$$\frac{\partial u}{\partial t} = \alpha \Delta u - \beta \left(u - \frac{1}{|\Omega|} \int u \, dx \right)$$

$u(t=0) = u_0$ 2D domain - $[0,1]^2$

Periodic B.C.

<u>Weak formulation</u> :

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v = \alpha \int_\Omega \Delta u \cdot v - \beta \int_\Omega u \cdot v + \frac{\beta}{|\Omega|} \int \left[\int_\Omega u \, dx \cdot v \right]$$

$$\nabla \cdot (\nabla u v) = \Delta u v + \nabla u \cdot \nabla v$$

$$\Rightarrow \int_\Omega \frac{\partial u}{\partial t} \cdot v = -\alpha \int_\Omega \nabla u \cdot \nabla v - \beta \int_\Omega u \cdot v + \frac{\beta}{|\Omega|} \int_\Omega \left(\int u \, dx \cdot v \right)$$

<u>Time discretization</u> (Crank-Nicolson scheme)

$$\int_\Omega \frac{u^{n+1} - u^n}{\Delta t} \cdot v = -\alpha \int_\Omega \frac{\nabla u^{n+1} + \nabla u^n}{2} \cdot \nabla v - \beta \int_\Omega \frac{u^{n+1} + u^n}{2} \cdot v$$

$$+ \frac{\beta}{|\Omega|} \int_\Omega \left(\int \frac{u^{n+1} + u^n}{2} dx \cdot v \right)$$

$$\boxed{\text{Let } N(u^{n+1}, u^n) = \frac{1}{|\Omega|} \int_\Omega \frac{u^{n+1} + u^n}{2} dx}$$

where u^n is known & u^{n+1} is unknown

$$\int_\Omega \frac{u^{n+1}-u^n}{\Delta t}\cdot v = -\alpha \int_\Omega \frac{\nabla u^{n+1}+\nabla u^n}{2}\cdot \nabla v - \beta \int_\Omega \frac{u^{n+1}+u^n}{2}\cdot v$$

$$+ \beta \int_\Omega N(u^{n+1}, u^n)\cdot v$$

<u>Gateaux derivative</u>

$N : V \times M \to X$

$N : V \times M \times X^* \to \mathbb{R}$

In our case $V = M$ & $X = \mathbb{R}$

Let $N = \frac{1}{|\Omega|}\int_\Omega \frac{u+f}{2}\,dx$

Directional derivative in the direction $\psi \in V$

$$\delta N(u, f; \psi) = \lim_{t\to 0} \frac{N(u+t\psi) - N(u)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{1}{|\Omega|}\int_\Omega \frac{u+t\psi+f}{2}\,dx - \frac{1}{|\Omega|}\int_\Omega \frac{u+f}{2}\,dx}{t}$$

$$= \lim_{t\to 0} \frac{\frac{1}{|\Omega|}\int_\Omega \frac{t\psi}{2}\,dx}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{t}}{\cancel{t}} \frac{1}{|\Omega|}\int_\Omega \frac{\psi}{2}\,dx = \frac{1}{|\Omega|}\int_\Omega \frac{\psi}{2}\,dx$$

$\delta N(u, f; \varphi)$ exists for all $\varphi \in V$ and it is linear and bounded.

Therefore the Gateaux derivative of N is

$$\delta N(u, f; \varphi) = \frac{1}{|\Omega|} \int_{\Omega} \frac{\varphi}{2} \, dx$$