Literature Review on Resource Theory of non-Gaussian Operations

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Supervision

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Abstract

This paper does a literature review of the Resource Theory of Non-Gaussian Operations within the framework of Quantum Resource Theories (QRTs). While Gaussian states and operations are foundational in continuous-variable quantum information processing, they fall short for critical tasks such as universal quantum computation and entanglement distillation, necessitating a focus on non-Gaussian resources. However, QRTs for non-Gaussianity often lack convexity and finite-dimensional characterizations, making their analysis challenging. By defining non-Gaussian monotones and providing upper and lower bounds, this work establishes a robust framework to quantify the non-Gaussianity of quantum operations. The results from the various papers that are reviewed, offer insights into non-Gaussian resource manipulation and pave the way for advancements in quantum computation and communication systems.

1 Introduction

Quantum Resource Theories (QRTs) provide a rigorous framework to classify and quantify quantum resources, offering insights into their role in various quantum information processing tasks. While QRTs have been extensively studied for Gaussian states and operations due to their mathematical simplicity and experimental relevance, their limitations in addressing universal quantum computation and entanglement distillation have led to increasing interest in non-Gaussian resources. Non-Gaussian states and operations, characterized by their deviation from Gaussianity, are critical for overcoming these limitations. However, existing QRTs for non-Gaussianity face significant challenges: they are neither convex nor finite-dimensional, making analytical results and operational characterizations difficult to obtain.

This paper addresses these issues by showcasing the development of a comprehensive Resource Theory for non-Gaussian operations. We begin by reviewing the foundational concepts of Gaussianity and non-Gaussianity in quantum systems, followed by a formal definition of non-Gaussian operations. By identifying free operations as Gaussian operations and exploring their closure under superoperators, we construct a monotone based on the entanglement-assisted generating power of quantum channels. Upper and lower bounds for this monotone are derived, using measures such as distance-based metrics and partial trace techniques. The properties of these monotones are rigorously analyzed to ensure their validity as measures of non-Gaussianity. Finally, we discuss potential applications of our findings, including their relevance to quantum computation and communication systems.

2 Theoretical Background

2.1 Quantum Resource Theory

Quantum Information Resource Theory (QRT) is a foundational framework to quantify and classify the resources necessary for various quantum information processing tasks [1]. At its core, QRT identifies three key components: free states, free operations, and resource states. Free states and operations are easily accessible within the constraints of the system. On the other hand, resource states provide a computational or operational advantage and are typically harder to generate or maintain. The utility of a resource is quantified using monotones—mathematical functions that remain invariant or weaken through free operations. More formally, we define $\mathcal{O}$ as the mapping that assigns to two input/output physical systems A and B, with corresponding Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$, a unique set of Completely Positive Trace Preserving (CPTP) operations $\mathcal{O}(A \to B) \equiv \mathcal{O}(\mathcal{H}^A \to \mathcal{H}^B) \subset \mathcal{Q}(A \to B)$. Then let $\mathcal{F}$ be the induced mapping $\mathcal{F}(\mathcal{H}) := \mathcal{O}(\mathcal{C} \to \mathcal{H})$. Then the tuple $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ is called a Quantum Resource Theory (QRT) if the following two criteria are met[1]:

1. For any physical system A the set $\mathcal{O}(A) := \mathcal{O}(A \to A)$ contains the identity map id^A
2. For any three physical systems, A, B, and C if $\Phi \in \mathcal{O}(A \to B)$ and $\Lambda \in (B \to C)$ the $\Phi \circ \Lambda \in \mathcal{O}(A \to C)$

In QRT convention we let the set $\mathcal{F}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ be the free acting states of $\mathcal{H}$ and the elements $\mathcal{S}(\mathcal{H}) \backslash \mathcal{F}(\mathcal{H})$ be the resource states. Similarly, CPTP maps in $\mathcal{O}(A \to B)$ are classified as free operations while the CPTP maps which do not adhere to $\mathcal{O}(A \to B)$ are called dynamical sources[1].

The first condition in Definition 1 states that the identity map (i.e., not operate) is considered free, a fundamental requirement for any valid QRT. The second condition specifies the composition of two free operations, Λ and Φ, which are also free. This guarantees that operations within ($\mathcal{O}$ remain free, regardless of how many times or in what sequence they are applied. A key implication of this second condition is that free operations cannot transform a state from the set $\mathcal{F}$ into a state outside of $\mathcal{F}$. This is often called the golden rule of QRTs and can be formally expressed as:

For any two physical systems A and B, if $\Phi \in \mathcal{O}(A \to B)$ and $\rho \in \mathcal{F}(A)$, then $\Phi(\rho) \in \mathcal{F}(B)$

2.2 Gaussianity

Gaussian states and Gaussian operations are foundational concepts in continuous-variable quantum information theory. A quantum state is classified as Gaussian if its Wigner function—a quasi-probability distribution in phase space—takes a Gaussian form. For example, coherent states and thermal states. These states are fully characterized by their first and second statistical moments, namely the mean and covariance matrix of the quadrature operators. More formally, a quantum state ρ can be considered Gaussian if its characteristic function has the form:

$$\chi(\zeta) = exp(-\frac{1}{2}\zeta^T(\Omega\Lambda\Omega^T)\zeta - i(\Omega\bar{x})^T\zeta) \tag{1}$$

Where ζ is a vector of $2n$ real numbers, $D(\zeta)$ is the Weyl operator, $\bar{x}$ is the state's mean, $\Omega = i \oplus_{k=1}^{n} \boldsymbol{Y}$ ($\boldsymbol{Y}$ is the Pauli Matrix) and the covariance matrix is defined as

$$\Lambda_{ij} = \frac{1}{2}\left\langle \{x_i - d_i, x_j - d_j\} \right\rangle_\rho \tag{2}$$

Furthermore, Gaussian operations are quantum operations that preserve the Gaussian nature of states[2]. Examples include displacement operations, squeezing, beam splitters, and phase shifters, all of which correspond to linear transformations on the quadrature operators. Mathematically, a Gaussian operation applies a symplectic transformation $\mathcal{S}$ on the covariance matrix Λ, combined with displacement d such that the covariance matrix transforms as $\Lambda' = \mathcal{S}\Lambda\mathcal{S}^T$ and the displacement $d' = \mathcal{S}d + d_0$. In our QRT, Gaussian states will defined as the free states and the Gaussian channels as free operations[3].

2.3 Non-Gaussianity

While Gaussian states play a vital role in many quantum systems, they are insufficient for tasks like universal quantum computation and entanglement distillation where non-Gaussian states must be considered[3]. Non-Gaussian states deviate from the Gaussian formalism by exhibiting non-Gaussian Wigner functions, often possessing higher-order correlations or non-trivial structures beyond the statistical moments described by their mean and covariance matrix. For example photon-subtracted or photon-added states, Fock states, Schrödinger-cat states, and cubic-phase states. Mathematically speaking, a state ρ is non-Gaussian if its characteristic function $\chi(\zeta)$ cannot be expressed in a Gaussian form, i.e:

$$\chi(\zeta) \neq exp(-\frac{1}{2}\zeta^T(\Omega\Lambda\Omega^T)\zeta - i(\Omega\bar{x})^T\zeta) \quad (3)$$

In the context of our resource theory, non-Gaussianity can be treated as a resource. Figure 1 explains the connection between the Gaussian and non-Gaussian states and operations under set theory.

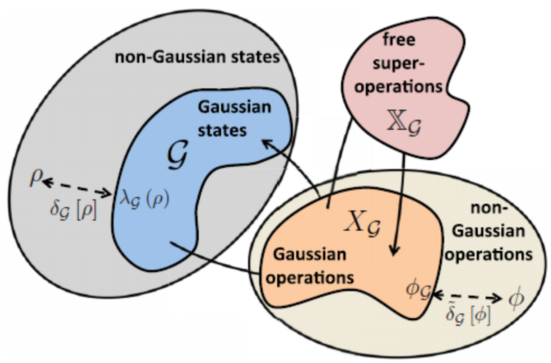

Figure 1: Diagrammatic explanation of the framework for non-Gaussian resource theory. The set of free Gaussian states $\mathcal{G}$ is closed under the set of free operations $X_{\mathcal{G}}$. The free operations are subsequently closed under the set of free super operations $\mathbb{X}_{\mathcal{G}}$. The monotone for the quantum state ρ is $\delta_{\mathcal{G}}[\rho]$[3]

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3 Resource Theory of Non-Gaussian Operations

To re-iterate, we define free operations as Gaussian operations $X_{\mathcal{G}}$. To begin the formulation we need to find a set of superoperators under which $X_{\mathcal{G}}$ is closed. Let's consider

$$\mathbb{X}_{\mathcal{G}} = \{\otimes\phi_{\mathcal{G}}, \circ\phi_{\mathcal{G}}, \phi_{\mathcal{G}}\circ\} \quad (4)$$

where we have tensored with a Gaussian channel ($\otimes\phi_{\mathcal{G}}$), pre-concatenated with a Gaussian channel ($\circ\phi_{\mathcal{G}}$) and post-concatenated with a Gaussian channel ($\phi_{\mathcal{G}}\circ$). The above superoperators map one Gaussian operation to another. But $\mathbb{X}_{\mathcal{G}}$ does not include general probabilistic mixing which can be non-Gaussian[3]. We will also remove $\mathbb{X}_{\mathcal{G}}$ from the action of taking the complement because if non-Gaussian is non-increasing under the complement then it must also be invariant when the complement is taken. Nonetheless, we can construct a channel by swapping the incoming state with a non-Gaussian pure state. The channel would be very clearly non-Gaussian but its complementary channel would be Gaussian i.e., the identity.

This brings us to the vital process of characterizing non-Gaussian operations by finding a monotone. We need a monotone that is non-increasing under the set of free operations $\mathbb{X}_{\mathcal{G}}$. To do so the paper presents a monotone based on entanglement-assisted generating power of quantum channels $\tilde{\delta}_{\mathcal{G}}[\cdot]$. We will then proceed to find a lower-bound $d_{\mathcal{G}}[\cdot]$. We will then also find an upper bound $D_{\mathcal{G}}[\cdot]$ based on the distance measure between quantum operations. We then propose a theorem that for all conditional quantum maps ϕ, $d_{\mathcal{G}}[\phi] \leq \tilde{\delta}[\phi] \leq D_{\mathcal{G}}[\phi]$[3].

3.1 Entanglement-assisted generating power as a monotone

We define that for the input Gaussian state $\rho_{A'} \in \mathcal{G}[n_\phi]$ to conditionally map its purification $\psi_{AA'} \in \mathcal{G}[2n_\phi]$. We posit that the entanglement-assisted Non-Gaussian generating power is as follows:

$$\tilde{\delta}_{\mathcal{G}}[\phi] = \max_{\rho_{A'} \in \mathcal{G}[n_\phi]} \delta_{\mathcal{G}}[(\mathcal{I}_{n_\phi} \otimes \phi)(\psi_{AA'})] \tag{5}$$

Before discussing the properties that allow $\tilde{\delta}_{\mathcal{G}}[\cdot]$ to be monotone for Non-Gaussian, we justify the number of ancilla modes by stating the following lemma: $\tilde{\delta}_{\mathcal{G}}[\cdot]$ is invariant under local isometry on ancilla A and giving ancilla A extra modes[4]. This can be proven using a phase space Schmidt decomposition. For our case, we have chosen an ancilla with the least possible amount of modes. Also note that due to the symmetry of the purification, maximizing over $\rho_{A'}$ is the same as maximization over the pure state $\psi_{AA'}$[3]. This symmetry also allows us to ensure that the pure states are optimum which means for $\mathcal{H}[n+n_\phi]$ with $n \geq n_\phi$ we have:

$$\tilde{\delta}_{\mathcal{G}}[\phi] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}[n+n_\phi]} \delta_{\mathcal{G}}[(\mathcal{I}_{n_\phi} \otimes \phi)(\rho_{\mathcal{G}})]. \tag{6}$$

This implies that non-Gaussianity does not increase when we go into an arbitrary mixed state with an arbitrary amount of modes. Utilizing equations 5 and 6 as well as the lemma that $\tilde{\delta_{\mathcal{G}}}[\cdot]$ is monotonically decreasing under partial trace[5] we can bound $\delta_{\mathcal{G}}[\cdot]$ as:

$$\max_{\rho\mathcal{G} \in \mathcal{G}[n+n_p hi]} \delta_{\mathcal{G}}[(\mathcal{I}_n \otimes \phi)(\rho\mathcal{G})] \leq \max_{\rho\mathcal{G} \in \mathcal{G}[2n+2n_p hi]} \delta_{\mathcal{G}}[(\mathcal{I}_{2n+n_p hi} \otimes \phi)(\psi_{\rho\mathcal{G}})] \tag{7}$$

For the sake of keeping this review concise, we will list the following properties of $\tilde{\delta}_{\mathcal{G}}[\cdot]$ that are derived in reference [3].

1. Non-negativity. $\tilde{\delta}_{\mathcal{G}}[\phi] \geq 0$, with equality if and only if $\phi \in \mathrm{X}_{\mathcal{G}}$.

2. Invariance under tensoring with Gaussian channels. $\forall \phi_{\mathcal{G}} \in \mathrm{X}_{\mathcal{G}}^{L}$ we have $\tilde{\delta}_{\mathcal{G}}[\phi \otimes \phi_{\mathcal{G}}] = \tilde{\delta}_{\mathcal{G}}[\phi]$.

3. Invariance under concatenation with a Gaussian unitary. $\forall U_{\mathcal{G}} \in \mathrm{X}_{\mathcal{G}}^{U}$, $\tilde{\delta}_{\mathcal{G}}[U_{\mathcal{G}} \circ \phi] = \tilde{\delta}_{\mathcal{G}}[\phi \circ U_{\mathcal{G}}] = \tilde{\delta}_{\mathcal{G}}[\phi]$.

4. Monotonically decreasing under concatenation with partial trace.

5. Monotonically increasing under Stinespring dilation with a vacuum environment (only valid for channels and not general operations).

6. Nonincreasing under concatenation with a Gaussian channel.

7. Superaddivity. $\tilde{\delta}_{\mathcal{G}}[\phi_1 \otimes \phi_2] \geq \tilde{\delta}_{\mathcal{G}}[\phi_1] + \tilde{\delta}_{\mathcal{G}}[\phi 2]$.

Due to property 7, one can introduce a regularization such that $\tilde{\delta}_{\mathcal{G}}^{\infty}[\phi^{\otimes 2}] = \tilde{\delta}_{\mathcal{G}}^{\infty}[\phi]$. However, note that, unlike cases in communication capacity, we can regard ϕ and $\phi^{\otimes 2}$ as two different quantum operations making the need for regularization optional.

3.2 Generating power as a lower bound

Let us now trace out the ancilla in equation 6, we can define another function as follows:

$$d_{\mathcal{G}}[\phi] = \max_{\rho\mathcal{G} \in \mathcal{G}_{[n_\phi]}} \tag{8}$$

According to reference [6], this can be used as a measure for the Non-Gaussianity of quantum operations. By taking input as a product state with the ancilla, it becomes easy to see that $\tilde{\delta}_{\mathcal{G}} \equiv d_{\mathcal{G}}[(\mathcal{I}_n \otimes \phi)] \geq d_{\mathcal{G}}[\phi]$ by the monotonically decreasing nature of $\tilde{\delta}_{\mathcal{G}}[\cdot]$ under a partial trace. This proves the first part of the theorem proposed previously. Note that it may be possible to make the inequality strict because the identity function is a Gaussian channel (invariance or non-increasing under tensoring can not be proved with Gaussian channels.) Furthermore, $d_{\mathcal{G}}[\phi] = 0$ can only suggest $\forall \rho_{\mathcal{G}} \in \mathcal{G}, \phi(\rho_{\mathcal{G}}) \in \mathcal{G}$ which does not prove that $\phi \in \mathrm{X}_{\mathcal{G}}$ because a conditional quantum map $\phi \in \mathrm{X}_{\mathcal{G}}$, if and only if $\forall \rho_{\mathcal{G}} \in \mathcal{G}[n_\phi + n], n \in \{0, 1, \dots\}$, we have $(\mathcal{I}_n \otimes \phi)(\rho_{\mathcal{G}}) \in \mathcal{G}$. This means that only properties 3 to 7 are satisfied[3].

3.3 Upper bounds: Distance as a monotone

For this section, we will use a geometric approach to finding the upper bound. Consider conditional maps ϕ_1 and ϕ_2 each with n input modes. We define difference measurement as:

$$D_{\mathcal{G}}(\phi_1, \phi_2) = \max_{\psi_{\mathcal{G}} \in \mathcal{G}[2n]} S[(\mathcal{I}_n \otimes \phi_1)(\psi_{\mathcal{G}}) || (I_n \otimes \phi_2)(\psi_{\mathcal{G}})] \tag{9}$$

Which equates to:

$$D_{\mathcal{G}}(\phi_1, \phi_2) = \max_{\rho_{\mathcal{G}} \in \mathcal{G}[n]} S[(\mathcal{I}_n \otimes \phi_1)(\rho_{\mathcal{G}}) || (I_n \otimes \phi_2)(\rho_{\mathcal{G}})] \tag{10}$$

Equation 9 restricts the state to a pure state and within $\mathcal{G}[2n]$. We can now characterize a measure of Non-Gaussianity by the distance from the closest Gaussian conditional map with the same amount of input modes. Define $D_{\mathcal{G}}[\phi] \equiv min_{\phi_{\mathcal{G}} \in \mathrm{X}_{\mathcal{G}}} D_{\mathcal{G}}(\phi, \phi_{\mathcal{G}})$. To showcase how the second part of the theorem is true we use the following analysis:

$$\begin{aligned} D_{\mathcal{G}}[\phi] &= \min_{\phi_g \in \mathcal{X}_{\mathcal{G}}} \max_{\psi_{\mathcal{G}} \in \mathcal{G}} S[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi_g)(\psi_{\mathcal{G}})] \\ &\geq \max_{\psi_{\mathcal{G}} \in \mathcal{G}} \min_{\phi_g \in \mathcal{X}_{\mathcal{G}}} S[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi_g)(\psi_{\mathcal{G}})] \\ &\geq \max_{\psi_{\mathcal{G}} \in \mathcal{G}} \min_{\rho_{\mathcal{G}} \in \mathcal{G}} S[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}}) \| \rho_{\mathcal{G}}] \\ &= \max_{\psi_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]. \end{aligned}$$

Note that the first inequality arises from the max-min inequality[7] and the second inequality is because of $(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}})(\psi_{\mathcal{G}}) \in \mathcal{G}$. Finally, the last equality can be obtained from the purification definition in subsection 3.1 and the fact that:

$$\delta_{\mathcal{G}}[\rho] = \min_{\rho_{\mathcal{G}} \in \mathcal{G}} S(\rho \| \rho_{\mathcal{G}}) = S(\rho \| \lambda_{\mathcal{G}}(\rho)) = S[\lambda_{\mathcal{G}}(\rho)] - S(\rho), \tag{11}$$

where $S(\rho||\sigma) \equiv Tr[\rho(\log_2 \rho - \log_2 \sigma)]$ is the relative quantum entropy. Using the above formalism it is possible to show that $D_{\mathcal{G}}[\cdot]$ satisfies properties 1 through 6 which allows it to be a measure of Non-Gaussianity for quantum operations[3].

4 Conclusion

The development of a Resource Theory for non-Gaussian operations presented in this paper addresses the inherent challenges of analyzing non-Gaussian resources within a QRT framework. By introducing a monotone based on entanglement-assisted generating power and establishing its upper and lower bounds, we provide a robust mathematical foundation for quantifying non-Gaussianity in quantum operations.

The proposed monotone is shown to satisfy key properties such as invariance under tensoring with Gaussian channels, non-increasing behavior under concatenation with free operations, and monotonicity under partial trace, thus ensuring its applicability across diverse quantum systems. Additionally, the bounds derived using distance-based metrics offer valuable tools for analyzing the non-Gaussianity of quantum maps, paving the way for further theoretical and experimental advancements.

These results have significant implications for quantum information processing. Non-Gaussian resources are indispensable for tasks such as universal quantum computation with a focus on coherence[8], quantum error correction, and optimal cloning[9]. By providing a rigorous framework to quantify and manipulate these resources, this work contributes to the foundational understanding of continuous-variable quantum systems and their applications in future quantum technologies.

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