Inverse of $x \mapsto x \log_b x$

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Let $b > 1$. Let $f : \mathbb{R}_{\ge 0} \mapsto \mathbb{R}_{\ge 1}$ be a function such that $f(x)\log_b(f(x)) = x$; i.e. it is the inverse of $x \mapsto x\log_b(x)$. Our objective is to determine the asymptotic behavior of $f(x)$.

Theorem 1. *f is a bijective function.*

Proof. Let $h : \mathbb{R}_{\ge 1} \mapsto \mathbb{R}_{\ge 0}$ where $h(x) = x \log_b x$.

$x \ge 1 \implies x \log_b x \ge 0$. So h is a function. $h'(x) = \log_b x + \frac{1}{\ln b} > 0$. So $h(x)$ is strictly increasing.

Since h is a strictly increasing function from $[1, \infty)$ to $[0, \infty)$ and $h(1) = 0$, h is a bijective function (proof beyond the scope of this document).

Since f is the inverse of h, f is also bijective (proof beyond the scope of this document). □

Define functions ℓ and u as

$$\ell(x) = \frac{x}{\log_b x} \qquad\qquad u(x) = \left(1 + \frac{1}{e \ln b}\right)\frac{x}{\log_b x}$$

Theorem 2. $y \ge b \implies \ell(y) \le f(y)$

Proof. Let $y \ge b$ and $x = f(y)$. Therefore, $y = x\log_b x$ and $x \ge b$.

$$\begin{aligned}
\ell(y) &= \ell(x \log_b x) \\
&= \frac{x \log_b x}{\log_b x + \log_b \log_b x} \\
&\le \frac{x \log_b x}{\log_b x} && (x \log_b x \ge 0 \text{ and } \log_b \log_b x \ge 0) \\
&= x = f(y)
\end{aligned}$$

□

Lemma 3. *Let $g(x) = \frac{\log_b x}{x}$. Then $g(x) \le \frac{1}{e \ln b}$.*

Proof.

$$g(x) = \frac{1}{\ln b}\frac{\ln x}{x} \qquad\qquad g'(x) = \frac{1}{\ln b}\frac{1 - \ln x}{x^2}$$

$$x \ge e \implies 1 - \ln x \le 0 \implies g'(x) \le 0$$

$$0 < x \le e \implies 1 - \ln x \ge 0 \implies g'(x) \ge 0$$

Therefore, $g(x)$ attains its maximum value at $x = e$.

$$g(x) \le g(e) = \frac{1}{e \ln b}$$

□

Theorem 4. $y \ge 1 \implies f(y) \le u(y)$

Proof. Let $y \ge 1$ and $x = f(y)$. Therefore, $y = x \log_b x$ and $x \ge b$. Let $\frac{1}{e \ln b} = c$.

$$\begin{aligned}
u(y) &= u(x \log_b x) \\
&= (1 + c)\frac{x \log_b x}{\log_b x + \log_b \log_b x} \\
&= (1 + c)\frac{x}{1 + g(\log_b x)} \\
&\ge x && (x \ge 0 \text{ and } g(\log_b x) \le c) \\
&= f(y)
\end{aligned}$$

□

Therefore,

$$f(x) \in \Theta\left(\frac{x}{\log_b x}\right)$$