Linear Programming Counterexamples

Eklavya Sharma

1 Polyhedra

Example 1 (Square Pyramid). *Let*

$$\begin{aligned} P &:= \{(x,y,z) : \max(|x|,|y|) \le z, z \le 1\} \\ &= \{(x,y,z) : x \le z, -x \le z, y \le z, -y \le z, z \le 1\}. \end{aligned}$$

Then P is an (inverted) square pyramid. The base of the pyramid is $[-1,1]^2 \times \{1\}$. Its vertices are $(0,0,0)$, $(-1,-1,1)$, $(-1,1,1)$, $(1,-1,1)$, $(1,1,1)$.

Example 2 (Square Pyramid). *Let*

$$P := \{(x,y,z) : x \le z, y \le z, x \ge 0, y \ge 0, z \ge 0\}.$$

Then P is an (inverted) square pyramid. The base of the pyramid is $[0,1]^2 \times \{1\}$. Its vertices are $(0,0,0)$, $(0,0,1)$, $(0,1,1)$, $(1,0,1)$, $(1,1,1)$.

Example 3 (Introducing degeneracy by changing RHS to 0). *Let $P := \{x : (a_i^T x = b_i \forall i \in E)$ and $(a_i^T x \le b_i \forall i \in E)\}$. Let $D := \{x : (a_i^T x = 0 \forall i \in E)$ and $(a_i^T x \le 0 \forall i \in E)\}$. Then D is the set of directions of P. If P is bounded, then $D = \{0\}$. Now all bases of D correspond to the same point. If P has multiple bases, then 0 is a degenerate point of D. Furthermore, the simplex method can be made to run on P and D with the same pivots.*

Example 4 (Non-extreme point with n active constraints). *Let $P := \{(x,y) : x + y \ge 1, x + y \le 1, x \ge 0, y \ge 0\}$. No constraint is redundant. $(1,0)$ and $(0,1)$ are degenerate BFSes of P. Their midpoint, $(1/2,1/2)$, is not an extreme point, but has 2 active constraints (which are linearly dependent).*

1.1 Degeneracy vs Redundancy

Example 5 (Redundancy $\not\Rightarrow$ Degeneracy). *Let $P := \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Then every extreme-point is non-degenerate. Adding the constraint $y \le 2$ doesn't add degeneracy but adds redundancy.*

Example 6 (Degeneracy $\not\Rightarrow$ Redundancy). *Let $P := \{(x,y) : y \le x, y \le -x, y \ge 0\}$ (so $P = \{(0,0)\}$). Then $(0,0)$ is a degenerate extreme point, but no constraint is redundant.*

Example 7 (Degeneracy $\not\Rightarrow$ Redundancy). *Let P be a square pyramid (c.f. Example 1). Then there is a degenerate extreme point but no constraint is redundant.*

2 Simplex Method

Example 8. *Let $b \in \mathbb{R}_{\geq 0}$ and $a \in \mathbb{R}^n$, where $0 < a_1 < a_2 < \ldots < a_n$. Consider the LP*

$$\max_{x \geq 0} \quad a^T x \quad \textit{where} \quad \sum_{i=1}^{n} x_i \leq b.$$

Clearly, the optimal solution is $[0, 0, \ldots, 0, b]$. In standard form, the LP becomes

$$\min_{x \geq 0} \quad \sum_{i=1}^{n} (-a_i) x_i \quad \textit{where} \quad \sum_{i=1}^{n+1} x_i = b.$$

Suppose our initial basis is $\{x_{n+1}\}$. For ease of notation, let $x_0 := x_{n+1}$.

If we run the simplex method with Bland's rule (variable of lowest index enters basis), then there will be n iterations, where in the i^{th} iteration, x_i enters the basis and x_{i-1} leaves the basis. Hence, we visit each of the $n+1$ bases. If $b > 0$, each basis corresponds to a unique BFS. If $b = 0$, all bases correspond to the same BFS $\mathbf{0}$.

If we run the simplex method with Dantzig's rule (variable of most negative reduced cost enters basis), then there will be just 1 iteration where x_n enters and x_{n+1} leaves.

Example 9. *For the following LP, where $b \in \mathbb{R}_{\geq 0}$, the optimal solution is $(b, 0)$.*

$$\max_{x \geq 0, y \geq 0} \quad 2x + 3y \quad \textit{where} \quad x + 2y \leq b.$$

Bland's rule takes 1 iteration but Dantzig's rule takes 2 iterations. There are 3 bases. When $b > 0$, each base corresponds to a different BFS. When $b = 0$, all bases correspond to the same BFS $\mathbf{0}$.