

# CMO 1: Preliminaries

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## 1 Central problem and algorithm template

**Central Problem of the course ‘Computational Methods of Optimization’:**  
Given an objective function  $f : \mathbb{R}^d \mapsto \mathbb{R}$  and a constraint set  $S \subseteq \mathbb{R}^d$ , find  $x^* = \operatorname{argmin}_{x \in S} f(x)$  and  $f^* = f(x^*)$ .

Example: for  $\min_{x \in \mathbb{R}} (x - t)^2$ ,  $x^* = t$  and  $f^* = 0$ .

All algorithms we develop to find  $x^*$  will follow this template:

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Pick  $x \in S$ .  
while  $x$  is not optimal do  
    Pick another  $x \in S$  such that  $f(x)$  decreases.  
end while  
return  $x$ 
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## 2 Metric space

For any set  $S$  (we’ll usually consider  $S = \mathbb{R}^d$ ),  $D : S \times S \mapsto \mathbb{R}$  is a distance function iff all of the following are true:

- $D(x, y) = 0 \iff x = y$ .
- $D(x, y) \geq 0$ .
- Symmetry:  $D(x, y) = D(y, x)$ .
- Triangle inequality:  $D(x, y) + D(y, z) \geq D(x, z)$ .

**Theorem 1.**  $D(x, y) = \|x - y\|$  is a distance function. Here

$$\|x\| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^d x_i^2}$$

**Theorem 2.**  $D(x, y) = \sum_{i=1}^d |x_i - y_i|$  is a distance function.

### 3 Neighborhood function and Open sets

**Definition 1.** For  $r > 0$  and  $x \in \mathbb{R}^d$ ,  $N_r(x) = \{z : D(x, z) < r\}$  is called a neighborhood of  $x$  of radius  $r$ .

**Definition 2.**  $x \in \mathbb{R}^d$  is an interior point of  $S$  iff  $\exists r > 0, N_r(x) \subseteq S$ .

**Definition 3.** Let  $x, y \in \mathbb{R}$ .

- $(x, y) = \{z : x < z < y\}$ .
- $(x, y] = \{z : x < z \leq y\}$ .
- $[x, y) = \{z : x \leq z < y\}$ .
- $[x, y] = \{z : x \leq z \leq y\}$ .

**Definition 4.**  $S$  is an open set iff  $\forall x \in S$ ,  $x$  is an interior point of  $S$ .

**Example 1.**  $(0, 1)$  is an open set but  $[0, 1]$  is not.

**Definition 5.**  $x \in \mathbb{R}^d$  is a limit point of  $S$  iff  $N_r(x) \cap S \neq \emptyset$ .

**Example 2.**  $0, \frac{1}{2}, 1$  are 3 of the limit points of  $(0, 1]$ .

**Definition 6.** Closure of a set  $S$  is the set of all limit points of  $S$ .

**Definition 7.** A set  $S$  is closed iff all limit points of  $S$  lie in  $S$ .

**Example 3.**  $[0, 1]$  is a closed set.

### 4 Limit and Bounds

**Definition 8.** Let  $[x_i]_{i \in \mathbb{N}}$  be an infinite sequence where  $x \in \mathbb{R}^d$ . Then

$$\lim_{i \rightarrow \infty} x_i = x \iff \forall \epsilon > 0, \exists n, \forall i \geq n, \|x - x_i\| < \epsilon$$

**Definition 9.**  $S \subseteq \mathbb{R}^d$  is a bounded set iff  $\exists M, \forall x \in S, \|x\| \leq M$ .

**Definition 10.** For  $x_i \in \mathbb{R}$ ,  $M$  is an upper bound of  $[x_i]_{i \in \mathbb{N}}$  iff  $\forall i, x_i \leq M$ . A sequence with an upper bound is called an upper-bounded sequence.

**Definition 11.**  $g$  is a least upper bound (LUB) (of  $[x_i]_{i \in \mathbb{N}}$ ) iff  $g$  is an upper bound and for every upper bound  $h$ ,  $g \leq h$ .

**Example 4.** For  $x_i = 1 - \frac{1}{i}$ , LUB is 1.

**Theorem 3.** A monotonic bounded sequence has a limit.

## 5 Continuity

**Definition 12.**

$$\lim_{x \rightarrow p} f(x) = q \iff \forall \epsilon > 0, \exists \delta > 0, \forall x \in N_\delta(p), f(x) \in N_\epsilon(q)$$

**Definition 13.**  $f$  is continuous at  $x \iff \lim_{x \rightarrow p} f(x) = f(p)$ .  $f$  is continuous over  $S \iff f$  is continuous at all points  $x \in S$ .

**Theorem 4.** Let  $S \subseteq \mathbb{R}^d$  be closed and bounded. Let  $f(S) = \{f(x) : x \in S\}$ . Let  $f$  be continuous over  $S$ . Then  $f(S)$  is closed and bounded.

For optimization problems,  $x^*$  is guaranteed to exist iff  $f$  is continuous and  $S$  is closed and bounded. Henceforth, we will assume  $S$  to be closed and bounded and assume functions to be continuous.

## 6 Asymptotics

$$a(x) \in o(b(x)) \iff \lim_{x \rightarrow x_0} \left| \frac{a(x)}{b(x)} \right| = 0$$

For example, at  $x = 0$ ,  $x^3 \in o(x^2)$ .

If  $f$  is continuous at  $x = p$ ,  $f(x) = f(p) + o(1)$ .