Classical Laminate Theory

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This document provides a summarized walk-through of Classical Laminate Theory (CLT) using George Staab's *Laminar Composites* [1] as a reference.

1 Lamina Properties

Basic material properties are determined empirically, and must be known in order to determine the lamnia's characteristic matrices and the resulting loads and stresses. These values are:

Symbol	Description
$\boldsymbol{E_{11}}$	elastic modulus in the ply's 0°-direction
$\boldsymbol{E_{22}}$	elastic modulus in the ply's 90°-direction
$\boldsymbol{\nu_{12}}$	Poisson's ratio in the 2-direction when the lamina is loaded in the 1-direction
$\boldsymbol{G_{12}}$	Shear modulus in the 12-plane
$\boldsymbol{t_k}$	thickness of the lamina
$\boldsymbol{\alpha_{11}}$	thermal expansion coefficient in the in the ply's 0°-direction
$\boldsymbol{\alpha_{12}}$	thermal expansion coefficient in the in the ply's 90°-direction
$\boldsymbol{\beta_{11}}$	hygral expansion coefficient in the in the ply's 0°-direction
$\boldsymbol{\beta_{12}}$	hygral expansion coefficient in the in the ply's 90°-direction
$\boldsymbol{\theta}$	the ply orientation (used for off-axis transformations)

Note that, in general, the subscript k is used to denote that the a value is a lamina value.

2 Characteristic Matrices of the Lamina

The characteristic matrices for each lamina can be calculated as follows:

1. The *reduced stiffness matrix* $\boldsymbol{Q_k}$ describes the elastic behavior of the ply in in-plane loading.[i]

$$\boldsymbol{Q_k} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \tag{1}$$

where

$$Q_{11} = \frac{E_{11}^2}{(E_{11} - \nu_{12} \cdot E_{22})}$$

$$Q_{22} = \frac{E_{11}E_{22}}{(E_{11} - \nu_{12}^2 E_{22})}$$

$$Q_{12} = \frac{\nu_{12}E_{11}E_{22}}{(E_{11} - \nu_{12}^2 E_{22})}$$

$$Q_{66} = G_{12}$$

[i] Q_k is derived from Staab [1], Eq. 3.9, while the values for Q_{11}, Q_{22}, and Q_{12} are from Nettles [2], Eq. (10).

2. The *strain transformation matrix* $\boldsymbol{T_\varepsilon}$ is used to transform other characteristic matrices into the laminate coordinate system.[ii]

$$\boldsymbol{T_\varepsilon} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \tag{2}$$

   where

$$m = \cos\theta$$

$$n = \sin\theta$$

   θ is the relative orientation of the lamina with respect to the laminate coordinate system shown in Figure 1[iii].

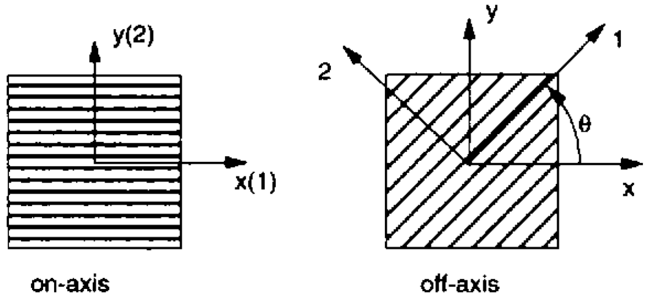

Figure 1: On- and Off-axis Ply Orientations

3. The *stress transformation matrix* $\boldsymbol{T_\sigma}$, by comparison, is calculated by the following equation.[iv]

$$\boldsymbol{T_\sigma} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \tag{3}$$

   It should be noted that if tensor notation is used for both strains, then the stress and strain transformation matrices should be equal, $\boldsymbol{T_\varepsilon} = \boldsymbol{T_\sigma}$.

4. The *transformed reduced stiffness matrix* $\overline{\boldsymbol{Q}}_{\boldsymbol{k}}$ is calculated by modifying Q_k by T_ε [v].

$$\overline{\boldsymbol{Q}}_{\boldsymbol{k}} = \boldsymbol{T_\sigma^{-1} Q_k T_\varepsilon} \tag{4}$$

3 Determining the $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ Matrices

When lamina are bonded together to form a laminate, there exist three matrices that characterize the stiffness of the laminate. These are the *extensional stiffness matrix* $\boldsymbol{A}$, the *extension-bending stiffness matrix* $\boldsymbol{B}$, and the *bending stiffness matrix* $\boldsymbol{D}$.

3.1 The Extensional Stiffness Matrix

The *extensional stiffness matrix* $\boldsymbol{A}$ characterizes the axial, in-plane stiffness of the laminate and is defined

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{16} \end{bmatrix} \tag{5}$$

where

$$A_{ij} = \sum_{k=1}^{n} \left[\overline{Q}_{ij}\right]_k (z_k - z_{k-1}) \tag{6}$$

[ii] Staab [1], Eq 2.1
[iii] *ibid.*, Fig. 3.3
[iv] *ibid.*, Eq. 2.3
[v] *ibid.* Section 3.2.2

3.2 The Extension-Bending Coupling Matrix

The *extension-bending coupling matrix* $\boldsymbol{B}$ couples the extensional stiffness and the bending stiffness matrices. It is defined:

$$\boldsymbol{B} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{16} \end{bmatrix} \tag{7}$$

where

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left[\overline{Q}_{ij}\right]_k \left(z_k^2 - z_{k-1}^2\right) \tag{8}$$

3.3 The Bending Stiffness Matrix

The *bending stiffness matrix* $\boldsymbol{B}$ characterizes the stiffness of the laminate when subjected to bending loads and is defined:

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{16} \end{bmatrix} \tag{9}$$

where

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left[\overline{Q}_{ij}\right]_k \left(z_k^3 - z_{k-1}^3\right) \tag{10}$$

3.4 The *ABD* Matrix

Together, all three stiffness matrices fully characterize the laminate stiffness and can be used to relate applied loads to the resulting strains on a laminate and in its lamina. This relationship is defined as

$$\left\{\begin{array}{c} \boldsymbol{N} \\ \hline \boldsymbol{M} \end{array}\right\} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{\varepsilon^0} \\ \hline \boldsymbol{\kappa} \end{array}\right\} \tag{11}$$

which, when expanded, becomes

$$\left\{\begin{array}{c} N'_{xx} \\ N'_{yy} \\ N'_{xy} \\ \hline M'_{xx} \\ M'_{yy} \\ M'_{xy} \end{array}\right\} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ A_{61} & A_{62} & A_{66} & B_{61} & B_{62} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{21} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} \\ B_{61} & B_{62} & B_{66} & D_{61} & D_{62} & D_{66} \end{array}\right] \left\{\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right\} \tag{12}$$

References

[1] G. H. Staab, *Laminar Composites.* 225 Wildwood Avenue, Woburn, MA 01801: Butterworth-Heinemann, 1999.

[2] A. T. Nettles, “Basic mechanics of laminated composite plates,” Tech. Rep. NASA-RP-1351, NASA, NASA Marshall Space Flight Center, Huntsville, AL USA, October 1994.

4 Creating the ABD Matrix

The ABD matrix is a 6x6 matrix that serves as a connection between the applied loads and the associated strains in the laminate. It essentially defines the elastic properties of the entire laminate. To assemble the ABD matrix, follow these steps:

1. Calculate reduced stiffness matrix $\boldsymbol{Q_k}$ for each material used in the laminate (if a laminate uses only one type of composite material, there will be only 1 stiffness matrix). The stiffness matrix describes the elastic behavior of the ply in plane loading

$$\boldsymbol{Q_k} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \tag{13}$$

where

$$Q_{11} = \frac{E_{11}^2}{(E_{11} - \nu_{12} \cdot E_{22})}$$

$$Q_{12} = \frac{\nu_{12} E_{11} E_{22}}{(E_{11} - \nu_{12}^2 E_{22})}$$

$$Q_{22} = \frac{E_{11} E_{22}}{(E_{11} - \nu_{12}^2 E_{22})}$$

$$Q_{66} = G_{12}$$

where

$$Q_{11} = \frac{E_{11}^2}{(E_{11} - \nu_{12} \cdot E_{22})}$$

$$Q_{12} = \frac{\nu_{12} E_{11} E_{22}}{(E_{11} - \nu_{12}^2 E_{22})}$$

$$Q_{22} = \frac{E_{11} E_{22}}{(E_{11} - \nu_{12}^2 E_{22})}$$

$$Q_{66} = G_{12}$$

2. Calculate the transformed reduced stiffness matrix $\overline{\boldsymbol{Q}}_{\boldsymbol{k}}$ for each ply based on the reduced stiffness matrix and fiber angle.

$$\overline{\boldsymbol{Q}}_{\boldsymbol{k}} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix} \tag{14}$$

where

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11}\cos^4(\theta) + 2\big(Q_{12} + Q_{66}\big)\cos^2(\theta)\cdot\sin^2(\theta) + Q_{22}\sin^4(\theta) \\
\overline{Q}_{12} = \overline{Q}_{21} &= Q_{12}\big(\cos^4(\theta) + \sin^4(\theta)\big) + \big(Q_{11} + Q_{22} - 4Q_{66}\big)\cos^2(\theta)\sin^2(\theta) \\
\overline{Q}_{16} = \overline{Q}_{61} &= \big(Q_{11} - Q_{12} - 2Q_{66}\big)\cos^3(\theta)\sin(\theta) - \big(Q_{22} - Q_{12} - 2Q_{66}\big)\cos(\theta)\sin^3(\theta) \\
\overline{Q}_{22} &= Q_{11}\sin^4(\theta) + 2\big(Q_{12} + Q_{66}\big)\cos^2(\theta)\cdot\sin^2(\theta) + Q_{22}\cos^4(\theta) \\
\overline{Q}_{26} = \overline{Q}_{62} &= \big(Q_{11} - Q_{12} - 2Q_{66}\big)\cos(\theta)\sin^3(\theta) - \big(Q_{22} - Q_{12} - 2Q_{66}\big)\cos^3(\theta)\sin(\theta) \\
\overline{Q}_{66} &= \big(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\big)\cos^2(\theta)\sin^2(\theta) + Q_{66}\big(\cos^4(\theta) + \sin^4(\theta)\big)
\end{aligned}$$

3. Calculate the laminate *extensional stiffness matrix*, $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix} \tag{15}$$

The individual terms of $\boldsymbol{A}$ are calculated by

$$A_{ij} = \sum_{k=1}^{n} \left\{Q_{ij}\right\}_n (z_k - z_{k-1}) \tag{16}$$

where z is the vertical position in the ply from the midplane, and k is for each ply.

4. Calculate the laminate *coupling stiffness matrix*, $\boldsymbol{B}$:

$$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{bmatrix} \tag{17}$$

   where

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n} \left\{Q_{ij}\right\}_n \left(z_k^2 - z_{k-1}^2\right) \tag{18}$$

5. Calculate the laminate *bending stiffness matrix*, $\boldsymbol{D_{ij}}$:

$$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix} \tag{19}$$

   where

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} \left\{Q_{ij}\right\}_n \left(z_k^3 - z_{k-1}^3\right) \tag{20}$$

6. Assemble the $\boldsymbol{ABD}$ matrix:

$$\boldsymbol{ABD} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \tag{21}$$

5 Laminate Properties

Overall laminate properties can be calculated from the $\boldsymbol{ABD}$ matrix.

$$\left\{\begin{array}{c} \boldsymbol{N} \\ \hline \boldsymbol{M} \end{array}\right\} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{\varepsilon^0} \\ \hline \boldsymbol{\kappa} \end{array}\right\} \tag{22}$$

$$\left\{\begin{array}{c} Q_x \\ Q_y \end{array}\right\} = \begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix}_k \left\{\begin{array}{c} \gamma_{xz} \\ \gamma_{yz} \end{array}\right\}_k \tag{23}$$

where

$$A_{ij} = c\sum_{k=1}^{N} \left[\overline{Q}_{ij}\right]_k \left\{\left(z_k - z_{k-1}\right) - \frac{4}{3h^2}\left(z_k^3 - z_{k-1}^3\right)\right\} \tag{24}$$

where $i, j = 4, 5$ and $c = 6/5$ for a rectangular section. Generally speaking, the stiffness terms ($\overline{Q}_{44}$, $\overline{Q}_{55}$, etc.) associated with Q_x and Q_y are difficult to experimentally determine and are, therefore, approximated.

$$\left\{\begin{array}{c} N'_{xx} \\ N'_{yy} \\ N'_{xy} \\ \hline M'_{xx} \\ M'_{yy} \\ M'_{xy} \end{array}\right\} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ A_{61} & A_{62} & A_{66} & B_{61} & B_{62} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{21} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} \\ B_{61} & B_{62} & B_{66} & D_{61} & D_{62} & D_{66} \end{array}\right] \left\{\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right\} \tag{25}$$

where N' and M' are the total running loads, including thermal and hygral effects:

$$N'_{ij} = N_{ij} + N^T_{ij} + N^M_{ij} \tag{26}$$

$$M'_{ij} = M_{ij} + M^T_{ij} + M^M_{ij} \tag{27}$$

6 Calculating Individual Ply Strains and Stresses

Having calculated the $\boldsymbol{ABD}$ matrix for the laminate, it is possible to then calculate individual ply strains and stresses based on the loads and temperature applied to the laminate.

1. Calculate the thermal expansion coefficients for each ply:

$$\alpha_{xx} = \alpha_{11}\cos^2(\theta) + \alpha_{22}\sin^2(\theta) \tag{28}$$

$$\alpha_{yy} = \alpha_{11}\sin^2(\theta) + \alpha_{22}\cos^2(\theta) \tag{29}$$

$$\alpha_{xy} = 2\cos(\theta)\sin(\theta)\big(\alpha_{11} - \alpha_{22}\big) \tag{30}$$

2. Calculate the hygral expansion coefficients for each ply:

$$\beta_{xx} = \beta_{11}\cos^2(\theta) + \beta_{22}\sin^2(\theta) \tag{31}$$

$$\beta_{yy} = \beta_{11}\sin^2(\theta) + \beta_{22}\cos^2(\theta) \tag{32}$$

$$\beta_{xy} = 2\cos(\theta)\sin(\theta)\big(\beta_{11} - \beta_{22}\big) \tag{33}$$

3. Calculate the thermal running loads:

$$N_{xx}^{T} = \Delta T \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{11}}\alpha_{xx} + \overline{Q_{12}}\alpha_{yy} + \overline{Q_{16}}\alpha_{xy}\big]_k \big(z_k - z_{k-1}\big) \Big\} \tag{34}$$

$$N_{yy}^{T} = \Delta T \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{12}}\alpha_{xx} + \overline{Q_{22}}\alpha_{yy} + \overline{Q_{26}}\alpha_{xy}\big]_k \big(z_k - z_{k-1}\big) \Big\} \tag{35}$$

$$N_{xy}^{T} = \Delta T \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{16}}\alpha_{xx} + \overline{Q_{26}}\alpha_{yy} + \overline{Q_{66}}\alpha_{xy}\big]_k \big(z_k - z_{k-1}\big) \Big\} \tag{36}$$

$$M_{xx}^{T} = \frac{\Delta T}{2} \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{11}}\alpha_{xx} + \overline{Q_{12}}\alpha_{yy} + \overline{Q_{16}}\alpha_{xy}\big]_k \big(z_k^2 - z_{k-1}^2\big) \Big\} \tag{37}$$

$$M_{yy}^{T} = \frac{\Delta T}{2} \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{12}}\alpha_{xx} + \overline{Q_{22}}\alpha_{yy} + \overline{Q_{26}}\alpha_{xy}\big]_k \big(z_k^2 - z_{k-1}^2\big) \Big\} \tag{38}$$

$$M_{xy}^{T} = \frac{\Delta T}{2} \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{16}}\alpha_{xx} + \overline{Q_{26}}\alpha_{yy} + \overline{Q_{66}}\alpha_{xy}\big]_k \big(z_k^2 - z_{k-1}^2\big) \Big\} \tag{39}$$

4. Calculate the hygral expansion running loads:

$$N_{xx}^{M} = \Delta T \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{11}}\beta_{xx} + \overline{Q_{12}}\beta_{yy} + \overline{Q_{16}}\beta_{xy}\big]_k \big(z_k - z_{k-1}\big) \Big\} \tag{40}$$

$$N_{yy}^{M} = \Delta T \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{12}}\beta_{xx} + \overline{Q_{22}}\beta_{yy} + \overline{Q_{26}}\beta_{xy}\big]_k \big(z_k - z_{k-1}\big) \Big\} \tag{41}$$

$$N_{xy}^{M} = \Delta T \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{16}}\beta_{xx} + \overline{Q_{26}}\beta_{yy} + \overline{Q_{66}}\beta_{xy}\big]_k \big(z_k - z_{k-1}\big) \Big\} \tag{42}$$

$$M_{xx}^{M} = \frac{\Delta T}{2} \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{11}}\beta_{xx} + \overline{Q_{12}}\beta_{yy} + \overline{Q_{16}}\beta_{xy}\big]_k \big(z_k^2 - z_{k-1}^2\big) \Big\} \tag{43}$$

$$M_{yy}^{M} = \frac{\Delta T}{2} \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{12}}\beta_{xx} + \overline{Q_{22}}\beta_{yy} + \overline{Q_{26}}\beta_{xy}\big]_k \big(z_k^2 - z_{k-1}^2\big) \Big\} \tag{44}$$

$$M_{xy}^{M} = \frac{\Delta T}{2} \sum_{k=1}^{n} \Big\{ \big[\overline{Q_{16}}\beta_{xx} + \overline{Q_{26}}\beta_{yy} + \overline{Q_{66}}\beta_{xy}\big]_k \big(z_k^2 - z_{k-1}^2\big) \Big\} \tag{45}$$

5. Calculate the inverse $\boldsymbol{ABD}$ matrix, $\boldsymbol{abd}$:

$$\boldsymbol{abd} = \left\{ \begin{array}{c|c} \boldsymbol{A^{-1}} & \boldsymbol{B^{-1}} \\ \hline \boldsymbol{B^{-1}} & \boldsymbol{D^{-1}} \end{array} \right\} \tag{46}$$

   This inverse matrix is the *stiffness matrix* of the laminate.

6. Calculate midplane strains and curvatures induced in the laminate using the relationship between strain, stiffness, and the applied load.

$$\boldsymbol{\varepsilon} = \boldsymbol{abd} \cdot \boldsymbol{N} \tag{47}$$

   Expanded, this equation becomes:

$$\left[\begin{array}{c} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array} \right] = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{21} & a_{22} & a_{26} & b_{21} & b_{22} & b_{26} \\ a_{61} & a_{62} & a_{66} & b_{61} & b_{62} & b_{66} \\ \hline b_{11} & b_{12} & b_{16} & d_{11} & d_{12} & d_{16} \\ b_{21} & b_{22} & b_{26} & d_{21} & d_{22} & d_{26} \\ b_{61} & b_{62} & b_{66} & d_{61} & d_{62} & d_{66} \end{array} \right] \cdot \left[\begin{array}{c} N_{xx} + N_{xx}^T + N_{xx}^M \\ N_{yy} + N_{yy}^T + N_{yy}^M \\ N_{xy} + N_{xy}^T + N_{xy}^M \\ \hline M_{xx} + M_{xx}^T + M_{xx}^M \\ M_{yy} + M_{yy}^T + M_{yy}^M \\ M_{xy} + M_{xy}^T + M_{xy}^M \end{array} \right]$$

7. Individual ply strains are then calculated in the laminate xy-coordinate system by the equation:

$$\left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \varepsilon_{xx}^0 \\ \varepsilon_{xy}^0 \\ \gamma_{xy}^0 \end{array} \right\} + z \left\{ \begin{array}{c} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array} \right\} \tag{48}$$

8. Individual ply stresses are similarly calculated in the xy-coordinate system by the corresponding equation:

$$\left[\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{array} \right] = \left[\begin{array}{ccc} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{array} \right] \cdot \left\{ \begin{array}{c} \varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta T \beta_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta T \beta_{yy} \\ \gamma_{xy} - \Delta T \alpha_{xy} - \Delta T \beta_{xy} \end{array} \right\} \tag{49}$$

9. Interlaminar shear forces on each ply are calculated