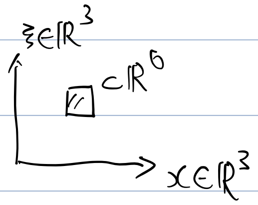
Boltzmann eqn

$f(t,x,\xi)\,\Delta x\,\Delta\xi$: amount of gas located in $x + \Delta x$ with microscopic velocity $\xi + \Delta\xi$



$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)\,d\xi$$

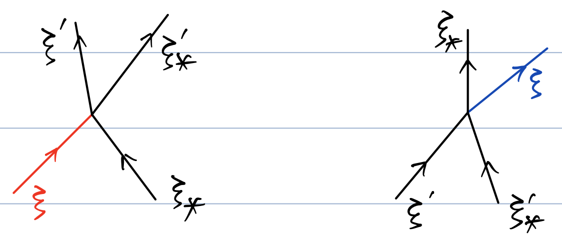
$$\rho v(t,x) = \int_{\mathbb{R}^3} \xi f(t,x,\xi)\,d\xi$$

$$\rho e(t,x) = \int_{\mathbb{R}^3} \frac{1}{2}|\xi - v|^2 f\,d\xi$$

etc.

$$\partial_t f + \xi\cdot\partial_x f = \frac{1}{Kn} Q(f,f) = \text{Gain} - \text{loss due to collision}$$

$$Kn = \frac{\text{M.F.P}}{L}$$



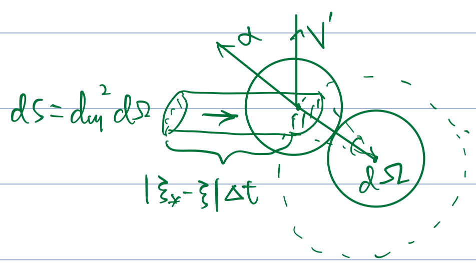
$-f(\xi)f(\xi_*)$ $\qquad$ $+f(\xi')f(\xi'_*)$

Molecular chaos: two particles are independent before going to collide.

$$\xi' = \xi - [(\xi-\xi_*)\cdot\Omega]\,\Omega, \qquad \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\Omega]\,\Omega$$

$$Q(f,f) = \int_{\substack{\mathbb{R}^3\times S^2 \\ (\xi-\xi_*)\cdot\Omega \geq 0}} \left(-f(\xi)f(\xi_*) + f(\xi')f(\xi'_*)\right) B(|\xi-\xi_*|,\Omega)\,d\xi_*\,d\Omega$$

Rmk. For hard sphere, $B(|\xi-\xi_*|,\Omega) = |(\xi-\xi_*)\cdot\Omega|$



$$\frac{|\xi_*-\xi|\Delta t \cdot \frac{|\alpha\cdot V|}{|V|}\cdot d_m^2\,d\Omega}{\Delta t} = |(\xi-\xi_*)\cdot\Omega|\,d_m^2\,d\Omega$$

Formal derivation

$$f^N(t, \underline{T^t z^N}) = f_0^N(z^N)$$

N-particle distribution $f^N(t, x_1,\cdots,x_N, \xi_1,\cdots,\xi_N)$ $\qquad x_i, \xi_i \in \mathbb{R}^3$

$\underline{T_d^t z^N}$

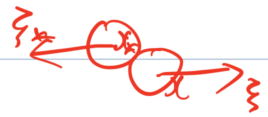
$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^{N} \left(\xi_i \cdot \frac{\partial f^N}{\partial x_i} + \cancel{F_i \cdot \frac{\partial f^N}{\partial \xi_i}} \right) = 0$$ hard sphere dynamics

$f^N(t, x_{\sigma(1)}, \dots, x_{\sigma(N)}, \xi_{\sigma(1)}, \dots, \xi_{\sigma(N)})$ $\sigma \in S_N$: permutation

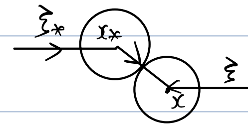
$$f^{(s)}(t, x_1, \dots, x_s, \xi_1, \dots, \xi_s) := \int f^N(t, x_1, \dots, x_N, \xi_1, \dots, \xi_N)\, dz^{N-s}$$ $|x_i - x_j| \geq d$ $\underline{i \neq j}$

$$z^{N-s} = (x_{s+1}, \dots, x_N, \xi_{s+1}, \dots, \xi_N)$$

$$\frac{\partial f^{(1)}}{\partial t} + \xi \cdot \frac{\partial f^{(1)}}{\partial x} = (N-1) d_m^2 \left(\int_{\substack{\mathbb{R}^3 \times S^2 \\ (\xi - \xi_*)\cdot(x - x_*) > 0}} f^{(2)}(\xi, \xi_*) \,|(\xi - \xi_*)\cdot \Omega|\, d\xi_* d\Omega - \int_{\substack{\mathbb{R}^3 \times S^2 \\ (\xi - \xi_*)\cdot(x - x_*) < 0}} \underbrace{f^{(2)}(\xi, \xi_*)}_{f^{(1)}(\xi)} \,|(\xi - \xi_*)\cdot \Omega|\, d\xi_* d\Omega \right)$$



$$f^{(2)}(t^+, \xi, \xi_*) = f^{(2)}(t^-, \xi', \xi_*')$$



Replace post-collision by pre-collision + molecular chaos.

Rigorous derivation

BBGKY for hard sphere with diameter σ.

$$\frac{d}{dt} P^{(s)}(T_\sigma^t z^s, t) = Q^\sigma_{s+1} P^{(s+1)}(T_\sigma^t z^s, t)$$ weak form

Hamiltonian dynamics for hard sphere

$$Q^\sigma_{s+1} P^{(s+1)}(t, x_1, \dots, x_s, \xi_1, \dots, \xi_s)$$
$$= \sum_{j=1}^{s} (N-s)\sigma^2 \int d\xi_{s+1} \int_{n\cdot(\xi_j - \xi_{s+1}) \geq 0} dn\; n\cdot(\xi_j - \xi_{s+1}) \Big\{ P^{(s+1)}(t, x_1, \dots, x_j, \dots, x_j - \sigma n, \xi_1, \dots, \xi_j', \dots, \xi_{s+1}') - P^{(s+1)}(t, x_1, \dots, x_j, \dots, x_j + \sigma n, \xi_1, \dots, \xi_j, \dots, \xi_{s+1}) \Big\}$$

Boltzmann-Grad limit : $N\sigma^2 = \alpha$ $Q^\sigma \to Q$

$$\text{MFP} \sim \frac{L^3}{N\sigma^2}$$ $P_0(z^N) \lesssim e^{-\sum |\xi_i|^2}$

$$P^{(s)}(T_\sigma^t z^s, t) = P_0^{(s)}(z^s) + \int_0^t dt_1 \left(Q^\sigma_{s+1} P^{(s+1)}\right)(T_\sigma^{t_1} z^s, t)$$ $\left| P^{(s)} e^{\beta |\xi_s|^2} \right|_\infty \leq \underline{P_0^{(1)} e^{2\beta|\xi|^2}}$

$$P^{(s)}(z^s, t) = \left(S_\sigma(t) P_0^{(s)}\right)(z^s) + \int_0^t dt_1\, S_\sigma(t - t_1) Q^\sigma_{s+1} P^{(s+1)}(z^s, t_1),$$ $S_\sigma(t) f(z^s) := f(T_\sigma^{-t} z^s)$

$$f^{(s)} = \dots = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, S_\sigma(t - t_1) Q^\sigma_{s+1} S_\sigma(t_1 - t_2) \cdots Q^\sigma_{s+n} S_\sigma(t_n) P_0(z^s)$$

$$P^{(N)}(z, t) = S_\sigma(t) P_0^{(N)}(s)$$ $P^{(s)}(z^s, t) \leq e^{-\beta' E_s} (Cb)^s \sum_{n \geq 0} t^n (C\alpha A b)^n$

Prove convergence (Cauchy-Kovalevskaya methodology) $0 < t < t_0 \ll 1$

• $P^{(s)} \longrightarrow f^{(s)}$ a.e. $\quad f^{(s)}(z^s,t) = \prod_{i=1}^{s} f(t,x_i,\xi_i) \longrightarrow$ sol to BE.

under certain condition.

Cf. Lanford 1975

Cercignani–Illner–Pulvirenti 1994 mathematical theory of dilute

Saint-Raymond

$\ln f \sim C + \vec{a}\cdot\vec{v} + |\vec{v}|^2$

Quantum Boltzmann eqn

$f(v')f(v_*') = f(v)f(v_*)$

$$\left[f_{k_1}(s) f_{k_2}(s) \tilde{f}_{k_3}(s) \tilde{f}_{k_4}(s) - f_{k_4}(s) f_{k_3}(s) \tilde{f}_{k_2}(s) \tilde{f}_{k_1}(s)\right]$$

$$\partial_t f + \frac{p}{P}\cdot\nabla_x f = \int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)\Big(f(v')f(v_*')(1+\lambda h^3 f(v))(1+\lambda h^3 f(v_*))$$
$$- f(v)f(v_*)(1+\lambda h^3 f(v'))(1+\lambda h^3 f(v_*'))\Big)\, d\omega\, dv_*$$

$\lambda = \{0, -1, 1\}$ $\qquad$ h: Planck constant

classical, Fermion, Boson

$v' = v - ((v-v_*)\cdot\omega)\omega, \qquad v_*' = v_* + ((v-v_*)\cdot\omega)\omega$

$$\frac{f'}{1+\lambda h^3 f'}\,\frac{f_*'}{1+\lambda h^3 f_*'} = \frac{f}{1+\lambda h^3 f}\,\frac{f_*}{1+\lambda h^3 f_*} \Rightarrow \ln\left(\frac{f}{1+\lambda h^3 f}\right) = \beta(\mu - E) + C \;= C + \vec{a}\cdot\vec{v} + |\vec{v}|^2$$

(No $\vec{v}$, $[\vec{P}, H] \neq 0$)

$$\Rightarrow f \propto \frac{1}{e^{\beta(E-\mu)} - \lambda}$$

$\lambda = 1$ Boson–Einstein

$\lambda = -1$ Fermi–Dirac

From Schrödinger to QBE.

$\frac{p^2}{2m}$

$V \sim ML^2T^{-2}$

NO rigorous derivation so far.

Two formal approaches

— Benedetto–Castella–Esposito–Pulvirenti (JSP '04, M³AS '05, CMP '08)

1° many body Schrödinger

$$i\partial_t \Psi(X_N,t) = -\frac{1}{2}\Delta_N \Psi(X_N,t) + U(X_N)\Psi(X_N,t)$$

$$X_N = (x_1,\dots,x_N), \qquad U(X_N) = \sum_{i<j}\phi(x_i - x_j)$$

$i\hbar\partial_t \quad \frac{\hbar^2\Delta}{m} \quad V$

$[\hbar] = M\cdot LT^{-2} L\, T = ML^2T^{-1}$

ML^2T^{-1}

Weak coupling scaling: $x = \frac{\tilde{x}}{\varepsilon} \quad t = \frac{\tilde{t}}{\varepsilon}, \quad \phi(x) = \frac{\tilde{\phi}(x)}{\sqrt{\varepsilon}}$

$$i\varepsilon\partial_{\tilde{t}}\Psi^\varepsilon(\tilde{X}_N,\tilde{t}) = -\frac{1}{2}\varepsilon^2\Delta_N\Psi^\varepsilon(\tilde{X}_N,\tilde{t}) + U_\varepsilon(\tilde{X}_N)\Psi^\varepsilon(\tilde{X}_N,\tilde{t})$$

drop out "~" $\quad U_\varepsilon(\tilde{X}_N) = \sum_{i<j} \sqrt{\varepsilon}\, \tilde{\phi}\left(\frac{\tilde{x}_i - \tilde{x}_j}{\varepsilon}\right)$

2°. Wigner transform

$$W^N(X_N, V_N) := \left(\frac{1}{2\pi}\right)^{3N} \int dY_N\, e^{iY_N \cdot V_N}\, \overline{\Psi^\varepsilon}(X_N + \tfrac{\varepsilon}{2} Y_N)\, \Psi^\varepsilon(X_N - \tfrac{\varepsilon}{2} Y_N)$$

$$\int W^N(X_N, V_N)\, dV_N = \rho$$

3°. Partial trace

$$f_j^N(X_j, V_j) := \int dx_{j+1} \cdots dx_N \int dv_{j+1} \cdots dv_N\, W^N(X_j, x_{j+1}, \dots, x_N; V_j, v_{j+1}, \dots, v_N)$$

$$f_N^N = W^N$$

4°. BBGKY for f_j^N $(j = 1, 2, \dots, N)$

Try Cauchy-Kovalevskaya, CANNOT prove $\begin{cases} \text{uniform bound for series.} \\ \text{convergence term by term.} \end{cases}$ ('08 ok?)

— Erdös-Salmhofer-Yau JSP '04

Field-theoretical, analyze correlation function.

Set up: Fermion $\quad L^d$

Configuration space: $\Lambda = \mathbb{Z}^d/(L\mathbb{Z})^d \quad L \in \mathbb{N},\ L \gg 1$

Fock space: $\mathcal{F}_\Lambda := \bigoplus_{n \geq 0} \underbrace{\Lambda^n \mathcal{H}_\Lambda}_{\text{exterior product}}, \quad \underbrace{\mathcal{H}_\Lambda := \ell^2(\Lambda, \mathbb{C})}_{\text{one particle state space}}$

$\Lambda^* := \left(\frac{2\pi}{L}\mathbb{Z}\right)^d / (2\pi\mathbb{Z})^d, \quad 2\pi \frac{l}{L}, \quad l = 0 \dots L-1$

$\{w_p(x) = e^{ip\cdot x},\ p \in \Lambda^*\}$ orthogonal basis of $\mathcal{H}_\Lambda$.

$$\langle w_p | w_q \rangle := \int_\Lambda \overline{w_p}\, w_q\, dx = \sum_{x \in \Lambda} e^{i(q-p)\cdot x}$$

$$= \left(\sum_{x_i \in \mathbb{Z}/L\mathbb{Z}} e^{i(q_i - p_i) x_i}\right)^d = \begin{cases} L^d, & p = q \in \Lambda^* \\ 0, & p \neq q \end{cases}$$

$$\sum_{j=0}^{L-1} e^{ikj} = \frac{1 - e^{ikL}}{1 - e^{ik}} \quad \left(k = \frac{2\pi l}{L}\right) = \begin{cases} L & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Second quantization

Fermion $\psi_F(x_1, \cdots, x_N)$ antisymmetric

quantum particles are indistinguishable! $\psi_B(\cdots)$ symmetry

If insist on using wave function, Need impose symmetries!

occupation number representation $|n_1, n_2 \cdots n_k \cdots \rangle$

$\rightsquigarrow$

Annihilation / creation op.

$a_p^\dagger$: $\mathcal{F}_\Lambda \to \mathcal{F}_\Lambda$ create a particle in momentum state $|p\rangle$

a_p: " annihilate " " $|p\rangle$.

Fermion satisfies Canonical anticommutation relation (CAR)

$\{a_p, a_q\} = \{a_p^\dagger, a_q^\dagger\} = 0$ $\qquad \{A,B\} \equiv AB+BA$

$$\{a_p, a_q^\dagger\} = \delta(p,q) = \begin{cases} L^d, & p=q \\ 0, & \text{else} \end{cases}$$

$2\pi \frac{l}{L}$ $\quad l = 0 \ldots L-1$

If $F \in C(\mathcal{B})$
$\quad \mathcal{B} = \mathbb{R}^d/(2\pi\mathbb{Z})^d$

$$L^{-d} \sum_{p \in \Lambda^*} F(p) \longrightarrow \int_{\mathcal{B}} \frac{d^d p}{(2\pi)^d} F(p). \quad (L \to \infty)$$

$\Lambda^* = (\frac{2\pi}{L}\mathbb{Z})^d/(2\pi\mathbb{Z})^d$

$\mathcal{H} = \ell^2(\Lambda, \mathbb{C}^2)$

$\Lambda \to \mathbb{C}$

$$\int_{\Lambda^*} dp\, F(p) \equiv L^{-d} \sum_{p \in \Lambda^*} F(p)$$

$\mathcal{A} \equiv$ C^* algebra generated by $\{a_p^\dagger, a_p, p \in \Lambda^*\}$.

$H \in \mathcal{A}$, H self-adjoint, (Hamiltonian)

for $A \in \mathcal{A}$, $A_t \equiv e^{-itH} A e^{+itH}$ Heisenberg picture

$\Big[$ $i \frac{d}{dt} |\psi\rangle = H|\psi\rangle$ Schrödinger $\qquad i\partial_t A_t = [H, A_t]$

$$\langle \psi(t) | A | \psi(t) \rangle = \langle \psi(0) | \underbrace{e^{itH} A e^{-itH}}_{\text{Heisenberg}} | \psi(0) \rangle \quad \Big]$$

$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$

A state ρ on the algebra $\mathcal{A}$.

$$\rho(A) = \mathrm{Tr}(\hat{\rho} A)$$ $\hookrightarrow$ density operator.

$$\rho_t(A) := \rho(A_t)$$

$$i\frac{\partial}{\partial t}\rho_t(A) = i\frac{\partial}{\partial t}\rho(A_t) = \rho(i\partial_t A_t) = \rho([H, A_t]) = \rho_t([H,A])$$

Hamiltonian: $\quad H := \underbrace{H_0}_{\text{free}} + \underbrace{\lambda \Phi}_{\text{interaction}}$ $\quad \lambda \sim \sqrt{\varepsilon}$

$$H_0 := \int dp\, e(p)\, a_p^\dagger a_p \qquad \text{(Spectral decomposition)}$$

$e(p)$ $\downarrow$ energy for state $|p\rangle$

$\Big[$ $H_0|q\rangle = H_0 a_q^\dagger |0\rangle = \int dp\, e(p)\, a_p^\dagger a_p a_q^\dagger |0\rangle$

$$= \int dp\, e(p)\, a_p^\dagger (\{a_p, a_q^\dagger\} - a_q^\dagger a_p)|0\rangle = \int dp\, e(p)\, a_p^\dagger \delta(p,q)|0\rangle = e(q)|q\rangle \Big]$$

$\hat{\Phi}$: two particle interaction $V(x-y)$

$$\hat{\Phi} = \int dx\,dy\, \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(y) V(x,y) \hat{\psi}(y)\hat{\psi}(x)$$

$\Big[$ $|\alpha\rangle$, $\quad \langle x|\alpha\rangle =: \psi_\alpha(x) \qquad \langle p|\alpha\rangle = \int \langle p|x\rangle\langle x|\alpha\rangle = \int e^{-ip\cdot x}\psi_\alpha(x)$

$$\langle x|\alpha\rangle = \int \langle x|p\rangle\langle p|\alpha\rangle = \int e^{ip\cdot x}\psi_\alpha(p)$$

$$\hat{\psi}(x) = \int e^{ip\cdot x} a_p \Big]$$

$$= \int dx\,dy \int_{k_1\cdots k_4} e^{-ik_1 x} a_{k_1}^\dagger\, e^{-ik_2 y} a_{k_2}^\dagger\, V(x,y)\, e^{ik_3 y} a_{k_3}\, e^{ik_4 x} a_{k_4}$$

$$= \int dk_1\cdots dk_4 \left(\int dx\,dy\, V(x,y)\, e^{i(k_3-k_2)y + i(k_4-k_1)x}\right) a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$$

$V(x,y) = V(x-y)$

$$\Rightarrow \int dx\,dy\, V(x-y)\, e^{i(k_3-k_2)y + i(k_4-k_1)x}$$

$$= \int dz\,dy\, V(z)\, e^{i(k_3-k_2)y + i(k_4-k_1)(y+z)}$$

$$= \hat{V}(k_1-k_4)\cdot\delta(k_3-k_2,\, k_1-k_4) = \hat{V}(k_1-k_4)\,\delta(k_1+k_2,\, k_3+k_4)$$

$k_3 \leftrightarrow k_4, \quad k_1 \leftrightarrow k_2$

$$\hat{V}(k_2 - k_3)\, \delta(k_1+k_2, k_3+k_4)$$

$k_1 \leftrightarrow k_2 \qquad -\hat{V}(k_2 - k_4)\, \delta(k_1+k_2, k_3+k_4)$

$k_3 \leftrightarrow k_4 \qquad -\hat{V}(k_1 - k_3)\, \delta(k_1+k_2, k_3+k_4)$

$$\hat{\Phi} = \int dk_1 \cdots dk_4 \underbrace{\langle k_1 k_2 | \Phi | k_3 k_4 \rangle}_{= \delta(k_1+k_2, k_3+k_4) \frac{1}{4}\left[\hat{V}(k_1-k_4) - \hat{V}(k_2-k_4) - \hat{V}(k_1-k_3) + \hat{V}(k_2-k_3)\right]} a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$

$$\hat{\Phi}^\dagger = \int dk_1 \cdots dk_4 \overline{\langle k_1 k_2 | \Phi | k_3 k_4 \rangle}\, a^\dagger_{k_4} a^\dagger_{k_3} a_{k_2} a_{k_1}$$

$$= \int dk_1 \cdots dk_4 \overline{\langle k_4 k_3 | \Phi | k_2 k_1 \rangle}\, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$

$\hat{\Phi}^\dagger = \hat{\Phi} \implies \langle k_4 k_3 | \Phi | k_2 k_1 \rangle = \overline{\langle k_1 k_2 | \Phi | k_3 k_4 \rangle} \implies V$ is real

also assume $V(-x) = V(x)$

$$H_0 = \int dp\, e(p)\, a^\dagger_p a_p$$

operator F is quartic if

$$F = \int dk_1 \cdots dk_4 \langle k_1 k_2 | F | k_3 k_4 \rangle a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$

Two point (correlation) function in Fourier space

$$\gamma_{pq}(t) := \rho_t(a^\dagger_p a_q) \qquad \rho_t(A) := \mathrm{Tr}\left(\hat{\rho}\, e^{iHt} a^\dagger_p a_q e^{-iHt}\right)$$

$$i\partial_t \rho_t(A) = \rho_t([H, A])$$

$$i\partial_t \gamma_{pq}(t) = i\partial_t \rho_t(a^\dagger_p a_q) = \rho_t\left([H_0 + \lambda\Phi, a^\dagger_p a_q]\right)$$

$$[H_0, a^\dagger_p a_q] = a^\dagger_p [H_0, a_q] + [H_0, a^\dagger_p] a_q$$

$$= a^\dagger_p \int dk\, e(k)\, [a^\dagger_k a_k, a_q] + \int dk\, e(k)\, [a^\dagger_k a_k, a^\dagger_p]\, a_q$$

$$\Big[\; a^\dagger_k a_k a_q - a_q a^\dagger_k a_k = -a^\dagger_k a_q a_k - a_q a^\dagger_k a_k = -\{a_q, a^\dagger_k\} a_k = -\delta(q,k) a_k$$

$$a^\dagger_k a_k a^\dagger_p - a^\dagger_p a^\dagger_k a_k = a^\dagger_k a_k a^\dagger_p + a^\dagger_k a^\dagger_p a_k = a^\dagger_k \{a_k, a^\dagger_p\} = a^\dagger_k \delta(k,p) \;\Big]$$

$$= -e(q)\, a_p^\dagger a_q + e(p)\, a_p^\dagger a_q$$

$$\partial_t \left([H_0, a_p^\dagger a_q] \right) = \left[-e(q) + e(p) \right] \nu_{pq}(t)$$

$$[\Phi, a_p^\dagger a_q] = \int dk_1 \cdots dk_4 \, \langle k_1 k_2 | \Phi | k_3 k_4 \rangle \, [a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}, a_p^\dagger a_q]$$

$$\underbrace{a_p^\dagger \, [a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}, a_q]}_{\mathrm{I}} + \underbrace{[a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}, a_p^\dagger]\, a_q}_{\mathrm{II}}$$

$$\mathrm{I} = a_p^\dagger \left(a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} a_q - a_q a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \right)$$

$$= a_p^\dagger \left(a_{k_1}^\dagger a_{k_2}^\dagger a_q a_{k_3} a_{k_4} - a_q a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \right)$$

$$= a_p^\dagger \left(a_{k_1}^\dagger \{a_q, a_{k_2}^\dagger\} a_{k_3} a_{k_4} - \{a_q, a_{k_1}^\dagger\} a_{k_2}^\dagger a_{k_3} a_{k_4} \right)$$

$$= \delta(q, k_2)\, a_p^\dagger a_{k_1}^\dagger a_{k_3} a_{k_4} - \delta(q, k_1)\, a_p^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$$

$$\mathrm{II} = \left(a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} a_p^\dagger - a_p^\dagger a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \right) a_q$$

$$= \left(a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} a_p^\dagger - a_{k_1}^\dagger a_{k_2}^\dagger \{a_p^\dagger, a_{k_3}\} a_{k_4} + a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_p^\dagger a_{k_4} \right) a_q$$

$$= \left(a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} \{a_{k_4}, a_p^\dagger\} - a_{k_1}^\dagger a_{k_2}^\dagger \{a_p^\dagger, a_{k_3}\} a_{k_4} \right) a_q$$

$$= \delta(k_4, p)\, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_q - \delta(k_3, p)\, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_4} a_q$$

$$[\Phi, a_p^\dagger a_q] = \int dk_1 \cdots dk_4 \, \langle k_1 k_2 | \Phi | k_3 k_4 \rangle$$
$$\Big[\delta(q, k_2)\, a_p^\dagger a_{k_1}^\dagger a_{k_3} a_{k_4} - \delta(q, k_1)\, a_p^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$$
$$+ \delta(k_4, p)\, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_q - \delta(k_3, p)\, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_4} a_q \Big]$$

$$= \int dk_{1,3,4} \, \langle k_1 q | \Phi | k_3 k_4 \rangle \, a_p^\dagger a_{k_1}^\dagger a_{k_3} a_{k_4}$$
$$- \int dk_{2,3,4} \, \langle q k_2 | \Phi | k_3 k_4 \rangle \, a_p^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}$$
$$+ \int dk_{1,2,3} \, \langle k_1 k_2 | \Phi | k_3 p \rangle \, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_q$$

$$- \int dk_{124} \, \langle k_1 k_2 | \Phi | p k_4 \rangle \, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_4} a_q$$

$$= \int dk_1 \cdots dk_4 \Big[\langle k_1 q | \Phi | k_3 k_4 \rangle \, \delta(p, k_2) \, (- a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4})$$
$$- \langle q k_2 | \Phi | k_3 k_4 \rangle \, \delta(p, k_1) \, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$
$$+ \langle k_1 k_2 | \Phi | k_3 p \rangle \, \delta(q, k_4) \, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$
$$- \langle k_1 k_2 | \Phi | p k_4 \rangle \, \delta(q, k_3) \, (-1) \, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} \Big]$$

$a^\dagger_{k_1} a^\dagger_{k_2} a_{k_4} a_q$

$$= \int dk_1 \cdots dk_4 \, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} \underbrace{\Big[- \delta(p, k_2) \langle k_1 q | \Phi | k_3 k_4 \rangle + \delta(p, k_1) \langle k_2 q | \Phi | k_3 k_4 \rangle - \delta(q, k_4) \langle k_1 k_2 | \Phi | p k_3 \rangle + \delta(q, k_3) \langle k_1 k_2 | \Phi | p k_4 \rangle \Big]}_{\langle k_1 k_2 | F_{pq} | k_3 k_4 \rangle}$$

$$=: F_{pq} \qquad \underline{\text{quartic}}$$

$$F_{pq} = [\Phi, \, a^\dagger_p a_q]$$

$$\Rightarrow \; [i\partial_t \underbrace{- e(p) + e(q)}_{[H_0, a^\dagger_p a_q]}] \, \nu_{pq}(t) = \lambda \, S_t(F_{pq})$$

$$\nu_{pq}(t) = \nu_{pq}(0) \, e^{-it(e(p) - e(q))} - i\lambda \int_0^t ds \, e^{-i(t-s)(e(p)-e(q))} \, S_s(F_{pq})$$

$$S_t(F_{pq}) = \int dk_1 \cdots dk_4 \, \langle k_1 k_2 | F | k_3 k_4 \rangle \, \underbrace{S_t(a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4})}_{\varphi_{k_1 \cdots k_4}(t)}$$

$$a^\dagger_{k_1} \cdots a^\dagger_{k_m} a_{l_1} \cdots a_{l_m}$$

$$(i\partial_t - \Delta e(r_1 \cdots r_4)) \, \varphi_{r_1 \cdots r_4}(t) = \lambda \, S_t([\Phi, \, a^\dagger_{r_1} a^\dagger_{r_2} a_{r_3} a_{r_4}])$$

$$\Delta e(r_1, \ldots, r_4) = e(r_1) + e(r_2) - e(r_3) - e(r_4)$$

$$\varphi_{r_1 \cdots r_4}(t) = \varphi_{r_1 \cdots r_4}(0) \, e^{-it \Delta e(r_1 \cdots, r_4)}$$
$$- i\lambda \int_0^t ds \, e^{-i(t-s) \Delta e(r_1 \cdots r_4)} \, S_s([\Phi, a^\dagger_{r_1} a^\dagger_{r_2} a_{r_3} a_{r_4}])$$

$$S_t(F_{pq}) = S_0(F_{pq}) \qquad \cdots$$

$\cdots$

$$(i\partial_t - e(p) + e(q))\, \nu_{pq}(t) = \lambda \rho_0(\underbrace{F_{pq}}_{4\text{-pt}}) - i\lambda^2 \int_0^t ds\, \rho_s(\underbrace{[\Phi, G_{pq}(t-s)]}_{8\text{-pt}}) \qquad (*)$$

$F_{pq} = [\Phi, a_p^\dagger a_q]$ quartic

$G_{pq}(t-s)$ quartic

$\langle k_1 k_2 | G_{pq} | k_3 k_4 \rangle = e^{-i(t-s)\Delta e(k_1 \cdots k_4)}$ . . .

(18)

$$\partial_{J_i}\partial_{J_j} C \cdot e^{-J^T H^{-1} J} \quad (H^{-1})_{ij}$$

$\|$

So far rigorous & exact.

$$\mathrm{Tr}(\hat{\rho} A) =: \rho(A)$$

$$\partial_{J_i}\partial_{J_j} \cdots \int e^{-x^T H x + J x}\, dx$$

$\|$

$$\int e^{-x^T H x}\, x_i x_j \cdots x_{i_n}\, dx$$

Key assumption: ρ_s is quasi-free state

$$\rho_s(\underbrace{a^\dagger a^\dagger \cdots a^\dagger}_{l}\ \underbrace{a \cdots a}_{l}) = \text{product of } \rho_s(a^\dagger a)$$

Rmk. $L < \infty \Rightarrow A$ finite-dim

ρ is quasi-free if $\rho(A) = \frac{1}{Z} \mathrm{tr}(e^{-H_0} A)$

H_0 is quadratic $\sim$ free Hamiltonian

? If space homogeneous

$$\nu_{pq}(t) = \delta(p,q) f_p(t)$$

$(*) \rightsquigarrow$

$$\partial_t f_p(t) = -\lambda^2 \int_0^t ds \int dk_1 \cdots dk_4\, \delta(k_1+k_2, k_3+k_4)\, e^{-i(t-s)\Delta e(k_1 \cdots k_4)}$$

$$\times\ 2(\delta(k_4, p) - \delta(k_1, p))\, |\hat{v}(k_1-k_4) - \hat{v}(k_1-k_3)|^2$$

$$[f_{k_1}(s) f_{k_2}(s) \tilde{f}_{k_3}(s) \tilde{f}_{k_4}(s) - f_{k_4}(s) f_{k_3}(s) \tilde{f}_{k_2}(s) \tilde{f}_{k_1}(s)]$$

$\tilde{f}_p = 1 - f_p$

$\lambda \sim \sqrt{\varepsilon}$

$$-\lambda^{-2}\partial_t f_p(t) = \int_{-\infty}^{+\infty} dE \int_0^t ds \; e^{-iE(t-s)} \quad \beta(E,p,s)$$ (17)

$$\beta(E,p,s) = \int dk_1 \cdots dk_4 \; \delta(k_1+k_2, k_3+k_4) \; 2\left(\delta(k_4,p) - \delta(k_1,p)\right)$$
$$\left|\hat{V}(k_1-k_4) - \hat{V}(k_1-k_3)\right|^2 \; \delta(E - \Delta e(k_1 \cdots k_4))$$
$$\times\left[f^\lambda_{k_1}(s) f^\lambda_{k_2}(s) \tilde{f}^\lambda_{k_3}(s) \tilde{f}_{k_4}(s) - f_{k_4}(s) f_{k_3}(s) \tilde{f}_{k_2}(s) \tilde{f}_{k_1}(s)\right]$$

$$\frac{x}{\varepsilon} \quad \frac{t}{\varepsilon} \qquad \phi \to \sqrt{\varepsilon}\,\phi$$

f, β depends on λ $\qquad H = H_0 + \lambda \Phi$

Assume

$$\lim_{\lambda\to 0} f^\lambda_p\left(\frac{T}{\lambda^2}\right) = F(T,p), \quad \lim_{\lambda\to 0} \beta^\lambda\left(E,p,\frac{T}{\lambda^2}\right) = B(E,p,T) \quad \text{exists.}$$

$$(17) \Rightarrow \quad -\partial_T F(T,p) = \lim_{\lambda\to 0} \int_{-\infty}^{+\infty} dE \int_0^T \frac{ds}{\lambda^2} \, e^{-iE(T-S)/\lambda^2} \beta^\lambda\left(E,p,\frac{S}{\lambda^2}\right)$$

$$= \lim_{\lambda\to 0} \int_{-\infty}^{+\infty} dE \int_0^T \frac{ds}{\lambda^2} \, e^{-iE(T-S)/\lambda^2} \quad B(E,p,S)$$

$$= \lim_{\lambda\to 0} \int_0^T \frac{ds}{\lambda^2} \, \hat{B}\left(\frac{T-S}{\lambda^2}, p, S\right)$$

$$\frac{T-S}{\lambda^2} = u \qquad = \lim_{\lambda\to 0} \int_0^{+\infty} du \, \hat{B}(u,p, T-\lambda^2 u)$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \hat{B}(u,p,T) = B(0,p,T)$$

$$\left[\begin{array}{ll} \delta(E - \Delta e(k_1 \cdots k_4)) & \text{energy conservation} \\ \delta(k_1+k_2, k_3+k_4) & \text{momentum conservation.} \end{array}\right.$$