Stochastic Processes

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Definition 1 (Stochastic Process). *Let $\mathcal{T} \subseteq \mathbb{R}$. For any $t \in \mathcal{T}$, let X_t (or $X(t)$) be a random variable with support D. Then $X := \{X_t : t \in \mathcal{T}\}$ is called a* stochastic process *on state-space D and time $\mathcal{T}$. Usually, $\mathcal{T}$ is either $\mathbb{Z}_{\geq 0}$ (discrete-time) or $\mathbb{R}_{\geq 0}$ (continuous-time).*

1 Discrete-Time Markov Chains

Definition 2 (Markov Chain). *Let $X := [X_0, X_1, \ldots]$ be a stochastic process on state-space D and time $\mathbb{Z}_{\geq 0}$. X is called a discrete-time markov chain if $\Pr(X_{t+1} = d \mid X_t, X_{t-1}, \ldots, X_0) = \Pr(X_{t+1} = d \mid X_t)$. If $\Pr(X_{t+1} = v \mid X_t = u) = \Pr(X_1 = v \mid X_0 = u)$ for all t, u, v, then X is called* time-homogeneous.

Definition 3 (Transition function). *Let X be a markov chain on state space D. Define $P^{(k)} : D \times D \mapsto [0, 1]$ as $P^{(k)}(i, j) = \Pr(X_k = j \mid X_0 = i)$. Then $P^{(k)}$ is called the k-step transition function of X. For $k = 1$, we simply write P instead of $P^{(1)}$. For a finite state space, we can represent P as a matrix.*

Lemma 1 (Chapman-Kolmogorov Equation). $P^{(m+n)}(i, j) = \sum_k P^{(m)}(i, k)P^{(n)}(k, j)$.

1.1 Classification of States, Recurrence, Limiting Probabilities

Definition 4. *Let $f_{i,j} := \Pr\left(\bigvee_{t\geq 1}(X_t = j) \,\middle|\, X_0 = i\right)$. Then $f_{i,j}$ is called the* eventual transition probability *from i to j. If $i = j$, then we write $f_{i,i}$ as f_i, and call it the* recurrence probability *of state i.*

Definition 5. *For a state i, let N_i be the random variable that counts the number of times we are in state i, i.e., $N_i := \sum_{t=0}^{\infty} \mathbf{1}(X_t = i)$. Then N_i is called the visit-count of i.*

Definition 6. *A state i of a markov chain is* recurrent *iff (the following are equivalent):*

- *the recurrence probability (f_i) of i is 1.*
- *i is visited infinitely often, i.e., $\Pr(N_i = \infty \mid X_0 = i) = 1$.*
- *i is visited infinitely often in expectation, i.e., $\mathrm{E}(N_i \mid X_0 = i) = \infty$.*

A non-recurrent state is called a transient *state.*

Lemma 2. $\Pr(N_i = k \mid X_0 = i) = f_i^{k-1}(1 - f_i)$.

Lemma 3. $\mathrm{E}(N_i \mid X_0 = i) = 1/(1 - f_i) = \sum_{t=0}^{\infty} P^{(t)}(i, i)$.

Definition 7. *State j is* accessible *from state i if $P^{(t)}(i,j) > 0$ for some t. States i and j* communicate *(denoted as $i \leftrightarrow j$) if i and j are both accessible from each other.*

Lemma 4. *Accessibility is reflexive and transitive. Communication is an equivalence relation. The equivalence classes of communicability are called* state classes*. A markov chain is* irreducible *if it has just one state class.*

Definition 8. *Let T_i be the time when a markov chain moves to state i, i.e., $T_i := \min_{t \geq 1}(X_t = i)$. When conditioned on $X_0 = i$, T_i is called the* recurrence time *of i. State i is called* positive recurrent *if $\mathrm{E}(T_i \mid X_0 = i)$ is finite, otherwise it is* null recurrent*.*

Lemma 5. *Recurrence and positive recurrence are class properties, i.e., they are same for all states in a class.*

Lemma 6. *In a finite-state markov chain, all recurrent states are positive recurrent, and there is at least one recurrent state.*

Definition 9 (Periodicity)**.** *For a state i, its period is defined as $\gcd(\{t : \Pr(T_i = t \mid X_0 = i) > 0\})$. A state is* aperiodic *if its period is 1.*

Lemma 7. *Periodicity is a class property.*

Definition 10 (Ergodicity)**.** *A state is ergodic if it is positive recurrent and aperiodic. A markov chain is ergodic if all its states are ergodic.*

Lemma 8. *In an irreducible ergodic markov chain, for every state j, $\lim_{t\to\infty} P^{(t)}(j,i) = \pi_i$ for a unique real number π_i. π_i is called the* limiting probability *of state i. Furthermore, π_i is the unique solution to this system of equations: $\pi_i = \sum_j \pi_j P(j,i)$ for all i ($\pi = P^T \pi$ in matrix form) and $\sum_i \pi_i = 1$.*

Lemma 9. *In an irreducible ergodic markov chain, $\mathrm{E}(T_i \mid X_0 = i) = 1/\pi_i$.*

Corollary 9.1. *A state i is null recurrent iff $\pi_i = 0$.*

Theorem 10. *If the transition function of markov chain X is doubly-stochastic (i.e., each row and each column sums to 1), then the limiting probability of each state is $1/n$, where n is the number of states.*

1.2 Time-Reversibility

Definition 11. *For an irreducible ergodic markov chain X with limiting probabilities π. Let Y be a markov chain whose transition function is $Q(i,j) = P(j,i)(\pi_j/\pi_i)$. Then Y is called the* time-reversed *markov chain of X. X is called* time-reversible *if $Q = P$.*

Theorem 11. *Let X be a time-reversible markov chain with limiting probabilities π. Then π is the unique solution to this system of equations: $x_j P(j,i) = x_i P(i,j)$ for all states i and j, and $\sum_i x_i = 1$.*

Theorem 12. *If the transition function of markov chain X is symmetric, then X is time-reversible.*

1.3 Simple Random Walk

Let X be a TH MC on state space $S = I \subseteq \mathbb{Z}$, where I is an interval of $\mathbb{R}$ and

$$\Pr(X_1 = j \mid X_0 = i) = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 13 (Martingale property)**.** *Let* $\forall t \geq 1$*. If* $p = 1/2$*, then* $\mathrm{E}(X_t \mid X_{t-1}) = X_{t-1}$*. Otherwise, for* $q = (1-p)/p$*,* $\mathrm{E}(q^{X_t} \mid X_{t-1}) = q^{X_{t-1}}$*.*

Lemma 14. *Let* $I = [0, b]$*. Let* $T := \min_{t \geq 0}(X_t \in \{0, b\})$*. Let* $q = (1-p)/p$*.*

$$\Pr(X_T = b \mid X_0 = i) = \begin{cases} \dfrac{i}{b} & \textit{if } p = 1/2 \\ \dfrac{q^i - 1}{q^b - 1} & \textit{if } p \neq 1/2 \end{cases}$$

$$\mathrm{E}(T \mid X_0 = i) = \begin{cases} i(b-i) & \textit{if } p = 1/2 \\ \dfrac{1}{1-2p}\left(i - b\dfrac{q^i - 1}{q^b - 1}\right) & \textit{if } p \neq 1/2 \end{cases}$$

Proof sketch. Use Lemma 13 to compute $\Pr(X_T \mid X_0 = i)$. Let $\mu_i := \mathrm{E}(X_T \mid X_0 = i)$. Then $\mu_0 = \mu_b = 0$ and $\mu_i = 1 + (1-p)\mu_{i-1} + p\mu_i$ for $i \in [1, b-1]$. Let $d_i := \mu_i - \mu_{i-1}$ for $i \in [1, b]$. Then $qd_i - d_{i+1} = 1/p$ for $i \in [1, b-1]$. Solve the recurrences. □

Lemma 15. *Let* $I = [0, \infty)$*. Let* $T := \min_{t \geq 0}(X_t = 0)$*. Then for* $i > 0$*,*

$$\mu_i := \mathrm{E}(T \mid X_0 = i) = \begin{cases} i/(1-2p) & \textit{if } p < 1/2 \\ \infty & \textit{if } p \geq 1/2 \end{cases}.$$

$$p_i := \Pr(T \neq \infty \mid X_0 = i) = \begin{cases} 1 & \textit{if } p \leq 1/2 \\ q^i & \textit{if } p > 1/2 \end{cases}.$$

Proof sketch. $\mu_i = i\mu_1$ and $\mu_1 = 1 + p\mu_2$. $p_i = p_1^i$ and $p_1 = (1-p) + p_2$. This gives us $p_1 \in \{1, q\}$. (TODO) □

2 Counting Process

Definition 12 (Counting Process)**.** *Let* N *be a stochastic process on state space* $\mathbb{Z}_{\geq 0}$ *and time* $\mathbb{R}_{\geq 0}$*. Then* N *is called a counting process if* $N(0) = 0$ *and* $N(t)$ *is monotone in* t*, i.e.,* $t_1 < t_2 \implies N(t_1) \leq N(t_2)$*.*

Definition 13 (Independent increments)**.** *A counting process* N *has* independent increments *iff for any two disjoint intervals* $(u_1, v_1]$ *and* $(u_2, v_2]$ *in* $\mathbb{R}_{\geq 0}$*, the random variables* $N(v_1) - N(u_1)$ *and* $N(v_2) - N(u_2)$ *are independent.*

Definition 14 (Stationary increments)**.** *A counting process* N *has* stationary increments *iff for any* $u \leq v$*, the random variables* $N(v) - N(u)$ *and* $N(v-u)$ *have the same distribution.*

Definition 15 (Arrival and interarrival times). *For a counting process N, for $i \in \mathbb{Z}_{\geq 0}$, define the i^{th} arrival time $S_i := \min_{t \geq 0}(N(t) = i)$. For $i \in \mathbb{Z}_{\geq 1}$, define the i^{th} interarrival time $T_i := S_i - S_{i-1}$.*

Lemma 16. *For a counting process N with arrival times S, $N(t) \geq n \iff S_n \leq t$.*

Definition 16 (Stopping time). *Let $X = [X_1, X_2, \ldots]$ be a sequence of random variables. The random variable N is called a stopping time for X if for all $n \geq 0$, (the following two definitions are equivalent):*

- *$N = n$ is independent of $X_{n+1}, X_{n+2}, \ldots$.*
- *$N \leq n$ is independent of $X_{n+1}, X_{n+2}, \ldots$.*

Theorem 17 (Wald's identity). *Let $X = [X_1, X_2, \ldots]$ be a sequence of random variables where $\mathrm{E}(X_i) = \mu$ for all i. Let N be a stopping time for X. Then*

$$\mathrm{E}\left(\sum_{i=1}^{N} X_i\right) = \mu\, \mathrm{E}(N).$$

Proof sketch. For all i, $N \geq i$ is independent of X_i, and $\displaystyle\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i \,\mathbf{1}(N \geq i)$. $\square$

3 Poisson Process

Definition 17 (Poisson process). *A counting process N is a Poisson process with rate function $\lambda : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ if N has independent increments and $N(t_2) - N(t_1) \sim \mathrm{Poisson}(\mu)$, where $\mu := \int_{t_1}^{t_2} \lambda(t) dt$. N is called* homogeneous *if $\lambda(t) = \lambda(0)$ for all t, otherwise it is called* inhomogeneous*. For a homogeneous process, we denote $\lambda(0)$ by λ.*

Lemma 18. *A Poisson process N is homogeneous iff it has stationary increments.*

Theorem 19 (Alternative definition of Poisson process). *A counting process N is a Poisson process with continuous rate function λ iff N has independent and stationary increments and $\Pr(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$ and $\Pr(N(t+h) - N(t) \geq 2) = o(h)$.*

Proof sketch for homogeneous. Let $g(u, t) := \mathrm{MGF}_u(N(t)) = \mathrm{E}(e^{uN(t)})$. Show $g(u, t) = 1 + \lambda t(e^u - 1) + o(t)$ straightforwardly. Use calculus to show that $g(u, t) = \exp(e^{\lambda t}(e^u - 1))$ (find derivative w.r.t t by computing $\lim_{h \to 0}(g(u, t+h) - g(u, t))/h$; this gets rid of $o(h)$). Conclude that $N(t) \sim \mathrm{Poisson}(\lambda t)$ since $g(u, t)$ is MGF of $\mathrm{Poisson}(\lambda t)$. $\square$

Lemma 20. *For a homogeneous Poisson process N,*

$$\Pr(N(s) = a \mid N(s+t) = a+b) = \binom{a+b}{a} \left(\frac{s}{s+t}\right)^a \left(\frac{t}{s+t}\right)^b.$$

Theorem 21. *Let N be a counting process. Then N is a homogeneous Poisson process with rate λ iff all interarrival times are independent and distributed $\mathrm{Expo}(\lambda)$.*

Theorem 22 (Decomposition theorem 1). *Let K be a finite set, and let $\{N_i : i \in K\}$ be independent Poisson processes, where N_i has rate function λ_i. Let $N := \sum_{i \in K} N_i$. Then N is a Poisson process with rate function $\sum_{i \in K} \lambda_i$.*

Theorem 23 (Decomposition theorem 2)**.** *Let N be a Poisson process with rate function λ. Let K be a finite set (called set of labels). Suppose the j^{th} event receives label $L_j \in K$, where $\Pr(L_j = i) = p_i(S_j)$ for some function $p_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$, and $\{N, L_1, L_2, \ldots\}$ are independent. For $i \in K$, let $N_i(t)$ be the number of events having label i, i.e, $N_i(t) = \sum_{j=1}^{N(t)} \mathbf{1}(L_j = i)$. Then N_i is a Poisson process with rate function $p_i\lambda$. Furthermore, all N_i are independent and if all p_i are constant, then $N_i(t) \mid N(t) \sim \operatorname{Binom}(N(t), p_i)$.*

Lemma 24. *Let $N^{(1)}$ and $N^{(2)}$ be independent homogeneous Poisson processes with rates λ_1 and λ_2. Then*

$$\Pr(S_n^{(1)} < S_m^{(2)}) = \sum_{i=n}^{n+m-1} \binom{n+m-1}{i} \frac{\lambda_1^i \lambda_2^{n+m-1-i}}{(\lambda_1 + \lambda_2)^{n+m-1}}.$$

Proof sketch. Model as a continuous markov chain with state space (n_1, n_2), where n_i is the number of events of $N^{(i)}$ that have occurred. □

Theorem 25 (arrival times distributed as order statistics)**.** *Let $X = [X_1, X_2, \ldots, X_n]$ be IID uniform variables over $[0, t]$. Let $U = \operatorname{sorted}(X)$. Let N be a homogeneous Poisson process. Let S_i be the i^{th} arrival time of N. Then conditioned on $N(t) = n$, the distribution of $[S_1, \ldots, S_n]$ and U are identical.*

Lemma 26 (Excess and Residual)**.** *Let N be a Poisson process with rate function λ. Let S_i be the i^{th} arrival time. Let $Y(t) := S_{N(t)+1} - t$ and $R(t) := t - S_{N(t)}$. Then $Y(t) > s \iff N(t+s) - N(t) = 0$ and $R(t) > r \iff N(t) - N(t-r) = 0$. If N is homogeneous, we get $Y(t) \sim \operatorname{Expo}(\lambda)$ and $R(t) \sim \operatorname{Expo}(\lambda)$.*

4 Continuous-Time Markov Chain

Definition 18 (CTMC)**.** *Let $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$ be a stochastic process on discrete state-space D. X is called a continuous-time markov chain (CTMC) if $\Pr(X(t+s) = d \mid \{X(u) : 0 \leq u \leq s\}) = \Pr(X(t+s) = d \mid X(s))$ for all $s, t \in \mathbb{R}_{\geq 0}$. If $\Pr(X(t+s) = v \mid X(s) = u) = \Pr(X(t) = v \mid X(0) = u)$ for all u, v, s, t, then X is called* time-homogeneous *(TH) or* stationary.

Theorem 27 (Equiv defn of TH CTMC)**.** *Let $X := \{X(t) : t \in \mathbb{R}_{\geq 0}\}$ be a stochastic process on discrete state-space D. Let $Y(t) := \{X(u) : 0 \leq u < t\}$. Let $T_i^{(s)} := \min_{t \geq 0}(X(t+s) \neq i)$. Let $P_{i,j}^{(s)} := \Pr(X(s + T_i^{(s)}) = j \mid X(s) = i, Y(s))$. X is TH CTMC iff $(T_i^{(s)} \mid X(s) = i, Y(s)) \sim \operatorname{Expo}(\nu_i)$, where ν_i is a constant that doesn't depend on s or $Y(s)$, and $P_{i,j}^{(s)}$ is a constant that doesn't depend on s or $Y(s)$.*

Since $T_i^{(s)}$ and $P_{i,j}^{(s)}$ don't depend on s, we simply write T_i and $P_{i,j}$. T_i is called the transition time *out of state i, ν_i is called the* transition rate *out of state i, and $P_{i,j}$ is the* probability of transitioning *from state i to state j.*

Let $q_{i,j} := \nu_i P_{i,j}$. Then $\nu_i = \sum_j q_{i,j}$.

Theorem 28 (Chapman-Kolmogorov DiffEqs)**.** *For a TH CTMC X, let $P_{i,j}(t) := \Pr(X(t) = j \mid X(0) = i)$. Then*

- *Backward DiffEqs:* $\dfrac{dP_{i,j}(t)}{dt} = \sum_{k \neq i} q_{i,k} P_{k,j}(t) - \nu_i P_{i,j}(t)$.

- *Forward DiffEqs:* $\dfrac{dP_{i,j}(t)}{dt} = \sum_{k \neq j} P_{i,k}(t) q_{k,j} - P_{i,j}(t)\nu_j$.

Let $r_{i,j} := \begin{cases} q_{i,j} & \text{if } i \neq j \\ -\nu_i & \text{if } i = j \end{cases}$. *Let the state space be* $[n]$. *Let* R *be a matrix where* $R[i,j] = r_{i,j}$. *Then CBKE becomes* $P'(t) = RP(t)$ *and CFKE becomes* $P'(t) = P(t)R$.

Lemma 29. *CKBE* $P'(t) = RP(t)$ *solves to* $P(t) = e^{Rt}$, *where* $e^A := \sum_{i=0}^{\infty} A^i/i!$ *for any square matrix* A. *Suppose* R *has* n *eigenpairs* $\{(\lambda_1, v_i) : i \in [n]\}$. *Let* P *be a square matrix whose* i^{th} *column is* v_i, *and* D *be a diagonal matrix whose* i^{th} *diagonal entry is* λ_i. *Then* $R = PDP^{-1}$, $e^{Rt} = Pe^{Dt}P^{-1}$, *and* $e^{Dt} = \operatorname{diag}([e^{\lambda_1 t}, \ldots, e^{\lambda_n t}])$.

Lemma 30. *Let* X *be a TH CTMC.*

$$\lim_{h \to 0} \frac{1 - P_{i,i}(h)}{h} = \nu_i \quad \forall i \qquad\qquad \lim_{h \to 0} \frac{P_{i,j}(h)}{h} = q_{i,j} \quad \forall i \neq j$$

Lemma 31 (Limiting probability)**.** *In an irreducible positive-recurrent TH CTMC* X, *for every state* j, $\lim_{t \to \infty} P_{j,i}(t) = P_i$ *for a unique real number* P_i. P_i *is called the* limiting probability *of state* i. *Furthermore,* P_i *is the unique solution to* $\sum_i P_i = 1$ *and CK forward equations, i.e.,* $P_i \nu_i = \sum_{j \neq i} P_j q_{j,i}$.

Lemma 32 (Limiting probability of embedded chain)**.** *Let* X *be an irreducible positive-recurrent TH CTMC. Let* Y *be the sequence of states visited by* X. *Then* Y *is a discrete MC. Let* P *and* π *be the limiting probabilities of* X *and* Y, *respectively. Then* $P_i = (\pi_i/\nu_i)/(\sum_j \pi_j/\nu_j)$ *and* $\pi_i = P_i \nu_i/(\sum_j P_j \nu_j)$.

Definition 19. *A CTMC is time-reversible iff the corresponding embedded discrete-time MC is time-reversible.*

Lemma 33 (2-state)**.** *For a CTMC on states* $\{0, 1\}$, *where* $q_{0,1} = \lambda$ *and* $q_{1,0} = \mu$, *we get*

$$P(t) = \frac{1}{\lambda + \mu} \left(\begin{bmatrix} \mu & \lambda \\ \mu & \lambda \end{bmatrix} + e^{-(\mu+\lambda)t} \begin{bmatrix} \lambda & -\lambda \\ -\mu & \mu \end{bmatrix} \right).$$

4.1 Birth and Death Process

Definition 20. *A birth-and-death (B&D) process is a TH CTMC* X *on state space* $\mathbb{Z}_{\geq 0}$ *where* $q_{i,j} = 0$ *if* $j \notin \{i-1, i+1\}$. *Let* $\lambda_i := q_{i,i+1}$ *for* $i \geq 0$, $\mu_i := q_{i,i-1}$ *for* $i \geq 1$, $\mu_0 := 0$.

$X(t)$ *is called the* population *at time* t, λ_i *is called the* birth rate *at population* i, *and* μ_i *is called the* death rate *at population* i.

Lemma 34. *Let* X *be a B&D process where* $X(0) = n$. *Let* T_n *be the time to reach state* $n+1$, *i.e.,* $T_n := \min_{t \geq 0}(X(t) = n+1)$. *Then*

$$\mathrm{E}(T_n) = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} \mathrm{E}(T_{n-1}) = \frac{1}{\lambda_n} \sum_{i=0}^{n} \prod_{j=1}^{i} \frac{\mu_{n-j+1}}{\lambda_{n-j}}.$$

$$\mathrm{Var}(T_n) = \frac{1}{\lambda_n(\lambda_n + \mu_n)^2} + \frac{\mu_n}{\lambda_n} \mathrm{Var}(T_{i-1}) + \frac{\mu_n}{\lambda_n + \mu_n} (\mathrm{E}(T_{n-1}) + \mathrm{E}(T_n))^2$$

Proof sketch. Let $I_i = \mathbf{1}(\text{next transition goes to state } i+1)$. Let X_i be the transition time out of state i. Then $I_i \sim \text{Bernouilli}(\lambda_i/(\mu_i+\lambda_i))$, $X_i \sim \text{Expo}(\lambda_i+\mu_i)$, and

$$\begin{aligned}
\mathrm{E}(T_i \mid I_i) &= \mathrm{E}(X_i) + (1-I_i)(\mathrm{E}(T_{i-1}) + \mathrm{E}(T_i))), \\
\mathrm{Var}(T_i \mid I_i) &= \mathrm{Var}(X_i) + (1-I_i)(\mathrm{Var}(T_{i-1}) + \mathrm{Var}(T_i)).
\end{aligned}$$ □

CKBE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_i P_{i-1,j}(t) + \lambda_i P_{i+1,j}(t) - (\lambda_i+\mu_i)P_{i,j}(t).$$

CKFE for B&D:

$$\frac{dP_{i,j}(t)}{dt} = \mu_{j+1} P_{i,j+1}(t) + \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j+\mu_j)P_{i,j}(t).$$

Theorem 35 (Limiting Probabilities)**.** *Let X be an irreducible B&D process on state space $D \subseteq \mathbb{Z}_{\geq 0}$ where $0 \in D$. For $n \in D$, let $\alpha_n := \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$. If $\sum_{i \in D} \alpha_i$ is finite, then $P_i = \alpha_i P_0$, and $P_0 = 1/\sum_{i \in D} \alpha_i$.*

Proof sketch. Use Lemma 31 and add adjacent equations. □

5 Renewal Theory

Definition 21. *Let $[X_1, X_2, \ldots]$ be a sequence of IID non-negative randvars, called* interarrival times*, such that $\Pr(X_1 = 0) < 1$ and $\Pr(X_1 = \infty) = 0$. Let $S_n := \sum_{i=1}^{n} X_i$ (called* arrival times*). Let $N(t) := \max_n(S_n \leq t)$. Then N is called a* renewal process *(note that it is a counting process).*

We let F and f denote the CDF and PDF/PMF of X_1, respectively. We let $F^{(n)}$ and $f^{(n)}$ denote the CDF and PDF/PMF of S_n, respectively.

Let R_i be the reward *obtained at time X_i for all $i \geq 1$, where all R_i are independent. Let $R(t) := \sum_{i=1}^{N(t)} R_i$. Then R is called a* renewal reward process*.*

Lemma 36. *For all $t \geq 0$, $\Pr(N(t) = \infty) = 0$. $\Pr(\lim_{t\to\infty} N(t) = \infty) = 1$.*

Proof. Let $\mu := \mathrm{E}(X_1)$. $\mu > 0$ since $\Pr(X_n = 0) < 1$.

$$\Pr\left(\lim_{t\to\infty} \frac{S_n}{n} = \mu\right) = 1. \qquad \text{(strong law of large numbers)}$$

$$N(t) = \infty \iff (\forall n, S_n \leq t) \implies \lim_{t\to\infty} \frac{S_n}{n} = 0.$$

$\Pr(N(\infty) = \infty) = 1$ since $\Pr(X_1 = \infty) = 0$. □

Definition 22. *For a renewal process N, let $m_N(t) := \mathrm{E}(N(t))$. Then m_N is called the* mean-value function *of N. (If N is clear from context, we will write m instead of m_N.)*

Lemma 37. *$m(t) = \sum_{n=1}^{\infty} \Pr(S_n \leq t) = \sum_{n=1}^{\infty} F^{(n)}(t)$.*

Theorem 38. *m uniquely characterizes F.*

Lemma 39. *$m(t)$ is finite for all t.*

Theorem 40 (Renewal equation)**.** *When interarrival times are continuous randvars,*

$$m(t) = F(t) + \int_0^t m(t-x)f(x)dx.$$

Proof sketch. Let $N'(t) := \max_n \left(\sum_{i=2}^{n+1} X_i \le t\right)$. Then N and N' are identically distributed and

$$N(t) = \begin{cases} 1 + N'(t - X_1) & \text{if } X_1 \le t \\ 0 & \text{if } X_1 > t \end{cases}.$$

Finally, $m(t) = \mathrm{E}(\mathrm{E}(N(t) \mid X_1))$. ☐

Corollary 40.1. *Let N be a renewal process where interarrival times are distributed* $\mathrm{Uniform}(0,1)$*. Then for $0 \le t \le 1$, $m(t) = e^t - 1$.*

Theorem 41 (Limit theorems)**.** *For a renewal process N with $\mu := \mathrm{E}(X_1)$,*

$$\Pr\left(\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}\right) = 1. \qquad \lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Theorem 42 (Limit theorems for rewards)**.** *For a renewal process N with rewards $\{R_i : i \in \mathbb{Z}_{\ge 1}\}$, let $\alpha := \mathrm{E}(R_1)$ and $\mu := \mathrm{E}(X_1)$. Then*

$$\Pr\left(\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\alpha}{\mu}\right) = 1. \qquad \lim_{t\to\infty} \frac{\mathrm{E}(R(t))}{t} = \frac{\alpha}{\mu}.$$

Theorem 43 (Central limit theorem for renewals)**.** *For a renewal process N with $\mu := \mathrm{E}(X_1)$ and $\sigma^2 := \mathrm{Var}(X_1)$, the random variable*

$$\lim_{t\to\infty} \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}}$$

tends to the standard normal distribution.

Lemma 44 (Stopping time)**.** *Let $X = [X_1, X_2, \ldots]$ be the sequence of interarrival times for renewal process N. Then $N(t) + 1$ is a stopping time for X.*

Proof sketch. $N(t) + 1 \le n \iff S_n > t$. ☐

Definition 23. *For a renewal process N with arrival times $S_1, S_2, \ldots$:*

- *Let $Y(t) := S_{N(t)+1} - t$. $Y(t)$ is called the* excess *at time t.*
- *Let $L(t) := t - S_{N(t)}$. $L(t)$ is called the* remaining life *at time t.*

Lemma 45. *Let N be a renewal process with interarrival times $X = [X_1, X_2, \ldots]$. Then* $\mathrm{E}(S_{N(t)+1}) = t + \mathrm{E}(Y(t)) = \mathrm{E}(X_1)(m(t) + 1)$*.*

Proof. $N(t) + 1$ is a stopping time for X. ☐