Recurrence Relations

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Contents

1 Divide in half and Conquer

We will look at recurrence relations of the form:

$$f(n) = 2f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + g(n)$$

$$f(n) = 2f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$$

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$$

Here f and g are functions from $\mathbb{N}$ to $\mathbb{N}$ ($\mathbb{N} = \{0, 1, \ldots\}$). We define the recurrence relations only for $n \geq 2$. Therefore, $f(0)$ and $f(1)$ are boundary values. (The recurrence relations can be made to hold true for $n = 0$ and $n = 1$ as well, for example, by setting $f(0) = f(1) = g(0) = g(1) = 0$).

We will assume that g is non-negative and monotonic and $0 \leq f(0) \leq f(1)$.

We will find an exact closed-form solution for f and simple lower and upper bounds on f.

1.1 Mathematical background

Lemma 1.

$$\forall a, b \in \mathbb{N}, \left\lceil \frac{a}{b} \right\rceil = \left\lfloor \frac{a-1}{b} \right\rfloor + 1$$

Definition 1. $\lg x = \log_2(x)$

Lemma 2. $n \in \mathbb{N}$

$$\forall n \geq 1, \lfloor \lg n \rfloor = \lceil \lg(n+1) \rceil - 1$$

Lemma 3. $n, k \in \mathbb{N}$

$$\frac{n}{2^k} \in [1, 2) \iff k = \lfloor \lg n \rfloor$$

$$\frac{n}{2^k} \in \left(\frac{1}{2}, 1\right] \iff k = \lceil \lg n \rceil$$

Lemma 4. $n \in \mathbb{N}$

$$2^{\lfloor \lg n \rfloor} \in \left[\frac{n+1}{2}, n\right] \qquad 2^{\lceil \lg n \rceil} \in [n, 2(n-1)]$$

1.2 Type 1: $f(n) = 2f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + g(n)$

Theorem 5 (Monotonicity)**.** $i \leq j \implies f(i) \leq f(j)$

Proof. Use induction and monotonicity of g:

$$f(n) = 2f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + g(n) \geq 2f\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + g(n-1) = f(n-1)$$

$\square$

Theorem 6. $\forall k \geq 0$:

$$f(n) = 2^k f\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) + \sum_{i=0}^{k-1} 2^i g\left(\left\lfloor \frac{n}{2^i} \right\rfloor\right)$$

Set $k = \lfloor \lg n \rfloor$ in the above theorem to get

$$f(n) = 2^{\lfloor \lg n \rfloor} f(1) + \sum_{i=0}^{\lfloor \lg n \rfloor - 1} 2^i g\left(\left\lfloor \frac{n}{2^i} \right\rfloor\right)$$

1.3 Type 2: $f(n) = 2f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$

Theorem 7 (Monotonicity)**.** $i \leq j \implies f(i) \leq f(j)$

Proof. Use induction and monotonicity of g $\square$

Theorem 8. $\forall k \geq 0$:

$$f(n) = 2^k f\left(\left\lceil \frac{n}{2^k} \right\rceil\right) + \sum_{i=0}^{k-1} 2^i g\left(\left\lceil \frac{n}{2^i} \right\rceil\right)$$

Set $k = \lceil \lg n \rceil$ in the above theorem to get

$$f(n) = 2^{\lceil \lg n \rceil} f(1) + \sum_{i=0}^{\lceil \lg n \rceil - 1} 2^i g\left(\left\lceil \frac{n}{2^i} \right\rceil\right)$$

1.4 Type 3: $f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$

Theorem 9 (Monotonicity)**.** $i \leq j \implies f(i) \leq f(j)$

Proof. Use induction and monotonicity of g $\square$

We will also look at a special instance of this recurrence where $g(n) = n-1$ and $f(1) = 0$. (This is the recurrence for the number of comparisons in merge sort)

1.4.1 Weak bounds

Let

$$f_l(n) = 2f_l\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + g(n)$$

$$f_u(n) = 2f_u\left(\left\lceil\frac{n}{2}\right\rceil\right) + g(n)$$

where $f_l(0) = f(0) = f_u(0)$ and $f_l(1) = f(1) = f_u(1)$.

Theorem 10. $\forall n \geq 0, f_l(n) \leq f(n) \leq f_u(n)$

Proof. Use induction and monotonicity of f:

$$f(n) - f_l(n) = 2\left[f\left(\left\lfloor\frac{n}{2}\right\rfloor\right) - f_l\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right] + \left[f\left(\left\lceil\frac{n}{2}\right\rceil\right) - f\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right]$$

$$f_u(n) - f(n) = 2\left[f_u\left(\left\lceil\frac{n}{2}\right\rceil\right) - f\left(\left\lceil\frac{n}{2}\right\rceil\right)\right] + \left[f\left(\left\lceil\frac{n}{2}\right\rceil\right) - f\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\right]$$

□

For $g(n) = n - 1$ and $f(1) = 0$, we get

$$\begin{aligned}
f_l(n) &= \sum_{i=0}^{\lfloor\lg n\rfloor - 1} 2^i\left(\left\lfloor\frac{n}{2^i}\right\rfloor - 1\right) \\
&= \sum_{i=0}^{\lfloor\lg n\rfloor - 1} 2^i\left(\left\lceil\frac{n+1}{2^i}\right\rceil - 2\right) \\
&\geq \sum_{i=0}^{\lfloor\lg n\rfloor - 1} 2^i\left(\frac{n+1}{2^i} - 2\right) \\
&= \lfloor\lg n\rfloor(n+1) - 2^{\lfloor\lg n\rfloor + 1} + 2 \\
&\geq \lfloor\lg n\rfloor(n+1) - 2n + 2
\end{aligned}$$

$$\begin{aligned}
f_u(n) &= \sum_{i=0}^{\lceil\lg n\rceil - 1} 2^i\left(\left\lceil\frac{n}{2^i}\right\rceil - 1\right) \\
&= \sum_{i=0}^{\lceil\lg n\rceil - 1} 2^i\left(\left\lfloor\frac{n-1}{2^i}\right\rfloor\right) \\
&\leq \sum_{i=0}^{\lceil\lg n\rceil - 1} 2^i\left(\frac{n-1}{2^i}\right) \\
&= \lceil\lg n\rceil(n-1)
\end{aligned}$$

Therefore, $\lfloor\lg n\rfloor(n+1) - 2n + 2 \leq f(n) \leq \lceil\lg n\rceil(n-1)$.

1.4.2 Exact solution

The exact solution is based on [1]. This approach requires the recurrence to hold at $n = 0$ and $n = 1$. Therefore, $f(0) = -g(0) = -g(1)$.

Extend the domain of f and g by linear interpolation:

$$f(x) = (1 - \{x\})f(\lfloor x \rfloor) + \{x\}f(\lfloor x \rfloor + 1)$$

$$g(x) = (1 - \{x\})g(\lfloor x \rfloor) + \{x\}g(\lfloor x \rfloor + 1)$$

where $\{x\} = x - \lfloor x \rfloor$.

Theorem 11. $\forall x \geq 0, f(x) = 2f(\frac{x}{2}) + g(x)$

Proof. Let $n = \lfloor x \rfloor$ and $h = \{x\}$.

$$\begin{aligned}
f(x) &= (1-h)f(n) + hf(n+1) \\
&= (1-h)\left(f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)\right) \\
&+ h\left(f\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n+1}{2} \right\rceil\right) + g(n+1)\right) \\
&= g(x) + (1-h)\left(f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right)\right) \\
&+ h\left(f\left(\left\lceil \frac{n}{2} \right\rceil\right) + f\left(\left\lfloor \frac{n}{2} + 1 \right\rfloor\right)\right) \\
&= g(x) + f\left(\left\lfloor \frac{n}{2} \right\rfloor + h\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right)
\end{aligned}$$

$$\left\lfloor \frac{x}{2} \right\rfloor = \left\lfloor \frac{n+h}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$$

$$\left\{ \frac{x}{2} \right\} = \frac{n+h}{2} - \left\lfloor \frac{n}{2} \right\rfloor = \frac{(n\%2) + h}{2}$$

where '%' is the remainder operator $\left(n\%k = n - k\left\lfloor \frac{n}{k} \right\rfloor\right)$.

$$1 - \left\{ \frac{x}{2} \right\} = 1 - \frac{(n\%2) + h}{2} = \frac{(1 - n\%2) + (1-h)}{2}$$

$$\begin{aligned}
2f\left(\frac{x}{2}\right) &= 2\left(1 - \left\{\frac{x}{2}\right\}\right) f\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + 2\left\{\frac{x}{2}\right\} f\left(\left\lfloor \frac{x}{2} \right\rfloor + 1\right) \\
&= ((1 - n\%2) + (1-h))f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + (n\%2 + h)f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \\
&= f\left(\left\lfloor \frac{n}{2} \right\rfloor + h\right) + (1 - n\%2)f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + (n\%2)f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \\
&= f\left(\left\lfloor \frac{n}{2} \right\rfloor + h\right) + \begin{cases} f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) & n\%2 = 0 \\ f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) & n\%2 = 1 \end{cases} \\
&= f\left(\left\lfloor \frac{n}{2} \right\rfloor + h\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) \\
&= f(x) - g(x)
\end{aligned}$$

□

Theorem 12.

$$f(n) = 2^k f\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} 2^i g\left(\frac{n}{2^i}\right)$$

Let's consider the special instance of this recurrence:

$$g(n) = \begin{cases} n-1 & n \geq 1 \\ 0 & n = 0 \end{cases}$$

Also assume $f(1) = 0$. Therefore, $f(0) = f(1) = g(0) = g(1) = 0$.

After linear interpolation, we get

$$g(x) = \begin{cases} x-1 & x \geq 1 \\ 0 & 0 \leq x \leq 1 \end{cases}$$

and $f(x) = 0$ when $0 \leq x \leq 1$.

$$k = \lfloor \lg n \rfloor + 1 \implies \frac{n}{2^k} \in \left[\frac{1}{2}, 1\right) \implies f\left(\frac{n}{2^k}\right) = 0$$

$$\begin{aligned} f(n) &= \sum_{i=0}^{\lfloor \lg n \rfloor} 2^i \left(\frac{n}{2^i} - 1\right) \\ &= n(\lfloor \lg n \rfloor + 1) - 2^{\lfloor \lg n \rfloor + 1} + 1 \\ &\in n \lfloor \lg n \rfloor - [0, n-1] \end{aligned}$$

$$k = \lceil \lg n \rceil \implies \frac{n}{2^k} \in \left(\frac{1}{2}, 1\right] \implies f\left(\frac{n}{2^k}\right) = 0$$

$$f(n) = \sum_{i=0}^{\lceil \lg n \rceil - 1} 2^i \left(\frac{n}{2^i} - 1\right) = n \lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1$$

References

[1] Hsien-Kuei Hwang, Svante Janson, and Tsung-Hsi Tsai. Exact and asymptotic solutions of a divide-and-conquer recurrence dividing at half: Theory and applications. *ACM Trans. Algorithms*, 13(4):47:1–47:43, October 2017. URL: http://doi.acm.org/10.1145/3127585, doi:10.1145/3127585.