Chapter 8: Black and Scholes option pricing

Exercises - solutions

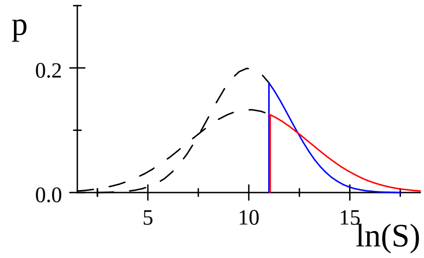
1. The statement is not correct. The share price is expected to increase during the option's life time, whereas the exercise price remains constant. So if the current share price equals the exercise price, the expected share price at maturity will be larger than the exercise price. The growth rate of the stock depends on the probability measure that is used to model stock prices. Under the equivalent martingale measure, returns are equalized into the risk free interest rate, so that the share grows with r_f. The example in section 8.3.3 of the book illustrates this case. It values an at the money call option on a non-dividend paying share with a current price of €100 and a volatility of 20%. The option's time to maturity is 1 and the risk free interest rate is 10%. The calculations in the book (easily verified with the option price calculator accompanying the book) show that the equivalent martingale probability that the option will be exercised, or $N(d_2)$, is 0.655. Under the real probability measure the share will grow with μ, the expected, continuously compounded return of the share.

2. Under the iid assumption, variance increases with time, so that standard deviation increases with the square root of time. An annual standard deviation of 34% thus gives a daily standard deviation of $\sigma\sqrt{T} = 0.34\sqrt{1/252} = 0.02142$ or 2.14%. In section 3.2.2 (Home-made portfolio optimization) we used the same procedure 'in reverse' to calculate the annual volatility of the stocks in uncle Bob's portfolio from their daily returns.

3. As was pointed out by Fama, if *stock returns* are iid, then *stock prices* will not follow a random walk since price changes will depend on the price level.

4. Stability over time means that past returns are the best information to assess the distributional properties of the returns. However, they *cannot* be used the predict future returns because they contain no information on the *sequence* of future returns.

5. Volatility increases the probability of large price changes, both price increases and price decreases. For stockholders these price movements will tend to cancel out. But option holders have an exposure to only one of these price movements: call holders to price increases and put holders to price decreases. For option holders it doesn't matter how far out of the money the option ends, the option is worthless if it ends out of the money. So options profit from the upward potential of price movements but have a limited exposure to the downside risk, hence their values increase with volatility. This is illustrated in the graph below with the truncated normal distribution we used in the derivation of the Black and Scholes formula. The truncated distribution with the higher standard deviation has a higher expectation than the one with a lower standard deviation (14.5 vs. 11.8).



Lognormally distributed stock prices ($ln(S) \backsim N(10, 2$ and $3)$, dashed), and their left truncations at $ln(S) = 11$ (solid)

6. Holders of a long call have a claim on the upward potential of stock, but they don't pay for it until maturity, if at all (the option may also expire out of the money). The possibility to delay payment is more valuable the higher the interest rate is (the holder can earn interest over the exercise price). Holders of a long put can sell the stock at the fixed exercise price in the future. The value of a future payment decreases with the interest rate.

7. The put-call parity states:
$$put = call + PV(X) - S$$
If the stock price increases with 1, the call increases (by definition) with $\Delta_c \times 1 = \Delta_c$. The $PV(X)$ is unaffected by changes in the stock price. So the right hand side of the equation changes with
$$\Delta_c \times 1 + 0 - 1 = \Delta_c - 1$$
This must be the delta of the put: $\Delta_p = \Delta_c - 1$.

8. Economically, holding an extremely far in the money call option is equivalent to holding a share that is not yet paid for, i.e. $S - PV(X)$. Differences in volatility (practically) do not matter any more as the options are (almost) certain to be exercised anyway. The options in left hand figure are paid for on the same date, so their PV(X) are the same and, consequently, they have a common value as a function of the stock price. The options in right hand figure are not paid for on the same date, so their PV(X) are different and they do not have a common value as a function of the stock price.
Technically, as the stock price S get larger and larger, both the option delta N(d_1) and the probability of exercise N(d_2) approach 1. The Black and Scholes price then approaches $O_{c,0} = S_0 - Xe^{-rT}$. This price is independent of σ, but not of T, hence options with different volatilities converge to a common value, but options with different maturities do not.

9. We use the Black and Scholes option pricing formula:
$$O_{c,0} = S_0 \times N(d_1) - X \times e^{-rT} \times N(d_2)$$
with
$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2) \times T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$
The input data are: $S = 240$, $X = 250$, $\sigma = 25\%$, $r = 6\%$, $T = 1$
$$d_1 = \frac{\ln(240/250) + (0.06 + 0.5 \times 0.25^2) \times 1}{0.25 \times \sqrt{1}} = 0.20171$$
$N(d_1) \rightarrow \text{NormalDist}(0.20171) = 0.579\,93$
$$d_2 = 0.20171 - 0.25 \times \sqrt{1} = -0.04829$$

$N(d_2) \to \text{NormalDist}(-0.04829) = 0.480\,74$

$$O_{c,0} = 240 \times 0.57993 - 250 \times e^{-0.06\times 1} \times 0.48074 = 25.997$$

10. (a) The parameters of the binomial process are:

$$u = 1.075 \qquad d = .93 \qquad r = 1.025 \qquad p = \frac{1.025 - .93}{1.075 - .93} = .655$$

With these parameter the binomial tree can be constructed as in Lattice 1. Note that the tree no longer recombines after a fixed amount of dividend payments. The option values are obtained by calculating their values at maturity, taking their risk neutral expectation and discounting this back in time with the risk neutral rate. This gives the values of the option alive, and since this is an American option the values alive have to be compared with the values dead, i.e. if exercised. Lattice 2 gives the results, the calculations are below.

$$\begin{array}{ccccc}
 & & 107.5 & & 104.8 \\
 & \nearrow & \downarrow & \nearrow & \\
 & & 97.5 & & \\
100 & & & \searrow & \\
 & \searrow & & & 90.7 \\
 & & 93 & & 89.2 \\
 & & \downarrow & \nearrow & \\
 & & 83 & & \\
 & & & \searrow & \\
 & & & & 77.2 \\
t=0 & & t=1 & & t=2
\end{array}$$

Lattice 1 Value NSG stock

The options only ends in the money in the upper node at *t=2*. Its value at expiration in that node is

$$104.8 - 102.5 = 2.3$$

This gives a *t=1* value alive of:

$$(.655 \times 2.3)/1.025 = 1.47$$

The value dead is $107.5 - 102.5 = 5$ so the option should be exercised which gives a *t=0* value of

$$(.655 \times 5)/1.025 = 3.2$$

$$\begin{array}{ccccc}
 & & & & O_{uu} = 2.3 \\
 & & & \nearrow & \\
 & & O_u = \max[\underset{alive}{(1.47)}, \underset{dead}{\max[0,(107.5-102.5)]}] = 5 & & \\
 & \nearrow & & \searrow & \\
3.2 & & & & O_{ud} = 0 \\
 & \searrow & & & O_{du} = 0 \\
 & & & \nearrow & \\
 & & O_d = \max[\underset{alive}{(0)}, \underset{dead}{\max[0,(93-102.5)]}] = 0 & & \\
 & & & \searrow & \\
 & & & & O_{dd} = 0 \\
t=0 & & t=1 & & t=2
\end{array}$$

Lattice 2 Value call option on NSG stock

(b) To construct the hedge portfolio we first calculate Δ and D:

$$\Delta = \frac{O_u - O_d}{S_u - S_d} = \frac{5-0}{107.5-93} = .345$$

$$D = \frac{uO_d - dO_u}{(u-d)r} = \frac{1.075 \times 0 - .93 \times 5}{(1.075 - .93)1.025} = -31.287$$

As always, the hedging portfolio for a call is a levered long position in the stock. We sell the option at 3.2 and we buy the hedge portfolio to cover the obligations from the call. The fraction of the stock costs: $.345 \times 100 = 34.5$. We borrow 31.287 and the difference of $34.5 - 31.287 = 3.2$ is covered by the received premium from selling the call. If at *t=1*:

i. The stock price is 107.5: the option will be exercised and we have to sell the stock at the exercise price of 102.5. We have .345 stock in the hedge portfolio, so we have to buy $1 - .345 = .665$ stock, which costs $.665 \times 107.5 = 70.41$. Hence, we receive $102.5 - 70.41 = 32.09$. This is exactly enough to pay off the debt, which now amounts to $1.025 \times 31.287 = 32.07$. We have a perfect hedge.

ii. The stock price is 93: the option is worthless. The debt amounts to the same 32.07, which is exactly covered by the fraction of the share: $.345 \times 93 = 32.09$. Again, we have a perfect hedge.

(c) The call option delta increases with the stock price, all other things equal. As the stock price increases, the option becomes more likely to be exercised and becomes more like a stock that has not yet been paid for. Ultimately, when the option is so far in the money that it is certain to be exercised the call option delta becomes 1. Conversely, when the stock price falls and the call is farther and farther out of the money, it becomes less and less likely to be exercised. Ultimately, when the option is so far out of the money that it is certain not to be exercised the call option delta becomes 0. The call has lost its value, no matter what happens to the stock price. This is the case in the lower at *t=1* in Lattice 2: the option cannot get in the money from this node and is worthless and no longer sensitive to stock price changes.

11. (a) We use the put-call parity to construct a synthetic put and check for any mispricing:

$$long\ put = long\ call + PV(X) - share\ price$$

First we calculate the PV(X) using the appropriate NIBOR rate and period:
NHY nov.5: $620e^{-.02235 \times 2/12} = 617.69$
NHY nov.5: $680e^{-.02235 \times 2/12} = 677.47$
NHY feb.6: $620e^{-.02337 \times 5/12} = 613.99$
NHY feb.6: $680e^{-.02337 \times 5/12} = 673.41$
ORK jan.6: $240e^{-.02313 \times 4/12} = 238.16$
NSG dec.5: $100e^{-.0229 \times 3/12} = 99.429$
We construct a synthetic put by buying a call, putting PV(X) in the bank and selling the share. So we use the ask price for the call and the bid price for the share.

Synthetic and market option prices

Ticker	T	X	call ask	PV(X)	share bid	syn. put	put option bid	put option ask
NHY	nov.5	620	70.00	617.69	677.00	10.69	9.25	10.00
„	„	680	30.25	677.47	677.00	30.72	29.00	31.25
„	feb.6	620	83.75	613.99	677.00	20.74	19.00	20.75
„	„	680	47.00	673.41	677.00	43.41	41.50	44.25
ORK	jan.6	240	26.75	238.16	259.00	5.91	5.00	5.50
NSG	dec.5	100	11.00	99.429	108.75	1.68	2.85	3.35

The rows for NHY nov.5-620, ORK jan.6 and NSG dec.5 in Table 4 show that the prices of the synthetic puts for these options lie outside the bid-ask spread on the market. But the first two are not arbitrage opportunities: you can buy a synthetic put NHY nov.5-620 at 10.69 or an ordinary put at 10, but you cannot sell at 10.69, only at 9.25. The same applies to ORK jan.6. They are outside the bid-ask spread, but on wrong side from an arbitrage point of view. The NSG option is outside the bid-ask spread on the other side and that offers an arbitrage opportunity: we can buy the synthetic put at 1.68 and sell the ordinary put at 2.85. This gives an arbitrage profit of 1.17. If we manage to close a million of these contracts we have become millionaires overnight.

Such arbitrage opportunities are in practice not open to investors who get their information from a newspaper and on closer examination it appears that we made a typing error in Table 2: the bid-ask prices for NSG are 106.75 and 107.5. With the proper stock price, the price of the synthetic put becomes:

call + PV(X) - share = 11+99.429-106.75=3.679

The price of the synthetic put now lies outside the bid-ask spread on the other side and the arbitrage opportunity has disappeared. We can make sure by checking the relation the other way around by constructing a synthetic short put. Then we write a call, borrow pv(X) and buy the share. That costs: -10 + (-99.429) + 107.5 = -1.929 i.e. brings in 1.929. Note that we use the bid price for the call and the ask price for the share. If we sell the put on the market we get 2.85, so the synthetic price is on the wrong side of the bid-ask spread from a an arbitrage point of view.

(b) Differences between implied and observed prices can occur because we apply the put-call parity, which is only valid for European options on non-dividend paying stocks, to traded American options on stocks that may pay dividends. Further, price differences can occur because of nonsynchronous trading. The prices in newspapers are generally closing prices, but we do not know when the last trade of the day took place. If the last option trade was at 13.00 hours and the last stock trade at 15.00 hours, the option trade could be based on a different stock price than the one we read in the newspaper.