Stochastic Processes

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Definition 1 (Stochastic Process). *Let $\mathcal{T} \subseteq \mathbb{R}$. For any $t \in \mathcal{T}$, let X_t be a random variable with support D. Then $X := \{X_t : t \in \mathcal{T}\}$ is called a* stochastic process *on state-space D and time $\mathcal{T}$. Usually, $\mathcal{T}$ is either $\mathbb{Z}_{\geq 0}$ (discrete-time) or $\mathbb{R}_{\geq 0}$ (continuous-time).*

1 Discrete-Time Markov Chains

Definition 2 (Markov Chain). *Let $X := [X_0, X_1, \ldots]$ be a stochastic process on state-space D and time $\mathbb{Z}_{\geq 0}$. X is called a discrete-time markov chain if $\Pr(X_{t+1} = d \mid X_t, X_{t-1}, \ldots, X_0) = \Pr(X_{t+1} = d \mid X_t)$. If $\Pr(X_{t+1} = d \mid X_t) = \Pr(X_1 = d \mid X_0)$, then X is called* time-homogeneous.

Definition 3 (Transition function). *Let X be a markov chain on state space D. Define $P^{(k)} : D \times D \mapsto [0, 1]$ as $P^{(k)}(i, j) = \Pr(X_k = j \mid X_0 = i)$. Then $P^{(k)}$ is called the k-step transition function of X. For $k = 1$, we simply write P instead of $P^{(1)}$. For a finite state space, we can represent P as a matrix.*

Lemma 1 (Chapman-Kolmogorov Equation). $P^{(m+n)}(i, j) = \sum_k P^{(m)}(i, k) P^{(n)}(k, j)$.

1.1 Classification of States, Recurrence, Limiting Probabilities

Definition 4. *Let $f_{i,j} := \Pr\left(\bigvee_{t\geq 1}(X_t = j) \,\middle|\, X_0 = i\right)$. Then $f_{i,j}$ is called the* eventual transition probability *from i to j. If $i = j$, then we write $f_{i,i}$ as f_i, and call it the* recurrence probability *of state i.*

Definition 5. *For a state i, let N_i be the random variable that counts the number of times we are in state i, i.e., $N_i := \sum_{t=0}^{\infty} \mathbf{1}(X_t = i)$. Then N_i is called the visit-count of i.*

Definition 6. *A state i of a markov chain is* recurrent *iff (the following are equivalent):*

- *the recurrence probability (f_i) of i is 1.*
- *i is visited infinitely often, i.e., $\Pr(N_i = \infty \mid X_0 = i) = 1$.*
- *i is visited infinitely often in expectation, i.e., $\mathrm{E}(N_i \mid X_0 = i) = \infty$.*

A non-recurrent state is called a transient *state.*

Lemma 2. $\Pr(N_i = k \mid X_0 = i) = f_i^{k-1}(1 - f_i)$.

Lemma 3. $\mathrm{E}(N_i \mid X_0 = i) = 1/(1 - f_i) = \sum_{t=0}^{\infty} P^{(t)}(i, i)$.

Definition 7. *State j is* accessible *from state i if $P^{(t)}(i,j) > 0$ for some t. States i and j* communicate *(denoted as $i \leftrightarrow j$) if i and j are both accessible from each other.*

Lemma 4. *Accessibility is reflexive and transitive. Communication is an equivalence relation. The equivalence classes of communicability are called* state classes*. A markov chain is* irreducible *if it has just one state class.*

Definition 8. *Let T_i be the time when a markov chain moves to state i, i.e., $T_i := \min_{t \geq 1}(X_t = i)$. When conditioned on $X_0 = i$, T_i is called the* recurrence time *of i. State i is called* positive recurrent *if $\mathrm{E}(T_i \mid X_0 = i)$ is finite, otherwise it is* null recurrent*.*

Lemma 5. *Recurrence and positive recurrence are class properties, i.e., they are same for all states in a class.*

Lemma 6. *In a finite-state markov chain, all recurrent states are positive recurrent, and there is at least one recurrent state.*

Definition 9 (Periodicity)**.** *For a state i, its period is defined as $\gcd(\{t : \Pr(T_i = t \mid X_0 = i) > 0\})$. A state is* aperiodic *if its period is 1.*

Lemma 7. *Periodicity is a class property.*

Definition 10 (Ergodicity)**.** *A state is ergodic if it is positive recurrent and aperiodic. A markov chain is ergodic if all its states are ergodic.*

Lemma 8. *In an irreducible ergodic markov chain, for every state j, $\lim_{t\to\infty} P^{(t)}(j,i) = \pi_i$ for a unique real number π_i. π_i is called the* limiting probability *of state i. Furthermore, π_i is the unique solution to this system of equations: $\pi_i = \sum_j \pi_j P(j,i)$ for all i ($\pi = P^T\pi$ in matrix form) and $\sum_i \pi_i = 1$.*

Lemma 9. *In an irreducible ergodic markov chain, $\mathrm{E}(T_i \mid X_0 = i) = 1/\pi_i$.*

Corollary 9.1. *A state i is null recurrent iff $\pi_i = 0$.*

Theorem 10. *If the transition function of markov chain X is doubly-stochastic (i.e., each row and each column sums to 1), then the limiting probability of each state is $1/n$, where n is the number of states.*

1.2 Time-Reversibility

Definition 11. *For an irreducible ergodic markov chain X with limiting probabilities π. Let Y be a markov chain whose transition function is $Q(i,j) = P(j,i)(\pi_j/\pi_i)$. Then Y is called the* time-reversed *markov chain of X. X is called* time-reversible *if $Q = P$.*

Theorem 11. *Let X be a time-reversible markov chain with limiting probabilities π. Then π is the unique solution to this system of equations: $x_j P(j,i) = x_i P(i,j)$ for all states i and j, and $\sum_i x_i = 1$.*

Theorem 12. *If the transition function of markov chain X is symmetric, then X is time-reversible.*