Bernstein-concentration

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Definition 1 (γ-bernstein-concentration)**.** *A random variable $X \in \mathbb{R}^n$ is γ-bernstein-concentrated iff*

$$\Pr[a^T X - \mathrm{E}(a^T X) \geq t] \leq \exp\left(\frac{-1}{2\gamma} \frac{t^2}{\sum_{i=1}^n a_i^2 \operatorname{Var}(X_i) + \frac{Mt}{3}}\right)$$

where $M = \max_{i=1}^n |a_i(X_i - \mathrm{E}(X_i))|$.

Theorem 1 (Bernstein's inequality)**.** *Let $X \in \mathbb{R}^n$ be a random variable where each coordinate of X is independent. Then X is 1-bernstein-concentrated.*

1 Bernouilli variables

Theorem 2. *Let $X \in \{0,1\}^n$ be a random variable with $\mathrm{E}(X) = x$. If X is γ-bernstein-concentrated, then*

$$\Pr[a^T X - a^T x \geq t] \leq \exp\left(\frac{-1}{2\gamma} \frac{t^2}{\|a\|_\infty (a^T x + t/3)}\right)$$

Proof. $\operatorname{Var}(X_i) = x_i(1 - x_i)$.

$$\sum_{i=1}^n a_i^2 x_i(1 - x_i) \leq \|a\|_\infty \sum_{i=1}^n a_i x_i = \|a\|_\infty a^T x$$

$$M = \max_{i=1}^n |a_i| \, |X_i - \mathrm{E}(X_i)| \leq \|a\|_\infty \max_{i=1}^n |X_i - \mathrm{E}(X_i)| \leq \|a\|_\infty$$

□

Theorem 3. *Let $X \in \{0,1\}^n$ be a random variable with $\mathrm{E}(X) = x$. Let X be γ-bernstein-concentrated. Let $\alpha \leq a^T x \leq \beta$. Then*

$$\Pr[a^T X \geq (1 + \delta)\beta] \leq \exp\left(\frac{-1}{2\gamma} \frac{\beta}{\|a\|_\infty} \frac{\delta^2}{1 + \delta/3}\right)$$

$$\Pr[a^T X \leq (1 - \delta)\alpha] \leq \exp\left(\frac{-1}{2\gamma} \frac{\alpha}{\|a\|_\infty} \frac{\delta^2}{1 + \delta/3}\right)$$

Proof. Let $c = (1+\delta)\beta$.

$$\begin{aligned}
\Pr[a^T X \geq c] &= \Pr[a^T X - a^T x \geq c - a^T x] \\
&\leq \exp\left(-\frac{1}{2\gamma}\,\frac{(c-a^T x)^2}{\|a\|_\infty\left(a^T x + \frac{1}{3}(c - a^T x)\right)}\right) && \text{(by theorem 2)} \\
&\leq \exp\left(-\frac{1}{2\gamma}\frac{3}{\|a\|_\infty}\frac{(c-a^T x)^2}{c + 2a^T x}\right)
\end{aligned}$$

Let $f(y) = (c-y)^2/(c+2y)$, where $y \in [\alpha, \beta]$. $f(y) > 0$.

$$\ln(f(y)) = 2\ln(c-y) - \ln(c+2y)$$

$$\frac{f'(y)}{f(y)} = -\frac{2}{c-y} - \frac{2}{c+2y} < 0$$

Since $f'(y) < 0$, $f(y)$ is minimized at $y = \beta$. $f(\beta) = (\delta^2\beta)/(3+\delta)$. Therefore,

$$\Pr[a^T X \geq (1+\delta)\beta] \leq \exp\left(\frac{-1}{2\gamma}\frac{\beta}{\|a\|_\infty}\frac{\delta^2}{1+\delta/3}\right)$$

The other inequality can be proven similarly. $\square$