Modern Portfolio Theory

an example of home-made portfolio optimization

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1. Small scale portfolio management
   - Background
   - Problem
2. Portfolio optimization
   - Return estimates
   - Var-covar matrix
   - Portfolio optimization
   - Implementation in excel
3. Further analyses
   - Efficient frontier
   - Short selling

Markowitz portfolio selection illustrated with small example

many investors hold small portfolios

- cannot spread small amounts over many shares
- private investors cannot stay informed about many companies
- like to make small scale investment decisions
- often personal interest in companies

You probably have a (distant) relative active on the stock market

- boasts about good results on birthday parties
- silent when results are not so good

What advice can you give a distant relative?
(don't try this with your parents until you are financially independent)

Portfolio theory advises against small portfolios:

- investors should be fully diversified
- unsystematic risk is unnecessary, should be avoided
- risk can be increased or decreased by varying the proportion of risk free asset

Most research shows: financial markets are efficient

- the chances of beating the market are very slim
- very little evidence that even professionals:
    - can pick stocks that outperform the market
    - can time the stock market (buy before a rise, sell before a decline)

Best advice would be:

- Place risky investments in index funds:
  - an index fund follows (replicates) some index
  - without ambition to do better (passive strategy)
- Spread over funds and indices according to preference
- Select on costs and reliability
- reduces risk of fraud and collapse

Many people hold small portfolios anyway

- should be for the fun of it
- Markowitz' portfolio optimization
  - can also be applied to small portfolios
  - all you need is a spreadsheet program (alas!)
  - and some data
  - more interesting on birthday parties than index funds

Illustrate how it is done

- using a hypothetical ('Uncle Bob's') portfolio
- with real life data, 5 stocks
- look at following historical returns:

Returns October 2010 - October 2011

Stock	ticker	weight	return	weight×return
Google	GOOG	0.1	0.000	0.0000
Cisco Systems	CSCO	0.1	-0.205	-0.0205
Logitech International	LOGI	0.6	-0.525	-0.3150
Amazon.com	AMZN	0.1	0.560	0.0560
Apple	AAPL	0.1	0.314	0.0314
Total		1		-0.2481

- portfolio reflects personal interests (former computer engineer)
- How do you remix (optimize) this portfolio for next year?

First step: come up with return estimates

- Problem: stock returns unpredictable
- Problem: cannot use last year's returns
    - 2 of the 5 stocks had negative returns
    - nobody invests in stocks expected to lose money
    - tempting to extrapolate good results!
- Could use long term averages + subjective expectations

Have to 'extract' expectations from your uncle Bob

- make sure estimates are provided by uncle Bob!
- you offer portfolio management tools
- not predictions

Uncle Bob gives these estimates (under some pressure):

Next year's expected returns

Stock	ticker	weight	return	weight×return
Google	GOOG	0.1	0.08	0.0080
Cisco Systems	CSCO	0.1	0.075	0.0075
Logitech International	LOGI	0.6	0.06	0.0360
Amazon.com	AMZN	0.1	0.125	0.0125
Apple	AAPL	0.1	0.10	0.0100
Total		1		0.0740

- Expected portfolio return below uncle Bob's target of 10%
- reluctant to allocate more money to 'risky' (unknown) stocks
- you want to show him risk-return trade-off

Second step: come up with variance-covariance matrix

- Problem: matrix refers to expectations
  - could press your relative to give return estimates in different future scenarios
  - seldom done, use historical var-covar. instead
- Problem: var-covar not constant over time

To calculate historical variance-covariance matrix:

1. download prices from e.g. yahoo.com
   example: daily closing prices last year
2. transform them into returns: $r_{it} = (P_{it+1} - P_{it})/P_{it}$
3. calculate statistics:
   - variance for each stock
   - correlation matrix

Have to match time-scale of variance and return:

- Your relative estimated yearly returns
- variance is daily
- 'scaling up' variance requires assumptions about distribution

Assuming that returns are identically and independently distributed:

- variance increases with time
- standard deviation increases with square root of time
- multiply daily standard deviation with $\sqrt{252} = 15.875$

Ingredients for portfolio optimization

	Correlation matrix					Re-	Ann.
	GOOG	CSCO	LOGI	AMZN	AAPL	turn	st.dev.
GOOG	1					0.08	0.287
CSCO	0.43	1				0.075	0.363
LOGI	0.34	0.28	1			0.06	0.462
AMZN	0.55	0.34	0.35	1		0.125	0.340
AAPL	0.62	0.43	0.39	0.58	1	0.10	0.250

With these statistics, fill the var-covar matrix

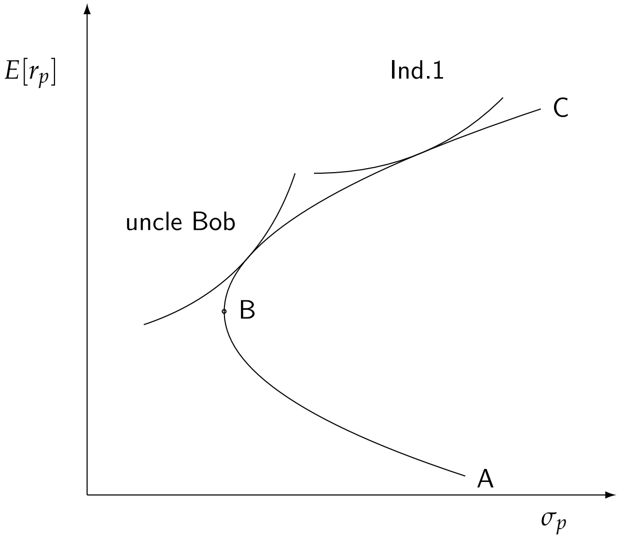
- $x_i x_j \sigma_i \sigma_j \rho_{i,j}$ in each cell
- portfolio variance is the sum of all cells

Now we can calculate the risk and return of portfolios!

But how do we choose an optimal portfolio?

In this setting, portfolio selection is done in 2 steps:

1. the preferred risk return combination is chosen
2. portfolio variance is minimized subject to the restrictions that
   1. the return is not less than the chosen return
   2. the portfolio weights sum to 1
   3. the portfolio weights are positive: uncle Bob is if not allowed to sell short



Choices along the efficient frontier

In our example, portfolio variance is minimized:

- by using the solver in spreadsheet
- to manipulate the cells containing the weights
- subject to the restrictions that:
    1. weights sum to 1
    2. weights $\geqslant 0$ (no short selling)
    3. portfolio return $\geqslant .10$ your uncle's minimum required return

Before you get enthusiastic about the results, recall the procedure:

- empirical basis are daily returns during 1 year
- variances scaled up to annual figures
- using them with uncle Bob's estimates assumes:
  - covariance structure constant over time
  - estimates 'match' the distribution

Would be better to use longer time series and better estimation techniques

But uncle Bob's return estimates are probably the most important, and weakest, elements

You press the 'solve' button. The results are in column 3 (Min.var., $r_{p \geq 0.1}$)

Portfolio:	Original	Min. var. $r_p \geq 0.1$
Google	0.1	0.12
Cisco Systems	0.1	0.10
Logitech International	0.6	0.0
Amazon.com	0.1	0.19
Apple	0.1	0.59
Total (sum weights)	1	1
Portf. return	0.074	0.10
Portf. stand. dev.	0.33	0.236

You are pleased with the results:

- the expected portfolio return increased
  - from 7.4% to 10% (min. return restriction binding)
- the risk of the portfolio decreased
  - from 33% to 23.6%

Re-mixed portfolio has a lower risk than any of its stocks

Uncle Bob agrees, but dislikes zero weight of his familiar Logitech

Why does Logitech get weight 0?
The restriction $E\left(r_p\right) \geqslant 0.1$ works against including a stock with $E(r) = .06$

To demonstrate effect of min. return restriction:

- re-calculate minimum variance portfolio
- without restriction on the portfolio return

Results are in column 4 (Min.var.)

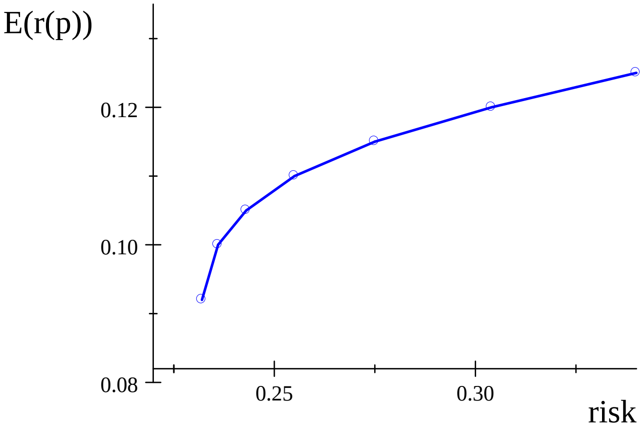
Portfolio:	Original	Min. var. $r_p \geq 0.1$	Min. var.
Google	0.1	0.12	0.22
Cisco Systems	0.1	0.10	0.14
Logitech International	0.6	0.0	0.05
Amazon.com	0.1	0.19	0.06
Apple	0.1	0.59	0.53
Total (sum weights)	1	1	1
Portf. return	0.074	0.10	0.092
Portf. stand. dev.	0.33	0.236	0.232

How can you construct the efficient frontier?

- Can be done analytically (with "Black's result")
- Simple practical approximation:
  - Calculate different points on the frontier
  - Connect the dots

Which three points on the frontier do we already know?

- minimum variance portfolio
- Uncle Bob's portfolio, and
- maximum return portfolio
  (no short selling $\Rightarrow$ 100% in max return stock (AMZN)



Efficient frontier

You observe: starting from minimum variance portfolio, small risk increase gives large return increase

- Uncle Bob's portfolio gives
  - 0.8 perc. points higher return than minimum variance portfolio
  - its stand. dev. is only 0.4 percentage points higher
- The 0.5 perc. point increase in return from 12% to 12.5%
  - gives increase in stand. dev. of 3.6 percentage points (from 30.4% to 34%)

You also observe that:

- all five stocks are included in minimum variance portfolio
- four in uncle Bob's portfolio
- fewer and fewer stocks in the higher return portfolios
- only one stock in maximum return portfolio

That is the diversification effect working in reverse!

How do you investigate the effects of short selling?

- Simple: remove non-negativity restrictions portfolio weights
- recalculate the minimum variance portfolio

You find exactly the same portfolio as without short selling restriction

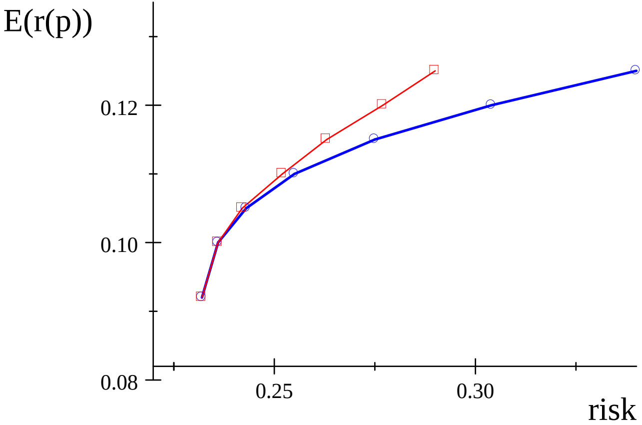
- not a general result
- happens to be the case in this dataset

Recalculate uncle Bob's portfolio without short selling restriction

- first negative weight appears (-0.01 for LOGI)
- too small to have effect on portfolio risk and return

Further increase minimum return restriction in steps of 0.5 pp

- more and larger negative weights appear
- efficient frontier becomes:



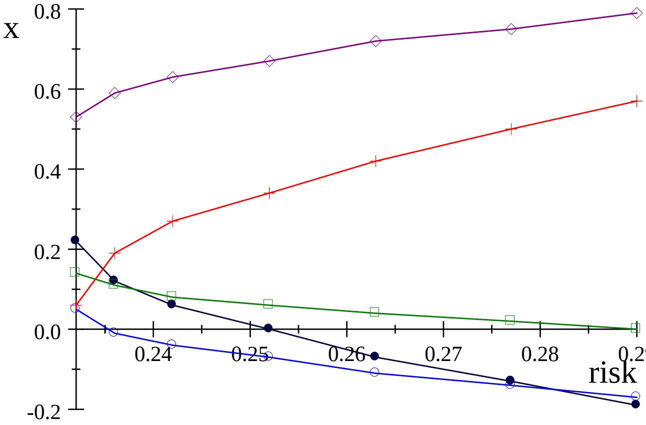
Efficient frontier with (lower) and without (upper) short selling restriction

You observe: short selling has risk reducing effect

- Higher return portfolios constructed more efficiently with short selling
- Without short selling
  - $E(r_p) \geqslant 0.125$ means all money in highest return stock
  - Amazon has a standard deviation of 34%
- With short selling
  - optimal portfolio with $E(r_p) \geqslant 0.125$ is:
  - -0.19 Google, -0.17 LogiTech, 0.57 Amazon and 0.79 Apple
  - portfolio standard deviation is 29%
  - 5 percentage points lower

Removing the short selling restriction

- expands the investment opportunity set
- moves the efficient frontier upwards and to the left



Portfolio composition (weights x) versus risk along the efficient frontier (GOOG= •, LOGI= ○, AMZN= +, CSCO=□, AAPL= ◊)