Chapter 7: Option pricing foundations

Exercises - solutions

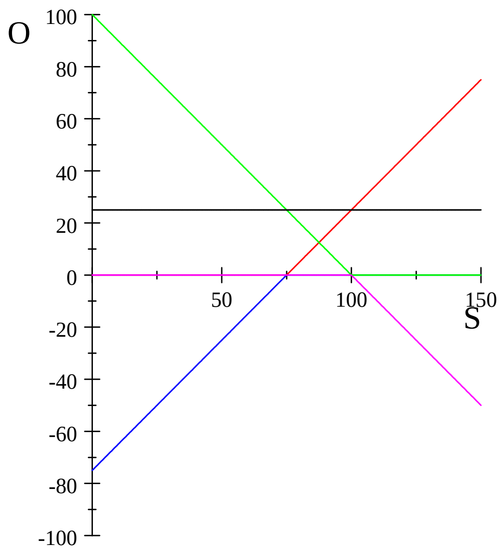
1. (a) We use the put-call parity: Share + Put = Call + PV(X) or
Share + Put - Call $= 97.70 + 4.16 - 23.20 = 78.66$
and $PV(X) = 80 \times e^{-0.0315} = 77.519$.
We see that the put-call parity did not precisely hold. This probably does not mean that are any arbitrage opportunities since:

    - PCP is only valid for European options, and most traded options are American options.
    - ZX Co. may pay a dividend before expiry.
    - We ignored bid – ask spreads in trading and any other transaction costs.
    - There may have been non synchronous trading, i.e. put price may refer to a different time than the call and/or the share.

2. (a) It may be helpful to calculate the options' pay-offs on different points in the interval:

Share price (S)	0	50	75	100	125	150
short put (X=75)	- 75	- 25	0	0	0	0
long put (X=100)	100	50	25	0	0	0
long call (X=75)	0	0	0	25	50	75
short call (X=100)	0	0	0	0	- 25	- 50
Total position (O)	+ 25	+ 25	+ 25	+ 25	+ 25	+ 25

The positions are plotted in the figure below; the plotted functions are $\min[0, x - 75]$, $\max[0, 100 - x]$, $\max[0, x - 75]$, $\min[0, 100 - x]$ and their sum.



3. No, puts and calls do not cancel out, but buying a put is cancelled out by selling a put. Similarly, a short call is cancelled out by a long call. Also, the prices of otherwise identical at-the-money puts and calls are not the same, as is easily verified with the put-call parity: share + long put = long call + PV(X) or long put - long call = PV(X) - share. If the put and call have the same value, the share price has to be equal to the present value of the exercise price (so not the exercise price).

4. (a) A butterfly with calls consists of 1 long call with exercise price x1, 2 short calls with exercise price x2 and 1 long call with exercise price x3.
   (b) The same position with puts consists of 1 long put with exercise price x3, 2 short puts with exercise price x2 and 1 long put with exercise price x1. The positions are depicted in Figure 1 below.

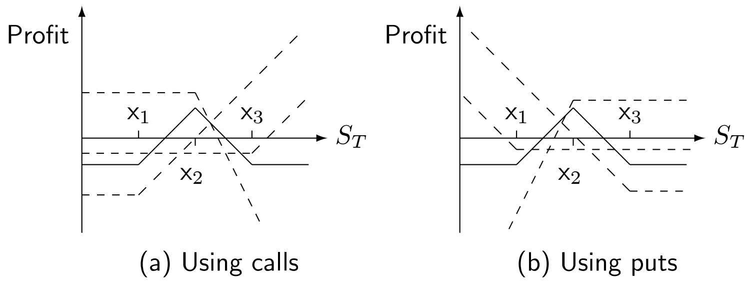


Figure 1: Profit diagrams for a butterfly spread

5. A butterfly spread involves buying 1 call with a low exercise price, selling two calls with a higher exercise price and buying 1 call with an even higher exercise price.

   (a) The initial investment required is:

	strike	price	amount
buy 1	460	20.75	-20.75
sell 2	480	11.75	23.50
buy 1	500	6.00	-6
total			-3.25

   (b) We first have to calculate the put prices using the put-call parity:

call	+PV(X)	-share	=price put
20.75	460/1.015=453.2	-462.50	=11.45
11.75	480/1.015=472.9	-462.50	=22.15
6	500/1.015=492.6	-462.50	=36.10

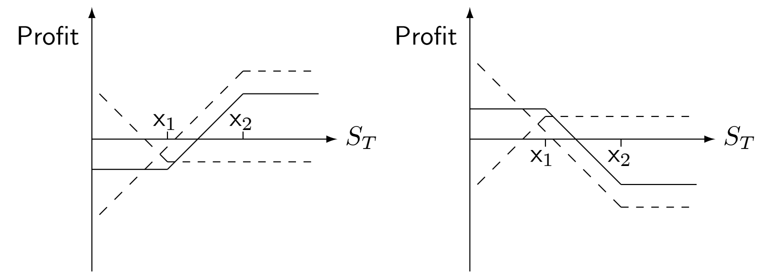
Then we can set up the butterfly with puts:

	strike	price	amount
buy 1	500	36.10	-36.10
sell 2	480	22.15	44.30
buy 1	460	11.45	-11.45
total			-3.25

which gives the same initial investment.

(c) These prices are different because traded options are (almost) always American options and by using the put-call parity we calculated prices of European options. European puts trade at a lower price range than American puts because the right of early exercise is valuable (as well as frequently used).

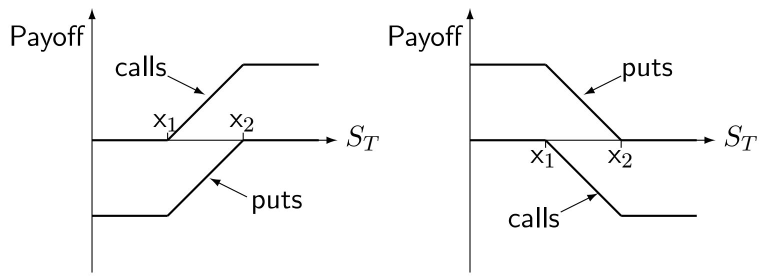
6. (a) The bull spread is constructed by selling 1 put with exercise price x2 and buying 1 put with exercise price x1. In the bear spread the positions are reversed, i.e. a put with exercise prise x1 is sold and a put with exercise price x2 is bought. The positions are depicted in Figure 2 below



(a) Bull spread (b) Bear spread

Figure 2: Profit diagrams for spreads

(b) The initial balances of option premiums (initial investments) are reversed compared with the same positions constructed with calls. The price of a put increases with the exercise price (call prices decrease). In a bull spread constructed with puts, the more expensive put with a high exercise price is sold and the cheaper one is bought so that the initial balance is positive (negative investment). On the other hand, the payoff at maturity is either zero or negative. In a bear spread constructed with puts the more expensive option is bought so that the initial balance is negative (initial investment required). However, the payoff at maturity is either zero or positive. The payoffs at maturity at depicted in the payoff diagram below. The differences



(a) Bull spread (b) Bear spread

Figure 3: Payoff diagrams of spreads

are summarized in the table below.

	Bull	Bear
Calls	buy low X (costs more)	sell low X (costs more)
decrease	sell high X (costs less)	buy high X (costs less)
with X	price < 0	price > 0
	payoff at maturity > 0	payoff at maturity < 0
Puts	buy low X (costs less)	sell low X (costs less)
increase	sell high X (costs more)	buy high X (costs more)
with X	price > 0	price < 0
	payoff at maturity < 0	payoff at maturity > 0

7. The bound states that a European put option cannot be worth more than the present value of the exercise price and this bound is *not* valid for American puts. The arbitrage strategy that enforces the bound for European puts does not always hold for American puts because of early exercise. If the put is exercised early the interest earned on PV(X) is not enough to cover the exercise price so that additional cash is required. The easiest way to illustrate this is by assuming that the stock price drops to zero right after the put was sold:

	Now	At exercise $S_T = 0 < X$
Sell put	$+O_p^A$	$-X$
Lend $PV(X)$	$-PV(X)$	$PV(X)$
Total position	> 0	$PV(X) - X < 0$

8. (a) We calculate the values as the expected pay-offs discounted at the risk adjusted rate of return; since both states are equally likely:

$$\begin{aligned} A &= \frac{0.5 \times 6 + 0.5 \times 12}{1.08} = 8.33 \\ B &= \frac{0.5 \times 4 + 0.5 \times 14}{1.12} = 8.04 \end{aligned}$$

(b) We can calculate state price by making portfolios of A and B that pay off 1 in only one state and 0 in the other. For the first state this means solving the equations:

$$\begin{aligned} 6 \times a + 4 \times b &= 1 \\ 12 \times a + 14 \times b &= 0 \end{aligned}$$

for the weights a and b. The solution is: $a = \frac{7}{18},\ b = -\frac{1}{3}$. This gives a state price of $\frac{7}{18} \times 8.33 - \frac{1}{3} \times 8.04 = 0.559$. For the second state price we solve the equations (re-using the symbols a and b):

$$\begin{aligned} 6 \times a + 4 \times b &= 0 \\ 12 \times a + 14 \times b &= 1 \end{aligned}$$

The solution is: $a = -\frac{1}{9},\ b = \frac{1}{6}$, so the second state price is $-\frac{1}{9} \times 8.33 + \frac{1}{6} \times 8.04 = 0.414$. We can also use matrix algebra:

$$\left[\begin{array}{cc} 8.33 & 8.04 \end{array}\right] \left[\begin{array}{cc} 6 & 4 \\ 12 & 14 \end{array}\right]^{-1} = \left[\begin{array}{cc} 0.559 & 0.414 \end{array}\right]$$

To calculate the risk free interest rate we have to make a portfolio of A and B that pays off the same in both states. This means solving the equations:

$$\begin{aligned} 6 \times a + 4 \times b &= 1 \\ 12 \times a + 14 \times b &= 1 \end{aligned}$$

which gives $a = \frac{5}{18}$, $b = -\frac{1}{6}$. The present value of this portfolio is $\frac{5}{18} \times 8.33 - \frac{1}{6} \times 8.04 = 0.9739$. We can check this with the sum of the state prices, which has to be the same: $0.559 + 0.414 = 0.973$. The risk free interest rate is $1/0.9739 = 1.027$.

(c) The values of the securities A and B are re-calculated as follows:

i. using state prices:
$A = 6 \times 0.559 + 12 \times 0.414 = 8.32$
$B = 4 \times 0.559 + 14 \times 0.414 = 8.03$

ii. to use the risk neutral valuation formula we first have to calculate the risk neutral probabilities. The are defined as the state prices $\times(1 + r_f)$ so $p_1 = 0.559 \times 1.027 = 0.574$ and $p_2 = 0.414 \times 1.027 = 0.426$. The risk neutral valuation formula then gives:

$$\begin{aligned} A &= \frac{0.574 \times 6 + 0.426 \times 12}{1.027} = 8.33 \\ B &= \frac{0.574 \times 4 + 0.426 \times 14}{1.027} = 8.04 \end{aligned}$$

(d) At maturity the options pay off the maximum of zero and the difference between the security price and the exercise price: $\max[0, S - X]$, so:

$$\begin{array}{lcc} & A & B \\ State1 & \max[0, 6 - 10] = 0 & \max[0, 4 - 10] = 0 \\ State2 & \max[0, 12 - 10] = 2 & \max[0, 14 - 10] = 4 \end{array}$$

The state price for the second state is 0.414, so the options prices are $2 \times 0.414 = 0.828$ and $4 \times 0.414 = 1.656$.
We can also use the risk neutral valuation formula and the risk neutral probabilities we calculated above (which is, of course, basically the same thing):

$$\begin{aligned} Option\ on\ A &= \frac{0.574 \times 0 + 0.426 \times 2}{1.027} = 0.830 \\ Option\ on\ B &= \frac{0.574 \times 0 + 0.426 \times 4}{1.027} = 1.659 \end{aligned}$$

9. State prices can be calculated in 3 ways (which all boil down to the same thing, of course):

(I) We can construct the payoff matrix for the securities in this market, take its inverse to get the weights of the portfolios that produce the state securities and then multiply by the values of the securities now to get the state prices.

(II) We can write out the system of 2 equations with 2 unknowns and solve it.

(III) We can calculate the risk neutral probabilities and then discount them to get the state prices.
We present all 3 methods.

(I) The payoff matrix is:

$$\Psi = \begin{bmatrix} 1.05 & 80 \\ 1.05 & 120 \end{bmatrix}$$

and its inverse

$$\Psi^{-1} = \begin{bmatrix} 2.8571 & -1.9048 \\ -0.025 & 0.025 \end{bmatrix}$$

The prices of the securities now are:

$$v = \begin{bmatrix} 1 & 100 \end{bmatrix}$$

so that the state prices are:

$$v\Psi^{-1} = \left[\begin{array}{cc} 0.357\,14 & 0.595\,24 \end{array}\right]$$

The option's pay-offs are

$$O = \left[\begin{array}{c} 0 \\ 20 \end{array}\right]$$

multiplying these by the state prices gives the option price:

$$v\Psi^{-1}O = 11.\,905$$

(II) The system of 2 equations with 2 unknowns is:

$$\begin{array}{rcl} 1 & = & \psi_1 \times 1.05 + \psi_2 \times 1.05 \\ 100 & = & \psi_1 \times 80 + \psi_2 \times 120 \end{array}$$

and its solution is:

$$[\psi_1 = 0.357\,14, \psi_2 = 0.595\,24]$$

so that the option price becomes

$$0.357\,14 \times 0 + 0.595\,24 \times 20 = 11.\,905$$

(III) State prices are discounted risk neutral probabilities:

$$p = \frac{r-d}{u-d} = \frac{1.05 - .8}{1.2 - .8} = 0.625\,/1.05 = 0.595\,24$$

$$1 - p = 1 - 0.625 = 0.375/1.05 = 0.35714$$

and the calculation proceeds as under (b).

10. (a) Since $u \neq r \neq d$ the stock and risk free debt are linearly independent, so there are as many independent securities as there are states of the world, which means that the market is complete. Since $d < r < u$ there are no arbitrage opportunities on this market.

(b) We first calculate the value tree for the stock without and with dividends:

$$\begin{array}{ccccc} & & & & 225 \\ & & & \nearrow & \\ & & 150 & & \\ & \nearrow & & \searrow & \\ 100 & & & & 75 \\ & \searrow & & \nearrow & \\ & & 50 & & \\ & & & \searrow & \\ & & & & 25 \end{array} \qquad \begin{array}{ccccc} & & & & 168.75 \\ & & 150 & \nearrow & \\ & \nearrow & 112.5 & & \\ 100 & & & \searrow & \\ & \searrow & & & 56.25 \\ & & 50 & \nearrow & \\ & & 37.5 & & \\ & & & \searrow & \\ & & & & 18.75 \end{array}$$

The parameters of the binomial process are: $u = 1.5$, $d = 0.5$, $r = 1.25$, $p = (r-d)/(u-d) = 0.75$

i. European call, no dividends:
$O_{uu} = \max[0, 225 - 110] = 115$
$O_{ud} = \max[0, 75 - 110] = 0$
$O_{dd} = \max[0, 25 - 110] = 0$
$O_u = (.75 \times 115) + (.25 \times 0)/1.25 = 69$
$O_d = (.75 \times 0) + (.25 \times 0)/1.25 = 0$
$O = (.75 \times 69) + (.25 \times 0)/1.25 = 41.4$

ii. European call, 25% dividends:
$O_{uu} = \max[0, 168.75 - 110] = 58.75$
$O_{ud} = \max[0, 56.25 - 110] = 0$
$O_{dd} = \max[0, 18.75 - 110] = 0$
$O_u = (.75 \times 58.75) + (.25 \times 0)/1.25 = 35.25$
$O_d = (.75 \times 0) + (.25 \times 0)/1.25 = 0$
$O = (.75 \times 35.25) + (.25 \times 0)/1.25 = 21.15$

iii. American call on a non dividend paying stock is the same as a European call

iv. American call, 25% dividend:
$O_{uu} = \max[0, 168.75 - 110] = 58.75$
$O_{ud} = \max[0, 56.25 - 110] = 0$
$O_{dd} = \max[0, 18.75 - 110] = 0$
$O_u - alive = (.75 \times 58.75) + (.25 \times 0)/1.25 = 35.25$
$O_u - dead = \max[0, 150 - 110] = 40$
$O_u = \max[alive, dead] = 40$
$O_d = 0$ both dead and alive
$O - alive = (.75 \times 40) + (.25 \times 0)/1.25 = 24$
$O - dead = \max[0, 100 - 110] = 0$
$O = \max[alive, dead] = 24$

v. European put, no dividends:
$O_{uu} = \max[0, 110 - 225] = 0$
$O_{ud} = \max[0, 110 - 75] = 35$
$O_{dd} = \max[0, 110 - 25] = 85$
$O_u = (.75 \times 0) + (.25 \times 35)/1.25 = 7$
$O_d = (.75 \times 35) + (.25 \times 85)/1.25 = 38$
$O = (.75 \times 7) + (.25 \times 38)/1.25 = 11.80$

vi. European put, 25% dividends:
$O_{uu} = \max[0, 110 - 168.75] = 0$
$O_{ud} = \max[0, 110 - 56.25] = 53.75$
$O_{dd} = \max[0, 110 - 18.75] = 91.25$
$O_u = (.75 \times 0) + (.25 \times 53.75)/1.25 = 10.75$
$O_d = (.75 \times 53.75) + (.25 \times 91.25)/1.25 = 50.50$
$O = (.75 \times 10.75) + (.25 \times 50.50)/1.25 = 16.55$

11. Next period the stock price is either $50 \times 1.3 = 65$ or $50 \times .75 = 37.5$. The value trees for the options are:

$$
\begin{array}{lll}
 & & \max[0, 65 - 45] = 20 \\
 & \nearrow & \\
\text{O}_c & & \\
 & \searrow & \\
 & & \max[0, 37.5 - 45] = 0
\end{array}
\qquad
\begin{array}{lll}
 & & \max[0, 45 - 65] = 0 \\
 & \nearrow & \\
\text{O}_p & & \\
 & \searrow & \\
 & & \max[0, 45 - 37.5] = 7.5
\end{array}
$$

(a) For a call option, the option Δ and D are:

$$
\begin{aligned}
\Delta &= \frac{O_u - O_d}{(u-d) \times S} = \frac{20 - 0}{(1.3 - 0.75) \times 50} = 0.72727 \\
D &= \frac{u \times O_d - d \times O_u}{(u-d) \times r} = \frac{1.3 \times 0 - 0.75 \times 20}{(1.3 - 0.75) \times 1.08} = -25.253
\end{aligned}
$$

The price of the call is $O_c = S \times \Delta + D = 50 \times 0.72727 - 25.253 = 11.111$

(b) For a put option, the option Δ and D are:

$$\begin{aligned}
\Delta &= \frac{O_u - O_d}{(u-d)\times S} = \frac{0-7.5}{(1.3-0.75)\times 50} = -0.272\,73 \\
D &= \frac{u \times O_d - d \times O_u}{(u-d)\times r} = \frac{1.3\times 7.5 - 0.75\times 0}{(1.3-0.75)\times 1.08} = 16.414
\end{aligned}$$

The price of the put $O_P = S \times \Delta + D = 50 \times -0.27273 + 16.414 = 2.7775$

12. The self-financing property means that, when the portfolio is rebalanced, the new portfolio has exactly the same value as the old one. So we could sell the old portfolio and use the money to buy the new one without needing any additional cash. In the two-period example on page 212-213 we start with a hedging portfolio of 0.753 shares and 212.11 in borrowing. If the stock price rises to 500 at t_1 the value of this old portfolio is:

$$0.753 \times 500 + (-212.11 \times 1.07) = 149.54$$

The amount of borrowing has increased with the risk free interest rate over the period. With this stock price the new hedge ratio becomes 1, so we need to buy $1-0.753 = 0.247$ of the stock at a price of 500, which costs $0.247\times 500 = 123.50$. We borrow this amount, so the value of the new portfolio is:

$$1 \times 500 - (123.50 + 212.11 \times 1.07) = 149.54$$

exactly the same as the old portfolio.
If the stock price falls to 320 at t_1 the value of the old portfolio is:

$$0.753 \times 320 + (-212.11 \times 1.07) = 14$$

With this stock price the new hedge ratio becomes 0.174, so we have to sell $0.753 - 0.174 = 0.579$ of the stock at a price of 320. This brings in $0.579\times 320 = 185.28$ and we use this amount to reduce the borrowing. The value of the new porfolio thus becomes:

$$0.174 \times 320 - ((212.11 \times 1.07) - 185.28) = 14$$

again exactly the same as the old porfolio.