Dawson $\sigma = \sigma_c \leftarrow$ 噪声强度.

$\sigma > \sigma_c$ Usual Central Limit Theory

$\sigma = \sigma_c$ Non-Gaussian Fluctuation

$$U_N(t,\cdot) = N^{1/4}\left[X_N(N^{1/2}t,\cdot) - \underline{\underline{P_0(x)}}\,dx\right].$$

$$Y_N(t,\cdot) = N^{1/2}\left[X_N(t,\cdot) - \underline{X_\infty(t,\cdot)}\right]$$

Critical Dynamics and Fluctuations for a Mean-Field Model

anharmonic oscillator $x(t) \in \mathbb{R}$.

$$dx(t) = [-x^3(t) + x(t)]\,dt + \sigma\, \underbrace{dW_t}_{BM} + \text{interaction}$$

here underlying space $x_t \in \mathbb{R}$

$\{x_t\}_{t\geq 0}$ is a Markov process

$\downarrow$ its corresponding Fokker-Planck.

$$\frac{\partial}{\partial t} p(t; x, y) = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} p(t; x, y)$$

$$- \frac{\partial}{\partial y}\left[(-y^3 + y)\, p(t; x, y)\right]$$

velocity field.

initial condition:

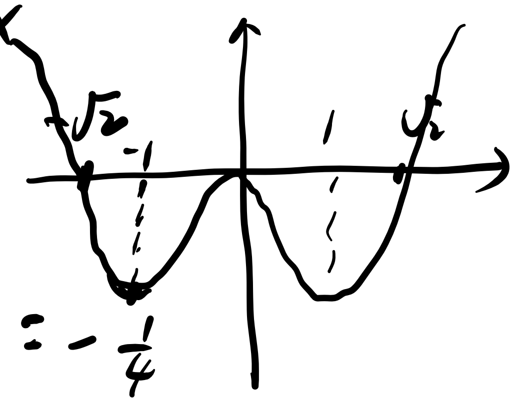
$$\lim_{t\downarrow 0} \underbrace{\int f(y)\, p(t;x,y)\, dy}_{\| \atop \mathbb{E}[f(X(t)) \mid X(0)=x]} = f(x) \quad \forall f \in C_b.$$

Previously:

$$dX_t = \overset{=-x^3+x}{\boxed{-U'(X_t)}}\, dt + \sigma\, dW_t.$$

where $U(x) = \boxed{\frac{1}{4}x^4 - \frac{1}{2}x^2}$

$U'(x) = x^3 - x$



$U(x)$

$= \frac{1}{4}x^2(x^2-2)$

$$\min_{x\in\mathbb{R}} U(x) = \frac{1}{4}\cdot 1\cdot(-1) = -\frac{1}{4}$$

$$\arg\min_{x\in\mathbb{R}} U(x) = \pm 1$$

Fokker-Planck: $\partial_t p(y) = \frac{1}{2}\sigma^2 \partial_y^2 p$

$$+\ \partial_y\left(p\cdot(y^3-y)\right)=0$$

i.e.

$$\frac{1}{2}\sigma^2\partial_y^2 p+\partial_y\left(p\cdot(y^3-y)\right)=0$$

$$p(x)=\frac{1}{Z}\exp\left(\frac{1}{\sigma^2}\left(x^2-\frac{1}{2}x^4\right)\right)$$

$$=\frac{1}{Z}\exp\left(-\frac{1}{\frac{\sigma^2}{2}}\underbrace{u(x)}_{\frac{1}{4}x^4-\frac{1}{2}x^2}\right).$$

$\boxed{\text{Going to } N\text{-particle system}}$

Consider IPS: N-oscillators.

$$\begin{cases} dx_j=(-x_j^3+x_j)\,dt+\underline{\underline{\sigma\,dW_j(t)}}-\underline{\theta(x_j-\bar{x})\,dt}. \quad \boxed{\theta>0} \\ \text{where } W_j \text{ are independent BMs.} \\ \bar{x}=\frac{1}{N}\sum_{i=1}^{N}x_i \end{cases}$$

Question: The role of σ, given fixed $\theta>0$.

$$\dot{x}_j = -u'(x_j) \quad - \theta\left(\frac{1}{2N}\sum_k (x_j - x_k)^2\right)'_{x_j}$$

$$\dot{x}_j = -u'(x_j) - \theta\left[\nabla_{x_j}\frac{1}{2N}\sum_{j,k=1}^{N}(x_j - x_k)^2\right]$$

Langevian Dynamics

$$H_N(x_1, \dots x_N) = \sum_{j=1}^{N} u(x_j) + \frac{1}{2N}\sum_{j,k=1}^{N}(x_j - x_k)^2.$$

Fokker-Planck (N-particle)

$$\partial_t P(t; x, \underline{y}) = \frac{\sigma^2}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial y_j^2}P(t; x, y)$$

$$- \sum_{j=1}^{N}\frac{\partial}{\partial y_j}\left[(-y_j^3 + y_j)\,P(t; x, y)\right]$$

unique equilibrium.

$$+ \frac{\theta}{N}\sum_{j=1}^{N}\sum_{k=2}^{N}\frac{\partial}{\partial y_j}\left[(y_j - y_k)\,P(t; x, y)\right].$$

$$P_N(x) = \frac{1}{Z_N}\exp\left[\frac{2}{\sigma^2}H_I(x_1, \dots x_N)\right].$$

$$\cdot \prod_{j=1}^{N} p(x_j)$$

where $H_I(x_1, \ldots, x_N) = \frac{\theta}{2N} \sum_{j,k=1}^{N} x_j x_k \sim N^2$

$$p(x_j) = \exp\left(\frac{1}{\sigma^2}\left[(1-\theta)x_j^2 - \frac{1}{2}x_j^4\right]\right)$$ $\boxed{??}$

$$\sum_{j=1}^{N}\left(\partial_{y_j}\left(p \cdot \left[\underbrace{\frac{\sigma^2}{2}\partial_j \log p}_{} + (u'(y_j)) + \frac{\theta}{N}\sum_{k=1}^{N}(y_j - y_k)\right]\right)\right) = 0$$

$$\partial_j \log p = -\frac{2}{\sigma^2}\left(u'(y_j) + \frac{\theta}{N}\sum_{k=1}^{N}(y_j - y_k)\right)$$

$$\log p \sim -\frac{2}{\sigma^2}\left(u(y_j) + \frac{\theta}{2N}\sum_{k=1}^{N}(y_j - y_k)^2\right)$$

$$p \propto \exp\left(-\frac{2}{\sigma^2}\sum_j u(y_j) + \frac{\theta}{2N}\sum_{j,k}(y_j - y_k)^2\right)$$

$$\sum_{j,k=1}^{N}(x_j - x_k)^2 = \sum_{j,k=1}^{N}\left(x_j^2 + x_k^2 - 2x_j x_k\right)$$

$$\underset{T}{u(x_j)} + \frac{1}{N}\sum_{k \neq j} \nabla W(x_j - x_k)$$

self-interaction

Xie: Meyer 展开 (2体, 3体 或多体相互作用).

$$H(x_1, \dots, x_N) = \sum_{j=1}^{N} U_1(x_j) + \frac{1}{N}\sum_{v_1, v_2 = 1}^{N} U_2(x_{v_1}, x_{v_2}) + \cdots + \frac{1}{N^{k-1}} \sum_{v_1, \dots, v_k = 1}^{N} U_k(x_{v_1}, \dots, x_{v_k}).$$

$$\boxed{\begin{gathered} PV = nRT \\ (P + \cdots)(V + \cdots) = nRT \end{gathered}}$$

Math. Problem.
Equilibrium Statistical Physics

$$\frac{1}{N} \log \int_{E^N} \bar{\rho}^{\otimes N} \exp\underbrace{\left(N \cdot \int_{E^2} \phi(x,y) \, (d\mu_N - d\bar{\rho})^{\otimes 2}(x,y) \right)}_{< \infty} dX^N < \frac{C}{N}$$

只需 $\|\phi\|_{L^\infty} \le c_0 < \frac{1}{3}$

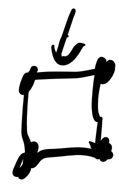
$$\boxed{\frac{\beta}{N} \sum_{i,j=1}^{N} \bar{\phi}(x_i, x_j)}, \quad \|\phi\|_{L^\infty} < \infty$$

$\beta \ll 1$. 则有.

$$\frac{1}{N}\log\int \bar{\rho}^{\otimes N}\exp\left(\frac{\beta}{N}\sum_{i,j=1}^{N}\bar{\phi}(x_i,x_j)\right)dx^N < \frac{C}{N}$$

$$P_N(x) = \frac{1}{Z_N}\exp\left[\underset{\beta}{\underset{\|}{\frac{2}{\sigma^2}}}\, H_I(x_1,\dots,x_N)\right]\prod_{j=1}^{N}\rho(x_j)$$

$$H_I(x_1,\dots,x_N) = \frac{\theta}{2N}\sum_{j,k=1}^{N}x_j x_k$$



ϕ^4: Interaction energy

$$H(x_1,\dots,x_N) = \sum_{j\in\Lambda}\sum_{k\in\Lambda} J_{jk}\cdot x_j x_k$$

Λ = Finite box in $\mathbb{Z}^d$ $\qquad J_{jk} = \phi(|x_j - x_k|)$

$J_{jk} \geq 0$ Ferromagnetic

Main Results: θ - Fixed. $\theta > 0$

① Mean-Field Limit.

$\exists$ critical σ_c, $\begin{cases} \sigma > \sigma_c & \text{Usual CLT.} \quad ② \\ \sigma = \sigma_c & \text{Fluctuation.} \quad ③ \end{cases}$

Fix the notations :

$$\mathbb{R} \ni X_N(t; A) := \frac{1}{N} \sum_{j=1}^{N} 1_A[X_j(t)].$$

$A \in \mathcal{B}(\mathbb{R})$

Borel σ-alg.

$$X_N(t, \cdot) \in M_1(\mathbb{R}^2) .$$

① As $N \to \infty$, $X_N(t, \cdot) \to X_\infty(t)$,

in the sense of weak convergence of measure-valued stochastic process.

Limit : $\{X_\infty(t) \mid t \geq 0\}$.

$$X_\infty \sim p(t, \cdot)$$

satisfies Nonlinear PDE (MFD).

$$\frac{\partial_t p}{\partial t} = \frac{\sigma^2}{2} \partial_x^2 p - \frac{\partial}{\partial x} \left\{ \left[(1-\theta) x - x^3 \right] p \right\}$$

$$- \theta m(t) \frac{\partial p}{}$$

$$- \theta\, a(t) \cdot \frac{\partial}{\partial x}$$

$$a(t) = \int x\, p(t,x)\, dx$$

② Phase Transition $\mathbb{R}$.

Look at the equilibrium of MF-D

$$\begin{cases} \underline{\frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \frac{\partial}{\partial x}\left\{\left[(1-\theta)x - x^3\right]p\right\}} - \theta a \frac{\partial p}{\partial x} = 0 & ① \\ a = m(a) = \int_{\mathbb{R}} x\, p(x)\, dx & ② \end{cases}$$

$(\pm a_0, \underline{P_{\pm a_0}(x_0)})$

$$\boxed{a = 0, \ \cancel{\mathbb{R}}\ p = 0}$$

$$P_0(x) = \frac{1}{Z}\exp\left(\sigma^{-2}\left((1-\theta)x^2 - \frac{1}{2}x^4\right)\right)$$

$$\Downarrow$$

$$\int x\, p(x)\, dx = 0$$

$\underline{\underline{a \neq 0}}$

$$P_{\pm a_0}(x) = Z_{a_0}^{-1} \exp\left(\sigma^{-2}\left[(1-\theta)x^2 - \frac{1}{2}x^4 \pm 2a_0\theta x\right]\right)$$

Results: $\exists\ \sigma_c,\quad 0 < \sigma_c < \infty$

Eq (2)

Case I: $\sigma \geq \sigma_c$, $0 \overset{!}{=} m(a) = a$ $\Leftarrow$ of A-Eq.

$a = 0 \Rightarrow \rho = \rho_0(x)$

$$= \frac{1}{Z} \exp\left(\sigma^{-2}\left[(1-\theta)x^2 - \frac{1}{2}x^4\right]\right)$$

Case II: $\sigma < \sigma_c$

Eq ② have nonzero solutions $\pm a_0$.

$$\rho_{\pm a_0}(x) = Z_{a_0}^{-1} \exp\left\{\sigma^{-2}\left[(1-\theta)x^2 - \frac{1}{2}x^4 \pm 2a_0\theta x\right]\right\}$$

$$\rho_0(x) \propto \arg\min_{\rho \in P(\mathbb{R})} F(\rho)$$

N - Fixed Only 1 equilibrium.

Result III Fluctuations $\begin{cases} \sigma > \sigma_c \\ \sigma = \sigma_c \text{ Critical fluctuations} \end{cases}$

$\sigma > \sigma_c$: CLT

$$Y_N(t, \cdot) := N^{1/2}\left[X_N(t, \cdot) - X_\infty(t, \cdot)\right]$$

$N \to \infty$

Theorem: $\{Y_N(t,\cdot) \xrightarrow{N} Y(t,\cdot)\}_{t\in[0,\infty)}$

weak convergence of probability measures on

$C([0,\infty), \mathcal{D}')$

where $Y(\cdot,\cdot)$ is Gaussian s.t.

$$\frac{\partial Y}{\partial t} = \mathcal{L}_t^* Y + W(t)$$

$$\boxed{\partial_t \bar{\rho} + \mathrm{div}(\bar{\rho}\, K * \bar{\rho}) = \sigma \Delta \bar{\rho}}$$

$$\partial_t \mu_N + \mathrm{div}(\mu_N K * \mu_N) = \sigma \Delta_x \mu_N + \text{martingale}$$

$$Y_N \sim \eta_N = \sqrt{N}(\mu_N - \rho)$$

$\downarrow$

η Gaussian.

$$\partial_t \eta = \sigma \Delta \eta - \nabla\cdot(\bar{\rho} K * \eta) - \nabla\cdot(\eta K * \rho) - \sqrt{2\sigma}\, \nabla\cdot(\sqrt{\bar{\rho}}\, \xi)$$

Classical

$\boxed{\sigma < \sigma_c}$ phase Transition

$Y_N(t,\cdot) := N^{1/2}\left[X_N(t,\cdot) - X_\infty(t,\cdot)\right].$

$$\partial_t Y = \mathcal{L}_t^* Y + W'(t)$$

$\{W(t) \mid t \geq 0\}$ Gaussian Markov Process

in $\mathcal{D}'$ with covariance:

$$\mathrm{Cov}\left(\langle W(t), \phi\rangle, \langle W(t), \psi\rangle\right)$$

$$= \sigma^2 \int_0^t \int \phi'(x)\,\psi'(x)\, X_\infty(s,x)\,dx\,ds.$$

$$\mathcal{L}_t^* Y = \frac{1}{2}\sigma^2 \frac{\partial^2 Y}{\partial x^2} - \frac{\partial}{\partial x}\left\{\left[(1+\theta)x - x^3\right] Y\right\}$$

$$- \theta\left[\int y X_\infty(t,y)\,dy\right]\frac{\partial Y}{\partial x}$$

$$- \theta \langle Y, y\rangle \frac{\partial X_\infty}{\partial x}$$

Weak convergence: $\forall \phi \in C_c^\infty$, then $\left(Y_N(t,\cdot) \to Y_\infty(t,\cdot)\right)_{t\in(0,\infty)}$

$\sum_{j=1}^{N} \phi(X_j(t)) \sim N$ (weakly dependent)

Fix t.

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left[\phi(x_j(t)) - \int \phi(x) X_\infty(t,x)\,dx\right] \to \langle Y(t), \phi \rangle \quad \text{in law.}$$

$\sigma < \sigma_c \qquad P_N(x) =$ unique equilibrium

$\boxed{t = \infty}$

<u>Thm</u> (Sznitman. Topics in Propagation of Chaos)

$\rho_N \in P_{sym}((\mathbb{R}^d)^N)$ i.e.

exchangeable / symmetric $\qquad \rho_N(x_1, \dots x_N) \overset{\tau\in S_N}{=} \rho_N(x_{\tau(1)}, \dots x_{\tau(N)})$

TFAE:

1. $\rho_{N,2} \longrightarrow \bar{\rho}^{\otimes 2} \qquad \left(\int \phi(x_1)\phi(x_2)\rho_{N,2}(x_1,x_2) \to \left(\int \phi \bar{\rho}\right)^2\right)$

② $\forall k = 1, 2, 3, \ldots, \quad \rho_{N,k} \to \bar{\rho}^{\otimes k}$.

③ $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i}, \quad \mathrm{Law}(X_1, \ldots X_N) = \rho_N$

$\mu_N \to \delta_{\bar{\rho}}$ in law

$$\begin{cases} \rho_N = \underline{\frac{1}{2}(\rho_1)^{\otimes N} + \frac{1}{2}(\rho_2)^{\otimes N}} \in P_{sym}(E^N) \\ \rho_{N,1} = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 = \rho \text{ (converges)}. \end{cases}$$

$\rho_{N,2} = \frac{1}{2}\rho_1^{\otimes 2} + \frac{1}{2}\rho_2^{\otimes 2}$ (converges).

$\neq \rho^{\otimes 2}$.

$\sigma > \sigma_*$, $\mathcal{L}^*$ (around $X_{\infty}^{(t;i)}$ or over $\bar{\rho}_+$)

$\mathcal{L}^*$ is stable: $\quad \lambda_0 = 0$ simple.

all others $\quad \lambda_n < 0$

$\boxed{\partial_t Y_t = \mathcal{L}^*_t Y_t + N(t)}$ ☆

space-time.

$$\mathrm{Cov}\left(\langle W(t),\phi\rangle, \langle W(t),\psi\rangle\right) = \sigma^2 \int_0^t \int \phi'(x)\psi'(x)\, \bar\rho(s,x)\,dx\,ds$$

$$\star \quad \boxed{\partial_t \bar\rho + \partial_x\left(\bar\rho\left[-U'(x) + \theta\left(x - \int y\bar\rho(y)\,dy\right)\right]\right) = \frac{\sigma^2}{2}\Delta_x \bar\rho}$$

Well-posedness.

<u>Question:</u>

$$\partial_t \bar\rho + \mathrm{div}(\bar\rho\, K * \bar\rho) = \sigma \Delta_x \bar\rho$$

- $\mathrm{div}\, K \geq 0$ no phase transition.

uniform-in-time POC.

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