

# Stochastic Processes

Eklavya Sharma

**Definition 1** (Stochastic Process). Let  $\mathcal{T} \subseteq \mathbb{R}$ . For any  $t \in \mathcal{T}$ , let  $X_t$  (or  $X(t)$ ) be a random variable with support  $D$ . Then  $X := \{X_t : t \in \mathcal{T}\}$  is called a stochastic process on state-space  $D$  and time  $\mathcal{T}$ . Usually,  $\mathcal{T}$  is either  $\mathbb{Z}_{\geq 0}$  (discrete-time) or  $\mathbb{R}_{\geq 0}$  (continuous-time).

## 1 Discrete-Time Markov Chains

**Definition 2** (Markov Chain). Let  $X := [X_0, X_1, \dots]$  be a stochastic process on state-space  $D$  and time  $\mathbb{Z}_{\geq 0}$ .  $X$  is called a discrete-time markov chain if  $\Pr(X_{t+1} = d \mid X_t, X_{t-1}, \dots, X_0) = \Pr(X_{t+1} = d \mid X_t)$ . If  $\Pr(X_{t+1} = v \mid X_t = u) = \Pr(X_1 = v \mid X_0 = u)$  for all  $t, u, v$ , then  $X$  is called time-homogeneous.

**Definition 3** (Transition function). Let  $X$  be a markov chain on state space  $D$ . Define  $P^{(k)} : D \times D \mapsto [0, 1]$  as  $P^{(k)}(i, j) = \Pr(X_k = j \mid X_0 = i)$ . Then  $P^{(k)}$  is called the  $k$ -step transition function of  $X$ . For  $k = 1$ , we simply write  $P$  instead of  $P^{(1)}$ . For a finite state space, we can represent  $P$  as a matrix.

**Lemma 1** (Chapman-Kolmogorov Equation).  $P^{(m+n)}(i, j) = \sum_k P^{(m)}(i, k) P^{(n)}(k, j)$ .

### 1.1 Classification of States, Recurrence, Limiting Probabilities

**Definition 4.** Let  $f_{i,j} := \Pr\left(\bigvee_{t \geq 1} (X_t = j) \mid X_0 = i\right)$ . Then  $f_{i,j}$  is called the eventual transition probability from  $i$  to  $j$ . If  $i = j$ , then we write  $f_{i,i}$  as  $f_i$ , and call it the recurrence probability of state  $i$ .

**Definition 5.** For a state  $i$ , let  $N_i$  be the random variable that counts the number of times we are in state  $i$ , i.e.,  $N_i := \sum_{t=0}^{\infty} \mathbf{1}(X_t = i)$ . Then  $N_i$  is called the visit-count of  $i$ .

**Definition 6.** A state  $i$  of a markov chain is recurrent iff (the following are equivalent):

- the recurrence probability ( $f_i$ ) of  $i$  is 1.
- $i$  is visited infinitely often, i.e.,  $\Pr(N_i = \infty \mid X_0 = i) = 1$ .
- $i$  is visited infinitely often in expectation, i.e.,  $E(N_i \mid X_0 = i) = \infty$ .

A non-recurrent state is called a transient state.

**Lemma 2.**  $\Pr(N_i = k \mid X_0 = i) = f_i^{k-1}(1 - f_i)$ .

**Lemma 3.**  $E(N_i \mid X_0 = i) = 1/(1 - f_i) = \sum_{t=0}^{\infty} P^{(t)}(i, i)$ .

**Definition 7.** State  $j$  is accessible from state  $i$  if  $P^{(t)}(i, j) > 0$  for some  $t$ . States  $i$  and  $j$  communicate (denoted as  $i \leftrightarrow j$ ) if  $i$  and  $j$  are both accessible from each other.

**Lemma 4.** Accessibility is reflexive and transitive. Communication is an equivalence relation. The equivalence classes of communicability are called state classes. A markov chain is irreducible if it has just one state class.

**Definition 8.** Let  $T_i$  be the time when a markov chain moves to state  $i$ , i.e.,  $T_i := \min_{t \geq 1} (X_t = i)$ . When conditioned on  $X_0 = i$ ,  $T_i$  is called the recurrence time of  $i$ . State  $i$  is called positive recurrent if  $E(T_i | X_0 = i)$  is finite, otherwise it is null recurrent.

**Lemma 5.** Recurrence and positive recurrence are class properties, i.e., they are same for all states in a class.

**Lemma 6.** In a finite-state markov chain, all recurrent states are positive recurrent, and there is at least one recurrent state.

**Definition 9** (Periodicity). For a state  $i$ , its period is defined as  $\gcd(\{t : \Pr(T_i = t | X_0 = i) > 0\})$ . A state is aperiodic if its period is 1.

**Lemma 7.** Periodicity is a class property.

**Definition 10** (Ergodicity). A state is ergodic if it is positive recurrent and aperiodic. A markov chain is ergodic if all its states are ergodic.

**Lemma 8.** In an irreducible ergodic markov chain, for every state  $j$ ,  $\lim_{t \rightarrow \infty} P^{(t)}(j, i) = \pi_i$  for a unique real number  $\pi_i$ .  $\pi_i$  is called the limiting probability of state  $i$ . Furthermore,  $\pi_i$  is the unique solution to this system of equations:  $\pi_i = \sum_j \pi_j P(j, i)$  for all  $i$  ( $\pi = P^T \pi$  in matrix form) and  $\sum_i \pi_i = 1$ .

**Lemma 9.** In an irreducible ergodic markov chain,  $E(T_i | X_0 = i) = 1/\pi_i$ .

**Corollary 9.1.** A state  $i$  is null recurrent iff  $\pi_i = 0$ .

**Theorem 10.** If the transition function of markov chain  $X$  is doubly-stochastic (i.e., each row and each column sums to 1), then the limiting probability of each state is  $1/n$ , where  $n$  is the number of states.

## 1.2 Time-Reversibility

**Definition 11.** For an irreducible ergodic markov chain  $X$  with limiting probabilities  $\pi$ . Let  $Y$  be a markov chain whose transition function is  $Q(i, j) = P(j, i)(\pi_j/\pi_i)$ . Then  $Y$  is called the time-reversed markov chain of  $X$ .  $X$  is called time-reversible if  $Q = P$ .

**Theorem 11.** Let  $X$  be a time-reversible markov chain with limiting probabilities  $\pi$ . Then  $\pi$  is the unique solution to this system of equations:  $x_j P(j, i) = x_i P(i, j)$  for all states  $i$  and  $j$ , and  $\sum_i x_i = 1$ .

**Theorem 12.** If the transition function of markov chain  $X$  is symmetric, then  $X$  is time-reversible.

## 2 Counting Process

**Definition 12** (Counting Process). *Let  $N$  be a stochastic process on state space  $\mathbb{Z}_{\geq 0}$  and time  $\mathbb{R}_{\geq 0}$ . Then  $N$  is called a counting process if  $N(0) = 0$  and  $N(t)$  is monotone in  $t$ , i.e.,  $t_1 < t_2 \implies N(t_1) \leq N(t_2)$ .*

**Definition 13** (Independent increments). *A counting process  $N$  has independent increments iff for any two disjoint intervals  $(u_1, v_1]$  and  $(u_2, v_2]$  in  $\mathbb{R}_{\geq 0}$ , the random variables  $N(v_1) - N(u_1)$  and  $N(v_2) - N(u_2)$  are independent.*

**Definition 14** (Stationary increments). *A counting process  $N$  has stationary increments iff for any  $u \leq v$ , the random variables  $N(v) - N(u)$  and  $N(v - u)$  have the same distribution.*

**Definition 15** (Arrival and interarrival times). *For a counting process  $N$ , for  $i \in \mathbb{Z}_{\geq 0}$ , define the  $i^{\text{th}}$  arrival time  $S_i := \min_{t \geq 0} (N(t) = i)$ . For  $i \in \mathbb{Z}_{\geq 1}$ , define the  $i^{\text{th}}$  interarrival time  $T_i := S_i - S_{i-1}$ .*

**Lemma 13.** *For a counting process  $N$  with arrival times  $S$ ,  $N(t) \geq n \iff S_n \leq t$ .*

**Definition 16** (Stopping time). *Let  $X = [X_1, X_2, \dots]$  be a sequence of random variables. The random variable  $N$  is called a stopping time for  $X$  if for all  $n \geq 0$ , (the following two definitions are equivalent):*

- $N = n$  is independent of  $X_{n+1}, X_{n+2}, \dots$
- $N \leq n$  is independent of  $X_{n+1}, X_{n+2}, \dots$

**Theorem 14** (Wald's identity). *Let  $X = [X_1, X_2, \dots]$  be a sequence of random variables where  $E(X_i) = \mu$  for all  $i$ . Let  $N$  be a stopping time for  $X$ . Then*

$$E\left(\sum_{i=1}^N X_i\right) = \mu E(N).$$

*Proof sketch.* For all  $i$ ,  $N \geq i$  is independent of  $X_i$ , and  $\sum_{i=1}^N X_i = \sum_{i=1}^{\infty} X_i \mathbf{1}(N \geq i)$ . □

## 3 Poisson Process

**Definition 17** (Poisson process). *A counting process  $N$  is a Poisson process with rate function  $\lambda : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  if  $N$  has independent increments and  $N(t_2) - N(t_1) \sim \text{Poisson}(\mu)$ , where  $\mu := \int_{t_1}^{t_2} \lambda(t) dt$ .  $N$  is called homogeneous if  $\lambda(t) = \lambda(0)$  for all  $t$ , otherwise it is called inhomogeneous. For a homogeneous process, we denote  $\lambda(0)$  by  $\lambda$ .*

**Lemma 15.** *A Poisson process  $N$  is homogeneous iff it has stationary increments.*

**Theorem 16** (Alternative definition of Poisson process). *A counting process  $N$  is a Poisson process with continuous rate function  $\lambda$  iff  $N$  has independent and stationary increments and  $\Pr(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$  and  $\Pr(N(t+h) - N(t) \geq 2) = o(h)$ .*

*Proof sketch for homogeneous.* Let  $g(u, t) := \text{MGF}_u(N(t)) = E(e^{uN(t)})$ . Show  $g(u, t) = 1 + \lambda t(e^u - 1) + o(t)$  straightforwardly. Use calculus to show that  $g(u, t) = \exp(e^{\lambda t}(e^u - 1))$  (find derivative w.r.t  $t$  by computing  $\lim_{h \rightarrow 0} (g(u, t+h) - g(u, t))/h$ ; this gets rid of  $o(h)$ ). Conclude that  $N(t) \sim \text{Poisson}(\lambda t)$  since  $g(u, t)$  is MGF of  $\text{Poisson}(\lambda t)$ .  $\square$

**Lemma 17.** *For a homogeneous Poisson process  $N$ ,*

$$\Pr(N(s) = a \mid N(s+t) = a+b) = \binom{a+b}{a} \left(\frac{s}{s+t}\right)^a \left(\frac{t}{s+t}\right)^b.$$

**Theorem 18.** *Let  $N$  be a counting process. Then  $N$  is a homogeneous Poisson process with rate  $\lambda$  iff all interarrival times are independent and distributed  $\text{Expo}(\lambda)$ .*

**Theorem 19** (Decomposition theorem 1). *Let  $K$  be a finite set, and let  $\{N_i : i \in K\}$  be independent Poisson processes, where  $N_i$  has rate function  $\lambda_i$ . Let  $N := \sum_{i \in K} N_i$ . Then  $N$  is a Poisson process with rate function  $\sum_{i \in K} \lambda_i$ .*

**Theorem 20** (Decomposition theorem 2). *Let  $N$  be a Poisson process with rate function  $\lambda$ . Let  $K$  be a finite set (called set of labels). Suppose the  $j^{\text{th}}$  event receives label  $L_j \in K$ , where  $\Pr(L_j = i) = p_i(S_j)$  for some function  $p_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ , and  $\{N, L_1, L_2, \dots\}$  are independent. For  $i \in K$ , let  $N_i(t)$  be the number of events having label  $i$ , i.e.,  $N_i(t) = \sum_{j=1}^{N(t)} \mathbf{1}(L_j = i)$ . Then  $N_i$  is a Poisson process with rate function  $p_i \lambda$ . Furthermore, all  $N_i$  are independent and if all  $p_i$  are constant, then  $N_i(t) \mid N(t) \sim \text{Binom}(N(t), p_i)$ .*

**Lemma 21.** *Let  $N^{(1)}$  and  $N^{(2)}$  be independent homogeneous Poisson processes with rates  $\lambda_1$  and  $\lambda_2$ . Then*

$$\Pr(S_n^{(1)} < S_m^{(2)}) = \sum_{i=n}^{n+m-1} \binom{n+m-1}{i} \frac{\lambda_1^i \lambda_2^{n+m-1-i}}{(\lambda_1 + \lambda_2)^{n+m-1}}.$$

*Proof sketch.* Model as a continuous markov chain with state space  $(n_1, n_2)$ , where  $n_i$  is the number of events of  $N^{(i)}$  that have occurred.  $\square$

**Theorem 22** (arrival times distributed as order statistics). *Let  $X = [X_1, X_2, \dots, X_n]$  be IID uniform variables over  $[0, t]$ . Let  $U = \text{sorted}(X)$ . Let  $S = [S_1, S_2, \dots, S_n]$ . Then conditioned on  $N(t) = n$ , the distribution of  $S$  and  $U$  are identical.*

## 4 Continuous-Time Markov Chain

**Definition 18** (CTMC). *Let  $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$  be a stochastic process on discrete state-space  $D$ .  $X$  is called a continuous-time markov chain (CTMC) if  $\Pr(X(t+s) = d \mid \{X(u) : 0 \leq u \leq s\}) = \Pr(X(t+s) = d \mid X(s))$  for all  $s, t \in \mathbb{R}_{\geq 0}$ . If  $\Pr(X(t+s) = v \mid X(s) = u) = \Pr(X(t) = v \mid X(0) = u)$  for all  $u, v, s, t$ , then  $X$  is called time-homogeneous (TH) or stationary.*

**Theorem 23** (Equiv defn of TH CTMC). *Let  $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$  be a stochastic process on discrete state-space  $D$ . Let  $Y(t) := \{X(u) : 0 \leq u < t\}$ . Let  $T_i^{(s)} := \min_{t \geq 0} (X(t+s) \neq i)$ . Let  $P_{i,j}^{(s)} := \Pr(X(s+T_i^{(s)}) = j \mid X(s) = i, Y(s))$ .  $X$  is TH CTMC iff  $(T_i^{(s)} \mid X(s) = i, Y(s)) \sim \text{Expo}(\nu_i)$ , where  $\nu_i$  is a constant that doesn't depend on  $s$  or  $Y(s)$ , and  $P_{i,j}^{(s)}$  is a constant that doesn't depend on  $s$  or  $Y(s)$ .*

Since  $T_i^{(s)}$  and  $P_{i,j}^{(s)}$  don't depend on  $s$ , we simply write  $T_i$  and  $P_{i,j}$ .  $T_i$  is called the transition time out of state  $i$ ,  $\nu_i$  is called the transition rate out of state  $i$ , and  $P_{i,j}$  is the probability of transitioning from state  $i$  to state  $j$ .

Let  $q_{i,j} := \nu_i P_{i,j}$ . Then  $\nu_i = \sum_j q_{i,j}$ .

**Theorem 24** (Chapman-Kolmogorov DiffEqs). *For a TH CTMC  $X$ , let  $P_{i,j}(t) := \Pr(X(t) = j \mid X(0) = i)$ . Then*

- *Backward DiffEqs:  $\frac{dP_{i,j}(t)}{dt} = \sum_{k \neq i} q_{i,k} P_{k,j}(t) - \nu_i P_{i,j}(t)$ .*
- *Forward DiffEqs:  $\frac{dP_{i,j}(t)}{dt} = \sum_{k \neq j} P_{i,k}(t) q_{k,j} - P_{i,j}(t) \nu_j$ .*

**Lemma 25.** *Let  $X$  be a TH CTMC.*

$$\lim_{h \rightarrow 0} \frac{1 - P_{i,i}(h)}{h} = \nu_i \quad \forall i \quad \quad \quad \lim_{h \rightarrow 0} \frac{P_{i,j}(h)}{h} = q_{i,j} \quad \forall i \neq j$$

**Lemma 26** (Limiting probability). *In an irreducible positive-recurrent TH CTMC  $X$ , for every state  $j$ ,  $\lim_{t \rightarrow \infty} P_{j,i}(t) = P_i$  for a unique real number  $P_i$ .  $P_i$  is called the limiting probability of state  $i$ . Furthermore,  $P_i$  is the unique solution to  $\sum_i P_i = 1$  and CK forward equations, i.e.,  $P_i \nu_i = \sum_{j \neq i} P_j q_{j,i}$ .*

**Lemma 27** (Limiting probability of embedded chain). *Let  $X$  be an irreducible positive-recurrent TH CTMC. Let  $Y$  be the sequence of states visited by  $X$ . Then  $Y$  is a discrete MC. Let  $P$  and  $\pi$  be the limiting probabilities of  $X$  and  $Y$ , respectively. Then  $P_i = (\pi_i / \nu_i) / (\sum_j \pi_j / \nu_j)$  and  $\pi_i = P_i \nu_i / (\sum_j P_j \nu_j)$ .*

**Definition 19.** *A CTMC is time-reversible iff the corresponding embedded discrete-time MC is time-reversible.*

## 4.1 Birth and Death Process

**Definition 20.** *A birth-and-death (B&D) process is a TH CTMC  $X$  on state space  $\mathbb{Z}_{\geq 0}$  where  $q_{i,j} = 0$  if  $j \notin \{i-1, i+1\}$ . Let  $\lambda_i := q_{i,i+1}$  for  $i \geq 0$ ,  $\mu_i := q_{i,i-1}$  for  $i \geq 1$ ,  $\mu_0 := 0$ .*

*$X(t)$  is called the population at time  $t$ ,  $\lambda_i$  is called the birth rate at population  $i$ , and  $\mu_i$  is called the death rate at population  $i$ .*

**Lemma 28.** *Let  $X$  be a B&D process where  $X(0) = n$ . Let  $T_n$  be the time to reach state  $n+1$ , i.e.,  $T_n := \min_{t \geq 0} (X(t) = n+1)$ . Then*

$$\begin{aligned} \mathbb{E}(T_n) &= \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} \mathbb{E}(T_{n-1}) = \frac{1}{\lambda_n} \sum_{i=0}^n \prod_{j=1}^i \frac{\mu_{n-j+1}}{\lambda_{n-j}}. \\ \text{Var}(T_n) &= \frac{1}{\lambda_n(\lambda_n + \mu_n)^2} + \frac{\mu_n}{\lambda_n} \text{Var}(T_{n-1}) + \frac{\mu_n}{\lambda_n + \mu_n} (\mathbb{E}(T_{n-1}) + \mathbb{E}(T_n))^2 \end{aligned}$$

*Proof sketch.* Let  $I_i = \mathbf{1}(\text{next transition goes to state } i+1)$ . Let  $X_i$  be the transition time out of state  $i$ . Then  $I_i \sim \text{Bernoulli}(\lambda_i / (\mu_i + \lambda_i))$ ,  $X_i \sim \text{Expo}(\lambda_i + \mu_i)$ , and

$$\begin{aligned} \mathbb{E}(T_i \mid I_i) &= \mathbb{E}(X_i) + (1 - I_i)(\mathbb{E}(T_{i-1}) + \mathbb{E}(T_i)), \\ \text{Var}(T_i \mid I_i) &= \text{Var}(X_i) + (1 - I_i)(\text{Var}(T_{i-1}) + \text{Var}(T_i)). \end{aligned} \quad \square$$

CKBE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_i P_{i-1,j}(t) + \lambda_i P_{i+1,j}(t) - (\lambda_i + \mu_i) P_{i,j}(t).$$

CKFE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_{j+1} P_{i,j+1}(t) + \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{i,j}(t).$$

**Theorem 29** (Limiting Probabilities). *Let  $X$  be an irreducible B&D process on state space  $D \subseteq \mathbb{Z}_{\geq 0}$  where  $0 \in D$ . For  $n \in D$ , let  $\alpha_n := \prod_{i=1}^n \frac{\lambda_{i-1}}{\mu_i}$ . If  $\sum_{i \in D} \alpha_i$  is finite, then  $P_i = \alpha_i P_0$ , and  $P_0 = 1 / \sum_{i \in D} \alpha_i$ .*

*Proof sketch.* Use Lemma 26 and add adjacent equations.  $\square$

## 5 Renewal Theory

**Definition 21.** *Let  $[X_1, X_2, \dots]$  be a sequence of IID non-negative randvars, called interarrival times, such that  $\Pr(X_1 = 0) < 1$  and  $\Pr(X_1 = \infty) = 0$ . Let  $S_n := \sum_{i=1}^n X_i$  (called arrival times). Let  $N(t) := \max_n (S_n \leq t)$ . Then  $N$  is called a renewal process (note that it is a counting process).*

*We let  $F$  and  $f$  denote the CDF and PDF/PMF of  $X_1$ , respectively. We let  $F^{(n)}$  and  $f^{(n)}$  denote the CDF and PDF/PMF of  $S_n$ , respectively.*

*Let  $R_i$  be the reward obtained at time  $X_i$  for all  $i \geq 1$ , where all  $R_i$  are independent. Let  $R(t) := \sum_{i=1}^{N(t)} R_i$ . Then  $R$  is called a renewal reward process.*

**Lemma 30.** *For all  $t \geq 0$ ,  $\Pr(N(t) = \infty) = 0$ .  $\Pr(\lim_{t \rightarrow \infty} N(t) = \infty) = 1$ .*

*Proof.* Let  $\mu := E(X_1)$ .  $\mu > 0$  since  $\Pr(X_n = 0) < 1$ .

$$\Pr\left(\lim_{t \rightarrow \infty} \frac{S_n}{n} = \mu\right) = 1. \quad (\text{strong law of large numbers})$$

$$N(t) = \infty \iff (\forall n, S_n \leq t) \implies \lim_{t \rightarrow \infty} \frac{S_n}{n} = 0.$$

$\Pr(N(\infty) = \infty) = 1$  since  $\Pr(X_1 = \infty) = 0$ .  $\square$

**Definition 22.** *For a renewal process  $N$ , let  $m_N(t) := E(N(t))$ . Then  $m_N$  is called the mean-value function of  $N$ . (If  $N$  is clear from context, we will write  $m$  instead of  $m_N$ .)*

**Lemma 31.**  $m(t) = \sum_{n=1}^{\infty} \Pr(S_n \leq t) = \sum_{n=1}^{\infty} F^{(n)}(t)$ .

**Theorem 32.**  $m$  uniquely characterizes  $F$ .

**Lemma 33.**  $m(t)$  is finite for all  $t$ .

**Theorem 34** (Renewal equation). *When interarrival times are continuous randvars,*

$$m(t) = F(t) + \int_0^t m(t-x)f(x)dx.$$

*Proof sketch.* Let  $N'(t) := \max_n (\sum_{i=2}^{n+1} X_i \leq t)$ . Then  $N$  and  $N'$  are identically distributed and

$$N(t) = \begin{cases} 1 + N'(t - X_1) & \text{if } X_1 \leq t \\ 0 & \text{if } X_1 > t \end{cases}.$$

Finally,  $m(t) = E(E(N(t) | X_1))$ . □

**Corollary 34.1.** *Let  $N$  be a renewal process where interarrival times are distributed  $\text{Uniform}(0, 1)$ . Then for  $0 \leq t \leq 1$ ,  $m(t) = e^t - 1$ .*

**Theorem 35** (Limit theorems). *For a renewal process  $N$  with  $\mu := E(X_1)$ ,*

$$\Pr \left( \lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mu} \right) = 1. \qquad \lim_{t \rightarrow \infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

**Theorem 36** (Limit theorems for rewards). *For a renewal process  $N$  with rewards  $\{R_i : i \in \mathbb{Z}_{\geq 1}\}$ , let  $\alpha := E(R_1)$  and  $\mu := E(X_1)$ . Then*

$$\Pr \left( \lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{\alpha}{\mu} \right) = 1. \qquad \lim_{t \rightarrow \infty} \frac{E(R(t))}{t} = \frac{\alpha}{\mu}.$$

**Theorem 37** (Central limit theorem for renewals). *For a renewal process  $N$  with  $\mu := E(X_1)$  and  $\sigma^2 := \text{Var}(X_1)$ , the random variable*

$$\lim_{t \rightarrow \infty} \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}}$$

*tends to the standard normal distribution.*

**Lemma 38** (Stopping time). *Let  $X = [X_1, X_2, \dots]$  be the sequence of interarrival times for renewal process  $N$ . Then  $N(t) + 1$  is a stopping time for  $X$ .*

*Proof sketch.*  $N(t) + 1 \leq n \iff S_n > t$ . □

**Definition 23.** *For a renewal process  $N$  with arrival times  $S_1, S_2, \dots$ :*

- *Let  $Y(t) := S_{N(t)+1} - t$ .  $Y(t)$  is called the excess at time  $t$ .*
- *Let  $L(t) := t - S_{N(t)}$ .  $L(t)$  is called the remaining life at time  $t$ .*

**Lemma 39.** *Let  $N$  be a renewal process with interarrival times  $X = [X_1, X_2, \dots]$ . Then  $E(S_{N(t)+1}) = t + E(Y(t)) = E(X_1)(m(t) + 1)$ .*

*Proof.*  $N(t) + 1$  is a stopping time for  $X$ . □