# Stochastic Processes

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**Definition 1** (Stochastic Process). *Let $\mathcal{T} \subseteq \mathbb{R}$. For any $t \in \mathcal{T}$, let $X_t$ (or $X(t)$) be a random variable with support $D$. Then $X := \{X_t : t \in \mathcal{T}\}$ is called a* stochastic process *on state-space $D$ and time $\mathcal{T}$. Usually, $\mathcal{T}$ is either $\mathbb{Z}_{\geq 0}$ (discrete-time) or $\mathbb{R}_{\geq 0}$ (continuous-time).*

# 1 Discrete-Time Markov Chains

**Definition 2** (Markov Chain). *Let $X := [X_0, X_1, \ldots]$ be a stochastic process on state-space $D$ and time $\mathbb{Z}_{\geq 0}$. $X$ is called a discrete-time markov chain if $\Pr(X_{t+1} = d \mid X_t, X_{t-1}, \ldots, X_0) = \Pr(X_{t+1} = d \mid X_t)$. If $\Pr(X_{t+1} = v \mid X_t = u) = \Pr(X_1 = v \mid X_0 = u)$ for all $t, u, v$, then $X$ is called* time-homogeneous.

**Definition 3** (Transition function). *Let $X$ be a markov chain on state space $D$. Define $P^{(k)} : D \times D \mapsto [0, 1]$ as $P^{(k)}(i, j) = \Pr(X_k = j \mid X_0 = i)$. Then $P^{(k)}$ is called the $k$-step transition function of $X$. For $k = 1$, we simply write $P$ instead of $P^{(1)}$. For a finite state space, we can represent $P$ as a matrix.*

**Lemma 1** (Chapman-Kolmogorov Equation). *$P^{(m+n)}(i, j) = \sum_k P^{(m)}(i, k) P^{(n)}(k, j)$.*

## 1.1 Classification of States, Recurrence, Limiting Probabilities

**Definition 4.** *Let $f_{i,j} := \Pr\left(\bigvee_{t \geq 1}(X_t = j) \mid X_0 = i\right)$. Then $f_{i,j}$ is called the* eventual transition probability *from $i$ to $j$. If $i = j$, then we write $f_{i,i}$ as $f_i$, and call it the* recurrence probability *of state $i$.*

**Definition 5.** *For a state $i$, let $N_i$ be the random variable that counts the number of times we are in state $i$, i.e., $N_i := \sum_{t=0}^{\infty} \mathbf{1}(X_t = i)$. Then $N_i$ is called the* visit-count *of $i$.*

**Definition 6.** *A state $i$ of a markov chain is* recurrent *iff (the following are equivalent):*

- *the recurrence probability ($f_i$) of $i$ is 1.*

- *$i$ is visited infinitely often, i.e., $\Pr(N_i = \infty \mid X_0 = i) = 1$.*

- *$i$ is visited infinitely often in expectation, i.e., $\mathrm{E}(N_i \mid X_0 = i) = \infty$.*

*A non-recurrent state is called a* transient *state.*

**Lemma 2.** *$\Pr(N_i = k \mid X_0 = i) = f_i^{k-1}(1 - f_i)$.*

**Lemma 3.** *$\mathrm{E}(N_i \mid X_0 = i) = 1/(1 - f_i) = \sum_{t=0}^{\infty} P^{(t)}(i, i)$.*

**Definition 7.** *State $j$ is* accessible *from state $i$ if $P^{(t)}(i,j) > 0$ for some $t$. States $i$ and $j$* communicate *(denoted as $i \leftrightarrow j$) if $i$ and $j$ are both accessible from each other.*

**Lemma 4.** *Accessibility is reflexive and transitive. Communication is an equivalence relation. The equivalence classes of communicability are called* state classes. *A markov chain is* irreducible *if it has just one state class.*

**Definition 8.** *Let $T_i$ be the time when a markov chain moves to state $i$, i.e., $T_i := \min_{t \geq 1}(X_t = i)$. When conditioned on $X_0 = i$, $T_i$ is called the* recurrence time *of $i$. State $i$ is called* positive recurrent *if $\mathrm{E}(T_i \mid X_0 = i)$ is finite, otherwise it is* null recurrent.

**Lemma 5.** *Recurrence and positive recurrence are class properties, i.e., they are same for all states in a class.*

**Lemma 6.** *In a finite-state markov chain, all recurrent states are positive recurrent, and there is at least one recurrent state.*

**Definition 9** (Periodicity)**.** *For a state $i$, its period is defined as $\gcd(\{t : \Pr(T_i = t \mid X_0 = i) > 0\})$. A state is* aperiodic *if its period is 1.*

**Lemma 7.** *Periodicity is a class property.*

**Definition 10** (Ergodicity)**.** *A state is ergodic if it is positive recurrent and aperiodic. A markov chain is ergodic if all its states are ergodic.*

**Lemma 8.** *In an irreducible ergodic markov chain, for every state $j$, $\lim_{t \to \infty} P^{(t)}(j,i) = \pi_i$ for a unique real number $\pi_i$. $\pi_i$ is called the* limiting probability *of state $i$. Furthermore, $\pi_i$ is the unique solution to this system of equations: $\pi_i = \sum_j \pi_j P(j,i)$ for all $i$ ($\pi = P^T \pi$ in matrix form) and $\sum_i \pi_i = 1$.*

**Lemma 9.** *In an irreducible ergodic markov chain, $\mathrm{E}(T_i \mid X_0 = i) = 1/\pi_i$.*

**Corollary 9.1.** *A state $i$ is null recurrent iff $\pi_i = 0$.*

**Theorem 10.** *If the transition function of markov chain $X$ is doubly-stochastic (i.e., each row and each column sums to 1), then the limiting probability of each state is $1/n$, where $n$ is the number of states.*

## 1.2 Time-Reversibility

**Definition 11.** *For an irreducible ergodic markov chain $X$ with limiting probabilities $\pi$. Let $Y$ be a markov chain whose transition function is $Q(i,j) = P(j,i)(\pi_j/\pi_i)$. Then $Y$ is called the* time-reversed *markov chain of $X$. $X$ is called* time-reversible *if $Q = P$.*

**Theorem 11.** *Let $X$ be a time-reversible markov chain with limiting probabilities $\pi$. Then $\pi$ is the unique solution to this system of equations: $x_j P(j,i) = x_i P(i,j)$ for all states $i$ and $j$, and $\sum_i x_i = 1$.*

**Theorem 12.** *If the transition function of markov chain $X$ is symmetric, then $X$ is time-reversible.*

## 2 Counting Process

**Definition 12** (Counting Process). *Let $N$ be a stochastic process on state space $\mathbb{Z}_{\geq 0}$ and time $\mathbb{R}_{\geq 0}$. Then $N$ is called a counting process if $N(0) = 0$ and $N(t)$ is monotone in $t$, i.e., $t_1 < t_2 \implies N(t_1) \leq N(t_2)$.*

**Definition 13** (Independent increments). *A counting process $N$ has* independent increments *iff for any two disjoint intervals $(u_1, v_1]$ and $(u_2, v_2]$ in $\mathbb{R}_{\geq 0}$, the random variables $N(v_1) - N(u_1)$ and $N(v_2) - N(u_2)$ are independent.*

**Definition 14** (Stationary increments). *A counting process $N$ has* stationary increments *iff for any $u \leq v$, the random variables $N(v) - N(u)$ and $N(v - u)$ have the same distribution.*

**Definition 15** (Arrival and interarrival times). *For a counting process $N$, for $i \in \mathbb{Z}_{\geq 0}$, define the $i^{th}$ arrival time $S_i := \min_{t \geq 0}(N(t) = i)$. For $i \in \mathbb{Z}_{\geq 1}$, define the $i^{th}$ interarrival time $T_i := S_i - S_{i-1}$.*

**Lemma 13.** *For a counting process $N$ with arrival times $S$, $N(t) \geq n \iff S_n \leq t$.*

**Definition 16** (Stopping time). *Let $X = [X_1, X_2, \ldots]$ be a sequence of random variables. The random variable $N$ is called a stopping time for $X$ if for all $n \geq 0$, (the following two definitions are equivalent):*

- *$N = n$ is independent of $X_{n+1}, X_{n+2}, \ldots$.*

- *$N \leq n$ is independent of $X_{n+1}, X_{n+2}, \ldots$.*

**Theorem 14** (Wald's identity). *Let $X = [X_1, X_2, \ldots]$ be a sequence of random variables where $\mathrm{E}(X_i) = \mu$ for all $i$. Let $N$ be a stopping time for $X$. Then*

$$\mathrm{E}\left(\sum_{i=1}^{N} X_i\right) = \mu \, \mathrm{E}(N).$$

*Proof sketch.* For all $i$, $N \geq i$ is independent of $X_i$, and $\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i \, \mathbf{1}(N \geq i)$. $\qquad \square$

## 3 Poisson Process

**Definition 17** (Poisson process). *A counting process $N$ is a Poisson process with rate function $\lambda : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ if $N$ has independent increments and $N(t_2) - N(t_1) \sim \mathrm{Poisson}(\mu)$, where $\mu := \int_{t_1}^{t_2} \lambda(t)dt$. $N$ is called* homogeneous *if $\lambda(t) = \lambda(0)$ for all $t$, otherwise it is called* inhomogeneous. *For a homogeneous process, we denote $\lambda(0)$ by $\lambda$.*

**Lemma 15.** *A Poisson process $N$ is homogeneous iff it has stationary increments.*

**Theorem 16** (Alternative definition of Poisson process). *A counting process $N$ is a Poisson process with continuous rate function $\lambda$ iff $N$ has independent and stationary increments and $\Pr(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$ and $\Pr(N(t+h) - N(t) \geq 2) = o(h)$.*

*Proof sketch for homogeneous.* Let $g(u,t) := \mathrm{MGF}_u(N(t)) = \mathrm{E}(e^{uN(t)})$. Show $g(u,t) = 1 + \lambda t(e^u - 1) + o(t)$ straightforwardly. Use calculus to show that $g(u,t) = \exp(e^{\lambda t}(e^u - 1))$ (find derivative w.r.t $t$ by computing $\lim_{h \to 0}(g(u,t+h) - g(u,t))/h$; this gets rid of $o(h)$). Conclude that $N(t) \sim \mathrm{Poisson}(\lambda t)$ since $g(u,t)$ is MGF of $\mathrm{Poisson}(\lambda t)$. $\qquad\square$

**Lemma 17.** *For a homogeneous Poisson process $N$,*

$$\Pr(N(s) = a \mid N(s+t) = a + b) = \binom{a+b}{a} \left(\frac{s}{s+t}\right)^a \left(\frac{t}{s+t}\right)^b.$$

**Theorem 18.** *Let $N$ be a counting process. Then $N$ is a homogeneous Poisson process with rate $\lambda$ iff all interarrival times are independent and distributed $\mathrm{Expo}(\lambda)$.*

**Theorem 19** (Decomposition theorem 1)**.** *Let $K$ be a finite set, and let $\{N_i : i \in K\}$ be independent Poisson processes, where $N_i$ has rate function $\lambda_i$. Let $N := \sum_{i \in K} N_i$. Then $N$ is a Poisson process with rate function $\sum_{i \in K} \lambda_i$.*

**Theorem 20** (Decomposition theorem 2)**.** *Let $N$ be a Poisson process with rate function $\lambda$. Let $K$ be a finite set (called set of labels). Suppose the $j^{th}$ event receives label $L_j \in K$, where $\Pr(L_j = i) = p_i(S_j)$ for some function $p_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$, and $\{N, L_1, L_2, \ldots\}$ are independent. For $i \in K$, let $N_i(t)$ be the number of events having label $i$, i.e, $N_i(t) = \sum_{j=1}^{N(t)} \mathbf{1}(L_j = i)$. Then $N_i$ is a Poisson process with rate function $p_i\lambda$. Furthermore, all $N_i$ are independent and if all $p_i$ are constant, then $N_i(t) \mid N(t) \sim \mathrm{Binom}(N(t), p_i)$.*

**Lemma 21.** *Let $N^{(1)}$ and $N^{(2)}$ be independent homogeneous Poisson processes with rates $\lambda_1$ and $\lambda_2$. Then*

$$\Pr(S_n^{(1)} < S_m^{(2)}) = \sum_{i=n}^{n+m-1} \binom{n+m-1}{i} \frac{\lambda_1^i \lambda_2^{n+m-1-i}}{(\lambda_1 + \lambda_2)^{n+m-1}}.$$

*Proof sketch.* Model as a continuous markov chain with state space $(n_1, n_2)$, where $n_i$ is the number of events of $N^{(i)}$ that have occurred. $\qquad\square$

**Theorem 22** (arrival times distributed as order statistics)**.** *Let $X = [X_1, X_2, \ldots, X_n]$ be IID uniform variables over $[0,t]$. Let $U = \mathrm{sorted}(X)$. Let $N$ be a homogeneous Poisson process. Let $S_i$ be the $i^{th}$ arrival time of $N$. Then conditioned on $N(t) = n$, the distribution of $[S_1, \ldots, S_n]$ and $U$ are identical.*

**Lemma 23** (Excess and Residual)**.** *Let $N$ be a Poisson process with rate function $\lambda$. Let $S_i$ be the $i^{th}$ arrival time. Let $Y(t) := S_{N(t)+1} - t$ and $R(t) := t - S_{N(t)}$. Then $Y(t) > s \iff N(t+s) - N(t) = 0$ and $R(t) > r \iff N(t) - N(t-r) = 0$. If $N$ is homogeneous, we get $Y(t) \sim \mathrm{Expo}(\lambda)$ and $R(t) \sim \mathrm{Expo}(\lambda)$.*

# 4 Continuous-Time Markov Chain

**Definition 18** (CTMC)**.** *Let $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$ be a stochastic process on discrete state-space $D$. $X$ is called a continuous-time markov chain (CTMC) if $\Pr(X(t+s) = d \mid \{X(u) : 0 \leq u \leq s\}) = \Pr(X(t+s) = d \mid X(s))$ for all $s, t \in \mathbb{R}_{\geq 0}$. If $\Pr(X(t+s) = v \mid X(s) = u) = \Pr(X(t) = v \mid X(0) = u)$ for all $u, v, s, t$, then $X$ is called* time-homogeneous *(TH) or* stationary.

**Theorem 24** (Equiv defn of TH CTMC). *Let $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$ be a stochastic process on discrete state-space $D$. Let $Y(t) := \{X(u) : 0 \leq u < t\}$. Let $T_i^{(s)} := \min_{t \geq 0}(X(t + s) \neq i)$. Let $P_{i,j}^{(s)} := \Pr(X(s + T_i^{(s)}) = j \mid X(s) = i, Y(s))$. $X$ is TH CTMC iff $(T_i^{(s)} \mid X(s) = i, Y(s)) \sim \mathrm{Expo}(\nu_i)$, where $\nu_i$ is a constant that doesn't depend on $s$ or $Y(s)$, and $P_{i,j}^{(s)}$ is a constant that doesn't depend on $s$ or $Y(s)$.*

Since $T_i^{(s)}$ and $P_{i,j}^{(s)}$ don't depend on $s$, we simply write $T_i$ and $P_{i,j}$. $T_i$ is called the transition time *out of state $i$*, $\nu_i$ is called the transition rate *out of state $i$*, and $P_{i,j}$ is the probability of transitioning *from state $i$ to state $j$*.

Let $q_{i,j} := \nu_i P_{i,j}$. Then $\nu_i = \sum_j q_{i,j}$.

**Theorem 25** (Chapman-Kolmogorov DiffEqs). *For a TH CTMC $X$, let $P_{i,j}(t) := \Pr(X(t) = j \mid X(0) = i)$. Then*

- *Backward DiffEqs:* $\dfrac{dP_{i,j}(t)}{dt} = \sum_{k \neq i} q_{i,k} P_{k,j}(t) - \nu_i P_{i,j}(t).$

- *Forward DiffEqs:* $\dfrac{dP_{i,j}(t)}{dt} = \sum_{k \neq j} P_{i,k}(t) q_{k,j} - P_{i,j}(t) \nu_j.$

**Lemma 26.** *Let $X$ be a TH CTMC.*

$$\lim_{h \to 0} \frac{1 - P_{i,i}(h)}{h} = \nu_i \quad \forall i \qquad\qquad \lim_{h \to 0} \frac{P_{i,j}(h)}{h} = q_{i,j} \quad \forall i \neq j$$

**Lemma 27** (Limiting probability). *In an irreducible positive-recurrent TH CTMC $X$, for every state $j$, $\lim_{t \to \infty} P_{j,i}(t) = P_i$ for a unique real number $P_i$. $P_i$ is called the limiting probability of state $i$. Furthermore, $P_i$ is the unique solution to $\sum_i P_i = 1$ and CK forward equations, i.e., $P_i \nu_i = \sum_{j \neq i} P_j q_{j,i}$.*

**Lemma 28** (Limiting probability of embedded chain). *Let $X$ be an irreducible positive-recurrent TH CTMC. Let $Y$ be the sequence of states visited by $X$. Then $Y$ is a discrete MC. Let $P$ and $\pi$ be the limiting probabilities of $X$ and $Y$, respectively. Then $P_i = (\pi_i/\nu_i)/(\sum_j \pi_j/\nu_j)$ and $\pi_i = P_i\nu_i/(\sum_j P_j\nu_j)$.*

**Definition 19.** *A CTMC is time-reversible iff the corresponding embedded discrete-time MC is time-reversible.*

## 4.1 Birth and Death Process

**Definition 20.** *A birth-and-death (B&D) process is a TH CTMC $X$ on state space $\mathbb{Z}_{\geq 0}$ where $q_{i,j} = 0$ if $j \notin \{i-1, i+1\}$. Let $\lambda_i := q_{i,i+1}$ for $i \geq 0$, $\mu_i := q_{i,i-1}$ for $i \geq 1$, $\mu_0 := 0$.*

$X(t)$ is called the population *at time $t$*, $\lambda_i$ is called the birth rate *at population $i$*, and $\mu_i$ is called the death rate *at population $i$*.

**Lemma 29.** *Let $X$ be a B&D process where $X(0) = n$. Let $T_n$ be the time to reach state $n+1$, i.e., $T_n := \min_{t \geq 0}(X(t) = n+1)$. Then*

$$\mathrm{E}(T_n) = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} \mathrm{E}(T_{n-1}) = \frac{1}{\lambda_n} \sum_{i=0}^{n} \prod_{j=1}^{i} \frac{\mu_{n-j+1}}{\lambda_{n-j}}.$$

$$\mathrm{Var}(T_n) = \frac{1}{\lambda_n(\lambda_n + \mu_n)^2} + \frac{\mu_n}{\lambda_n} \mathrm{Var}(T_{i-1}) + \frac{\mu_n}{\lambda_n + \mu_n}(\mathrm{E}(T_{n-1}) + \mathrm{E}(T_n))^2$$

*Proof sketch.* Let $I_i = \mathbf{1}(\text{next transition goes to state } i + 1)$. Let $X_i$ be the transition time out of state $i$. Then $I_i \sim \text{Bernouilli}(\lambda_i/(\mu_i + \lambda_i))$, $X_i \sim \text{Expo}(\lambda_i + \mu_i)$, and

$$\text{E}(T_i \mid I_i) = \text{E}(X_i) + (1 - I_i)(\text{E}(T_{i-1}) + \text{E}(T_i))),$$
$$\text{Var}(T_i \mid I_i) = \text{Var}(X_i) + (1 - I_i)(\text{Var}(T_{i-1}) + \text{Var}(T_i)). \qquad \square$$

CKBE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_i P_{i-1,j}(t) + \lambda_i P_{i+1,j}(t) - (\lambda_i + \mu_i) P_{i,j}(t).$$

CKFE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_{j+1} P_{i,j+1}(t) + \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{i,j}(t).$$

**Theorem 30** (Limiting Probabilities). *Let $X$ be an irreducible B&D process on state space $D \subseteq \mathbb{Z}_{\geq 0}$ where $0 \in D$. For $n \in D$, let $\alpha_n := \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$. If $\sum_{i \in D} \alpha_i$ is finite, then $P_i = \alpha_i P_0$, and $P_0 = 1 / \sum_{i \in D} \alpha_i$.*

*Proof sketch.* Use Lemma 27 and add adjacent equations. $\qquad \square$

# 5 Renewal Theory

**Definition 21.** *Let $[X_1, X_2, \ldots]$ be a sequence of IID non-negative randvars, called* interarrival times, *such that $\Pr(X_1 = 0) < 1$ and $\Pr(X_1 = \infty) = 0$. Let $S_n := \sum_{i=1}^{n} X_i$ (called* arrival times*). Let $N(t) := \max_n(S_n \leq t)$. Then $N$ is called a* renewal process *(note that it is a counting process).*

*We let $F$ and $f$ denote the CDF and PDF/PMF of $X_1$, respectively. We let $F^{(n)}$ and $f^{(n)}$ denote the CDF and PDF/PMF of $S_n$, respectively.*

*Let $R_i$ be the* reward *obtained at time $X_i$ for all $i \geq 1$, where all $R_i$ are independent. Let $R(t) := \sum_{i=1}^{N(t)} R_i$. Then $R$ is called a* renewal reward process.

**Lemma 31.** *For all $t \geq 0$, $\Pr(N(t) = \infty) = 0$. $\Pr(\lim_{t \to \infty} N(t) = \infty) = 1$.*

*Proof.* Let $\mu := \text{E}(X_1)$. $\mu > 0$ since $\Pr(X_n = 0) < 1$.

$$\Pr\left(\lim_{t \to \infty} \frac{S_n}{n} = \mu\right) = 1. \qquad \text{(strong law of large numbers)}$$

$$N(t) = \infty \iff (\forall n, S_n \leq t) \implies \lim_{t \to \infty} \frac{S_n}{n} = 0.$$

$\Pr(N(\infty) = \infty) = 1$ since $\Pr(X_1 = \infty) = 0$. $\qquad \square$

**Definition 22.** *For a renewal process $N$, let $m_N(t) := \text{E}(N(t))$. Then $m_N$ is called the* mean-value function *of $N$. (If $N$ is clear from context, we will write $m$ instead of $m_N$.)*

**Lemma 32.** $m(t) = \sum_{n=1}^{\infty} \Pr(S_n \leq t) = \sum_{n=1}^{\infty} F^{(n)}(t)$.

**Theorem 33.** *$m$ uniquely characterizes $F$.*

**Lemma 34.** *$m(t)$ is finite for all $t$.*

**Theorem 35** (Renewal equation)**.** *When interarrival times are continuous randvars,*

$$m(t) = F(t) + \int_0^t m(t-x)f(x)dx.$$

*Proof sketch.* Let $N'(t) := \max_n \left( \sum_{i=2}^{n+1} X_i \le t \right)$. Then $N$ and $N'$ are identically distributed and

$$N(t) = \begin{cases} 1 + N'(t - X_1) & \text{if } X_1 \le t \\ 0 & \text{if } X_1 > t \end{cases}.$$

Finally, $m(t) = \mathrm{E}(\mathrm{E}(N(t) \mid X_1))$. $\qquad\square$

**Corollary 35.1.** *Let $N$ be a renewal process where interarrival times are distributed* Uniform$(0, 1)$*. Then for $0 \le t \le 1$, $m(t) = e^t - 1$.*

**Theorem 36** (Limit theorems)**.** *For a renewal process $N$ with $\mu := \mathrm{E}(X_1)$,*

$$\Pr\left( \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \right) = 1. \qquad\qquad \lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

**Theorem 37** (Limit theorems for rewards)**.** *For a renewal process $N$ with rewards $\{R_i : i \in \mathbb{Z}_{\ge 1}\}$, let $\alpha := \mathrm{E}(R_1)$ and $\mu := \mathrm{E}(X_1)$. Then*

$$\Pr\left( \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\alpha}{\mu} \right) = 1. \qquad\qquad \lim_{t\to\infty} \frac{\mathrm{E}(R(t))}{t} = \frac{\alpha}{\mu}.$$

**Theorem 38** (Central limit theorem for renewals)**.** *For a renewal process $N$ with $\mu := \mathrm{E}(X_1)$ and $\sigma^2 := \mathrm{Var}(X_1)$, the random variable*

$$\lim_{t\to\infty} \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}}$$

*tends to the standard normal distribution.*

**Lemma 39** (Stopping time)**.** *Let $X = [X_1, X_2, \ldots]$ be the sequence of interarrival times for renewal process $N$. Then $N(t) + 1$ is a stopping time for $X$.*

*Proof sketch.* $N(t) + 1 \le n \iff S_n > t$. $\qquad\square$

**Definition 23.** *For a renewal process $N$ with arrival times $S_1, S_2, \ldots$:*

- *Let $Y(t) := S_{N(t)+1} - t$. $Y(t)$ is called the* excess *at time $t$.*

- *Let $L(t) := t - S_{N(t)}$. $L(t)$ is called the* remaining life *at time $t$.*

**Lemma 40.** *Let $N$ be a renewal process with interarrival times $X = [X_1, X_2, \ldots]$. Then $\mathrm{E}(S_{N(t)+1}) = t + \mathrm{E}(Y(t)) = \mathrm{E}(X_1)(m(t) + 1)$.*

*Proof.* $N(t) + 1$ is a stopping time for $X$. $\qquad\square$