Homework 2

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Exercise 1

TO PROVE:

$$\sum_i (y_i - \overline{y})^2 = \sum_i (y_i - \hat{y}_i)^2 + \sum_i (\hat{y}_i - \overline{y})^2$$

$$\text{TSS} = \sum_i (y_i - \overline{y})^2$$

$$= \sum_i (y_i - \hat{y}_i + \hat{y}_i - \overline{y})^2$$

$$= \sum_i ((y_i - \hat{y}_i) + (\hat{y}_i - \overline{y}))^2$$

$$= \sum_i [(y_i - \hat{y}_i)^2 + (\hat{y}_i - \overline{y})^2 + 2[(y_i - \hat{y}_i)(\hat{y}_i - \overline{y})]]$$

$$= \sum_i (y_i - \hat{y}_i)^2 + \sum_i (\hat{y}_i - \overline{y})^2 + \sum_i 2[(y_i - \hat{y}_i)(\hat{y}_i - \overline{y})]$$

We need to show the following, to prove the answer:

$$2 \sum_i [(y_i - \hat{y}_i)(\hat{y}_i - \overline{y})] = 0 \tag{1.1}$$

By definition: $\hat{y}_i = \beta^T x_i$

RSS $= \sum_i (y_i - \beta^T x_i)^2$

$\frac{d}{d\beta} = -2 \sum_i (y_i - \beta^T x_i) x_i = 0$

$$\frac{d}{d\beta^{(j)}} = \sum_i (y_i - \beta^T x_i) x_i^{(j)} = 0 \tag{1.2}$$

We will split up (1.1) into 2 parts and show each of them are 0:

$$\sum_i [(y_i - \hat{y}_i)(\hat{y}_i - \overline{y})]$$

$$= \sum_i \hat{y}_i(y_i - \hat{y}_i) - \sum_i \overline{y}(y_i - \hat{y}_i)$$

First we show $\sum_i \overline{y}(y_i - \hat{y}_i) = 0$ by evaluating (1.2) at $j = 0$

$$\frac{d}{d\beta^{(0)}} = \sum_i (y_i - \beta^T x_i) x_i^{(0)}$$

$$= \sum_i (y_i - \beta^T x_i) \cdot 1$$

$$= \sum_i (y_i - \hat{y}_i) \cdot 1$$

$$= 0$$

Next, we show that $\sum_i \hat{y}_i(y_i - \hat{y}_i) = 0$

We know the following:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}^T x_i \tag{1.3}$$

$$\sum_i x_i(y_i - \hat{y}_i) = 0 \tag{1.4}$$

Using (1.4), we can do the following:

$$\rightarrow \sum_i \hat{\beta}^T x_i(y_i - \hat{y}_i) = 0$$

$$\rightarrow \sum_i \hat{\beta}^T x_i(y_i - \hat{y}_i) + \sum_i \hat{\beta}_0(y_i - \hat{y}_i) = 0$$

$$\rightarrow \sum_i (\hat{\beta}_0 + \hat{\beta}^T x_i) \cdot (y_i - \hat{y}_i) = 0$$

$$\rightarrow \sum_i \hat{y}_i(y_i - \hat{y}_i) = 0$$

Thus we have shown that $\sum_i (y_i - \overline{y})^2 = \sum_i (y_i - \hat{y}_i)^2 + \sum_i (\hat{y}_i - \overline{y})^2$

Exercise 2

TO PROVE:

$$E_y[(y - \overline{\beta}^T x)^2] = E_y[(y - f(x))^2] + (f(x) - \overline{\beta}^T x)^2$$

$$= E_y[(y - \overline{\beta}^T x)^2]$$

$$= E_y[(y - f(x) + f(x) + \overline{\beta}^T x)^2]$$

$$= E_y[((y - f(x)) + (f(x) + \overline{\beta}^T x))^2]$$

$$= E_y[(y - f(x))^2 + (f(x) + \overline{\beta}^T x)^2 + 2[(y - f(x))(f(x) + \overline{\beta}^T x)]]$$

$$= E_y[(y - f(x))^2] + E_y[(f(x) + \overline{\beta}^T x)^2] + E_y[2[(y - f(x))(f(x) + \overline{\beta}^T x)]]$$

$$= E_y[(y - f(x))^2] + (f(x) + \overline{\beta}^T x)^2 + E_y[2[(y - f(x))(f(x) + \overline{\beta}^T x)]]$$

Now we need to show that $E_y[2[(y - f(x))(f(x) + \overline{\beta}^T x)]] = 0$

$$E_y[2[(y - f(x))(f(x) + \overline{\beta}^T x)]]$$

$$= 2 \cdot (f(x) + \overline{\beta}^T x) \cdot E_y[(y - f(x))]$$

We know the following:

$$y = f(x) + \varepsilon$$

$$\varepsilon \sim \mathcal{N}(0, \sigma^2)$$

$$\therefore y \sim \mathcal{N}(f(x), \sigma^2)$$

Thus,

$$2 \cdot (f(x) + \overline{\beta}^T x) \cdot E_y[(f(x) - f(x))] = 2 \cdot (f(x) + \overline{\beta}^T x) \cdot 0 = 0$$