Path Integral Quantization

Ali Shahryar Khokhar and Sharjeel Ahmad[1]

LUMS, Lahore, Pakistan

E-mail: 24100266@lums.edu.pk, 24100083@lums.edu.pk

Abstract: This paper comprehensively investigates the path integral quantization method in quantum field theory, applied to scalar, electromagnetic, and spinor fields. We commence with an introduction to functional methods, setting the foundation for understanding versatile path integral formalism. Following this, we delve into the principles of path integrals in quantum mechanics, elucidating the underlying concepts that enable their extension to quantum field theory.

In our exploration, we perform functional quantization of scalar fields and evaluate the associated correlation functions, which leads to the formulation of Feynman's rules for scalar field theory. As the study progresses, we extend the path integral framework to encompass electromagnetic and spinor fields. This expansion allows for the derivation of Feynman rules for Dirac fields and quantum electrodynamics, further showcasing the adaptability of the path integral approach in the context of diverse quantum fields.

Throughout this paper, we emphasize the integrative nature of the path integral method, which facilitates a more profound understanding of the fundamental forces governing interactions among subatomic particles. The knowledge obtained from this work is expected to pave the way for future research on more complex systems, including non-Abelian gauge theories and quantum gravity. Moreover, the findings may prove valuable in interdisciplinary applications within other branches of physics, such as condensed matter and cosmology.

Contents

1 Introduction

Quantum Field Theory (QFT) provides a powerful and versatile framework to understand the behavior of subatomic particles and their interactions. This paper focuses on the path integral quantization method and its application to various quantum fields, such as scalar, electromagnetic, and spinor fields. The study aims to deepen our understanding of the fundamental principles of QFT, thereby enhancing our comprehension of subatomic particles and their interactions.

We introduce the functional techniques that underlie the path integral formalism, which facilitates a deeper understanding of the mathematical structures involved. Subsequently, we delve into the principles of path integrals in quantum mechanics, setting the stage for their application to quantum field theory. The path integral approach offers an alternative to canonical quantization, allowing for a more intuitive representation of field theories and enabling the visualization of particle interactions through Feynman diagrams.

Following the groundwork on path integrals, we quantize scalar fields, which describe particles such as the Higgs boson. Scalar fields obey the Klein-Gordon equation, and we analyze their behavior through the functional integral approach, evaluating correlation functions and deriving the Feynman rules for scalar field theory.

Next, we extend the path integral quantization method to Dirac fields, which characterize fermions like electrons and protons that adhere to Fermi-Dirac statistics. The Dirac field obeys the Dirac equation, and we explore the anti-commutation relations that arise from Fermi-Dirac statistics. Moreover, we examine the Dirac propagator, which explains how fermions move and interact, and investigate the symmetries within the Dirac theory.

Subsequently, we delve into Quantum Electrodynamics (QED), which describes the interactions between electromagnetic fields and fermions. We study gauge symmetry, a fundamental property that governs the behavior of electromagnetic fields, and learn to quantify the electromagnetic field while maintaining gauge symmetry using covariant derivatives. We also analyze the coupling between the electromagnetic field and fermions, visualizing their interactions through Feynman diagrams.

Throughout this paper, we emphasize the integrative and adaptable nature of the path integral approach in quantum field theory, which facilitates a more comprehensive understanding of the fundamental forces that govern interactions between subatomic particles. The insights from this research are anticipated to contribute to future investigations on more intricate systems and inform applications in other physics domains.

2 Introduction to Functional Methods

Functional methods are powerful mathematical techniques that have been applied to various areas of physics, including quantum mechanics, quantum field theory, and statistical mechanics ([1], [2]). The core idea behind these methods is to extend standard calculus on functions to an "infinite-dimensional calculus" that works with functionals instead. A functional is a mapping from a function space to a scalar, meaning it takes a function as input and produces a scalar as output.

In this introduction, we will discuss the key components of functional methods, including functionals, functional derivatives, and functional integrals. We will also provide an example of how these methods can be applied to solve a physics problem.

2.1 Functionals

A functional F is a mapping from a function space to a scalar, i.e., $F : \mathcal{F} \to \mathbb{R}$. As an example, consider the functional that maps a function $f(x)$ to its definite integral over a given interval $[a, b]$:

$$F[f(x)] = \int_a^b f(x)\, dx. \tag{2.1}$$

2.2 Functional Derivatives

The concept of functional derivatives is an extension of the ordinary derivatives for functions. For a functional $F[f(x)]$, the functional derivative $\frac{\delta F}{\delta f(x)}$ is defined as:

$$\frac{\delta F}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\zeta(x)] - F[f(x)]}{\epsilon}. \tag{2.2}$$

Here, $\zeta(x)$ is an arbitrary function, and ϵ is a small scalar parameter. The functional derivative measures how the functional changes when the function $f(x)$ is perturbed by an infinitesimal amount $\epsilon\zeta(x)$ ([3]).

2.3 Functional Integrals

Functional integrals, also known as path integrals, generalize the standard integrals to the space of functions. Essentially, they represent an integration over all possible configurations of a given function. The functional integral is defined as [**Feynman:1948ur**]:

$$\int \mathcal{D}f(x)\, e^{iS[f(x)]} \tag{2.3}$$

Here, $\mathcal{D}f(x)$ denotes the integration over all possible functions $f(x)$, and $S[f(x)]$ is the action functional, which plays a crucial role in the dynamics of the system. In quantum mechanics, the action functional is related to the Lagrangian of the system.

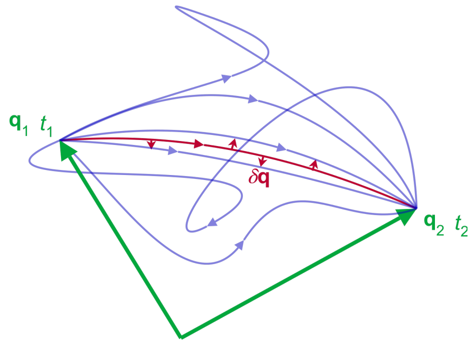


Figure 1. As the system evolves, q traces a path through configuration space. The red path the system takes has a stationary action under small changes in the system's configuration $\delta\mathbf{q}$[5].

Functional methods are especially valuable in the path integral formulation of quantum mechanics. In this approach, the transition amplitude between two states is represented by the functional integral over all possible connecting paths, with each path weighted by the action functional ([1]). This creates a profound link between classical and quantum mechanics, as the paths with the least action, or the classical paths, contribute most significantly to the transition amplitude.

2.4 Example: The Principle of Least Action

Let us further consider the Principle of Least Action in classical mechanics to motivate the use and understanding of functional methods. This principle states that the actual path taken by a particle between two points in space and time is the one that minimizes the action functional $S[f(x)]$, where S is given by the integral of the Lagrangian L over time ([4]):

$$S[f(x)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, dt \tag{2.4}$$

Here, $q(t)$ represents the generalized coordinates of the system, $\dot{q}(t)$ denotes their time derivatives (velocities), and $L(q(t), \dot{q}(t), t)$ is the Lagrangian of the system, which depends on the coordinates, velocities, and time. The action functional $S[f(x)]$ is minimized when the following Euler-Lagrange equation is satisfied ([3]):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} - \frac{\partial L}{\partial q(t)} = 0 \tag{2.5}$$

This equation describes the equations of motion for the system and is derived by finding the functional derivative of the action with respect to the generalized coordinates $q(t)$ and setting it to zero. A classical way of understanding this is given in figure 1.

To understand the connection between the Principle of Least Action and functional methods, consider a simple harmonic oscillator with mass m and spring constant k. The Lagrangian for this system is given by ([4]):

$$L(q(t), \dot{q}(t), t) = \frac{1}{2}m\dot{q}^2(t) - \frac{1}{2}kq^2(t) \tag{2.6}$$

Applying the Euler-Lagrange equation to this Lagrangian yields the equation of motion for the simple harmonic oscillator:

$$m\ddot{q}(t) + kq(t) = 0 \tag{2.7}$$

This example illustrates the application of functional methods to derive equations of motion from the Principle of Least Action in classical mechanics. As we progress in this project, we will examine the application of functional methods in quantum mechanics and quantum field theory. This will lead to robust techniques for determining observables within these theories.

3 Path Integrals in Quantum Mechanics

The path integral formulation, developed by Richard Feynman, offers a different and intuitive perspective on quantum mechanics ([1]). This approach is grounded in the idea that the probability amplitude for a particle to travel between two points in spacetime is determined by a sum (integral) over all potential paths connecting these points. The path integral method is closely linked to classical mechanics and emphasizes the importance of the principle of least action.

3.1 Feynman's Path Integral

In the path integral formulation, the transition amplitude for a particle to move from position x_i at time t_i to position x_f at time t_f is given by ([1]):

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x(t)]} \tag{3.1}$$

Here, $\mathcal{D}x(t)$ denotes the integration over all possible paths $x(t)$ connecting the initial and final points, and $S[x(t)]$ is the action functional, defined as:

$$S[x(t)] = \int_{t_i}^{t_f} L(x, \dot{x}, t)\, dt, \tag{3.2}$$

where $L(x, \dot{x}, t)$ is the Lagrangian of the system. The factor $\frac{i}{\hbar}$ in the exponent ensures the correct units for the action and introduces the crucial quantum phase.

3.2 Propagator and Time Evolution

The quantity $\langle x_f, t_f | x_i, t_i \rangle$ is called the propagator or the kernel, and it plays a central role in the path integral formulation ([1]). The propagator contains all the information about the system's dynamics and can be used to compute the time evolution of a given initial state. For a given wave function $\psi(x, t_i)$, the wave function at a later time t_f can be obtained as:

$$\psi(x_f, t_f) = \int dx_i \, \langle x_f, t_f | x_i, t_i \rangle \, \psi(x_i, t_i). \tag{3.3}$$

This equation expresses the time evolution of the wave function as a superposition of amplitudes for all possible initial positions x_i, weighted by the propagator.

3.3 Path Integral for the Harmonic Oscillator

As an example, let's consider the quantum harmonic oscillator. Its Lagrangian is given by ([4]):

$$L(x, \dot{x}, t) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \tag{3.4}$$

The action functional for the harmonic oscillator can then be written as:

$$S[x(t)] = \int_{t_i}^{t_f} \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right] dt \tag{3.5}$$

Although the path integral for the harmonic oscillator cannot be computed precisely, various approximation techniques, such as the stationary phase approximation or the saddle-point method, can be employed to solve it ([1]). These methods emphasize the close relationship between the classical path and the dominant contribution to the path integral, further reinforcing the connection between classical and quantum mechanics.

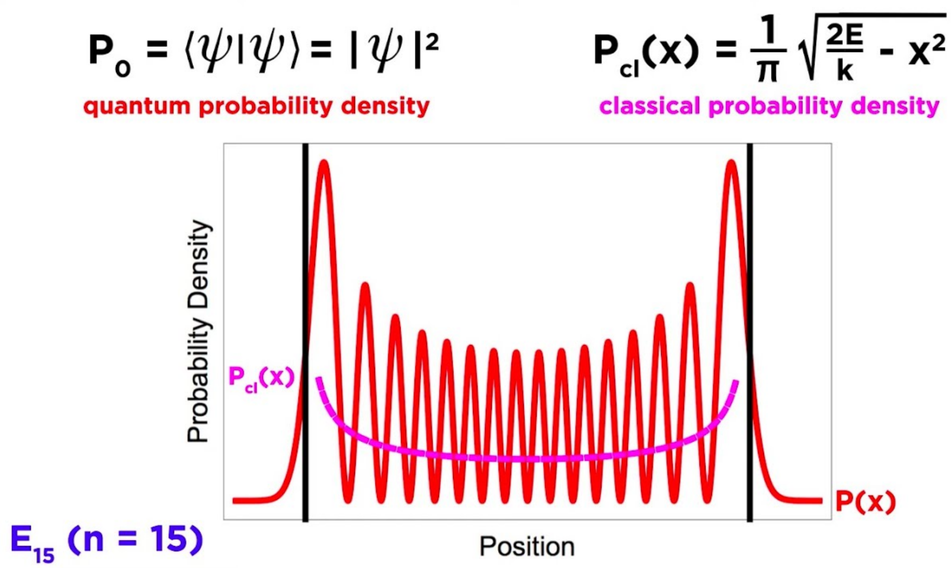


Figure 2. Quantum Harmonic Oscillator

3.4 Path Integral for the Free Particle: Step-by-Step Derivation

To compute the path integral for the free particle, we will first express the propagator in terms of a discretized path integral and then evaluate it in the limit as the time steps become infinitesimally small ([1]). This will yield the propagator for a free particle in terms of an integral expression.

Divide the time interval (t_i, t_f) into N equal intervals of size $\epsilon = \frac{t_f - t_i}{N}$, and label the intermediate times as $t_0 = t_i, t_1, \ldots, t_N = t_f$.

The action functional for the free particle discretized over the time interval can be written as ([1]):

$$S[x(t)] \approx \sum_{j=0}^{N-1} \frac{1}{2} m \left(\frac{x_{j+1} - x_j}{\epsilon} \right)^2 \epsilon, \tag{3.6}$$

where $x_j = x(t_j)$.

The path integral is then given by ([1]):

$$\langle x_f, t_f | x_i, t_i \rangle \approx \lim_{N\to\infty} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{\frac{N}{2}} \int dx_1 \ldots dx_{N-1}, e^{\frac{i}{\hbar} \sum_{j=0}^{N-1} \frac{1}{2} m \left(\frac{x_{j+1} - x_j}{\epsilon} \right)^2 \epsilon}. \tag{3.7}$$

We can introduce the Wiener measure, a probability measure associated with the Wiener process (a continuous-time random walk) to evaluate this integral ([6]). The Wiener measure can be expressed as:

$$\mathcal{D}W(x) = \lim_{N\to\infty} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{\frac{N}{2}} e^{-\frac{m}{2\hbar\epsilon} \sum_{j=0}^{N-1} (x_{j+1} - x_j)^2} \prod_{j=1}^{N-1} dx_j, \tag{3.8}$$

where the factor $\frac{m}{2\hbar\epsilon}$ in the exponent ensures the correct scaling of the Wiener measure.

Using the Wiener measure, the path integral for the free particle becomes ([1]):

$$\langle x_f, t_f | x_i, t_i \rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}W(x), e^{\frac{i}{\hbar} \sum_{j=0}^{N-1} \frac{1}{2} m \left(\frac{x_{j+1} - x_j}{\epsilon} \right)^2 \epsilon}. \tag{3.9}$$

Finally, we can evaluate this path integral by considering the limit as $N \to \infty$ ([1]). The result is the propagator for the free particle:

$$\langle x_f, t_f | x_i, t_i \rangle = \sqrt{\frac{m}{2\pi i \hbar (t_f - t_i)}}, e^{\frac{im(x_f - x_i)^2}{2\hbar(t_f - t_i)}}. \tag{3.10}$$

This expression gives the propagator for the free particle, representing the probability amplitude for the particle to transition from the initial position x_i at time t_i to the final position x_f at time t_f. Note that the propagator is a Gaussian function centered around the classical trajectory $x_f - x_i = \dot{x}(t_f - t_i)$, where $\dot{x}$ is the constant velocity of the free particle ([1]). This highlights the connection between the path integral formulation and classical mechanics, as the dominant contribution to the path integral comes from paths close to the classical trajectory.

3.5 Path Integral Applications and Connections to Other Areas of Physics

The path integral formulation of quantum mechanics has far-reaching implications and connections to other areas of physics. For instance, it is a powerful tool in statistical mechanics that can be used to compute partition functions and thermodynamic quantities. The method is also employed in quantum field theory, which helps describe the interactions of fields and particles, such as electroweak and strong forces.

The path integral formulation has also inspired various mathematical techniques and concepts, such as the Wiener measure and the Wiener process in stochastic calculus. Furthermore, it has shed light on the geometric and topological aspects of quantum mechanics and classical mechanics, leading to new insights and discoveries.

In conclusion, the path integral formulation of quantum mechanics provides a versatile and profound framework for understanding quantum phenomena. Its applications span numerous areas of physics, and it forges deep connections between classical and quantum mechanics. As we continue to study quantum field theory and other advanced topics in physics, the path integral approach will serve as a crucial foundation for our understanding.

4 Functional Quantization of Scalar Fields and Evaluation of Correlation Functions

In this section, we will perform the functional quantization of scalar fields and evaluate the correlation functions. To start, consider a real scalar field $\phi(x)$ in d-dimensional Minkowski space-time, governed by the classical Lagrangian density:

$$\mathcal{L}[\phi] = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2. \tag{4.1}$$

The goal of functional quantization is to compute the generating functional $Z[J]$ of Green's functions:

$$Z[J] = \int \mathcal{D}\phi \, e^{i\int d^dx(\mathcal{L}[\phi]+J\phi)}. \tag{4.2}$$

Here, $J(x)$ is an external source, and the path integral is taken over all possible field configurations $\phi(x)$. The correlation functions, also known as Green's functions, can be obtained by taking functional derivatives of $Z[J]$ with respect to $J(x)$ and setting $J(x) = 0$ afterward.

To compute $Z[J]$, we first need to rewrite the Lagrangian density in Fourier space. Defining the Fourier transform of the fields and external source as follows:

$$\phi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ikx} \tilde{\phi}(k) J(x), \tag{4.3}$$

$$\phi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ikx} \tilde{J}(k), \tag{4.4}$$

the Lagrangian density can be rewritten as:

$$\mathcal{L}[\phi] = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left[(k^2 + m^2)\tilde{\phi}(-k)\tilde{\phi}(k) \right]. \tag{4.5}$$

Now, the generating functional becomes:

$$Z[J] = \int \mathcal{D}\tilde{\phi}\, e^{i\int d^d k \left[\frac{1}{2}(k^2+m^2)\tilde{\phi}(-k)\tilde{\phi}(k) + \tilde{J}(-k)\tilde{\phi}(k) \right]}. \tag{4.6}$$

This is a Gaussian integral, which can be computed using the well-known formula:

$$\int d^n x\, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{J}^T \boldsymbol{x}} = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2}\boldsymbol{J}^T A^{-1} \boldsymbol{J}}, \tag{4.7}$$

where A is a symmetric matrix, and $\boldsymbol{J}$ is an n-dimensional vector.

Applying this formula to the path integral, we find:

$$Z[J] = \mathcal{N} \exp\left[i\frac{1}{2} \int d^d k\, \tilde{J}(-k) \frac{1}{k^2 + m^2} \tilde{J(k)} \right], \tag{4.8}$$

where $\mathcal{N}$ is a normalization factor that ensures $Z[0] = 1$.

Now, we can obtain the n-point correlation functions by taking functional derivatives of $Z[J]$ with respect to the source $J(x)$ and setting $J(x) = 0$ afterward. For instance, the two-point correlation function, also known as the propagator, is given by:

$$G(x_1, x_2) = \left. \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} \right|_{J=0}. \tag{4.9}$$

Using the chain rule, we obtain the following:

$$G(x_1, x_2) = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} e^{-ik_1 x_1} e^{-ik_2 x_2} \frac{\delta^2}{\delta \tilde{J}(-k_1)\delta \tilde{J}(k_2)} \exp^H, \tag{4.10}$$

where H is equal to $[i\frac{1}{2} \int d^d k, \tilde{J}(-k) \frac{1}{k^2+m^2} \tilde{J}(k)]|_{J=0}$

$$G(x_1, x_2) = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} e^{-ik_1 x_1} e^{-ik_2 x_2} \frac{1}{k_1^2 + m^2} (2\pi)^d \delta(k_1 - k_2), \tag{4.11}$$

$$G(x_1, x_2) = \int \frac{d^d k}{(2\pi)^d} e^{-ik(x_1 - x_2)} \frac{1}{k^2 + m^2}, \tag{4.12}$$

which is the well-known expression for the propagator of a free scalar field in momentum space.

In summary, we have performed the functional quantization of scalar fields using the path integral formalism and derived the expression for the two-point correlation function. This procedure can be extended to compute higher-point correlation functions and can be used as a foundation for deriving Feynman's rules in scalar field theory.

5 Feynman Rules for a Scalar Field Theory

This section will derive the Feynman rules for a scalar field theory using path integral formalism. We start by the generating functional $Z[J]$ for the interacting scalar field theory with the Lagrangian density:

$$\mathcal{L}[\phi] = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{5.1}$$

where λ is the coupling constant.

The generating functional $Z[J]$ is given by:

$$Z[J] = \int \mathcal{D}\phi, e^{i\int d^dx(\mathcal{L}[\phi]+J\phi)}. \tag{5.2}$$

Now, we introduce a new auxiliary field $\sigma(x)$ and rewrite the interaction term as:

$$e^{i\int d^dx\frac{\lambda}{4!}\phi^4} = \int \mathcal{D}\sigma\, e^{i\int d^dx\left[\frac{1}{2}\sigma^2 - \frac{1}{\sqrt{2\lambda}}\sigma\phi^2\right]}. \tag{5.3}$$

The generating functional can now be expressed as:

$$Z[J] = \int \mathcal{D}\phi\mathcal{D}\sigma\, e^{i\int d^dx\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}(m^2+\sigma)\phi^2 + \frac{1}{2}\sigma^2 + J\phi\right]}. \tag{5.4}$$

To evaluate this functional integral, we first integrate over ϕ. The result is a Gaussian integral that can be computed, as shown in the previous section. After integrating over ϕ, we get:

$$Z[J] = \int \mathcal{D}\sigma\, e^{i\int d^dx\left[\frac{1}{2}\sigma^2 + iW_\sigma[J]\right]}, \tag{5.5}$$

where $W_\sigma[J]$ is the effective action for the auxiliary field $\sigma(x)$, which is given by:

$$W_\sigma[J] = -\frac{1}{2}\ln\left[\det\left(-\partial^2 + m^2 + \sigma\right)\right] + \int d^dx J(x)\phi_\sigma(x), \tag{5.6}$$

and $\phi_\sigma(x)$ is the classical field that satisfies the equation of motion in the presence of $\sigma(x)$ and the source $J(x)$:

$$\left(-\partial^2 + m^2 + \sigma(x)\right)\phi_\sigma(x) = J(x). \tag{5.7}$$

Now, we can expand the effective action $W_\sigma[J]$ in powers of σ and J using perturbation theory. The Feynman rules can be derived by identifying the terms in the perturbative expansion that correspond to the vertices and propagators of the theory.

The propagator for the scalar field ϕ is given by the inverse of the operator $(-\partial^2 + m^2 + \sigma)$, which in momentum space is:

$$G(k) = \frac{1}{k^2 + m^2 + \sigma}. \tag{5.8}$$

This is the propagator for the interacting scalar field, represented by a line in Feynman diagrams. The interaction term in the effective action is given by:

$$-\frac{1}{\sqrt{2\lambda}}\int d^dx\sigma(x)\phi^2(x), \tag{5.9}$$

which, in momentum space, can be written as:

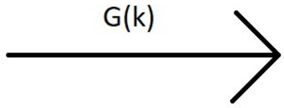
$$-\frac{1}{\sqrt{2\lambda}}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\tilde{\sigma}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)(2\pi)^d\delta(k_1+k_2+k_3). \tag{5.10}$$

This term corresponds to a three-point vertex in the Feynman diagrams with one σ and two ϕ lines, which has a coupling constant $\frac{1}{\sqrt{2\lambda}}$.

Now, we can summarize the Feynman rules for the scalar field theory:

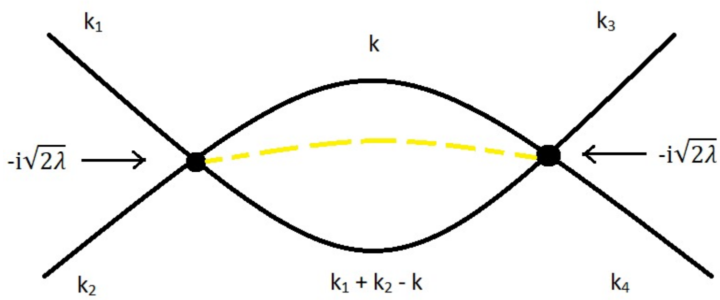
1. Each internal line (propagator) corresponds to the momentum-space propagator $G(k)$, which is given by:

$$G(k)=\frac{1}{k^2+m^2+\sigma}. \tag{5.11}$$

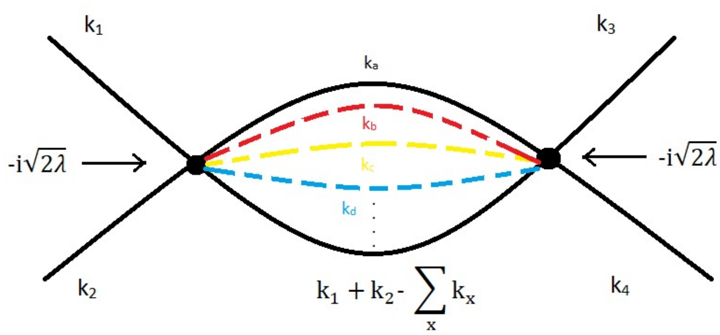


2. Each vertex with one σ and two ϕ lines corresponds to a factor of $-i\sqrt{2\lambda}$ and conserves momentum at the vertex. In other words, the sum of the incoming momenta equals the sum of the outgoing momenta:

$$k_1+k_2=k_3+k_4. \tag{5.12}$$



3. Integrate overall internal momenta k_i with the measure $\frac{d^dk_i}{(2\pi)^d}$.



4. Impose momentum conservation at each vertex using a δ-function, $(2\pi)^d\delta(k_1+k_2-k_3-k_4)$.

5. Multiply by a symmetry factor for each diagram, which is the inverse of the number of ways to rearrange the external lines without changing the diagram's topology.

In conclusion, we have derived the Feynman rules for a scalar field theory using path integral formalism. These rules can be used to compute scattering amplitudes and correlation functions for the interacting scalar field theory, providing a systematic way to study its properties and behavior.

6 Quantization of the Electromagnetic Field

Let's begin with the following function integral:

$$\int \mathcal{D}Ae^{iS[A]}, \tag{6.1}$$

where $S[A]$ is the action of the free electromagnetic field, and the integral is done over all four space-time components, i.e., $\mathcal{D}A \equiv \mathcal{D}A^0\mathcal{D}A^1\mathcal{D}A^2\mathcal{D}A^3$. We can integrate the expression in parts and expand the electromagnetic field as a Fourier integral to write:

$$S = \int d^4x[-\frac{1}{4}(F_{\mu v})^2], \tag{6.2}$$

$$S = \frac{1}{2}\int d^4xA_\mu(x)(\partial^2 g^{\mu v} - \partial^\mu\partial^v)A_v(x), \tag{6.3}$$

$$S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2g^{\mu v} + k^\mu k^v)\tilde{A}_v(-k). \tag{6.4}$$

Note that the above equation vanishes when $\tilde{A}_\mu(k) = k_\mu\alpha(k)$, when $\alpha(k)$ is an arbitrary scalar function. The integrand for equation 6.1 is 1 for this sizeable set of field configurations which means that the functional integral diverges as there is no Gaussian damping. This means the 4x4 matrix $(-k^2g^{\mu v} + k^\mu k^v)$ is singular and the Feynman Propagator which is defined as $(-k^2g^{\mu v} + k^\mu k^v)\tilde{D}_F^{v\rho}(k) = i\delta_\mu^\rho$ has no solution. We face this issue because of the gauge invariance of $F_\mu v$ and because our integral is poorly configured to sum over a continuum of physically equivalent field configurations. We then seek to isolate only the part of the integral that only integrates over each field configuration once.

In order to achieve this, we use the Faddeev–Popov ghost fields (see appendix)[7]. Start by defining a function $G(A)$, which we want to equate to zero as part of the gauge fixing condition. This dummy function is an infinite product of delta functions, one for each point in a given field. To remain mathematically sound, we insert 1 under Equation 6.1 and get:

$$1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha))\det(\frac{\delta G(A^\alpha)}{\delta\alpha}), \tag{6.5}$$

where A^α is the gauge-transformed field denoted by:

$$A^\alpha_\mu = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x). \tag{6.6}$$

Also, note that equation 6.5 is just the continuum generalization of the mathematical identity:

$$1 = (\prod_i \int da_i)\delta^{(n)}(\mathbf{g}(\mathbf{a}))\det(\frac{\partial g_i}{\partial a_j}) \tag{6.7}$$

for discrete n-dimensional vectors.

In the Lorentz gauge we know that $G(A^\alpha) = \partial^\mu A_\mu + (1/e)\partial^2\alpha$ so the functional determinant $\det(\delta G(A^\alpha)/\delta\alpha)$ is equivalent to $\det(\partial^2/e)$. In our case, $\frac{\delta G(A^\alpha)}{\delta\alpha}$ is independent of A and hence can be viewed as a constant. We now insert Equation 6.5 into Equation 6.1 to obtain the following:

$$\det(\frac{\delta G(A^\alpha)}{\delta\alpha})\int \mathcal{D}\alpha \int \mathcal{D}Ae^{iS[A]}\delta(G(A^\alpha)). \tag{6.8}$$

Changing variables from A to A^α since it is just a dummy index due to gauge invariance, we get:

$$\int \mathcal{D}Ae^{iS[A]} = \det(\frac{\delta G(A^\alpha)}{\delta\alpha})\int \mathcal{D}\alpha \int \mathcal{D}Ae^{iS[A]}\delta(G(A)). \tag{6.9}$$

We have successfully restricted our integral to physically different field configurations via the delta function. We obtain an infinite multiplicative factor by the divergent integral over $\alpha(x)$. It is now time to set our gauge-fixing function as follows:

$$G(A) = \partial^\mu A_\mu(x) - \omega(x), \tag{6.10}$$

where $\omega(x)$ can be any given scalar function. We then set $G(A)$ equal to 0 to get a generalized Lorentz gauge condition. Since the Lorentz gauge is equivalent to the functional determinant, our integral becomes:

$$\int \mathcal{D}Ae^{iS[A]} = \det(\frac{1}{e}\partial^2)(\int \mathcal{D}\alpha)\int \mathcal{D}Ae^{iS[A]}\delta(\partial^\mu A_\mu - \omega(x)), \tag{6.11}$$

The above equality will be true for any arbitrary selection of ω. Finally, we integrate over all $\omega(x)$, with ω centered at 0. The right-hand side of the above equation then becomes:

$$N(\zeta)\det(\frac{1}{e}\partial^2)(\int \mathcal{D}\alpha)\int \mathcal{D}Ae^{iS[A]}exp[-i\int d^4x\frac{1}{2\zeta}(\partial^\mu A_\mu)^2], \tag{6.12}$$

$N(\zeta)$ is just a normalization constant, and the delta function has been used to integrate over ω. This has allowed adding a new term to our Lagrangian: $-(\partial^\mu A_\mu)^2/2\zeta$.

Using the new ζ term, we are able to obtain a sensible photon propagator from the $S[A]$ action function (see appendix). Our equation then becomes:

$$(-k^2 g_{\mu v} + (1 - \frac{1}{\zeta})k_\mu k_v)\tilde{D}_F^{v\rho}(k) = i\delta_\mu^\rho, \tag{6.13}$$

solving for the Feynman Propagator, we get:

$$\tilde{D}_F^{\mu v}(k) = \frac{-i}{k^2 + i\epsilon}(g^{\mu v} - (1 - \zeta)\frac{k^\mu k^v}{k^2}. \tag{6.14}$$

The $i\epsilon$ term is the exact same denominator one would arrive at in the Klein-Gordon propagator, which attests to the validity of the solution [8]. During computation, ζ is usually chosen to be 1, which is known as the Feynman Gauge.

To wrap up our quantization of the electromagnetic field, we must ensure that the S-matrix (see appendix) elements computed by this procedure are correct. By adiabatically turning off the coupling constant in the far past and future, we can compute S-matrix elements between asymptotic states. When the coupling constant is zero, we can distinguish between gauge-invariant and gauge-variant states cleanly.

Single-particle states with one electron, one positron, or one transversely polarized photon are gauge-invariant, while states with time-like and longitudinal photon polarizations change under gauge transformations. As a result, we can define a gauge-invariant S-matrix by computing S_{FP}, which is the S'-matrix between general asymptotic states, using the above procedure. Although this matrix is unitary, it is not gauge-invariant. To obtain a gauge-invariant S-matrix, we can use a projection P_0 to select the subspace of the space of asymptotic states where all particles are either electrons, positrons, or transverse photons. Then let

$$S = P_0 S_{FP} P_0 \tag{6.15}$$

Of course, the above matrix is gauge invariant because it is projected onto gauge invariant states. Also, note that the S-matrix above is unitary because it is in line with the photon emission equation, which says:

$$\sum_{i=1,2} \epsilon_i^* \epsilon_{iv} \mathcal{M}^\mu \mathcal{M}^{*v} = (-g_{\mu v})\mathcal{M}^\mu \mathcal{M}^{*\nu}, \tag{6.16}$$

on the left-hand side, the sum only includes transverse polarizations. This reasoning holds true for cases where $\mathcal{M}^{*\nu}$ and $\mathcal{M}^\mu$ represent different amplitudes as long as they meet the requirements of the Ward identity. Using this information, we can see that:

$$SS^\dagger = P_0 S_{FP} P_0 S_{FP}^\dagger P_0 = P_0 S_{FP} S_{FP}^\dagger P_0. \tag{6.17}$$

Using the unitarity of S_{FP}, we can see that S is also unitary hence $SS^\dagger = 1$ in the subspace of gauge invariant states. Also, note that the S-matrix is independent of ζ as the

Ward Identity suggests that any Quantum Electro-Dynamic matrix is unaffected if we add any term proportional to the photon propagator any term proportional to q^μ as long as all the external fermions are on-shell.

7 Feynman Rules for Dirac Fields and Quantum Electrodynamics

This section will derive the Feynman quantum electrodynamics (QED) rules using path integral formalism. QED is a quantum field theory describing charged particles (such as electrons) interacting with the electromagnetic field. The fundamental fields in QED are the Dirac spinor field $\psi(x)$ and the photon field $A_\mu(x)$.

The Lagrangian density for QED is given by:

$$\mathcal{L}[\psi, A] = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x), \tag{7.1}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, γ^μ are the Dirac matrices, m is the mass of the charged particle, and e is its electric charge.

The generating functional $Z[J, \bar{J}]$ for QED can be expressed as a path integral over the Dirac field $\psi(x)$, its conjugate $\bar{\psi}(x)$, and the photon field $A_\mu(x)$:

$$Z[J, \bar{J}] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu \, e^{i\int d^dx(\mathcal{L}[\psi,A]+\bar{J}\psi+\bar{\psi}J)}. \tag{7.2}$$

In order to compute the path integral, we first introduce the gauge-fixing term and the corresponding Faddeev-Popov ghost term in the Lagrangian. We choose the Lorentz gauge-fixing condition $\partial^\mu A_\mu = 0$ and add a gauge-fixing term with a parameter ξ:

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial^\mu A\mu)^2. \tag{7.3}$$

The total Lagrangian density, including the gauge-fixing and ghost terms, is given by:

$$\mathcal{L}_{\mathrm{total}} = \mathcal{L}[\psi, A] + \mathcal{L}\mathrm{GF} + \mathcal{L}_{\mathrm{ghost}}. \tag{7.4}$$

We can now compute the generating functional $Z[J, \bar{J}]$ using the total Lagrangian density. This calculation uses perturbation theory, which involves expanding the path integral in powers of the interaction terms and integrating over the fields.

From the expansion, we can derive the Feynman rules for QED. The basic ingredients are the propagators for the fermion and photon fields and the interaction vertices. In momentum space, the fermion propagator is given by:

$$S(k) = \frac{i(\gamma^\mu k_\mu + m)}{k^2 - m^2 + i\epsilon}, \tag{7.5}$$

and the photon propagator is given by:

$$D^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{7.6}$$

The interaction term in the Lagrangian is given by:

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}(x)\gamma^\mu\psi(x)A\mu(x). \tag{7.7}$$

This term corresponds to a three-point vertex in Feynman diagrams with one photon line and two fermion lines, with a coupling constant $-ie$. In momentum space, the vertex factor is given by $-ie\gamma^\mu$.

Now, we can summarize the Feynman rules for QED:

1. Each internal fermion line (propagator) corresponds to the momentum-space fermion propagator $S(k)$:

$$S(k) = \frac{i(\gamma^\mu k_\mu + m)}{k^2 - m^2 + i\epsilon}. \tag{7.8}$$

2. Each internal photon line (propagator) corresponds to the momentum-space photon propagator $D^{\mu\nu}(k)$:

$$D^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{7.9}$$

3. Each vertex with one photon line and two fermion lines corresponds to a factor of $-ie\gamma^\mu$ and conserves momentum at the vertex. In other words, the sum of the incoming momenta equals the sum of the outgoing momenta:

$$k_1 + k_2 = k_3 + k_4. \tag{7.10}$$

4. Integrate overall internal momenta k_i with the measure $\frac{d^d k_i}{(2\pi)^d}$.

5. Impose momentum conservation at each vertex using a δ-function, $(2\pi)^d\delta(k_1 + k_2 - k_3 - k_4)$.

6. Multiply by a symmetry factor for each diagram, which is the inverse of the number of ways to rearrange the external lines without changing the diagram's topology.

In summary, we have derived the Feynman rules for quantum electrodynamics (QED) using path integral formalism. These rules can be used to compute scattering amplitudes and correlation functions for QED, providing a systematic way to study the properties and behavior of the electromagnetic interactions between charged particles and the photon field.

8 Quantization of the Spinor Field

8.1 Anticommunting Numbers

Spinor fields obey canonical anticommutation relations; we must express classical fields by anticommuting numbers to generalize our quantization methods over spinor fields. The basic rule behind such numbers is that they anticommunte, which means that for any two numbers α and β, we have:

$$\alpha\beta = -\beta\alpha. \tag{8.1}$$

To be more precise, the square of such numbers always equates to zero. We can introduce an anticommutation field, a function of space-time consisting of anticommuting values. Barring the case of determinants being in the numerator instead of the denominator, Gaussian integrals over anticommuting variables act just like Gaussian integrals over normal variables.

8.2 The Dirac Propagator

Let's define a field $\phi(x)$ in terms of an arbitrary set of orthonormal basis functions:

$$\phi(x) = \sum_i \phi_i \psi_i(x), \tag{8.2}$$

where the $\psi_i(x)$ and ϕ_i are c-number functions (see appendix) and anticommuting numbers, respectively. To derive the Dirac field, we take the c-number functions as a basis of the four-component spinors. Using the above formalism, we can now evaluate any functional integrals and the correlation functions containing fermions. For example, we can derive the Feynman propagator by taking the inverse in Fourier space for the two-point function (see appendix). The result is as follows:

$$\langle 0| T\phi(x_1)\bar{\phi}(x_2) |0\rangle = S_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x_1-x_2)}}{k - m + i\epsilon}. \tag{8.3}$$

We can use the same technique to calculate higher correlation functions for free Dirac fields.

8.3 Generating Functional for the Dirac Field

Next, we can derive the Feynman Rules for the free Dirac theory using a generating functional. Let's define the functions as follows:

$$Z[\bar{\eta}, \eta] = Z_0 \cdot exp[-\int d^4x d^4y \bar{\eta}(x) S_F(x-y) \eta(y)], \tag{8.4}$$

where Z_0 is the value of the generating functional with external sources set at zero and $\eta(x)$ is an anticommutation source field. In order to deliver correlation functions, we simply differentiate Z with respect to η and $\bar{\eta}$ using the sign convention for anticommuting numbers. The two-point function is now given by:

$$\langle 0| T\phi(x_1)\bar{\phi}(x_2) |0\rangle = Z_0^{-1}(-i\frac{\delta}{\delta\eta(\bar{x_1})})(+i\frac{\delta}{\delta\eta_(x_2)})Z[\bar{\eta}, \eta]|_{\bar{\eta},\eta=0}. \tag{8.5}$$

The above expression is equal to the Feynman Propagator when we plug in equation 8.3, and higher correlation functions follow the same methodology.

8.4 Quantum Electrodynamics

For Quantum Electrodynamics, the full Lagrangian is:

$$\mathcal{L}_{QED} = \mathcal{L}_0 - e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{8.6}$$

where $\mathcal{L}_0$ is equal to $\bar{\psi}(i\partial - m)\psi - \frac{1}{4}(F_{\mu v})^2$. We can then expand the exponential of the interaction term and obtain the correlation functions. The two terms yield the Dirac and electromagnetic propagators, while the interaction term yields the QED vertex: $\mu = -ie\gamma^\mu \int d^4x$. To wrap up the QED Feynman rules for spinors, note that we can permute the rules and perform integrals over the vertex points to obtain the delta functions that conserve momentum. These delta functions can then integrate most of the momentum propagators.

8.5 Functional Determinants

Lastly, let's also observe how we can use Feynmann diagrams to write determinants explicitly. As a base case, consider:

$$\int \mathcal{D}\bar{\phi}\mathcal{D}\phi exp[i\int d^4x\bar{\phi}(i\not{D} - m)\phi], \tag{8.7}$$

where $D_\mu = ieA_\mu$ and $A_\mu(x)$ is an external background field. This expression can be expressed as the functional determinant:

$$\det(i\not{D} - m) = \det(i\not{\partial} - m)\cdot\det(1 - \frac{i(-ie\not{A})}{i\not{\partial} - m}). \tag{8.8}$$

The first time on the right-hand side is an infinite constant. The second term is dependent on the determinant of the external field A. We will now see that this dependence equals the sum of the vacuum diagrams. To do this, note that we can write our determinant as:

$$\det(1 - \frac{i(-ie\not{A})}{i\not{\partial} - m}) = exp[\sum_{n=1}^{\infty} -\frac{1}{n}Tr[\frac{i(-ie\not{A})}{i\not{\partial} - m}^n]]. \tag{8.9}$$

The following identity has been used to re-write the determinant:

$$\det B = exp\,[Tr(\log B)], \tag{8.10}$$

where the matrix B has eigenvalues b. We can also evaluate this determinant by observing equation 8.7 and using the vertex rule (discussed in section 6) to expand the interaction term. This allows us to equate our determinant to the following sum of Feynmann diagrams shown in figure 3.

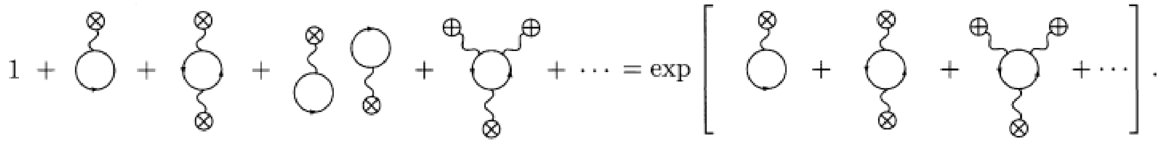


Figure 3. Feynmann Diagrams for Vacuum

The exponential comes from the fact that the diagrams are disconnected but products of connected pieces with appropriate symmetry rules when any given piece is repeated. An illustration is given in figure 4.

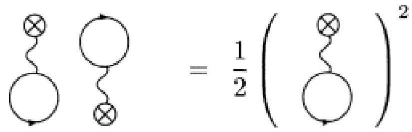


Figure 4. Exponentiation of the diagram due to repeats.

Finally, if we evaluate the Feynmann Diagram to the nth component of figure 3, we get the diagram shown in 5:

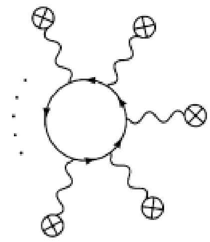


Figure 5. nth diagram in the exponential of figure 3.

Note that there will be a factor of -1 because of the fermion loop and a symmetry of $1/n$ as we could rotate the interactions n times before altering the diagram. Hence our mathematical expression becomes:

$$-\frac{1}{n}\int dx_1 \dots dx_n tr[(-ie\not{A}(x_1))S_F(x_2 - x_1)\dots(-ie\not{A}(x_n))S_F(x_1 - x_n)], \tag{8.11}$$

which is exactly equal to:

$$-\frac{1}{n}Tr[\frac{i(-ie\not{A})}{i\not{\partial} - m}^n]. \tag{8.12}$$

9 Conclusion

In conclusion, we have effectively applied the path integral quantization approach to various quantum fields, such as scalar, electromagnetic, and spinor. This thorough method has allowed us to derive the Feynman rules for both scalar field theory and quantum electrodynamics (QED), presenting a robust and elegant means of calculating correlation functions and examining particle interactions.

The path integral quantization technique provides a unifying view of quantum field theory (QFT), exposing profound connections between classical and quantum physics. Our research highlights the versatility of this framework and its ability to manage diverse systems. Moreover, the derived Feynman rules are invaluable for analyzing scattering amplitudes and particle decay processes, illuminating the fundamental forces that govern subatomic particle behavior.

This project has enriched our understanding of the path integral approach and contributed to the broader quantum field theory research field. The outcomes presented here may set the stage for future investigations into more complex systems, like non-Abelian gauge theories and quantum gravity. Furthermore, the methods used in this work can be applied to challenges in condensed matter physics, cosmology, and other areas, emphasizing the far-reaching impact of our findings.

Our exploration of path integral quantization has led to significant discoveries and enhanced our comprehension of quantum field theories. We have established the groundwork for continued inquiry and innovation in theoretical physics by utilizing this potent formalism.

10 Appendix

10.1 Gaussian Integrals

A Gaussian integral is an integral of the form:

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2}\, dx, \tag{10.1}$$

where $a > 0$ is a constant. The value of a Gaussian integral can be computed using polar coordinates:

$$(I(a))^2 = \int_{-\infty}^{\infty} e^{-ax^2}\, dx \int_{-\infty}^{\infty} e^{-ay^2}\, dy \tag{10.2}$$

$$(I(a))^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\, dx\, dy \tag{10.3}$$

$$(I(a))^2 = \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r \, dr \, d\theta \tag{10.4}$$

$$(I(a))^2 = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\infty} e^{-ar^2} r \, dr \right) = 2\pi \left[-\frac{1}{2a} e^{-ar^2} \right]_0^{\infty} = \frac{\pi}{a}. \tag{10.5}$$

Taking the square root of both sides, we find:

$$I(a) = \sqrt{\frac{\pi}{a}}. \tag{10.6}$$

10.2 Fourier Transform

The Fourier transform of a function $f(x)$ is defined as:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx. \tag{10.7}$$

The inverse Fourier transform can be used to recover the original function from its Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x} \, dk. \tag{10.8}$$

The Fourier transform can be used to simplify certain integrals and differential equations by transforming them to the Fourier space. In particular, it can greatly simplify the computation of path integrals in quantum mechanics.

10.3 Stationary Phase Approximation

The stationary phase approximation is a technique used to approximate integrals of the form:

$$I[f(x)] = \int_{-\infty}^{\infty} g(x) e^{iS(x)} \, dx, \tag{10.9}$$

where $g(x)$ and $S(x)$ are real-valued functions, and $S(x)$ varies rapidly compared to $g(x)$. The stationary phase approximation states that the main contribution to the integral comes from the points where the phase $S(x)$ is stationary, i.e., where its derivative with respect to x vanishes:

$$\frac{dS(x)}{dx} = 0. \tag{10.10}$$

The phase changes slowly at these stationary points, and the contributions from nearby points do not cancel each other out. The stationary phase approximation can compute path integrals in quantum mechanics when the action varies rapidly along the paths. In such cases, the main contributions to the path integral come from the points where the action is stationary, i.e., the classical paths.

To apply the stationary phase approximation to a path integral, follow these steps:

Identify the stationary points of the phase $S(x)$ by solving the equation $\frac{dS(x)}{dx} = 0$.

Expand the phase $S(x)$ in a Taylor series around the stationary points, keeping only the first non-vanishing term, typically the quadratic term:

$$S(x) \approx S(x_0) + \frac{1}{2}S''(x_0)(x - x_0)^2, \tag{10.11}$$

where x_0 is a stationary point, and $S''(x_0)$ is the second derivative of $S(x)$ evaluated at x_0.

Approximate the integral by a Gaussian integral around each stationary point:

$$I[f(x)] \approx \sum_{x_0} g(x_0) \int_{-\infty}^{\infty} e^{i\left[S(x_0)+\frac{1}{2}S''(x_0)(x-x_0)^2\right]}, dx. \tag{10.12}$$

Use the result for Gaussian integrals to evaluate the integral:

$$I[f(x)] \approx \sum_{x_0} g(x_0) e^{iS(x_0)} \sqrt{\frac{2\pi i}{S''(x_0)}}. \tag{10.13}$$

This approximation provides a good estimate of the path integral when the action varies rapidly along the paths, and the contributions from non-stationary paths effectively cancel each other out.

10.4 Faddeev–Popov Ghost Fields

In physics, Faddeev–Popov ghosts are extraneous fields introduced into gauge quantum field theories to sustain the coherency of the path integral formulation because there is over counting of fields when a gauge symmetry is used.

10.5 Photon and Dirac Propogator

The Dirac propagator is given by the equation:

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \tag{10.14}$$

The photon propagator is given by:

$$\frac{ig_{\mu\nu}}{p^2 + i\epsilon}. \tag{10.15}$$

10.6 S-matrix

S-matrices are unitary matrices used in quantum mechanics that relate the initial transition to the final state of a given system. This is done through the absolute values of the squares whose elements are equal to the transition probabilities between different states.

10.7 Two-point function

The two-point function is given as:

$$\langle 0| T\psi(x_1)\bar{\psi}(x_2) |0\rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, exp[i \int d^4x\bar{\psi}(i\not{\partial} - m)\psi]\psi(x_1)\bar{\psi}(x_2)}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, exp[i \int d^4x\bar{\psi}(i\not{\partial} - m)\psi]\psi}. \tag{10.16}$$

10.8 C-Number

"C-number" refers to a "classical" number, which denotes any quantity that isn't a quantum operator applied to elements of a quantum system's Hilbert space. The purpose of this term is to differentiate it from "q-numbers" or "quantum" numbers, which are quantum operators. It is commonly used in quantum field theory to differentiate between classical and quantum elements.

Acknowledgments

We thank Dr Rizwan Khalid for teaching the course on Quantum Field Theory and motivating us to prepare this project report on Path Integral Quantization. We also thank Muhammad Abdullah Mutahar for his oversight and our peers for the feedback.

11 Bibliography

[1] R. P. Feynman, A. R. Hibbs, and George H. Weiss. “Quantum Mechanics and Path Integrals”. In: *Physics Today* 19.6 (June 1966), pp. 89–89. ISSN: 0031-9228. DOI: 10.1063/1.3048320. eprint: https://pubs.aip.org/physicstoday/article-pdf/19/6/89/8265270/89_1_online.pdf. URL: https://doi.org/10.1063/1.3048320.

[2] Jean Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. Oxford University Press, June 2002. ISBN: 9780198509233. DOI: 10.1093/acprof:oso/9780198509233.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780198509233.001.0001.

[3] I.M. Gelfand and S.V. Fomin. *Calculus of Variations*. Dover Books on Mathematics. Dover Publications, 2012. ISBN: 9780486135014. URL: https://books.google.com.pk/books?id=CeC7AQAAQBAJ.

[4] Herbert Goldstein, Charles Poole, and John Safko. *Classical Mechanics*. 3rd ed. Addison Wesley, 2001.

[5] R. Penrose. *The Road to Reality: A Complete Guide to the Laws of the Universe*. Knopf Doubleday Publishing Group, 2007. ISBN: 9780679776314. URL: https://books.google.com.pk/books?id=coahAAAACAAJ.

[6] M. A. Stephens. “Introduction to Kolmogorov (1933) On the Empirical Determination of a Distribution”. In: *Breakthroughs in Statistics: Methodology and Distribution*. Ed. by Samuel Kotz and Norman L. Johnson. New York, NY: Springer New York, 1992, pp. 93–105. ISBN: 978-1-4612-4380-9. DOI: 10.1007/978-1-4612-4380-9_9. URL: https://doi.org/10.1007/978-1-4612-4380-9_9.

[7] L.D. Faddeev and V.N. Popov. “Feynman diagrams for the Yang-Mills field”. In: *Physics Letters B* 25.1 (1967), pp. 29–30. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(67)90067-6. URL: https://www.sciencedirect.com/science/article/pii/0370269367900676.

[8] Michael E. Peskin and Daniel V. Schroeder. “An Introduction to Quantum Field Theory”. In: (1995).