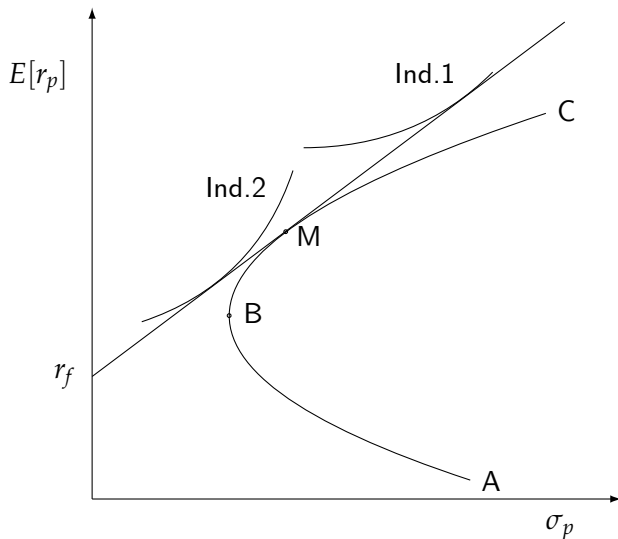


Capital Asset Pricing Model and Arbitrage Pricing Theory

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- 1 Capital Asset Pricing Model
- 2 A summarizing digression
- 3 Arbitrage Pricing Theory



The Capital Market Line

Capital Asset Pricing Model CAPM

Capital Market Line only valid for efficient portfolios, i.e.

- combinations of risk free asset and market portfolio M
⇒ all risk comes from market portfolio

What about inefficient portfolios or individual stocks?

- don't lie on the CML, cannot be priced with it
need a different model for that

What needs to be changed in the model:

the market price of risk $((E(r_m) - r_f) / \sigma_m)$,
or the measure of risk σ_p ?

CAPM is more general model, developed by Sharpe

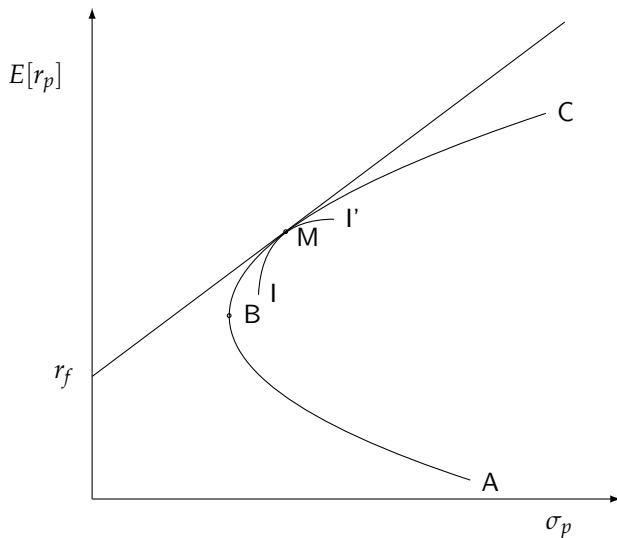
Consider a two asset portfolio:

- one asset is market portfolio M, weight $(1 - x)$
- other asset is individual stock i, weight x

Is this an efficient portfolio?

Analyse what happens if we vary proportion x invested in i

- begin in point I, 100% in i, $x=1$
- in point M, $x=0$, but asset i is included in M with its market value weight
- to point I', $x<0$ to eliminate market value weight of i



Portfolios of asset i and market portfolio M

Risk-return characteristics of this 2-asset portfolio:

$$E(r_p) = xE(r_i) + (1 - x)E(r_m)$$

$$\sigma_p = \sqrt{[x^2\sigma_i^2 + (1 - x)^2\sigma_m^2 + 2x(1 - x)\sigma_{i,m}]}$$

Expected return and risk of a marginal change in x are:

$$\frac{\partial E(r_p)}{\partial x} = E(r_i) - E(r_m)$$

$$\begin{aligned} \frac{\partial \sigma_p}{\partial x} &= \frac{1}{2} \left[x^2\sigma_i^2 + (1 - x)^2\sigma_m^2 + 2x(1 - x)\sigma_{i,m} \right]^{-\frac{1}{2}} \\ &\quad \times \left[2x\sigma_i^2 - 2\sigma_m^2 + 2x\sigma_m^2 + 2\sigma_{i,m} - 4x\sigma_{i,m} \right] \end{aligned}$$

First term of $\partial\sigma_p/\partial x$ is $\frac{1}{2\sigma_p}$, so:

$$\begin{aligned}\frac{\partial\sigma_p}{\partial x} &= \frac{2x\sigma_i^2 - 2\sigma_m^2 + 2x\sigma_m^2 + 2\sigma_{i,m} - 4x\sigma_{i,m}}{2\sigma_p} \\ &= \frac{x\sigma_i^2 - \sigma_m^2 + x\sigma_m^2 + \sigma_{i,m} - 2x\sigma_{i,m}}{\sigma_p}\end{aligned}$$

Isolating x gives:

$$\frac{\partial\sigma_p}{\partial x} = \frac{x(\sigma_i^2 + \sigma_m^2 - 2\sigma_{i,m}) + \sigma_{i,m} - \sigma_m^2}{\sigma_p}$$

At point M all funds are invested in M so that:

- $x = 0$ and $\sigma_p = \sigma_m$

Note also that:

- i is already included in M with its market value weight
- economically x represents excess demand for i
- in equilibrium M excess demand is zero

This simplifies marginal risk to:

$$\left. \frac{\partial \sigma_p}{\partial x} \right|_{x=0} = \frac{\sigma_{i,m} - \sigma_m^2}{\sigma_p} = \frac{\sigma_{i,m} - \sigma_m^2}{\sigma_m}$$

So the slope of the risk-return trade-off at equilibrium point M is:

$$\left. \frac{\partial E(r_p) / \partial x}{\partial \sigma_p / \partial x} \right|_{x=0} = \frac{E(r_i) - E(r_m)}{(\sigma_{i,m} - \sigma_m^2) / \sigma_m}$$

But at point M the slope of the risk-return trade-off is also the slope of the CML, so:

$$\frac{E(r_i) - E(r_m)}{(\sigma_{i,m} - \sigma_m^2) / \sigma_m} = \frac{E(r_m) - r_f}{\sigma_m}$$

Solving for $E(r_i)$ gives:

$$\begin{aligned} E(r_i) &= r_f + (E(r_m) - r_f) \frac{\sigma_{i,m}}{\sigma_m^2} \\ &= r_f + (E(r_m) - r_f) \beta_i \end{aligned}$$

$$E(r_i) = r_f + (E(r_m) - r_f)\beta_i$$

This is the *Capital Asset Pricing Model*

- Sharpe was awarded the Nobel prize for this result
- Its graphical representation is known as the
 - *Security Market Line*
- Pricing relation for entire investment universe
 - including inefficient portfolios
 - including individual assets
- clear price of risk: $E(r_m) - r_f$
- clear measure of risk: β

CAPM formalizes risk-return relationship:

- well-diversified investors value assets according to their contribution to portfolio risk
 - if asset i increases portf. risk $E(r_i) > E(r_p)$
 - if asset i decreases portf. risk $E(r_i) < E(r_p)$
- expected risk premium proportional to β

Offers other insights as well. Look at 4 of them:

- ① Systematic and unsystematic risk
- ② Risk adjusted discount rates
- ③ Certainty equivalents
- ④ Performance measures

1. Systematic & unsystematic risk

- The CML is pricing relation for *efficient* portfolios:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m} \sigma_p$$

- $\frac{E(r_m) - r_f}{\sigma_m}$ is the price per unit of risk
- σ_p is the volume of risk.

The SML valid for all investments, incl. inefficient portfolios and individual stocks:

$$E(r_p) = r_f + (E(r_m) - r_f) \beta_p$$

we can write β as:

$$\beta_p = \frac{cov_{p,m}}{\sigma_m^2} = \frac{\sigma_p \sigma_m \rho_{p,m}}{\sigma_m^2} = \frac{\sigma_p \rho_{p,m}}{\sigma_m}$$

so that the SML becomes:

$$E(r_p) = r_f + (E(r_m) - r_f) \frac{\sigma_p \rho_{p,m}}{\sigma_m},$$

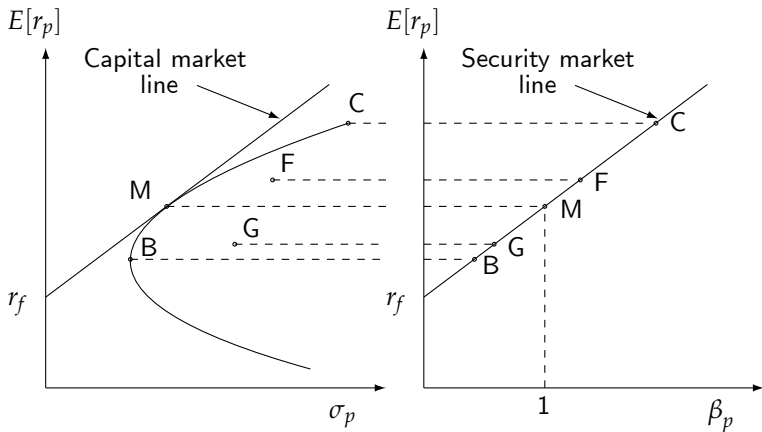
$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m} \sigma_p \rho_{p,m}$$

Compare with CML:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m} \sigma_p$$

The difference between CML and SML is in volume part:

- SML only prices the *systematic risk*
 - is therefore valid for all investment objects.
- CML prices *all risks*
 - only valid when all risk is systematic risk, i.e. for efficient portfolios
 - otherwise, CML uses 'wrong' risk measure
- difference is correlation term, that is ignored in CML
 - efficient portfolios only differ in proportion M in it
 - so all efficient portfolios are perfectly positively correlated:
$$\rho_{M,(1-x)M} = 1$$
 - if $\rho_{p,m} = 1 \Rightarrow \sigma_p \rho_{p,m} = \sigma_p$ and $CML = SML$



Systematic and unsystematic risk

2. CAPM and discount rates

Recall general valuation formula for investments:

$$Value = \sum^t \frac{Exp [Cash\ flows_t]}{(1 + discount\ rate_t)^t}$$

Uncertainty can be accounted for in 3 different ways:

- ① Adjust discount rate to *risk adjusted discount rate*
- ② Adjust cash flows to *certainty equivalent cash flows*
- ③ Adjust probabilities (expectations operator) from normal to *risk neutral or equivalent martingale probabilities*

Use of CAPM as risk adjusted discount rate is easy
CAPM gives expected (=required) return on portfolio P as:

$$E(r_p) = r_f + (E(r_m) - r_f)\beta_p$$

But return is also:

$$E(r_p) = \frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}}$$

Discount rate:

- links expected end-of-period value, $E(V_{p,T})$, to value now, $V_{p,0}$
- found by equating the two expressions:

$$\frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}} = r_f + (E(r_m) - r_f)\beta_p$$

solving for $V_{p,0}$ gives:

$$V_{p,0} = \frac{E(V_{p,T})}{1 + r_f + (E(r_m) - r_f)\beta_p}$$

- r_f is the time value of money
- $(E(r_m) - r_f)\beta_p$ is the adjustment for risk
- together they form the risk adjusted discount rate

3. Certainty equivalent formulation

The second way to account for risk:

- adjust uncertain cash flow to a *certainty equivalent*
- can (and should) be discounted with risk free rate

Requires some calculations, omitted here

$$\frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}} = r_f + (E(r_m) - r_f)\beta_p$$

can be written as:

$$V_{p,0} = \frac{E(V_{p,T}) - \lambda \text{cov}(V_{p,T}, r_m)}{1 + r_f}$$

This is the *certainty equivalent formulation* of the CAPM:

- uncertain end-of-period value is adjusted by
 - the market price of risk, λ :

$$\lambda = \frac{E(r_m) - r_f}{\sigma_m^2}$$

- \times the volume of risk, i.e. $\text{cov.}(\text{end-of-period value, return on market portfolio})$
- The resulting certainty equivalent value is discounted at the risk free rate to find the present value.

4. Performance measures

CML and SML relate expected return to risk

- can be reformulated as *ex post performance measures*
- relate realized returns to observed risk

Sharpe uses slope of CML for this:

$$\begin{aligned} E(r_p) &= r_f + \frac{E(r_m) - r_f}{\sigma_m} \sigma_p \Rightarrow \\ \frac{E(r_p) - r_f}{\sigma_p} &= \frac{E(r_m) - r_f}{\sigma_m} \end{aligned}$$

Left hand side is *return-to-variability ratio* or *Sharpe ratio*

In ex post formulation:

$$\text{Sharpe ratio: } SR_p = \frac{\bar{r}_p - \bar{r}_f}{\hat{\sigma}_p}$$

- SR_p is Sharpe ratio of portfolio p
- \bar{r}_p is portfolio's historical average return $\bar{r}_p = \sum_t r_{pt} / T$
- \bar{r}_f is historical average risk free interest rate
- $\hat{\sigma}_p$ is stand. dev.portf. returns: $\hat{\sigma}_p = \sqrt{\sum_t (r_{pt} - \bar{r}_p)^2 / T}$
- T is number of observations (periods)

Sharpe ratios widely used to:

- rank portfolios, funds or managers
- identify poorly diversified portfolios (too high $\hat{\sigma}_p$)
- identify funds that charged too high fees (\bar{r}_p too low)

Sharpe ratio can be adapted:

- measure the risk premium over other benchmark than r_f
 - also known as the *information ratio*
- measure risk as semi-deviation (downward risk)
 - known as *Sortino ratio*

WKN	Name	Volatilität	Sharpe-Ratio	Ausgabe-Aufschlag	Mgmt.-gebühr
<input type="checkbox"/> A0DPXX	DB Platinum IV Europ...	2,00	1,88	4,00%	1,00%
<input type="checkbox"/> 593060	Nordea 1 Danish Bond...	4,50	1,25	3,00%	1,35%
<input type="checkbox"/> 532652	KCD-Union-Renten Plu...	3,96	1,19	0,00%	1,25%
<input type="checkbox"/> A0LCFH	Fortis L Fund Bond S...	5,09	0,97	5,00%	0,30%
<input type="checkbox"/> A0ML43	CAAM Volatility Equi...	16,23	0,95	4,50%	1,00%
<input type="checkbox"/> A0NG05	BNY Mellon Vietnam,L...	44,32	0,95	5,00%	1,00%
<input type="checkbox"/> A0MVST	pulse invest - ABSOL...	21,84	0,95	--	--
<input type="checkbox"/> A0H1AL	Dexia Sustainable Eu...	5,79	0,94	2,50%	0,60%
<input type="checkbox"/> A0J4YL	Dexia Sustainable Eu...	5,79	0,94	2,50%	0,60%
<input type="checkbox"/> A0MVSS	pulse invest - ABSOL...	21,85	0,94	--	--
<input type="checkbox"/> A0NG03	BNY Mellon Vietnam,L...	44,39	0,92	5,00%	2,00%
<input type="checkbox"/> 849625	Allianz ExxonMobil-M...	5,47	0,91	0,00%	0,60%

Example from www.handelsblatt.com, Sonntag, 02.08.2009



Norske fond	Minstetegning
1 Holberg Norge	3000
2 Fondsfinans Spar	10.000
3 Storebrand Verdi	100
4 SKAGEN Vekst	1000
5 DnB NOR Norge Selektiv (II)	5.000.000
6 DnB NOR Norge Selektiv (III)	10.000.000
7 Danske Inv. Norge Aksj. Inst 1	3.000.000
8 DnB NOR Norge (III)	2.500.000
9 WarrenWicklund Norge	5000
10 Danske Invest Norge II	50.000

Norwegian example (stavanger aftenblad, 14.03.09)
(Sharpe ratio too difficult?)

Treynor ratio uses security market line, β as risk measure:

$$\text{Treynor ratio: } TR_p = \frac{\bar{r}_p - \bar{r}_f}{\hat{B}_p}$$

\hat{B}_p is estimated from historical returns

- Treynor ratio usually compared with risk premium market portfolio
- is TR for portfolio with β of 1

What does the CAPM predict about the TR of different assets and portfolios?

All assets lie on SML \Rightarrow all have same TR

Jensen's alpha also based on CAPM

- measures portfolio return in excess of CAPM
- found by regressing portfolio risk-premium on market portfolio's risk-premium:

$$r_{pt} - r_{ft} = \hat{\alpha}_p + \hat{B}_p(r_{mt} - r_{ft}) + \hat{\varepsilon}_{pt}$$

- taking averages and re-writing gives Jensen's alpha:

$$\text{Jensen's alpha : } \hat{\alpha}_p = \bar{r}_p - (\bar{r}_f + \hat{B}_p(\bar{r}_m - \bar{r}_f))$$

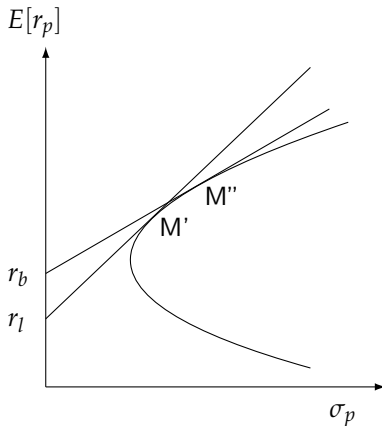
We will meet these performance measures again in market efficiency tests

Assumptions CAPM is based on:

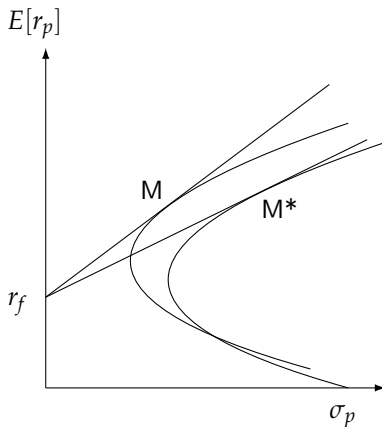
- Financial markets are perfect and competitive:
 - no taxes or transaction costs, all assets are marketable and perfectly divisible, no limitations on short selling and risk free borrowing and lending
 - large numbers of buyers and sellers, none large enough to individually influence prices, all information simultaneously and costlessly available to all investors
- Investors
 - maximize expected utility of their end wealth by choosing investments based on their mean-variance characteristics over a single holding period
 - have homogeneous expectations w.r.t. returns (i.e. they observe same efficient frontier)

Assumptions have different backgrounds and importance

- Some make modelling easy, model doesn't break down if we include phenomena now 'assumed away':
 - no taxes or transaction costs, all assets are marketable and divisible
- Another points at unresolved shortcoming of the model:
 - single holding period unrealistic, multi-period variants not very satisfactory
- Still others have important consequences:
 - different borrowing and lending rates invalidate same risk-return trade-off for all (see picture)
 - if investors see different frontiers, effect comparable to restriction, e.g. ethical and unethical investments (see picture)



CML with different borrowing and lending rates



CML with heterogeneous expectations

Key assumption is:

Investors maximize expected utility of their end wealth by choosing investments based on their mean-variance characteristics

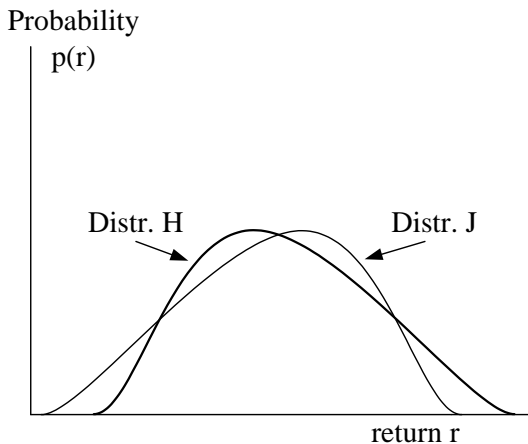
- Is the 'behavioural assumption' (assertion):
- the behaviour (force) that drives the model into equilibrium
- Mean variance optimization *must* take place for the model to work

We did not explicitly say anything about mean-variance in utility theory. Is that special for Markowitz' analysis? Not quite

Mean variance optimization fits in with general economic theory under 2 possible scenario's (assumptions):

- ① Asset returns are jointly normally distributed
 - ① means, variances and covariances completely describe return distributions (higher moments zero)
 - ② no other information required for investment decisions
- ② Investors have quadratic utility functions
 - ① If $U(W) = \alpha + \beta W - \gamma W^2$; choosing a portfolio to maximize U only depends on $E[W]$ and $E[W^2]$, i.e. expected returns and their (co-)variances
 - ② means investors only care about first 2 moments

Do investors ignore higher moments? Which would you chose?



2 mirrored distributions with identical mean and stand.dev.

Empirical tests of the CAPM

Require approximations and assumptions:

- model formulated in expectations
- has to be tested with historical data
- gives returns a function of β , not directly observable

Tested with a two pass regression procedure:

- ① time series regression of individual assets
- ② cross section regression of assets' β s on returns

First pass, time series regression estimates β s:

$$r_{it} - r_{ft} = \hat{\alpha}_i + \hat{\beta}_i(r_{mt} - r_{ft}) + \hat{\varepsilon}_{it}$$

- regresses asset risk premiums on market risk premiums for each asset separately
- market approximated by some index usually short observation periods (weeks, months)
- result is called *characteristic line*
- slope coefficient is estimated beta of asset i , $\hat{\beta}_i$

Second pass, cross section regression estimates risk premiums:

$$\overline{rp}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \gamma_{2n}(\text{testvar}_n) + \hat{u}_i$$

- regresses average risk premiums on $\hat{\beta}$
- rp averaged over observation period $\overline{rp}_i = \sum_t (r_{it} - r_{ft}) / T$
- β can also be estimated over prior period

Some more details:

- usually done with portfolios, not individual assets
- over longer periods (years)
- with rolling time window (drop oldest year, add new year)
- often includes other variables (testvars)

What does the CAPM predict about the coefficients of 2nd pass regression?

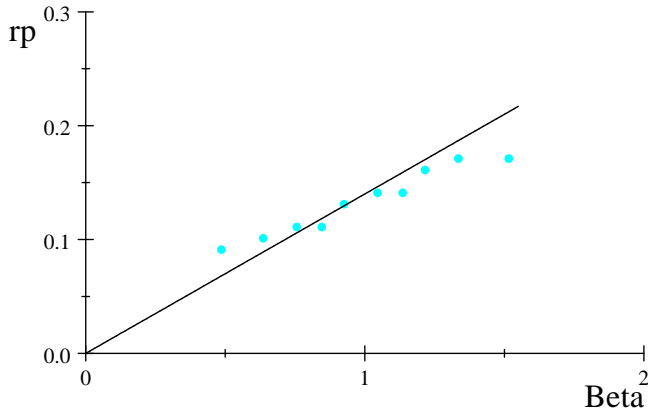
$$\overline{rp}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \gamma_{2n}(\text{testvar}_n) + \hat{u}_i$$

- ① $\gamma_0 = 0$
- ② $\gamma_1 = \overline{rp}_m$
- ③ $\gamma_2 = 0$
- ④ and relation should be linear in β
e.g. β^2 as testvar should not be significant
- ⑤ R^2 should be reasonably high

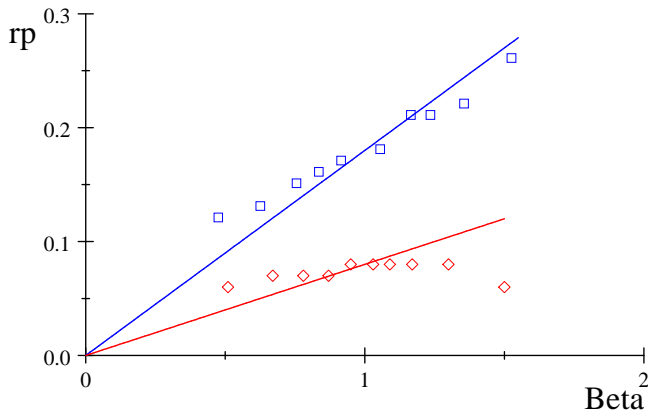
Example: Fischer Black: Return and Beta, *Journal of Portfolio Management*, vol.20 no.1, fall 1993

- uses all stocks on NYSE 1926-1991, monthly data
 - 1931: 592 stocks, 1991: 1505 stocks
- starting 1931, makes yearly β portfolios:
 - estimates individual β s over previous 60 months
 - by regressing risk premium on market risk premium
 - makes 10 portfolios, after β deciles (high - low β)
 - calculates portfolio average β , \bar{r}_{p_i} etc.
- repeats 'rolling' for 1932, 1933, etc. yearly rebalancing
- calculates portfolio averages whole period + sub-periods

For 10 portfolios, β plotted against risk premium:



Black, 1931-1991, line is $\bar{r}_p \times \beta$



Black, 1931-1965 (blue) and 1966-1991 (red), lines are $\overline{rp}_m \times \beta$

Black's results are typical for many other studies:

- ① $\gamma_0 > 0$ (i.e. too high)
- ② $\gamma_1 < \overline{rp}_m$ but $\gamma_1 > 0$ (i.e. too low)
 - ① in recent data, γ_1 is lower than before
 - ② even close to zero ('Beta is dead')
- ③ linearity generally not rejected
- ④ other variables are significantly $\neq 0$, so other factors play a role:
 - ① small firm effect
 - ② book-to-market effect
 - ③ P/E ratio effect
- ⑤ R^2 ?

Roll's critique: can CAPM be tested at all?

Roll argues: CAPM produces only 1 testable hypothesis:

the market portfolio is mean-variance efficient

Argument based on following elements:

- There is only 1 ex ante efficient market portfolio using the whole investment universe
- includes investments in human capital, venture idea's, collectors' items as wine, old masters' paintings etc.
- is unobservable
- tested with ex post sample of market portfolio, e.g. S&P 500 index, MSCI, Oslo Børs Benchmark Index

Gives rise to benchmark problem:

- sample may be mean-variance efficient, while the market portfolio is not
- or the other way around

But if sample is ex post mean-variance efficient:

- mathematics dictate that β 's calculated relative to sample portfolio will satisfy the CAPM
- means: all securities will plot on the SML

Only test is whether portfolio we use is really the market portfolio
⇒ untestable

A simple practical application of what we have learned so far

Suppose you are very risk averse, what would you choose:

- ① A very risky share of €250 in a company you expect to perform badly in the near future
- ② A risk free bond of €235

What would you chose:

- ① €250 today
- ② €235 today

What do we learn from this?

- Financial markets provide information needed to value alternatives
 - nature of the bond and stock already reflected in price
 - nobody needs stocks or bonds to allocate consumption over time
 - everybody prefers more to less
- Financial decisions can be made rationally by maximizing value regardless of risk preferences or expectations
 - risky share and risk free bond have the same value for risk averse student and rich businessman
 - doesn't matter where the money comes from
 - simply choose highest PV, reallocate later

Financial markets give the opportunity to:

- expose to risk / eliminate risk
- move consumption back and forth in time

On well functioning financial markets:

- prices are 'fair', i.e. arbitrage free
- arbitrage brings about the 'Law of one price':
 - same assets have same price
 - asset value comes from its cash flow pattern over time/scenario's
 - if same pattern can be constructed with different combination of assets, price must be the same
 - if not, buying what is cheap and selling what is expensive will drive prices to same level

Arbitrage

Arbitrage is strategy to profit from mispricing in markets

Formally, an arbitrage strategy:

- either requires
 - investment ≤ 0 today, while
 - all future pay-offs ≥ 0 and
 - at least one payoff > 0
- or requires
 - investment < 0 today (=profit) and
 - all future pay-offs ≥ 0

Less formally:

- either costs nothing today + payoff later
- or payoff today without obligations later

Arbitrage

Example:

If gold costs

- \$670/ounce in New York
- ¥80.000/ounce in Tokyo
- then this implies ¥119 for \$1

At ¥115/\$1 there is this arbitrage opportunity:

- buy gold in New York, costs \$670
- sell gold in Tokyo, gives ¥80.000
- change $¥80.000/115 = \$696$ or \$26 riskless, instantaneous arbitrage profit
- and then you do it again, and again..

In practice, you and I cannot do this, and certainly not again and again

- Deals are done electronically with very large amounts (measured in trillions - 10^9 per day) and very low transaction costs
 - makes even small price differences profitable
 - profiting makes them disappear quickly
- Real arbitrage opportunities are few and far between
 - takes a lot of research to find them (usually)
 - are not scalable (cannot do them again and again)

Ross (2005) estimates arbitrage opportunities at less than 0.1%, and many people look out for them:

Power of arbitrage: a horror story

- Thursday 8 Dec. 2005, 9:27 am, a trader at Japanese brokerage unit of Mizuho Financial Group (2nd largest bank in Japan) wrongly put in an order to sell 610,000 shares of J-Com for ¥1 each.
- The intention was to sell 1 share for ¥610,000 for a client.
- Was first day of J-Com's listing. Order was 42 times larger than 14,500 outstanding J-Com shares, which had a total market value of 11.2 billion yen (\$93 million).
- Within the 11 minutes before Mizuho could cancel the order, 607,957 shares traded, generating \$3.5 billion of trades in a company the market valued at \$93 million.
- Mizuho Securities lost about \$347 million on the mistake

Arbitrage Pricing Theory

- Introduced by Ross (1976)
- Does not assume that investors maximize utility based on stocks' mean-variance characteristics
- Instead, assumes stock returns are generated by a multi-index, or multi-factor, process
- More general than CAPM, gives room for more than 1 risk factor
- Widely used, e.g. Fama-French 3 factor model

Introduce with detour over *single index model*

Single index model

So far, we used whole variance-covariance matrix

- With I stocks, calls for $\frac{1}{2}I(I-1)$ covariances
- Gives practical problems for large I
- plus: non marked related part of covariance low/erratic

Single index model is practical way around this:

- Assumes there is *only 1* reason why stocks covary: they all respond to changes in market as a whole
- Stocks respond in different degrees (measured by β)
- But stocks do not respond to unsystematic (not marked related) changes in other stocks' values

Can be formalized by writing return on stock i as:

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

r_i, r_m = return stock i , market

α = expected value non marked related return

ε = random element of non marked related return, with $E(\varepsilon) = 0$
and variance = σ_ε^2

β = beta coefficient (sensitivity for changes in the market)

Single factor model makes 2 assumptions:

- ① $cov(r_m, \varepsilon_i) = 0$: random element of non marked related return not correlated with market return
- ② $cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$: random elements of non marked related returns are uncorrelated

Means that variance, covariance of stocks is:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2 \quad \sigma_{i,j} = \beta_i \beta_j \sigma_m^2$$

covar determined by stocks' responses to changes in marked

Simplifies analysis of large portfolios drastically:

- have to calculate each stock's α , β and σ_{ε}^2
- plus r_m and σ_m^2 , i.e. $3I + 2 < I + \frac{1}{2}I(I-1)$
- for 100 stock portfolio
 - full var-covar has $100 \times 99/2 = 4950$ covar's + 100 var's
 - index model uses $3 \times 100 + 2 = 302$

The single index model

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

can also be looked upon as a *return generating process* :

The returns on any investment consist of:

- α_i expected return not related to the return on the market
- $\beta_i r_m$ return that is related to the return on the market
- ε_i random element

Return generating process easily extended to more indices (or factors):

- 'split' market index in several industry indices (industrials, shipping, financial,...)
- general economic factors (interest rate, oil price,...)

Expression for stock returns then becomes:

$$r_i = \alpha_i + b_{1i}F_1 + b_{2i}F_2 + \dots + b_{Ki}F_K + \varepsilon_i$$

b_{1i} = sensitivity of stock i for changes in factor F_1

F_1 = return on factor 1, etc.

The multi-factor (-index) model assumes that:

- factors are uncorrelated: $cov(F_m, F_k) = 0$ for all $m \neq k$
- residuals uncorrelated with factors $cov(F_k, \varepsilon_i) = 0$
- residuals of different stocks uncorrelated $cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$

Arbitrage Pricing Theory

- Arbitrage pricing theory builds on such a multi-factor return generating process
- Distinguishes between
 - *expected* part of stock returns
 - *unexpected* part
- Unexpected part (risk) consists of
 - systematic (or market) risk
 - and unsystematic (or idiosyncratic) risk
- Market risk not expressed as covar with market but as sensitivity to (any) number of risk factors

To derive pricing relation, start with return generating process:

$$r_i = \alpha_i + b_{1i}F_1 + b_{2i}F_2 + \dots + b_{Ki}F_K + \varepsilon_i$$

taking expectations:

$$E(r_i) = \alpha_i + b_{1i}E(F_1) + b_{2i}E(F_2) + \dots + b_{Ki}E(F_K)$$

subtracting lower from upper gives:

$$\begin{aligned} r_i - E(r_i) &= (\alpha_i + b_{1i}F_1 + \dots + b_{Ki}F_K + \varepsilon_i) \\ &\quad - (\alpha_i + b_{1i}E(F_1) + \dots + b_{Ki}E(F_K)) \end{aligned}$$

which can be re-written as:

$$r_i = E(r_i) + \sum_{k=1}^K b_{ik}(F_k - E(F_k)) + \varepsilon_i$$

- $E(r_i)$ = is expected return of stock i
- b_{ik} = is sensitivity of stock i to factor k
- F_k = return of factor k, with $E(F_k - E(F_k)) = 0$
(\Rightarrow fair game: expectations accurate in long run)
- ε_i = idiosyncratic return stock i, $E(\varepsilon_i) = 0$

Terms after $E(r_i)$ are 'error' part of process:

- describe deviation from expected return
- b_{ik} is sensitivity for *unexpected* factor changes
- expected part included in $E(r_i)$

Next, construct portfolio, I assets, weights x_i , then portfolio return is:

$$r_p = \sum_{i=1}^I x_i r_i$$

substituting expression for r_i gives:

$$r_p = \sum_{i=1}^I x_i E(r_i) + \sum_{i=1}^I \sum_{k=1}^K x_i b_{ik} (F_k - E(F_k)) + \sum_{i=1}^I x_i \varepsilon_i$$

In well diversified portfolios, idiosyncratic risk (last term) disappears

APT's equilibrium condition is:

the absence of arbitrage opportunities

Means if you make a well diversified portfolio ($\sum_i x_i \varepsilon_i = 0$):

- ① that requires no net investment
 - sum portfolio weights is zero: $\sum_i x_i = 0$
- ② that involves no risks
 - weighted sum of all b_{ik} is zero : $\sum_i x_i b_{ik} = 0$ for all k
- ③ then

what?

the expected return must be zero:

- $\sum_i x_i E(r_i) = 0$

These three no-arbitrage conditions can be interpreted as orthogonality conditions from linear algebra:

- ① $\sum_i x_i = 0$ means:
 - vector of weights is orthogonal to a vector of 1's
- ② $\sum_i x_i b_{ik} = 0$ means:
 - vector of weights orthogonal to vectors of sensitivities
- ③ $\sum_i x_i E(r_i) = 0$ means:
 - vector weights orthogonal to vector expected returns

This means that the last vector, $E(r_i)$, must be a linear combination of the other 2:

$$E(r_i) = \lambda_0 + \lambda_1 b_{1i} + \lambda_2 b_{2i} + \dots + \lambda_k b_{ki}$$

To give lambda's economic meaning:

- construct risk free portfolio:
 - earns risk free rate
 - has zero value for all b_{ij}
 - $r_f = \lambda_0 + \lambda_1 0 + \dots + \lambda_k 0 \Rightarrow \lambda_0 = r_f$
- construct pure factor portfolio for factor 1:
 - sensitivity 1 for factor 1 and zero value for all other b_{ij} :
 - earns expected return of factor 1
 - $E(F_1) = r_f + \lambda_1 1 + \lambda_1 0 + \dots + \lambda_k 0 \Rightarrow \lambda_1 = E(F_1) - r_f$
- repeat for all factors

Gives usual form of APT as equilibrium relation:

$$E(r_i) = r_f + \sum_{k=1}^K b_{ik}(E(F_k) - r_f)$$

Example

Illustrates APT with 3 well diversified portfolios and their sensitivities to 2 factors, priced to give these returns:

	P ₁	P ₂	P ₃
r_p	.18	.15	.12
b_1	1.5	0.5	0.6
b_2	0.5	1.5	0.3

Portfolio returns are functions of

- risk free rate and 2 factor returns (risk premiums)
- portfolios' sensitivities

Factor returns and r_f found by solving 3 APT equations:

$$.18 = \lambda_0 + \lambda_1 \times 1.5 + \lambda_2 \times .5$$

$$.15 = \lambda_0 + \lambda_1 \times .5 + \lambda_2 \times 1.5$$

$$.12 = \lambda_0 + \lambda_1 \times .6 + \lambda_2 \times .3$$

which gives $\lambda_0 = 0.075$, $\lambda_1 = 0.06$ and $\lambda_2 = 0.03$

Equilibrium relation $E(r_i) = .075 + .06b_{1i} + .03b_{2i}$

- defines return plane in 2 risk dimensions
- all investments must lie on this plane
- otherwise arbitrage opportunities exist

Suppose you make a portfolio:

- with $b_1=.75$ and $b_2=.7$
- you figure it is somewhere between P_1 and P_2
- price it to offer a .16 return, also between P_1 and P_2

What happens?

You go bankrupt quickly! You offer this arbitrage opportunity:

- construct arbitrage portfolio of $.2P_1 + .3P_2 + .5P_3$, has:
- $b_1 = .2 \times 1.5 + .3 \times .5 + .5 \times .6 = .75$
- $b_2 = .2 \times .5 + .3 \times 1.5 + .5 \times .3 = .7$
- return of $.2 \times .18 + .3 \times .15 + .5 \times .12 = .141$

Arbitrage strategy:

- buy what is cheap (your portfolio)
- sell what is expensive (arbitrage portfolio)

	Cfl_{now}	Cfl_{later}	b_1	b_2
buy your portfolio	-1	1.160	.75	.7
sell arbitrage.portfolio	1	-1,141	-.75	-.7
net result	0	.019	0	0

Profit of .019 is risk free, zero sensitivity to both factors

Empirical tests of APT

- require same assumptions & approximations as CAPM
- done with similar two pass regression procedure:
 - time series regression to estimate sensitivities
 - cross section analysis to estimate risk premiums

Example: split total market in 2 industry indices:

- manufacturing (F_{man})
- trade (F_{trad})

① First pass regression: estimate sensitivities

$$r_{it} - r_{ft} = \hat{\alpha}_i + \hat{\beta}_{man,i}(F_{man,t} - r_{ft}) + \hat{\beta}_{trad,i}(F_{trad,t} - r_{ft}) + \hat{\varepsilon}_{it}$$

for all individual assets

② Then calculate average risk premiums (\overline{rp}_i) etc. over same/subsequent period and estimate risk factor premiums in second pass regression:

$$\overline{rp}_i = \gamma_0 + \gamma_1 \hat{\beta}_{man,i} + \gamma_2 \hat{\beta}_{trad,i} + \hat{u}_i$$

③ APT predictions tested by:

- γ_0 should be zero
- γ_1 should be $\overline{F_{man} - r_f}$
- γ_2 should be $\overline{F_{trad} - r_f}$

Industry indices are easy:

- readily observable, also their risk premiums
- describe market completely:
market = manufacturing + trade

More difficult if we use standard ways of getting factors:

- business characteristics
 - size, book-to-market value, price-earnings ratio, etc.
- general economic variables
 - interest rate, oil price, exchange rates, etc.
- statistical technique called factor analysis
 - reduces dimensions of data matrix

Factor analysis:

Starts with a data matrix of:

- T ($t=1,2,..,T$) returns
- for I ($i=1,2,..,I$) assets

Variance of T returns can be completely described by

- T new variables (the factors) which are
- linear combinations of the original variables (=returns):

$$F_1 = \alpha_{11}r_1 + \alpha_{12}r_2 + .. + \alpha_{1T}r_T$$

$$F_2 = \alpha_{21}r_1 + \alpha_{22}r_2 + .. + \alpha_{2T}r_T$$

.... ..

$$F_T = \alpha_{T1}r_1 + \alpha_{T2}r_2 + .. + \alpha_{TT}r_T$$

F = new factors, r = returns, α = coefficients to be calculated

Point is the way the factors are constructed (i.e. how the coefficients α are calculated)

Most common variant of factor analyses:

- called principal components analysis
- constructs first factor such that it makes maximal contribution to the sum of the variances of the T returns
- second factor constructed such that it
 - is orthogonal to the first
 - makes a maximal contribution to remaining sum of variances
- etc.

Result:

- first few factors describe the bulk of the variance
- the other factors can be omitted without much loss of information

With all three methods:

- APT does not specify number of factors
 - difficult to be 'complete'
 - model can be under-specified (omitted viable bias)
 - model can be over-specified
- APT does not specify risk premiums
 - are no observed risk premiums either
 - have to be estimated

Example 1: Fama-French three factor model

- estimated on monthly data 1963-1991
- all stocks on all US exchanges (NYSE, ASE, NASDAQ)
- for each year, different portfolios are made:
 - size: small and big stocks
 - each month portfolio returns calculated
 - difference: SMB, small minus big
 - approximates premium size related risk factor
 - book-to-market: high (top 30%), middle, low (bottom 30%)
 - each month portfolio returns calculated
 - difference: HML, high minus low
 - approximates premium book-to-market related risk factor

- Third factor is total market: return value weighted portfolio of all stocks minus r_f

First pass estimates portfolios' sensitivities (factor loadings):

$$r_{it} - r_{ft} = \hat{a}_i + \hat{b}_i(r_{mt} - r_{ft}) + \hat{s}_iSMB_t + \hat{h}_iHML_t + \hat{\varepsilon}_{it}$$

- \hat{b}_i , \hat{s}_i and \hat{h}_i are sensitivities of portfolio i
- SMB, HML are risk premiums (small-big, high-low), no $-r_f$

Fama-French three factor model formulated as:

$$E(r_i) - r_f = \hat{a}_i + \hat{b}_i[E(r_m) - r_f] + \hat{s}_iE(SMB) + \hat{h}_iE(HML)$$

APT predicts intercept, \hat{a}_i , should be zero

- Fama and French find \hat{a}_i close to 0 in most cases
- also claim model explains much ($>90\%$) of variance in average stock returns

Fama-French model widely used

- to calculate $E(r_p)$ when size and value effects can play a role
- see examples in market efficiency

But: more recent research shows that the model's relevance has diminished over time.

Example 2: Chen, Roll and Ross' *Economic forces and the stock market*

- uses simple arguments to choose economic variable
- tests their role as sources of systematic risk
- with long time series (1958 - 1983)
- of monthly data
- also tests 3 sub-periods

Variable capture broad spectrum, refer to:

- real economy (industrial production, consumption)
- financial markets (stock index, interest rates)
- commodity markets (oil price)

Systematic risk factors they found risk premiums for:

- growth industrial production
- Δ risk premium
(return risky corporate bonds — return government bonds)
- unanticipated changes term structure
(long — short government debt)
- (more weakly) unanticipated inflation

They found no (or limited) pricing influence of:

- stock market index NYSE (was significant in first pass)
- unanticipated changes in real per capita consumption
- oil price changes

Summarizing, Arbitrage Pricing Theory:

- Rests on different assumptions than CAPM
- Is more general than CAPM
 - makes less restrictive assumptions
 - allows more factors, more realistic
- Is less precise than CAPM
 - does not give a volume of risk (what or even how many factors to use)
 - does not give a price of risk (no expression for factor risk premiums, have to be estimated empirically)
- has interesting applications in risk management, default prediction, etc.