Binomial Option Pricing

The wonderful Cox Ross Rubinstein model

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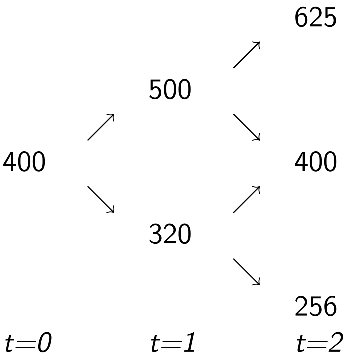
Recall earlier discussion of risk modelling:

2 ways to model future time and uncertain future variables:

1. Discretely
   enumerate (list) all possible:
   - points in time
   - outcomes of variables in each point with their probabilities

   example: binomial tree
2. Continuously
   use dynamic process with infinitesimal time steps
   - number of time steps $\rightarrow \infty$
   - probabilistic changes in variables (drawn from a distribution)

   example: geometric Brownian motion



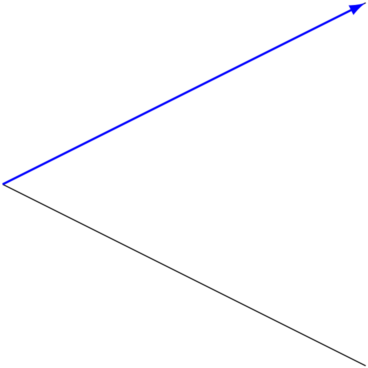
Binomial tree for a stock price

Each period, stock price can :
- go up with 25%, probability q
- go down with 20%, probability 1-q

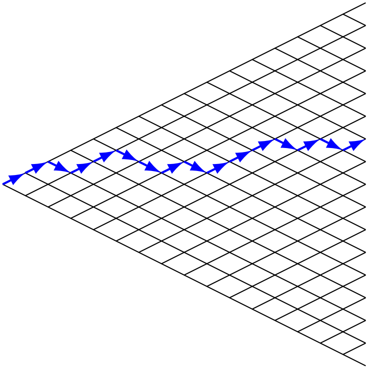
In binomial trees risk is expressed as:

- difference between up and down
- probability q

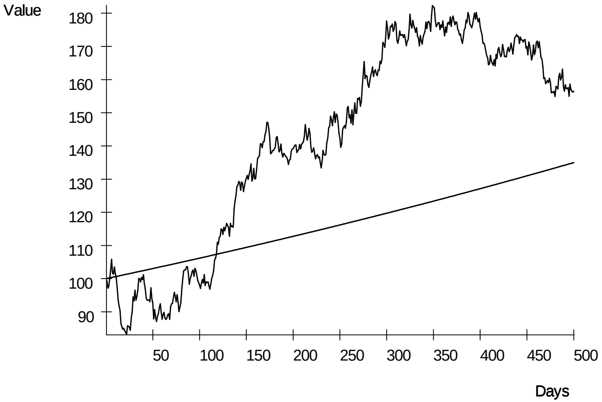
$$
\begin{array}{lll}
 & q = .5 & 500 \\
 & \nearrow & \\
400 & & \\
 & \searrow & \\
 & 1-q & 320
\end{array}
\qquad
\begin{array}{lll}
 & q = .5 & 415 \\
 & \nearrow & \\
400 & & \\
 & \searrow & \\
 & 1-q & 387
\end{array}
\qquad
\begin{array}{lll}
 & q = .95 & 500 \\
 & \nearrow & \\
400 & & \\
 & \searrow & \\
 & 1-q & 320
\end{array}
$$



1 period of 1 year



16 periods of 3 weeks



Sample path geometric Brownian motion, $\mu = .15$, $\sigma = .3$, t=500 days; smooth line is deterministic part of the motion

Risk is expressed as continuous time standard deviation

Turned 90°, vertical binomial tree can be seen as randomizer

- Francis Galton's quincunx (1873-74)
- physical model of 'theory of errors'

Demonstrates how succession of small changes

- can lead to big changes
- but with low frequency
- used to explain evolution theory

Also model of normal distribution (for large n)

Binomial Option Pricing model

- Introduced by Cox, Ross and Rubinstein (1979)
- elegant and easy way of demonstrating the economic intuition behind option pricing and its principal techniques
- not a simple approximation of a complex problem: is a powerful tool for valuing quite general derivative securities
- can be used when no analytical closed form solution of the continuous time models exists
- In the limit, when the discrete steps converge to a continuum, the model converges to an exact formula in continuous time

Setting of the model:

- Similar to State-Preference Theory, bit more specific
- The binomial method uses discrete time and discrete variables
- Time modelled as a series of points in time at which
  - the uncertainty over the previous period is resolved
  - new decisions are made
  - and in between 'nothing happens'
- Uncertainty in variables modelled by distinguishing only 2 different future states of the world
  - usually called an 'up' state and a 'down' state

- Both states have a return factor for the underlying variable (usually the stock price)
  - for which u and d are used as symbols
- Customary to use return & interest rate *factors* not *rates:*
  - interest rate of 8% expressed as $r = 1.08$
  - up & down factors also $1 +$ the rate
- state 'up' occurs with prob. q, so 'down' with $(1 - q)$
- Since $u \times d = d \times u$ result is a re-combining binomial tree (or lattice) and underlying variable follows a multiplicative binomial process

$$
\begin{tikzcd}
 & & u^2A \\
 & uA \arrow[ur] \arrow[dr] & \\
A \arrow[ur, "q"] \arrow[dr, "1\text{-}q"'] & & udA \\
 & dA \arrow[ur] \arrow[dr] & \\
 & & d^2A \\
t=0 & t=1 & t=2
\end{tikzcd}
$$

Lattice 1: Binomial tree for security priced A

Looks excessively restrictive to model uncertainty by only 2 discrete changes
Not necessarily so:

- Many variables move in discrete, albeit small, steps:
  - The prices of stocks and most other securities change with *ticks*, i.e. a minimum allowed amount
  - Changes in interest rates are expressed in discrete basis points of one hundredth of a percent
- Number of states increases with the number of time steps
  - grid becomes finer
  - number of possible end states increases

In practice, over a short period of time, many stockprices indeed change only with one or two ticks

A simple 1 period model:

Introduce binomial method in 1 period - 2 moment setting

- Assume a perfect financial market without taxes, transaction costs, margin requirements, etc

Traded on the market are 3 securities:

1. A stock with current price S
    - stock pays no dividend
    - stock price follows a multiplicative binomial process:
        - up factor u
        - downward factor d.
        - prob. of an upward movement is q
        - prob. of a downward movement is $1-q$

2. A European call option on the stock with unknown current price of O
   - option has exercise price of X
   - matures at the end of the period
   - pays off the maximum of null and the stock price minus the exercise price
3. Riskless debt with an interest rate factor of r

(recall that r, u, and d and defined as 1 + the rate)

What does 'no-arbitrage' imply for u, d and r?
the interest rate has to be: $d < r < u$ to avoid arbitrage

Is the market complete?
Yes: we have 2 states and 2 securities, stock and risk free debt

The payoffs of the stock and the option are:

$$\begin{array}{ccc} & q & uS \\ & \nearrow & \\ S & & \\ & \searrow & \\ & 1\text{-}q & dS \end{array} \qquad \begin{array}{ccc} & q & O_u = \max[0, uS - X] \\ & \nearrow & \\ O & & \\ & \searrow & \\ & 1\text{-}q & O_d = \max[0, dS - X] \end{array}$$

Lattice 2: Binomial trees for a stock (S) and an option (O)

The question is, of course, to find the current price of the option O

Cox Ross Rubinstein approach to pricing options:

- Construct a *replicating* portfolio of existing and, thus, priced securities that gives the same payoffs as the option
- Option price then has to be the same as the price of the portfolio, otherwise there are arbitrage opportunities

The existing securities are the stock and risk free debt
So we form a portfolio with:

- a fraction Δ of the stock
- a risk free loan of D

Δ and D can be positive or negative:
positions can be long or short

This gives the following payoff tree for the portfolio:

$$
\begin{array}{ccc}
 & q & uS\Delta + rD \\
 & \nearrow & \\
\Delta S + D & & \\
 & \searrow & \\
 & 1\text{-}q & dS\Delta + rD
\end{array}
$$

Lattice 3: Binomial tree for the replicating portfolio

Δ and D can be chosen freely on perfect markets

Choose Δ and D such that they make the end of period value of the porfolio equal to the end of period value of the option:

$$\begin{aligned} uS\Delta + rD &= O_u \\ dS\Delta + rD &= O_d \end{aligned} \tag{1}$$

The two equations in (1) can be solved for Δ and D which gives:

$$\Delta = \frac{O_u - O_d}{(u-d)S} \tag{2}$$

and:

$$D = \frac{uO_d - dO_u}{(u-d)r} \tag{3}$$

Delta, Δ, is often used in finance

- is the number (fraction) of shares needed to replicate the option
- called the 'hedge ratio' or the 'option delta'
- is measured as spread in option values divided by spread in stock values

Portfolio with these Δ and D called the *hedging portfolio* or the *option equivalent portfolio*.

- Equivalent to option $\Rightarrow$ generates same payoffs as option
- Means they must have same current price to avoid arbitrage opportunities. So:

$$O = \Delta S + D \tag{4}$$

Substituting expression for Δ (2) and D (3) into (4) gives:

$$O = \frac{O_u - O_d}{(u-d)} + \frac{uO_d - dO_u}{(u-d)r} = \frac{\left[\frac{r-d}{u-d}\right] O_u + \left[\frac{u-r}{u-d}\right] O_d}{r} \tag{5}$$

To simplify equation (5) we define:

$$p_1 = \frac{r-d}{u-d} \Rightarrow p_2 = (1-p_1) = \frac{u-r}{u-d} \tag{6}$$

With this definition, $p_{1,2}$ behave as probabilities:

- $0 < p_{1,2} < 1$
- $p_1 + p_2 = 1$

Can be used as probabilities associated with the 2 states of nature

- They are different from the true prob. q and (1-q)
- are called:
  - *risk neutral probabilities* or
  - *equivalent martingale probabilities*
- we saw them before in state-preference theory $(\psi(1+r))$

By using p instead of q we have changed the *probability measure* and, thus, the expectation operator!

Term *risk neutral probabilities* is a bit misleading

- refers to fact that p is value q would have if investors were risk neutral
- risk neutral investors would require stock return of r :

  $$rS = qSu + (1-q)Sd$$

  solving for q gives:

  $$q = \frac{r-d}{u-d} = p$$

- Does not mean we assume investors to be risk neutral:
  - 'real life' required return of stock $> r$
  - $q \neq p$
  - risk neutral probabilities are only for pricing, not for describing probabilities of real world events

Substituting these probabilities (6) into the pricing formula for the option (5) we get:

$$O = \frac{pO_u + (1-p)O_d}{r} \tag{7}$$

This is an exact formula to price the option!

Same as the risk neutral valuation formula
Says true value of a *risky* asset can be found by

- taking expected payoff and
- discounting it with the risk free interest rate
- *if expectation is calculated with risk neutral probabilities*
- recall: these probabilities contain pricing information (was easier to see in state-preference theory)

Characteristics of the model

Easier explained in option context:

- Risk accounted for by adjusting probabilities (probability measure), not discount rate or cash flows
- What does not appear in the formula:
  - Investors' attitudes toward risk
  - Other securities or portfolio's (market portfolio)
  - Real probabilities q and $1 - q$
- Reason: conditional nature of the pricing approach
  - Option pricing models do not 'explain' prices of existing securities, like CAPM and APT
  - They 'translate' prices of existing securities into option prices

- Of course, price of a stock option depends on stock price
- but real probabilities q and $1 - q$ not needed to price the option:
  - even if investors have different subjective expectations regarding q
  - they still agree on value of the option relative to the share
- Investors' greed is modelled explicitly: have to chose max[0, ..]
- But greedy investors are implicit in all arbitrage arguments, otherwise arbitrage opportunities will not used

A 2-period example

Binomial option pricing best illustrated with examples
Start by extending time period to 2 periods and 3 moments
On a perfect financial market are traded:

- A stock with a current price of 400
  - stock price follows a binomial process
  - can go up with a factor 1.25 or go down with 0.8
  - stock pays no dividends
- Risk free debt is available at 7% interest
- A European call option on the stock
  - exercise price is 375
  - option matures at the end of the second period

The parameters of the binomial process are:

- u = 1.25
- d = 0.8
- r = 1.07
- so

$$
\begin{aligned}
p &= \frac{r-d}{u-d} = \frac{1.07-0.8}{1.25-0.8} = 0.6 \\
(1-p) &= 0.4
\end{aligned}
$$

Gives following binomial tree (or lattice)

$$
\begin{tikzcd}[row sep=small]
 & & 625 \\
 & 500 \arrow[ur] \arrow[dr] & \\
400 \arrow[ur] \arrow[dr] & & 400 \\
 & 320 \arrow[ur] \arrow[dr] & \\
 & & 256 \\
t{=}0 & t{=}1 & t{=}2
\end{tikzcd}
$$

Lattice 4 Binomial tree for a stock

We can write out the tree for the option as well and fill in the values at maturity:

$$
\begin{array}{ccccc}
 & & & & O_{uu} = \max[0, 625 - 375] = 250 \\
 & & & \nearrow & \\
 & & O_u = 149.53 & & \\
 & \nearrow & & \searrow & \\
O = 89.09 & & & & O_{ud} = \max[0, 400 - 375] = 25 \\
 & \searrow & & \nearrow & \\
 & & O_d = 14.02 & & \\
 & & & \searrow & \\
 & & & & O_{dd} = \max[0, 256 - 375] = 0 \\
\textit{t=0} & & \textit{t=1} & & \textit{t=2}
\end{array}
$$

Lattice 5 Binomial tree for an option

How did we get the values for O_u, O_d and O?

The tree solved from the end backwards:

- O_u and O_d can be found by applying the binomial option pricing formula (7)

$$O_u = \frac{0.6 \times 250 + 0.4 \times 25}{1.07} = 149.53$$

$$O_d = \frac{0.6 \times 25 + 0.4 \times 0}{1.07} = 14.02$$

- O can then be found by repeating the procedure:

$$O = \frac{0.6 \times 149.53 + 0.4 \times 14.02}{1.07} = 89.09$$

Can use binomial option pricing formula for O_u and O_d substitute these in the formula for O to get a 2-period formula:

$$O = \frac{p^2 O_{uu} + 2p(1-p)O_{ud} + (1-p)^2 O_{dd}}{r^2}$$

more usual to write out the recursive procedure:

- allows for events in the nodes O_u and O_d
- e.g. dividends and early exercise of American options

How well does hedge portfolio replicate?

Binomial option pricing formula based on arbitrage arguments:

- if we can make a hedging portfolio with the same payoffs
- then we can price the option

Check how well hedging portfolio performs:

- assume we sell the option
- hedge the obligations from the option with hedging portfolio of stock and debt
- hedge is dynamic: adjust portfolio as stock price changes

Hedge portfolio needs to be self financing. Means: no new cash along the way, all payments part of portfolio

$$\begin{array}{ccccc}
 & & & & O_{uu} = 250 \\
 & & & & S_{uu} = 625 \\
 & & & \nearrow & \\
 & & O_u = 149.53 & & \\
 & & S_u = 500 & & \\
 & \nearrow & & \searrow & \\
O = 89.09 & & & & O_{ud} = 25 \\
S = 400 & & & & S_{ud} = 400 \\
 & \searrow & & \nearrow & \\
 & & O_d = 14.02 & & \\
 & & S_d = 320 & & \\
 & & & \searrow & \\
 & & & & O_{dd} = 0 \\
 & & & & S_{dd} = 256 \\
t = 0 & & t = 1 & & t = 2
\end{array}$$

Price at t_0 of option is 89.09. The t_0 option delta is:

$$\Delta = \frac{O_u - O_d}{(u-d)S} = \frac{149.53 - 14.02}{500 - 320} = \frac{135.51}{180} = 0.753$$

- t_0 hedging portfolio contains 0.753 shares with a price of 400 $0.753 \times 400 = 301.2$
- Received 89.09 for the option $\Rightarrow$ have to borrow: $89.09 - 301.2 = -212.11$
- (using formula for D would give same result)
- Hedging portfolio is leveraged long position in the stock

If stock price rises to 500 at t_1 :

$$
\begin{array}{ccccc}
 & & & & \begin{array}{l} O_{uu} = 250 \\ S_{uu} = 625 \end{array} \\
 & & & \nearrow & \\
 & & \begin{array}{l} O_u = 149.53 \\ S_u = 500 \end{array} & & \\
 & \nearrow & & \searrow & \\
\begin{array}{l} O = 89.09 \\ S = 400 \end{array} & & & & \begin{array}{l} O_{ud} = 25 \\ S_{ud} = 400 \end{array} \\
 & \searrow & & \nearrow & \\
 & & \begin{array}{l} O_d = 14.02 \\ S_d = 320 \end{array} & & \\
 & & & \searrow & \\
 & & & & \begin{array}{l} O_{dd} = 0 \\ S_{dd} = 256 \end{array} \\
t = 0 & & t = 1 & & t = 2
\end{array}
$$

New hedge ratio becomes:

$$\Delta = \frac{O_u - O_d}{(u-d)S} = \frac{250 - 25}{625 - 400} = \frac{225}{225} = 1$$

- Have to buy $1 - 0.753 = 0.247$ stock extra at a cost of $0.247 \times 500 = 123.50$
- Borrow the extra 123.50 so that
  - total debt now is $123.50 + 1.07 \times 212.11 = 350.46$
- using formula gives same result:

$$D = \frac{uO_d - dO_u}{(u-d)r} = \frac{1.25 \times 25 - .8 \times 250}{(1.25 - .8) \times 1.07} = -350.47$$

From 500 at t_1, stock price can

- rise to 625 at t_2
- fall to 400 at t_2.

Either way, option is in-the-money.
If the option ends in the money:

- We are required to give up the stock in portfolio against exercise price of 375
- this 375 exactly enough to pay off the debt, which now amounts $1.07 \times 350.46 = 374.99$

So net position is zero, a perfect hedge!

Now look at lower half of the tree, where stock price falls to 320 at t_1

$$
\begin{array}{ccccc}
 & & & & \begin{array}{l} O_{uu} = 250 \\ S_{uu} = 625 \end{array} \\
 & & & \nearrow & \\
 & & \begin{array}{l} O_u = 149.53 \\ S_u = 500 \end{array} & & \\
 & \nearrow & & \searrow & \\
\begin{array}{l} O = 89.09 \\ S = 400 \end{array} & & & & \begin{array}{l} O_{ud} = 25 \\ S_{ud} = 400 \end{array} \\
 & \searrow & & \nearrow & \\
 & & \begin{array}{l} O_d = 14.02 \\ S_d = 320 \end{array} & & \\
 & & & \searrow & \\
 & & & & \begin{array}{l} O_{dd} = 0 \\ S_{dd} = 256 \end{array} \\
t = 0 & & t = 1 & & t = 2
\end{array}
$$

The hedge ratio becomes:

$$\Delta = \frac{O_u - O_d}{(u-d)S} = \frac{25-0}{400-256} = \frac{25}{144} = 0.174$$

- Have to sell $0.753 - 0.174 = 0.579$ stock at 320, gives $0.579 \times 320 = 185.28$
- Use 185.28 to pay off debt
  - new mount of debt becomes: $-185.28 + 1.07 \times 212.11 = 41.68.$
- using formula $D = (uO_d - dO_u)/(u-d)r$ gives (almost) same result:
  - $(1.25 \times 0) - (0.8 \times 25)/((1.25 - .8) \times 1.07) = -41.54$

From the lower node at t_1 the stock price can increase to 400 or fall to 256 at t_2

If the stock price increases to 400:

- option is in the money
- have to deliver stock against exercise price 375
- only have .174 stock in portfolio $\Rightarrow$ have to buy 1-.174=.826 stock at a price of 400, costs $.826 \times 400 = 330.40$
- net amount we get from the stock is $375 - 330.40 = 44.60$
- just enough to pay debt, which is now $41.68 \times 1.07 = 44.598$

So we have a perfect hedge!

If the stock price falls to 256 at t_2 :

- option is out of the money, expires worthlessly
- left in portfolio is .174 stock, value of $.174 \times 256 = 44.544$
- just enough to pay off the debt of 44.598

Again, a perfect hedge!

Note the following:

- When we adjust the portfolio at t_1, we do not know what is going to happen at t_2
- We make our adjustments on the basis of the current, observed stock price
    - hedging portfolio does not require predictions

We do know, however, that there is no risk: we always have a perfect hedge with a self financing strategy

Dividends

Assume stock pays out 25% of its value at t_1

- Irrelevant for the stockholders in perfect capital markets
- Does matter to the option holders
  - they only have right to buy the stock at maturity
  - do not receive any dividends before exercise

Extreme case:

- firm sells all its assets, pays out proceeds as dividends to stockholders
- leaves option holders with right to buy worthless stock

Assume stock value drops with dividend amount right after payment. Binomial tree for the stock becomes:

$$
\begin{array}{ccccc}
 & & 500\text{ (cum)} & & 468.75 \\
 & \nearrow & \downarrow & \nearrow & \\
400 & & 375\text{ (ex)} & & \\
 & \searrow & & \searrow & \\
 & & 320\text{ (cum)} & & 300 \\
 & & \downarrow & \nearrow & \\
 & & 240\text{ (ex)} & & \\
 & & & \searrow & \\
 & & & & 192 \\
t=0 & & t=1 & & t=2
\end{array}
$$

Lattice 6 Binomial tree for a dividend paying stock

- Loss in stock value represented by ex-dividend value below cum-dividend value
- Parameters of binomial process refer to all values
    - u, d and p do not change
    - binomial tree continues from the ex-dividend value

Calculate option price as before, starting with values at maturity then solve tree backwards

$$
\begin{array}{ccccc}
 & & & & O_{uu} = \max[0, 468.75 - 375] = 93.75 \\
 & & & \nearrow & \\
 & & O_u = 52.57 & & \\
 & \nearrow & & \searrow & \\
O = 29.48 & & & & O_{ud} = \max[0, 300 - 375] = 0 \\
 & \searrow & & \nearrow & \\
 & & O_d = 0 & & \\
 & & & \searrow & \\
 & & & & O_{dd} = \max[0, 192 - 375] = 0 \\
\text{t=0} & & \text{t=1} & & \text{t=2}
\end{array}
$$

Lattice 7 Binomial tree for an option

The calculations are as before:

$$O_u = \frac{0.6 \times 93.75 + 0.4 \times 0}{1.07} = 52.57$$

and

$$O_d = 0$$

so that

$$O = \frac{0.6 \times 52.57 + 0.4 \times 0}{1.07} = 29.48$$

A considerable reduction from the option value without dividends, 89.09

An American call

American options can be exercised early

- without dividends, early exercise of call not profitable
- with dividends early execise may be optimal (cf. extreme case just mentioned)

Have to test whether option should be exercised or not, easily done in binomial model:

- include in all relevant nodes this condition:
- max[exercing, keeping] (or popularly max[dead, alive])

Values at maturity remain unchanged

- option cannot be kept at maturity
- is either exercised or expires

So the end nodes in tree remain the same. Then:

- Calculate t_1 values of payoffs at maturity

  = option values 'alive'

- Compare those with t_1 values 'dead'

  = difference between *cum-dividend value* and exercise price.

Whole point of exercising early is to receive the dividends

- exercise before dividends are paid

In the case of O_u at t_1 early exercise is profitable:

- value dead is 125 $>$ value alive of 52.57
- higher value reflected in higher t_0 value $(.6 \times 125)/1.07 = 70.09$

Latter value should be checked against value dead

- $400 - 375 = 25 < 70.09$
- makes sense: nobody sells option that should be exercised immediately

$$O_{uu} = 93.75$$

$$\max[\underset{alive}{52.57}, \underset{dead}{(500 - 375)}]$$
$$O_u = 125$$

$$O = 70.09 \qquad\qquad O_{ud} = 0$$

$$\max[\underset{alive}{0}, \max[0, \underset{dead}{(320 - 375)}]]$$
$$O_d = 0$$

$$O_{dd} = 0$$

t=0 *t=1* *t=2*

Lattice 8 Binomial tree for an option

A put option:

Binomial option pricing works just as well for put options

- we did not explicitly model the nature of the option
- formulated option's payoff as $O_{u,d} = \max[0, S_{u,d} - X]$
- can change this to $O_{u,d} = \max[0, X - S_{u,d}]$
- does not change the derivation

Use original two period example to illustrate, redefine option as a European put with the same exercise price of 375

Stock price development remains as in Lattice 4, option values depicted in Lattice 9

$$
\begin{array}{ccccc}
 & & & & O_{uu} = \max[0, 375 - 625] = 0 \\
 & & & \nearrow & \\
 & & O_u = 0 & & \\
 & \nearrow & & \searrow & \\
O = 16.63 & & & & O_{ud} = \max[0, 375 - 400] = 0 \\
 & \searrow & & \nearrow & \\
 & & O_d = 44.49 & & \\
 & & & \searrow & \\
 & & & & O_{dd} = \max[0, 375 - 256] = 119 \\
\text{t=0} & & \text{t=1} & & \text{t=2}
\end{array}
$$

Lattice 9 A put option

Working out the tree

- $O_d = ((0.6 \times 0) + (0.4 \times 119))/1.07 = 44.49$
- $Ou = 0$
- so that $O = ((0.6 \times 0) + (0.4 \times 44.49))/1.07 = 16.63$

What other way is there to calculate the value of this put?
Put call parity gives same result:

- $call + PV(X) = S + put$ or $put = call + PV(X) - S$
  - call price is 89.09, stock price is 400
  - PV(X) is $375/1.07^2 = 327.54$
- Price of the put is: put $= 89.09 + 327.54 - 400 = 16.63$

Remember: only for European puts on stocks without dividends

What does the hedging portfolio for this put look like?
Calculating the put's Δ and D at t_0 we get:

$$\begin{aligned}
\Delta &= \frac{O_u - O_d}{(u-d)S} = \frac{0 - 44.49}{500 - 320} = -0.247 \text{ and} \\
D &= \frac{uO_d - dO_u}{(u-d)r} = \frac{1.25 \times 44.49 - 0.8 \times 0}{1.25 \times 1.07 - 0.8 \times 1.07} = 115.50
\end{aligned}$$

Hedging portfolio for put

- is short position in the stock and long position in risk free debt

The opposite of what we calculated for the call

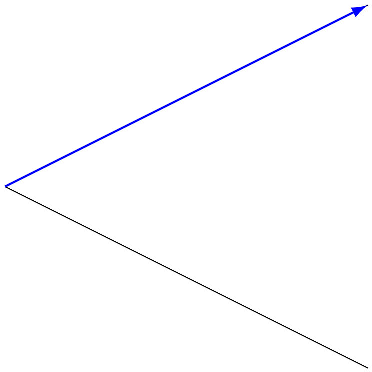
- leveraged long position in the stock

Asymptotic properties

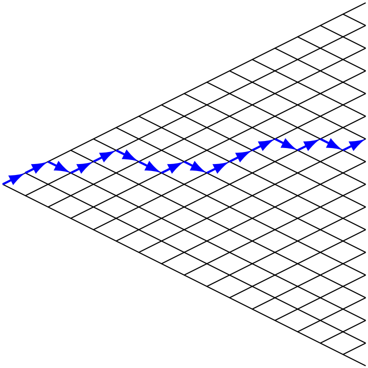
Sofar, we used only few time steps

- keeps calculations easy to follow
- not necessay, can make time grid a fine as we want
- 1 calender year can modelled as
  - 6 periods of 2 months
  - 12 months
  - 52 weeks
  - 250 days, etc.
- Cox, Ross, Rubinstein show that:
  - under certain parameter assumptions
  - model converges to Black and Scholes model

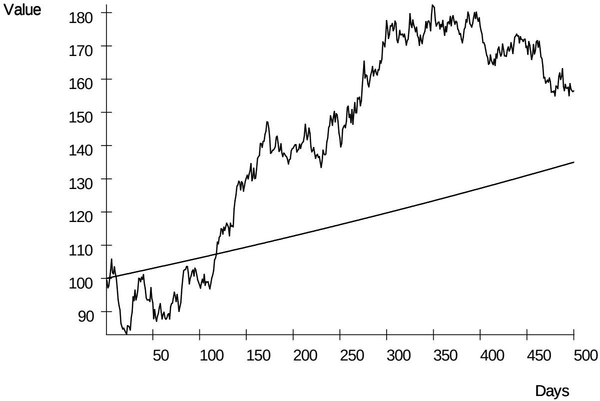
Illustrated graphically as in beginning:



1 period of 1 year



16 periods of 3 weeks



Sample path geometric Brownian motion, $\mu = .15$, $\sigma = .3$, t=500 days; smooth line is deterministic part of the motion