Option Pricing in Continuous Time

the famous Black and Scholes option pricing formula

Nico van der Wijst

Logarithmic stock returns

Recall from second chapter: discretely compounded returns:

$$r = (S_t - S_{t-1})/S_{t-1}$$

r is return, $S_{t,t-1}$ stock prices end - begin period.
Discretely compounded stock returns

- are *easily aggregated across investments:*
  - attractive in portfolio analysis
- but *non-additive over time:*
  - 5% p. year over 10 years = 62,9% return (1.05^{10}) not 50%

Option pricing uses individual returns over time

- makes continuously compounded returns convenient

Continuously compounded returns calculated as:

$$\frac{S_T}{S_0} = e^{rT} \quad or \quad S_T = S_0 e^{rT}$$

Taking natural log's gives the log returns:

$$\ln \frac{S_T}{S_0} = \ln e^{rT} = rT$$

Log returns additive over time:

- $\ln\left(\frac{S_1}{S_0} \times \frac{S_2}{S_1}\right) \Rightarrow \ln \frac{S_1}{S_0} + \ln \frac{S_2}{S_1} = \ln e^{r_1} + \ln e^{r_2} = r_1 + r_2$
- convenient to use in continuous time models

But: non-additive across investments:

- log is non-linear $\longrightarrow$ ln of sum $\neq$ sum of ln's

Properties of log returns

Have to describe return behaviour over time
Done by making one critical assumption:

log returns are independently and identically distributed (iid)

- Looks innocent assumption for convenience
- Has far reaching consequences:
    - iid assumption means we can invoke Central Limit Theorem: *sum of n iid variables is $\pm$ normally distributed*

Consequences of normally distributed returns:

- returns = ln stock prices
  - if returns $\backsim$ N $\Rightarrow$ stock prices $\backsim \log N$.
- sum 2 indep. normal variables is also normal with
  - mean = sum 2 means
  - variance = sum 2 variances
- extend to many (T) time periods $\Rightarrow$ mean & variance grow linearly with time:
  - so $R_T \backsim N(\mu T, \sigma^2 T)$
  - $R_T =$ continuously compounded return time $[0, T]$
  - expectation $E[R_T] = \mu T$
  - variance $var[R_T] = \sigma^2 T$
  - instantaneous return $= \mu$

Some more consequences:

- iid returns follow a random walk
- random walks have *Markov property* of memorylessness
  - past returns & patterns useless to predict future returns
  - means market is weak form efficient.

Assumptions & consequences fit the real world well but real life stock returns have:

- fatter tails
- more skewness,
- more kurtosis

than normal distribution
Fatter tails give underpricing of financial risks

Transforming probabilities: loading a die

In discrete time, risk neutral probabilities followed 'naturally' from analysis (discounted state prices)
In continuous time specific action is needed:

- change of probability measure

Idea of changing probabilities is counter-intuitive, illustrate with example of loading a die

Simple die game: sixes bet

- aka de Méré's problem
- even money bet that player will roll a 'six' at least once in four rolls of a single die

What is the probability the player will win?

- NOT: $4/6$; often used wrong calculation:
  - prob. of rolling 'six' is $1/6$
  - 4 rolls, so total prob. is $4/6$
- Different formulation of same mistake (all over internet):
  - game should be played with 3 rolls
  - to make it fair, even money bet

Correct calculation:

- prob. player wins in first roll is $1/6 = 0.16667$
- prob. in the second roll is $5/6 \times 1/6 = 0.13889$, etc.
- total probability of winning is:

$$\frac{1}{6} + \frac{5}{6} \times \frac{1}{6} + \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^3 \times \frac{1}{6} = 0.51775$$

Simpler calculation:

- $1-$ the probability of losing:

$$1 - (5/6)^4 = 0.51775$$

Player has 'edge' of 0.01775, can expected to earn money in the long run

We now turn the problem around:

- what must the probability of rolling a 'six' be
- to make three-roll sixes bet an even money game?

Mathematically reformulated:

- how must we transform the die's probability measure
- to make the probability of winning 50%?

A *probability measure* is a rule (or function) that assigns probabilities to set of events.

Probability measure for fair die:

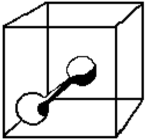
- each side equal probability of coming up
- each outcome has probability of $1/6$

Problem assumes probabilities that specific die faces (1,2,..6) come up can be manipulated
Crooked gamblers found several ways to do that

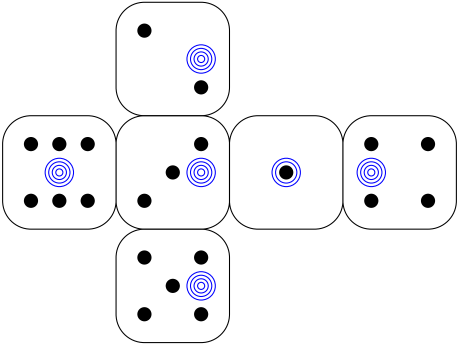
- Dice that are not perfect cubes ('shapes')

Notice: This is for educational purposes only

- Dice with sticky substance on the side with one spot
  - glue activated by blowing on it ('for luck') is harder to detect
- Dice hollowed out on one side, weight is removed
  - called 'floats' or 'floaters'
- 'Tappers', dice with small mercury reservoir at the center
  - connected to other reservoir at side by thin tube
  - tapping makes mercury flow to side reservoir: die becomes loaded



- Loaded dice, with a weight on one side
  - placing the weight towards side with one spot
  - increases probability die will land on side with one spot, so that 'six' comes up



A loaded die, the concentric circles represent the weight
This loading increases prob. die will land on side with one spot
('six' comes up) and decreases prob. 'one' comes up

Assume we can load a die very accurately

- can move probability mass from 'one' to 'six' in any degree

We can then calculate what probability of rolling a 'six' will make three-roll sixes bet an even money game

- Calculate it from the probability of losing, p, which is also 0.5
- so in a three roll game:

$$p^3 = 0.5 \Rightarrow p = \sqrt[3]{0.5} = 0.7937$$

- probability of rolling a 'six' is then:

$$1 - 0.7937 = 0.2063$$

What must prob. of rolling 'six' be to give player same 'edge' of 0.01775 as in four-roll game with fair die?

- Probability of winning then is 0.51775
- probability of losing becomes $1 - 0.51775 = 0.48225$
- reusing symbol p we get:

$$p^3 = 0.48225 \Rightarrow p = \sqrt[3]{0.48225} = 0.7842$$

- so probability of rolling a 'six' is

$$1 - 0.7842 = 0.2158$$

A more complex gambling game with a die:

- you have to pay to get in
- payoff = number of spots turning up: 1, 2,.., 6

What is a fair price to enter the game?

- With a fair die, all outcomes equal probability $1/6$
- expected payoff $\Sigma p_i R_i = 3.5$, (payoffs $= R$, prob.$= p$)
- variance $= \Sigma p_i (R_i - (\Sigma p_i R_i))^2 = 2.917$

With 3.5 entry price:

- both players have equal expected gain - loss (zero)
- game is fair

But organizers want to make money, not to have fair games
Can be done in several ways:

- raise the entrance price:
  - 4.5 gives exp. payoff 1 for organizer, same loss for player
  - looks silly, but is basis of all lotteries
- adjust spots:
  - blot out 6 (replacing with 0) reduces exp. payoff to 2.5, variance same 2.917
  - also looks silly, but is done in roulette
- change the probabilities:
  - by tampering with the die, e.g. loading it

How should the die be loaded to give the organizer an exp. payoff of 1?

Reformulated as scientific problem:

can probability measure for a die be transformed by a formula that affects all probabilities in such a way that expected payoff = 2.5 and variance left unchanged?

Restrictions:

- measures must be equivalent
  (means they assign positive prob. to same events)
- $0 <$ probabilities < 1, and sum to 1
- for convenience, additional 'smoothness' restriction:
  probability of 1 spot $\geq$ prob. 2 spots $\geq$ prob. 3 spots, etc.

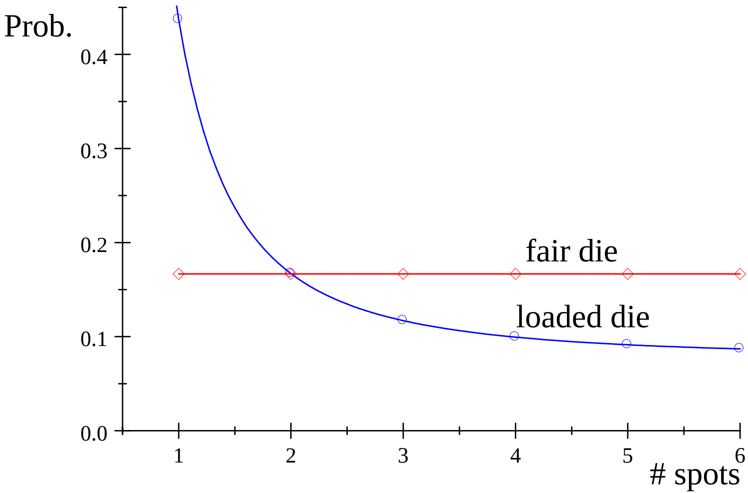
Probabilities for fair die are: $p_{fair} = 1/6 = .1667$
We want to load die so that:

- sides with few spots get higher probability
- sides with many spots get lower probability

probabilities = f(no.spots X)
function $\pm$ hyperbola, increase curvature with a power:

$$p_{loaded} = \left(\frac{\alpha}{X}\right)^{\beta} + \gamma$$

coefficients α, β and γ easily found by solver spreadsheet:
$\alpha = 0.6, \quad \beta = 2$ and $\gamma = 0.077$ Gives:



Probabilities of a fair and a loaded die

Or in table form:

spots	prob.	expectation	variance (contr.)
1	.437	.437	.9833
2	.167	.334	.0418
3	.117	.351	.0293
4	.100	.400	.2250
5	.091	.455	.5688
6	.087	.522	1.0658
sum:	.999	2.499	2.914

Transformed probabilities for a die

We can express one measure as a function of the other:
(the likelihood ratio of two probability measures is called their *Radon-Nikodym* derivative)

$$\frac{p_{loaded}}{p_{fair}} = \frac{\left(\frac{.6}{X}\right)^2 - .0897}{.1667} = \frac{2.16}{X^2} + .462$$

then write them as 'measure transformation functions':

$$p_{loaded} = \left(\frac{2.16}{X^2} + .462\right) p_{fair}$$

$$p_{fair} = \frac{p_{loaded}}{\left(\frac{2.16}{X^2} + .462\right)}$$

ensures equivalence: zero p_{fair} cannot be transformed in positive p_{loaded} and vice versa

What have we accomplished?

- changed probability measure (loaded the die)
- with a formula (also works in reverse)
- left 'probability process' in tact (we still roll the die)
- process now produces different expectation (2.5 instead of 3.5)
- variance remains 2.9

Apply same idea to model of stock prices by *changing probability measure*

Modelling stock returns: Brownian motion

Have to model properties of stock return in a forward looking way

- In discrete time - variables:
  - we list all possibilities as:
    - states of the world or
    - values in binomial tree
- In continuous time - variables:
  - infinite number of possibilities, cannot be listed
  - have to express in probabilistic way.

Standard equipment: *stochastic process*

Most used process is *Brownian motion*, or Wiener process

- Discovered ± 1825 by botanist Robert Brown
- looked through microscope at pollen floating on water
- observed pollen moving around

Physics described by Albert Einstein in 1905

Mathematical process described by Norbert Wiener in 1923

We use the

- term *Brownian motion*
- and the symbol W or $\widetilde{W}$ (for Wiener)

Standard Brownian motion = continuous time analogue of random walk

- can be thought of as series of very small steps
- each drawn randomly from standard normal distribution

Definition

Process $\widetilde{W}$ is standard Brownian motion if:

- $\widetilde{W}_t$ is continuous and $\widetilde{W}_0 = 0$,
- has independent increments
- increments $\widetilde{W}_{s+t} - \widetilde{W}_s \backsim N(0, \sqrt{t})$, which implies:
- increments are stationary: only function of length of time interval t, not of location s.

From definition follows:

- Brownian motion has Markov property

Discrete representation over short period δt :

- $\epsilon\sqrt{\delta t}$, $\epsilon =$ random drawing from standard normal distribution

Brownian motion has remarkable properties:

- wild: no upper - lower bounds, will eventually hit any barrier
- continuous everywhere, differentiable nowhere:
  - never 'smooths out' if scale is compressed or stretched
  - that why special, stochastic calculus is required
- is a fractal

Standard Brownian motion poor model of stock price behavior:

- Catches only random element
- Misses individual parameter for stock's volatility
- Misses expected positive return (positive *drift*)
- Misses proportionality: changes should be in % not in amounts

Missing elements expressed by adding:

- deterministic drift term for expected return
- parameter for stock's volatility
- proportionality: return and random movements (or volatility) in proportion to stock's value

Standard model is *geometric Brownian motion* in a stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t d\widetilde{W}_t \tag{1}$$
$$S_0 > 0$$

- $d =$ next instant's incremental change
- $S_t =$ stock price at time t
- $\mu =$ drift coefficient, exp. instantaneous stock return
- $\sigma =$ diffusion coefficient, stock's volatility (stand. dev. returns), 'scales' random term
- $\widetilde{W} =$ standard Brownian motion, stochastic disturbance term
- $S_0 =$ initial condition (a process has to start somewhere)
- μ, σ are assumed to be constants

Geometric Brownian motion has all the properties we set out to model
But is also restricted:

- constant volatility
- no jumps or 'catastophes'

Formula (1) is stochastic differential equation (SDE)

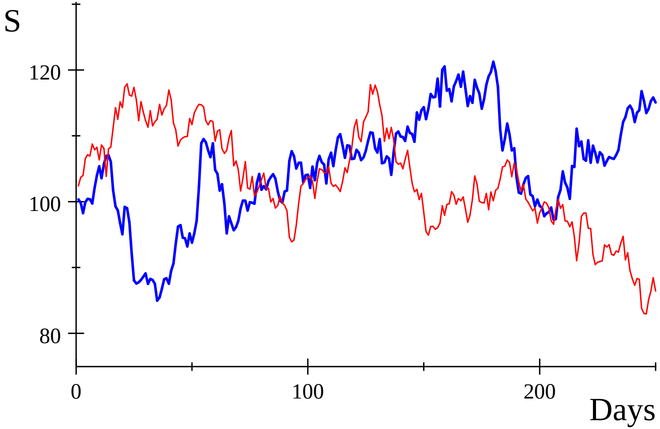
- is a differential equation with a stochastic process in it
- Need a special, stochastic calculus to manipulate SDEs

Financial market also contains risk free debt, D

- defined in similar, but simpler, manner:

$$dD_t = rD_t dt \tag{2}$$

- r is short for r_f, risk free rate (also called money market account or bond)
- risk free $\rightarrow$ no stochastic disturbance term
- natural interpretation for r is short interest rate
- r is assumed to be constant



Sample paths of geometric Brownian motion with $\mu = 0.15$, $\sigma = 0.3$ and T=250

Technique of changing measure

Want to change probabilities such that they embed market price of risk

- so that all assets can be discounted at risk free rate

Mathematical instrument for that is *Girsanov's theorem*:

- Transforms stochastic process, that is a Brownian motion under one probability measure
- into another stochastic process that is a Brownian motion under another probability measure;
- transformation done with third process, Girsanov kernel

The expression for Girsanov kernel is:

$$d\widetilde{W}_t = \theta_t dt + dW_t \tag{3}$$

- $\widetilde{W}$ = original process, Brownian motion under original, real probability measure called Q
- W = transformed process, Brownian motion under new probability measure called P
- θ = Girsanov kernel

Inserting (3) into (1) gives stock price dynamics under P measure:

$$dS_t = \mu S_t dt + \sigma S_t(\theta_t dt + dW_t)$$

Collecting terms:

$$dS_t = (\mu + \sigma\theta_t)S_t dt + \sigma S_t dW_t \tag{4}$$

original process $\widetilde{W}$ replaced with new process W

- we have changed measure!

Looks futile:

- switched from Q-Brownian motion with drift μ
- to P-Brownian motion with drift $(\mu + \sigma\theta_t)$

But latter contains process θ, is not yet defined

We know desired result from definition:

- process should contain pricing information
- similar to state prices in binomial model
- so that proper discount rate = drift = risk free rate r

Solution: define θ as minus the market price of risk:

$$\theta = -\frac{\mu - r}{\sigma}$$

We have seen θ before:

- price of risk in CML and SML
- also used in Sharpe ratio

The Girsanov kernel $-\frac{\mu - r}{\sigma}$ is very simple:

- it is deterministic (no stochastic term)
- it is constant (μ, σ and r are constants)

Substituting for θ in the drift term we get:

$$\mu + \sigma\theta_t = \mu + \sigma\left(-\frac{\mu - r}{\sigma}\right) = r \tag{5}$$

So we have a dynamic process with drift of risk free rate and, under measure P, BM disturbance term:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{6}$$

Solving the sde

- SDEs are notoriously difficult to solve
- Deterministic equivalent of (6) simplified by taking logarithms
- Try same transformation here
    - that is how it is done, trial & error

Have to use stochastic calculus (Ito's lemma), result:

$$d(\ln S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t \tag{7}$$

changes ln(stock price) follow BM, drift $(r - \frac{1}{2}\sigma^2)$, diffusion σ

Term $-\frac{1}{2}\sigma^2$ in drift follows from stochastic nature of returns
Illustrate intuition with example:

- security has return (1+r) over 2 periods
- plus random term of ε in one period, $-\varepsilon$ in other
- Compound return:

$$((1+r)+\varepsilon) \times ((1+r)-\varepsilon) = (1+r)^2 - \varepsilon^2$$

- cross terms + and $-(1+\text{r})\varepsilon$ cancel out, $-\varepsilon \times +\varepsilon = -\varepsilon^2$ not
- In Brownian motion $E[\varepsilon^2] = \sigma^2$
- over 2 periods average reduction is $-\frac{1}{2}\sigma^2$
- We see: volatility reduces compound return
- that is why geometric average < arithmetic average

Recall: increments Brownian motion normally distibuted
and notice: drift and diffusion of

$$d(\ln S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

are constants $\Rightarrow d(\ln S_t)$ also normally distributed:

$$\begin{aligned} \ln S_T - \ln S_0 &\sim N((r - \frac{1}{2}\sigma^2)T, \sigma^2 T) \\ or \quad \ln S_T &\sim N(\ln S_0 + (r - \frac{1}{2}\sigma^2)T, \sigma^2 T) \end{aligned}$$

We use this property later on

Constant drift and diffusion make process for $d(\ln S_t)$ very simple
SDE

can be integrated directly over time interval $[0, T]$, result:

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T} \tag{8}$$

- since $\ln S_T$ is normally distributed
- S_T must be lognormally distributed

$E[S_t]$ follows from properties lognormal distribution:

- expectation of lognormally distributed variable is

$$e^{m + \frac{1}{2}s^2}$$

- m and s are mean and variance of corresponding normal distribution

We have

$$\ln S_T \sim N(\ln S_0 + (r - \frac{1}{2}\sigma^2)T, \sigma^2 T)$$

So expectaion of S_T is:

$$E[S_T] = e^{\ln S_0 + (r - \frac{1}{2}\sigma^2)T + \frac{1}{2}\sigma^2 T} = S_0 e^{rT}$$

$$E[S_T] = S_0 e^{rT} \qquad \text{means} \qquad e^{-rT} E[S_T] = S_0$$

discounted future exp. stock price = current stock price under prob. measure P

- risky assets can be discounted with risk free rate
- as long as expectations are under measure P

The exact equivalent of Binomial model

The Black & Scholes formula

Formula can be obtained in several ways:

1. Black & Scholes original work uses partial differential equations (outline in appendix)
2. Cox, Ross Rubinstein show that binomial approach converges to B&S formula
3. Martingale method (used here)
    - prices by directly calculating expectation under probability measure Q
    - discount result with risk free rate

Problem:

- price now (t=0) of European call option $O^E_{c,0}$,
  - exercise price X,
  - matures at time T,
  - written on non-dividend paying stock

Using martingale method:

$$O_{c,0} = e^{-rT} E\left[O_{c,T}\right] \tag{9}$$

r is the risk free rate

Option's payoff at maturity:

$$O_{c,T} = \begin{cases} S_T - X & \text{if} \quad S_T > X \\ 0 & \text{if} \quad S_T \leq X \end{cases}$$

can be written as:

$$O_{c,T} = (S_T - X)1_{S_T > X} \tag{10}$$

$1_{S_T > X}$ is step function:

$$1_{S_T > X} = \begin{cases} 1 & \text{if} \quad S_T > X \\ 0 & \text{if} \quad S_T \leq X \end{cases}$$

Substituting step function (10) into option value (9):

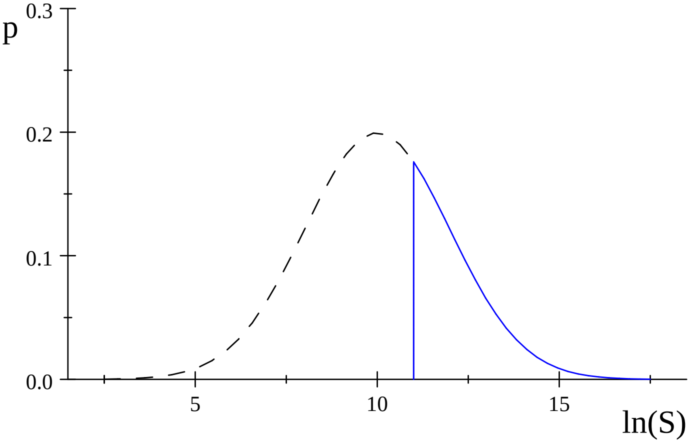
$$O_{c,0} = e^{-rT} E\left[(S_T - X) 1_{S_T > X}\right] \tag{11}$$

To prepare for rest of derivation, we write option value (11) as:

$$O_{c,0} = e^{-rT} E\left[(e^{\ln S_T} - e^{\ln X}) 1_{\ln S_T > \ln X}\right] \tag{12}$$

We use two key elements:

1. $\ln S_T$ is normally distributed, mean $= (\ln S_0 + (r - \frac{1}{2}\sigma^2)T)$, var.$= \sigma^2 T$
2. We can regard step function as truncation of distribution of S_T on left: values $< X$ replaced by zero (truncated distributions are well researched, formula for truncated normal distribution available)



Lognormally distributed stock price ($ln(S) \backsim N(10, 2)$, dashed), and its left truncation at $ln(S) = 11$ (solid)

We use following step function for normally distributed variable Y with mean M and variance v^2 truncated at A:

$$
\begin{aligned}
E\left[\left(e^Y - e^A\right) 1_{Y>A}\right] &= e^{M+\frac{1}{2}v^2} N\left(\frac{M+v^2-A}{v}\right) \\
&\quad - e^A N\left(\frac{M-A}{v}\right) \qquad (13)
\end{aligned}
$$

$N(.)$ is cum. standard normal distr.
Has same form as (12), apply to option pricing problem :
$M = \ln S_0 + (r - \frac{1}{2}\sigma^2)T$
$v^2 = \sigma^2 T \rightarrow v = \sigma\sqrt{T}$
$Y = \ln S_T$
$A = \ln X$

Substituting:

- Details of our problem (M, v^2, Y, A) into formula (13) for the expectation of truncated distribution
- that expectation formula in our option pricing formula

and collecting terms we get the famous Black and Scholes formula:

$$
\begin{aligned}
O_{c,0} \quad = \quad & S_0 N\left(\frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& -Xe^{-rT} N\left(\frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
\tag{14}
$$

Defining, as is commonly done:

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{15}$$

and

$$d_2 = \frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{16}$$

we get the usual form of the *Black & Scholes option pricing formula*:

$$O_{c,0} = S_0 N(d_1) - Xe^{-rT} N(d_2) \tag{17}$$

with the corresponding value of a European put:

$$O_{p,0} = Xe^{-rT} N(-d_2) - S_0 N(-d_1) \tag{18}$$

Interpretation:

$$O_{c,0} = \underbrace{(S_0)}_{\text{stock price}} \underbrace{N(d_1)}_{\text{option delta}} - \underbrace{\left(Xe^{-rT}\right)}_{\text{PV (exerc.p.)}} \underbrace{N(d_2)}_{\text{prob. of exercise}}$$

$N(d_1)$ = option delta, has different interpretations:

- *hedge ratio*: # shares needed to replicate option
- *sensitivity:* of option price for changes in stock price
- technical: partial derivative w.r.t. stock price: $\partial O_{c,0}/\partial S_0 = N(d_1)$
- not just prob. of exercise, also encompasses in-the-moneyness

What is *not* in the Black and Scholes formula:

- real drift parameter μ
- investors' attitudes toward risk
- other securities or portfolios

Greediness, in $max[]$ expressions, implicit in analysis.

Reflects conditional nature of B&S:
As the binomial model, B&S only translates existing security prices on a market into prices for additional securities.

Determinants of option prices

In B&S, stock price + four other variables
Option price sensitivity for these 4 derived in same way as Δ (partial derivatives), called 'the Greeks'

Determinant	Greek	Effect on call option	Effect on put option
Exercise price		< 0	> 0
Stock price	Delta	$0 < \Delta_c < 1$	$-1 < \Delta_p < 0$
Volatility	Vega	$\nu_c > 0$	$\nu_p > 0$
Time to maturity	Theta	$-\Theta_c < 0$	$-\Theta_p <> 0$
Interest rate	Rho	$\rho_c > 0$	$\rho_p < 0$
	Gamma	$\Gamma_c > 0$	$\Gamma_p > 0$

'The Greeks' is a bit of a misnomer

- X is determinant without Greek
- Vega is not a Greek letter
- Gamma is Greek without determinant, gamma is:
  - effect of increase in stock price on delta
  - second derivative option price w.r.t. stock price

Generally, option value increases with time to maturity

- American options always do
- European call on dividend paying stock may decrease with time to maturity if dividends are paid in 'extra' time.
- Value of deep in the money European puts can decrease with time to maturity: means longer waiting time before exercise money is received

An example:

Calculate value of at the money European call

- matures in one year
- strike price of 100
- underlying stock pays no dividends
- has annual volatility of 20%
- risk free interest rate is 10% per year.

We have our five determinants:
$S_0 = 100$, $X = 100$, $r = .1$, $\sigma = .2$ and $T = 1$.

$$\begin{aligned} d_1 &= \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ &= \frac{\ln(100/100) + (.1 + \frac{1}{2}.2^2)1}{.2\sqrt{1}} = .6 \end{aligned}$$

$$d_2 = d_1 - \sigma\sqrt{T} = .6 - .2\sqrt{1} = .4$$

Areas under normal curve for values of d_1 and d_2 can be found:

- table in compendium (good enough for this course), calculator, spread sheet, etc.:

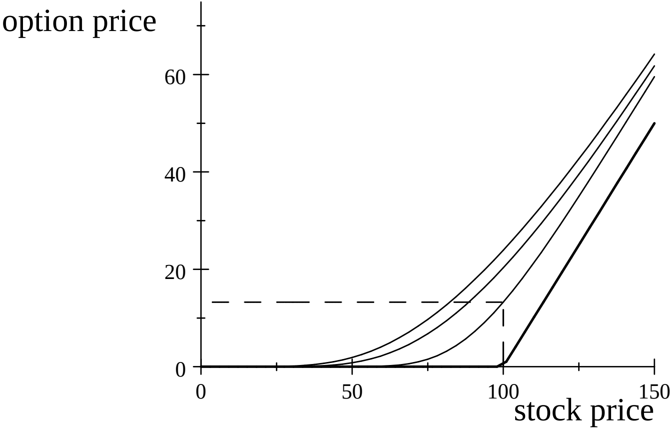
d=	0	0.01	0.02	0.03	0.04	0.05	0.06	0.09
0	0.500	0.504	0.508	0.512	0.516	0.520	0.524	0.536
0.1	0.540	0.544	0.548	0.552	0.556	0.560	0.564	0.575
0.2	0.579	0.583	0.587	0.591	0.595	0.599	0.603	0.614
0.3	0.618	0.622	0.626	0.629	0.633	0.637	0.641	0.652
0.4	0.655	0.659	0.663	0.666	0.670	0.674	0.677	0.688
0.5	0.691	0.695	0.698	0.702	0.705	0.709	0.712	0.722
0.6	0.726	0.729	0.732	0.736	0.739	0.742	0.745	0.755
0.7	0.758	0.761	0.764	0.767	0.770	0.773	0.776	0.785
0.8	0.788	0.791	0.794	0.797	0.800	0.802	0.805	0.813
0.9	0.816	0.819	0.821	0.824	0.826	0.829	0.831	0.839
1	0.841	0.844	0.846	0.848	0.851	0.853	0.855	0.862
2.5	0.994	0.994	0.994	0.994	0.994	0.995	0.995	0.995

$\text{NormalDist}(.6) = 0.72575$, $\text{NormalDist}(.4) = 0.65542$,
Option price becomes:

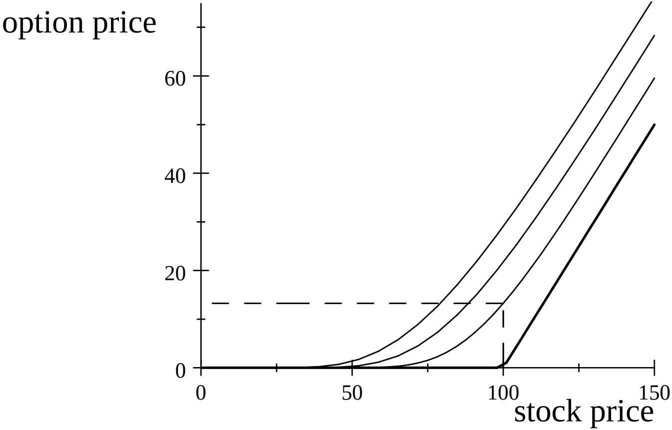
$$O_{c,0} = 100 \times (0.72575) - 100e^{-.1}(0.65542) = 13.27$$

Value put option calculated with equation or the put call parity:

$$\begin{aligned} O_{p,0} &= O_{c,0} + Xe^{-rT} - S_0 \\ &= 13.27 + 100e^{-.1} - 100 = 3.75 \end{aligned}$$



Call option prices for $\sigma = 0.5$ (top), 0.4 and 0.2 (bottom)



Call option prices for $T = 3$ (top), 2 and 1 (bottom)

Dividends

Black & Scholes assumes

- European options
- on non dividend paying stocks

Can be adapted to allow for deterministic (non-stochastic) dividends (can be predicted with certainty)
Dividends:

- stream of value out of the stock
- stream accrues to stockholders
- not option holders

Stock price for stockholders has:

- stochastic part (stock without dividends)
- deterministic part (PV dividends)

Stock price for option holders:

- only stochastic part relevant

Adaptation Black & Scholes formula:

- subtract PV(dividends) from stock price (S_0)
- dividends certain $\rightarrow$ discount with risk free rate
- (implicitly redefines volatility parameter σ for stochastic part only)

Other determinants (X, T and r) unaffected by dividends

Example:

- same stock used before
- pays semi-annual dividends of 2.625
  - first after 3 months
  - then after 9 months

Stock price = 100, volatility 20%, risk free interest rate 10% per year.

> *What is value European call, maturity 1 year, strike price = 100?*

$S_0 = 100$, $X = 100$, $r = .1$, $\sigma = .2$ and $T = 1$.
Start by calculating PV dividends:

- $2.625e^{-.25\times.1} + 2.625e^{-.75\times.1} = 5.$
- makes adjusted stock price $S_0 = 100 - 5 = 95$

Then we can proceed as before:

$$\begin{aligned} d_1 &= \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ &= \frac{\ln(95/100) + (.1 + \frac{1}{2}.2^2)1}{.2\sqrt{1}} = 0.34353 \end{aligned}$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.34353 - .2\sqrt{1} = 0.14353$$

Areas under normal curve for values d_1 and d_2 are:

- $\text{NormalDist}(0.34353) = 0.6344$ and
- $\text{NormalDist}(0.14353) = 0.5571.$

So the option price becomes:

$$O_{c,0} = 95 \times (0.6344) - 100e^{-.1}\,(0.5571) = 9.86$$

- value call lowered by dividends
- from 13.27 to 9.86

Value of a put (same specifications) calculated with equation

$$O_{p,0} = Xe^{-rT}N\left(-d_2\right) - S_0N\left(-d_1\right)$$

Just calculated that $d_1 = 0.34353$ and $d_2 = 0.14353$
$\text{NormalDist}(-0.34353) = 0.3656$ and
$\text{NormalDist}(-0.14353) = 0.44294$

- In table use symmetric property $N(-d) = 1 - N(d)$

Value of the put is:

$$O_{p,0} = 100 \times e^{-.1}\left(0.44294\right) - 95 \times \left(0.3656\right) = 5.35$$

- value of put increased by dividends
- from 3.75 to 5.35

Matching discrete and continuous time volatility

We have expressed volatility in 2 ways:

- In binomial model:
  - difference betwee up and down movement
- In Black and Scholes model:
  - volatility parameter σ used to scale $\widetilde{W}$

If we want to switch models

- we have match the parameters
- recalculate μ and σ into u, d and p

Looking at small time interval δt

- we can equate the return expressions:

$$e^{r\delta t} = pu + (1-p)d$$

r = risk free rate and p = risk neutral probability

- we can also equate variance expressions:

$$\sigma^2\delta t = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2$$

notice:

  - continuous variance increases with time (δt)
  - discrete variance uses definition:
    variance of a variable A is $E(A^2) - [E(A)]^2$

This gives us 2 expressions:

- 1 for return, 1 for variance
- for 3 unknowns: p, u and d
- need additional assumption for third equation

Most common assumption is:

$$u = \frac{1}{d}$$

three equations give (after much algebra):

$$u = e^{\sigma\sqrt{\delta t}}, \quad d = e^{-\sigma\sqrt{\delta t}} \quad and \quad p = \frac{e^{r\delta t} - d}{u - d}$$

Same definition of p we found in binomial model

Implied volatility

Black & Scholes formula has 5 determinants of option prices:

- X, T, S, r, σ are model inputs
- 6 if dividends are included

4 of then are easy to obtain:

- X, T, S, r are, at least in principle, observable:
  - X and T are determined in option contract
  - S and r are market determined
- σ is not observable

There are 2 ways of obtaining numerical value for σ:

1. Estimate from historical values and extrapolate into future;
   1. assumes, like Black & Scholes, that volatility is constant
   2. known not to be the case
      (volalitity peaks around events as quarterly reports)
2. Estimate from prices of other options;
   1. given X, T, S, r each value for σ corresponds to 1 B&S price and vice-versa
   2. for given price, run B&S in reverse (numerically) and find σ
   3. called *implied volatility*

Implied volatility is commonly used:

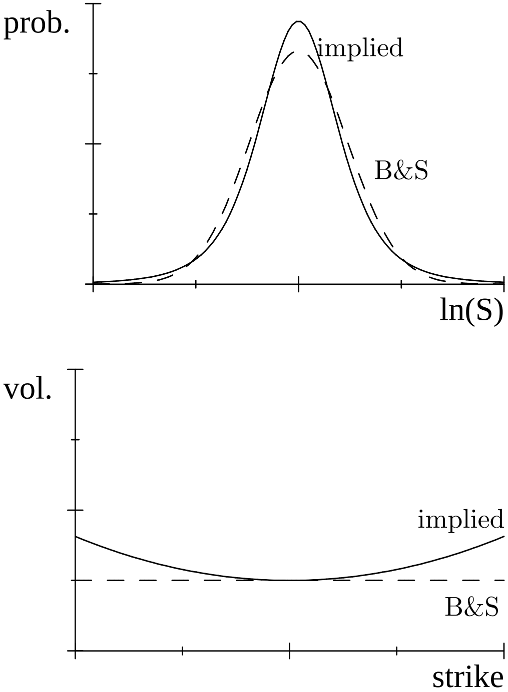
- option traders quote option prices in volatilities
- not \$ or € amounts.

Can also be used to test validity of B&S model

How do you use implied volatility to test B&S?
Black & Scholes assumes constant volatility:
Options with different X and T should give same implied volatility.

Implied volatility typically not constant:

- far in- and out-of the money options give higher implied volatilities than at the money options
  - called *volatility smile* after its graphical representation
  - implies more kurtosis (peakedness) of stock prices than lognormal distribution
  - also fatter tails, but intermediate values less likely
- Stock options may also imply volatility skewness:
  - far out of the money calls priced lower than far out of the money puts (or far in the money calls)
  - implies skewed distribution of stock prices
  - left tail fatter than right tail
- Implied volatility may also increase with time to maturity


prob.
implied
B&S
ln(S)
vol.
implied
B&S
strike