Three Phase Distribution Network Optimal Power Flow Formulation*

Swastik Sharma[†] Swathi Battula[‡] Sri Niwas Singh[§]

1 Introduction

There are very few repositories or open-source programs in Python that can help researchers run OPF on Unbalanced Distribution Networks; this program bridges the gap. It provides a powerful open-source program for that purpose. We use the modelling techniques presented in [1–3] to model the OPF. This document will help you understand the formulation employed in the software, and instructions on how to run the Pyomo-based T-DOPF application and provide an in-depth overview of the repository's organization.

2 Nomenclature

Acronyms and Parameters

(i, j)	Line segment connecting buses i and j;
0	Head bus of the radial network at which IDSO's substation is located, linking Transmission and Distribution systems;
$\tilde{\boldsymbol{C}}$	Branch-node incidence matrix for a 3-phase radial distribution network;
$\boldsymbol{C}$	Branch-node incidence matrix excluding the head bus;
$\boldsymbol{c}_0$	Branch-node incidence matrix of the head bus;
$\boldsymbol{D_r}, \boldsymbol{D_x}$	Diagonal matrices of resistances and reactances (p.u.) of all lines;
$\boldsymbol{R}_{(i,j)}, \boldsymbol{X}_{(i,j)}$	3-phase resistance and reactance matrix of a line (i, j) in p.u.;
$\boldsymbol{W}$	Phase coupling matrix;
$\omega = e^{j\frac{2\pi}{3}}$	Phase shift operator;
ϕ	Phase connection of a line, bus or DER;
N	Number of non-head buses in distribution network;
p_i^ϕ, q_i^ϕ	Real and reactive power injection at a bus i and phase ϕ in p.u.;

Sets and Vectors

$\boldsymbol{\Phi}$	Set of valid phases;
$\mathcal{L}$	Set containing the N distinct line segments of distribution system;
$\mathcal{N}$	Set containing all the non-head buses of distribution system;

Variables

$\boldsymbol{P}, \boldsymbol{Q}$	3-phase real and reactive power flow (p.u.) column vector across all lines in the distribution network;
$\boldsymbol{p}, \boldsymbol{q}$	3-phase real and reactive power injections (p.u.) column vector across all buses in the distribution network;

*Latest Revision: 11$^{\text{th}}$ March, 2025

[†]swastiks21@iitk.ac.in, Dept. of Electrical Engineering, Indian Institute of Technology Kanpur 208016
[‡]swathi@iitk.ac.in, Dept. of Electrical Engineering, Indian Institute of Technology Kanpur 208016
[§]snsingh@iitk.ac.in, Dept. of Electrical Engineering, Indian Institute of Technology Kanpur 208016

$\boldsymbol{p}_0, \boldsymbol{q}_0$	3-phase real and reactive power injection (p.u.) column vector of the head bus;
$\boldsymbol{p}_i, \boldsymbol{q}_i$	3-phase real and reactive power injection (p.u.) column vector of a bus i;
$\boldsymbol{P}_{(i,j)}, \boldsymbol{Q}_{(i,j)}$	3-phase real and reactive power flow (p.u.) column vector of a line (i,j);
$\boldsymbol{p}_{1:N}, \boldsymbol{q}_{1:N}$	3-phase real and reactive power injection (p.u.) column vector at all non-head buses;
$\boldsymbol{v}$	3-phase square voltage magnitudes (p.u.) column vector of all buses in the distribution network;
$\boldsymbol{v}_0$	3-phase squared voltage magnitude of head bus in p.u.;
$\boldsymbol{v}_i$	3 phase squared voltage column vector for a bus i in p.u.;
$\boldsymbol{v}_{1:N}$	3-phase squared voltage magnitude (p.u.) of all non-head buses;

3 Formulation of Three Phase Distribution Optimal Power Flow

This study considers an $N+1$ bus unbalanced radial distribution network, characterized by a graph $\mathcal{G}(\mathcal{N} \cup \{0\}, \mathcal{L})$, with phases $\phi \in \boldsymbol{\Phi} : \{a, b, c\}$. The index of the head bus in the feeder where the IDSO's substation is located is denoted by 0, and all other non-head buses are the elements of the set $\mathcal{N} : \{1, 2, \cdots, N\}$. The distribution network has N distinct line segments denoted by set $\mathcal{L} \subseteq \{\mathcal{N} \cup \{0\}\} \times \{\mathcal{N} \cup \{0\}\}$, with $(i,j) \in \mathcal{L}$ denoting a line segment between buses i and j. For each bus i, its squared voltage magnitude is denoted by v_i and the real and reactive power injection by p_i and q_i, respectively. For a line (i,j), the resistance and reactances are denoted by $R_{(i,j)}$ and $X_{(i,j)}$, respectively, and the real and reactive power flow by $P_{(i,j)}$ and $Q_{(i,j)}$, respectively. The node voltages, injections and line flows can easily be represented as 3×1 column vectors as shown in (1), where all items are in p.u.

$$\begin{gathered} \boldsymbol{v}_i = [v_i^\phi]_{\phi \in \boldsymbol{\Phi}}; \quad \boldsymbol{p}_i = [p_i^\phi]_{\phi \in \boldsymbol{\Phi}}; \quad \boldsymbol{q}_i = [q_i^\phi]_{\phi \in \boldsymbol{\Phi}}; \\ \boldsymbol{P}_{(i,j)} = [P_{(i,j)}^\phi]_{\phi \in \boldsymbol{\Phi}}; \quad \boldsymbol{Q}_{(i,j)} = [Q_{(i,j)}^\phi]_{\phi \in \boldsymbol{\Phi}}. \end{gathered}$$

Also, line resistances and reactances in p.u. for a line (i,j) can be equivalently represented in a 3×3 matrix form as:

$$\boldsymbol{R}_{(i,j)} = [R_{(i,j)}^{\phi\phi}]_{\phi \in \boldsymbol{\Phi}}; \quad \boldsymbol{X}_{(i,j)} = [X_{(i,j)}^{\phi\phi}]_{\phi \in \boldsymbol{\Phi}}.$$

Then the LinDistFlow equation from [4] can be represented as follows, where node k is the parent of node i, $\forall k, i \in \mathcal{N}, \forall (i \to j) \in \mathcal{L}$:

$$\sum_{j: i \to j} \boldsymbol{P}_{(i,j)} = \boldsymbol{P}_{(k,i)} + \boldsymbol{p}_i \ , \tag{3a}$$

$$\sum_{j: i \to j} \boldsymbol{Q}_{(i,j)} = \boldsymbol{Q}_{(k,i)} + \boldsymbol{q}_i \ , \tag{3b}$$

$$\boldsymbol{v}_i - \boldsymbol{v}_j = 2 \cdot (\bar{\boldsymbol{R}}_{(i,j)} \boldsymbol{P}_{(i,j)} + \bar{\boldsymbol{X}}_{(i,j)} \boldsymbol{Q}_{(i,j)}). \tag{3c}$$

Where,

$$\bar{\boldsymbol{R}}_{(i,j)} = \Re(\boldsymbol{W}) \odot \boldsymbol{R}_{(i,j)} + \Im(\boldsymbol{W}) \odot \boldsymbol{X}_{(i,j)}, \tag{4a}$$

$$\bar{\boldsymbol{X}}_{(i,j)} = \Re(\boldsymbol{W}) \odot \boldsymbol{X}_{(i,j)} - \Im(\boldsymbol{W}) \odot \boldsymbol{R}_{(i,j)}. \tag{4b}$$

Here, $\boldsymbol{W}$ is the phase coupling matrix:

$$\boldsymbol{W} = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}; \quad \omega = e^{j\frac{2\pi}{3}},$$

$\odot$ denotes the Hadamard Product.

Using [1], the above linear power flow equations can be conveniently represented by a graph-based matrix formalism. Let the branch-by-node incidence matrix for $\mathcal{G}$ be denoted by $\tilde{\boldsymbol{C}} \in \{-\boldsymbol{I}^3, 0, \boldsymbol{I}^3\}^{(N)\times(N+1)}$ representing a three-phase connection structure of the lines and buses, the entries can then be expressed as $\forall i \in \{\mathcal{N} \cup \{0\}\}, \forall l \in \mathcal{L}$:

$$\tilde{\boldsymbol{C}}_{il} = \begin{cases} \boldsymbol{I}^3, & \text{if line } l \text{ originates from node } i, \\ -\boldsymbol{I}^3, & \text{if line } l \text{ feeds node } i, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Where $\boldsymbol{I}^3$ is an identity matrix of size 3. The LinDistFlow equations in (3) can therefore be represented as:

$$\tilde{\boldsymbol{C}}^T \boldsymbol{P} = \boldsymbol{p}; \quad \tilde{\boldsymbol{C}}^T \boldsymbol{Q} = \boldsymbol{q}; \tag{6a}$$

$$\tilde{\boldsymbol{C}} \boldsymbol{v} = 2\left[\boldsymbol{D_r} \boldsymbol{P} + \boldsymbol{D_x} \boldsymbol{Q}\right]. \tag{6b}$$

where, $\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^{3(N+1)\times 1}$, and $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{3N\times 1}$ defined as

$$\boldsymbol{v} = \left[\boldsymbol{v}_i\right]_{i\in\mathcal{N}\cup\{0\}}, \quad \boldsymbol{p} = \left[\boldsymbol{p}_i\right]_{i\in\mathcal{N}\cup\{0\}}, \quad \boldsymbol{q} = \left[\boldsymbol{q}_i\right]_{i\in\mathcal{N}\cup\{0\}},$$
$$\boldsymbol{P} = \left[\boldsymbol{P}_{(i,j)}\right]_{(i,j)\in\mathcal{L}}, \quad \boldsymbol{Q} = \left[\boldsymbol{Q}_{(i,j)}\right]_{(i,j)\in\mathcal{L}}.$$

$\boldsymbol{D_r}$ and $\boldsymbol{D_x} \in \mathbb{R}^{3N\times 3N}$ are diagonal matrices, defined as:

$$\boldsymbol{D_r} = \operatorname{diag}((\bar{\boldsymbol{R}}_{(i,j)})_{(i,j)\in\mathcal{L}}), \tag{7a}$$

$$\boldsymbol{D_x} = \operatorname{diag}((\bar{\boldsymbol{X}}_{(i,j)})_{(i,j)\in\mathcal{L}}). \tag{7b}$$

Let the first column of $\tilde{\boldsymbol{C}}$ be denoted by the column vector $\boldsymbol{c_0}$ that corresponds to the head bus, while the rest of the matrix as $\boldsymbol{C}$. Therefore, the incidence matrix can be represented more compactly as:

$$\tilde{\boldsymbol{C}} = \left[\boldsymbol{c}_0 \quad \boldsymbol{C}.\right] \tag{8}$$

Therefore, the equation (6) can be written as:

$$\boldsymbol{c}_0^T \boldsymbol{P} = \boldsymbol{p}_0; \quad \boldsymbol{c}_0^T \boldsymbol{Q} = \boldsymbol{q}_0; \tag{9a}$$

$$\boldsymbol{C}^T \boldsymbol{P} = \boldsymbol{p}_{1:N}; \quad \boldsymbol{C}^T \boldsymbol{Q} = \boldsymbol{q}_{1:N}; \tag{9b}$$

$$\boldsymbol{c}_0 \boldsymbol{v}_0 + \boldsymbol{C} \boldsymbol{v}_{1:N} = 2\left[\boldsymbol{D_r} \boldsymbol{P} + \boldsymbol{D_x} \boldsymbol{Q}\right]. \tag{9c}$$

Since, $\tilde{\boldsymbol{C}}$ is invertible [2],(9c) can be written as

$$\boldsymbol{v}_{1:N} = -\boldsymbol{C}^{-1} \boldsymbol{c}_0 \boldsymbol{v}_0 + 2\boldsymbol{C}^{-1}\left[\boldsymbol{D_r} \boldsymbol{P} + \boldsymbol{D_x} \boldsymbol{Q}\right]. \tag{10}$$

For a tree topology network, the following holds[1],

$$\tilde{\boldsymbol{C}} \boldsymbol{I}^3_{N+1} = \boldsymbol{0}, \tag{11a}$$

$$\boldsymbol{c}_0 \boldsymbol{I}^3 + \boldsymbol{C} \boldsymbol{I}^3_N = 0, \tag{11b}$$

$$\boldsymbol{I}^3_N = -\boldsymbol{C}^{-1} \boldsymbol{c}_0. \tag{11c}$$

Using (9b) and (11c), (10) can be re-written as:

$$\boldsymbol{v}_{1:N} = \boldsymbol{I}^3_N \boldsymbol{v}_0 + 2\boldsymbol{C}^{-1}\left[\boldsymbol{D_r} \boldsymbol{P} + \boldsymbol{D_x} \boldsymbol{Q}\right]. \tag{12}$$

This concludes the Three Phase LinDistFlow power flow model adopted in this software.

[1] Here, $\boldsymbol{I}^3_{N+1}$ denotes a column vector of size $N+1$ with individual entries being $\boldsymbol{I}^3$

References

[1] S. H. Low, "Lecture Notes for EE/CS/EST 135: Power System Analysis - A Mathematical Approach," 2024, accessed: 2024-09-19. [Online]. Available: https://netlab.caltech.edu/assets/pdf/Low-201909-ee135Notes-Ch1toCh17AppNoSol-20240430.pdf

[2] H. Zhu and H. J. Liu, "Fast Local Voltage Control Under Limited Reactive Power: Optimality and Stability Analysis," *IEEE Transactions on Power Systems*, vol. 31, no. 5, pp. 3794–3803, 2016.

[3] R. Cheng, L. Tesfatsion, and Z. Wang, "A Consensus-Based Transactive Energy Design for Unbalanced Distribution Networks," *IEEE Transactions on Power Systems*, vol. 38, no. 1, pp. 114–128, 2023.

[4] M. Baran and F. Wu, "Network reconfiguration in distribution systems for loss reduction and load balancing," *IEEE Transactions on Power Delivery*, vol. 4, no. 2, pp. 1401–1407, 1989.