

Technical Talk - PhD Project

Kamal Sharma

Department of Mathematics, University of Hamburg

February 13, 2026



Introduction

What is it about?

Development of a **structure-preserving idealized stochastic climate model**

- ▶ **Climate model:** set of equations modelling coupled ocean-atmosphere dynamics
- ▶ **Idealized:** 2D model → solves for velocity, temperature, and pressure
- ▶ **Stochastic:** has stochastic/random terms (think of Brownian motion)
- ▶ **Structure-preserving:** preserves underlying geometric and physical structure

What is it about?

Development of a **structure-preserving idealized stochastic climate model**

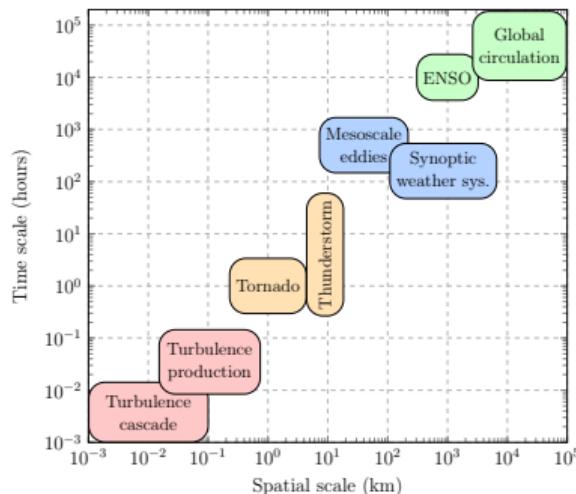
- ▶ **Climate model:** set of equations modelling coupled ocean-atmosphere dynamics
- ▶ **Idealized:** 2D model → solves for velocity, temperature, and pressure
- ▶ **Stochastic:** has stochastic/random terms (think of Brownian motion)
- ▶ **Structure-preserving:** preserves underlying geometric and physical structure

We are solving stochastic PDEs which model ocean-atmosphere interactions!

Introduction

Why add stochastic terms?

Multiscale phenomena

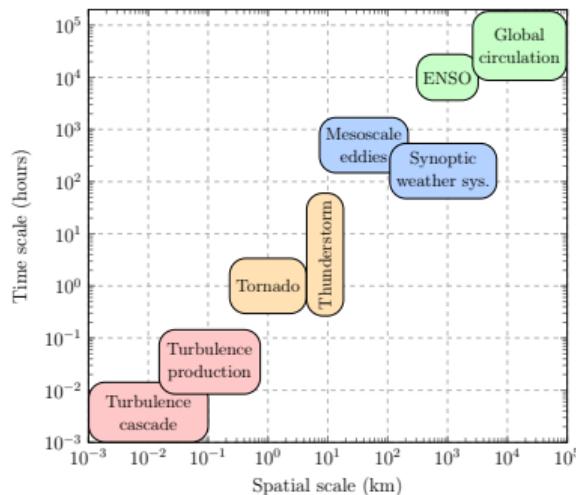


Spatial and temporal scales of various meteorological phenomena¹

- ▶ Numerical models have fixed resolution
- ▶ Processes occurring below the model resolution are not captured → leads to prediction error!

¹Roland B. Stull. *Practical Meteorology: An Algebra-Based Survey of Atmospheric Science*. 2017.

Multiscale phenomena



Spatial and temporal scales of various meteorological phenomena¹

- ▶ Numerical models have fixed resolution
- ▶ Processes occurring below the model resolution are not captured → leads to prediction error!
- ▶ **Solution:** Parameterization or subgrid-scale modeling

¹Roland B. Stull. *Practical Meteorology: An Algebra-Based Survey of Atmospheric Science*. 2017.

Parameterization

- ▶ Additional terms are added to account for the missing effect of unresolved/small scales
- ▶ Parameterization techniques can be divided into two categories:
 1. Deterministic
 2. Stochastic
- ▶ We explored **stochastic parameterization** : stochastic advection by Lie transport (SALT)

Parameterization

- ▶ Additional terms are added to account for the missing effect of unresolved/small scales
- ▶ Parameterization techniques can be divided into two categories:
 1. Deterministic
 2. Stochastic
- ▶ We explored **stochastic parameterization** : stochastic advection by Lie transport (SALT)
- ▶ Why SALT?
 - ▶ preserves conservation properties → reliable long simulations
 - ▶ better forecast skills → uncertainty due to unresolved transport processes can be quantified
 - ▶ conducive to data-driven approaches → can be estimated from observations/satellite data

Research objectives

Main objectives

1. Explore the efficacy of stochastic parameterization (in terms of UQ skills) for a climate model
 - ▶ Previous studies focused on Euler, quasi-geostrophic, and shallow water equations
2. Numerically solve the idealized stochastic climate model

The climate model

Model equations

2D climate model equations:

$$\text{Atmosphere : } d\mathbf{u}^a + ((\mathbf{u}^a dt + \sum_i \boldsymbol{\xi}_i \circ dW^i) \cdot \nabla) \mathbf{u}^a + \frac{1}{Ro^a} \hat{\mathbf{z}} \times (\mathbf{u}^a dt + \sum_i \boldsymbol{\xi}_i \circ dW^i)$$

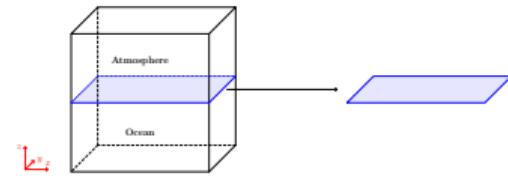
$$+ \sum_i (u_1^a \nabla \xi_{i,1} + u_2^a \nabla \xi_{i,2}) \circ dW^i = (-\frac{1}{C^a} \nabla \theta + \frac{1}{Re^a} \Delta \mathbf{u}^a) dt,$$

$$d\theta^a + \nabla \cdot (\theta^a (\mathbf{u}^a dt + \sum_i \boldsymbol{\xi}_i \circ dW^i)) = (\gamma(\theta^a - \theta^o) + \frac{1}{Pe^a} \Delta \theta^a) dt,$$

$$\text{Ocean : } \frac{\partial \mathbf{u}^o}{\partial t} + (\mathbf{u}^o \cdot \nabla) \mathbf{u}^o + \frac{1}{Ro^o} \hat{\mathbf{z}} \times \mathbf{u}^o + \frac{1}{Ro^o} \nabla p^a = \sigma(\mathbf{u}^o - \mathbb{E}\mathbf{u}_{sol}^a) + \frac{1}{Re^o} \Delta \mathbf{u}^o,$$

$$\nabla \cdot \mathbf{u}^o = 0,$$

$$\frac{\partial \theta^o}{\partial t} + \mathbf{u}^o \cdot \nabla \theta^o = \frac{1}{Pe^o} \Delta \theta^o.$$



Model domain

- \mathbf{u} : velocity,
- θ : temperature,
- p : pressure

Incompressible/compressible Navier-Stokes + Advection-diffusion equations

How to get the stochastic terms?

- ▶ Difference between Lagrangian trajectories at different resolutions:

$$\sum_i \xi_i dW^i \approx \mathbf{u}_{true} dt - \mathbf{u} dt$$

\mathbf{u}_{true} : true velocity

\mathbf{u} : mean flow velocity i.e. the large-scale component of \mathbf{u}_{true}

- ▶ Ways to estimate the true velocity \mathbf{u}_{true} :

1. Observation data (for example, from satellites)
2. Synthetic data (from high-resolution numerical simulations)

How to get the stochastic terms?

- ▶ Difference between Lagrangian trajectories at different resolutions:

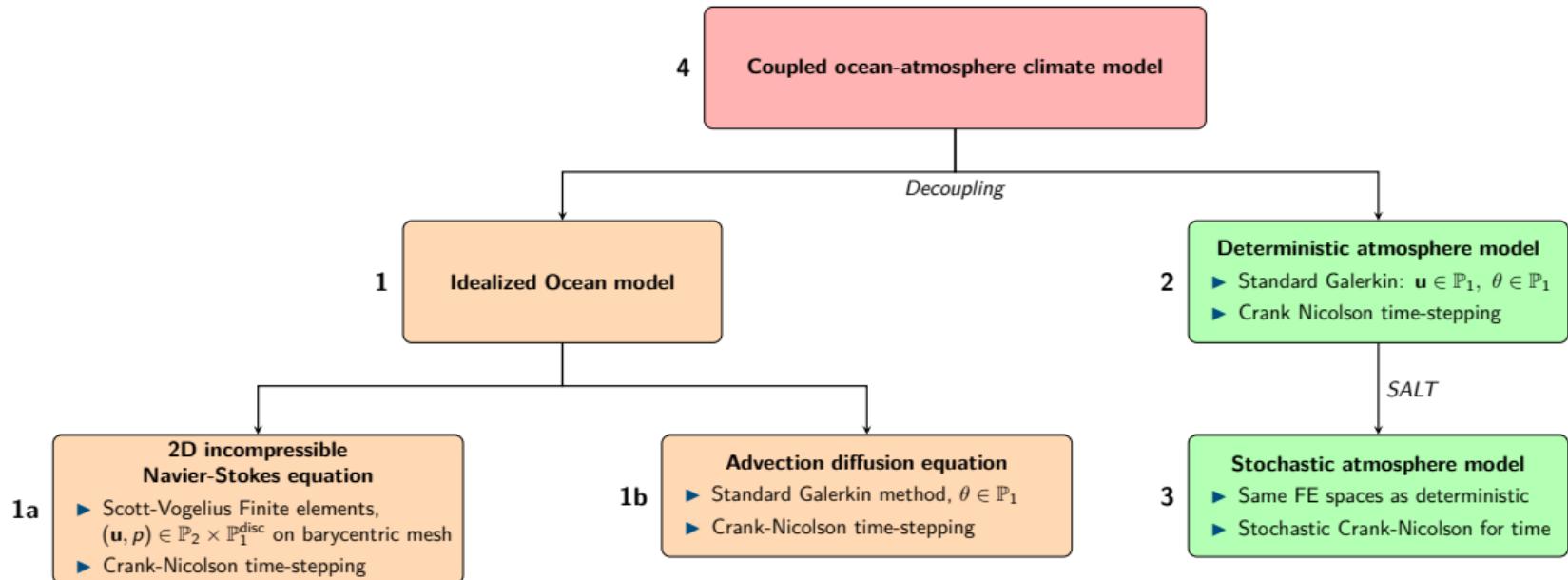
$$\sum_i \xi_i dW^i \approx \mathbf{u}_{true} dt - \mathbf{u} dt$$

\mathbf{u}_{true} : true velocity

\mathbf{u} : mean flow velocity i.e. the large-scale component of \mathbf{u}_{true}

- ▶ Ways to estimate the true velocity \mathbf{u}_{true} :
 1. Observation data (for example, from satellites)
 2. Synthetic data (from high-resolution numerical simulations)
- ▶ **We use synthetic data (proof of concept)!**

Our methodology

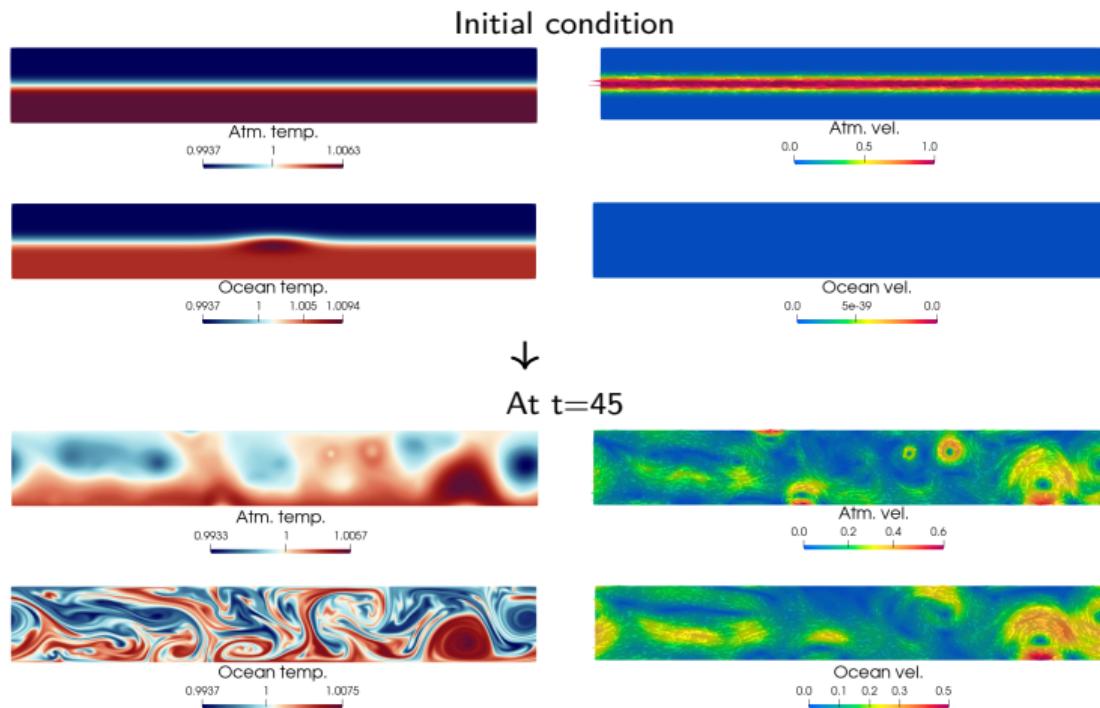


Incremental validation approach - solve simpler components first, verify at each step!

Numerical simulation of the climate model

Data collection and model calibration

- We run the deterministic climate model in high-res. ($\Delta x = 1/128 \sim 30 \text{ km}$)



- Analyze data \rightarrow extract small-scale features (using statistical algorithms) \rightarrow obtain $\sum_i \xi_i \, dW^i$

Numerical simulation of the climate model

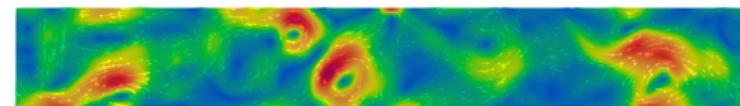
Stochastic model simulation

Stochastic model simulation

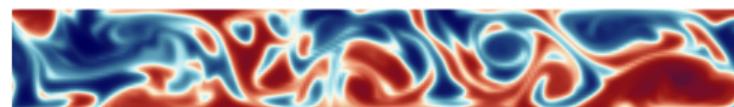
- We run the stochastic model on a coarse grid ($\Delta x = 1/32 \sim 120$ km)
- Initial conditions correspond to coarse grained velocity and temperature fields at $t = 25$



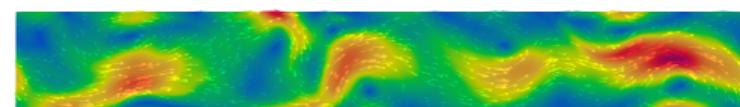
Atm. temp.
0.9934 1 1.0054



Atm. vel.
0.0 0.2 0.4 0.7

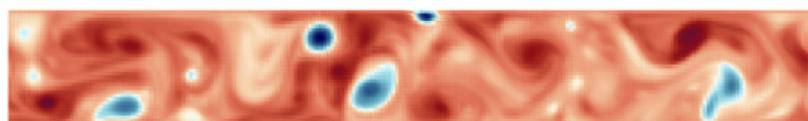


Ocean temp.
0.9938 1 1.0072



Ocean vel.
0.0 0.1 0.2 0.3 0.4

Atmospheric vorticity at $t = 25$

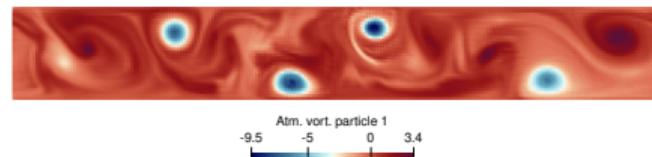


Atm. vort.
-12.1 -5 0 6.4

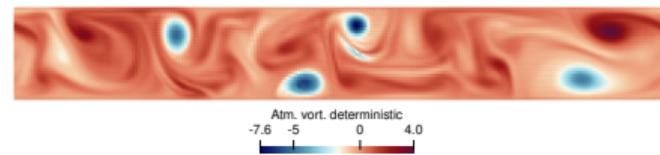
Simulation results

- ▶ Stochastic model is simulated for 20 time units ($t = 25$ to $t = 45$)
- ▶ Stochastic vs. deterministic (without parameterization) model results at $t = 35$

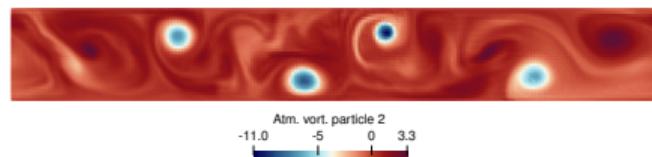
Particle 1



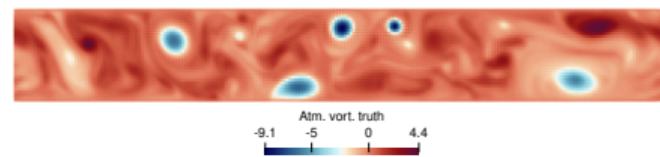
Deterministic sol. (w/o parameterization)



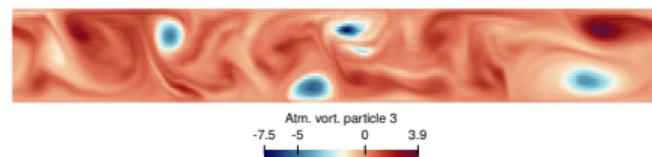
Particle 2



True solution



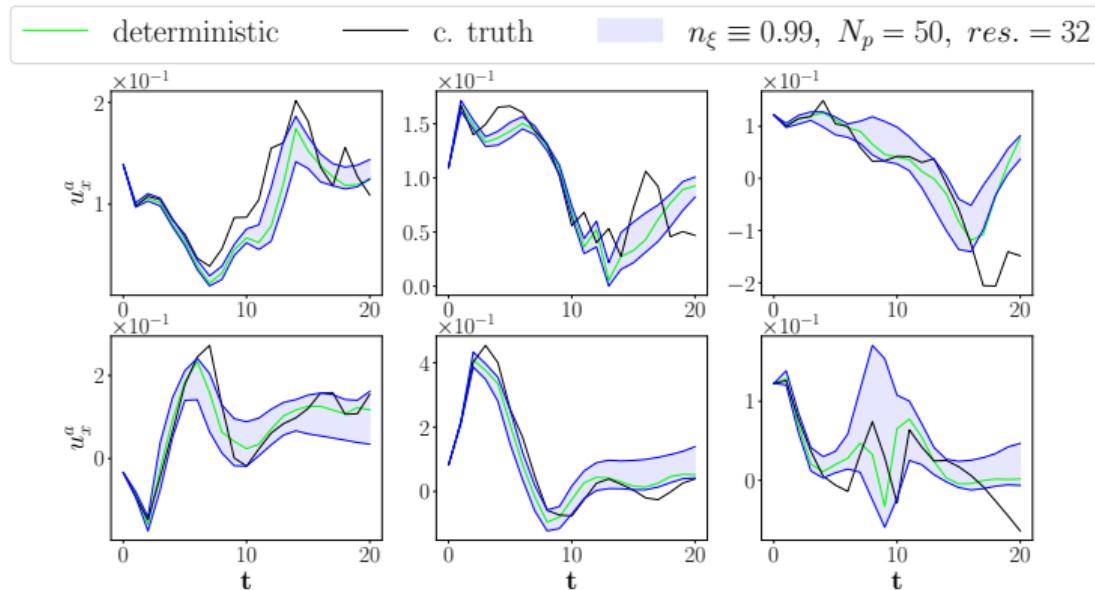
Particle 3



Simulation results

- ▶ Comparison between the stochastic ($\mu \pm 1\sigma$) and deterministic sol. at 6 grid points:

Atmospheric velocity (x component)

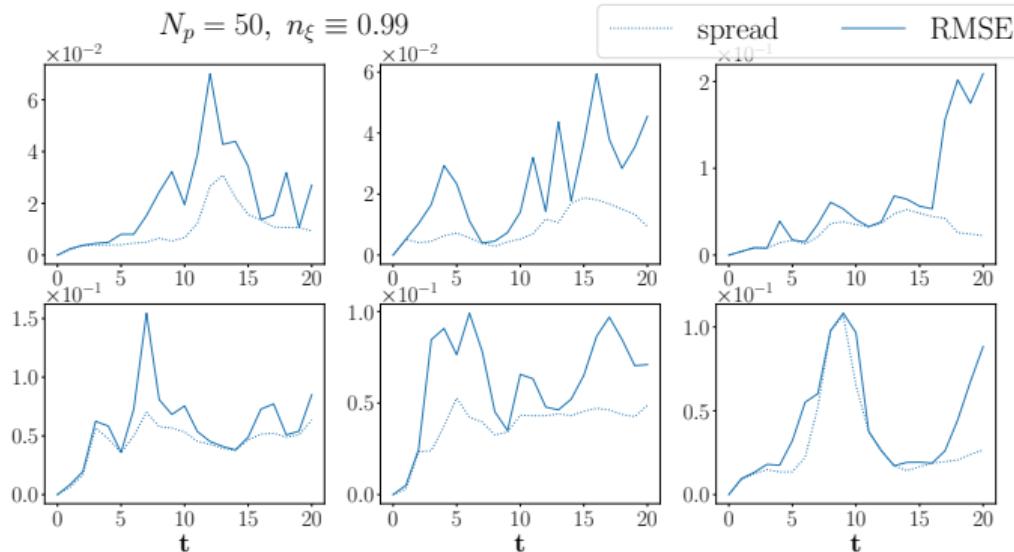


1. Ensemble spread increases over time
2. Stochastic model captures the truth for 5 to 10 time units

Uncertainty quantification

- ▶ Plots for the evolution of ensemble spread and RMSE at six different locations on the grid

RMSE vs. spread for u_x^a



- ▶ Ensemble spread \propto RMSE (for at least 15 time units)
- ▶ Error in the stochastic solution can be estimated by its own spread

Summary

Summary & outlook

- ▶ **Aim:** solve a stochastic coupled ocean-atmosphere climate model
- ▶ **Novelty:** data-driven model
 - ▶ Stochastic terms (modelling unresolved processes) are obtained from observation data
 - ▶ Exhibit desired conservation properties
 - ▶ High-res. model sim. data is used to estimate $\xi_i \rightarrow$ data analysis & statistical modelling
- ▶ **Methodology** for solving the model equations:
 - ▶ Solve simpler models first
 - ▶ build the climate model step-by-step
 - ▶ verify numerical schemes at each step
- ▶ **Simulation results:**
 - ▶ Stochastic model solution captures the truth for some time units
 - ▶ Shows promising UQ test results: ensemble spread size \propto RMSE
- ▶ **Outlook:** first step towards using SALT in a highly complex coupled system!

Thank you for your attention!

Additional slides

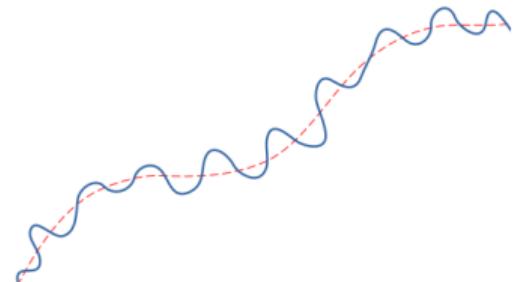
What is Stochastic Advection by Lie Transport (SALT)?

- ▶ Assumption (in the deterministic approach): fluid particles follow

$$\frac{d\mathbf{X}(a, t)}{dt} = \mathbf{u}(\mathbf{X}(a, t), t), \quad \mathbf{X}(a, 0) = a \in \mathbb{R}^2 \text{ or } \mathbb{R}^3,$$

$\mathbf{X}(a, t)$: particle trajectory (with Lagrangian label a)

\mathbf{u} : velocity field



- ▶ In the SALT approach, Lagrangian particle paths follow a Stratonovich stochastic process

$$d\mathbf{X}(a, t) = \bar{\mathbf{u}}(\mathbf{X}(a, t))dt + \sum_i \xi_i(\mathbf{X}(a, t)) \circ dW_t^i,$$

$\bar{\mathbf{u}}$: mean flow velocity $\sum_i \xi_i(\mathbf{X}(a, t)) \circ dW_t^i$: stochastic perturbation about the mean flow
 W_t^i : Ind. Wiener processes ξ_i : spatial correlation of the small-scale velocity fluctuations

- ▶ Fluid dynamics equations are derived based on this assumption

Estimation of ξ_i from high resolution data

$$(\mathbf{u} - \bar{\mathbf{u}})\Delta t \approx \sum_{i=1}^N \xi_i \Delta W_m^i$$

- W^i : Brownian motion $\rightarrow \Delta W_m^i$: normal distribution with $\mu = 0$ and $\sigma^2 = \Delta t$

$$\begin{bmatrix} \Delta X_{11} & \Delta X_{12} & \cdots & \Delta X_{1n} \\ \Delta X_{21} & \Delta X_{22} & \cdots & \Delta X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta X_{m1} & \Delta X_{m2} & \cdots & \Delta X_{mn} \end{bmatrix} \xrightarrow{\text{SVD}} \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_r \\ | & | & & | \\ \Delta X_{m1} & \Delta X_{m2} & \cdots & \Delta X_{mn} \end{bmatrix} \begin{bmatrix} \cdots & v_1^T & \cdots \\ \cdots & v_2^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & v_r^T & \cdots \end{bmatrix}$$

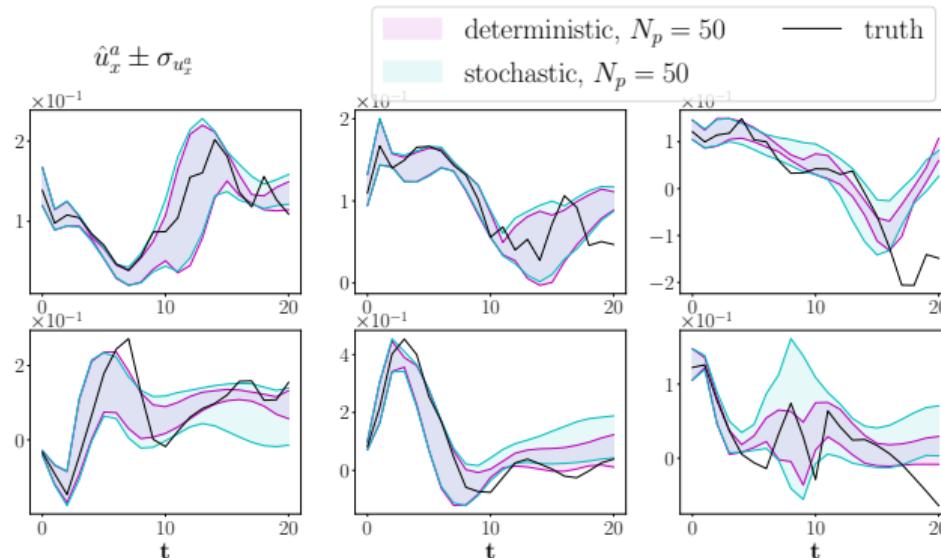
- Elements of time-series a_i are generally modeled using a normal distribution ($\mu = 0$, $\sigma^2 = \Delta t$)
- v_i^T represent the spatial correlation of the unresolved small-scale dynamics and hence are the ξ_i
- Only a few ξ 's are needed to simulate most of the variance in $(\mathbf{u} - \bar{\mathbf{u}})\Delta t$ data

Stochastic versus deterministic ensemble

- ▶ How to quantify uncertainty due to unresolved scales with deterministic models?
- ▶ Perturb the atmosphere velocity field at $t = 25$ to create an ensemble of initial conditions

$$\mathbf{u}_{\text{pert}} = \mathbf{u}_0 + 0.2 \times r \times \mathbf{u}_0, \quad \mathbf{u}_0 : \text{Initial velocity field}, \quad r \in \mathcal{N}(0, 1)$$

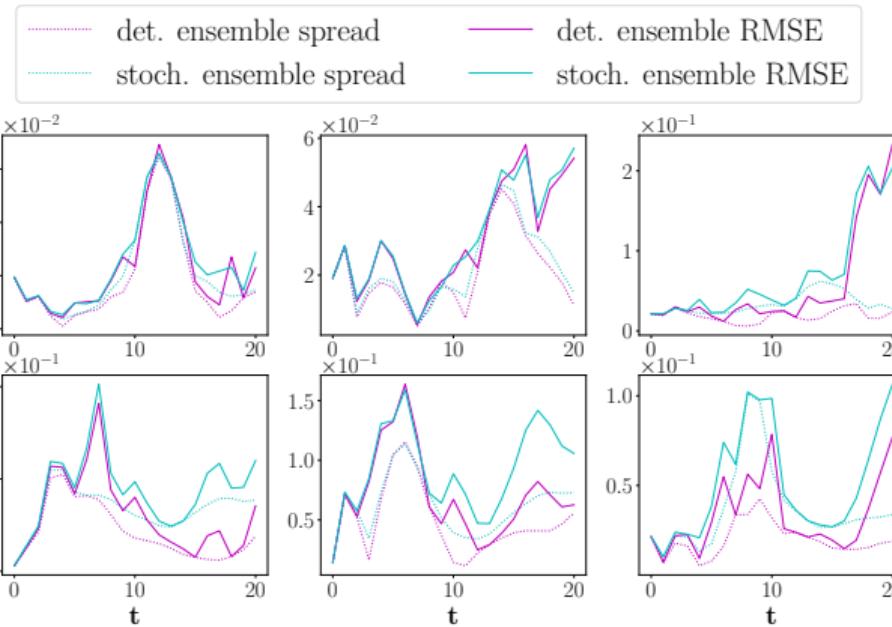
- ▶ Comparison between 50 independent realizations of the deterministic and stochastic model



- ▶ Stochastic model exhibits bigger spreads → true solution stays inside the spread for longer

Uncertainty Quantification (UQ) skills

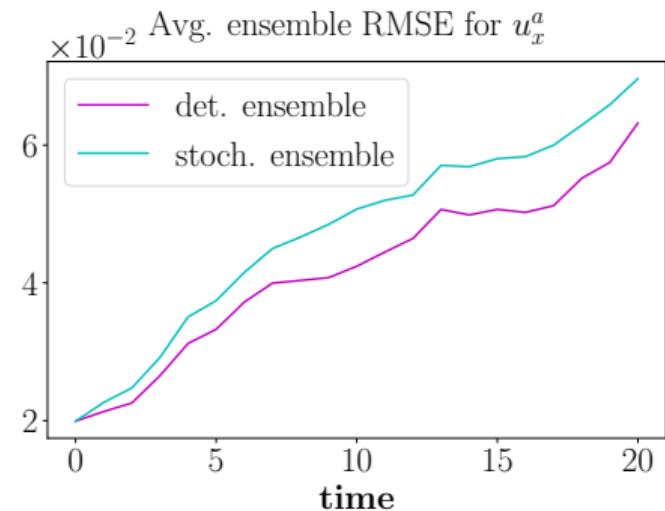
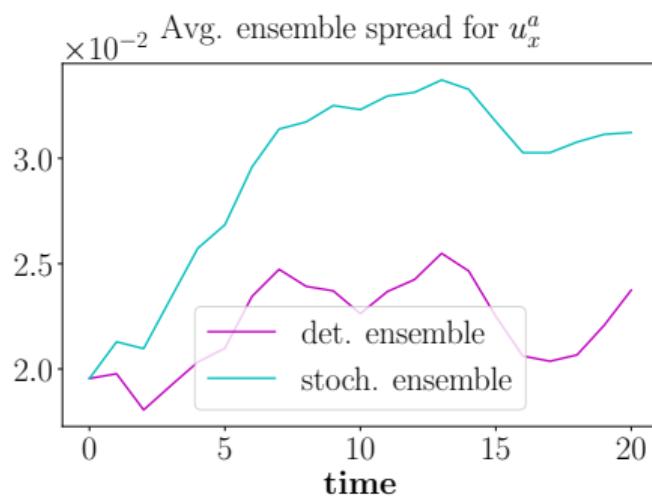
- ▶ Comparison between the ensemble spread and RMSE for atmospheric velocity (x component)



- ▶ Both models have good UQ skills (for at least 10 to 15 time units)
- ▶ Stochastic model indeed exhibits larger spreads
- ▶ RMSE of stochastic > RMSE of deterministic

UQ skill comparison

- ▶ Average spread size and average RMSE comparison:



- ▶ Stochastic model exhibits bigger spread size but also higher RMSE!
- ▶ **Which model has better prediction skills?**

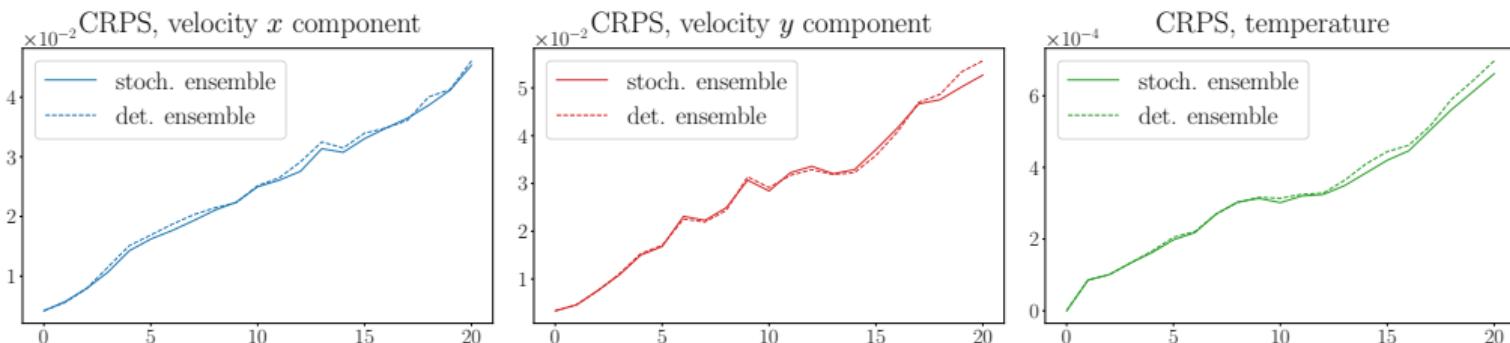
Continuous Ranked Probability Score

- ▶ Continuous Ranked Probability Score (CRPS) evaluates the accuracy of the entire predicted probability distribution against the observed value

$$\text{crps}(F, y) = \int_{-\infty}^{\infty} (F(x) - \mathbb{1}(x \geq y))^2 dx,$$

$F(x)$: Forecast CDF, y : Observation, $\mathbb{1}$: Heavyside step function

- ▶ CRPS plots comparing the performance of the stochastic and the deterministic ensemble:



- ▶ **Stochastic parameterization produces better forecasts than the deterministic approach**
 - ▶ Physics-based model
 - ▶ Produces bigger spreads, captures the truth for longer → more suitable for data assimilation