Technical Talk - PhD Project

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Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Introduction

What is it about?

Development of a structure-preserving idealized stochastic climate model

- Climate model: set of equations modelling coupled ocean-atmosphere dynamics
- Idealized: 2D model $\rightarrow$ solves for velocity, temperature, and pressure
- Stochastic: has stochastic/random terms (think of Brownian motion)
- Structure-preserving: preserves underlying geometric and physical structure

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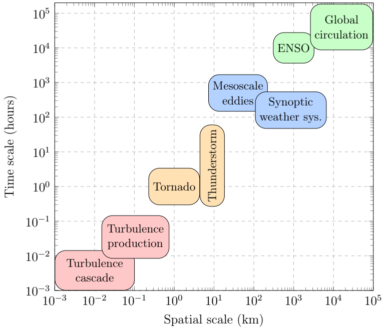
We are solving stochastic PDEs which model ocean-atmosphere interactions!

Introduction

Why add stochastic terms?

Multiscale phenomena

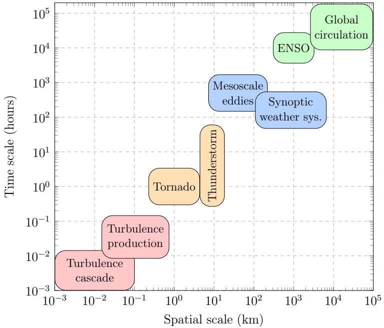


Spatial and temporal scales of various meteorological phenomena[1]

- Numerical models have fixed resolution
- Processes occurring below the model resolution are not captured $\rightarrow$ leads to prediction error!

[1] Roland B. Stull. *Practical Meteorology: An Algebra-Based Survey of Atmospheric Science*. 2017.

Multiscale phenomena



Spatial and temporal scales of various meteorological phenomena[1]

- Numerical models have fixed resolution
- Processes occurring below the model resolution are not captured $\rightarrow$ leads to prediction error!
- **Solution**: Parameterization or subgrid-scale modeling

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Parameterization

- Additional terms are added to account for the missing effect of unresolved/small scales
- Parameterization techniques can be divided into two categories:
    1. Deterministic
    2. Stochastic

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- We explored **stochastic parameterization** : stochastic advection by Lie transport (SALT)
- Why SALT?
  - preserves physical structure $\rightarrow$ reliable long simulations
  - better forecast skills $\rightarrow$ uncertainty due to unresolved transport processes can be quantified
  - conducive to data-driven approaches $\rightarrow$ can be estimated from observations/satellite data

Research objectives

Main objectives

1. Explore the efficacy of stochastic parameterization (in terms of UQ skills) for a climate model
   - Previous studies focused on Euler, quasi-geostrophic, and shallow water equations
2. Numerically solve the idealized stochastic climate model

The climate model

Model equations

2D climate model equations:

$$\text{Atmosphere}: \mathrm{d}\mathbf{u}^a + ((\mathbf{u}^a \mathrm{d}t + \sum_i \xi_i \circ \mathrm{d}W^i) \cdot \nabla)\mathbf{u}^a + \frac{1}{Ro^a}\hat{\mathbf{z}} \times (\mathbf{u}^a \mathrm{d}t + \sum_i \xi_i \circ \mathrm{d}W^i)$$

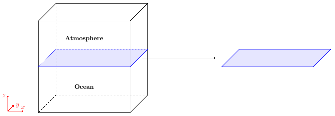
$$+ \sum_i (u_1^a \nabla \xi_{i,1} + u_2^a \nabla \xi_{i,2}) \circ \mathrm{d}W^i = (-\frac{1}{C^a}\nabla\theta + \frac{1}{Re^a}\Delta \mathbf{u}^a)\mathrm{d}t,$$

$$\mathrm{d}\theta^a + \nabla \cdot (\theta^a(\mathbf{u}^a \mathrm{d}t + \sum_i \xi_i \circ \mathrm{d}W^i)) = (\gamma(\theta^a - \theta^o) + \frac{1}{Pe^a}\Delta\theta^a)\mathrm{d}t,$$

$$\text{Ocean}: \frac{\partial \mathbf{u}^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\mathbf{u}^o + \frac{1}{Ro^o}\hat{\mathbf{z}} \times \mathbf{u}^o + \frac{1}{Ro^o}\nabla p^a = \sigma(\mathbf{u}^o - \mathbb{E}\mathbf{u}^a_{sol}) + \frac{1}{Re^o}\Delta \mathbf{u}^o,$$

$$\nabla \cdot \mathbf{u}^o = 0,$$

$$\frac{\partial \theta^o}{\partial t} + \mathbf{u}^o \cdot \nabla\theta^o = \frac{1}{Pe^o}\Delta\theta^o.$$



Model domain

$\mathbf{u}$: velocity,

θ : temperature,

p : pressure

Incompressible/compressible Navier-Stokes + Advection-diffusion equations

How to get the stochastic terms?

- Difference between Lagrangian trajectories at different resolutions:

$$\sum_i \xi_i \, \mathrm{d}W^i \approx \mathbf{u}_{true} \, \mathrm{d}t - \mathbf{u} \, \mathrm{d}t$$

$\mathbf{u}_{true}$: true velocity
$\mathbf{u}$: mean flow velocity i.e. the large-scale component of $\mathbf{u}_{true}$

- Ways to estimate the true velocity $\mathbf{u}_{true}$:
  1. Observation data (for example, from satellites)
  2. Synthetic data (from high-resolution numerical simulations)

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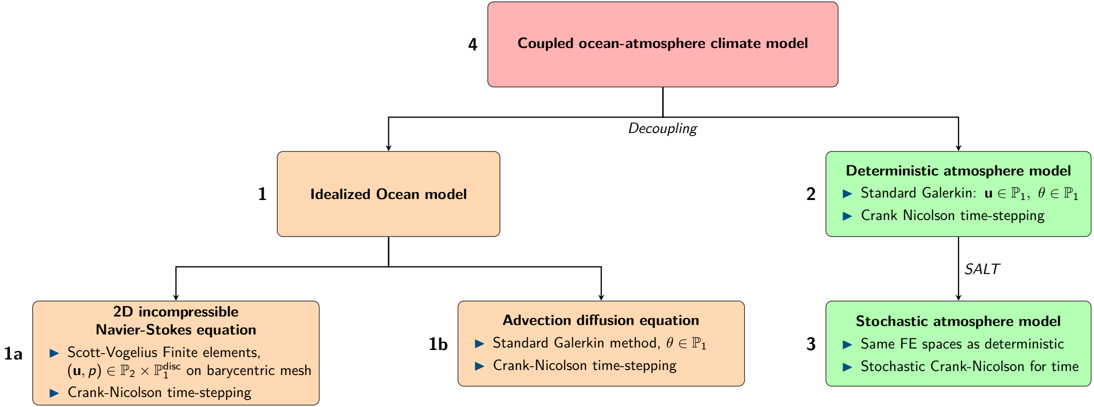
- **We use synthetic data (proof of concept)!**

Our methodology



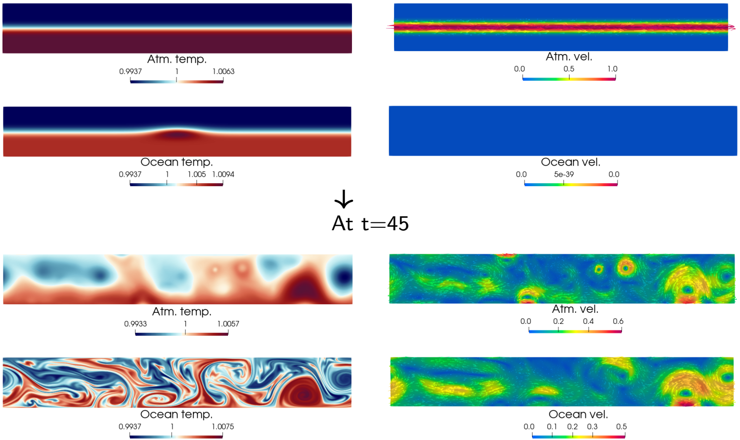
Incremental validation approach - solve simpler components first, verify at each step!

Numerical simulation of the climate model

Data collection and model calibration

- We run the deterministic climate model in high-res. ($\Delta x = 1/128 \sim 30$ km)

Initial condition



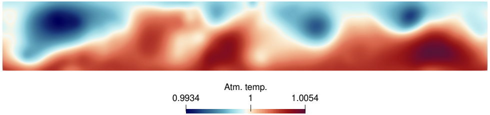
- Analyze data $\rightarrow$ extract small-scale features (using statistical algorithms) $\rightarrow$ obtain $\sum_i \xi_i \, \mathrm{d}W^i$

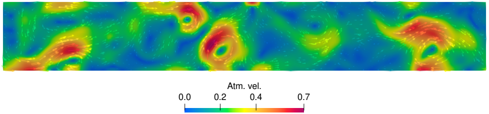
Numerical simulation of the climate model

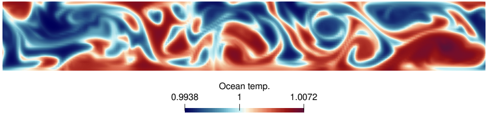
Stochastic model simulation

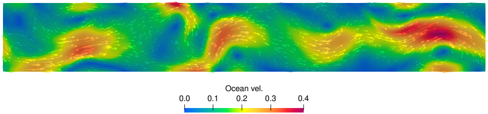
Stochastic model simulation

- We run the stochastic model on a coarse grid ($\Delta x = 1/32\ \sim 120$ km)
- Initial conditions correspond to coarse grained velocity and temperature fields at $t = 25$

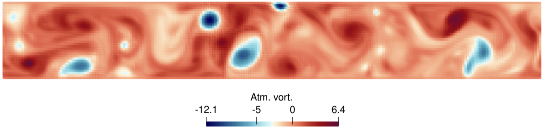








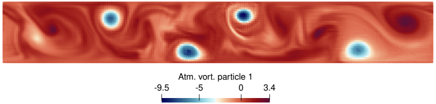
Atmospheric vorticity at $t = 25$



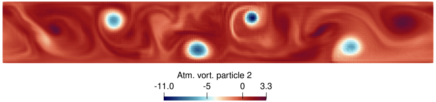
Simulation results

- Stochastic model is simulated for 20 time units ($t = 25$ to $t = 45$)
- Stochastic vs. deterministic (without parameterization) model results at $t = 35$

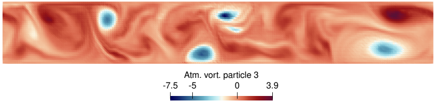
Particle 1



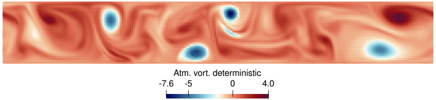
Particle 2



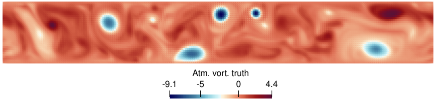
Particle 3



Deterministic solution

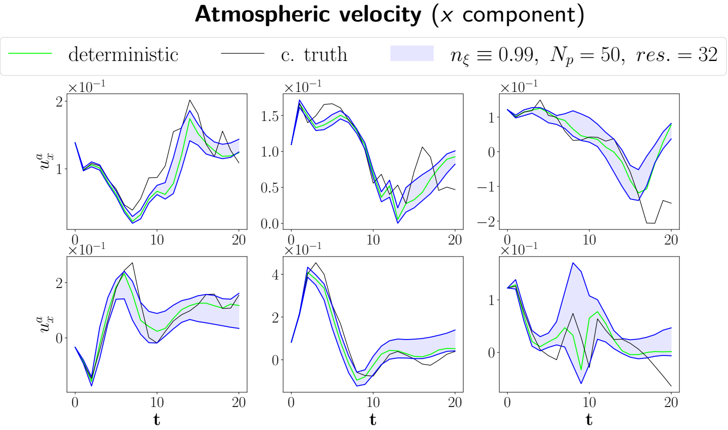


True solution



Simulation results

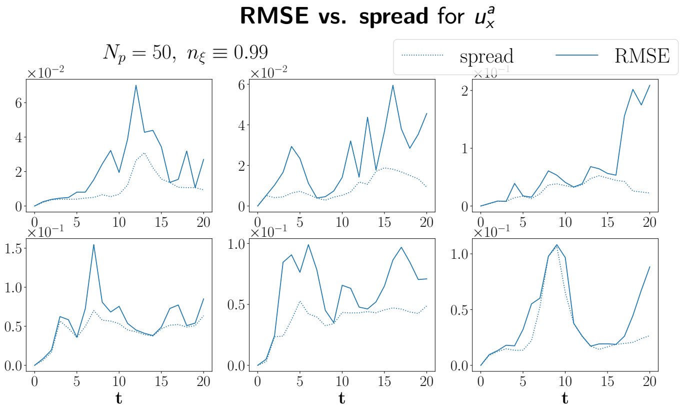
- Comparison between the stochastic ($\mu \pm 1\sigma$) and deterministic sol. at 6 grid points:



1. Ensemble spread increases over time
2. Stochastic model captures the truth for 5 to 10 time units

Uncertainty quantification

- Plots for the evolution of ensemble spread and RMSE at six different locations on the grid



- Ensemble spread $\propto$ RMSE (for at least 15 time units)
- Error in the stochastic solution can be estimated by its own spread

Summary

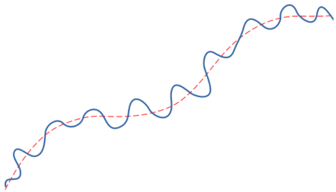
Summary & outcomes

- Aim: solve a stochastic coupled ocean-atmosphere climate model
- Novelty: data-driven model
  - Stochastic terms (modelling unresolved processes) are obtained from observation data
  - Exhibit desired conservation properties
  - High-res. model sim. data is used to estimate $\xi_i \to$ data analysis & statistical modelling
- Methodology for solving the model equations:
  - Solve simpler models first
  - build the climate model step-by-step
  - verify numerical schemes at each step
- Results:
  - Stochastic model solution captures the truth for some time units
  - Shows promising UQ test results: ensemble spread size $\propto$ RMSE
- Outcome: first step towards using SALT in a highly complex coupled system!

Additional slides

What is Stochastic Advection by Lie Transport (SALT)?

- Assumption (in the deterministic approach): fluid particles follow



$$\frac{\mathrm{d}\boldsymbol{X}(a,t)}{\mathrm{d}t} = \mathbf{u}(\boldsymbol{X}(a,t),t), \quad \boldsymbol{X}(a,0) = a \in \mathbb{R}^2 \text{ or } \mathbb{R}^3,$$

$\boldsymbol{X}(a,t)$: particle trajectory (with Lagrangian label a)
$\mathbf{u}$: velocity field

- In the SALT approach, Lagrangian particle paths follow a Stratonovich stochastic process

$$\mathrm{d}\boldsymbol{X}(a,t) = \overline{\mathbf{u}}(\boldsymbol{X}(a,t))\mathrm{d}t + \sum_i \xi_i(\boldsymbol{X}(a,t)) \circ \mathrm{d}W_t^i,$$

$\overline{\mathbf{u}}$: mean flow velocity
$\sum_i \xi_i(\boldsymbol{X}(a,t)) \circ \mathrm{d}W_t^i$: stochastic perturbation about the mean flow
W_t^i: Ind. Wiener processes
$\boldsymbol{\xi_i}$: spatial correlation of the small-scale velocity fluctuations

- Fluid dynamics equations are derived based on this assumption

Estimation of ξ_i from high resolution data

$$(\mathbf{u}-\overline{\mathbf{u}})\Delta t \approx \sum_{i=1}^{N} \xi_i \Delta W_m^i$$

- W^i: Brownian motion $\rightarrow \Delta W_m^i$: normal distribution with $\mu = 0$ and $\sigma^2 = \Delta t$

$$\begin{bmatrix} \Delta X_{11} & \Delta X_{12} & \cdots & \Delta X_{1n} \\ \Delta X_{21} & \Delta X_{22} & \cdots & \Delta X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta X_{m1} & \Delta X_{m2} & \cdots & \Delta X_{mn} \end{bmatrix} \xrightarrow{\textbf{SVD}} \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_r \\ | & | & & | \end{bmatrix} \begin{bmatrix} \text{—} & v_1^T & \text{—} \\ \text{—} & v_2^T & \text{—} \\ & \vdots & \\ \text{—} & v_r^T & \text{—} \end{bmatrix}$$

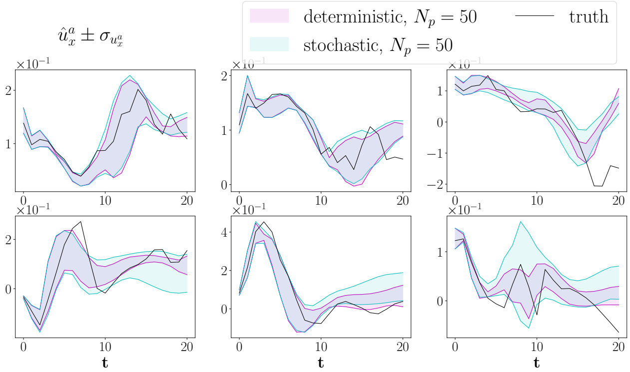
- Elements of time-series a_i are generally modeled using a normal distribution ($\mu = 0$, $\sigma^2 = \Delta t$)
- v_i^T represent the spatial correlation of the unresolved small-scale dynamics and hence are the ξ_i
- Only a few ξ's are needed to simulate most of the variance in $(\mathbf{u}-\overline{\mathbf{u}})\Delta t$ data

Stochastic versus deterministic ensemble

- How to quantify uncertainty due to unresolved scales with deterministic models?
- Perturb the atmosphere velocity field at $t = 25$ to create an ensemble of initial conditions

$$\mathbf{u}_{\text{pert}} = \mathbf{u}_0 + 0.2 \times r \times \mathbf{u}_0, \qquad \mathbf{u}_0 : \text{Initial velocity field}, \quad r \in \mathcal{N}(0,1)$$

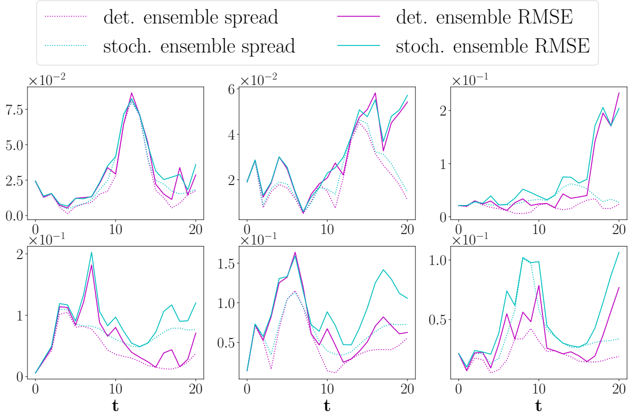
- Comparison between 50 independent realizations of the deterministic and stochastic model



- Stochastic model exhibits bigger spreads → true solution stays inside the spread for longer

Uncertainty Quantification (UQ) skills

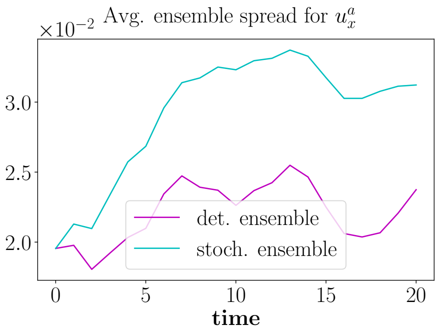
- Comparison between the ensemble spread and RMSE for atmospheric velocity (x component)

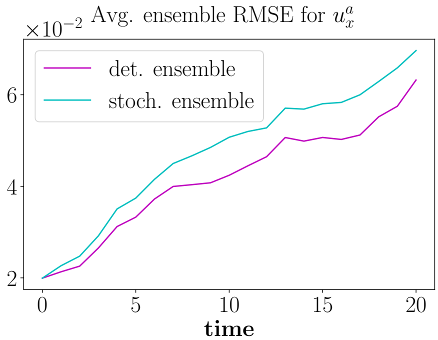


- Both models have good UQ skills (for at least 10 to 15 time units)
- Stochastic model indeed exhibits larger spreads
- RMSE of stochastic > RMSE of deterministic

UQ skill comparison

- Average spread size and average RMSE comparison:





- Stochastic model exhibits bigger spread size but also higher RMSE!
- **Which model has better prediction skills?**

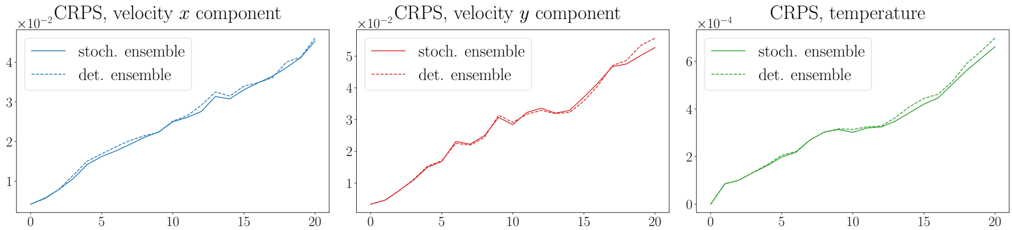
Continuous Ranked Probability Score

- Continuous Ranked Probability Score (CRPS) evaluates the accuracy of the entire predicted probability distribution against the observed value

$$\mathrm{crps}(F, y) = \int_{-\infty}^{\infty} \left(F(x) - \mathbb{1}(x \geq y)\right)^2 \, \mathrm{d}x,$$

$F(x)$: Forecast CDF, y : Observation, $\mathbb{1}$: Heavyside step function

- CRPS plots comparing the performance of the stochastic and the deterministic ensemble:



- **Stochastic parameterization produces better forecasts** than the deterministic approach
  - Physics-based model
  - Produces bigger spreads, captures the truth for longer $\rightarrow$ more suitable for data assimilation