INFERENTIAL STATISTICS

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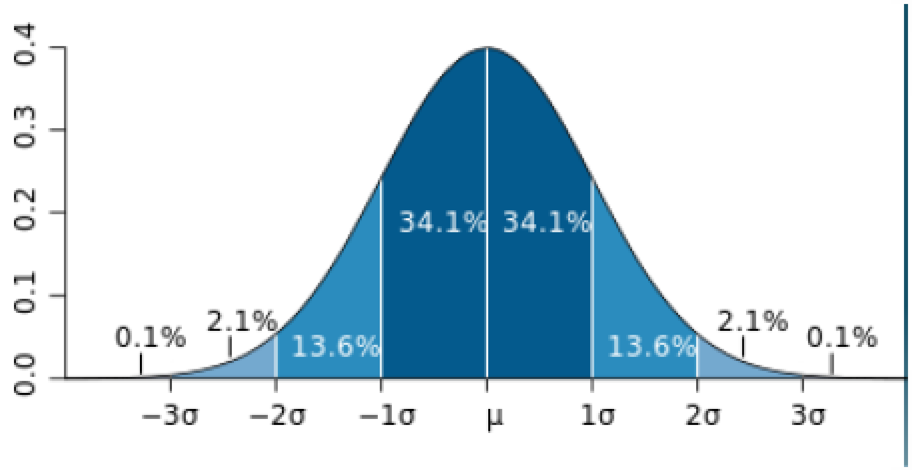
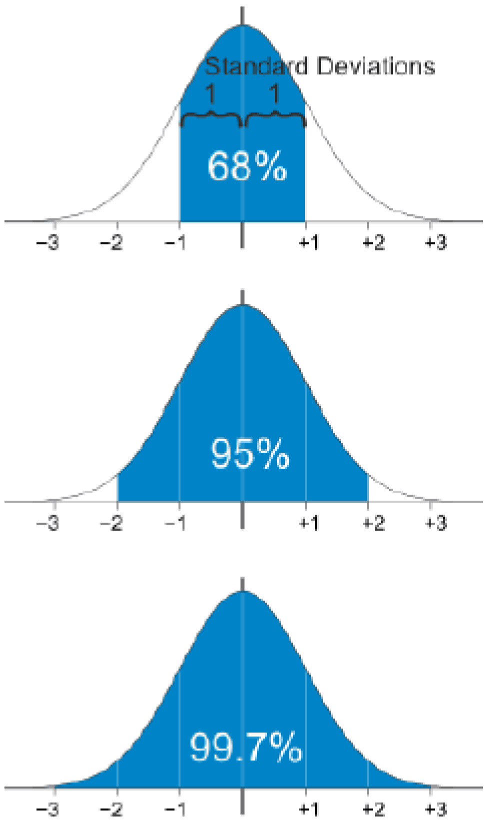
Normal Distribution

Mean = Median = Mode



Normal Distribution

68-95-99.7 empirical rule

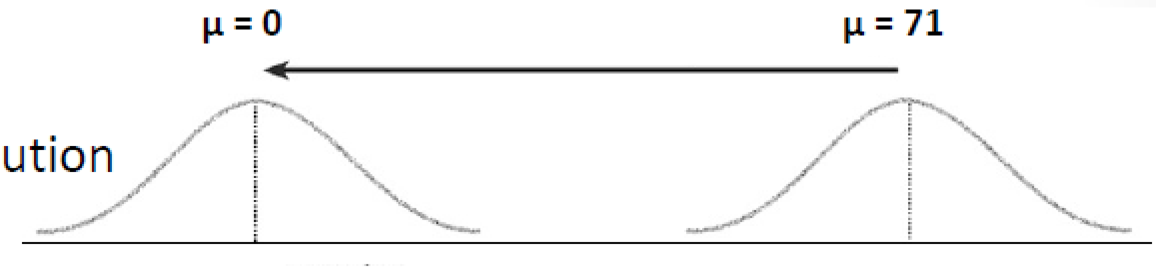


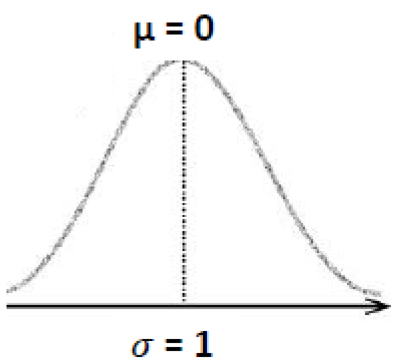
Standard Normal Distribution

Move the mean

This gives a new distribution

X-71 ~ N(0,20.25)





$Z = \frac{X-\mu}{\sigma}$ is called the Standard Score or the z-score.

Central Limit Theorem

The Central Limit Theorem is the sampling distribution of the sampling means approaches a normal distribution as the sample size gets larger, no matter what the shape of the data distribution

Key Points

1. Mean of sample is same as mean of the population. $\mu_{\overline{x}} = \mu$

2. Standard deviation of the sample is equal to standard deviation of the population divided by square root of sample size. $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$

Where,

μ = Population mean

σ = Population standard deviation

$\mu_{\overline{x}}$ = Sample mean

$\sigma_{\overline{x}}$ = Sample standard deviation

n = Sample size

Calculating Probability using Z table

Julie wants to marry a person taller than her and is going on blind dates. The mean height of the 'available' guys is 71" and the variance is 20.25 $inch^2$

Calculating probability using Z table

Julie wants to marry a person taller than her and is going on blind dates. The mean height of the 'available' guys is 71" and the variance is 20.25 $inch^2$

By the way, Julie is 64" tall.

Solution

$Z = \frac{64-71}{4.5} = -1.56$ in the case of our problem.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916

Note the tables give P(Z<z).

$z = \frac{64-71}{4.5} = -1.56$ in the case of our problem.

P(Z>-1.56) = 1- P(Z<-1.56) = 1- 0.0594 = 0.9406

Deviate z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-4.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
-3.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
-3.8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
-3.7	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
-3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379

Note:

Z table gives probability less than Z value and to find the probability more than the Z value, subtract 1 from the probability found in the Z table.

Continuation..

Q. Julie just realized that she wants her date to be taller when she is wearing her heels, which are 5” high. Find the new probability that her date will be taller.

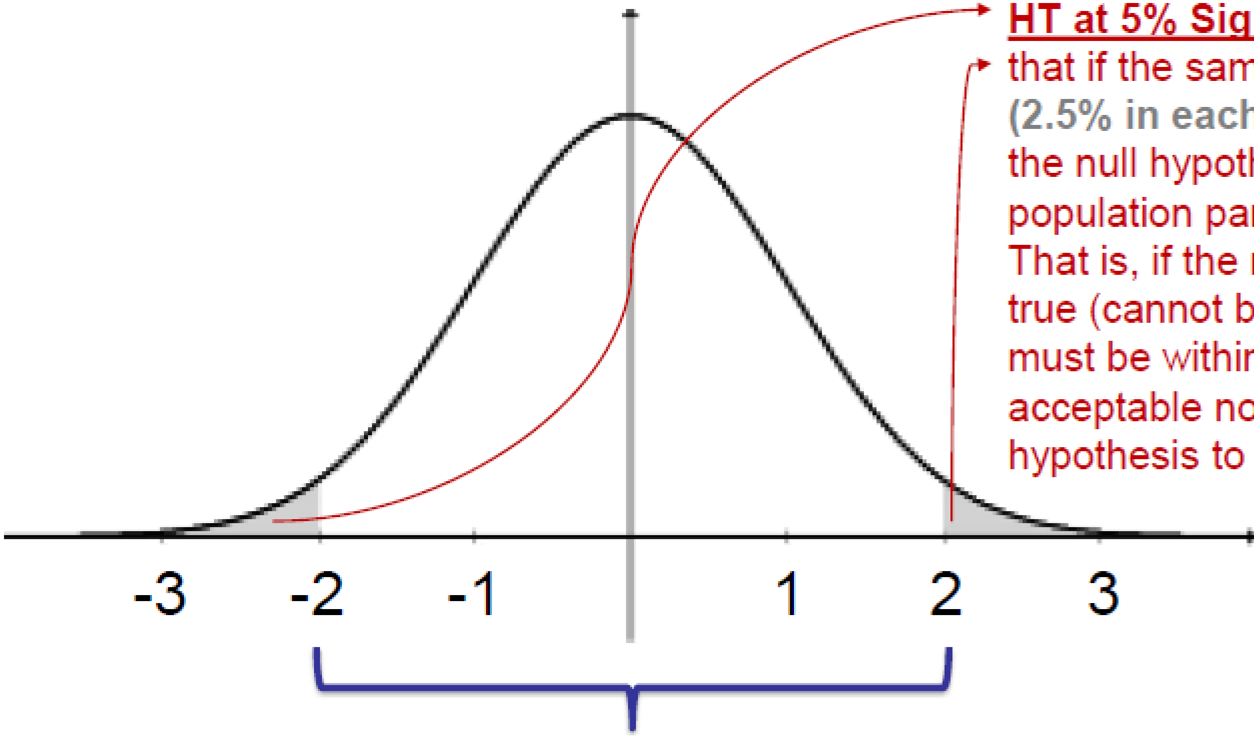
Will this impact on the existing probability ?

New Probability

Q. Julie just realized that she wants her date to be taller when she is wearing her heels, which are 5” high. Find the new probability that her date will be taller.

$$z = \frac{69-71}{4.5} = -0.44;\ P(Z<-0.44) = 0.33,\ \therefore\ P(Z>-0.44) = 0.67 \text{ or } 67\%$$

Confidence Intervals and Hypothesis Testing – Two Ways of Inferring the Same



HT at 5% Significance Level: Implies that if the sample is in the 5% zone (2.5% in each tail shown in gray), then the null hypothesis regarding the population parameter can be rejected. That is, if the null hypothesis has to be true (cannot be rejected), the sample must be within $\pm 2SE$ limits from it (the acceptable normal variation for the null hypothesis to be true).

95% CI: Implies that the true population parameter (e.g., mean) will lie within this range ($\pm 2SE$) for 95% of the samples. If the sample is in the 5% zone (2.5% in each tail shown in gray), then the true population parameter will not lie in the range $\bar{x} \pm 2SE$.

Critical Region & Significance level

Critical region:

The region in the tail of the distribution which corresponds to the rejection of the null hypothesis at some chosen significance level.

Z Critical Value:

The Z value which separates the critical region from the rest of the region in the distribution. Any Z value higher than Z critical value means that the value is in the critical region.

Significance Level:

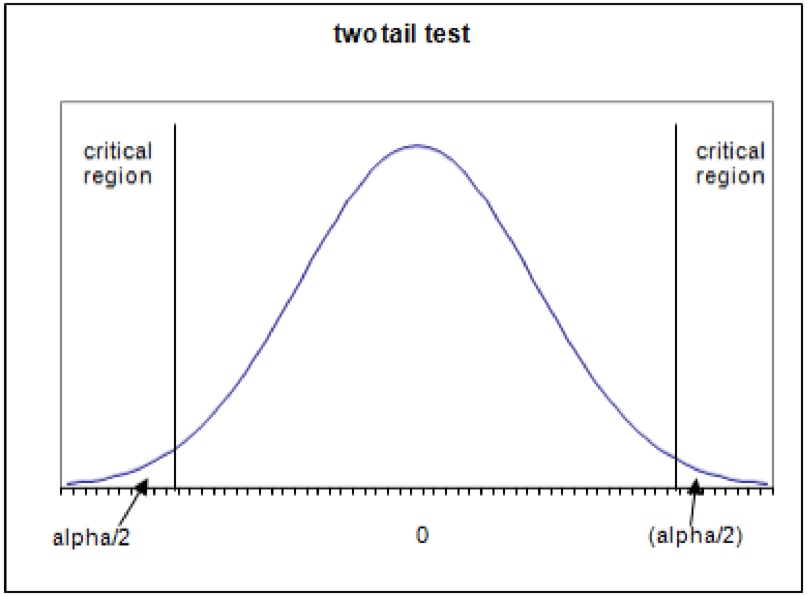
The probability level of that is chosen to test the hypothesis testing in statistics. They are 3 levels - 10%, 5%, 1% and normally if this is not provided during testing then **5% is what chosen as a standard**.

One tailed & two tailed test

The statistical tests used will be **one tailed or two tailed** depending on the nature of the null hypothesis and the alternative hypothesis

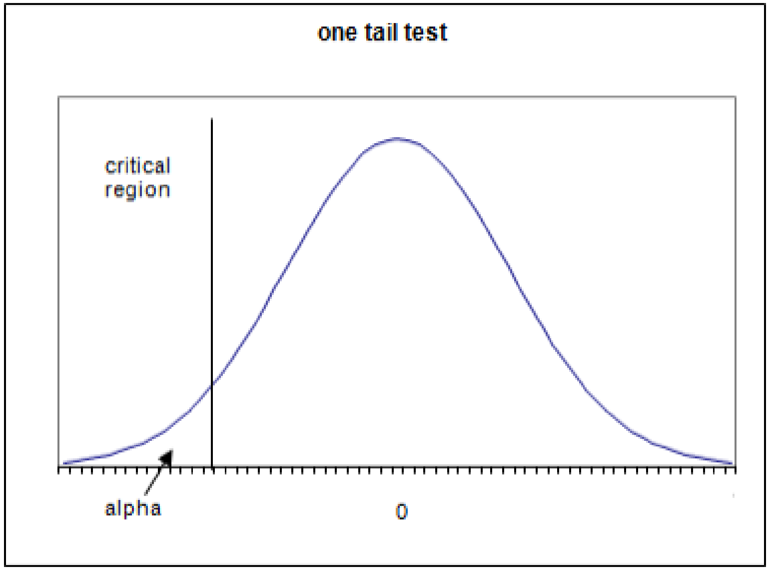
1. Two Tail test

$\mathbf{H_0}: \mu = \mu_0$ $\mathbf{H_1}: \mu \neq \mu_0$;

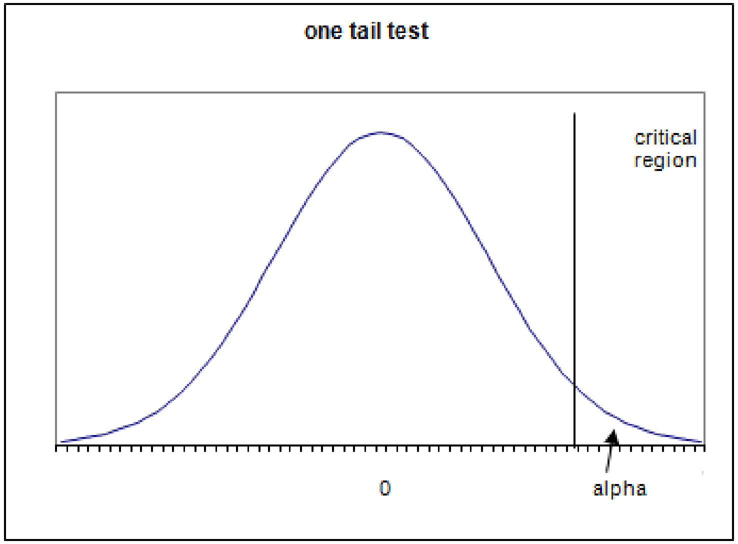


2. One Tail tests

$\mathbf{H_0}: \mu = \mu_0$ $\mathbf{H_1}: \mu < \mu_0;$



$\mathbf{H_0}: \mu = \mu_0$ $\mathbf{H_1}: \mu > \mu_0;$



One tail test – Z critical values

1. α (Significance level) = 10 %

   Z = 1.28

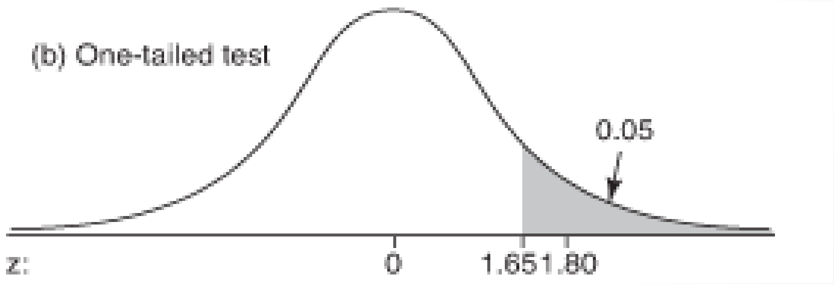
2. α (Significance level) = 5 %

   Z = 1.64

3. α (Significance level) = 1 %

   Z = 2.29

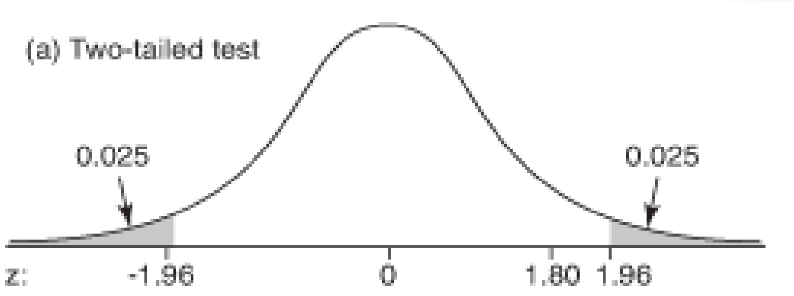
Sample of 1 tail test



Two Tail Test

α level	Z critical values
0.10	1.645
0.05	1.960
0.010	2.576

Sample of 2 tail test



Hypothesis Testing

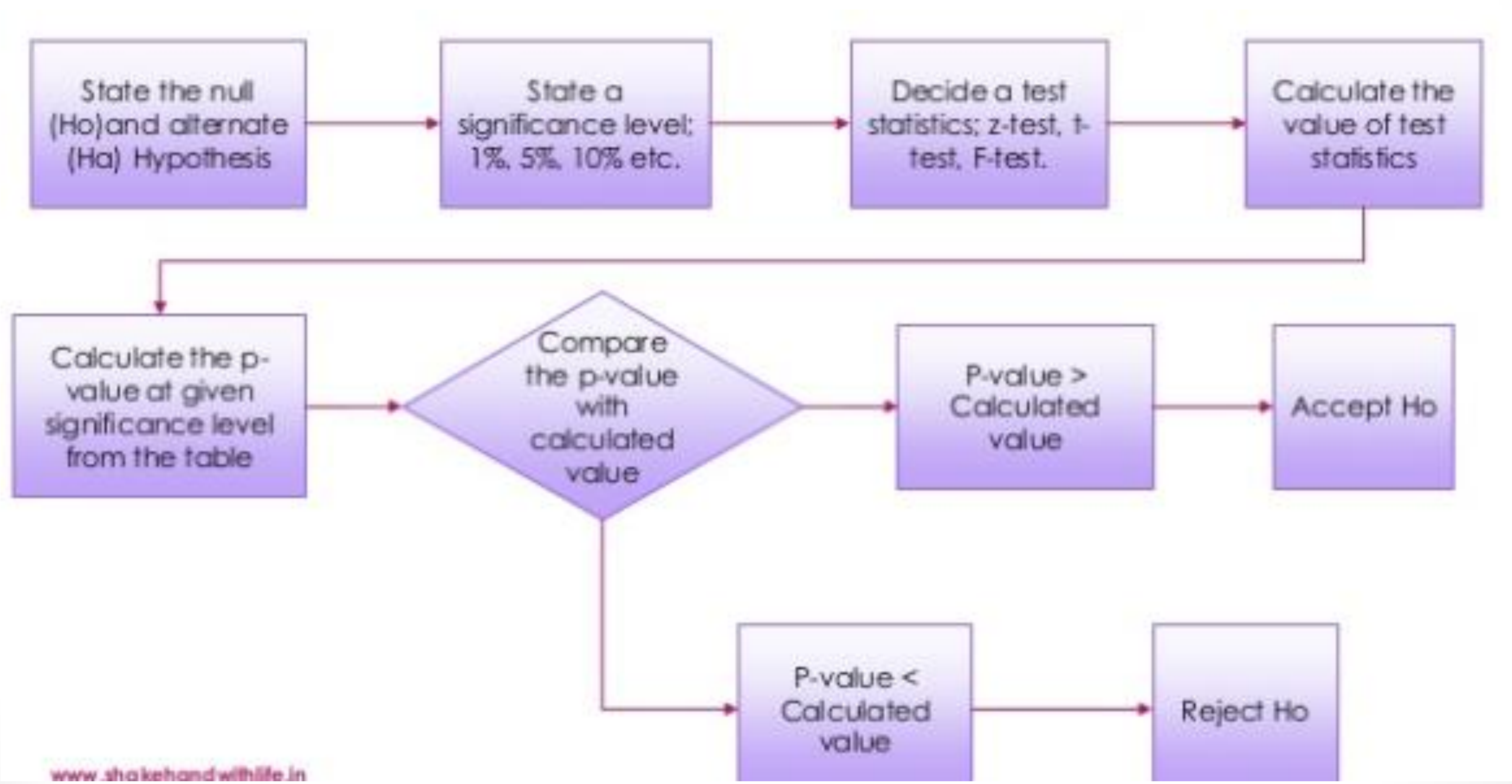
Hypothesis testing is the explanation of the phenomenon - scientific proof of concept about the event

1. Null Hypothesis (H_0)
2. Alternate Hypothesis (H_a)

Hypothesis Testing Steps

1. State null (H_0) and alternative (H_1) hypothesis
2. Choose level of significance (α)
3. Find critical values
4. Find test statistic
5. Draw your conclusion

Steps - Flowchart



Identify Null & Alternate Hypothesis

It is believed that a candy machine makes chocolate bars that are 5g on average. A worker claims that after maintenance it no longer makes 5g bar.

Write Null and alternate hypothesis ?

NOTE:

Both are mathematical opposites

Identify Null & Alternate Hypothesis

It is believed that a candy machine makes chocolate bars that are 5g on average. A worker claims that after maintenance it no longer makes 5g bar.

Write Null and alternate hypothesis ?

$H_0 = 5$
$H_a \neq 5$

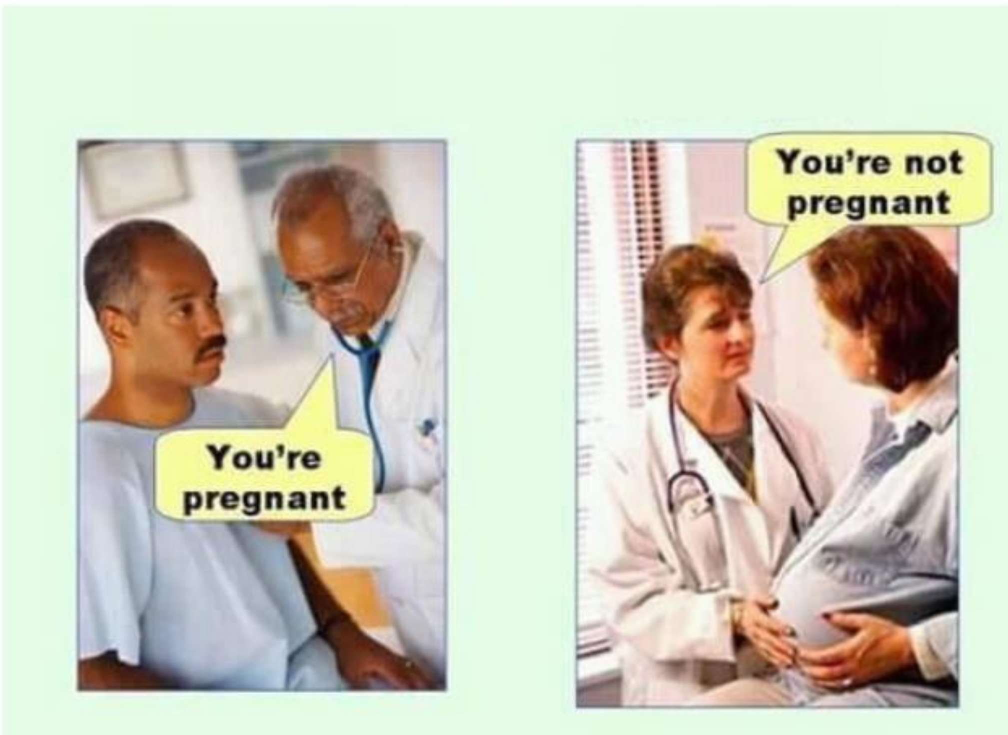
NOTE:

Both are mathematical opposites

Hypothesis Errors

- Type I: We reject the NULL hypothesis incorrectly
- Type II: We “accept” it incorrectly

Type 1 & Type 2 Errors



Common Test Statistics for Inferential Techniques

Inferential techniques (Confidence Intervals and Hypothesis Testing) most commonly use 4 test statistics:

- z
- t
- χ^2 (Chi-squared)
- F

Closely related to Sampling Distribution of **Means**

- Closely related to Sampling Distribution of **Variances**
- Derived from Normal Distribution

Z Test

In a survey conducted for the psychological test about the students attitudes towards studying with the range of score between 0-200. The mean score is 115 and SD is 30. John suspects that older students have better attitudes and selects 35 students more than 30 years to test and the mean score is 118.6

Carry out the significance test at 0.05 ?

Solution

1. **State hypothesis**

   $H_0 = 115$

   $H_a > \mu$ *(115)*

2. Significance level $\alpha = 0.05$

3. Z critical value Z = 1.64

Given parameters

$\mu = 115$, $\sigma = 30$, $n = 35$, **sample mean** = 118.6

Find Z ?

Z - Test Statistic

$$Z = \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}$$

Substituting the parameters

Z=0.74

Using Z table – find probability

P(Z<0.74) = 0.77

To find probability of Z > than 0.74

P(Z>0.74) = 1-0.77= 0.23

Conclusion

1. The Z value is not in the critical region

2. There is no strong evidence to reject the Null hypothesis and therefore the mean score older students is same as everyone

i.e 115

Case Study

School	Mean	Standard deviation (pop.)	N
Private	110	15	60
Public	104	15	60

Do students from private school obtain significantly higher scores at exams than students from public schools ?

- Assignment

Z Test Vs T Test

1. **Sample size**

   Z test when n > 30

   T test when n < 30

2. Z test when population standard deviations are Known

NOTE:

P value denotes – Probability of getting the result we expect if the Null hypothesis is true

T Test

The average IQ of the adult population is 100. A researcher believes that the average IQ of adults is lower. He takes a random sample of 5 adults score and tests it (69,79,89,99,109) with sample SD 15.81

Is there enough evidence to suggest that the average IQ in adults is lower ?

Solution

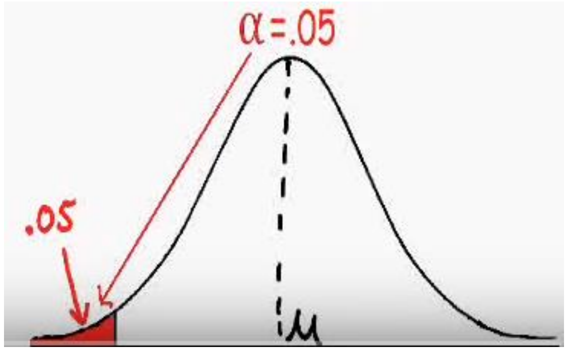
1. **State hypothesis**

$H_0 = 100$

$H_a < \mu$ *(100)*

2. Significance level $\alpha = 0.05$

3. T critical value T = -2.132



Given parameters

μ = 100, s = 15.81, n= 5, **sample mean** = 89

Find T ?

T Table - How to use T table to find critical value ?

1. Check for the degrees of freedom on Y axis
2. Check for the Alpha (α) level in X axis

Degrees of Freedom = n – 1

For < μ use Negative sign

***t* Table**

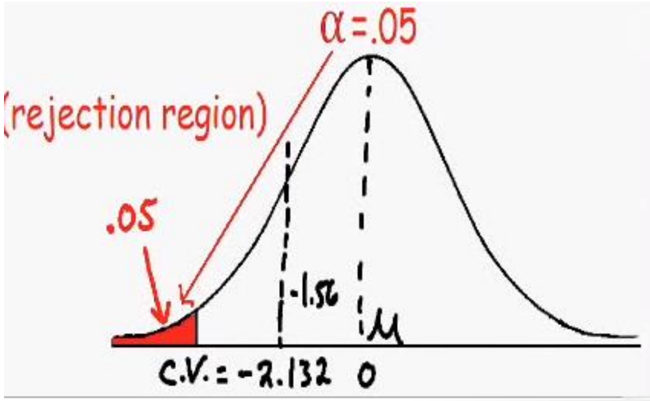
cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291

T - Test Statistic

$$t\ statistic\ (or\ t\ score), t = \frac{(\bar{x} - \mu)}{\frac{s}{\sqrt{n}}}$$

Substituting the parameters

T=-1.56



Conclusion

1. The calculated T value is not in the critical region
2. We do not have enough evidence to reject the NULL hypothesis and therefore the average IQ for adults is same as everyone i.e 100

χ^2 DISTRIBUTION

So you modeled a situation using a probability distribution and got a good idea of how things will shape up in the long run. But what if what you expected and what you observed are not the same? How would you know if the difference is due to normal fluctuations or if your model was incorrect?

Let us say you are running a casino and the slot machines are causing you headaches. You had designed them with the following expected probability distribution, with X being the net gain from each game played.

x	-2	23	48	73	98
P(X=x)	0.977	0.008	0.008	0.006	0.001

You collected some statistics and found the following frequency of peoples' winnings.

x	-2	23	48	73	98
Frequency	965	10	9	9	7

You want to compare the actual frequency with the expected frequency.

x	-2	23	48	73	98
P(X=x)	0.977	0.008	0.008	0.006	0.001

x	Observed Frequency	Expected Frequency
-2	965	977
23	10	8
48	9	8
73	9	6
98	7	1

Are these differences significant and if they are, is it just pure chance?

χ^2 test to the rescue

χ^2 distribution uses a test statistic to look at the difference between the expected and the actual, and then returns a probability of getting observed frequencies as extreme.

$X^2 = \sum \frac{(O-E)^2}{E}$, where O is the observed frequency and E the expected frequency.

x	Observed Frequency	Expected Frequency
-2	965	977
23	10	8
48	9	8
73	9	6
98	7	1

$$X^2 = 38.272$$

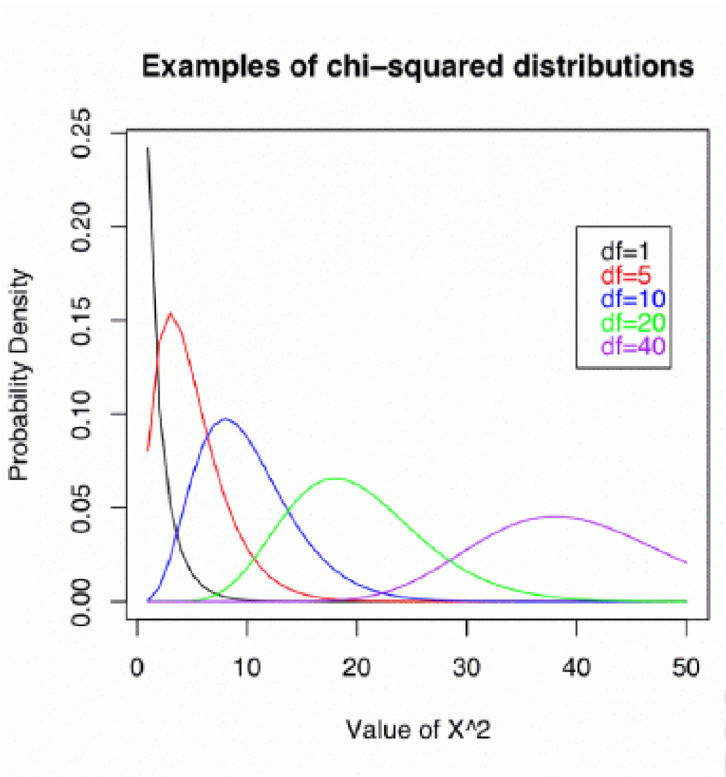
Is this high?

To find this, we need to look at the χ^2 distribution.

χ^2 distribution

$X^2 \sim \chi^2_{(\nu)}$, where ν represents the degrees of freedom.

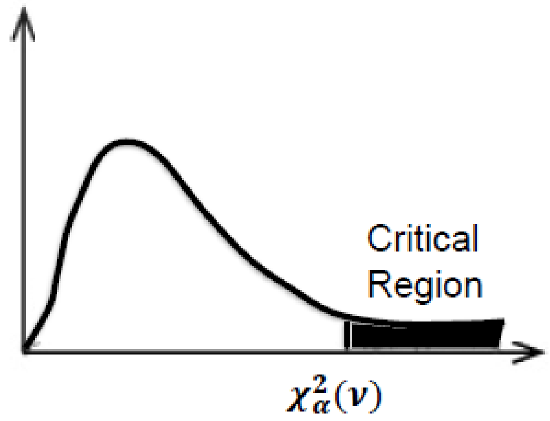
When ν is greater than 2, the shape of the distribution is skewed positively gradually becoming approximately normal for large ν.



How do we know the Significance of the difference?

One-tailed test using the upper tail of the distribution as the critical region.

A test at significance level α is written as $\chi^2_\alpha(\nu)$. The critical region is to its right.



Higher the value of the test statistic, the bigger the difference between observed and expected frequencies.

Uses of χ^2 distribution

- To test **goodness of fit**.
- To test **independence** of two variables.
- To test hypothesis about **variance** of a population.

Steps to test Goodness-of-fit

You want to see if there is sufficient evidence at the 5% significance level to say the slot machines have been rigged.

What are the null and alternate hypotheses?

H_0: The slot machine winnings per game follow the described probability distribution, i.e., they are not rigged.

H_1: The slot machine winnings per game do not follow this distribution.

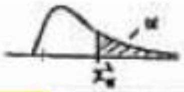
What are the expected frequencies and degrees of freedom?

x	Observed Frequency	Expected Frequency
-2	965	977
23	10	8
48	9	8
73	9	6
98	7	1

$\nu = 4$

What is the critical region?

TABLE OF CHI-SQUARE DISTRIBUTION



α / ν	0.995	0.99	0.98	0.975	0.95	0.90	0.80	0.20	0.10	0.05	0.025	0.02	0.01	0.005	0.001
1	$0.0^{4}393$	$0.0^{3}157$	$0.0^{3}628$	$0.0^{3}982$	0.00393	0.0158	0.0642	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.268
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.465
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.517
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.322
8	1.344	1.646	2.032	2.180	2.733	3.490	4.594	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.125
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	15.812	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

$\chi^2_{5\%}(4) = 9.488$. This means the critical region is $X^2 > 9.488$.

Is the test statistic inside or outside the critical region?

Since $X^2 = 38.27$ and the critical region is $X^2 > 9.488$, this means X^2 is inside the critical region.

Will you accept or reject the null hypothesis?

Reject. There is sufficient evidence to reject the hypothesis

This sort of hypothesis test is called a **goodness of fit** test. This test is used whenever you have a set of values that should fit a distribution, and you want to test whether the data actually does

Testing Hypotheses about a Variance

A manufacturing company produces bearings of 2.65 cm in diameter. A major customer requires that the variance in diameter be no more than 0.001 cm^2. The manufacturer tests 20 bearings using a precise instrument and gets the below values. Assuming the diameters are normally distributed, can the population of these bearings be rejected due to high variance at 1% significance level?

Data: 2.69, 2.66, 2.64, 2.59, 2.62, 2.63, 2.69, 2.66, 2.63, 2.65, 2.57, 2.63, 2.70, 2.71, 2.64, 2.65, 2.59, 2.66, 2.62, 2.57

Testing Hypotheses about a Variance

What are null and alternate hypotheses?

H_0: $\sigma^2 \leq 0.001$; H_1: $\sigma^2 > 0.001$

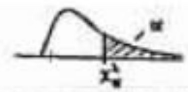
How many degrees of freedom?

Since n=20, df=19.

Testing Hypotheses about a Variance

What is the critical region?

TABLE OF CHI-SQUARE DISTRIBUTION



α / ν	0.995	0.99	0.98	0.975	0.95	0.90	0.80	0.20	0.10	0.05	0.025	0.02	0.01	0.005	0.001
1	0.0^4393	0.0^3157	0.0^3628	0.0^3982	0.00393	0.0158	0.0642	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
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13	3.565	4.107	4.765	5.009	5.892	7.042	8.634	16.985	19.812	22.362	24.736	25.472	27.688	29.819	34.528
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.123
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.697
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.790
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.820
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.315

$\chi^2_{0.01,19} = 36.191$

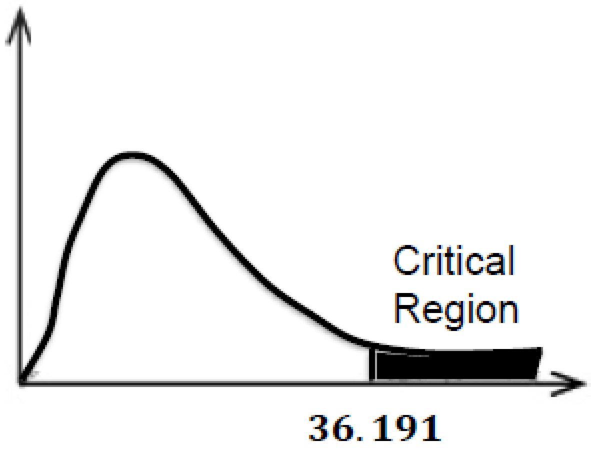
Testing Hypotheses about a Variance

What is the observed χ^2 value?

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{19 * 0.001621}{0.001} = 30.8$$

Is it in critical region? $\chi^2_{0.01,19} = 36.191$

No.



Will you reject or fail to reject the null hypothesis?

Fail to reject.

Business decision

The population variance is within specification limits required by the customer and hence the bearings can be shipped.

F distribution

- χ^2 was useful in testing hypotheses about a single population variance.
- Sometimes we want to test hypotheses about difference in variances of two populations:
  - Is the variance of 2 stocks the same?
  - Do parts manufactured in 2 shifts or on 2 different machines or in 2 batches have the same variance or not?
  - Is the powder mix for tablet granulations homogeneous?
  - Is there variability in assayed drug blood levels in a bioavailability study?

F distribution

- Ratio of 2 variance estimates: $F = \frac{s_1^2}{s_2^2} = \frac{est.\sigma_1^2}{est.\sigma_2^2}$
- Ideally, this ratio should be about 1 if 2 samples come from the same population or from 2 populations with same variance, but sampling errors cause variation.
- *Recall* $\chi^2 = \frac{(n-1)s^2}{\sigma^2}$. So, F is also a ratio of 2 chi-squares, each divided by its degrees of freedom, i.e.,

$$F = \frac{\frac{\chi^2_{\nu_1}}{\nu_1}}{\frac{\chi^2_{\nu_2}}{\nu_2}}$$

Hypothesis test for 2 sample variances

A machine produces metal sheets with 22mm thickness. There is variability in thickness due to machines, operators, manufacturing environment, raw material, etc. The company wants to know the consistency of two machines and randomly samples 10 sheets from machine 1 and 12 sheets from machine 2. Thickness measurements are taken. Assume sheet thickness is normally distributed in the population.

The company wants to know if the variance from each sample comes from the same population variance (population variances are equal) or from different population variances (population variances are unequal).

How do you test this?

Hypothesis test for 2 sample variances

Data

Machine 1		Machine 2	
22.3	21.9	22.0	21.7
21.8	22.4	22.1	21.9
22.3	22.5	21.8	22.0
21.6	22.2	21.9	22.1
21.8	21.6	22.2	21.9
		22.0	22.1
$s_1^2 = 0.11378$	$n = 10$	$s_2^2 = 0.02023$	$n = 12$

Ratio of sample variances, $F = \frac{s_1^2}{s_2^2} = \frac{0.11378}{0.02023} = 5.62$

Hypothesis test for 2 sample variances

What are null and alternate hypotheses?

H_0: $\sigma_1^2 = \sigma_2^2$; H_1: $\sigma_1^2 \neq \sigma_2^2$

Is it a one-tailed test or a two-tailed test?

Two-tailed.

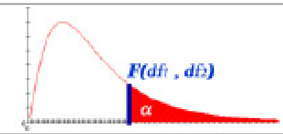
What are numerator and denominator degrees of freedom?

$\nu_1 = 10 - 1 = 9; \nu_2 = 12 - 1 = 11$

Hypothesis test for 2 sample variances

Reading an F-table.

F Table for α = 0.025



/	df₁=1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
df₂=1	647.7890	799.5000	864.1630	899.5833	921.8479	937.1111	948.2169	956.6562	963.2846	968.6274	976.7079	984.8668	993.1028	997.2492	1001.414	1005.598	1009.800	1014.020	1018.258
2	38.5063	39.0000	39.1655	39.2484	39.2982	39.3315	39.3552	39.3730	39.3869	39.3980	39.4146	39.4313	39.4479	39.4562	39.465	39.473	39.481	39.490	39.498
3	17.4434	16.0441	15.4392	15.1010	14.8848	14.7347	14.6244	14.5399	14.4731	14.4189	14.3366	14.2527	14.1674	14.1241	14.081	14.037	13.992	13.947	13.902
4	12.2179	10.6491	9.9792	9.6045	9.3645	9.1973	9.0741	8.9796	8.9047	8.8439	8.7512	8.6565	8.5599	8.5109	8.461	8.411	8.360	8.309	8.257
5	10.0070	8.4336	7.7636	7.3879	7.1464	6.9777	6.8531	6.7572	6.6811	6.6192	6.5245	6.4277	6.3286	6.2780	6.227	6.175	6.123	6.069	6.015
6	8.8131	7.2599	6.5988	6.2272	5.9876	5.8198	5.6955	5.5996	5.5234	5.4613	5.3662	5.2687	5.1684	5.1172	5.065	5.012	4.959	4.904	4.849
7	8.0727	6.5415	5.8898	5.5226	5.2852	5.1186	4.9949	4.8993	4.8232	4.7611	4.6658	4.5678	4.4667	4.4150	4.362	4.309	4.254	4.199	4.142
8	7.5709	6.0595	5.4160	5.0526	4.8173	4.6517	4.5286	4.4333	4.3572	4.2951	4.1997	4.1012	3.9995	3.9472	3.894	3.840	3.784	3.728	3.670
9	7.2093	5.7147	5.0781	4.7181	4.4844	4.3197	4.1970	4.1020	4.0260	3.9639	3.8682	3.7694	3.6669	3.6142	3.560	3.505	3.449	3.392	3.333
10	6.9367	5.4564	4.8256	4.4683	4.2361	4.0721	3.9498	3.8549	3.7790	3.7168	3.6209	3.5217	3.4185	3.3654	3.311	3.255	3.198	3.140	3.080
11	6.7241	5.2559	4.6300	4.2751	4.0440	3.8807	3.7586	3.6638	3.5879	3.5257	3.4296	3.3299	3.2261	3.1725	3.118	3.061	3.004	2.944	2.883
12	6.5538	5.0959	4.4742	4.1212	3.8911	3.7283	3.6065	3.5118	3.4358	3.3736	3.2773	3.1772	3.0728	3.0187	2.963	2.906	2.848	2.787	2.725

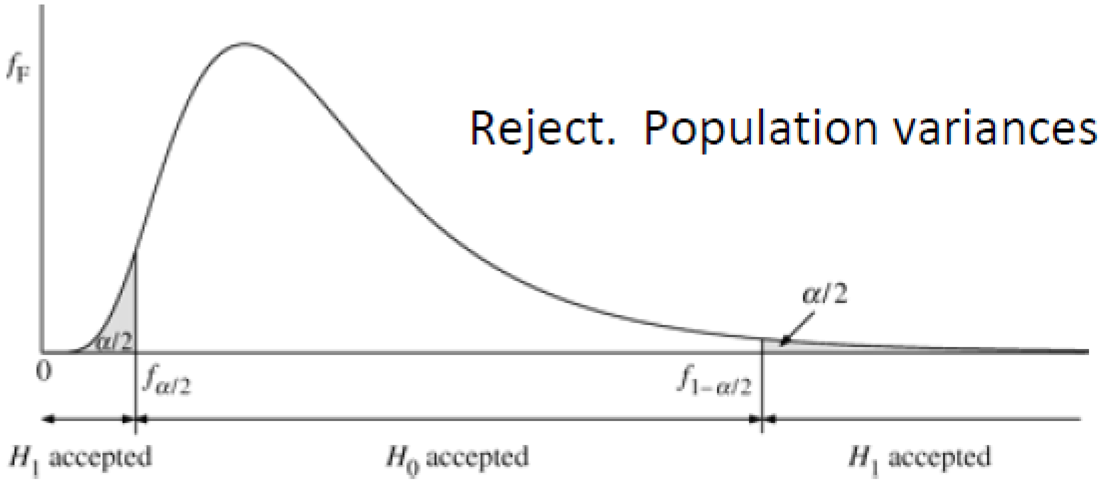
$$F_{0.025,9,11} = 3.5879$$

Hypothesis test for 2 sample variances

$$F_{0.025,9,11} = 3.5879$$

$$F_{observed} = 5.62$$

Will you reject the null hypothesis or not?



Reject. Population variances are not equal.

Business decision

Variance in machine 1 is higher than in machine 2. Machine 1 needs to be inspected for any issues.

Applications of F Distribution

- Test for equality of variances.
- Test for differences of means in ANOVA.
- Test for regression models (slopes relating one continuous variable to another, e.g., Entrance exam scores and GPA)