Chapter 1: Preliminaries

1 Sets

Definition 1 (Set basics)**.**

1. $A \subseteq B \;:\Longleftrightarrow\; (\forall x \in A, x \in B)$.
2. $A = B \;:\Longleftrightarrow\; (A \subseteq B \wedge B \subseteq A)$.
3. $A \subset B \;:\Longleftrightarrow\; (A \subseteq B \wedge B \not\subseteq A)$.
4. $A \cup B := \{x : x \in A \text{ or } x \in B\}$.
5. $A \cap B := \{x \in A : x \in B\}$.
6. $A \setminus B := \{x \in A : x \notin B\}$.
7. $\displaystyle\bigcup_{i \in I} A_i := \{x : \exists i \in I \text{ such that } x \in A_i\}$.
8. $\displaystyle\bigcap_{i \in I} A_i := \{x : \forall i \in I, x \in A_i\}$.
9. $A \times B := \{(x, y) : x \in A, y \in B\}$.
10. $\displaystyle\prod_{i=1}^{n} A_i := \{(x_1, x_2, \ldots, x_n) : x_i \in A_i \text{ for all } i\}$.

Theorem 1.

1. $A \subseteq B \iff A \cap B = A \iff A \cup B = B$.
2. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
3. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
4. $A \cup (B \cup C) = (A \cap B) \cup (A \cap C)$.
5. $A \cap (B \cap C) = (A \cup B) \cap (A \cup C)$.

2 Relations and Functions

Definition 2 (Relation and function). *A relation R between A and B is a subset of $A \times B$. A function $f : A \to B$ is a relation between A and B such that*

$$(a, b_1) \in f \text{ and } (a, b_2) \in f \implies b_1 = b_2.$$

$D(f) := A$ (called domain *of f), and $R(f) := B$ (called* range *of f).*

Lemma 2. *Let $f : A \to B$ and $g : A \to B$. Then $f = g \iff (\forall x \in A, f(x) = g(x))$.*

Definition 3 (Image and reverse image). *Let $f : A \to B$ be a function.*

1. *For $X \subseteq A$, $f(X) := \{f(x) : x \in X\}$ is called the* image *of X under f. Equivalently, $y \in f(X) \iff (\exists x \in X, f(x) = y)$.*
2. *For $Y \subseteq B$, $f^{-1}(Y) = \{x : f(x) \in Y\}$ is called the* inverse image *of Y under f. Equivalently, $x \in f^{-1}(Y) \iff f(x) \in Y$.*

Lemma 3. *Let $f : A \to B$. Let $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$.*

1. *$f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$.*
2. *$f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$.*
3. *$f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$.*
4. *$f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.*

Definition 4 (Composition). *For functions $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$ is the defined as $(g \circ f)(x) = g(f(x))$.*

Definition 5 (Injection and surjection). *Let $f : A \to B$.*

1. *f is injective (aka one-to-one) $:\iff \forall x_1 \in A, \forall x_2 \in A, (f(x_1) = f(x_2) \implies x_1 = x_2)$.*
2. *f is surjective (aka onto) $:\iff \forall y \in B, \exists x \in A, f(x) = y$.*

Lemma 4 (Composition). *Let $f : A \to B$ and $g : B \to C$.*

1. *If f and g are injective, then $g \circ f$ is injective.*
2. *If $g \circ f$ is injective, then f is injective.*
3. *If f and g are surjective, then $g \circ f$ is surjective.*
4. *If $g \circ f$ is surjective, then g is surjective.*

Definition 6 (Identity). *The identity function $\mathrm{id}_A : A \to A$ is given by $\mathrm{id}_A(x) = x$ for all $x \in A$.*

Definition 7 (Bijection). *A function $f : A \to B$ is a bijection iff (the following are equivalent):*

1. *f is injective and surjective.*
2. *$\exists g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. (Then g is called the* inverse *of f, and is denoted by f^{-1}.)*

Proof sketch of equivalence. If f is injective and surjective, for each $y \in B$, there is a unique $x \in A$ such that $f(x) = y$. Define $g(y) = x$ and show condition 2. To show that condition 2 implies condition 1, use Lemma 4. □

Definition 8 (Restriction)**.** *Let $f : A \to B$ be a function. Let $X \subseteq A$. Then $f|X$ is a function from X to B such that $(f|X)(x) = f(x)$ for all $x \in X$.*