Linear Programming Counterexamples

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1 Shapes

Example 1 (Square Pyramid)**.** *Let*

$$\begin{aligned} P &:= \{(x, y, z) : \max(|x|, |y|) \leq z, z \leq 1\} \\ &= \{(x, y, z) : x \leq z, -x \leq z, y \leq z, -y \leq z, z \leq 1\}. \end{aligned}$$

Then P is an (inverted) square pyramid. The base of the pyramid is $[-1, 1]^2 \times \{1\}$. Its vertices are $(0, 0, 0)$, $(-1, -1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, 1)$.

Example 2 (Square Pyramid)**.** *Let*

$$P := \{(x, y, z) : x \leq z, y \leq z, x \geq 0, y \geq 0, z \geq 0\}.$$

Then P is an (inverted) square pyramid. The base of the pyramid is $[0, 1]^2 \times \{1\}$. Its vertices are $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 1)$.

2 Degeneracy vs Redundancy

Example 3 (Redundancy $\not\Rightarrow$ Degeneracy)**.** *Let $P := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Then every extreme-point is non-degenerate. Adding the constraint $y \leq 2$ doesn't add degeneracy but adds redundancy.*

Example 4 (Degeneracy $\not\Rightarrow$ Redundancy)**.** *Let $P := \{(x, y) : y \leq x, y \leq -x, y \geq 0\}$ (so $P = \{(0, 0)\}$). Then $(0, 0)$ is a degenerate extreme point, but no constraint is redundant.*

Example 5 (Degeneracy $\not\Rightarrow$ Redundancy)**.** *Let P be a square pyramid (c.f. Example 1). Then there is a degenerate extreme point but no constraint is redundant.*

3 Simplex Method

Example 6. *Let $b \in \mathbb{R}_{\geq 0}$ and $a \in \mathbb{R}^n$, where $0 < a_1 < a_2 < \ldots < a_n$. Consider the LP*

$$\max_{x \geq 0} \quad a^T x \quad \textit{where} \quad \sum_{i=1}^{n} x_i \leq b.$$

Clearly, the optimal solution is $[0, 0, \ldots, 0, b]$. In standard form, the LP becomes

$$\min_{x \geq 0} \quad \sum_{i=1}^{n} (-a_i) x_i \quad \textit{where} \quad \sum_{i=1}^{n+1} x_i = b.$$

Suppose our initial basis is $\{x_{n+1}\}$. *For ease of notation, let* $x_0 := x_{n+1}$.

If we run the simplex method with Bland's rule (variable of lowest index enters basis), then there will be n *iterations, where in the* i^{th} *iteration,* x_i *enters the basis and* x_{i-1} *leaves the basis. Hence, we visit each of the* $n+1$ *bases. If* $b > 0$, *each basis corresponds to a unique BFS. If* $b = 0$, *all bases correspond to the same BFS* $\mathbf{0}$.

If we run the simplex method with Dantzig's rule (variable of most negative reduced cost enters basis), then there will be just 1 iteration where x_n *enters and* x_{n+1} *leaves.*

Example 7. *For the following LP, where* $b \in \mathbb{R}_{\geq 0}$, *the optimal solution is* $(b, 0)$.

$$\max_{x \geq 0, y \geq 0} \quad 2x + 3y \quad \textit{where} \quad x + 2y \leq b.$$

Bland's rule takes 1 iteration but Dantzig's rule takes 2 iterations. There are 3 bases. When $b > 0$, *each base corresponds to a different BFS. When* $b = 0$, *all bases correspond to the same BFS* $\mathbf{0}$.