Random Walks

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1 One-dimensional random walk with left bounce and right absorb

There are $n+1$ nodes, numbered from 0 to n, where $n \geq 1$. Let X_t be a random variable denoting our position at time t. Let $X_0 = 0$ (i.e., we start at node 0).

At node 0 we always move to node 1 in the next time step, i.e., $X_t = 0 \implies X_{t+1} = 1$. Node n is absorbing, i.e., $X_t = n \implies X_{t+1} = n$. At every other node i, we move to node $i+1$ with probability p and node $i-1$ with probability $1-p$, where $p \in (0,1)$ is a constant. So for $0 < i < n$, we have

$$X_t = i \implies X_{t+1} = \begin{cases} i+1 & \text{with probability } p \\ i-1 & \text{with probability } 1-p \end{cases}$$

Our aim is to find the expected number of moves to reach n from 0.

For $0 \leq i < n$, let s_i be the expected number of moves to reach $i+1$ from i. Then by linearity of expectation, the expected number of moves to reach n from 0 is $\sum_{i=0}^{n-1} s_i$.

Consider the sequence of nodes corresponding to a random walk that starts at i and ends at $i+1$. Suppose this sequence contains t occurrences of node i. This means that we moved $t-1$ times from node i to node $i-1$ and we moved once from i to $i+1$. The probability of observing such a sequence is $(1-p)^{t-1}p$, which means that t is a geometric random variable. Therefore, the expected number of times we will move from i to $i-1$ is $1/p - 1$. Whenever we move from i to $i-1$, we will have to random-walk our way back to i from $i-1$, which will take s_{i-1} steps in expectation. Therefore,

$$s_i = 1 + \left(\frac{1}{p} - 1\right)(s_{i-1} + 1) = \frac{1-p}{p} s_{i-1} + \frac{1}{p}$$

As a base case, we know that $s_0 = 1$. Therefore, we get

$$s_i = \begin{cases} 2i+1 & \text{if } p = 1/2 \\ \frac{2(1-p)}{1-2p}\left(\frac{1-p}{p}\right)^i - \frac{1}{1-2p} & \text{if } p \neq 1/2 \end{cases}$$

When $p = 1/2$, the expected number of steps to reach n from 0 is

$$\sum_{i=0}^{n-1} s_i = \sum_{i=0}^{n-1} (2i+1) = \sum_{i=0}^{n-1} ((i+1)^2 - i^2) = n^2$$

When $p \neq 1/2$, the expected number of steps to reach n from 0 is

$$\sum_{i=0}^{n-1} s_i = \frac{2p(1-p)}{(1-2p)^2}\left(\left(\frac{1-p}{p}\right)^n - 1\right) - \frac{n}{1-2p}$$

1.1 Alternative proof

Let Z_i be the expected number of steps needed to reach n from i. Then we have $Z_n = 0$ and $Z_0 = Z_1 + 1$. For all other i from 1 to $n-1$, we either move to $i+1$ with probability p and use Z_{i+1} steps to reach node n, or we move to $i-1$ with probability $1-p$ and use Z_{i-1} steps to reach node n. So, $Z_i = 1 + pZ_{i+1} + (1-p)Z_{i-1}$. Our aim is to find Z_0.

$$\begin{aligned}
&Z_i = 1 + pZ_{i+1} + (1-p)Z_{i-1} \\
&\implies Z_i - Z_{i+1} = \frac{1}{p} + \frac{1-p}{p}(Z_{i-1} - Z_i)
\end{aligned}$$

For $p = 1/2$, we get that for $0 < j < n$,

$$\begin{aligned}
&Z_j - Z_{j+1} = 2 + (Z_{j-1} - Z_j) \\
&\implies \sum_{j=1}^{i}(Z_j - Z_{j+1}) = \sum_{j=1}^{i} 2 + \sum_{j=1}^{i}(Z_{j-1} - Z_j) \\
&\implies Z_1 - Z_{i+1} = 2i + Z_0 - Z_i \\
&\implies Z_i - Z_{i+1} = 2i + 1 \\
&\implies \sum_{i=1}^{n-1}(Z_i - Z_{i+1}) = \sum_{i=1}^{n-1}(2i+1) \\
&\implies Z_1 - Z_n = \sum_{i=1}^{n-1}((i+1)^2 - i^2) = n^2 - 1 \\
&\implies Z_0 = n^2
\end{aligned}$$