Soccer Freekick Simulation

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1 Introduction

Initial conditions:

- position: (x_0, y_0, z_0) [m]
- velocity: v_0 $[\frac{m}{s}]$
- angular velocity: $\vec{\omega}_0$ $[\frac{rad}{s}]$
- Ball radius: R [m]
- Ball mass: m [kg]
- Air density: $\rho = 1.225[\frac{kg}{m^3}]$
- Coefficient of drag: $C_D = 0.5$

Forces acting on ball:

- Lift: $\vec{L} = \frac{4}{3}(4\pi^2 R^3 \rho)(\vec{\omega}_0 \times \vec{V})$
- Drag: $\vec{D} = -\frac{1}{2}\rho|\vec{V}|^2(\pi R^2)c_D(\frac{\vec{V}}{|\vec{V}|})$
- Gravity: $\vec{F} = -(mg)\hat{k}$

Assumptions:

- Angular velocity vector (ω_0) is constant in magnitude and direction for entire trajectory (no air resistance to slow down angular velocity)
- Simplified expressions for lift and drag
- No external collisions during trajectory
- Change in gravity is negligible

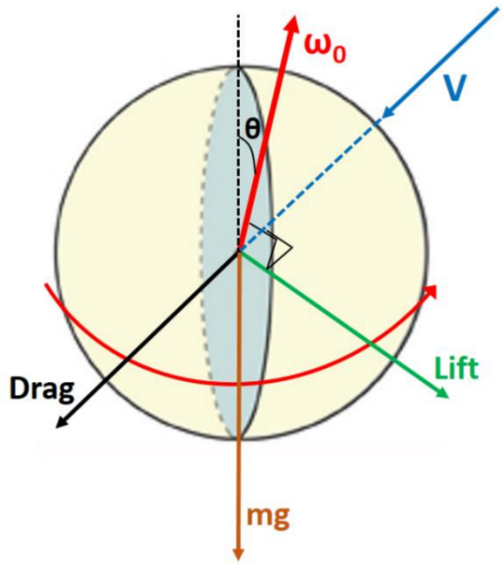

Figure 1: Diagram of Ball with forces and conditions

2 Differential Equations for Translational Motion

1. Begin with Newton's 2nd Law of Motion.

   $\sum \vec{F} = m\vec{a}$

   $\Rightarrow \vec{L} + \vec{D} + \vec{F} = m\vec{a}$
   $\Rightarrow \frac{4}{3}(4\pi^2 R^3 \rho)(\vec{\omega}_0 \times \vec{V}) - \frac{1}{2}\rho|\vec{V}|^2(\pi R^2)c_D(\frac{\vec{V}}{|\vec{V}|}) - (mg)\hat{k} = m\vec{a}$

2. Let $A = \frac{4}{3}(4\pi^2 R^3 \rho)$ and $B = \frac{1}{2}\rho(\pi R^2)C_D$ and $C = mg$.

   $\Rightarrow A(\vec{\omega}_0 \times \vec{V}) - B|\vec{V}|^2(\frac{\vec{V}}{|\vec{V}|}) - C\hat{k} = m\vec{a}$
   $\Rightarrow A(\vec{\omega}_0 \times \vec{V}) - B|\vec{V}|\vec{V} - C\hat{k} = m\vec{a}$

3. Expand and simplify further.
   $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{r}^2}{dt^2}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,
   $\vec{\omega}_0 = \omega_{0x}\hat{i} + \omega_{0y}\hat{j} + \omega_{0z}\hat{k}$ and $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

   $\Rightarrow A[(\omega_{0y}V_z - \omega_{0z}V_y)\hat{i} + (\omega_{0z}V_x - \omega_{0x}V_z)\hat{j} + (\omega_{0x}V_y - \omega_{0y}V_x)\hat{k}]$
   $- B|\vec{V}|[V_x\hat{i} + V_y\hat{j} + V_z\hat{k}] - C\hat{k} = m\frac{d\vec{r}^2}{dt^2}$

4. Split equation into 3 axes.

$$\Rightarrow \begin{cases} A(\omega_{0y}V_z - \omega_{0z}V_y) - B|\vec{V}|V_x = m\frac{d^2x}{dt^2} \\ \\ A(\omega_{0z}V_x - \omega_{0x}V_z) - B|\vec{V}|V_y = m\frac{d^2y}{dt^2} \\ \\ A(\omega_{0x}V_y - \omega_{0y}V_x) - B|\vec{V}|V_z - C = m\frac{d^2z}{dt^2} \end{cases}$$

5. Convert to derivatives to dot notation.

$$V_x = \frac{dx}{dt} = \dot{x}, \frac{d^2x}{dt^2} = \ddot{x}$$
$$V_y = \frac{dy}{dt} = \dot{y}, \frac{d^2y}{dt^2} = \ddot{y}$$
$$V_z = \frac{dz}{dt} = \dot{z}, \frac{d^2z}{dt^2} = \ddot{z}$$

$$\Rightarrow \begin{cases} A(\omega_{0y}\dot{z} - \omega_{0z}\dot{y}) - B|\vec{V}|\dot{x} = m\ddot{x} \\ A(\omega_{0z}\dot{x} - \omega_{0x}\dot{z}) - B|\vec{V}|\dot{y} = m\ddot{y} \\ A(\omega_{0x}\dot{y} - \omega_{0y}\dot{x}) - B|\vec{V}|\dot{z} - C = m\ddot{z} \end{cases}$$

6. Substitute constants back in ($A = \frac{4}{3}(4\pi^2 R^3 \rho)$, $B = \frac{1}{2}\rho(\pi R^2)C_D$, $C = mg$).

$$\Rightarrow \begin{cases} m\ddot{x} = \frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0y}\dot{z} - \omega_{0z}\dot{y}) - \frac{1}{2}\rho(\pi R^2)C_D|\vec{V}|\dot{x} \\ m\ddot{y} = \frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0z}\dot{x} - \omega_{0x}\dot{z}) - \frac{1}{2}\rho(\pi R^2)C_D|\vec{V}|\dot{y} \\ m\ddot{z} = \frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0x}\dot{y} - \omega_{0y}\dot{x}) - \frac{1}{2}\rho(\pi R^2)C_D|\vec{V}|\dot{z} - mg \end{cases}$$

7. Isolate 2nd order derivative terms.

$$\ddot{x} = \frac{\frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0y}\dot{z} - \omega_{0z}\dot{y}) - \frac{1}{2}\rho(\pi R^2)C_D|\vec{V}|\dot{x}}{m} \tag{1}$$
$$\ddot{y} = \frac{\frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0z}\dot{x} - \omega_{0x}\dot{z}) - \frac{1}{2}\rho(\pi R^2)C_D|\vec{V}|\dot{y}}{m} \tag{2}$$
$$\ddot{z} = \frac{\frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0x}\dot{y} - \omega_{0y}\dot{x}) - \frac{1}{2}\rho(\pi R^2)C_D|\vec{V}|\dot{z}}{m} - g \tag{3}$$

3 State Vector for Matlab ODE45

Function format: [t,y] = ode45(odefunc,tspan,r0,options)

Inputs:

- odefunc: A function returning a column vector containing the expressions to integrate
  - $$= \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \frac{\frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0z}\dot{y} - \omega_{0y}\dot{z}) - \frac{1}{2}\rho(\pi R^2) C_D |\vec{V}| \dot{x}}{m} \\ \frac{\frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0x}\dot{z} - \omega_{0z}\dot{x}) - \frac{1}{2}\rho(\pi R^2) C_D |\vec{V}| \dot{y}}{m} \\ \frac{\frac{4}{3}(4\pi^2 R^3 \rho)(\omega_{0y}\dot{x} - \omega_{0x}\dot{y}) - \frac{1}{2}\rho(\pi R^2) C_D |\vec{V}| \dot{z}}{m} - g \end{pmatrix}$$
- tspan: A row vector containing the time intervals for which you want the simulation to run on
- r0: A row vector containing the initial conditions for which would be used to solve the expressions in odefunc
  - r0 = $\begin{pmatrix} x(0) \\ y(0) \\ z(0) \\ \dot{x}(0) \\ \dot{y}(0) \\ \dot{z}(0) \end{pmatrix}$
- options: Settings that can be set using odeset() function
  - An event function that is triggered when the simulation trajectory reaches the 0 z-level

Outputs:

- t: A row vector containing the time intervals for the simulation results
- r: Array containing the solved column vectors for each expression in odefunc
  - r = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$