A Duality Method for Mean-Field Limits

Setting: $d \geq 1$

$$(1) \begin{cases} \dfrac{d}{dt} X_{i,N} = V_{i,N} \\[2ex] \dfrac{d}{dt} V_{i,N} = \dfrac{1}{N-1} \displaystyle\sum_{j \neq i} K(X_{i,N} - X_{j,N}) + \sqrt{2\sigma}\, dB_{i,N} \end{cases}$$

where $X_{i,N}, V_{i,N} \in \mathbb{R}^d$

The Liouville eq corresponds to (1)

$$\begin{cases} \partial_t F_N + \displaystyle\sum_{i=1}^{N} \Big(v_i \cdot \nabla_{x_i} F_N + \frac{1}{N-1} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{v_i} F_N \Big) = \sigma \sum_{i=1}^{N} \Delta_{v_i} F_N \\[2ex] F_N \big|_{t=0} = (f^0)^{\otimes N} \end{cases}$$

The expected limit MFD:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + K * f \cdot \nabla_x f = \sigma \Delta_v f \\ f|_{t=0} = f^0 \end{cases}$$

where $\displaystyle K * f(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x-x') f(t,x',v')\, dx' dv'$

Previously. Jabin & Wang.

$$\frac{d}{dt} H_N(F_N(t) \mid f_t^{\otimes N}) \lesssim C H_N(F_N(t) \mid f_t^{\otimes N}) + \frac{C}{N}$$

$\boxed{\text{(entropy solution)}}$

$$\Rightarrow H_N(F_N \mid f^{\otimes N})(t) \leq e^{Ct}\left(\frac{1}{N} + H_N(0)\right)$$

$$\| $$

$$\frac{1}{N} \int_{(\mathbb{R}^d \times \mathbb{R}^d)^N} F_N(t) \log \frac{F_N(t)}{f_t^{\otimes N}}$$

Key idea:

Use the dual eq. / the backward Kolmogorov eq

$$\Phi_N \in L^\infty([0,T]\times \mathbb{D}^N) \quad \mathbb{D} = \mathbb{R}^d \times \mathbb{R}^d$$ domain

solves the eq.

$$\begin{cases} \partial_t \Phi_N + \sum_{i=1}^N \left(v_i \cdot \nabla_{x_i} \Phi_N + \frac{1}{N-1} \sum_{j\neq i} K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N \right) = -\sigma \sum_{i=1}^N \Delta_{v_i} \Phi_N \\ \Phi_N(z_1, \dots, z_N)\Big|_{t=T} = \frac{1}{\binom{N}{k}} \sum_{1\le i_1 < \dots < i_k \le N} \psi(z_{i_1}) \cdots \psi(z_{i_k}) \end{cases}$$

<u>Prop 3.</u> The propagation of chaos holds

$$\int_{\mathbb{D}^k} \psi^{\otimes k} F_{N,k}(T) \xrightarrow{N\uparrow+\infty} \left(\int_{\mathbb{D}} \psi f(T) \right)^k$$

$$\Longleftrightarrow \quad N \int_0^T \left(\int_{\mathbb{D}^N} V_f(z_1, z_2)\, \Phi_N f^{\otimes N} \right) dt$$

$$\xrightarrow{N \uparrow \infty} 0$$

where

$$v_f(z_1, z_2) := \left(K(x_1 - x_2) - K * f(x_1)\right) \cdot \nabla_{v_1} \log f(z_1)$$

$$\int_0^T \boxed{\left| \int v_f(z_1, z_2)\, \underline{C_{N,2}}\, f^{\otimes 2} \right|} \longrightarrow 0$$

<u>Recall</u>: In relative entropy:

$$\frac{d}{dt} H_N \le -\sigma I_N + \underbrace{\frac{1}{\eta N}\int F_N \frac{1}{N^2} \sum_{i,j} v_f(z_i, z_j)\, \eta N}$$

$$\le \frac{1}{\eta N} H_N + \frac{1}{\eta N} \int f^{\otimes N} \exp\left(\frac{\eta}{N} \sum_{i,j} v_f\right)$$

Pf: $\frac{d}{dt} < \Phi_N, F_N > (t) \equiv 0.$

$$\begin{aligned} \frac{d}{dt} < \Phi_N, F_N > (t) &= < \partial_t \Phi_N, F_N > + < \Phi_N, \partial_t F_N > = 0 \end{aligned}$$

Thus

$$\int_{\mathbb{D}^N} \Phi_N(0) F_N(0) = \underbrace{\int_{\mathbb{D}^N} \Phi_N(T) F_N(T)}$$

$$= \int_{\mathbb{D}^N} \frac{1}{\binom{N}{n}} \sum_{\nu_1 < \cdots < \nu_n} \underbrace{\psi(x_{\nu_1}) \cdots \psi(x_{\nu_n})} F_N(T)$$

$$= \int_{\mathbb{D}^k} \psi^{\otimes n} F_{N,n}(T)$$

i.e. $\int_{\mathbb{D}^N} \Phi_N(0) F_N(0) = \int_{\mathbb{D}^k} \psi^{\otimes n} F_{N,n}(T)$

Now

$$\int_{\mathbb{D}^N} \Phi_N(0) \left(f^0\right)^{\otimes N} = \int_{\mathbb{D}^N} \Phi_N(T)\, f(T)^{\otimes N}$$

$$- \int_0^T \left(\partial_t \int_{\mathbb{D}^N} \Phi_N(t)\, f(t)^{\otimes N} \right) dt \quad \star$$

$$\left(\frac{d}{dt} \langle \Phi_N(t), \quad f(t)^{\otimes N} \rangle \right)$$

$\frac{1}{N-1} \sum_{j \neq v} K(x_v - x_j) \cdot D_{v_v} \Phi_N$

$K \circledast f(x_v) \cdot D_{v_v} f$

gap $\cdot\ V_f$.

Then $\int \psi^{\otimes k} F_{N,k}(T)$

$$\shortparallel$$

$$\int_{\mathbb{D}^N} \Phi_N(0) \underbrace{\left(f^0\right)^{\otimes N}}_{= F_N^0} = \int_{D^k} \psi^{\otimes k} f(T)^{\otimes k}$$

$$- \frac{1}{N-1} \sum_{v=1}^{N} \sum_{j \neq v} \int_0^T \int_{\mathbb{D}^N} \left(V_f(z_v, z_j) \right) \Phi_N f^{\otimes N}$$

$\boxed{\text{gap term}}$

So

$$\int \varphi^{\otimes k} \left(F_{N,k}(T) - f(T)^{\otimes k} \right)$$

$$\overset{\text{symmetry}}{=} \underbrace{-N \int_0^T \int_{\mathbb{D}^N} \left(V_f(z_1, z_2) \right) \Phi_N f^{\otimes N}}_{\downarrow\, 0} .$$

$\boxed{\text{Goal}}$

implies POC.

<u>Preliminary</u>:

Assume $\Phi_N(t, \cdot)$ solves backward Kolmogorov

$f^{\otimes N}$

Define dual marginals $\{M_{N,n}\}_{n=0}^{N}$

$$M_{N,n}(z_1, \cdots, z_n) = \int_{\mathbb{D}^{N-n}} \Phi_N(z_1, \cdots z_N)\, f^{\otimes N-n}(z_{n+1}, \cdots z_N)$$

$dz_{n+1} \cdots dz_N$

$n=0$,

$$M_{N,0} = \int_{\mathbb{D}^N} \Phi_N \, f^{\otimes N} = \mathbb{E}_{(z_1,\dots z_N) \sim f^{\otimes N}} \Phi_N(z_1, \dots z_N)$$

constant

$$M_{N,N} = \Phi_N$$

Cluster expansion:

$$\Phi_N(z_1, \dots z_N) = \sum_{n=0}^{N} \sum_{\sigma \in P_n^N} C_{N,n}(z_\sigma)$$

(symmetric)

choose n items from $\{1, 2, \dots N\}$

$$= C_{N,0} + \sum_{i=1}^{N} C_{N,1}(z_i)$$

$$+ \sum_{1 \le i_1 < i_2 \le N} C_{N,2}(z_{i_1}, z_{i_2})$$

$$+ \dots + \sum_{1 \le i_1 < \dots < i_N \le N} C_{N,N}(z_{i_1}, \dots, z_{i_N})$$

$+ \cdots + C_{N,N}(z_1, \cdots z_N)$

$\Phi_{N}, \longrightarrow \{M_{N,n}\}_{n=0}^{N}$ symmetric.

$M_{N,n}$ is symmetric w.r.t. $z_1, \cdots z_n$.

Hence:

$$M_{N,k}(z_1, \cdots z_k) = \sum_{l=0}^{k} \sum_{I \in P_l^k} C_{N,l}(z_I)$$

↑ a set

choose l indices from $\{1, 2, \ldots k\}$.

$$M_{N,0} = C_{N,0} = \int \Phi_N(z_1, \cdots z_N) f^{\otimes N}$$

$$M_{N,1}(z_1) = \int \Phi_N(z_1, z_2, \cdots z_N) f(z_2) \cdots f(z_N)$$

$$= C_{N,0} + C_{N,1}(z_1) \text{ (cluster expansion)}$$

So $C_{N,1}(z_1) = -M_{N,0} + M_{N,1}(z_1)$

$$\int_{ID} C_{N,1}(z) f(z)\, dz$$

$$= -M_{N,0} + \underbrace{\int_{ID} M_{N,1}(z_1) f(z_1)\, dz_1}_{\substack{\parallel \\ M_{N,0}}} = 0.$$

In general

$$\boxed{\int_{ID} C_{N,n}(z_1, \cdots, z_n) f(z_j)\, dz_j = 0, \quad \forall \ 1 \le j \le n}$$

$\boxed{\text{Some kind of orthogonality.}}$

Now $$N \int_0^T \Big(\underbrace{\int_{D^N} V_f(z_1, z_2) \Phi_N f^{\otimes N}}_{\parallel} \Big)\, dt$$

$$N \int_{ID^2} V_f(z_1, z_2) f(z_1) f(z_2) \underline{\int_{ID^{N-2}} \Phi_N(z_1, z_2, \cdots, z_N) f^{\otimes(N-2)}}$$

$$= N \int_{\mathbb{D}^2} V_f(z_1, z_2)\, f^{\otimes 2}(z_1, z_2)\, M_{N,2}(z_1, z_2)$$

$$= \boxed{N \int_{D^2} V_f\, f^{\otimes 2}\, C_{N,2}.}$$

USE THIS Formulation to conclude.

Why the last "=" is true?

$$M_{N,2}(z_1, z_2) = M_{N,0} + C_{N,1}(z_1) + C_{N,1}(z_2) + C_{N,2}(z_1, z_2)$$

$$\int_{\mathbb{D}^2} \underbrace{V_f(z_1, z_2)} f(z_1) f(z_2)\, M_{N,0}\, dz_1\, dz_2$$

$$= 0$$

$$\because \begin{cases} \int_{\mathbb{D}} V_f(z_1, z_2) f(z_1)\, dz_1 = 0 \\ \int_{\mathbb{D}} V_f(z_1, z_2) f(z_2)\, dz_2 = 0. \end{cases}$$

Recall $V_f(z_1, z_2) = \left(K(x_1 - x_2) - K * f(x_1)\right)$

$\cdot \nabla_{v_1} \log f(z_1)$

$$\int_{\mathbb{D}^2} V_f(z_1, z_2) f^{\otimes 2} C_{N,1}(z_1) \, dz_1 \, dz_2$$

$$= \int_{\mathbb{D}} dz_1 f(z_1) C_{N,1}(z_1) \underbrace{\int_{\mathbb{D}} V_f(z_1, z_2) f(z_2) \, dz_2}_{\overset{\shortparallel}{0}}$$

Later we only need to show:

$$N \int_0^T \left(\int_{\mathbb{D}^2} V_f \, C_{N,2} \, f^{\otimes 2} \right) dt \xrightarrow{N \uparrow +\infty} 0$$

Next step:

Obtain the "BBGKY" for "$C_{N,n}$"

Adjoint method to compute derivative

[20] M. Duerinckx. On the size of chaos via Glauber calculus in the classical mean-field dynamics. CMP 2021.