Modulated LSI and generation of chaos

by M. Rosenzweig & S. Serfaty

Previously, f_N = joint law μ = solution of limit PDE

$E_N(f_N, \mu)$

$= \frac{1}{\beta} H_N(f_N \mid \mu^{\otimes N}) + \mathbb{E}_{f_N}[F_N(X_N, \mu)]$

where

$$F_N(X_N, \mu) = \frac{1}{2}\int_{x \neq y} g(x,y)\, d(\mu_N - \mu)^{\otimes 2}(x,y)$$

$$\uparrow$$

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$$

$$dx_t^i = \frac{1}{N}\sum_{j \neq i} (-\nabla g)(x_i - x_j) + \sqrt{\frac{2}{\beta}}\, dW_t^i$$

$$\downarrow \qquad f_N = \text{law}(x_1^t, \dots, x_N^t)$$

$$\partial_t \mu + \operatorname{div}(\mu(-\nabla g * \mu)) = \frac{1}{\beta}\Delta\mu$$

$\frac{d}{dt} E_N(f_N, \mu)$ $\leftarrow$ modulated free energy

$$\le -\frac{1}{2}\int_{(\mathbb{R}^d)^N} d f_N^t \left[\int_{x\neq y} (u^t(x) - u^t(y)) \cdot \nabla g(x-y) \, d(\mu_N^t - \mu^t)^{\otimes 2}(x,y) \right]$$

$$- \frac{1}{\beta^2 N} \int \sum_{i=1}^{N} \left| \nabla_i \log\left(\frac{f_N^t}{(\mu^t)^{\otimes N}} \right) + \frac{\beta}{N} \sum_{j\neq i} \nabla_1 g(x_i, x_j) \right.$$

$$\left. - \beta \nabla g * \mu^t(x_i) \right|^2 d f_N^t$$

Fisher information

☆

Easiest case

$$\frac{d}{dt} H_N(f_N^t \mid \mu^{\otimes t})$$

$$\le - \boxed{\frac{1}{\beta} \int f_N^t \left| \nabla \log \frac{f_N^t}{\mu^{\otimes N}} \right|^2} \le -c \int$$

$$+ \int f_N^t \frac{1}{N^2} \sum_{i,j=1}^{N} \phi(x_i, x_j)$$

See Guillin et al. uniform-in-time for vortex model. (2 years ago). ($K = -\nabla W$ $\quad V$. W

$-\nabla V$ confine $\quad$ interaction

LSI (w. V. convex)

$$\boxed{\int_{(\mathbb{T}^d)^N} d\mu_t^N \log \frac{d\mu_t^N}{\mu_t^{\otimes N}} \le c \int_{(\mathbb{T}^d)^N} d\mu_t^N \left| \nabla \log \frac{d\mu_t^N}{\mu_t^{\otimes N}} \right|^2 \quad \text{if} \quad \mu_t \sim 1}$$

$\boxed{\text{Rosenzweig \& Serfaty}}$ $\mathbb{R}^d$

$$= \int d\mu_t^N \left| \nabla \log \frac{d\mu_t^N}{\mu^{\otimes N}} \right|^2$$

Basic idea (AS: Armstrong & Serfaty '19)

Energy

$$H_N(X_N) = \frac{1}{2N} \sum_{i \neq j} g(x_i, x_j) + \sum_{i=1}^N V(x_i)$$

cannonical Gibbs measure

$$d\mathbb{P}_{N,\beta}(X_N) = \frac{1}{Z_{N,\beta}} e^{-\beta H_N(X_N)} dx_1 \dots dx_N$$

stationary distribution of.

$$dx_i^t = -\underbrace{\nabla_i H_N(X_N)}\,dt + \sqrt{\frac{2}{\beta}}\,dW_t^i.$$

$$= \Big(-\frac{1}{N}\sum_{j\neq i}\nabla g(x_i - x_j) - \nabla V(x_i)\Big)dt$$

(relative energy)

$F_N(X_N,\mu)$ modulated energy

$$= \frac{1}{2}\int_{x\neq y} g(x-y)\, d(\mu_N^t - \mu^t)^{\otimes 2}(x,y)$$

(Serfaty, DMJ. 2020)

We can define the modulated Gibbs measure.

$$Q_{N,\beta}(\mu) = \frac{1}{K_{N,\beta}(\mu)}\, e^{-\beta N F_N(X_N,\mu)}\, d\mu^{\otimes N}(x_1,\cdots x_N)$$

$$\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \quad X^N = (x_1,\cdots x_N)$$

$\mu^{\otimes N}$ $(x_1, \cdots x_N) \rightsquigarrow$ $\mu_N \sim \mu^t$

Thermal equilibrium measure μ_β

defined as the minimizer of.

$$\mathcal{E}_\beta(\mu) = \frac{1}{2}\int g * \mu \, \mu + \int V \, d\mu + \frac{1}{\beta}\int \log \mu(x) \, d\mu(x)$$

$$= \frac{1}{2}\int g(x-y) \, d\mu^{\otimes 2}(x,y) + \int V \, d\mu + \frac{1}{\beta}\int \log \mu(x) \, d\mu(x)$$

Assuming V grows sufficiently fast at ∞,

then $\mathcal{E}_\beta$ has a unique minimizer

which satisfies. $\exists\, C_\beta \in \mathbb{R}$, st.

$$g * \mu_\beta + V + \frac{1}{\beta}\log \mu_\beta = C_\beta \quad \text{in } \mathbb{R}^d.$$

Splitting of the energy: μ_β

$$H_N(X_N) = \frac{1}{2}\int g \, d\mu_N^{\otimes 2} + \int V \, d\mu_N$$

$$= N\,\mathcal{E}_\beta(\mu_\beta) + N\,F_N(X_N, \mu_\beta) \}\}$$

$$- \frac{1}{\beta}\sum_{i=1}^{N} \log \mu_\beta(x_i)$$

Checking the equality:

$$\text{RHS} = N\cdot\frac{1}{2}\int g\,\mu_\beta^{\otimes 2} + N\int V\,d\mu_\beta$$

$$+ \underline{\frac{N}{\beta}\int \log\mu_\beta\,d\mu_\beta} \quad \checkmark$$

$$+ N\,\frac{1}{2}\int g\,(\mu_N - \mu_\beta)^2$$

$$- \frac{N}{\beta}\int \log\mu_\beta\,d\mu_N, \qquad \mu_N = \frac{1}{N}\sum_i \delta_{x_i}$$

Terms involving μ_β. $\checkmark$

$$N\langle \frac{1}{\beta}\log\mu_\beta,\ \mu_\beta - \mu_N\rangle \qquad \because \int \mu_\beta - \mu_N = 0$$

$$= N\langle \cancel{C_\beta} - g*\mu_\beta - V,\ \mu_\beta - \mu_N\rangle$$

Then

$$\text{RHS} = N\,\frac{1}{2}\int g\,\mu_\beta^{\otimes 2} + N\int V\mu_\beta$$

$$+ N\cdot\frac{1}{2}\int g\,(\mu_N - \mu_\beta)^2 - N\int V\,(\mu_\beta - \mu_N)$$

$$- N \int g * \mu_\beta \, (\mu_\beta - \mu_N)$$

$$= \frac{1}{2} \int g \, \mu_N^{\otimes 2} + \int V d\mu_N$$

Then

$$H_N(X_N) = N \mathcal{E}_\beta(\mu_\beta) + N F_N(X_N, \mu_\beta) - 1/\beta \sum_{i=1}^{N} \log \mu_\beta(x_i)$$

$$d \mathbb{P}^V_{N,\beta}(X_N)$$

$$= \frac{1}{\tilde{Z}^V_{N,\beta}} e^{-\beta H_N(X_N)} dx_1 \cdots dx_N$$

$$\overset{\text{代入等式}}{=\!=} \underbrace{\frac{e^{-\beta N \mathcal{E}_\beta(\mu_\beta)}}{Z_{N,\beta}}}_{= K_{N,\beta}(\mu_\beta)} e^{-\beta N F_N(X_N, \mu_\beta)} d \mu_\beta^{\otimes N}(x_1, \dots, x_N)$$

常数

$$\left((\mathrm{Leb}_{\mathbb{R}^d})^{\otimes N} \downarrow \mu_\beta^{\otimes N} \right)$$

Recall. modulated Gibbs.

$$\mathbb{Q}_{N,\beta} = \frac{1}{K_{N,\beta}(\mu)} e^{-\beta N F_N(X_N, \mu)} d\mu^{\otimes N}$$

We have: $\mathbb{P}^V_{N,\beta} = \mathbb{Q}_{N,\beta}(\mu_\beta)$

canonical Gibbs $\Longrightarrow$ a modulated Gibbs.

with μ_β which solves the stationary equation of binary PDE.

(Digression

Li Lei:

e^{-V} ← previously LDP

$$e^{-\beta H_N}$$

$$d\,\mathbb{P}^V_{N,\beta}(X_N) = C\, e^{-\beta N \overbrace{F_N(X_N, \mu_\beta)}^{o(N)}} d\mu_\beta^{\otimes N}$$

$$\left(e^{-\beta V}\right)^{\otimes N} e^{-\beta \frac{1}{2}\sum_{i \neq j} g(x_i, x_j)}$$

)

Sampling: don't know μ_β.
dPP but changing the reference measure from
$$(e^{-\beta V})^{\otimes N} \text{ to } (\mu_\beta)^{\otimes N}.$$

Serfaty minicourse in AMSS.

- do dynamical version of CLT as our work with Zhao & Zhu
- dPP

<u>A remark:</u>

$$Q_{N,\beta}(\mu) = \frac{1}{Z} e^{-N F(X_N, \mu)} \, d\mu^{\otimes N}$$

can be written as a canonical one with a potential

$$V_{\mu,\beta} = -g * \mu - \frac{1}{\beta} \log \mu$$

$$\mathbb{P}^{V}_{N,\beta} = \frac{1}{Z} e^{-\beta H_N^{V_{\mu,\beta}}}$$

Key estimate:

$$\begin{cases} Q_{N,\beta}(\mu) = \dfrac{1}{K_{N,\beta}(\mu)} e^{-\beta N F_N(X_N,\mu)} \, d\mu^{\otimes N} \\ \left| \log K_{N,\beta}(\mu) \right| = o(N) \\ \text{uniformly in } \beta \in [\tfrac{1}{2}\beta_0, 2\beta_0] \end{cases}$$

$\Updownarrow$

is equivalent to the following

$$\int \rho_N \Phi_N \leq \frac{1}{\eta N} \int \rho_N \log \frac{\rho_N}{\bar\rho^{\otimes N}} + \frac{1}{\eta N} \underbrace{\log \int \bar\rho^{\otimes N} \exp(\eta N \Phi_N)}_{\frac{1}{N} \log \tilde K_{N,\beta} = o(1)}$$

J.-W. 2018. Given $\|\phi\|_{L^\infty} < \infty$. η small enough

$$K^{\eta}_{N,\phi} = \int \bar\rho^{\otimes N} \exp(\eta N \underbrace{\langle \phi, (\mu_N - \rho)^{\otimes 2} \rangle}_{\text{same formula.}})$$

$$= O(1)$$

implies.

$$\frac{1}{N} \log K^{n}_{N,\phi} = O\left(\frac{1}{N}\right) = o(1)$$

$$Q_{N,\beta}(\mu) = \frac{1}{K_{N,\beta}(\mu)} e^{-\beta F_N(X_N, \mu)} d\mu^{\otimes N}$$

$$F_N(X_N, \mu) = \frac{1}{2}\int_{x \neq y} g \, d(\mu_N - \mu)^{\otimes 2}$$

As long as $g \in L^{\infty}$, $\beta < \beta_0$,

then $\sup_{N} K_{N,\beta}(\mu) \leq C$

$$\frac{1}{N} \log K_{N,\beta} = O(1/N)$$

$$\boxed{F_N(X^N, \mu) \geq -\frac{1}{N^{\theta}}} \quad \left(\frac{\hat{g} \geq 0}{\text{Serfaty's paper}}\right)$$

or $\mathbb{E} F_N(X^N, \mu) + \int \rho_N^t \log \frac{\rho_N^t}{\mu^{\otimes N}}$

$$\geq \theta \int \rho_N^t \log \frac{\rho_N^t}{\mu^{\otimes N}} - \frac{1}{N^\theta}$$

$$\log \mathbb{E}_{Q_{N,\beta}(\mu)}\left[\underbrace{e^{\frac{\beta}{2} N F_N(X_N, \mu)}} \right]$$

$$= \log \left(\frac{1}{K_{N,\beta}(\mu)} \underbrace{\int e^{-\frac{\beta}{2} N F_N(X_N, \mu)} \, d\mu^{\otimes N}}_{K_{N,\beta/2}(\mu)} \right)$$

$$= \log \frac{K_{N,\beta/2}(\mu)}{K_{N,\beta}(\mu)}$$

If $|\log K_{N,\beta}(\mu)| = o(N)$ for $\beta \in [\frac{\beta_0}{2}, \beta_0]$

then $\left| \log \mathbb{E}_{Q_{N,\beta}(\mu)} e^{\frac{\beta}{2} N F_N(X_N, \mu)} \right| \leq o(N)$

Then

$$\mathbb{E}_{Q_{N,\beta}(\mu)}\left[\|\mu_N - \mu\|^2_{H^{-\frac{d-s}{2}}}\right] \to 0.$$

Also we can obtain convergence in the following form

$$\left\{ \begin{array}{l} Q^{(k)}_{N,\beta}(\mu) \quad : \text{ its } k\text{-marginal} \\ \downarrow \\ \mu^{\otimes k}. \end{array} \right.$$

$$\underset{0}{f^{t}_{N}} \sim (\underset{\uparrow}{\mu^{t}})^{\otimes N}$$

limiting PDE

$$Q_{N,\beta}(\mu^{t})$$

Rewriting the modulated free energy

$$E_N(f_N, \mu) = \frac{1}{\beta} H_N(f_N \mid \mu_t^{\otimes N}) + \mathbb{E}_{f_N}\left(F_N(X_N, \mu_t) \right)$$

Another Form of modulated free energy

$$= \frac{1}{\beta}\left(H_N(f_N \mid Q_{N,\beta}(\mu)) - \frac{\log K_{N,\beta}(\mu)}{N} \right)$$

?

Recall $Q_{N,\beta}(\mu) = \underbrace{\frac{1}{K_{N,\beta}} e^{-\beta N F(X_N, \mu)} d\mu^{\otimes N}}$

$$\frac{1}{\beta N}\int f_N \log \frac{f_N}{Q_{N,\beta}}$$

$$= \frac{1}{\beta}\frac{\log K_{N,\beta}}{N} + \int f_N F(X_N, \mu) + \cdots$$

Modulated Fischer Information:

$$\star \; -\frac{1}{\beta^2 N}\int f_N^t \sum_{i=1}^{N}\Big|\nabla \log\Big(\frac{f_N^t}{(\mu^t)^{\otimes N}}\Big)+\frac{\beta}{N}\sum_{j\neq i}\nabla_1 g(x_i,x_j) - \nabla g * \mu^t(x_i)\Big|^2$$

$$= -\frac{1}{\beta^2 N}\int f_N^t \,\Big|\nabla \log \frac{f_N^t}{Q_{N,\beta}(\mu^t)}\Big|^2$$

$$= -\frac{1}{\beta^2 N}\int \Big|\nabla \sqrt{\frac{f_N^t}{Q_{N,\beta}(\mu^t)}}\Big|^2 dQ_{N,\beta}(\mu^t)$$

The evolution of modulated free energy

$$\frac{d}{dt}E_N(f_N^t,\mu^t)$$

$$\leq -\frac{1}{\beta^2 N}\int f_N^t \,\Big|\nabla \log \frac{f_N^t}{Q_{N,\beta}(\mu^t)}\Big|^2$$

$$-\frac{1}{2}\int df_N^t \underbrace{\int_{x\neq y} (u^t(x)-u^t(y))\cdot \nabla g(x-y)\, d(\mu_N^t-\mu^t)^{\otimes 2}(x,y)}$$

$$u^t = \frac{1}{\beta}\nabla \log \mu^t + \nabla U + \nabla g * \mu^t$$

Def 1.1 : LSI for P_N uniformly in N.

if $\exists\, C_{LS} > 0$, $N \geq 1$,

$f \in C^2((\mathbb{R}^d)^N)$,

we have.

$$\int_{(\mathbb{R}^d)^N} f^2 \log \frac{f^2}{\int f^2 \, dP_N} \, dP_N \leq C_{LS} \int_{(\mathbb{R}^d)^N} |\nabla f|^2 \, dP_N$$

If $Q_{N,\beta}(\mu) \sim$ uniform LSI,

apply ULSI to $f = \sqrt{\frac{f_N}{Q_{N,\beta}(\mu)}}$

$$\Big(\underbrace{\beta \cdot E_N(f_N, \mu) + \frac{1}{N} \log K_{N,\beta}(\mu)}_{= H_N(f_N \mid Q_{N,\beta}(\mu))}\Big).$$

$$\leq C_{LS} \frac{1}{N} \int f_N \log \frac{f_N}{Q_{N,\beta}}.$$

We need for $(\mu_t)_{t\in[0,\infty)}$ solution

to the limiting PDE that

$$\boxed{Q_{N,t}(\mu^t) \text{ satisfies LSI with uniform constant } C_{LS}}$$

$$\frac{1}{Z_{N,\beta}} e^{-\beta N F_N(X_N,\mu)} \, d\mu^{\otimes N}$$

Deng Ma. long time justification of wave turbulence

Landau: The Landau eq. does not blow up Fisher information

✓ (Landau - Coulomb.)

✓ $\partial_t f = \underset{\uparrow}{c} f^2 + \cdots$

✓ <u>Jingrui Chen</u>

$$\begin{cases} |\nabla \log \rho(x)| \leq C(1+|x|) \\ |\nabla^2 \log \rho(x)| \leq C(1+|x|^2) \end{cases}$$