Technical Talk - PhD Project

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Universität Hamburg
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Introduction

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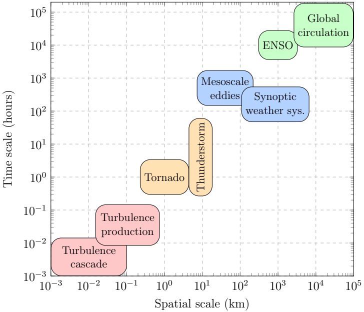
We are solving stochastic PDEs which model ocean-atmosphere interactions!

Introduction

Why add stochastic terms?

Multiscale phenomena

- Earth's oceans and atmosphere are characterized by a rich hierarchy of interacting processes



Spatial and temporal scales of various meteorological phenomena[1]

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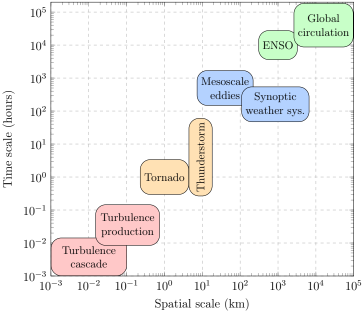

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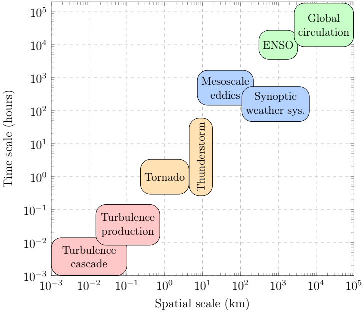

Spatial and temporal scales of various meteorological phenomena[1]

- Numerical models have fixed resolution
- Processes occurring below the model resolution are not captured $\rightarrow$ leads to prediction error!
- **Solution**: Parameterization or subgrid-scale modeling

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Parameterization

- Additional terms are added to account for the missing effect of unresolved/small scales

[2] Darryl D. Holm. "Variational Principles for Stochastic Fluid Dynamics". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* (Apr. 2015).

Parameterization

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- Parameterization techniques can be divided into two categories:
  1. Deterministic
  2. Stochastic

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- We explored **stochastic parameterization** : stochastic advection by Lie transport[2] (SALT)

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- Why **SALT**?
  - Structure preservation
  - Better uncertainty quantification skills
  - Data-driven approach

[2] Darryl D. Holm. "Variational Principles for Stochastic Fluid Dynamics". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* (Apr. 2015).

Research objectives

Objectives

1. Explore the efficacy of stochastic parameterization (in terms of UQ skills) for a climate model
   - Previous studies focused on Euler, quasi-geostrophic, and shallow water equations
2. Numerically solve the idealized stochastic climate model
3. How to model the noise/stochastic term effectively?
4. Compare the stochastic model results with other existing models/approaches

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4. ~~Compare the stochastic model results with other existing models~~

The climate model

Model equations

2D climate model equations[3]:

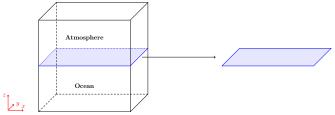
Atmosphere : $$\mathrm{d}\mathbf{u}^a + ((\mathbf{u}^a \mathrm{d}t + \sum_i \xi_i \circ \mathrm{d}W^i) \cdot \nabla)\mathbf{u}^a + \frac{1}{Ro^a}\hat{\mathbf{z}} \times (\mathbf{u}^a \mathrm{d}t + \sum_i \xi_i \circ \mathrm{d}W^i) + \sum_i (u_1^a \nabla \xi_{i,1} + u_2^a \nabla \xi_{i,2}) \circ \mathrm{d}W^i = (-\frac{1}{C^a}\nabla\theta + \frac{1}{Re^a}\Delta \mathbf{u}^a)\mathrm{d}t,$$

$$\mathrm{d}\theta^a + \nabla \cdot (\theta^a(\mathbf{u}^a \mathrm{d}t + \sum_i \xi_i \circ \mathrm{d}W^i)) = (\gamma(\theta^a - \theta^o) + \frac{1}{Pe^a}\Delta\theta^a)\mathrm{d}t,$$

Ocean : $$\frac{\partial \mathbf{u}^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\mathbf{u}^o + \frac{1}{Ro^o}\hat{\mathbf{z}} \times \mathbf{u}^o + \frac{1}{Ro^o}\nabla p^a = \sigma(\mathbf{u}^o - \mathbb{E}\mathbf{u}^a_{sol}) + \frac{1}{Re^o}\Delta \mathbf{u}^o,$$

$$\nabla \cdot \mathbf{u}^o = 0,$$

$$\frac{\partial \theta^o}{\partial t} + \mathbf{u}^o \cdot \nabla\theta^o = \frac{1}{Pe^o}\Delta\theta^o.$$



Model domain

$\mathbf{u}$: velocity, θ : temperature

p : pressure

Re : Reynolds number

Pe : Péclet number

Ro : Rossby number

σ, γ : coupling coefficients

[3] D Crisan, D D Holm, and P Korn. "An Implementation of Hasselmann's Paradigm for Stochastic Climate Modelling Based on Stochastic Lie Transport*". In: *Nonlinearity* (Sept. 2023).

How to get the stochastic terms?

- Difference between Lagrangian trajectories at different resolutions:

$$\sum_i \xi_i \,\mathrm{d}W^i \approx \mathbf{u}_{true}\,\mathrm{d}t - \mathbf{u}\,\mathrm{d}t$$

$\mathbf{u}_{true}$: true velocity
$\mathbf{u}$: mean flow velocity i.e. the large-scale component of $\mathbf{u}_{true}$

- Ways to estimate the true velocity $\mathbf{u}_{true}$:
  1. Observation data (for example, from satellites)
  2. Synthetic data (from high-resolution numerical simulations)

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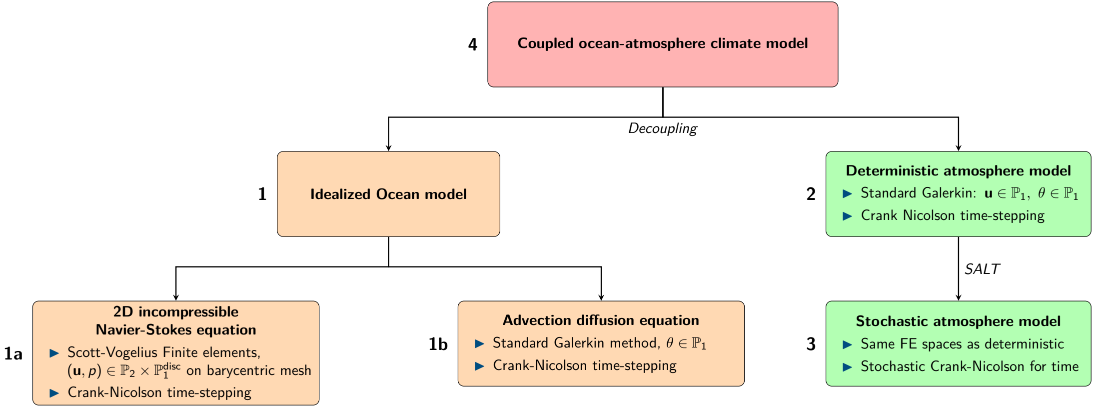
- **We use synthetic data!**

Our methodology

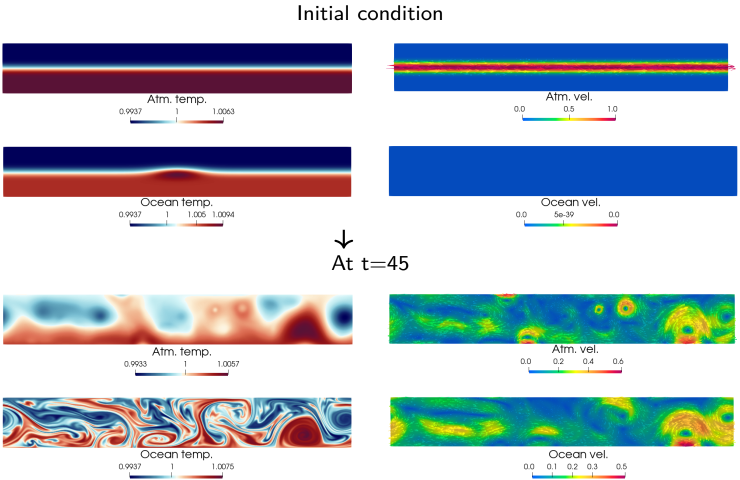


We used an open-source Python Finite Element package for numerical implementation and simulation

Numerical simulation of the climate model

Deterministic model simulation and model calibration

- We run the climate model in high-res. ($\Delta x = 1/128 \sim 30$ km) for 45 time units (~ 25 days)



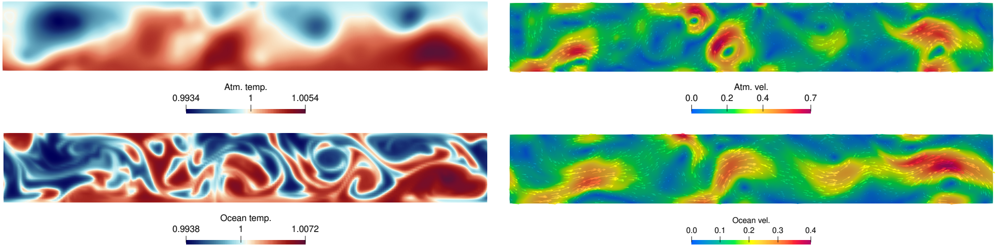
- Analyze data $\rightarrow$ extract small-scale features (using statistical algorithms) $\rightarrow$ obtain ξ_i

Numerical simulation of the climate model

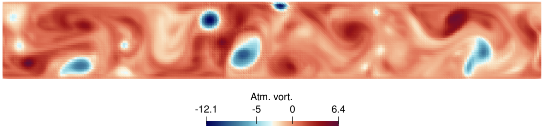
Stochastic model simulation

Stochastic model simulation

- We run the stochastic model on a coarse grid ($\Delta x = 1/32 \ \sim 120$ km)
- Initial conditions correspond to coarse grained velocity and temperature fields at $t = 25$



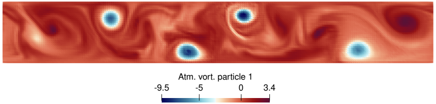
Atmospheric vorticity at $t = 25$



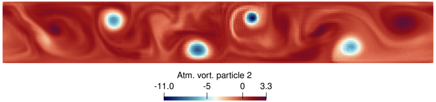
Simulation results

- Stochastic model is simulated for 20 time units ($t = 25$ to $t = 45$)
- Stochastic vs. deterministic (without parameterization) model results at $t = 35$

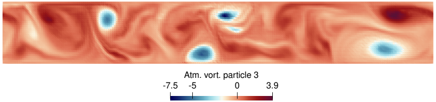
Particle 1



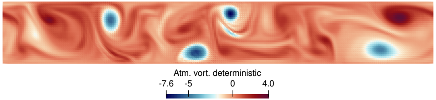
Particle 2



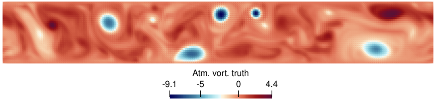
Particle 3



Deterministic solution



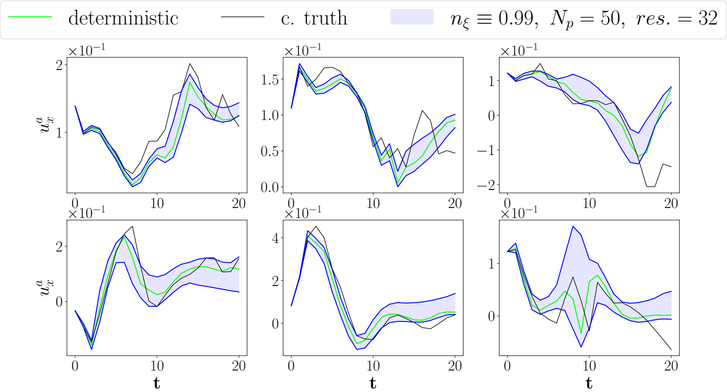
True solution



Simulation results

- Comparison between the stochastic ($\mu \pm 1\sigma$) and deterministic sol. at 6 grid points:

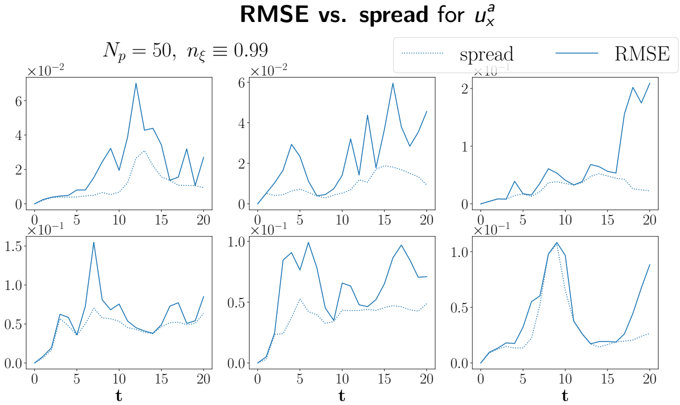
Atmospheric velocity (x component)



1. Ensemble spread increases over time
2. Stochastic model captures the truth for 5 to 10 time units

Uncertainty quantification

- Plots for the evolution of ensemble spread and RMSE at six different locations on the grid



- The size of ensemble spread is proportional to the RMSE (for at least 15 time units)
- Error in the stochastic solution can be estimated by its own spread
  - SPDE solution is suitable for data assimilation methods!

Summary

Summary & outcomes

- Aim: solve a stochastic coupled ocean-atmosphere climate model
- Novelty: Data-driven model
  - stochastic terms (modelling unresolved processes) are obtained from observation data
  - has desired properties: structure-preservation
  - We used high-res. model sim. data to estimate ξ_i $\rightarrow$ data analysis & statistical modelling
- Methodology for solving the model equations:
  - Solve simpler models first
  - build the climate model step-by-step
  - verify numerical schemes at each step
- Results:
  - Stochastic model solution capture the truth for some time units
  - Shows promising UQ test results: ensemble spread size $\propto$ RMSE
- Outcome: first step towards using SALT in a highly complex coupled system!

Thank you for your attention!