Phase Transition, Logarithmic Sobolev Inequalities (LSI)

$\underbrace{\text{Phase Transition}}_{①}$, $\underbrace{\text{Logarithmic Sobolev Inequalities (LSI)}}_{②}$

Uniform-in-time Propagation of Chaos

③

Consider IPS:

$$\begin{cases} dX_t^i = -\nabla V(X_t^i)\,dt - \frac{1}{N}\sum_{j=1}^{N} D_1 W(X_t^i, X_t^j)\,dt + \underbrace{\sqrt{2/\beta}\, dB_t^i}_{\text{Heat Bath}} \\ \mathrm{Law}(X_0^1, \cdots, X_0^N) = \rho_{in}^{\otimes N} \in \mathcal{P}_{2,sym}((\mathbb{R}^d)^N) \end{cases}$$

On Euclidean space

$(X_t^i \in \mathbb{R}^d)$

Classical Results : POC

$V, W \leftarrow$ semi-convex

as $N \to \infty$, tends to McKean-Vlasov PDE

(MFE) (Do $N \to \infty$ first)

$$\partial_t \rho = \frac{1}{\beta}\Delta\rho + \nabla\cdot(\rho(\nabla V + \nabla W * \rho))$$

$$\begin{cases} \partial_t \rho = \frac{1}{\beta} \Delta \rho + \nabla \cdot (\rho \nabla V + \rho \nabla W * \rho) \\ \rho(0) = \rho_{in} \end{cases}$$

(Lee-Yang) infinite volume

where $W * \rho(x) := \int_{\mathbb{R}^d} W(x, y)\, d\rho(y)$

$$M_N = \frac{1}{Z_N} e^{-\beta H_N}$$

Check: the uniqueness of the steady state of N-particle Fokker-Planck.

Take

$$\rho^N(t) = \mathrm{Law}(X_t^1, \dots, X_t^N) \in P_{sym}(\mathbb{R}^{dN})$$

$$\begin{cases} \partial_t \rho^N = \frac{1}{\beta} \Delta \rho^N + \nabla \cdot (\rho^N \underbrace{\nabla H_N}) \\ \rho^N(0) = \rho_{in}^{\otimes N} \end{cases}$$

where

$$H_N(x) = \sum_{i=1}^N V(x_i) + \frac{1}{2N} \sum_{i,j=1}^N W(x_i, x_j)$$

$x = (x_1, \dots, x_N)$

$(W(x,x)=0)$

N, β - fixed.

$$M_N = \frac{1}{Z_N} e^{-\beta H_N} \quad (\sim V_N)$$

$$\partial_t \rho^N = \nabla \cdot \left(\rho^N \left(\frac{1}{\beta} \nabla \log \rho^N + \nabla H_N \right) \right)$$

$$\bar{\mathcal{E}}(\rho^N(t) \,|\, M_N) = \frac{1}{N} \int_{\mathbb{R}^{dN}} \rho^N(t) \log \frac{\rho^N(t)}{M_N} \, dx$$

$$\frac{d}{dt} \bar{\mathcal{E}}(\rho^N(t) \,|\, M_N) = -\beta^{-1} \frac{1}{N} \int \left| \nabla \log \left(\frac{\rho^N(t)}{M_N} \right) \right|^2 \rho^N$$

(Exponential Decay) $\Leftarrow$

$$= -\beta^{-1} \bar{I}(\rho^N(t) \,|\, M_N)$$

$$\leq -\lambda_{LS}^N \bar{\mathcal{E}}(\rho^N(t) \,|\, M_N)$$

$\Rightarrow$ Uniqueness of the steady state

(if ρ_{in} = another steady state $\tilde{\rho} \neq M_N$

$\leftarrow$ $\boxed{\text{Free Energy}}$ Gibbs

$$\bar{\xi}(\rho^N(t) \mid M_N) \leftarrow$$

is one steady state,

$$= E^N[\rho^N] - E^N[M_N]$$

$$= \frac{1}{N}\left(\frac{1}{\beta}\int \rho^N \log \rho^N + \int \rho^N H_N \, dx\right) - \text{const.}$$

$$\rho_N(t) \equiv \tilde{\rho}_{in} \implies \frac{d}{dt}\bar{\xi}(\rho_N(t) \mid M_N) \equiv 0$$

while $RHS = -\beta^{-1}\bar{I}(\rho^N(t) \mid M_N) < 0$

$$= -\beta^{-1}\bar{I}(\rho_{in} \mid M_N) < 0 \;)$$

$$\bar{\xi}(\rho^N(t) \mid M_N) = -\frac{1}{\beta}\cdot \text{Entropy} + \text{Energy}$$

Temperature

$$-\text{Entropy} = \int f \log f.$$

Langevin: $dX_t = -\nabla V(X_t) + \sqrt{\frac{2}{\beta}}\, dW_t^{i}$

$$\text{Gibbs} = Z e^{-\beta V(x)}$$

$$\Big(\beta^{-1}\Delta\rho^N + \nabla\cdot(\rho^N \nabla H_N) = 0$$

$$\nabla\cdot\left(\rho^N\left(\tfrac{1}{\beta}\nabla\log\rho^N + \nabla H_N\right)\right) = 0$$

$$\nabla\cdot\left(\rho^N \quad \nabla\log\frac{\rho^N}{M_N}\right) = 0 \Big)$$

Fix β, Fix N.

uniqueness of N-particle system ✓

almost trivial.

Mean-Field PDE: V, W.

$\underbrace{\beta < \beta_c}_{\text{uniqueness}}$ $\qquad \beta = \beta_c \qquad \beta > \beta_c$

high temperature $\qquad$ multiply steady states

Digression:

Burgers $\begin{cases} \partial_t u + \frac{1}{2}\partial_x(u^2) = 0 \\ u_0 = a\delta_{x=0} \end{cases}$ time asymptotic N-wave / VF

Viscous $\quad \big\{ \partial_t u + \frac{1}{2}\partial_x(u^2) = \boxed{\varepsilon\partial_x^2 u}$ $\quad \varepsilon \to 0$

" L D D "

LDP

L $\quad u_0 = a\delta_{x=0}$ $\quad$ <u>EW</u>

$u \sim$ Heat Kernel

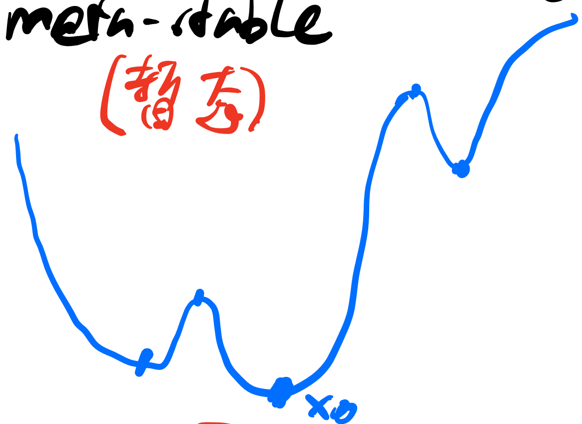
(For fixed, discuss the time interval according ε)

$\varepsilon \to 0$ inviscous limit $\quad$ (time-scale??)

N-wave: vs of meta-stable

(暂态)

Literature:



$$dX_t = -\nabla V(X_t) + \sqrt{\tfrac{2}{\beta}}\, dW_t$$

Estimate exit time ....

$\boxed{\text{Shock Waves}}$ Liu Taiping

Uniform-in-time POC

= Uniform approximation of Mean-field PDEs by N-particle system which is possible when the PDE admits unique stationary state.

$$\rho_{in}^{\otimes N} \xrightarrow[N\text{-particle}]{t \to \infty} \mu_N$$

$$\begin{tikzcd}
 & \rho_{in} \arrow[dl] \arrow[dr] & & \\
\rho_{in}^{1} \arrow[rrrd, bend right, "\text{MFE}"'] & & \rho_{in}^{2} \arrow[r, "\text{MFE}", "t \to \infty"'] & \hat{\rho}_1 \ \text{(steady state)} \\
 & & & \hat{\rho}_2 \ \text{(steady state)}
\end{tikzcd}$$

$$\rho_N(t) \approx \rho_t^{\otimes N}$$

$$\uparrow$$

$$\frac{1}{2}(\rho_{in}^{1})^{\otimes N} + \frac{1}{2}(\rho_{in}^{2})^{\otimes N}$$

"Study" the time scale of validity of mean-field approximation.

Question:

For MFE with two steady states, $\hat{\rho}_1$, $\hat{\rho}_2$ (Dawson 1983, JSP)

$$\rho_{in} = \underbrace{\tfrac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)}_{\text{NOT necessarily a steady state.}} \quad X$$

$$\rho_{in}^N = \tfrac{1}{2}\left(\hat{\rho}_1^{\otimes N} + \hat{\rho}_2^{\otimes N}\right) \in P_{sym}(\mathbb{R}^{dN})$$

N-particle

$M_N \leftarrow$ Gibbs

$T_{>0}$ chosen

Random Matrices

Dyson Brownian Motions. (linear Fokker-Planck)

Wishart : ??

POC

① $[0, T]$ Any Fix T

$$\| \mu_N(t) - \rho_t \|_{\square} \le e^{Ct} \| \mu_N(0) - \rho_0 \|$$

② Compactness

③ uniform-in-time

Time length of Validity Mean-field Approximation

Maybe look at derivation of Landau type kinetic eq. from physical point of view.

- $\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \varepsilon Q(f,f)$

Kernels & Potentials

MFE

$\varlimsup_{N \to \infty} \lambda_{LS}^N = c_0 > 0$

$V = 0$

$W =$

V, W – K-semi-convex

$K \in \mathbb{R}$ $\begin{cases} D^2 V \geq \underline{\underline{K_V}} \mathbb{I}^{d \times d} \\ D^2 W \geq \underline{\underline{K_W}} \mathbb{I}^{2d \times 2d} \end{cases}$

$V, W \geq -C$

(v) W. [illegible]

$$|V(x)| \geq |x|^{\delta} \quad \text{for} \quad |x| > R_0$$

$$|D_1 W(x,y)| \leq C\big(1 + |W(x,y)| + V(x) + V(y)\big)$$

Desai-Zwanzig model.

$$\begin{cases} V(x) = (1-|x|^2)^2 \\ W(x,y) = |x-y|^2 \end{cases}$$

Do formal asymptotic analysis first

Phase Transition by Analysis

$$dx(t) = \left[-x^3(t) + x(t)\right]dt + \sigma\, dw(t)$$

System: $(\phi^4_d \leftarrow \dots)$

$$dx_j = (-x_j^3 + x_j)\,dt + \sigma\, dw_j(t) - \theta(x_j - \bar{x})\,dt$$

Gibbs $\leftrightsquigarrow$ 类比 ϕ^4 - Euclidean lattice field on $\mathbb{Z}^d$.