Lecture 1: Intro to SK model

Sherrington - Kirkpatrick model (SK model)

Spin glass model

Hamiltonian $\sigma \in \{\pm 1\}^N$

$$H_N(\sigma) = \frac{1}{2\sqrt{N}} \sum_{i,j} g_{ij} \sigma_i \sigma_j = \frac{1}{2\sqrt{N}} \sigma^T G \sigma ;$$

$$g_{ij} = g_{ji} \overset{iid}{\sim} N(0,1) , \quad g_{ii} \overset{iid}{\sim} N(0,2)$$

$G :=$ Wigner matrix $\quad G = G^T ;$

Question: ground-state energy

$$\max_{\sigma} H_N(\sigma) \qquad \sigma_i \in \{\pm 1\}$$

Relaxation

$$H_N(\sigma) = \frac{1}{2\sqrt{N}} \sigma^T G \sigma \le \frac{1}{2\sqrt{N}} \|G\| \cdot \|\sigma\|^2$$

$$= \frac{1}{2\sqrt{N}} \cdot N \cdot 2\sqrt{N} (1+o(1)) = N(1+o(1))$$

$$\|G\|_{op} = N(1+o(1))$$

Relax: $\{\pm 1\}^N \to \sqrt{N} S^N$

$$\max_{\sigma} \frac{1}{N} H_N(\sigma)$$

Spectral relaxation

max ≤ 1 for N sufficiently large

Thm:

$$\lim_{N\to\infty} \frac{1}{N} \max_{\sigma} H_N(\sigma) = P^*$$

$P^* \approx 0.7632\cdots$ Spectral : upper bound 1.

Algo: poly-nomial time algo to approximate the maximizer, etc

SOS, AMP, ...

$\hookrightarrow$ ε-error approximate solution. (Montanari etc)

① Heuristic: Parisi formula.

② Talagrand 2006 Annals of Math.;

Guerra

③ Panchenko: ultra metricity, etc

To do: ① Parisi formula

② $\lim_{N\to\infty} \frac{1}{N} H_N(\sigma)$ exists.

$$H_N(\sigma) = \frac{1}{2\sqrt{N}} \sigma^T G \sigma \sim N(0, ?)$$

Gaussian field

G: Wigner $g_{ij} \sim N(0,1)$, $i \neq j$

$g_{ii} \sim N(0,2)$

$$\sigma^T G \sigma \overset{d}{=} \boxed{\sqrt{2}\, \sigma^T \tilde{G} \sigma}$$

$\tilde{G}$: $\tilde{g}_{ij} \overset{iid}{\sim} N(0,1)$

$$2\sum_{i<j} g_{ij}\sigma_i\sigma_j + \sum_{i=1}^N g_{ii}$$

$$\sqrt{2}\, \sigma^T \tilde{G} \sigma = \sqrt{2} \sum_{i,j} \tilde{g}_{ij}\sigma_i\sigma_j$$

$$= \sqrt{2} \underbrace{\sum_{i<j} (\tilde{g}_{ij} + \tilde{g}_{ji})}_{N(0,2)} \sigma_i\sigma_j + \sqrt{2} \sum_i \tilde{g}_{ii}$$

$$\mathbb{E} H_N(\sigma) H_N(\sigma')$$

$$= \frac{1}{4N} \mathbb{E}\; \sigma^T G \sigma \;\sigma'^T G \sigma' \quad = \frac{N}{2} R_{1,2}^2$$

$$= \frac{1}{2N} \mathbb{E} \boxed{\sigma^T \tilde{G} \sigma \;\sigma'^T \tilde{G} \sigma'} = \boxed{\frac{1}{2N} \langle \sigma, \tilde{\sigma} \rangle^2}$$

Def: $R_{1,2} = \frac{1}{N} \langle \sigma, \tilde{\sigma} \rangle \in [-1, 1]$

$$= \frac{N}{2} R_{1,2}^2$$

In particular, $\sigma = \sigma'$

$$\mathrm{Var}\, H_N(\sigma) = \frac{N}{2}$$

Note:

$\langle \sigma, G\sigma \rangle \langle \sigma', G\sigma' \rangle$ $\qquad$ ξ : G vectori zeta

$= \langle \xi, \sigma \otimes \sigma \rangle \langle \xi, \sigma' \otimes \sigma' \rangle$

$= (\sigma \otimes \sigma)^T \xi \xi^T (\sigma' \otimes \sigma') = |\langle \sigma, \sigma' \rangle|^2$

Partition function

$$Z_N(\beta) = \sum_\sigma \exp(\beta H_N(\sigma))$$

$\log Z_N(\beta)$

soft-max

$\geq \exp(\beta \max H_N(\sigma))$

$$\exp(\beta \max_\sigma H_N(\sigma)) \leq Z_N(\beta) \leq 2^N \exp(\beta \max_\sigma H_N(\sigma))$$

$$\frac{1}{N} \max_\sigma H_N(\sigma) \leq \frac{1}{N\beta} \log Z_N(\beta)$$

$$\leq \frac{1}{N\beta} \left(N \log 2 + \beta \max_\sigma H_N(\sigma) \right)$$

Then,

$$\left| \frac{1}{N} \max H_N(\sigma) - \frac{1}{N\beta} \log Z_N(\beta) \right| \leq \frac{\log 2}{\beta}, \quad \forall \beta > 0$$

It suffices to study the free energy

$$F_N(\beta) = \frac{1}{N} \mathbb{E} \log Z_N(\beta).$$

$\mathbb{E} \hookrightarrow$ over Gaussian.

$$\left| \frac{1}{N} \mathbb{E} \max_\sigma H_N(\sigma) - \frac{F_N(\beta)}{\beta} \right| \le \frac{\log 2}{\beta} \quad *$$

Explain. Since $\frac{1}{N} \max_\sigma H_N(\sigma)$: sub-gaussian,

it concentrates well around its expectation

Lemma:
$$\lim_{N\to\infty} \left| \frac{1}{N} \max_\sigma H_N(\sigma) - \frac{1}{N} \mathbb{E} \max_\sigma H_N(\sigma) \right| \overset{a.s.}{=} 0$$

— $X \sim N(0, I_n)$; f: $\underline{L}$-Lip

$$\mathbb{P}(\| f(x) - \mathbb{E} f(x) \| \ge a) \le \exp\left(-\frac{a^2}{4L^2}\right)$$

— Borel-Cantelli

Question ① $\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_\sigma H_N(\sigma)$ exists ?

② $\lim_{N\to\infty} F_N(\beta)$ exists ?

③ If ② holds, $F(\beta) = \lim_{N\to\infty} F_N(\beta)$,

whether $\lim_{\beta\to\infty} \frac{F(\beta)}{\beta}$ exists

③: If $F(\beta)$ exists, then

$$\lim_{\beta\to\infty}\frac{F(\beta)}{\beta} \text{ exists}$$

and

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\max_{\sigma}H_N(\sigma) = \lim_{\beta\to\infty}\frac{F(\beta)}{\beta}.$$

It suffices to show: the func below is non-decreasing

$$\frac{1}{\beta}\left(F_N(\beta)-\log 2\right)$$

$$= \frac{1}{\beta}\left(\frac{1}{N}\mathbb{E}\log Z_N(\beta) - \log 2\right)$$

$$= \frac{1}{\beta}\left(\frac{1}{N}\mathbb{E}\log\sum_{\sigma}\exp(\beta H_N(\sigma)) - \log 2\right)$$

$$= \frac{1}{\beta N}\,\mathbb{E}\log\frac{1}{2^N}\sum_{\sigma}\exp(\beta H_N(\sigma))$$

For $\beta' > \beta$, let $p = \frac{\beta'}{\beta}$, $q = \frac{\beta'}{\beta'-\beta}$, then apply Hölder inequality to

$$\frac{1}{2^N}\sum_{\sigma}\exp(\beta H_N(\sigma)) \le \left(\sum_{\sigma}\exp(\beta H_N(\sigma)\cdot p)\right)^{\frac{1}{p}}\left(\sum_{\sigma}\left(\frac{1}{2^N}\right)^{q}\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{2^N}Z_N(\beta')\right)^{\beta/\beta'}$$

Take log; divided by βN. $\rightarrow$ gives the result.

In conclusion,

$$\frac{1}{\beta}\left(\bar{F}_N(\beta) - \log 2\right) \quad \uparrow$$

Suppose $N \to \infty$, $\bar{F}_N(\beta)$ exists, $\frac{1}{\beta}\left(\bar{F}(\beta) - \log 2\right)$ is non-decreasing in β;

It implies $\underline{\frac{1}{\beta}\left(\bar{F}(\beta) - \log 2\right)}$ exists as $\beta \to \infty$.

Then $\frac{F(\beta)}{\beta}$ has a limit.

$$\lim_{\beta\to\infty} \left| \underline{\underline{\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_{\sigma} H_N(\sigma)}} - \frac{F(\beta)}{\beta} \right| \le \frac{\log 2}{\beta} \to 0$$

Why does $\lim_{N\to\infty} \underline{\underline{F_N(\beta)}}$ exist?

Lemma. If $\{X_n\}_{n\ge 1}$ is sup-additive, $X_{n+m} \ge X_m + X_n$, then

$$\lim_{n\to\infty} \frac{X_n}{n} = \sup_{n\ge 1} \frac{X_n}{n}.$$

It suffices to show $\underline{\underline{N \bar{F}_N(\beta)}}$ is sup-additive

Tool: interpolation

$$(M+N)\, \underline{\underline{F_{M+N}(\beta)}} \le M\, \underline{F_M(\beta)} + N\, \underline{F_N(\beta)}$$

$\sigma : \{\pm 1\}^{N+M}$ $\quad \rho : \{\pm 1\}^{N}$, $\tau \in \{\pm 1\}^{M}$

$\sigma = [\rho, \tau]$

$$H_t(\sigma) = \sqrt{t}\, H_{N+M}(\sigma) + \sqrt{1-t}\,\left(H_N(\rho) + H_M(\tau)\right)$$

Rk: H_{M+N}, H_M, H_N independent.

① $t=0$, $\quad H_0(\sigma) = H_N(\rho) + H_M(\tau)$

$t=1$ $\quad H_1(\sigma) = H_{N+M}(\sigma)$

② $\mathbb{E} H_t(\sigma)^2 = t\, \mathbb{E} H_{N+M}(\sigma)^2 + (1-t)\left(\mathbb{E} H_N^2 + \mathbb{E} H_M^2\right)$

$$= t\left(\frac{N+M}{2}\right) + (1-t)\left(\frac{N}{2} + \frac{M}{2}\right)$$

$$= \frac{N+M}{2}.$$

$g(t) = \mathbb{E} \log Z_t$ $\qquad Z_t = H_t(\sigma)$ - partition function

$$g(0) = \mathbb{E} \log Z_0 = \mathbb{E} \log \sum_{\rho,\tau} \exp\left(\beta\left(H_N(\rho) + H_M(\tau)\right)\right)$$

$$= \mathbb{E}\left[\log \sum_{\rho} \exp(\beta H_N(\rho)) + \log \sum_{\tau} \exp(\beta H_M(\tau))\right]$$

$$= N\, F_N(\beta) + M\, F_M(\beta)$$

$$g(1) = \mathbb{E} \log \sum_{\sigma} \exp\left(\beta H_{N+M}(\sigma)\right)$$

$$= (M+N)\, F_{N+M}(\beta)$$

$g(0) \le g(1)$ $\longrightarrow$ sup-add

It suffices to show $g'(t) > 0$

$$g(t) = \mathbb{E} \log Z_t$$

$$g'(t) = \frac{d}{dt} \mathbb{E} \log \sum_\sigma \exp(\beta H_t(\sigma))$$

Gaussian

$$= \sum_\sigma \mathbb{E} \boxed{\frac{1}{\underset{H_t(\sigma')}{Z_t}} \exp(\beta \underset{\text{Gaussian}}{\underline{H_t(\sigma)}}) \cdot \beta \cdot \underset{\ge 0}{\underline{\frac{\partial H_t(\sigma)}{\partial t}}}}$$

How to compute it?

Gaussian - integration by parts

$$\mathbb{E}\, g_1 F(g) = \sum_{\ell=1}^{n} \mathbb{E}\, g_1 g_\ell \cdot \mathbb{E} \frac{\partial F}{\partial x_\ell}(g)$$

$g \sim N(0, \Sigma)$

let $g_1 = \frac{\partial H_t(\sigma)}{\partial t}$, $\quad g_{\sigma'} = H_t(\sigma')$

$$F = \frac{\exp(\beta g_\sigma)}{\sum_{\sigma'} \exp(\beta g_{\sigma'})}$$

$\hookrightarrow$ It only depends on $R_{1,2}$.

Parisi formula.

What is the limit of $\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \max_{\sigma} H_N(\sigma)$?

↳ Free energy

Focus $\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log \boxed{Z_N(\beta)} = ?$ $\forall \beta$

① $0 = \xi_0 < \xi_1 < \cdots < \xi_{k+1} = 1$

② $0 = m_0 < m_1 < \cdots < m_k = 1$

m_k: a parameter attached to $[\xi_k, \xi_{k+1})$

Def: $\boxed{\xi(x) = x^2/2}$ → $\mathbb{E} H_N(\sigma) H_N(\sigma') = N \xi(R_{1,2})$

Formula by Parisi:

$$\lim_{N\to\infty} F_N(\beta) = P(\xi) ; \quad P(\xi) = \inf_{m, \xi, k} P_k(\xi, m, \xi)$$

$$P_k(\xi, m, \xi) = \log 2 + X_0 - \frac{1}{2} \sum_{\ell=1}^{k} \frac{\beta^2}{2} m_\ell (\xi_{\ell+1}^2 - \xi_\ell^2)$$

Start: k. define for each ℓ,

$$z_\ell \sim N(0, \beta^2(\xi_{\ell+1} - \xi_\ell)) \quad \text{independent}$$

$$X_{k+1} = \log\cosh\left(\sum_{\ell=0}^{k} z_\ell\right) \to X_\ell = \frac{1}{m_\ell} \log \mathbb{E}_\ell e^{m_\ell X_{\ell+1}}$$

$\ell : k \to 1$, $X_0 = \mathbb{E} X_1$ is a func of m, ξ, β