Distribution | α=0 |Heavy Tailed Distribution

3.2 The Effect of Asset Number

N=50, M=120, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

	ISR	OSR	Turnover
ew	0.143065	0.143059	0.046224
mkt	0.14002	0.140066	0
mv	0.813836	-0.02884	0.394494
u-mv	0.813836	-0.02884	0.225758
bs	0.709015	-0.00403	0.69895
min	0.129808	0.073668	0.383703
mv-c	0.304727	0.103429	0.194029
bs-c	0.243782	0.065349	0.367184
min-c	0.160341	0.143082	0.100464
r-m-1	0.726489	0.010393	0.288228
r-m-2	0.691866	0.014426	0.301535
dr-mv	0.720803	0.011247	0.354977

Table 5: N=50, M=120, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

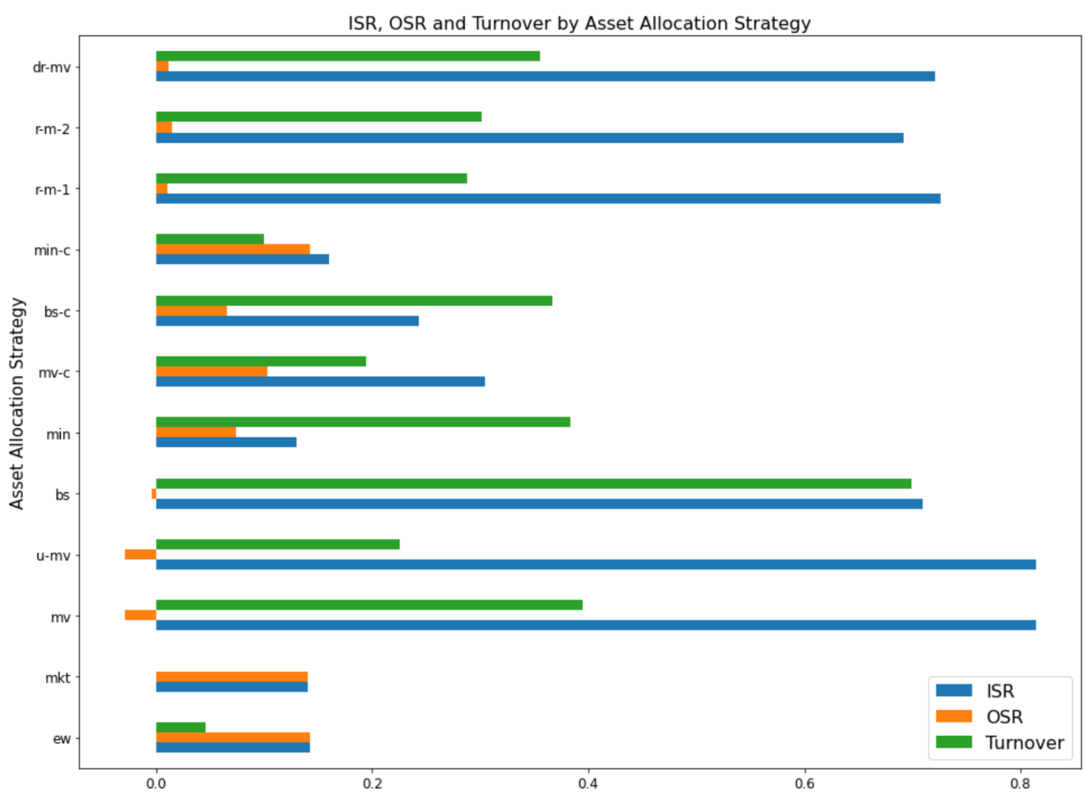

Figure 7: Bar Chart | N=50, M=120, T=2400 | Normal Distribution | α=0 | Light Tailed Distribution

N=50, M=120, T=2400 | Normal Distribution | α=0 | Heavy Tailed Distribution

	ISR	OSR	Turnover
ew	0.119936	0.119971	0.056323
mkt	0.12211	0.122201	0
mv	0.872251	0.039171	0.348463
u-mv	0.872251	0.039171	0.199456
bs	0.80283	0.04468	0.68774
min	0.036495	0.008122	0.453227
mv-c	0.244182	0.069053	0.170558
bs-c	0.237675	0.0878	0.261182
min-c	0.101777	0.087626	0.103039
r-m-1	0.781763	0.056776	0.258198
r-m-2	0.7751	0.054407	0.251641
dr-mv	0.758172	0.053839	0.298989

Table 6: N=50, M=120, T=2400 | Normal Distribution | α=0 | Heavy Tailed Distribution

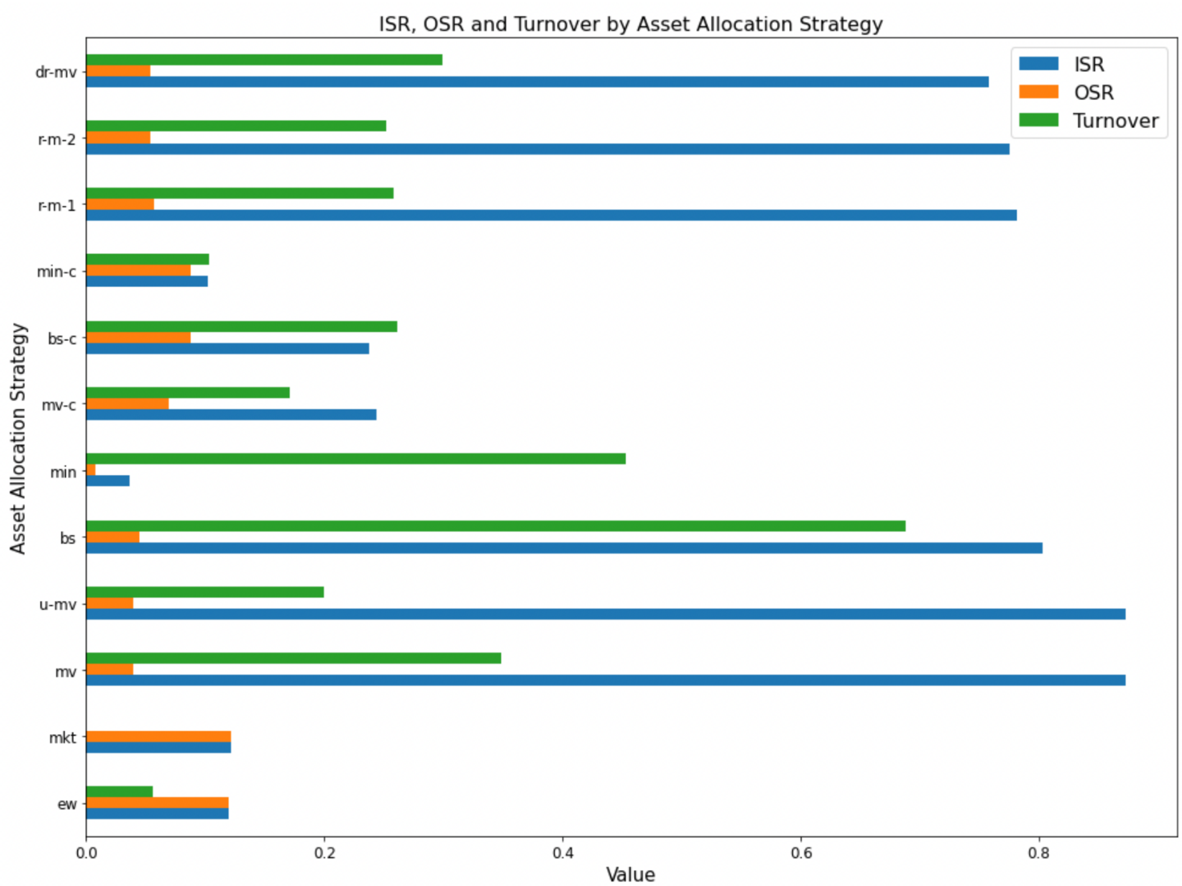

Figure 8: Bar Chart | N=50, M=120, T=2400 | Normal Distribution | α=0 | Heavy Tailed Distribution

3.3 The Effect of α

N=10, M=120, T=2400 | Normal Distribution | $\alpha \neq 0$ | Light Tailed Distribution

	ISR	OSR	Turnover
ew	0.163329	0.16478	0.04127
mkt	0.172342	0.173479	0
mv	0.371344	0.162888	0.1906
u-mv	0.371344	0.162888	0.172885
bs	0.32377	0.167702	0.2501
min	0.119299	0.112709	0.104305
mv-c	0.238818	0.167974	0.151572
bs-c	0.227598	0.160231	0.210593
min-c	0.14479	0.141088	0.084375
r-m-1	0.352986	0.175064	0.191962
r-m-2	0.348363	0.167667	0.194324
dr-mv	0.346721	0.178669	0.179168

Table 7: N=10, M=120, T=2400 | Normal Distribution | $\alpha \neq 0$ | Light Tailed Distribution

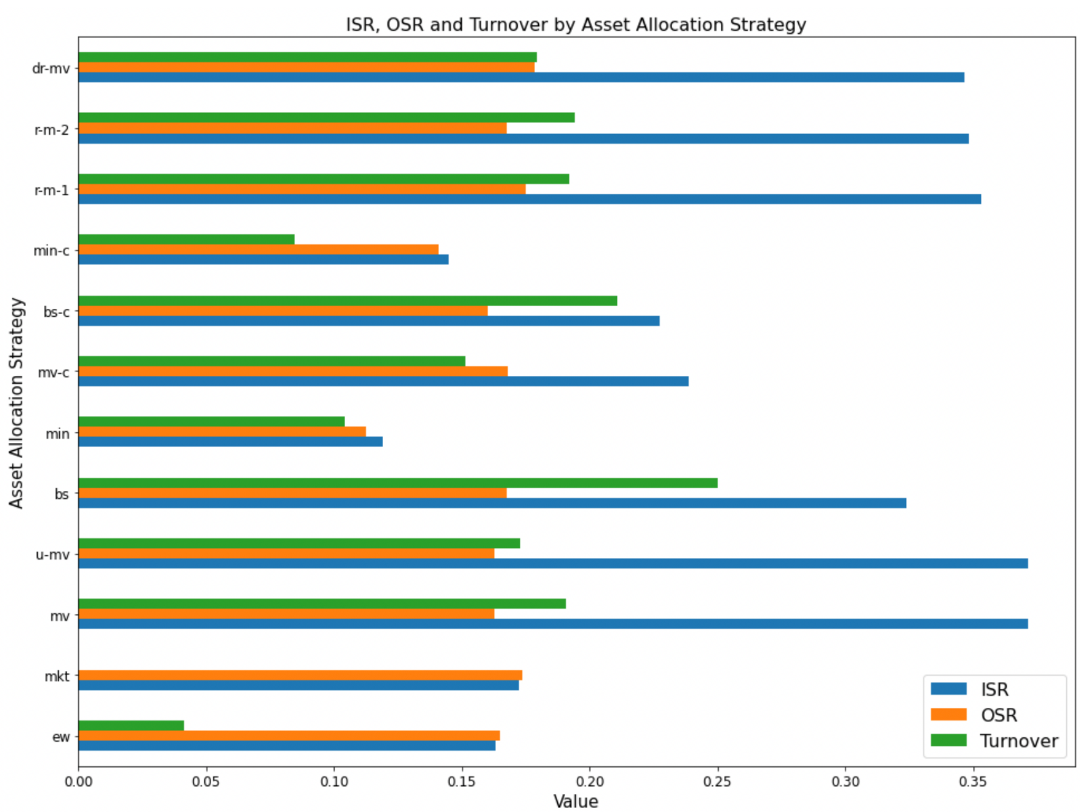

Figure 9: Bar Chart | N=10, M=120, T=2400 | Normal Distribution | $\alpha \neq 0$ | Light Tailed Distribution

N=10, M=120, T=2400 | Normal Distribution | $\alpha \neq 0$ | Heavy Tailed Distribution

	ISR	OSR	Turnover
ew	0.101126	0.101429	0.0503
mkt	0.130123	0.130554	0
mv	0.360067	0.124859	0.18926
u-mv	0.360067	0.124859	0.171533
bs	0.303374	0.116157	0.342231
min	0.10064	0.089387	0.127146
mv-c	0.175663	0.083941	0.160226
bs-c	0.173891	0.087709	0.19895
min-c	0.079964	0.072717	0.101584
r-m-1	0.335548	0.119127	0.203257
r-m-2	0.333933	0.11776	0.213188
dr-mv	0.328093	0.114909	0.199794

Table 8: N=10, M=120, T=2400 | Normal Distribution | $\alpha \neq 0$ | Heavy Tailed Distribution

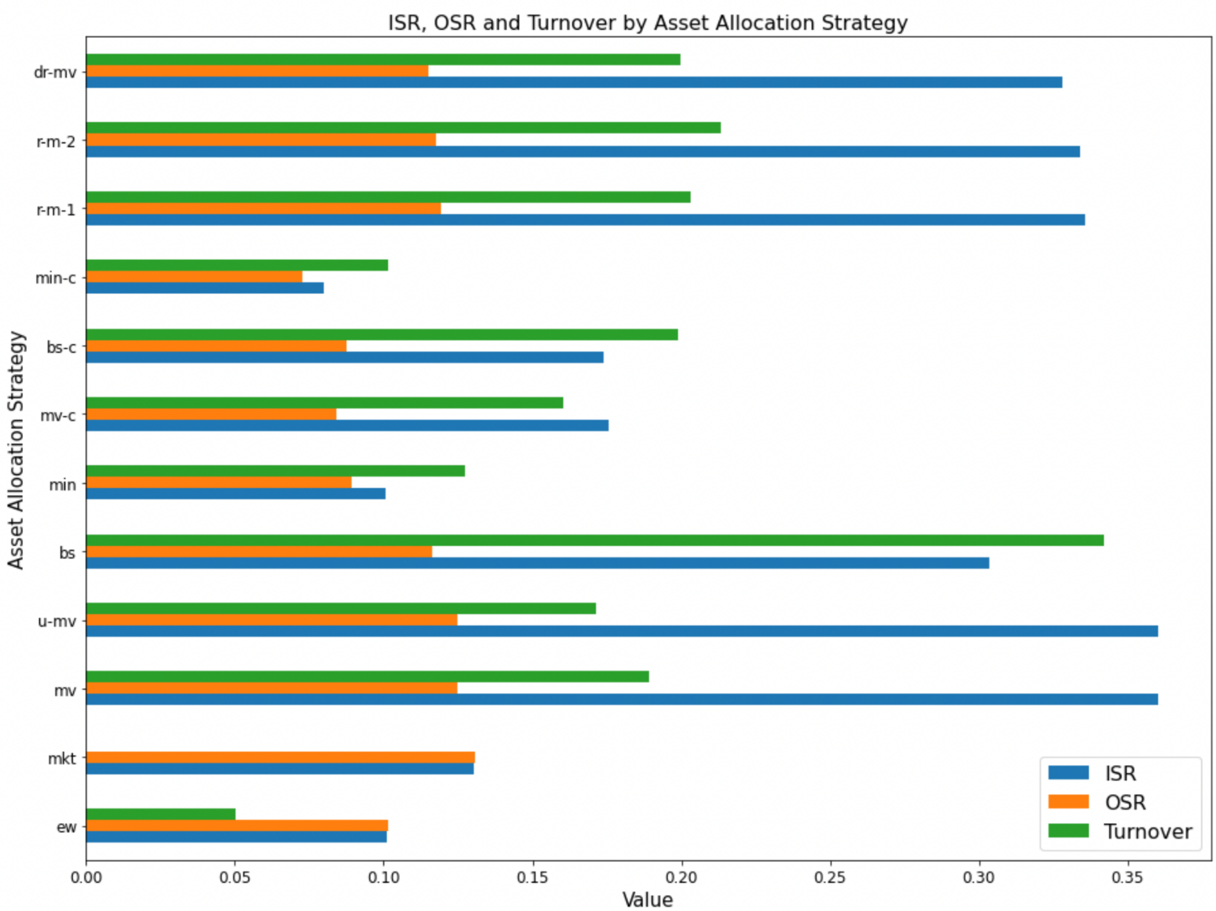

Figure 10: Bar Chart | N=10, M=120, T=2400 | Normal Distribution | $\alpha \neq 0$ | Heavy Tailed Distribution

4 Study on Intertemporal Correlation

N=10, M=120, T=2400 | α=0 | AR(1) | Light Tailed Distribution

	ISR	OSR	Turnover
ew	0.255299	0.255317	0.040756
mkt	0.286477	0.286366	0
mv	0.391722	0.201941	0.166424
u-mv	0.391722	0.201941	0.151045
bs	0.346977	0.254694	0.117221
min	0.304991	0.282136	0.065229
mv-c	0.313576	0.205434	0.177281
bs-c	0.295406	0.19857	0.216012
min-c	0.268851	0.257009	0.099843
r-m-1	0.353972	0.187461	0.178673
r-m-2	0.368653	0.184808	0.182898
dr-mv	0.348407	0.192993	0.198173

Table 9: N=10, M=120, T=2400 | Normal Distribution | α=0 | AR(1) | Light Tailed Distribution

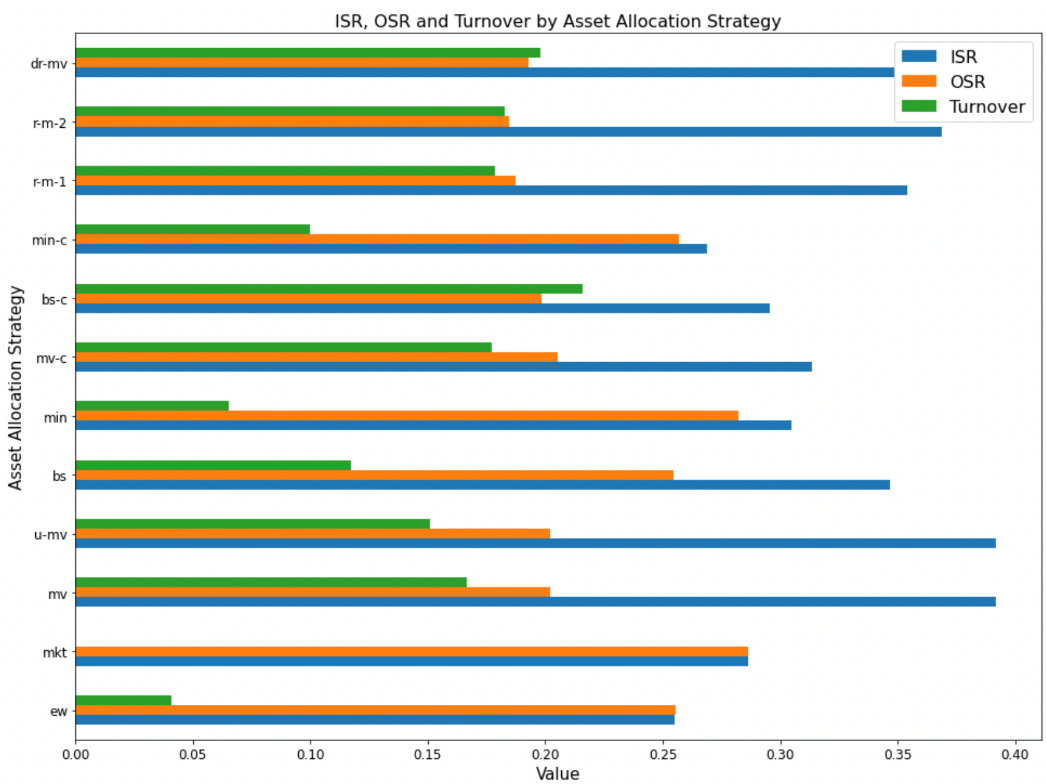

Figure 11: Bar Chart | N=10, M=120, T=2400 | Normal Distribution | α=0 | AR(1) | Light Tailed Distribution

5 Conclusion

In table 1 and table 2 we observe that in both light and heavy tailed distributions that market portfolio performs the best. This is inline with our expectations as alpha is zero.
Turnover is also 0 for market portfolio since weights do not update everything is allocated to market portfolio.

In table 3 and table 4 we observe that OSR have increased when we increase M (The rolling window length). Increasing M increases the data set over which the model is trained which improves the model estimation. One of the other reasons could be the model is less biased.
On the other hand, while the optimizer is showing better returns, this maybe at the cost of the number of times it iterates over the data, using which we are computing the mean of the estimates, which can end up having higher variance.

In table 5 and table 6 when we increase the number of assets to 50 the ISR of all strategies increases. This effect is observed due to the increase in data to train over. The larger dataset allows for better training and diversification, and as a result better optimization. This in turn improves the In-Sample ratio tremendously.
On the flipside, we also notice the dip in the Out-of-Sample ratios for most of the strategies. This means we have over trained the data and doesn't perform very well out of sample.

In table 7 and table 8, in case of non-zero alpha, we can see that the market portfolio is no longer the best performing portfolio. Mean-variance portfolio comfortably beating it. The reason for this is that there are risky assets which have a positive alpha, which in turn help the strategy to beat the market portfolio.

In table 9 observe that OSR in AR(1) is close to ISR values. This is expected as because in AR(1) next data is depended on previous value which reduces the variance of OSR.
Allocating the whole of the portfolio weight to the factor asset helps in minimizing the variance; is the fact that the returns of the min-c portfolio are almost identical to that of the factor portfolio. This happens as the variance of the portfolio is counterbalanced by the negative correlation. While we would hope to observe a similar effect on other strategies, we must remember the idiosyncratic noise that increases the variance. Moreover, it is also important to understand the significance of the loading vector, as auto-correlation may also lead to a higher variance. And thus, most of the portfolio weight is allocated to the factor asset in this case. This effect is only observed in min-c and not min is because in min-c, we are constrained by the fact that we are not allowed to short sale any assets in the allocation strategy.

Sample-based mean-variance portfolio: this portfolio strategy completely ignores the possibility of estimation error.

I would prefer 1/N and market asset allocation strategy. If I want to go for something simple I would go for naïve diversification strategy since it performs pretty well. 1/N performs modest in all asset allocation strategy. It means it can weather all types of conditions. One reason why it is performing the way it is could be due to the fact that the distribution is identical. But in real life the distribution is left skewed. So in that case we won't expect 1/N to perform that well.
In other cases I would go for Mean-Variance portfolio since in the real world, the value of alpha is non-zero, and the Mean-Variance portfolio has performed much better in the real-life scenarios.