A Duality Method for Mean field limit

Recall: SDE $\begin{cases} dX_i = V_i\,dt & K \in L^2_{loc} \quad \alpha \to 0 \\ dV_i = \frac{1}{N}\sum_{i=1}^{N} K(X_i - X_j)\,dt + \alpha\, dB_i \end{cases}$

① Liouville eq / Backward Liouville eq

$K \in L^\infty$

$$\begin{cases} \partial_t F_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} F_N + \frac{1}{N-1}\sum_{j\neq i} \boxed{K(x_i - x_j)} \cdot \nabla_{v_i} F_N = \alpha \sum_{i=1}^{N} \Delta_{v_i} F_N \quad (1) \end{cases}$$

$\begin{cases} N \text{ variables} \\ K \in L^2_{loc} \end{cases}$

$\boxed{\text{weak solution}}$ + entropy inequality 不唯一

$$\partial_t \Phi_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} \Phi_N + \frac{1}{N-1}\sum_{j\neq i} K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N = -\alpha \sum_{i=1}^{N} \Delta_{v_i} \Phi_N \quad \boxed{(2)}$$

$K * f$

$\boxed{K \in L^1_{loc}}$

✓ New definitions of solutions of Eq. (1). 不唯一

F_N (1): "$\boxed{\text{weak dual solutions}}$" weak solution + dual equality

Φ_N
- $\Phi_N \in L^\infty([0,T] \times D^N)$ $\quad D = \Omega \times \mathbb{R}^d$.
- $\|\Phi_N\|_{L^\infty((-\infty,T];D^N)} \le \|\Phi(T)\|_{L^\infty(D^N)}$ $\quad \|\psi\|^k_{L^\infty}$
- Final condition:

$$\Phi_N(t, z_1, \cdots, z_N)\Big|_{t=T} = \boxed{(C_N^k)^{-1} \sum_{1\le i_1 \cdots \le i_k \le N} \psi(z_{i_1}) \cdots \psi(z_{i_k})}$$

$\langle F_{N,k} \to f^{\otimes k}, \psi^{\otimes k} \rangle$

Remark: $K \in L^1_{loc}(\Omega; \mathbb{R}^d)$ $\quad F_N^0 \in L^1(D) \cap L^p(D)$

for some $1 < p \le \infty$, the "weak dual solutions" exist. 不唯一

Mean field equation:

$$\partial_t f + v \cdot \nabla_x f + K * \rho \cdot \nabla_v f = \alpha \Delta_v f.$$

② Dual Marignal $\{ M_{N,n} \}_{n=0}^{N}$

F_N

$$M_{N,n}(z_1, \cdots, z_n) = \int_{D^{N-n}} \underline{\Phi_N(z_1, \cdots, z_N)} f^{\otimes N-n}(z_{n+1}, \cdots z_N) \, dz_{n+1}, \cdots dz_N.$$

Remark: Corresponding to $\underline{\underline{F_{N,n}}}_{\,N-n}$.

③ (Dual) cluster expansion: $(F_{N,n})$

→ Correlation functio

$$\boxed{\Phi_N(z_1, \cdots z_N) = \sum_{n=0}^{N} \sum_{\sigma \in P_n^N} C_{N,n}(\underline{\underline{z_\sigma}})}$$ ☆

n个变量个展开.

$[1, \cdots, N] \rightsquigarrow$ n个变量 $|P_n^N| = C_N^n$.

$\underline{P_n^N}$: Set of all subset of $[N] = \{1, 2, \cdots, N\}$

选3n个变量. with n elements.

$$\boxed{|C_{N,n}| \sim \frac{1}{N^n}}$$

$\sigma = \{ i_1, \cdots i_n \}$, $\underline{z_\sigma = (z_{i_1}, \cdots, z_{i_n})}$.

Correlation function

① $C_{N,n}$ is Symmetric w.r.t n variables

$C_{N,n}(z_{i_1}, \cdots z_{i_n})$ 轮换不变.

② $\displaystyle\int_D C_{N,n}(z_1, \cdots, z_n) \underline{f(z_j)} \, dz_j = 0.$ $F_N \to f^{\otimes N}$

$\langle C_{N,n}, f(z_j) \rangle = 0$ $j \in \{1, \cdots, n\}$

Remark: Dual $\underline{\text{correlation functions } \{ C_{N,n} \}}$

are $\boxed{\text{unique}}$ if they satisfy above condition

• On the side of chaos via Clauber calculus in the classical MFD.

21, CMP, ~~Duerinckx~~. ~~$K \in C^\infty$~~

$$C_{N,n}(z_1, \dots, z_n) = \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma \in P_k^n} M_{N,k}(z_\sigma)$$

叫这一块定：

Proof:

$$\Phi_N(z_1, \dots, z_N) \qquad n=0, |P_0^n| = 1$$

$$C_N^0 \left\{ = \boxed{C_{N,0}}(\cdot) \right\} C_N^0 \qquad n=1, |P_1^n| = 1$$

$$C_N^1 \left\{ + C_{N,1}(z_1) + \dots + C_{N,1}(z_N) \right\} C_N^1$$

$$C_N^2 \left\{ \begin{array}{l} + C_{N,2}(z_1, z_2) + \dots + C_{N,2}(z_1, z_N) \\ \quad \dots \quad C_{N,2}(z_{N-1}, z_N) \end{array} \right\} C_N^2$$

$$+ C_{N,3}(z_1, z_2, z_3) \quad \Big\} C_N^3$$

$$\vdots$$

$$C_N^n \quad + C_{N,N}(z_1, \dots z_N) \quad \Big\} C_N^N$$

2^N.

$$C_N^0 \left\{ \int \Phi_N(z_1, \dots z_N) \boxed{f^{\otimes N}(z_1, \dots, z_N)} = C_{N,0} \right.$$

$j \neq 1$

$C_{N,1}(z_1, z_j)$

$$C_N^1 \left\{ \begin{array}{l} \int \Phi_N(z_1, \dots, z_N) \boxed{f^{\otimes N-1}(z_2 \dots z_N)} = \boxed{C_{N,1}(z_1)} + \boxed{C_{N,0}} \\ \qquad z_1 \quad \sigma \in \{2, \dots, N\} \quad C_{N,1}(z_2) + C_{N,0} \\ \int + \quad - \quad - \quad f^{\otimes N-1}(z_1, z_3, \dots z_N) \\ \quad \vdots \\ \int \Phi_N(z_1, \dots z_N) f^{\otimes N-1}(z_1, \dots, z_{N-1}) = C_{N,1}(z_N) + C_{N,0} \end{array} \right.$$

C_N^2 $=$

$$\int \Phi_N(z_1,\cdots,z_N) f^{\otimes N-2}(z_3,\cdots,z_N) = C_{N,0} + C_{N,1}(z_1) + C_{N,1}(z_2) + C_{N,2}(z_1,z_2)$$

$(\cancel{z_1,z_2})$

$$\vdots$$

$$\int \Phi_N(z_1,\cdots,z_N) f^{\otimes N-2}(z_1,\cdots,z_{N-3},z_{N-2}) = C_{N,0}+C_{N,1}(z_{N-1}) + C_{N,1}(z_N) + C_{N,2}(z_{N-1},z_N)$$

$C_N^3 \cdots C_N^N \sim 2^N$

We solve $\boxed{\{C_{N,n}\}_{n=0}^N}$ by $\boxed{\{M_{N,n}\}_{n=0}^N}$ iterately.

they both have 2^N terms,

(1), (2),

Φ_N, $f^{\otimes k}$

$$\boxed{C_{N,n}(z_1,\cdots,z_n) = \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma\in P_k^n} M_{N,k}(z_\sigma).}$$

$$F_N = \sum_{\ell=0}^{N} \sum_{P_\ell^N} \tilde{C}_{N,n}(z_1,\cdots z_\ell). \quad \tilde{C}_{N,n} = \cdots \quad F_{N,k}$$

$$M_{N,k}(z_1,\cdots z_k) = \sum_{\ell=0}^{k} \sum_{\tau\in P_\ell^k} C_{N,\ell}(z_\tau).$$

矢量时间: $\boxed{C_{N,n} \sim \frac{1}{N^n} \Rightarrow F_{N,1} \to f}$ (CMP)

④. (Proposition 3: New description of PoC by dual function Φ_N)

The propogation of chaos holds

$$\Phi = \frac{1}{C_N^k}\sum_{k=0} \psi^{\otimes k}.$$

$\Leftrightarrow$ dual

$$\int_{D^k} \psi^{\otimes k} F_{N,n}(T) \xrightarrow[N\to+\infty]{} \left(\int \psi f(T)\right)^k$$

$\Leftrightarrow$ eg

~~interaction term~~

$$\boxed{\int_0^T \left(\int_{D^N} V_f(z_1,z_2)\, \Phi_N f^{\otimes N} \right) dt \xrightarrow[N\to+\infty]{} 0}$$

Today, we start from here:

和 cluster expansion 无关

$$\succeq \boxed{\int_0^{T} \left(\int_{D^N} V_f(z_1,z_2)\, \Phi_N f^{\otimes N} \right) dt \xrightarrow[N\to+\infty]{} 0}$$

where

$$V_f(z_1,z_2) = \left(K(x_1-x_2) - K * f(x_1) \right) \cdot \nabla_{v_1} \log f(z_1)$$

$$\Phi_N = \sum_{n=0}^{N} \sum_{\sigma \in P_n^N} C_{N,n}(z_\sigma).$$

$$\int_0^T \left(\int_{D^N} \underline{V_f(z_1,z_2)}\, \Phi_N f^{\otimes N} \right) dt$$

$$= \sum_{n=0}^{N} \sum_{\sigma\in P_n^N} \int_0^T \left(\int_{D^N} V_f(z_1, z_2)\, \underbrace{C_{N,n} f^{\otimes N}}_{z_1 \cdots z_N} \right) dt$$

$$= \int_0^T \left(\int_{D^N} V_f(z_1,z_2) \left(\begin{array}{l} \cancel{C_{N,0}} \\ \cancel{C_{N,1}(z_1)} \quad \cancel{C_{N,1}(z_2)} \\ C_{N,2}(z_1,z_2) \end{array} \right) f^{\otimes 2} \right) dt$$

$$f^{\otimes 2} = f(z_1) f(z_2).$$

$$= \underline{C_{N,0} : \int_0^T \int_{D^N} V_f(z_1,z_2) f^{\otimes 2}\, dt = 0}$$

by
$$\begin{cases} \int_{\mathbb{D}^N} V_f(z_1, z_2) f(z_1) = 0 \quad ✓ \\ \int_{\mathbb{D}^N} V_f(z_1, z_2) f(z_2) = 0 \end{cases}$$

速度变量:
$$\int_{\mathbb{D}^N} \left(K(x_1 - x_2) - K * f(x_1)\right) \cdot \nabla_{v_1} \log f \; f(z_1)\, dz_1$$
$$z_1 = (x_1, v_1)$$
$$= \int_{\mathbb{D}^N} \left(K(x_1 - x_2) - K * f(x_1)\right) \nabla_{v_1} f(z_1)\, dz_1$$
$$= 0 \quad ✓$$

MF cancellation
$$\int_{\mathbb{D}^2} \left(K(x_1 - x_2) - K * f(x_1)\right) \cdot \nabla_{v_1} \log f(z_1) f(z_2)\, dz_2$$
$$= \int_{\mathbb{D}^2} \left(K * f(x_1) - K * f(x_1)\right) \cdot \nabla_{v_1} \log f(z_1)$$
$$= 0$$

$C_{N,1}(z_1)$:
$$\int_{\mathbb{D}^2} V_f(z_1, z_2)\, C_{N,1}(z_1) f(z_1) f(z_2)\, dz_1\, dz_2$$
$$= \int \left(\int_{\mathbb{D}^2} V_f(z_1, z_2) f(z_2)\, dz_2\right) C_{N,1}(z_1) f(z_1)\, dz_1$$
$$= 0$$

$$\overset{C_N^n}{\Longleftrightarrow} \boxed{N\int_0^T \left(\int_{\mathbb{D}} V_f \, C_{N,2} \, f^{\otimes 2}\right) dt \xrightarrow[N\to+\infty]{} 0.}$$

$$= \int_0^T \underline{\langle V_f , \underline{(N C_{N,2})} \rangle}_{L^2(\mathbb{D}^2, f^{\otimes 2})} dt \; \boxed{\to 0}$$

$$\Updownarrow$$

If: $N C_{N,2} \xrightarrow{*} 0$ in $\underline{L(0,T; L^2(\mathbb{D}^2, f^{\otimes 2}))}$

$$\Uparrow \quad \underline{\int |N C_{N,2}|^2 f^{\otimes 2}(z_1, z_2)\, dz_1 dz_2 \to 0.}$$

$$\boxed{\| N C_{N,2} \|_{L(0,T; L^2(\mathbb{D}^2, f^{\otimes 2}))} \xrightarrow{?} 0.}$$

$$\int_0^T \langle V_f , (N C_{N,2}) \rangle_{L^2(\mathbb{D}^2; f^{\otimes 2})} dt$$

$$\boxed{K \in L^2_{loc}(\Omega) + L^2(\Omega \setminus B_r(0))} \qquad \Omega = \mathbb{R}^d.$$

$$\leq \int_0^T \underbrace{\boxed{\| V_f \|_{L^2(\mathbb{D}^2; f^{\otimes 2})}}}_{< +\infty} \boxed{\| N C_{N,2} \|_{L^2(\mathbb{D}^2; f^{\otimes 2})}} dt$$

$$\boxed{\int |C_{N,2}|^2 f^{\otimes 2} = \underline{o\left(\frac{1}{N}\right)}} \quad \begin{cases} C_{N,2} \sim o\left(\frac{1}{N}\right) \\ \boxed{C_{N,2} \sim O\left(\frac{1}{N^2}\right)} \end{cases}$$

or ---

$$\Phi_N(z_1, \cdots z_N) = \sum_{n=0}^{N} \sum_{\sigma \in P_n^N} \boxed{C_{N,n}}$$

$$\underline{\int |\Phi_N|^2 f^{\otimes N}} = \underbrace{\int \Big| \underbrace{\sum_{n=0}^{N} \sum_{\sigma \in P_n^N}} C_{N,n} \Big|^2 f^{\otimes N}}$$

(z_1, z_2) (z_1, z_3)

$$\begin{cases} \int C_{N,n} C'_{N,n} f^{\otimes N} = 0 \\ \int C_{N,n} C_{N,m} f^{\otimes N} = 0 \qquad n \neq m. \end{cases}$$

$$= \underbrace{\sum_{n=0}^{N} C_N^n} \int |C_{N,n}|^2 f^{\otimes n}$$

$$\overline{C_N^n = |P_n^N|}$$

$$= \sum_{n=0}^{N} \left(C_N^n \int |C_{N,n}|^2 f^{\otimes n} \right) \underline{\|\Phi_N\|_{L^\infty} < +\infty}$$

$$C_N^n \int |C_{N,n}|^2 f^{\otimes 2} \leq \underline{\int \underline{|\Phi_N|}^2 f^{\otimes N} dz_1 \cdots dz_N}$$

$$< +\infty$$

$$\Rightarrow \int \underline{|(C_N^n)^{\frac{1}{2}} C_{N,n}|}^2 f^{\otimes 2} < +\infty$$

$$\| (C_N^n)^{\frac{1}{2}} C_{N,n} \|_{L^2(\mathbb{D}^2, f^{\otimes n})} < C$$

$n=2$: $\|(C_N^2)^{\frac{1}{2}} C_{N,n}\|_{L^2(\mathbb{D}^2, f^{\otimes n})} < C$

$\sim \boxed{\|N C_{N,n}\|_{L^2(\mathbb{D}^2, f^{\otimes n})} < C}$

① take subsequence of $\{N C_{N,2}\}_{N=1}^{\infty}$

$N C_{N,2} \xrightarrow{*} \bar{C}_2$

$H_n(t)$

② " $\bar{C}_2 \equiv 0$ " in distribution sense.

$\langle \bar{C}_2, C_{N,n} \rangle = 0$

☆ $H_n(t) = \{ g \in L^2(\mathbb{D}^n, f^{\otimes n}) : g$ is symmetric in its n varibles, $\boxed{\int g(z_1, \cdots z_n) f(t, z_j) dz_j = 0}$ for all $1 \le j \le n \}$

Step 1 : $\bar{C}_2 = ?$

$N C_{N,2}$ weak limit.

$\boxed{C_{N,2} \sim \boxed{(\phi, f)}}$

Backward Liouville / MFE

$\boxed{\text{What the equation}}$ of $C_{N,2}$.

$C_{N,2} \sim C_{N,3}\ C_{N,1}$

BBGKY

$C_{N,4} \sim \cdots$

1). Eq of $C_{N,2}$ is related with all correlation functions.

2). Eq of $C_{N,2}$ holds in the sense of distribution in the space $H_n(t)$.

Proposition: Assume $\nabla_v f \in L^1([0,T]\times \mathbb{D})$, For all $0 \le n \le N$, we have in the distributional sense on $[0,T]\times\mathbb{D}^n$:

Proposition 7. *Assume $\nabla_v f \in L^1([0,T]\times\mathbb{D})$. For all $0 \le n \le N$, we have in the distributional sense on $[0,T]\times\mathbb{D}^n$,*

(25)

$$\left\{\begin{aligned}
&\partial_t C_{N,n} + \sum_{i=1}^n v_i\cdot\nabla_{x_i} C_{N,n} + \alpha\sum_{i=1}^n \Delta_{v_i} C_{N,n} \\
&- \frac{N-n}{N-1}\sum_{j=1}^n \int_{\mathbb{D}} V_f(z_*,z_j)\, C_{N,n}(z_{[n]\setminus\{j\}},z_*)\, f(z_*)\, dz_* \\
&+ \frac{N-n}{N-1}\sum_{i=1}^n (K*f)(x_i)\cdot\nabla_{v_i} C_{N,n} + \frac{1}{N-1}\sum_{i\neq j}^n K(x_i-x_j)\cdot\nabla_{v_i} C_{N,n} \\
&- \frac{(N-n)(N-n-1)}{N-1}\int_{\mathbb{D}^2} V_f(z_*,z_*')\, C_{N,n+2}(z_{[n]},z_*,z_*')\, f(z_*)f(z_*')\, dz_*dz_*' \\
&+ \frac{N-n}{N-1}\sum_{i=1}^n \nabla_{v_i}\cdot\int_{\mathbb{D}} \left(K(x_i-x_*) - K*f(x_i)\right) C_{n+1}(z_{[n]},z_*)\, f(z_*)\, dz_* \\
&- \frac{N-n}{N-1}\sum_{j=1}^n \int_{\mathbb{D}} V_f(z_*,z_j)\, C_{N,n+1}(z_{[n]},z_*)\, f(z_*)\, dz_* \quad \cdot R_{N,n} = 0 \\
&+ \frac{1}{N-1}\sum_{i\neq j}^n K(x_i-x_j)\cdot\nabla_{v_i} C_{N,n-1}(z_{[n]\setminus\{j\}}) \quad = \quad R_{N,n},
\end{aligned}\right.$$

for some remainder term $R_{N,n} \in W^{-2,1}_{loc}([0,T]\times\mathbb{D}^n)$ that is orthogonal to H_n in the following weak sense,

$$\boxed{\int_0^T\int_{\mathbb{D}^n} h_n R_{N,n} = 0} \quad \text{for all } h_n \in C_c^\infty([0,T]\times\mathbb{D}^n)$$

$$\text{such that } \int_{\mathbb{D}} h_n(t,z_{[n]})\, dz_j = 0 \text{ a.e. for all } 1\le j\le n.$$

$$\underbrace{C_{N,n}(z_1, \cdots z_n)} = \sum_{k=0}^{n} \sum_{\sigma \in P_k^n} \underbrace{M_{N,k}(z_\sigma)}.$$

$$\begin{cases} \underbrace{M_{N,k}(\underbrace{z_1, \cdots z_k}))} = \int \Phi_N(z_1, \cdots z_N) f^{N-k}(z_{k+1}, \cdots z_N) \, dz_{k+1} \cdots dz_N \\ F_{N,k} : \text{BBGKY}. \end{cases}$$

Step 1.1 Eq of $M_{N,k}$ (Corresponding to $F_{N,k}$)

$$\partial_t M_{N,n} + \underline{\sum_{i=1}^{n} v_i \cdot \nabla_{x_i} M_{N,n}} + \alpha \underline{\sum_{i=1}^{n} \Delta_{v_i} M_{N,n}} \quad ① \quad ① \qquad F_{N,k}.$$

$$= \underline{-\frac{1}{N-1} \sum_{i \neq j}^{n} K(x_i - x_j) \cdot \nabla_{v_i} M_{N,n}} \quad ②$$

$$\underline{+ \frac{N-n}{N-1} \sum_{j=1}^{n} \int_{\mathbb{D}} V_f(z_*, z_j) \, \underline{M_{N,n+1}}(z_{[n]}, z_*) \, f(z_*) \, dz_*} \quad ③$$

$$- \frac{N-n}{N-1} \sum_{i=1}^{n} \int_{\mathbb{D}} K(x_i - x_*) \cdot \nabla_{v_i} M_{N,n+1}(z_{[n]}, z_*) \, f(z_*) \, dz_*$$

$$+ \frac{(N-n)(N-n-1)}{N-1} \quad ④$$

$$\times \int_{\mathbb{D}^2} V_f(z_*, z_*') \, \underline{M_{N,n+2}}(z_{[n]}, z_*, z_*') \, f(z_*) f(z_*') \, dz_* dz_*',$$

$$\partial_t M_{N,k} = \partial_t \int \Phi_N(z_1, \cdots z_N) f^{\otimes N-n}(z_{n+1}, \cdots z_N) \, dz_{n+1} \cdots dz_N$$

$$= \int \underline{\underline{\partial_t}} \Phi_N \underline{(z_1, \cdots z_N)} f^{\otimes N-n}(z_{n+1} \cdots z_N) \, dz_{n+1} \cdots dz_N$$

$$+ \left(\underline{\int \Phi_N(z_1, \cdots z_N) \, \underline{\partial_t f}^{\otimes N-n}(z_{n+1} \cdots dz_N) \, dz_{n+1} \cdots dz_N} \right)$$

$$\boxed{\partial_t \Phi_N} + \boxed{\sum_{i=1}^{N} v_i \cdot D_{x_i} \Phi_N} + \boxed{\frac{1}{N-1} \sum_{j \neq i} K(x_i - x_j) \cdot D_{v_i} \Phi_N}$$

$K * f$

$$= \boxed{-\alpha \sum_{i=1}^{N} \Delta_{v_i} \Phi_N}$$

$$-\int \sum_{i=1}^{N} \underline{v_i \cdot D_{x_i} \Phi_N} \; \underbrace{f^{\otimes N-n}}_{z_{k+1} \cdots z_N}$$

$$= \underline{-\int \sum_{i=1}^{k} v_i \cdot D_{v_i} \Phi_N f^{\otimes N-n}}$$

$$-\int \sum_{i=k+1}^{n} v_i \cdot D_{v_i} \Phi f^{\otimes N-n}$$

$$= -\sum_{i=1}^{k} v_i \cdot D_{v_i} \underbrace{\int \Phi_N f^{\otimes N-n}} + \sum_{i=k+1}^{n} \int \Phi_N v_i \cdot D_{v_i} f^{\otimes N-n}$$

$$= \underbrace{\boxed{-\sum_{i=1}^{k} v_i \cdot D_{v_i} M_{N,n}}} + \sum_{i=k+1}^{n} \int \Phi_N v_i \cdot D_{v_i} f^{\otimes N-n}$$

similarly:

$$-\int \sum_{i=1}^{N} \alpha \Delta_{v_i} \Phi_N f^{\otimes N-n}$$

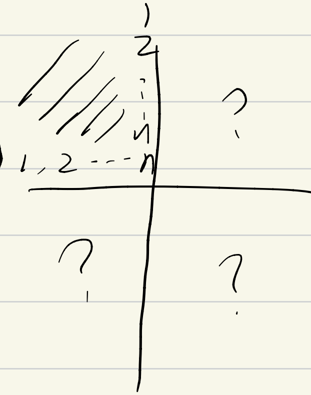
$$= \boxed{-\alpha \sum_{i=1}^{k} \Delta_{v_i} M_{N,k}} - \sum_{i=k+1}^{n} \int \Phi_N \Delta_{v_i} f^{\otimes N-n}$$

$$\partial_t M_{N,k} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} M_{N,n} + \alpha \sum_{i=1}^{k} \Delta_{v_i} M_{N,k}$$

$$= \sum_{i=k+1}^{n} \int \Phi_N v_i \cdot \cancel{\nabla_{x_i} f^{\otimes N-n}}$$

$$- \sum_{i=k+1}^{n} \int \Phi_N \cancel{\Delta_{v_i} f^{\otimes N-n}}$$

$$- \int \Phi_N \partial_t f^{\otimes N-n}$$



$$- \int \frac{1}{N} \sum_{i,j=1}^{N} K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N f^{\otimes N-n}$$

$$- \int \frac{1}{N} \sum_{i,j=1}^{N} K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N f^{\otimes N-n}$$

$$= - \int \frac{1}{N} \sum_{i,j=1}^{n} K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N f^{\otimes N-n} \quad ①$$

$$- \int \frac{1}{N} \sum_{i=1}^{n} \sum_{j=n+1}^{N} K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N f^{\otimes N-n} \quad ②$$

$$-\int \frac{1}{N}\sum_{i=n+1}^{N}\sum_{j=1}^{n} K(x_i-x_j)\cdot D_{v_i}\Phi_N f^{\otimes N-n} \quad ③$$

$$-\int \frac{1}{N}\sum_{i,j=n+1}^{N} K(x_i-x_j)\cdot D_{v_i}\Phi_N f^{\otimes N-n} \quad ④$$

$$\Rightarrow ① = \boxed{+\frac{1}{N}\sum_{i,j=1}^{n} K(x_i-x_j)\cdot \nabla_{v_i} M_{N,n}}$$

$$② := -\frac{1}{N}\sum_{i=1}^{n}\int \sum_{j=n+1}^{N} K(x_i-x_j)\cdot \nabla_{v_i}\Phi_N f^{\otimes N-n}$$

$$= -\frac{1}{N}\sum_{i=1}^{n}\nabla_{v_i}\int \sum_{j=n+1}^{N} K(x_i-x_j)\,\Phi_N f^{\otimes N-n}$$

$$= -\frac{N-n}{N}\sum_{i=1}^{n}\nabla_{v_i}\int K(x_i-x_j)\cdot \nabla_{v_i} M_{N,n+1}(z_{[n]}, z_*) f(z_*) f^{N-n-1}$$

$$= \boxed{-\frac{N-n}{N}\sum_{i=1}^{n}\nabla_{v_i}\cdot\int K(x_i-x_j)\cdot \nabla_{v_i} M_{N,n+1}(z_{[n]}, z_*) f(z_*) f^{N-n-1}}$$

similarly : 3

$$\frac{1}{N}\int \sum_{i=n+1}^{N}\sum_{j=1}^{n} K(x_i-x_j)\cdot\nabla_{v_i}\Phi_N f^{\otimes N-1}$$

$$=\frac{1}{N}\int \sum_{i=n+1}^{N}\sum_{j=1}^{n} K(x_i-x_j)\cdot\nabla_{v_i} f\,\Phi_N$$

$$-\frac{1}{N}\int \sum_{i=n+1}^{N}\sum_{j=1}^{n} K*f(x_i)\cdot\nabla_{v_i} f\,\Phi_N$$

$$d z_{n+1}\cdots dz_N$$

$$\Rightarrow \quad \frac{N-n}{N-1}\sum_{j=1}^{n}\int \underline{V_f(z_*, z_j)}\, M_{N,n+1}(z_{[n]}, z_*)\, f(z_*)\, dz_*.$$

4. $$\frac{1}{N}\int \underbrace{\sum_{i=n+1}^{N}}\ \sum_{\substack{j=n+1\\ i\neq j}}^{N} K(x_i-x_j)\cdot\nabla_{v_i}\Phi_N\, f^{\otimes N-2}$$

$$=\frac{\underbrace{(N-n)}(N-n-1)}{N}\int V_f(z_*, z_*')\,\underline{M_{N,n+2}}(z_{[n]}, z_*, z_*')$$

$f(z_*)\ f(z_*)$

dz_*

dz_*'

step 1.2.

$$\int \underline{h_n}\, \partial_t \underline{C_{N,n}} = \int \underline{h_n \boxed{\partial_t M_{N,n}}} \quad \text{in distribution sense.}$$

$$= T_1 + T_2 + \frac{N-n}{N-1}(T_3 + T_4) + \frac{(N-n)(N-n-1)}{N-1} T_5 .$$

$$
\begin{aligned}
T_1 &:= \int_{\mathbb{D}^n} M_{N,n}\Big(\sum_{i=1}^{n} v_i\cdot\nabla_{x_i} + \frac{1}{N-1}\sum_{i\neq j}^{n} K(x_i-x_j)\cdot\nabla_{v_i}\Big)h_n,\\
T_2 &:= -\alpha\sum_{i=1}^{n}\int_{\mathbb{D}^n} M_{N,n}\,\Delta_{v_i}h_n,\\
T_3 &:= \sum_{j=1}^{n}\int_{\mathbb{D}^{n+1}} V_f(z_{n+1},z_j)\,h_n(z_{[n]})M_{N,n+1}(z_{[n+1]})f(z_{n+1})\,dz_{[n+1]},\\
T_4 &:= \sum_{i=1}^{n}\int_{\mathbb{D}^{n+1}} K(x_i-x_{n+1})\cdot\nabla_{v_i}h_n(z_{[n]})\\
&\qquad\qquad \times M_{N,n+1}(z_{[n+1]})\,f(z_{n+1})\,dz_{[n+1]},\\
T_5 &:= \int_{\mathbb{D}^{n+2}} V_f(z_{n+1},z_{n+2})h_n(z_{[n]})M_{N,n+2}(z_{[n+2]})\,f(z_{n+1})f(z_{n+2})\,dz_{[n+2]}.
\end{aligned}
$$

$$T_1 = \int M_{N,n}\Big(\sum_{i=1}^{n} v_i\cdot\nabla_{x_i} + \frac{1}{N-1}\sum_{i\neq j}^{n} K(x_i-x_j)\cdot\nabla_{v_i}\Big)h_n$$

$$\Big(\text{by } \underbrace{\int\Big(\sum_{i=1}^{n} v_i\cdot\nabla_{x_i} + \frac{1}{N-1}\sum_{i\neq j} K(x_i-x_j)\cdot\nabla_{v_i}\Big)h_n\,dz_j}_{=0}\Big)$$

$$= \int C_{N,n} \left(\sum_{i=1}^{n} v_i \cdot \nabla_{x_i} + \frac{1}{N-1} \sum_{i \neq j}^{n} K(x_i - x_j) \cdot \nabla_{v_i} \right) h_n$$

T_2, T_3, T_4, T_5 - - -

Proposition 7. *Assume $\nabla_v f \in L^1([0,T] \times \mathbb{D})$. For all $0 \leq n \leq N$, we have in the distributional sense on $[0,T] \times \mathbb{D}^n$,*

orthogonal

$M_{N,n} \rightsquigarrow C_{N,n}$.

(25)

$$\begin{aligned}
&\partial_t C_{N,n} + \sum_{i=1}^{n} v_i \cdot \nabla_{x_i} C_{N,n} + \alpha \sum_{i=1}^{n} \Delta_{v_i} C_{N,n} \\
&- \frac{N-n}{N-1} \sum_{j=1}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, C_{N,n}(z_{[n]\setminus\{j\}}, z_*)\, f(z_*)\, dz_* \\
&+ \frac{N-n}{N-1} \sum_{i=1}^{n} (K * f)(x_i) \cdot \nabla_{v_i} C_{N,n} + \frac{1}{N-1} \sum_{i \neq j}^{n} K(x_i - x_j) \cdot \nabla_{v_i} C_{N,n} \\
&- \frac{(N-n)(N-n-1)}{N-1} \int_{\mathbb{D}^2} V_f(z_*, z_*')\, C_{N,n+2}(z_{[n]}, z_*, z_*')\, f(z_*) f(z_*')\, dz_* dz_*' \\
&+ \frac{N-n}{N-1} \sum_{i=1}^{n} \nabla_{v_i} \cdot \int_{\mathbb{D}} \left(K(x_i - x_*) - K * f(x_i) \right) C_{n+1}(z_{[n]}, z_*)\, f(z_*)\, dz_* \\
&- \frac{N-n}{N-1} \sum_{j=1}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, C_{N,n+1}(z_{[n]}, z_*)\, f(z_*)\, dz_* \\
&+ \frac{1}{N-1} \sum_{i \neq j}^{n} K(x_i - x_j) \cdot \nabla_{v_i} C_{N,n-1}(z_{[n]\setminus\{j\}}) \quad = \quad R_{N,n},
\end{aligned}$$

for some remainder term $R_{N,n} \in W^{-2,1}_{loc}([0,T] \times \mathbb{D}^n)$ that is orthogonal to H_n in the following weak sense,

$$\int_0^T \int_{\mathbb{D}^n} h_n R_{N,n} = 0 \quad \text{for all } h_n \in C_c^\infty([0,T] \times \mathbb{D}^n)$$

$$\text{such that } \int_{\mathbb{D}} h_n(t, z_{[n]})\, dz_j = 0 \text{ a.e. for all } 1 \leq j \leq n.$$

· Bogolyubov : Problems of a Dynamical Theory in statistical Physics.

C^∞	L^∞	$K \in L^2_{loc}$
21	23	24