Xiaoxin Liao
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Absolute Stability of Nonlinear Control Systems

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Second Edition

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Preface

The first edition of this book was published in 1993 by Kluwer Academic Publishers and the Science Press of China. In the preface of the first edition, we briefly introduced the history, main results and new developments of the Lurie control system, as well as the well-known Lurie problem. We also pointed out the importance of studying Lurie control systems in both theoretical development and applications. The materials presented in the first edition were chosen mainly from the author's results on the necessary and sufficient conditions, as well as simple, practically useful algebraic sufficient conditions for the absolute stability of various Lurie control systems. The characteristics of these results are: theoretically, to give many possible necessary and sufficient conditions of absolute stability of various nonlinear control systems; in applications, to derive simple enough and even constructive algebraic sufficient conditions from these theoretical necessary and sufficient conditions for use in practical work and in methodology. Whilst promoting the extensive use of modern methods and tools such as M-matrices, K-class functions, Dini-derivatives, partial stability, and set stability, we have not neglected traditional methods and results. Related works produced by other researchers have also been introduced.

In the ten years that have passed since the first edition of the book was published, new theories and methodologies have developed rapidly, and many new results have been obtained in this area. Also, applications have been extended to many frontier areas such as neural networks, chaos control, and chaos synchronization, etc. These developments have been the driving motivation behind the substantial revision and update this book has received for its second edition.

Since 1944, the study of the absolute stability of Lurie control systems, and its applications, has attracted many researchers. The well-known Lurie problem and the concept of absolute stability are presented, which is of universal significance both in theory and practice. The field of absolute stability was, until the end of the 1950s, monopolized mainly by Russian scholars such as A.I. Lurie, M.A. Aizeman, A.M. Letov, and others. Then, at the beginning of the 1960s, American mathematicians, such as J.P. LaSalle, S. Lefschetz, and R.E. Kalman, engaged themselves in this field. Meanwhile, the Romanian scholar Popov presented a well-known frequency criterion and consequently made a decisive breakthrough in the study of absolute stability. Since then, V.A. Yacobovich, R.E. Kalman, K.R. Meyer, and others have devoted themselves to the study of equivalent relations between Lurie's method (integral term and quadratic Lyapunov function method) and Popov's frequency method. In the first 30 years, this greatly stimulated the development of Lyapunov stability theory, and the importance of Lyapunov theory was finally recognized by the control society and mathematicians. The study of the Lurie problem has led to the development of new mathematical methods and techniques, such as Lurie–Lyapunov type V function, S program, the well-known Popov frequency method, and positive real function theory. It has also established various relations between complex function, linear algebra, calculus, and linear matrix inequality. The study of the Lurie control system has not only resulted in new mathematical theory and methodology, but also laid the foundations for the development of the modern control theory (mainly based on nonlinear controls) from the classical control theory (mainly based on time-invariant linear controls) leading to, in particular, the development of a new and important control area; robust control.

During the past two decades, more and more evidences have been found revealing the close relation between the absolute stability of the Lurie system and chaos control, chaos synchronization and the stability of neural networks. This has poured new vigor into this classical research area. In the early 1990s, Pecoron and Carroll were the first to use the chaos synchronization principal to design two chaotic circuits that could be synchronized, an accomplishment which was than applied to secure communications. This development was able to change opinion that chaos cannot be controlled, nor synchronized. This finding has, in turn, attracted many more researchers in to this challenging research area. Since then, despite more results being produced, a general theory of chaos synchronization has not been completely established. Curran and Chua were the first ones to suggest employing absolute stability to develop a more general theory and methodology for chaos synchronization, as the study of absolute stability has proved useful in providing new information and ideas.

Across the world, an increase in the study of neural networks began when a new neural model, now called Hopfield neural network, was proposed by Hopfield and Tank. They used electronic circuit simulations to solve nonlinear algebraic or transcendental equations, with automated process. This new method, due to its novel advantage, was immediately applied to many areas such as optimal computation, signal processing, and pattern recognition. The demanding of optimal computation has made it possible to relate it to the idea of the absolute stability of the Lurie control system. Following this, a new concept of the absolute stability of neural networks was proposed, establishing the relation between the existence and uniqueness of equilibrium points and the Lyapunov local stability and global attractive. Such properties are not dependent on the particular form of activation functions, or the strength of currents in circuits. This is indeed an extension of the absolute stability of the Lurie system to neural networks.

The developments in Chaos theory and neural networks have promoted new studies on absolute stability. In the last two decades, we have continuously studied the absolute stability of the Lurie system and obtained some new results. Therefore, we believed it necessary to revise the book and publish a second edition to catch up with the new developments in this area. Based on the six chapters of the first edition, the second edition has been expanded to 13 chapters. Amongst these, five chapters are completely new, and two chapters have been expanded by adding new results. We have also added an introductory chapter (Chap. 1) to give the reader a brief guide to the book. The new chapters are chapters 1, 8, 10, 12, and 13. On the following two pages, we have briefly described each chapter for you.

Chapter 1 is an introduction, presenting the Lurie problem, the relation between the Lyapunov stability and the absolute stability of the Lurie control system, as well as the recent developments in this area.

Chapter 2 describes the main tools and principal results, which play fundamental roles throughout the book.

Chapter 3 has been revised and expanded from the first edition. In particular, we describe the Lurie problem and Lurie system, and present three classical methods for studying absolute stability; the Lurie–Lyapunov V-function method (quadratic form plus integral terms); the Lurie method based on V function and S-program; and the classical Popov frequency criterion and the simplified Popov criterion.

Chapter 4 is devoted to the Lurie control systems described by ordinary differential equations. We obtain the necessary and sufficient conditions for absolute stability of various Lurie control systems. The absolute stability of these systems is equivalent to that of partial variables and the matrix Hurwitz stability.

Chapter 5 presents some necessary and sufficient algebraic conditions for the absolute stability of several special classes of Lurie-type control systems.

Non-autonomous systems are considered in Chapter 6.

In Chapter 7, we discuss the absolute stability of control systems with multiple nonlinear control terms.

The material presented in Chapters 2–7 and 11 are mainly taken from the first edition of the book, but have been improved and expanded by the addition of new results.

Chapter 8 presents the results for the robust absolute stability of interval control systems including the Lurie system and the Yocubovich system. Strictly speaking, for a control system, the form of feedback control function is not known exactly, but is known to belong to certain types of functions. Also, the information on the system coefficients is usually given in upper and lower bounds, not exact values. In the past two decades, the stability study for linear control systems with parameters varied within a finite closed interval has been a hot topic in control society. However, there has been a lack of results obtained on the stability of nonlinear control systems with varied parameters in an interval. Thus, we have added this chapter in order to present the new results obtained in this direction.

In Chapter 9, the theory and methodology for continuous Lurie control systems are generalized to study discrete Lurie control systems described by difference equations. This topic was only discussed in one section of the first edition. Due to the wide applications of discrete Lurie control systems in real applications, such as chaotic systems and neural networks, it became necessary to expand this into a chapter of its own. Moreover, the first edition only discussed the direct Lurie control system. In this new edition, we have added the results for Lurie control systems with loop feedback.

The absolute stability of the time-delayed and neutral Lurie control system is considered in Chapter 10. The first edition did not include the absolute stability of Lurie control systems, described by differential and difference equations, but instead jumped from ordinary differential equations straight to functional equations (FDE). Though FDE has general theoretical foundations, most practical problems and applications are based on differential and difference equations. Thus, we have added this chapter to introduce the results we recently obtained in this direction.

Chapter 11 considers the Lurie control systems described by functional differential equations. The results obtained by applying the ideas and methods described in Chapter 4 to abstract functional differential equations are presented here. Also, new results on control systems with multiple nonlinear feedback controls are given.

Chapter 12 introduces the concept of absolute stability for neural networks, and particular attention is given to the Hopfield neural network. The inherent relation between neural networks and Lurie control systems is discussed.

Finally, the theory of absolute stability is applied to consider the new area of chaos control and chaos synchronization in Chapter 13. The main attention is focused on the use of absolute stability of Lurie control systems to consider the synchronization of two Chua circuits. New concepts are proposed for the absolutely exponential stability of error systems, and the absolutely exponential stabilization using feedback controls when the error system is not absolutely stable.

The remaining four chapters of the first edition have also been modified, with new results added or new formulas used. We have omitted some parts of the old edition which are no longer useful in applications. Also, we corrected a few minor typographical errors from the first edition.

Finally, we would like to thank Mr. Z. Chen and Mr. F. Xu for their patience in typing a partial manuscript of the book, and we thank the support received from NNSF (No. 60274007, 60474011), NSERC (No. R2686A02), and PREA. Also our thanks go to the Department of Applied Mathematics, The University of Western Ontario, for hosting the visit of one of the authors (X.X. Liao) whilst the book was under preparation.

London, Canada, *Xiaoxin Liao*
November 2005 *Pei Yu*

Contents

1

Introduction

As we know, the stability is the most fundamental problem in the design of automatic control systems, since only a stable system can keep working properly under disturbances [3, 4, 6, 8–10]. In fact, automatic control theory began from Maxwell's study on the stability of Watt centrifugal governor. When one designs a control system, one first needs to consider some type of stability for the system and then investigate other problems.

Among various stability theories, the Lyapunov stability is still the most important one [56–63, 101–109, 128, 164, 186, 187]. However, the main difficulty in analyzing Lyapunov stability is how to determine a Lyapunov function for a given system. There does not exist general rules for constructing Lyapunov functions, but are merely based on a researcher or designer's experience and some particular techniques. The first-order approximation method and many results obtained for the first and second critical cases demand very restrictive requirements on nonlinear terms, which cause difficulties in applications. Moreover, it should be pointed out that the Lyapunov stability theory is mainly applicable for local stability, while many practical problems need to consider globally asymptotic stability or even globally exponential stability.

1.1 Lurie Control System

To solve the stability problem, in 1944, the former Soviet Union's scholar and control expert, A. I. Lurie, based on the study of many practical control systems including aircraft automatic control, proposed the following well-known Lurie nonlinear control system [10, 12, 13, 16–18, 100, 116]:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= Ax + bf(\sigma), \\ \sigma &= c^{\mathrm{T}}x, \end{aligned} \tag{1.1}$$

where $x \in R^n$, $b, c \in R^n$, $A \in R^{n\times n}$, and $\sigma f(\sigma) > 0$ when $\sigma \neq 0$. Lurie developed a new method to deal with the stability of nonlinear control system (1.1), which is now called *nonlinear isolation method* in the literature, that is, the nonlinear part of the system is isolated so that the system becomes one with closed-loop form. The principal of the isolation method is illustrated in Fig. 1.1, where the box containing L

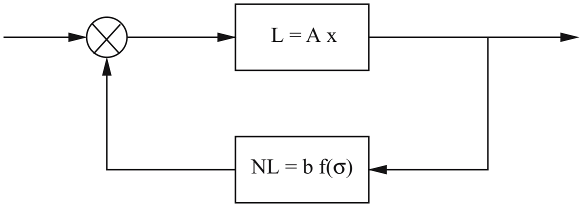

Fig. 1.1 Nonlinear isolation method

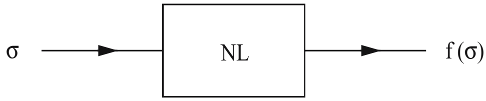

Fig. 1.2 Nonlinear part

denotes the linear part of the system, and the box having NL represents the nonlinear part of the system. Note that the L and NL parts in Fig. 1.1 do not necessarily correspond to the open-loop part and the closed-loop part of a real control system. In the early studies, the NL part was usually assumed to be a simple nonlinear function, but was later quickly extended to more complicated nonlinear cases [159].

Lurie et al. raised the following question: when the nonlinear part of a control system is isolated, but without having much information about the nonlinear characteristics, how to choose the parameters of the nonlinear part such that the system is globally asymptotically stable?

Let us consider a simple case [159]. Suppose that the NL in Fig. 1.1 represents a single-input single-output time-invariant nonlinear part. Let the input be σ and output $f(\sigma)$, and both σ and f are scalars (see Fig. 1.2). All we know about f is that $f(\sigma)$ is a continuous function, located in a sectional region in the first and third quadrants of the σ-f plane.

In many practical problems, the nonlinearity of a system is not exactly known or there exist many nondeterministic elements or disturbances, or the form of f is known but is very difficult to apply. In all these situations, the nonlinear isolation method is a useful approach.

1.2 Lurie Absolute Stability and Lyapunov Stability

With many research works, the isolation method became an important and particularly useful tool for the study of absolute stability [16–18, 48]. Later, the well-known control expert, V. M. Popov [120–123], developed a frequency criterion for absolute stability theory, which is a natural extension of the stability of linear feedback systems. Since then, the absolute stability theory played the main role in the stability analysis of nonlinear control systems.

Lurie control system or Lurie control problem not only plays an important role in nonlinear control systems, but also greatly influences other research areas. From the applied mathematics point of view, the Lurie system has proposed the important concept for the family of differential equations (containing an infinite number of equations), or multivalued differential equations, or nondeterministic systems. Many new mathematical methodologies and techniques have been generated from the study of absolute stability [2, 49, 62, 63, 69, 99, 105, 137, 138, 140, 144–146, 149–151, 159]. For instance, the positive real function theory and the Fourier transform are successfully employed in Popov principal; various methods and techniques of constructing Lyapunov functions promoted the development of matrix theory, leading to the most practically useful linear matrix inequality. General speaking, the study of absolute stability has greatly promoted the development of applied mathematics.

On the other hand, from the control theory point of view, the Lurie control system actually proposed the concept of robust control of nondeterministic systems, which is the pioneer work of robust stability. Moreover, many practical nonlinear control systems can be transformed to a unified Lurie control system to deal with. In the monograph [159], many applications using Lurie absolute stability have been discussed, such as transient stability of electrical network, stability of computer network, nonlinear capacity limit problem of optimal regulator, theory of nuclear reaction, etc.

The development of Lurie absolute stability also motivated the further development of the Lyapunov theory [97, 155–157]. As mentioned above, the Lyapunov stability is usually restricted to local behavior of a system, while the Lurie control system studies the globally asymptotic stability of equilibrium points, which is not for single equation, but for a family of differential equations consisting an infinite number of differential equations. This certainly poses new difficulty and challenges. Though still based on the Lyapunov function, the study of absolute stability has been greatly generalized. Because of the demanding of practical applications, the classical Lyapunov theory, which was developed 50 years earlier than the Lurie absolute stability theory, has reattracted many researchers' attention. In the 1960s, an American mathematician LaSalle pointed out that the stability theory is attracting the attention of mathematicians in the world, the Lyapunov direct method is widely used by engineers, and the stability theory has become a standard part in training control engineers. The stability theory started from the research of former Soviet Union's scholars finally became an area attracting all researchers in the world.

1.3 Recent Development of Absolute Stability Theory in New Areas

With the development of modern science and technology, the stability theory still plays very important and fundamental roles. For example, since 1990s, Pecoron et al. [117, 118] discovered that under certain conditions two chaotic systems can be synchronized, which changed the long time viewpoint: chaos cannot be controlled, nor can be synchronized. However, though many results for chaos control and chaos synchronization have been obtained, the general theory of chaos synchronization is still under development. Therefore, Curran and Chua have proposed [20, 141] that the general theory of chaos synchronization should be established under the frame of Lurie absolute stability. Chua and his coworkers did many pioneer work in this direction [142, 143]. We have also spent years in research in this direction [81–84].

This has made it possible for the Lurie absolute stability theory to be applied to a new research area.

Another new research area, which has been very active in the past two decades, is the study of dynamics of neural networks, which has resulted in new idea and methods for the development of optimal computation, connective memory, pattern recognition, signal processing, etc. To solve the optimal computation problem, many researchers have proposed the concept and methodology for the absolute stability of neural networks, which is basically motivated from the idea of the absolute stability of Lurie control systems.

From the above discussions, we have seen that the Lurie absolute stability is indeed important for further researches from both theoretical and application aspects.

1.4 The Lurie Problem

The basic Lurie problem is [62] What is the necessary and sufficient condition for the equilibrium point of system (1.1) to be globally asymptotically stable (i.e., absolutely stable)? There are many publications (papers, monographs, reports, etc.) appearing in the literature considering the absolute stability of the Lurie system and various generalized Lurie control systems. Recent results may be found in, for example, monographs [2, 62, 63, 99, 105, 159] and papers [31–36]. People hoped to obtain the necessary and sufficient condition to answer the well-known Lurie problem in the form of a, b, c in finite conditions. However, for a long time, whether Lurie or Popov, all were able to obtain only the sufficient condition for the absolute stability of single-valued nonlinear Lurie systems, yet these sufficient conditions depend upon the existence of unknown Lyapunov matrix solutions (satisfying Lyapunov matrix equation) and unknown parameters (satisfying some rational inequality).

The main difficulty of the Lurie problem is due to that the system has an infinite number of equations. Using a, b, c finite form to find the necessary and sufficient condition may not be possible (might be only possible for very specific systems).

The authors have been studying the absolute stability of Lurie control system for many years, and have found that the partial variable stability, which was developed by the former Soviet Union's mechanist, is useful in the study of the stability of Lurie systems. Also, many modern mathematical tools such as Dini derivative, K-function, M-matrix, linear matrix inequality [68], etc. can be efficiently applied to investigate the absolute stability of Lurie control systems. Most results of this book are based on the authors' research work in the past two decades.

1.5 The Aizerman Problem and Aizerman Conjecture

To this end, we would like to briefly introduce the Aizerman problem [2] (or the Aizerman conjecture), which made an important contribution in the study of global stability of nonlinear control systems and absolute stability of Lurie control systems. In 1949, the former Soviet Union control expert, M. A. Aizerman, proposed the global stability problem for the following nonlinear control system with separable variables:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} f_{ij}(x_j), \quad f_{ij}(0) = 0, \qquad i = 1, 2, \ldots, n. \tag{1.2}$$

This type of control systems has particular meaning in control systems, since a Lurie control system in the form of (1.1) can be transformed into (1.2) via a proper topological transform (or a full-rank linear transform). Further, system (1.2) is a natural generalization of linear systems, since a linear control system with some linear part replaced by nonlinear part becomes the system (1.2).

In early stage of stability study, when people encountered the difficulty in analyzing the global stability of nonlinear control systems, Aizerman proposed the following method for the global stability of system (1.2): rewrite (1.2) as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} f_{ij}(x_j) = \sum_{j=1}^{n} \frac{f_{ij}(x_j)}{x_j}\, x_j := \sum_{j=1}^{n} a_{ij}(x_j) x_j\,, \tag{1.3}$$

and consider the formal "characteristic equation":

$$\det(\lambda I - A\,(a_{ij}(x_i))_{n\times n}) = a_n(x)\lambda^n + a_{n-1}(x)\lambda^{n-1} + \cdots + a_1(x)\lambda + a_0(x) = 0, \tag{1.4}$$

where $(-1)^i a_i(x)$ is the summation of all the principal minor determinants of the function matrix $A(a_{ij}(x_j))_{n\times n}$ (which is the function of $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$). Denote the generalized Hurwitz determinants by

$$a_n(x) = 1, \quad D_1(x) := a_1(x),\ \ldots,$$

$$D_k(x) := \det \begin{bmatrix} a_1(x) & a_3(x) & \cdots & a_{2k-1}(x) \\ a_0(x) & a_2(x) & \cdots & a_{2k-2}(x) \\ 0 & a_1(x) & \cdots & a_{2k-3}(x) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_k(x) \end{bmatrix}, \quad k = 1, 2, \ldots, n,$$

where $a_j(x) = 0$ if $j > k$. Then Aizrman conjecture is as follows [1,2]: the necessary and sufficient conditions for the zero solution of (1.2) being globally stable are

$$D_k(x) > 0, \ \ 1 \le k \le n, \ \prod_{i=1}^{n} x_i \ne 0. \tag{1.5}$$

When (1.2) is a linear system, the Aizrman conjecture is the Hurwitz criterion [37]; when (1.2) is a nonlinear system but $n = 1$, the conjecture is true. However, when $n = 2$, mathematician Krossovskii found a counter example, and thus disproved the conjecture. However, in the earlier studies, because of very few methods that could be used, there were many scholars who investigated under what extra conditions added to (1.5) would the zero solution of system (1.2) became globally stable but restricted to cases $n = 2, 3$. Very little has been achieved for $n \ge 4$. Now, almost nobody continues to follow the Aizrman conjecture to study the global stability of the zero solution of system (1.2), but the proposal of the Aizrman conjecture had made an important contribution in putting forward the research of the stability problem.

1.6 Modern Mathematical Tools for Absolute Stability

In 1964, when most people were not familiar with M-matrix and Dini derivative, Li [64, 65] broke the limit of the Aizrman conjecture, and creatively constructed general nonsmooth linear type of Lyapunov function to obtain the sufficient condition for the equilibrium point of system (1.2), with arbitrary dimension n being globally stable. The results can be found in the papers [64, 65] and the monograph [127].

In 1979, the author [70, 71] further constructed general, variable-separable, radially unbounded, and positive definite function and obtained a series of constructive sufficient conditions for the equilibrium points of system (1.2) being globally stable. This has resulted in the development of many methods and techniques for studying the global stability and instability of system (1.2) which, used as the tools for this book, will be introduced in the next chapter.

The systems with separable variables, first proposed by Aizerman, not only have generality in control systems, but also are applicable to recently developed biomath models (e.g., Volterra model, Gilpin-Ayala model), neural network model (e.g., Cohen-Grossheig model), which can all be considered as the special cases of (1.2), but with new contents. Therefore, the tools described in Chap. 2 can also be used by the researchers who are working in the areas of biomath and neural networks.

2

Principal Theorems on Global Stability

In this chapter, we introduce the main tools and principal results that play fundamental roles for the whole book, such as Lyapunov function, K-class function (or wedge function), Dini-derivative, M-matrix, Hurwitz matrix, positive (negative) definite matrix; and the principal theorems on global stability, partial global stability, and global stability of sets.

Partial materials presented in this chapter are due to Lyapunov [97], Hahn [38], Malkin [102] (Sect. 2.1), Yoshizawa [170] (Sect. 2.2), Rumyantsev [133, 134] (Sect. 2.4.2), Liao [68,69] (Sect. 2.3–2.6), and Liao [69,70] (Sect. 2.7 and 2.8).

2.1 Lyapunov Function and K-Class Function

Suppose that $W(x) \in C[R^n, R^1]$, that is, $W\colon R^n \to R$ is continuous, $W(0) = 0$; $V(t,x) \in C[I \times R^n, R^1]$, $V(t,0) \equiv 0$, where $I = [t_0, +\infty)$.

Definition 2.1. *The function $W(x)$ is said to be positive (negative) definite if $W(x) \geq 0$ $(-W(x) \geq 0)$ and $W(x) = 0$ if and only if $x = 0$. The function $W(x)$ is said to be positive (negative) semi-definite if $W(x) \geq 0$ $(-W(x) \geq 0)$.* [41]

Definition 2.2. *The function $W(x)$ is said to be radially unbounded, positive definite if $W(x)$ is positive definite and $W(x) \to +\infty$ as $\|x\| \to +\infty$.*

Definition 2.3. *The function $V(t,x)$ is said to be positive definite if there is a positive function $W(x)$ such that $V(t,x) \geq W(x)$. The function $V(t,x)$ is said to be negative definite if $-V(t,x)$ is positive definite.*

Definition 2.4. *The function $V(t,x)$ is said to have infinitesimal upper bound if there exists a positive definite function $W_1(x)$ such that $\|V(t,x)\| \leq W_1(x)$. The function $V(t,x)$ is said to be radially unbounded, positive definite if there exists a radially unbounded, positive definite function $W_2(x)$ such that $V(t,x) \geq W_2(x)$.*

The positive or negative definite functions are usually called Lyapunov functions.

In the following, we introduce K-class function.

If a function $\varphi \in [R^+, R^+]$ (where $R^+ := [0, +\infty)$), φ is continuous, strictly monotone increasing, and $\varphi(0) = 0$, we call φ a *K-class function*, denoted by $\varphi \in K$.

If $\varphi \in K$ and $\lim_{r\to+\infty} \varphi(r) = +\infty$, then $\varphi(r)$ is called a *radially unbounded, K-class function*, denoted by $\varphi \in KR$.

Among the positive definite functions and the K-class functions, some essential equivalence relations hold.

Lemma 2.5. *Given a positive definite function $W(x)$, there exist two functions, φ_1, $\varphi_2 \in K$ such that*

$$\varphi_1(\|x\|) \le W(x) \le \varphi_2(\|x\|). \tag{2.1}$$

Proof. For any $R > 0$, we prove that (2.1) holds on $\|x\| \le R$. Let $\varphi(r) = \inf_{r\le\|x\|\le\le R} W(x)$. Evidently, we have $\varphi(0) = 0$, $\varphi(r) > 0$ for $r > 0$, and $\varphi(r)$ is a monotone nondecreasing function on $[0,R]$. (It may not be strictly monotonic.) Now, we proceed to prove that $\varphi(r)$ is continuous. Since $W(x)$ is continuous, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}
\varphi(r_2) - \varphi(r_1) &= \inf_{r_2\le\|x\|\le R} W(x) - \inf_{r_1\le\|x\|\le R} W(x) \\
&:= \inf_{r_2\le\|x\|\le R} W(x) - W(x_0) \\
&\le W(x_1) - W(x_0) \\
&< \varepsilon \qquad (\text{when } \|x_1 - x_0\| \le r_2 - r_1 < \delta),
\end{aligned}$$

we take $x_1 = x_0$ for $x_0 \in D_2 = \{x | r_2 \le \|x\| \le R\}$ and x_1 is an intersection point of the ray Ox_0 and $\|x\| = r_2$ for $x_0 \in D_1 = \{x | r_1 \le \|x\| \le R\}$.

Let $\varphi_1(r) := \frac{r\varphi(r)}{R} \le \varphi(r)$. Evidently, we have $\varphi_1(0) = 0$ and further if $0 \le r_1 < r_2 \le R$ we get

$$\varphi_1(r_1) = \frac{r_1\varphi(r_1)}{R} \le \frac{r_1\varphi(r_2)}{R} < \frac{r_2\varphi(r_2)}{R} = \varphi_1(r_2).$$

Thus $\varphi_1(r)$ is strictly monotone increasing and hence $\varphi_1 \in K$. If $\psi(r) := \max_{\|x\|\le r} W(x)$, then it follows that $\psi(0) = 0$. By the same argument, we can prove that ψ is a monotone nondecreasing and continuous function. Choosing $\varphi_2(r) := \psi(r) + kr$ $(k > 0)$ we have

$$\varphi_2(r_1) = \psi(r_1) + kr_1 \le \psi(r_2) + kr_1 < \psi(r_2) + kr_2 = \varphi_2(r_2).$$

Hence, $\varphi_2(r)$ is strictly monotone increasing and $\varphi_2(r) \in K$. From above, it is inferred that

$$\begin{aligned}
\varphi_1(\|x\|) \le \varphi(\|x\|) &:= \inf_{\|x\|\le\|\xi\|\le R} W(\xi) \le W(x) < \max_{\|\xi\|\le\|x\|} W(\xi) \\
&:= \psi(\|x\|) < \varphi_2(r_2).
\end{aligned}$$

Thus,

$$\varphi_1(\|x\|) \le W(x) \le \varphi_2(\|x\|).$$

This completes the proof of Lemma 2.5. □

Lemma 2.6. *For a given radially unbounded, positive definite function $W(x)$, there must exist two functions $\varphi_1(r)$, $\varphi_2(r) \in KR$ such that*

$$\varphi_1(\|x\|) \le W(x) \le \varphi_2(\|x\|).$$

Consequently, without loss of generality, the positive definite functions and the radially unbounded, positive definite functions can be replaced, respectively, by K-class functions and radially unbounded K-class functions.

2.2 Dini Derivative

Suppose $f(t) \in C[I, R^1]$, $I = [t_0, +\infty)$. For any $t \in I$, the following four derivatives

$$D^+ f(t) := \overline{\lim_{h \to 0^+}} \frac{1}{h}\Big(f(t+h) - f(t)\Big),$$

$$D_+ f(t) := \underline{\lim_{h \to 0^+}} \frac{1}{h}\Big(f(t+h) - f(t)\Big),$$

$$D^- f(t) := \overline{\lim_{h \to 0^-}} \frac{1}{h}\Big(f(t+h) - f(t)\Big),$$

$$D_- f(t) := \underline{\lim_{h \to 0^-}} \frac{1}{h}\Big(f(t+h) - f(t)\Big)$$

are called *right upper derivative, right lower derivative, left upper derivative, and left lower derivative* of $f(t)$ at t, respectively. They are all called Dini-derivative [170].

Sometimes the Dini-derivative may be $\pm\infty$, otherwise there always exists Dini-derivative. In particular, when $f(t)$ satisfies local Lipschitz condition, the four Dini-derivatives are finite. Moreover, the derivative of $f(t)$ exists if and only if the four derivatives are equal.

For a continuous function, the relation between the monotonicity and the definite sign of the Dini-derivative is as follows.

Theorem 2.7. *If $f(t) \in C[I, R^1]$, then $f(t)$ is monotone nondecreasing on I if and only if $D^+ f(t) \ge 0$ for $t \in I$.*

Proof. The *necessity* is obvious.

Sufficiency. First, we suppose $D^+ f(t) > 0$ on I. If there are two points $\alpha, \beta \in I$ and$\alpha < \beta$, $f(\alpha) > f(\beta)$, then there exist μ satisfying $f(\alpha) > \mu > f(\beta)$ and some point $t \in [\alpha, \beta]$ such that $f(t) > \mu$.

Let ξ be the supreme of those points. Then $\xi \in (\alpha, \beta)$, and the continuity of $f(t)$ leads to $f(\xi) = \mu$. Therefore, for $t \in [\xi, \beta]$, it follows

$$\frac{f(t) - f(\xi)}{t - \xi} < 0.$$

Hence, we obtain $D^+f(\xi) \leq 0$, which contradicts the hypothesis. Thus, $f(t)$ is monotone nondecreasing.

Next, assume that $D^+f(t) \geq 0$. For any $\varepsilon > 0$ we have

$$D^+\Big(f(t) + \varepsilon t\Big) = D^+f(t) + \varepsilon \geq \varepsilon > 0.$$

As a consequence, $f(t) + \varepsilon t$ is monotone nondecreasing on I. Since ε is arbitrary, $f(t)$ is monotone nondecreasing on I. □

Remark 2.8. If $D^+f(t) \geq 0$ is replaced by $D_+f(t) \geq 0$, the sufficient condition of Theorem 1.2.1 still holds because the latter implies the former.

Similarly, if we replace $D^+f(t) \geq 0$ by$D^-f(t) \geq 0$, it suffices to change the supreme of the points satisfying $f(t) > \mu$ to the infirm of the points satisfying $f(t) < \mu$. We may further intensify $D^-f(t) \geq 0$ to be $D^-f(t) \geq 0$, and thus any of the four derivatives is not less than zero, each of which implies that $f(t)$ is monotone nondecreasing.

In the following, we consider the Dini-derivative of a function along the solution of a differential equation.

Let a system of differential equations be given by

$$\dot{x} = f(t,x), \tag{2.2}$$

where the dot denotes differentiation with respect to time t, and $f(t,x) \in C[I \times R^n, R^n]$.

Theorem 2.9. [170] *Suppose that $V(t,x) \in C[I \times \Omega, R^1]$, where $\Omega \subset R^n$, Ω is a neighborhood containing the origin, and $V(t,x)$ satisfies local Lipschitz condition in x for t (expressed by $V(t,x) \in C_0(x)$), that is,*

$$|V(t,x) - V(t,y)| \leq L\|x-y\|.$$

Then the right upper derivative and the right lower derivative of $V(t,x)$ along the solution $x(t)$ of (2.2) have the forms

$$D^+V(t,x(t)) = \overline{\lim_{h\to 0^+}} \frac{1}{h}\Big[V(t+h, x+hf(t,x)) - V(t,x)\Big], \tag{2.3}$$

$$D_+V(t,x(t)) = \underline{\lim_{h\to 0^+}} \frac{1}{h}\Big[V(t+h, x+hf(t,x)) - V(t,x)\Big]. \tag{2.4}$$

Proof. Assume that $x(t)$ is the solution in the region $I \times \Omega$. For $(t,x) \in I \times \Omega$ and $0 < h \ll 1$, there exists a neighborhood U of (t,x) such that $U \subset I \times \Omega$, $(t+h, x+hf(t,x)) \in U$, and $(t+h, x(t,x)) \in U$. Let L be the Lipschitz constant of $V(t,x)$ in x on U. Making use of the Taylor expansion and the Lipschitz condition we arrive at

$$\begin{aligned}
&V(t+h, x(t,x)) - V(t,x(t)) \\
&= V(t+h, x+hf(t,x)+h\varepsilon) - V(t,x) \\
&< V(t+h, x+hf(t,x)) + Lh|\varepsilon| - V(t,x),
\end{aligned}$$

where $\varepsilon \to 0$ as $h \to 0$. Hence,

$$\begin{aligned} D^+V(t,x(t)) &:= \overline{\lim_{h\to 0^+}}\ \frac{1}{h}\left[V(t+h,x(t,x)) - V(t,x(t))\right] \\ &\le \overline{\lim_{h\to 0^+}}\ \frac{1}{h}\left[V(t+h,x+hf(t,x)) + Lh|\varepsilon| - V(t,x)\right] \\ &= \overline{\lim_{h\to 0^+}}\ \frac{1}{h}\left[V(t+h,x+hf(t,x)) - V(t,x)\right]. \end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned} & V(t+h,x(t,x+h)) - V(t,x(t)) \\ = {} & V(t+h,x+hf(t,x)+h\varepsilon) - V(t,x) \\ \ge {} & V(t+h,x+hf(t,x)) - Lh|\varepsilon| - V(t,x). \end{aligned}$$

Thus,

$$\begin{aligned} D^+V(t,x(t)) &:= \overline{\lim_{h\to 0^+}}\ \frac{1}{h}\left[V(t+h,x(t,x+h)) - V(t,x(t))\right] \\ &\ge \overline{\lim_{h\to 0^+}}\ \frac{1}{h}\left[V(t+h,x+hf(t,x)) - V(t,x)\right]. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we find that

$$D^+V(t,x(t)) = \overline{\lim_{h\to 0^+}}\ \frac{1}{h}\left[V(t+h,x+hf(t,x)) - V(t,x)\right].$$

Thus, (2.3) is true. The proof of (2.4) goes along the same line. □

If $V(t,x)$ has a continuous partial derivative of the first order, then along the solution $x(t)$ of (2.2)

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\cdot f(t,x) = \frac{\partial V}{\partial t} + \mathrm{grad}V \cdot f(t,x)$$

and

$$D^+V(t,x(t)) = D^+V(t,x(t)) = D^-V(t,x(t)) = D_-V(t,x(t)) = \frac{\mathrm{d}V(t,x(t))}{\mathrm{d}t}.$$

By Theorem 2.7, $V(t,x(t))$ is nondecreasing (nonincreasing) along the solution of (2.2) if and only if

$$D^+V(t,x(t)) \ge 0 \qquad (D^+V(t,x(t)) \le 0).$$

The significance of Theorem 2.9 lies in the fact that one does not need to know the solution when calculating the Dini-derivative of $V(t,x(t))$ along the solution of (2.2).

2.3 *M*-Matrix, Hurwitz Matrix, Positive (Negative) Definite Matrix

In this section, we introduce several useful matrices, including M matrix, Hurwitz matrix, and positive (negative) definite matrix. First we discuss M matrix, which is a practically applicable mathematical tool in the study of discrete or continuous dynamical systems, computational mathematics, and statistics. M matrix has many equivalent conditions. Here, we present only some of them which will be used in this book. More details about M matrix can be found in the book of Ortega and Rheinhold [113].

Definition 2.10. *A real symmetric matrix* $A\,(a_{ij})_{n\times n}$ *is called a nonsingular* M *matrix (simply* M *matrix) if the following conditions are satisfied:*

1. $a_{ii} > 0\ (i = 1, 2, \ldots, n),\ a_{ij} \le 0\ \ (i \ne j,\ \ i, j = 1, 2, \ldots, n)$.
2. The following n determinants are greater than zero:

$$\det\begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & \ldots & \vdots \\ a_{i1} & \cdots & a_{ii} \end{bmatrix} > 0 \qquad i = 1, 2, \ldots, n.$$

There are many equivalent conditions for M matrix, among them frequently used conditions are listed below.

I. $a_{ii} > 0\ (i = 1, 2, \ldots, n),\ a_{ij} \le 0\ \ (i \ne j,\ i, j = 1, 2, \ldots, n)$, and $A^{-1} \ge 0$, that is, A^{-1} is a nonnegative matrix.
II. $a_{ii} > 0\ (i = 1, 2, \ldots, n),\ a_{ij} \le 0\ \ (i \ne j,\ i, j = 1, 2, \ldots, n)$, and there exist constants $c_i > 0$ such that $\sum_{j=1}^{n} c_j a_{ij} > 0, i = 1, 2, \ldots, n$.
III. $a_{ii} > 0,\ a_{ij} \le 0,\ i, j = 1, 2, \ldots, n$, and there exist constants $d_j > 0$ such that $\sum_{i=1}^{n} d_i a_{ij} > 0, j = 1, 2, \ldots, n$.
IV. $a_{ii} > 0\ (i = 1, 2, \ldots, n),\ a_{ij} \le 0\ \ (i \ne j,\ i, j = 1, 2, \ldots, n)$, and for any positive real numbers $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}}$, the algebraic equations $A\boldsymbol{x} = \boldsymbol{\xi}$ has positive solution $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}}$.
V. $a_{ii} > 0,\ a_{ij} \le 0, i, j = 1, 2, \ldots, n, i \ne j$, and there exists a positive diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$, such that $PA + A^{\mathrm{T}}P$ is positive definite.
VI. $a_{ii} > 0,\ a_{ij} \le 0\ \ (i \ne j,\ i, j = 1, 2, \ldots, n)$, and $-A$ is a Hurwitz matrix, that is, all eigenvalues of A have negative real parts.
VII. $a_{ii} > 0\ (i = 1, 2, \ldots, n),\ a_{ij} \le 0\ \ (i \ne j,\ i, j = 1, 2, \ldots, n)$, and the spectral radius of the matrix $G := (I - D^{-1}A)$, $\rho(G) < 1$ (i.e., all the eigenvalues of G are located inside the unit circle on the complex plane, which is said that G is Schur stable [77]). Here, $D = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$, D^{-1} denotes the inverse of D.

The conditions I given in Definition 2.10 are relatively easy to verify since it has constructive computing program.

Next, we briefly discuss Hurwitz matrix. If all eigenvalues of $A\,(a_{ij})_{n\times n}$ have negative real parts, then A is called a Hurwitz matrix [68]. Let

$$\det(\lambda I - A) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \quad (a_n = 1).$$

Then the criterion for A being a Hurwitz matrix is the well-known Hurwitz criterion: If $a_i > 0,\ i = 0, 1, \ldots, n$, then A is a Hurwitz matrix if and only if $\Delta_i > 0,\ i = 1, 2, \ldots, n$, where

$$\Delta_1 := a_1,\ \ \Delta_2 := \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix},\ \ \cdots,\ \ \Delta_n := \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ & & & a_{n-1} & a_{n-2} \\ a_{2n-1} & \cdots & & & a_n \end{bmatrix} = \Delta_{n-1}\,a_n,$$

in which $a_s = 0$ for $s < 0$ or $s > n$.

Finally, we introduce the Sylvester condition for a matrix to be positive (or negative) definite. Let the corresponding quadratic form of the real symmetric matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

be $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} A\,\boldsymbol{x}$. Then the positive definite (negative definite), sign change, etc. can be determined by the Sylvester condition, simply called A positive (negative) definite, or positive semi-definite (negative semi-definite). Let

$$\Delta_i = \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & \cdots & \vdots \\ a_{i1} & \cdots & a_{ii} \end{bmatrix}.$$

The well-known Sylvester criterion can be stated as follows.

A is positive definite if and only if $\Delta_i > 0,\ i = 1, 2, \ldots, n$.

A is positive semi-definite if and only if $\Delta_i \geq 0,\ i = 1, 2, \ldots, n$.

A is negative definite if and only if $(-1)^i\Delta_i > 0,\ i = 1, 2, \ldots, n$.

A is semi-negative definite if and only if $(-1)^i\Delta_i \geq 0,\ i = 1, 2, \ldots, n$.

The above criteria or conditions show that when one wants to verify whether a matrix is an M matrix, or sign definite or Hurwitz matrix, one has to check all the signs of n determinants. This is certainly very tedious and time consuming. Now we introduce a method that needs only to check the last one of the n determinants. The calculations of the other determinants can be obtained in the process of computing the last determinant, implying that the first $n-1$ determinants are completely dependent upon the last determinant.

Now present a simple method [68, 69].

Definition 2.11. *The change of a determinant caused by multiplying a positive number to a row (column), or a row (column) multiplied by an arbitrary number and added to another row (column) is called a sign-invariant transform (SIT). The transformations keeping every sub-principal determinant invariant is called a series of sign-invariant transformations (SSIT).*

It is easy to see that a determinant can be always transformed to a triangle determinant via sign-invariant transform, that is,

$$|A| := \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \xrightarrow{\text{(SSIT)}} \quad \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & b_{nn} \end{vmatrix} := |B|$$

or

$$\xrightarrow{\text{(SSIT)}} \quad \begin{vmatrix} c_{11} & 0 & \cdots & 0 \\ c_{21} & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ c_{21} & \cdots & \cdots & c_{nn} \end{vmatrix} := |C|.$$

Theorem 2.12. *(1) Suppose $A = A^{\mathrm{T}}$. If $|A|$ can be transformed to $|B|$ or $|C|$ via a series of sign-invariant transformations, then $A > 0$ $(A \geq 0)$ if and only if $b_{ii} > 0$ or $c_{ii} > 0$ $(b_{ii} \geq 0$ or $c_{ii} \geq 0)$; $A < 0$ $(A \leq 0)$ if and only if $b_{ii} < 0$ or $c_{ii} < 0$ $(b_{ii} \leq 0$ or $c_{ii} \leq 0)$ $i = 1, 2, \ldots, n$.*

(2) If $a_{ii} > 0$, $a_{ij} \leq 0$ $i \neq j$, $i, j = 1, 2, \ldots, n$, A is an M matrix if and only if $b_{ii} > 0$ (or $c_{ii} > 0$), i = 1, 2, …, n.

(3) When $a_i > 0$ $(i = 0, 1, \ldots, n-1)$, $f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ is a Hurwitz polynomial if and only if

$$\Delta_{n-1} = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & \cdots & \cdots & \cdots \\ \vdots & & & \vdots & \\ & & & & a_{n-3} \\ a_{2n-3} & \cdots & \cdots & \cdots & a_{n-1} \end{vmatrix} \quad \xrightarrow{\text{(SSIT)}} \quad \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1(n-1)} \\ 0 & b_{22} & \cdots & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & b_{(n-1)(n-1)} \end{vmatrix}$$

or

$$\xrightarrow{\text{(SSIT)}} \quad \begin{vmatrix} c_{11} & 0 & \cdots & 0 & 0 \\ c_{21} & c_{22} & & & \vdots \\ \vdots & & & \ddots & \vdots \\ c_{(n-1)1} & \cdots & \cdots & \cdots & c_{(n-1)(n-1)} \end{vmatrix},$$

where $b_{ii} > 0$ ($rc_{ii} > 0$), $i = 1, 2, \ldots, n-1$.

Proof. We prove only the case when A is a symmetric, positive definite matrix. Other cases can be similarly proved and thus not repeated. Since

$$A > 0 \iff a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots \quad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0.$$

So from $a_{11} > 0$, we know $b_{11} > 0$. Then it follows from $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$ that $\begin{vmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{vmatrix} > 0$. Hence, from $a_{11} > 0, b_{11} > 0$, and $\begin{vmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{vmatrix} > 0$ we know that $b_{22} > 0$. Continuing this procedure shows that $b_{ii} > 0$ $(i = 1, 2, \ldots, n)$.

On the other hand, suppose $b_{ii} > 0$ $(i = 1, 2, \ldots, n)$, we can deduce that

$$\tilde{\Delta}_1 := b_{11} > 0, \quad \tilde{\Delta}_2 := \begin{vmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{vmatrix} > 0, \quad \cdots \quad \tilde{\Delta}_n := \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \\ 0 & \cdots & b_{nn} \end{vmatrix} > 0,$$

and then we can further prove that

$$\Delta_1 := a_{11} > 0, \quad \Delta_2 := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots \quad \Delta_n := \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0.$$

The proof is complete. □

Example 2.13. Verify if the following matrix is an M matrix:

$$A = \begin{bmatrix} 4 & -1 & -2 & -3 \\ -1 & 3 & -2 & -1 \\ -\frac{1}{2} & -1 & 4 & 0 \\ 0 & -1 & -1 & 5 \end{bmatrix}.$$

It is easy to see from the matrix that $a_{ii} > 0$, $a_{ij} \leq 0$, $i \neq j$, but the diagonal elements are not dominating. We can apply the above method to verify that A is an M matrix.

$$\begin{aligned} A &= \begin{bmatrix} 4 & -1 & -2 & -3 \\ -1 & 3 & -2 & -1 \\ -\frac{1}{2} & -1 & 4 & 0 \\ 0 & -1 & -1 & 5 \end{bmatrix} \quad (\text{2nd row} - \text{4th row} \times 2;\ \ \text{1st row} + \text{2nd row} \times 4) \\ &\Longrightarrow \begin{bmatrix} 4 & * & * & * \\ 0 & 11 & -10 & -7 \\ 0 & -5 & 10 & 1 \\ 0 & -1 & -1 & 5 \end{bmatrix} \quad (\text{3rd row} - \text{3rd row} \times 5;\ \ \text{2nd row} + \text{4th row} \times 11) \end{aligned}$$

$$\Longrightarrow \begin{bmatrix} 4 & * & * & * \\ 0 & 11 & * & * \\ 0 & 0 & 15 & -24 \\ 0 & 0 & -21 & 48 \end{bmatrix} \qquad (\text{3rd column} \times \frac{1}{3};\ \ \text{4th column} \times \frac{1}{24})$$

$$\Longrightarrow \begin{bmatrix} 4 & * & * & * \\ 0 & 11 & * & * \\ 0 & 0 & 5 & -1 \\ 0 & 0 & -7 & 2 \end{bmatrix} \qquad (\text{3rd column} + \text{4th column} \times \frac{7}{2})$$

$$\Longrightarrow \begin{bmatrix} 4 & * & * & * \\ 0 & 11 & * & * \\ 0 & 0 & \frac{3}{2} & * \\ 0 & 0 & 0 & 2 \end{bmatrix} := |B|,$$

which clearly shows that all the diagonal elements of the above matrix are positive, implying that A is an M matrix.

Remark 2.14. The elements marked by $*$ in the above procedure of calculation are not explicitly expressed since they do not affect the result. That is, they do not influence the calculation in the next step. When all a_{ij} $(i > j)$ are transformed to zero, a_{ij} no longer play any role and can be ignored.

2.4 Principal Theorems on Global Stability

In this section, we first introduce global stability with respect to all the variables of a system, and then partial global stability with respect to partial variables of the system.

2.4.1 Global Stability

Consider an n-dimensional autonomous system:

$$\dot{x} = f(x), \quad f(0) = 0, \tag{2.7}$$

where $x \in R^n$, $f \in C[R^n, R^n]$. Suppose that the solution $x(t, t_0; x_0)$ of the initial value problem (2.7) is unique.

Definition 2.15. *The zero solution of (2.7) is globally asymptotically stable (globally stable for short) if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\|x(t, t_0; x_0)\| < \varepsilon \quad \text{for all} \quad t \geq t_0 \quad \text{if} \quad \|x_0\| < \delta(\varepsilon),$$

and for any $x_0 \in R^n$,

$$\lim_{t \to +\infty} x(t, t_0; x_0) = 0.$$

Definition 2.16. *The set $E = \{x|x(t,t_0;x_0), t \geq t_0\}$ is called a positive semi-trajectory of (2.7) through x_0 at $t = t_0$; if $x_0 \neq 0$, then E is a nontrivial positive semi-trajectory. x^* is called an ω-limiting point of $x(t,t_0;x_0)$ if there is a sequence $\{t_k\}$ with $t_k \to +\infty$ as $k \to +\infty$ such that $x^* = \lim_{t_k \to +\infty} x(t_k,t_0;x_0)$.*

Note that $\Omega(x_0)$ is the set of ω-limiting points of the trajectory through x_0.

Lemma 2.17. *Suppose that x^* is an ω-limiting point of $x(t,t_0;x_0)$. Then the points on the positive semi-trajectory of $x(t,t_0;x^*)$ are all the ω-limiting points of $x(t,t_0;x_0)$.*

Proof. From the hypothesis, there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that

$$x^* = \lim_{n \to +\infty} x(t_n,t_0;x_0) = 0.$$

For an arbitrary point $x(t,t_0;x_0)$ on the trajectory through x^*, the property of the group for the solutions of autonomous systems and the relationship of continuous dependence of the solution on the initial value lead to

$$\lim_{t_n \to +\infty} x(t_n+t,t_0;x_0) = \lim_{n \to +\infty} x(t,t_0,x(t_n,t_0;x_0)) = x(t,t_0;x^*).$$

In other words, $x(t,t_0;x^*)$ is an ω-limiting point of $x(t,t_0;x_0)$. □

Theorem 2.18. [5] *If there exists a radially unbounded, positive definite differentiable function $V(x) \in C[R^n,R]$ such that*

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.7)} \leq 0, \tag{2.8}$$

and the set $M := \{x : \frac{dV}{dt} = 0\}$ does not contain any entire semi-trajectory of nonzero solutions of (2.7) except $x = 0$, then the zero solution of (2.7) is globally stable.

Proof. Since $V(x)$ is a radially unbounded, positive definite function, there exists $\varphi \in KR$ such that

$$V(x) \geq \varphi(\|x\|).$$

From the continuity of $V(x)$ and $V(0) = 0$, $V(x) \geq 0$, so for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$V(x_0) < \varphi(\varepsilon) \quad \text{if} \quad \|x_0\| < \delta(\varepsilon).$$

It follows from (2.8) that

$$\varphi(\|x(t,t_0;x_0)\|) \leq V(x(t,t_0;x_0)) \leq V(x_0) < \varphi(\varepsilon) \tag{2.9}$$

for all $t \geq t_0$. Since $\varphi \in KR$, (2.9) implies that

$$\|x(t,t_0;x_0)\| < \varepsilon.$$

Therefore, the zero solution of (2.7) is stable.

Similar to (2.9), for any $x_0 \in R^n$, we get

$$\varphi(\|x(t,t_0;x_0)\|) \leq V(x(t,t_0;x_0)) \leq V(x_0),$$

thus

$$\|x(t,t_0;x_0)\| \leq \varphi^{-1}(V(x_0)) := M.$$

Hence, according to the Weierstrass's accumulation principle, we see that the set $\Omega(x_0)$ is nonempty and bounded.

Now we proceed to prove that $\Omega(x_0) = \{0\}$. If this is not true, then there is a sequence $\{t_n\}$ satisfying $t_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} x(t_n,t_0;x_0) = x^* \neq 0.$$

In virtue of the positive definiteness of $V(x)$ and $\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leq 0$, we know that $V(x(t,t_0;x_0))$ is monotone nonincreasing, continuous, and nonnegative. In our case, this gives

$$\lim_{t\to+\infty} V(x(t),t_0;x_0)) = V(x^*) > 0. \tag{2.10}$$

Consider the trajectory $x(t,t_0;x^*)$ through x^*. Since

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.7)} \leq 0,$$

it follows that

$$V(x(t,t_0;x^*)) \leq V(x^*).$$

If for every $t \geq t_0$, $V(x(t,t_0;x^*)) = V(x^*)$, then there exists

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.7)} \equiv 0.$$

Thus the set M contains the entire positive semi-trajectory of the nonzero solution $x(t,t_0;x^*)$, which is inconsistent with the hypothesis. Then there exists $t_1 \geq t_0$ such that

$$V(x(t_1,t_0;x^*)) < V(x^*).$$

By Lemma 2.17, we find that $x(t_1,t_0;x^*)$ is an ω-limiting point of $x(t,t_0;x^*)$. Thus there exist a sequence $\{t_n\}$ with $\{t_n^*\} \to +\infty$ such that

$$\lim_{n\to+\infty} x(t_n^*,t_0;x^*) = x(t_1,t_0;x^*).$$

Hence, we obtain

$$\lim_{n\to+\infty} x(t_n^*,t_0;x^*) = V(x(t_1,t_0;x^*)) < V(x^*),$$

which leads to a contradiction with (2.10), and therefore, $\Omega = \{0\}$, that is,

$$\overline{\lim_{t\to+\infty}}\, x(t,t_0;x_0) = \underline{\lim_{t\to+\infty}}\, x(t,t_0;x_0) = 0 = \lim_{t\to+\infty} x(t,t_0;x_0). \qquad \square$$

Corollary 2.19. *If there exists a radially unbounded positive definite function $V(x) \in [R^n, R]$ such that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.7)}$ is negative definite, then the zero solution of (2.7) is globally stable.*

2.4.2 Partial Global Stability

In the following, a notion of partly global stability of zero solution for (2.7) will be introduced [133, 134].

Let $x = (y,z)^{\mathrm{T}}$, $y = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$, $z = (x_{m+1}, x_{m+2}, \ldots, x_n)^{\mathrm{T}}$.

Definition 2.20. *The zero solution of (2.7) is said to be globally stable with respect to (w.r.t.) the partial variable y if for any* $\varepsilon > 0$*, there exists* $\delta(\varepsilon) > 0$ *such that*

$$\|y(t,t_0;y_0)\| < \varepsilon \quad \text{for all} \quad t \geq t_0 \quad \text{if} \quad \|x_0\| < \delta(\varepsilon),$$

and for any $x_0 \in R^n$*, there exists*

$$\lim_{t \to +\infty} \|y(t,t_0;x_0)\| = 0.$$

Definition 2.21. *A function* $V(x) \in C[R^n, R^1]$ *is said to be radially unbounded positive definite w.r.t y if there exists a function* $\varphi \in KR$ *such that* $V(x) \geq \varphi(\|y\|)$*. A function* $V(x) \in C[R^n, R^1]$ *is negative definite w.r.t. y if there exists a function* $\varphi \in K$ *such that* $V(x) \leq -\varphi(\|y\|)$.

Theorem 2.22. *If there is a function* $V(x) \in C[R^n, R^1]$ *satisfying*

$$\varphi_1(\|y\|) \leq V(x) \leq \varphi_2\Big(\Big(\sum_{i=1}^{k} x_i^2\Big)^{1/2}\Big), \quad m \leq k \leq n \tag{2.11}$$

with φ_1, $\varphi_2 \in KR$*; and*

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(2.7)} \leq -\psi\Big(\Big(\sum_{i=1}^{k} x_i^2\Big)^{1/2}\Big), \quad \psi \in K, \tag{2.12}$$

then the zero solution of (2.7) is globally stable w.r.t. the partial variable y.

Proof. Since $V(x)$ is continuous and $V(0) = 0$, for given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$V(x_0) < \varphi_1(\varepsilon) \qquad \text{if} \quad \|x_0\| < \delta(\varepsilon).$$

Equations (2.11) and (2.12) yield

$$\varphi_1(\|y(t,t_0;x_0)\|) \leq V(x(t,t_0;x_0)) \leq V(x_0) < \varphi_1(\varepsilon).$$

Thus we have

$$\|y(t,t_0;x_0)\| < \varepsilon,$$

implying that the zero solution of (2.7) is stable w.r.t. the partial variable y.

Next, we prove that

$$\lim_{t \to +\infty} V(x(t,t_0;x_0)) = 0 \qquad \text{for any} \quad x_0 \in R^n;$$

thus the expression

$$\varphi_1(\|y(t,t_0;x_0)\|) \leq V(x(t,t_0;x_0)) \ \rightarrow \ 0 \quad \text{as} \quad t \rightarrow +\infty$$

implies

$$\|y(t,t_0;x_0)\| \rightarrow \ 0 \quad \text{as} \quad t \rightarrow +\infty.$$

If this is not true, then there exists $x_0 \in R^n$ such that

$$V(x(t,t_0;x_0)) \ \not\rightarrow \ 0 \quad \text{as} \quad t \rightarrow +\infty.$$

Using (2.12) we derive

$$\lim_{t\rightarrow+\infty} V(x(t,t_0;x_0)) = V_{+\infty},$$

then

$$V(x(t,t_0;x_0)) \geq V_{+\infty} > 0.$$

It follows from (2.11) that

$$\left(\sum_{i=1}^{k} x_i^2(t,t_0;x_0)\right)^{1/2} \geq \varphi_2^{-1}(V_\infty). \tag{2.13}$$

Applying the expressions (2.12) and (2.13), we write

$$0 \leq V(x(t,t_0;x_0)) \leq V(x(t_0)) - \psi(\varphi^{-1}(V_\infty))(t-t_0). \tag{2.14}$$

However, when $t \gg t_0$, the expression (2.14) does not hold; thus

$$\lim_{t\rightarrow+\infty} V(x(t,t_0;x_0)) = 0$$

and

$$\lim_{t\rightarrow+\infty} \|y(t,t_0;x_0)\| = 0.$$

The proof is complete. □

2.5 Global Stability of Sets

Let $M \subset R^n$ be a manifold or an arbitrary set of points.

For convenience, we define

$$d(x,M) := \inf_{y\in M} \|x-y\|,$$

that is, $d(x,M)$ is the distance from x to M.

Definition 2.23. *The solution of (2.7) is said to be globally stable w.r.t. the set M if for any $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that $d(x_0,M) < \delta(\varepsilon)$ implies*

$$d(x(t,t_0;x_0),M) < \varepsilon \qquad \text{for all } \ t \geq t_0,$$

and for any $x_0 \in R^n$,

$$\lim_{t\rightarrow+\infty} d(x(t,t_0;x_0),M) = 0.$$

Theorem 2.24. *Suppose that $V(x) \in C[R^n, R^1]$ and that $V(x)$ satisfies*

$$\varphi_1(d(x,M)) \le V(x) \le \varphi_2(d(x,M)), \qquad \varphi_1,\, \varphi_2 \in KR,$$

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.7)} \le -\psi(d(x,M)), \qquad \psi \in K.$$

Then the solution of (2.7) is globally stable w.r.t. the set M.

Proof. For any $\varepsilon > 0$, choosing $\delta(\varepsilon) := \varphi_2^{-1}(\varphi_1(\varepsilon))$, we write

$$\varphi_1(d(x(t,t_0;x_0),M)) \le V(x(t,t_0;x_0)) \le V(x_0) \le \varphi_2(d(x_0,M)) < \varphi_2(\delta(\varepsilon))$$

if $d(x_0,M) < \delta(\varepsilon)$. Thus

$$d(x(t,t_0;x_0),M)) \le \varphi_1^{-1}(\varphi_2(\partial(\varepsilon))) = \varepsilon \qquad \text{for all } \ t \ge t_0.$$

In the following, we prove the validity of

$$d(x(t,t_0;x_0),M)) := d(x(t),M) \ \to\ 0 \quad \text{as} \quad t \to +\infty.$$

For any $\varepsilon > 0$, any $x_0 \in R^n$, and any $\eta > 0$, since

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \le -\psi(d(x(t),M)),$$

and

$$\varphi_1(d(x(t),M)) \le V(x(t)) \le \varphi_2(d(x(t),M)),$$

we have

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \le -\psi(\varphi_2^{-1}(V(x(t)))) \le 0,$$

that is,

$$\frac{\mathrm{d}V(x(t))}{\psi(\varphi_2^{-1}(V(x(t))))} \le -\mathrm{d}t. \tag{2.15}$$

Therefore, (2.15) yields

$$\int_{V(x_0)}^{V(x(t))} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \le -(t-t_0),$$

or

$$\int_{V(x(t))}^{V(x_0)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \ge t-t_0.$$

Suppose $V(x_0) \le \varphi_2(d(x_0,M)) \le \varphi_2(\eta)$, then

$$\begin{aligned} t-t_0 \le \int_{V(x(t))}^{V(x_0)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} &\le \int_{\varphi_1(d(x(t),M))}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \\ &= \int_{\varphi_1(d(x(t),M))}^{\varphi_1(\varepsilon)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} + \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))}. \end{aligned}$$

Defining

$$T = T(\varepsilon,\eta) > \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))},$$

it is easy to see that

$$\int_{\varphi_1(d(x(t),M))}^{\varphi_1(\varepsilon)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \ge t - t_0 - \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} > t - t_0 - T \ge 0$$

if $t \ge t_0 + T$. Hence,

$$\varphi_1(\varepsilon) > \varphi_1(d(x(t),M)),$$

that is,

$$d(x(t),M) < \varepsilon \qquad \text{if } t \ge t_0 + T(\varepsilon,\eta).$$

The proof is complete. □

2.6 Nonautonomous Systems

Consider an n-dimensional nonautonomous system [68]:

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \tag{2.16}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $f \in C[I \times R^n, R^n]$, $I = [t_0, +\infty)$. Suppose that the solution of the initial value problem (2.16) is unique, and let $y = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$ and $z = (x_{m+1}, x_{m+2}, \ldots, x_n)^{\mathrm{T}}$.

In analogy with Definition 2.20 and 2.23, we can formulate the definition of globally uniform stability of the zero solution of (2.16) (see [68]).

Theorem 2.25. *If there exists a function $V(t,x) \in C[I \times R^n, R^n]$, satisfying*

$$\varphi_1(\|x\|) \le V(t,x) \le \varphi_2(\|x\|), \qquad \varphi_1,\, \varphi_2 \in KR$$

and

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.16)} \le -\psi(\|x\|), \qquad \psi \in K,$$

then the zero solution of (2.16) is globally uniformly stable.

We can follow the proof of Theorem 2.22 to prove Theorem 2.25.

Theorem 2.26. *If there exists a function $V(t,x) \in C[I \times R^n, R]$ satisfying*

$$\varphi_1(\|y\|) \le V(t,x) \le \varphi_2\Big(\Big(\sum_{i=1}^{k} x_i^2\Big)^{1/2}\Big), \quad m \le k \le n$$

and

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.16)} \le -\psi\Big(\Big(\sum_{i=1}^{k} x_i^2\Big)^{1/2}\Big), \quad \psi \in K,$$

then the zero solution of (2.16) is globally uniformly stable w.r.t. the partial variable y.

The proof of this theorem goes along the same line as in Theorem 2.22.

2.7 Systems with Separable Variables

It will be shown later that by topological transformations a number of automatic control systems of different forms can be reduced to systems with separable variables or to systems with generalized separable variables. In this section, we discuss the systems with separable variables in detail.

Consider a nonlinear system with separable variables [68]:

$$\dot{x} = \Big(\sum_{j=1}^{n} f_{1j}(x_j),\ \ldots,\ \sum_{j=1}^{n} f_{nj}(x_j) \Big)^{\mathrm{T}}, \tag{2.17}$$

where the dot denotes differentiation with respect to time t, $f_{ij}(x_j) \in C[R^1, R^1]$, $f_{ij}(0) = 0$, $i, j = 1, 2, \ldots, n$. Suppose that the solution of the initial value problem (2.17) is unique.

Let $y = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$ and $z = (x_{m+1}, x_{m+2}, \ldots, x_n)^{\mathrm{T}}$. Then (2.17) can be written as

$$\begin{aligned} \dot{y} &= \Big(\sum_{j=1}^{n} f_{1j}(x_j),\ \ldots,\ \sum_{j=1}^{n} f_{mj}(x_j) \Big)^{\mathrm{T}}, \\ \dot{z} &= \Big(\sum_{j=1}^{n} f_{(m+1)j}(x_j),\ \ldots,\ \sum_{j=1}^{n} f_{nj}(x_j) \Big)^{\mathrm{T}}. \end{aligned} \tag{2.18}$$

Similar to the Sylvester's condition, we first establish a criterion of positive definiteness and negative definiteness of quadratic forms w.r.t. partial variables [68].

Let us assume that $A(a_{ij})_{n\times n} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is symmetric, where A_{11}, A_{12}, A_{21}, and A_{22} are $m \times m$, $m \times p$, $p \times m$, and $p \times p$ matrices, respectively, and $m + p = n$.

Definition 2.27. *The quadratic form*

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

is said to be positive (negative) definite w.r.t. y if there are constants $\varepsilon_i > 0$ $(i = 1, 2, \ldots, m)$ such that

$$x^{\mathrm{T}} A x \geq \sum_{i=1}^{m} \varepsilon_i x_i^2 \qquad \Big(x^{\mathrm{T}} A x \leq -\sum_{i=1}^{m} \varepsilon_i x_i^2 \Big).$$

Lemma 2.28. *The quadratic form $\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$ is positive (negative) definite w.r.t. y if and only if there exists a constant $\varepsilon > 0$ such that $\begin{bmatrix} A_{11} - \varepsilon I_{m\times m} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is positive semi-definite $\Big(\begin{bmatrix} A_{11} + \varepsilon I_{m\times m} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is negative semi-definite $\Big)$, where $I_{m\times m}$ is an $m \times m$ matrix.*

Proof. For illustration we prove the positive semi-definite case. The proof of the negative semi-definite is similar and omitted.

Necessity. Since $\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$ is positive definite w.r.t. y, there exist some constants $\varepsilon_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq \sum_{i=1}^{m} \varepsilon_i x_i^2.$$

Let $\varepsilon = \min_{1 \leq i \leq n} \varepsilon_i$. Then we can find

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq \sum_{i=1}^{m} \varepsilon_i x_i^2 \geq \varepsilon \sum_{i=1}^{m} x_i^2 = \begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon I_{m\times m} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Thus

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} - \varepsilon I_{m\times m} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq 0,$$

which indicates that $\begin{bmatrix} A_{11} - \varepsilon I_{m\times m} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is positive semi-definite. In particular, A_{11} is positive definite.

Sufficiency. The assumptions can be reduced to

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} - \varepsilon I_{m\times m} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq 0.$$

Thus we have

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq \sum_{i=1}^{m} \varepsilon x_i^2.$$

This implies our claim. □

Lemma 2.29. *If there exist functions $\varphi_i(x_i)$ on $(-\infty, +\infty)$ $(i = 1, 2, \ldots, n)$, which are continuous or have only finite discontinuous points of the first or third kind (i.e., at the discontinuous points, the left- and right-hand limits of $\varphi(x_i)$ exist) such that*

1. *$\varphi_i(x_i)x_i > 0$ for $x_i \neq 0$, $i = 1, 2, \ldots, m$; $\varphi(x_i)x_i \geq 0$, $i = m+1, \ldots, n$;*
2. *$\int_0^{\pm\infty} \varphi_i(x_i)\,\mathrm{d}x_i = +\infty$, $i = 1, 2, \ldots, m$;*
3. *There is a positive definite function $\psi(y)$ satisfying*

$$G(x) := \sum_{i=1}^{m} \varphi_i(x_i \pm 0) \sum_{i=1}^{m} f_{ij}(x_j) \leq -\psi(y),$$

then the zero solution of (2.18) is globally stable w.r.t. the partial variable y.

Proof. First of all, we construct the Lyapunov function

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} \varphi_i(x_i)\,\mathrm{d}x_i.$$

Obviously, conditions (1) and (2) imply that

$$V(x) \geq \sum_{i=1}^{m} \int_0^{x_i} \varphi_i(x_i)\,\mathrm{d}x_i := \varphi(y) \;\to\; +\infty \qquad \text{as} \quad \|y\| \;\to\; +\infty.$$

Hence, $V(x)$ is radially unbounded, positive definite w.r.t. y, and along the solution of (2.18), the Dini-derivative of $V(x)$ is given by

$$D^+V(x)\big|_{(2.18)} = \begin{cases} \sum\limits_{i=1}^{n} \varphi_i(x_i) \sum\limits_{j=1}^{m} f_{ij}(x_j) \text{ at the continuous points of } \varphi_i(x_i), \\ \qquad\qquad\qquad\qquad i = 1,\ldots,n; \\ \max\Big\{ \sum\limits_{i=1}^{n} \varphi_i(x_i{+}0) \sum\limits_{j=1}^{m} f_{ij}(x_j),\ \sum\limits_{i=1}^{n} \varphi_i(x_i{-}0) \sum\limits_{j=1}^{m} f_{ij}(x_j) \Big\} \\ \quad \text{at the discontinuous points of } \varphi_i(x_i),\ i = 1,\ldots,n. \end{cases}$$

Therefore, the condition (3) implies that

$$D^+V(x)\big|_{(2.18)} \leq -\psi(y).$$

As a result, the zero solution of (2.18) is globally stable w.r.t. the partial variable y. $\square$

Remark 2.30. When $m = n$, the conditions of Lemma 2.29 imply that the zero solution of (2.17) is globally stable w.r.t. all variables. In Theorem 2.31 given below, for $m = n$, the statement follows from the global stability of all variables.

Theorem 2.31. *If system (2.18) satisfies*

1. $f_{ii}(x_i)x_i < 0$ *for* $x_i \neq 0$, $i = 1, 2, \ldots, m$; $f_{ii}(x_i)x_i \leq 0$, $i = m+1, \ldots, n$;
2. $\int_0^{\pm\infty} f_{ii}(x_i)\,\mathrm{d}x_i = -\infty$, $i = 1, 2, \ldots, m$;
3. *There are constants* $c_i > 0$ $(i = 1, 2, \ldots, m)$, $c_j \geq 0$ $(j = m+1, \ldots, n)$, $\varepsilon > 0$ *such that*

$$A(a_{ij})_{n\times n} + \begin{bmatrix} \varepsilon I_{m\times m} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n} \quad \text{is negative semi-definite},$$

where

$$a_{ij}(x) = \begin{cases} -\dfrac{1}{2}\Big(\dfrac{c_i f_{ij}(x_j)}{f_{jj}(x_j)} + \dfrac{c_j f_{ji}(x_i)}{f_{ii}(x_i)} \Big), & x_i x_j \neq 0, \\ 0, & x_i x_j = 0, \end{cases} \qquad i, j = 1, 2, \ldots, n,$$

then the zero solution of (2.18) is globally stable w.r.t. the partial variable y.

Proof. We construct the Lyapunov function

$$V(x) = -\sum_{i=1}^{n} \int_0^{x_i} c_i f_{ii}(x_i)\,\mathrm{d}x_i.$$

Clearly, $V(x)$ is radially unbounded, positive definite w.r.t. y. This is because

$$V(x) \geq -\sum_{i=1}^{m} \int_0^{x_i} c_i f_{ii}(x_i)\,\mathrm{d}x_i := \varphi(y) \ \to\ +\infty \qquad \text{as} \quad \|y\| \ \to\ +\infty.$$

Now we prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.18)} := G(x) = -\sum_{i=1}^{n} c_i f_{ii}(x_i) \sum_{j=1}^{n} f_{ij}(x_j)$$

is negative definite w.r.t. y.

For any $x = \xi \in R^n$, without loss of generality we can assume that

$$\prod_{i=1}^{k} \xi_i \neq 0, \qquad \sum_{i=k+1}^{n} \xi_i^2 = 0, \quad m \leq k \leq n.$$

Then, we obtain

$$\begin{aligned}
G(\xi) &= -\sum_{i=1}^{k} c_i f_{ii}(\xi_i) \sum_{j=1}^{k} f_{ij}(\xi_j),\\
&= -\frac{1}{2} \sum_{i,j=1, i\neq j}^{k} \Big[c_i f_{ii}(\xi_i) f_{ij}(\xi_j) + c_j f_{jj}(\xi_j) f_{ji}(\xi_i)\Big],\\
&= -\sum_{i,j=1}^{k} c_i f_{ii}^2(\xi_i) - \sum_{i,j=1, i\neq j}^{k} \frac{1}{2}\left[\frac{c_i f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{c_j f_{ji}(x_i)}{f_{ii}(x_i)}\right] f_{ii}(\xi_i) f_{jj}(\xi_j),\\
&= \sum_{i=1}^{k} a_{ii}(\xi) f_{ii}^2(\xi_i) + \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) + \sum_{i,j=1}^{k} a_{ij}(\xi) f_{ii}(\xi_i) f_{jj}(\xi_j) - \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i)\\
&\leq -\sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) < 0.
\end{aligned}$$

Since ξ is arbitrary, we obtain that $G(x)$ is negative definite w.r.t. the partial variable y. Then the zero solution of (2.18) is globally stable w.r.t. the partial variable y. □

Theorem 2.32. *If system (2.18) satisfies the following conditions:*

1. *The condition* (1) *of Theorem 2.31;*
2. *There exist n functions $c_i(x_i)$ $(i = 1, 2, \ldots, n)$, which are continuous or have only finite discontinuous points of the first or third kind, and satisfy*
   $c_i(x_i) > 0$ for $x_i \neq 0$ and $\int_0^{\pm\infty} c_i(x_i)\,\mathrm{d}x_i = +\infty,\ i = 1, 2, \ldots, m;$
   $c_i(x_i) \geq 0,\ i = m+1, \ldots, n;$

3. *There exist functions* $\varepsilon_i(x_i) > 0$ $(i = 1, 2, \ldots, n)$ *such that*

$$\tilde{A}(\tilde{a}_{ij})_{n\times n} + \begin{bmatrix} \mathrm{diag}(\varepsilon_1(x_1) \ \cdots \ \varepsilon_m(x_m)) & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$

is negative semi-definite, where

$$\tilde{a}_{ij}(x) = \begin{cases} \dfrac{c_i(x_i) f_{ij}(x_j) + c_j(x_j) f_{ji}(x_i)}{2\sqrt{|f_{ii}(x_i) f_{jj}(x_j)|}}, & x_i x_j \neq 0, \\ 0, & x_i x_j = 0, \end{cases} \quad i, j = 1, 2, \ldots, n,$$

then the zero solution of (2.18) is globally stable w.r.t. the partial variable y.

Proof. We can set

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} c_i(x_i)\,\mathrm{d}x_i,$$

and then proceed along the line of Theorem 2.26 to complete the proof. □

Theorem 2.33. *If system (2.18) satisfies*

1. *The condition* (1) *of Theorem 2.31;*
2. *There exist n functions* $c_i > 0$ $(i = 1, 2, \ldots, m)$ *and* $c_j \geq 0$ $(j = m+1, \ldots, n)$ *such that*

$$\begin{cases} -c_j|f_{jj}(x_j)| + \displaystyle\sum_{i=1, i\neq j}^{n} c_i|f_{jj}(x_j)| < 0 \quad \text{for } x_j \neq 0, & j = 1, \ldots, m, \\ -c_j|f_{jj}(x_j)| + \displaystyle\sum_{i=1, i\neq j}^{n} c_i|f_{jj}(x_j)| \leq 0, & j = m+1, \ldots, n, \end{cases}$$

then the zero solution of (2.18) is globally stable w.r.t. the partial variable y.

Proof. We construct the Lyapunov function

$$V(x) = \sum_{i=1}^{n} c_i|x_i|.$$

Clearly,

$$V(x) \geq \sum_{i=1}^{m} c_i|x_i| := \varphi(y) \ \to \ +\infty \qquad \text{as} \quad \|y\| \ \to \ +\infty,$$

and $\varphi(y)$ is positive definite. On the other hand, we have

$$\begin{aligned} D^+V(x)\big|_{(2.18)} &\leq \sum_{j=1}^{n} \Big[-c_j|f_{jj}(x_j)| + \sum_{i=1, i\neq j}^{n} c_i|f_{ij}(x_j)| \Big] \\ &\leq \sum_{j=1}^{m} \Big[-c_j|f_{jj}(x_j)| + \sum_{i=1, i\neq j}^{n} c_i|f_{ij}(x_j)| \Big] \\ &< 0 \qquad \text{if} \quad y \neq 0. \end{aligned}$$

Therefore, the zero solution of (2.18) is globally stable w.r.t. the partial variable y. □

Theorem 2.34. *Suppose that system (2.18) satisfies the following conditions:*

1. *The condition* (1) *of Theorem 2.31;*
2. $\left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)}\right| \le b_{ij} = const.,\ i \ne j,\ i,j = 1,\ldots,n;$
3. $$\tilde{A} := \begin{bmatrix} 1 & -b_{21} & \cdots & -b_{n1} \\ -b_{21} & 1 & \cdots & -b_{n2} \\ \vdots & \vdots & & \vdots \\ -b_{1n} & -b_{2n} & \cdots & 1 \end{bmatrix} := \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

   where $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$, and $\tilde{A}_{22}$ are $m \times m$, $m \times p$, $p \times m$, and $p \times p$ matrices, respectively, and $\tilde{A}_{11}$, $\tilde{A}_{22}$, $I - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ are all M *matrices.*

Then the zero solution of (2.18) is globally stable w.r.t. the partial variable y.

Proof. For any $\xi = (\xi_1,\ldots,\xi_m)^{\mathrm{T}} > 0$, $\eta = (\eta_1,\ldots,\eta_p)^{\mathrm{T}} \ge 0$, and $m + p = n$, we consider the linear algebraic equations w.r.t. $c = (c_1,\ldots,c_m)^{\mathrm{T}}$ and $\tilde{c} = (\tilde{c}_1,\ldots,\tilde{c}_p)^{\mathrm{T}}$:

$$\begin{aligned} \tilde{A}_{11}\,c + \tilde{A}_{12}\,\tilde{c} &= \xi, \\ \tilde{A}_{21}\,c + \tilde{A}_{22}\,\tilde{c} &= \eta, \end{aligned}$$

or the equivalent ones:

$$\begin{aligned} \tilde{c} &= -\tilde{A}_{22}^{-1}\tilde{A}_{21}\,c + \tilde{A}_{22}^{-1}\,\eta, \\ c &= \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}\,c - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\,\eta + \tilde{A}_{11}^{-1}\,\xi. \end{aligned} \tag{2.19}$$

Since $\tilde{A}_{11}$, $\tilde{A}_{22}$ are M matrices, we have $\tilde{A}_{11}^{-1} \ge 0$, $\tilde{A}_{22}^{-1} \ge 0$. But $\tilde{A}_{12} \le 0$ and $\xi > 0$, $\eta \ge 0$; therefore, there exist

$$-\tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\eta \ge 0 \qquad \text{and} \qquad \tilde{A}_{11}^{-1}\,\xi > 0.$$

Since $I - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ is an M matrix, the second equation in (2.19) has a positive solution w.r.t. c, and the first one in (2.19) has a nonnegative solution w.r.t. $\tilde{c}$. Thus, the conditions in Theorem 2.33 are satisfied. Therefore, we conclude that the zero solution of (2.18) is globally stable w.r.t. the partial variable y. □

In the following, we consider a more specific system given by

$$\begin{aligned} \dot{y} &= \Big(\sum_{j=1}^{n} a_{1j}\,f_j(x_j),\ \ldots,\ \sum_{j=1}^{n} a_{mj}\,f_j(x_j)\Big)^{\mathrm{T}}, \\ \dot{z} &= \Big(\sum_{j=1}^{n} a_{(m+1)j}\,f_j(x_j),\ \ldots,\ \sum_{j=1}^{n} a_{nj}\,f_j(x_j)\Big)^{\mathrm{T}}, \end{aligned} \tag{2.20}$$

where $f_j(x_j) \in C[R,R]$, $f_j(0) = 0$, $j = 1,\ldots,n$. Suppose that the solution of the initial value problem (2.20) is unique.

Theorem 2.35. *Suppose system (2.18) satisfies the following conditions:*

1. $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$ *and* $\int_0^{\pm\infty} f_i(x_i)\,\mathrm{d}x_i = +\infty$, $a_{ii} < 0$, $i = 1, 2, \ldots, m$, $f_i(x_i)x_i \geq 0$, $i = m+1, \ldots, n$;
2. *There exist constants* $c_i > 0$ $(i = 1, 2, \ldots, m)$, $c_j \geq 0$ $(j = m+1, \ldots, n)$, $\varepsilon > 0$ *such that*
$$B(b_{ij})_{n\times n} + \begin{bmatrix} \varepsilon I_{m\times m} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$
is negative semi-definite, where
$$b_{ij} = \begin{cases} -c_i\,|a_{ii}|, & i = j = 1, 2, \ldots, m; \\ -\dfrac{1}{2}\,(c_i a_{ij} + c_j a_{ji}), & i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases}$$
Then the zero solution of (2.20) is globally stable w.r.t. the partial variable y.

Proof. First we construct the radially unbounded, positive definite Lyapunov function w.r.t. the partial variable y:
$$V(x) = \sum_{i=1}^{n} c_i \int_0^{x_i} f_i(x_i)\,\mathrm{d}x_i.$$
Then the proof is analogous to that of Theorem 2.31, and is thus omitted. □

Theorem 2.36. *Let system (2.18) satisfy the following conditions:*

1. $f_i(x_i)x_i < 0$ *for* $x_i \neq 0$ $a_{ii} > 0$, $i = m+1, m+2, \ldots, n$, *and* $f_i(x_i)x_i \leq 0$, $a_{ii} \geq 0$, $i = 1, 2, \ldots, n$;
2. *There exist n functions* $c_i(x_i)$ $(i = 1, 2, \ldots, n)$, *which are continuous or have only finite discontinuous points of the first or third kind, and satisfy*
$c_i(x_i)x_i > 0$ *for* $x_i \neq 0$ *and* $\int_0^{\pm\infty} c_i(x_i)\,\mathrm{d}x_i = +\infty$, $i = 1, 2, \ldots, m$;
$c_i(x_i)x_i \geq 0$, $i = m+1, \ldots, n$;
3. *There exist functions* $\varepsilon_i(x_i) > 0$ $(i = 1, 2, \ldots, m)$ *such that*
$$\tilde{B}(\tilde{b}_{ij})_{n\times n} + \begin{bmatrix} \mathrm{diag}(\varepsilon_1(x_1)\ \cdots\ \varepsilon_m(x_m)) & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$
is negative semi-definite, where
$$\tilde{b}_{ij}(x) = \begin{cases} \dfrac{c_i(x_i)a_{ij}f_{ij}(x_j) + c_j(x_j)a_{ji}f_{ji}(x_i)}{2\sqrt{|f_i(x_i)\,f_j(x_j)|}}, & x_i x_j \neq 0, \\ 0, & x_i x_j = 0, \end{cases} \quad i, j = 1, \ldots, n,$$
then the zero solution of (2.20) is globally stable w.r.t. the partial variable y.

Proof. Construct the Lyapunov function
$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} c_i(x_i)\,\mathrm{d}x_i$$
which is radially unbounded, positive definite w.r.t. the partial variable y. The proof can be completed as in the case of Theorem 2.33. □

Theorem 2.37. *If system (2.18) satisfies*

1. $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$ $a_{ii} < 0$, $i = m+1, \ldots, n$, *and* $f_i(x_i)x_i \geq 0$, $a_{ii} \leq 0$, $i = 1, 2, \ldots, n$;
2. *There exist constants* $c_i > 0$ $(i = 1, 2, \ldots, m)$, $c_j \geq 0$ $(j = m+1, \ldots, n)$ *such that*

$$\begin{cases} -c_j|a_{jj}| + \sum\limits_{i=1, i\neq j}^{n} c_i|a_{ij}| < 0, & j = 1, \ldots, m, \\ -c_j|a_{jj}| + \sum\limits_{i=1, i\neq j}^{n} c_i|a_{ij}| \leq 0, & j = m+1, \ldots, n, \end{cases}$$

then the zero solution of (2.20) is globally stable w.r.t. the partial variable y.

Proof. We construct the Lyapunov function

$$V(x) = \sum_{i=1}^{n} c_i\, |x_i|.$$

Obviously, $V(x)$ is radially unbounded, positive definite w.r.t. y. Then the proof follows that of Theorem 2.34. □

Theorem 2.38. *Suppose system (2.20) satisfies the following conditions:*

1. $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $a_{ii} < 0$, $i = 1, 2, \ldots, n$, *and* $f_i(x_i)x_i \geq 0$, $i = m+1, \ldots, n$;
2.

$$\tilde{A} := \begin{bmatrix} 1 & -\dfrac{a_{21}}{a_{11}} & \cdots & -\dfrac{a_{n1}}{a_{11}} \\ -\dfrac{a_{21}}{a_{22}} & 1 & \cdots & -\dfrac{a_{n2}}{a_{22}} \\ \vdots & \vdots & & \vdots \\ -\dfrac{a_{1n}}{a_{nn}} & -\dfrac{a_{2n}}{a_{nn}} & \cdots & 1 \end{bmatrix} := \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

where $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$, *and* $\tilde{A}_{22}$ *are* $m \times m$, $m \times p$, $p \times m$, *and* $p \times p$ *matrices, respectively, and* $\tilde{A}_{11}$, $\tilde{A}_{22}$, $I - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ *are all M matrices. Then the zero solution of (2.20) is globally stable w.r.t. the partial variable y.*

Proof. It follows the same idea applied in the proof of Theorem 2.35. □

2.8 Autonomous Systems with Generalized Separable Variables

Consider the system with generalized separable variables:

$$\dot{x}_i = \sum_{i=1}^{n} F_{ij}(x)\, f_{ij}(x_j), \quad i = 1, 2, \ldots, n, \tag{2.21}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $F_{ij} \in C[R^n, R]$, $f_{ij} \in C[R, R]$, $f_{ij}(0) = 0$, $i, j = 1, 2, \ldots, n$. Suppose that the solution of the initial value problem (2.21) is unique.

Clearly, (2.18) is a special case of (2.21) with $F_{ij}(x) \equiv 1$.

In fact, we see in what follows that both the Lurie direct control system and the Lurie indirect one can be reduced to system (2.21).

We still let $y = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$ and $z = (x_{m+1}, x_{m+2}, \ldots, x_n)^{\mathrm{T}}$, and rewrite (2.21) as

$$\begin{aligned}
\dot{y} &= \Big(\sum_{j=1}^{n} F_{1j}(x) f_{1j}(x_j),\ \ldots,\ \sum_{j=1}^{n} F_{mj}(x) f_{mj}(x_j)\Big)^{\mathrm{T}}, \\
\dot{z} &= \Big(\sum_{j=1}^{n} F_{(m+1)j}(x) f_{(m+1)j}(x_j),\ \ldots,\ \sum_{j=1}^{n} F_{nj}(x) f_{nj}(x_j)\Big)^{\mathrm{T}}.
\end{aligned} \tag{2.22}$$

Theorem 2.39. *If system (2.22) satisfies the following conditions:*

1. $f_{ii}(x_i)x_i > 0$ *for* $x_i \neq 0$ *and* $\int_0^{\pm\infty} f_{ii}(x_i)\,\mathrm{d}x_i = +\infty$, $i = 1, 2, \ldots, m$, $f_{ii}(x_i)x_i \geq 0$, $i = m+1, \ldots, n$;
2. *There exist constants* $c_i > 0$ $(i = 1, 2, \ldots, m)$, $c_j \geq 0$ $(j = m+1, \ldots, n)$, $\varepsilon > 0$ *such that*

$$B(b_{ij})_{n\times n} + \begin{bmatrix} \varepsilon I_{m\times m} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n} \quad \text{is negative semi-definite,}$$

where

$$b_{ij}(x) = \begin{cases} \dfrac{1}{2}\Big(\dfrac{F_{ij}(x) f_{ij}(x_j)}{f_{jj}(x_j)} + \dfrac{F_{ji}(x) f_{ji}(x_i)}{f_{ii}(x_i)}\Big), & x_i x_j \neq 0, \\ 0, & x_i x_j = 0, \end{cases} \qquad i, j = 1, \ldots, n,$$

then the zero solution of (2.22) is globally stable w.r.t. the partial variable y.

Proof. Let

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} f_{ii}(x_i)\,\mathrm{d}x_i.$$

Clearly,

$$V(x) \geq \sum_{i=1}^{m} \int_0^{x_i} f_{ii}(x_i)\,\mathrm{d}x_i := \varphi(y) \ \to\ +\infty \qquad \text{as} \quad \|y\| \ \to\ +\infty.$$

Hence, $V(x)$ is radially unbounded, positive definite w.r.t. the partial variable y.

Now we proceed to prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.22)} := G(x) = \sum_{i=1}^{n} f_{ii}(x_i) \sum_{j=1}^{n} F_{ij}(x) f_{ij}(x_j)$$

is negative definite w.r.t. y.

For any $x = \xi \in R^n$, without loss of generality we can assume that

$$\prod_{i=1}^{k} \xi_i \neq 0, \qquad \sum_{i=k+1}^{n} \xi_i^2 = 0, \quad m \leq k \leq n.$$

Then, we obtain

$$
\begin{aligned}
G(\xi) &= \sum_{i=1}^{k} f_{ii}(\xi_i) \sum_{j=1}^{k} F_{ij}(\xi) f_{ij}(\xi_j), \\
&= \sum_{i=1}^{k} F_{ii}(x) f_{ii}^2(\xi_i) + \frac{1}{2} \sum_{i,j=1, i\neq j}^{k} \Big[f_{ii}(\xi_i) F_{ij}(\xi) f_{ij}(\xi_j) + f_{jj}(\xi_j) F_{ji}(\xi) f_{ji}(\xi_i) \Big], \\
&= \sum_{i=1}^{k} b_{ii}(\xi) f_{ii}^2(\xi_i) + \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) + \sum_{i,j=1, i\neq j}^{k} b_{ij}(\xi) f_{ii}(\xi_i) f_{jj}(x_j) - \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) \\
&\leq - \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) < 0.
\end{aligned}
$$

Since ξ is arbitrary, we have shown that $\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(2.22)}$ is negative definite w.r.t. the partial variable y. Hence the proof is finished. □

Theorem 2.40. *If system (2.22) satisfies the following conditions:*

1. $F_{ii}(x) f_{ii}(x_i) x_i < 0$ *for* $x_i \neq 0$ $i = 1, 2, \ldots, m$, *and* $F_{ii}(x) f_{ii}(x_i) x_i \leq 0$, $i = m + 1, 2, \ldots, n$*;*
2. *There exist constants* $c_i > 0$ $(i = 1, 2, \ldots, m)$, $c_j \geq 0$ $(j = m+1, \ldots, n)$ *such that*

$$
\begin{cases}
-c_j |F_{jj}(x) f_{jj}(x_j)| + \displaystyle\sum_{i=1, i\neq j}^{n} c_i |F_{ij}(x) f_{ij}(x_j)| < 0, \text{ for } x_j \neq 0,\ j = 1, \ldots, m, \\
-c_j |F_{jj}(x) f_{jj}(x_j)| + \displaystyle\sum_{i=1, i\neq j}^{n} c_i |F_{ij}(x) f_{ij}(x_j)| \leq 0,\ j = m+1, \ldots, n,
\end{cases}
$$

then the zero solution of (2.22) is globally stable w.r.t. the partial variable y.

Proof. We construct the radially unbounded, positive definite Lyapunov function w.r.t. y:

$$
V(x) = \sum_{i=1}^{n} c_i |x_i|,
$$

and complete the proof using a similar procedure as in Theorem 2.33. □

2.9 Nonautonomous Systems with Separable Variables

Consider the nonautonomous system with separable variables [71]:

$$
\begin{aligned}
\dot{y} &= \Big(\sum_{j=1}^{n} f_{1j}(t, x_j),\ \ldots,\ \sum_{j=1}^{n} f_{mj}(t, x_j) \Big)^{\mathrm{T}}, \\
\dot{z} &= \Big(\sum_{j=1}^{n} f_{(m+1)j}(t, x_j),\ \ldots,\ \sum_{j=1}^{n} f_{nj}(t, x_j) \Big)^{\mathrm{T}},
\end{aligned}
\tag{2.23}
$$

where $y = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$, $z = (x_{m+1}, x_{m+2}, \ldots, x_n)^{\mathrm{T}}$, $f_{ij}(t,x_j) \in C[I \times R, R]$, $f_{ij}(0) = 0$, $i, j = 1, 2, \ldots, n$. Suppose that the solution of the initial value problem (2.16) is unique.

Lemma 2.41. *If there exist functions $\varphi_i(x_i)$ on $(-\infty, +\infty)$ $(i = 1, 2, \ldots, n)$, which are continuous or have only finite discontinuous points of the first or third kind (i.e., at the discontinuous points, the left- and right-hand limits of $\varphi(x_i)$ exist) such that*

1. $\varphi_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $i = 1, 2, \ldots, m$; $\varphi(x_i)x_i \geq 0$, $i = m+1, \ldots, n$;
2. $\int_0^{\pm\infty} \varphi_i(x_i)\,\mathrm{d}x_i = +\infty$, $i = 1, 2, \ldots, m$;
3. *There is a positive definite function $\psi(y)$ satisfying*

$$G(x) := \sum_{i=1}^{m} \varphi_i(x_i \pm 0) \sum_{i=1}^{m} f_{ij}(x_j) \leq -\psi(y),$$

then the zero solution of (2.23) is globally stable w.r.t. the partial variable y.

Proof. The proof repeats the one for Lemma 2.29 and is omitted. □

Theorem 2.42. *Let system (2.23) satisfy the following conditions:*

1. $f_{ii}(t,x_i)x_i < 0$ *for* $x_i \neq 0$, $i = 1, 2, \ldots, m$, *and* $f_{ii}(t,x_i)x_i \leq 0$, $i = m+1, 2, \ldots, n$;
2. *There exist functions $F_{ii}(x_i)$ on $(-\infty, +\infty)$ $(i = 1, 2, \ldots, n)$, which are continuous or have only finite discontinuous points of the first or third kind such that*

$$\begin{cases} F_{ii}(x_i)x_i > 0 \;\; for \;\; x_i \neq 0, & i = 1, 2, \ldots, m, \\ F_{ii}(x_i)x_i \geq 0, & i = m+1, m+2, \ldots, n, \\ \int_0^{\pm\infty} F_{ii}(x_i)\,dx_i = +\infty, & i = 1, 2, \ldots, m, \\ |F_{ii}(x_i)| \leq |f_{ii}(t,x_i)|, & i = 1, 2, \ldots, n; \end{cases}$$

3. *The matrix* $A(a_{ij}(t,x))_{n\times n} + \begin{bmatrix} \varepsilon I_{m\times m} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$ is negative semi-definite, *where* $0 < \varepsilon \ll 1$, *and*

$$a_{ij}(t,x) = \begin{cases} -1, & i = j = 1, 2, \ldots, n, \\ \dfrac{1}{2}\left(\dfrac{f_{ij}(t,x_j)}{F_{jj}(x_j)} + \dfrac{f_{ji}(t,x_i)}{F_{ii}(x_i)}\right), & x_i x_j \neq 0, \;\; i, j = 1, 2, \ldots, n, \\ 0, & x_i x_j = 0, \;\; i, j = 1, 2, \ldots, n. \end{cases}$$

Then the zero solution of (2.23) is globally stable w.r.t. the partial variable y.

Proof. Consider the radially unbounded, positive definite Lyapunov function w.r.t. y:

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} F_{ii}(x_i)\,\mathrm{d}x_i.$$

Then

$$V(x) \geq \sum_{i=1}^{m} \int_0^{x_i} F_{ii}(x_i)\,\mathrm{d}x_i := \varphi(y).$$

Clearly we have $\varphi(y) \to +\infty$ as $\|y\| \to +\infty$.

We further prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.23)} := G(t,x) = \sum_{i=1}^{n} F_{ii}(x_i) \sum_{j=1}^{n} f_{ij}(t,x_j)$$

is negative definite w.r.t. y.

For any $x = \xi \in R^n$, without of loss of generality we can assume that

$$\prod_{i=1}^{k} \xi_i \neq 0, \qquad \sum_{i=k+1}^{n} \xi_i^2 = 0, \quad m \leq k \leq n.$$

Then, it follows that

$$\begin{aligned} G(t,\xi) &= \sum_{i=1}^{k} F_{ii}(\xi_i) \sum_{j=1}^{k} F_{ij}(t,\xi_j) \\ &\leq \sum_{i=1}^{k} a_{ii}(t,\xi) F_{ii}^2(\xi_i) + \sum_{i=1}^{m} \varepsilon F_{ii}^2(\xi_i) \\ &\quad + \sum_{i,j=1, i\neq j}^{k} a_{ij}(t,\xi) F_{ii}(\xi_i) F_{jj}(x_j) - \sum_{i=1}^{m} \varepsilon F_{ii}^2(\xi_i) \\ &\leq -\sum_{i=1}^{m} \varepsilon F_{ii}^2(\xi_i) < 0. \end{aligned}$$

Since ξ is arbitrary, we have derived that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.23)}$ is negative definite w.r.t. the partial variable y. Hence the zero solution of (2.23) is globally stable w.r.t. the partial variable y. $\square$

Theorem 2.43. *If system (2.23) satisfies the following conditions:*

1. $f_{ii}(x_i)\, x_i < 0$ *for* $x_i \neq 0$, $i = 1, 2, \ldots, m$, *and* $f_{ii}(x_i)\, x_i \leq 0$, $i = m+1, 2, \ldots, n$*;*
2. *There exist constants* $c_i > 0$ $(i = 1, 2, \ldots, m)$, $c_j \geq 0$ $(j = m+1, \ldots, n)$ *such that*

$$\begin{cases} -c_j |f_{jj}| + \displaystyle\sum_{i=1, i\neq j}^{n} c_i |f_{ij}(t,x_j)| < 0, & j = 1, \ldots, m, \\ -c_j |a_{jj}| + \displaystyle\sum_{i=1, i\neq j}^{n} c_i |f_{ij}(t,x_j)| \leq 0, & j = 1, \ldots, m, \end{cases}$$

then the zero solution of (2.23) is globally stable w.r.t. the partial variable y.

Proof. Let us choose

$$V(x) = \sum_{i=1}^{n} c_i |x_i|,$$

which is radially unbounded, positive definite w.r.t. y. Analogous to the proof of Theorem 2.33, we can verify the validity of this theorem. □

Theorem 2.44. *Let system (2.23) satisfy the following conditions:*

1. *There exist functions $\varphi_i(x_i)$ on $(-\infty, +\infty)$ $(i = 1, 2, \ldots, n)$, which are continuous or have only finite discontinuous points of the first or third kind (i.e., at the discontinuous points, the left- and right-hand limits of $\varphi(x_i)$ exist) such that $\varphi_i(x_i)x_i > 0$ for $x_i \neq 0$, and $\int_0^{\pm\infty} \varphi_i(x_i)\,\mathrm{d}x_i = +\infty$, $i = 1, 2, \ldots, m$, $\varphi_i(x_i)x_i \geq 0$, $i = m+1, m+2, \ldots, n$;*
2. *There are functions $a_i(x_i) > 0$ with $a_i(x_i) > 0$ for $x_i \neq 0$ $(i = 1, 2, \ldots, m)$ such that*

$$\begin{cases} \sum_{i=1}^{n} \varphi_i f_{ij}(x_j) \leq -a_j(x_j), & j = 1, \ldots, m, \\ \sum_{i=1}^{n} \varphi_i f_{ij}(x_j) \leq 0, & j = m+1, \ldots, n, \end{cases}$$

then the zero solution of (2.23) is globally stable w.r.t. the partial variable y.

Proof. We construct the Lyapunov function

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} \varphi_i(x_i)\,\mathrm{d}x_i,$$

which is radially unbounded, positive definite w.r.t. y. Then we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.23)} \leq \sum_{i=1}^{n} \varphi_i(x_i) \sum_{j=1}^{n} f_{ij}(t, x_j) \leq \sum_{j=1}^{m} \sum_{i=1}^{n} \varphi_i(x_i) f_{ij}(t, x_j) \leq -\sum_{j=1}^{m} a_j(x_j).$$

Thus $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.23)}$ is negative definite w.r.t. y. Therefore, we conclude that the zero solution of (2.23) is globally stable w.r.t. the partial variable y. □

3

Sufficient Conditions of Absolute Stability: Classical Methods

The development of control theory began from Maxwell's study on the stability of Watt centrifugal governor. In early 1940s, the former Soviet Union scholars Lurie, Postnikov, and others developed a method to deal with a class of nonlinear systems, now called nonlinear isolate method. Lurie and his coworkers studied many real control systems, including the Bulgakov problem of aircraft automatic control. They first isolated the nonlinear part from the system and considered it as a feedback control of the system so that the system has a closed-loop form. This problem is the well-known Lurie problem [99, 100]. The research on Lurie's problem has so far resulted in a number of monographs and several hundred of scientific publications. In fact, the pose of the Lurie problem actually initiated the research on the robust control and robust stability for nondeterministic systems or multivalued differential equations. It promoted the application and development of stability theory. In recent years, the developments in chaos control and chaos synchronization, neural networks, electrical systems have revealed that these research areas are closely related to Lurie control systems. Thus, it is useful and important to further study Lurie control systems, in particular, from the point of control theory and stability theory. [164, 173–179]

In this chapter, we briefly introduce the background of Lurie problem, the mathematical methodology for solving Lurie problem, and consider three classical methods for studying absolute stability: the Lyapunov–Lurie V-function method (having integrals and quadratic form), S-program, and Popov frequency-domain criterion and simplified Popov method. We also consider some equivalent conditions at which the derivative of the Lyapunov-Lurie V-function is negative definite.

Some materials are based on the results of Liao [68] (Sects. 3.1 and 3.2), Lurie [99, 100] (Sect. 3.3), Zhu [184] and Zhao [180] (Sect. 3.4), as well as Popov [120–123] and Zhang [178] (Sect. 3.5).

3.1 Absolute Stability of Lurie Control System

We first introduce the centrifugal governor as the earliest example of Lurie control system. As shown in Fig. 3.1, the angular velocity of the generator, ω, is measured by the centrifugal governor. The centrifugal governor is connected to the server 4 through the level 2 and the sliding valve 3. The server 4 makes the regulator 5 move

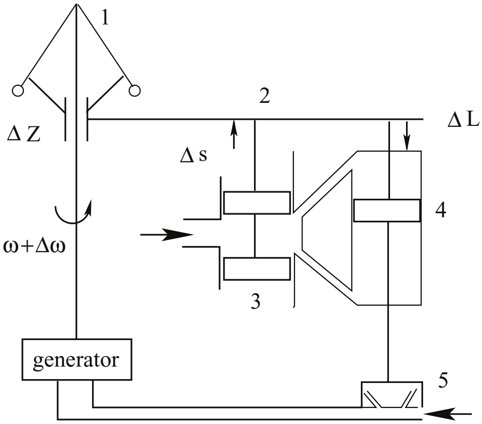

Fig. 3.1 Working principal of centrifugal governor

such that the generator rotates with a constant speed. When the load of the generator is reducing and thus the speed of the generator is increasing, the governor sleeve moves up to raise the sliding valve via the level. Thus high pressure gasoline enters the upper part of the server cylinder, and the gasoline left in the lower part of the cylinder is drained off through the lower narrow passage. Therefore, the piston descends to move the regulator to reduce the amount of gasoline so that the angular velocity of the generator is reduced. On the other hand, when the speed of the generator is below the normal speed, the server moves up to adjust the regulator to increase the amount of gasoline. Thus, the speed of the generator increases. Because of the negative feedback of the centrifugal governor, the generator's angular velocity can be kept a constant. As we know, a generator usually works in an environment where the end-users' loads are frequently varying. If the generator is not kept at a constant angular velocity, the users will not have a constant currency, which could cause disaster.

Now we consider the mathematical model of the centrifugal governor. The differential equation of the generator is given by

$$J\frac{\mathrm{d}\Delta\omega}{\mathrm{d}t} = k_1\Delta\omega + k_2\Delta L,$$

where J is the angular inertia, $\Delta\omega$ is the increment of the angular velocity, ΔL is the position increment of the regulator, k_1 and k_2 are the rates of the changes of the moments per unit.

The dynamic equation for the centrifugal governor is

$$M\frac{\mathrm{d}^2\Delta Z}{\mathrm{d}t^2} + C\frac{\mathrm{d}\Delta Z}{\mathrm{d}t} = F_1\,\Delta\omega + F_2\,\Delta Z,$$

where M is the generalized mass, C is the damping coefficient, ΔZ is the measurement of the position change of the governor sleeve, F_1 and F_2 represent the generalized forces per unit, respectively, for ω and Z.

The equation for the server is given by

$$A\frac{\mathrm{d}\Delta L}{\mathrm{d}t} = f(\Delta s),$$

where A is the cross-area of the governor cylinder, Δs is the amount of change of the sliding valve, $f(\Delta s)$ is the amount of gasoline entering the cylinder per unit time. The nonlinearity of f is determined by the shape of the sliding valve. In general, it is difficulty to know the exact form of f.

The kinematics of the feedback level is

$$\Delta s = a\Delta Z - b\Delta L,$$

where a and b are constants.

The above four equations can be transformed to the following dimensionless equations:

$$\begin{aligned}
&a_1\dot{\varphi} + a_2\varphi = -\mu,\\
&b_1\ddot{\eta} + b_2\dot{\eta} + b_3\eta = \varphi,\\
&\dot{\mu} = f(\sigma),\\
&\sigma = c_1\eta - c_2\mu,
\end{aligned}$$

where φ, η, μ, σ are, respectively, variables proportional to $\Delta\omega$, ΔZ, ΔL, Δs, while σ is the control signal, which determines the amount of the gasoline entering into the cylinder.

Let $\varphi = x_1$, $\eta = x_2$, $\dot{\eta} = x_3$, $\mu = x_4$. Then the above equations take the following standard form:

$$\begin{aligned}
\dot{x}_1 &= -\frac{a_2}{a_1}x_1 - \frac{1}{a_1}x_4,\\
\dot{x}_2 &= x_3,\\
\dot{x}_3 &= \frac{1}{b_1}x_1 - \frac{b_3}{b_1}x_2 - \frac{b_2}{b_1}x_3,\\
\dot{x}_4 &= f(\sigma),\\
\sigma &= c_1x_1 - c_2x_4.
\end{aligned} \tag{3.1}$$

When the generator is working in normal condition, we have

$$x_1 = x_2 = x_3 = x_4 = 0,$$

which is the equilibrium position of the system. It is required that the equilibrium point is globally stable.

Next, we introduce the Lurie-type nonlinear control system, that is, the so called Lurie problem and its mathematical description. Around 1944, the former Soviet Union mathematical control scholar A.I. Lurie, based on the study of aircraft automatic control system, proposed a control model described by the following general differential equations [99, 100]:

$$\begin{aligned}
&\dot{x} = Ax + bf(\sigma),\\
&\sigma = c^{\mathrm{T}}x = \sum_{i=1}^{n} c_i x_i,
\end{aligned} \tag{3.2}$$

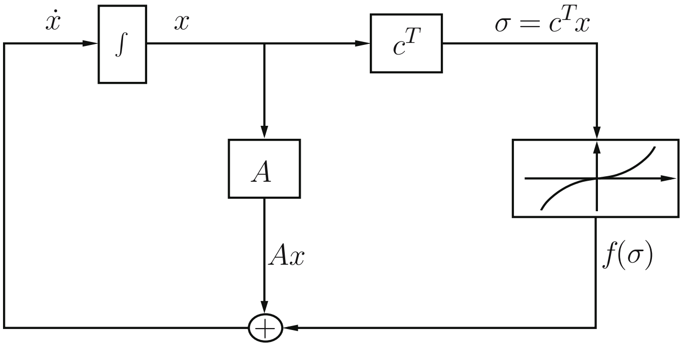

Fig. 3.2 Lurie control system

where $x \in R^n$ is the state variable, $b, c \in R^n$ are known constant vectors, σ is the feedback control variable, $f(\sigma)$ is a nonlinear function. System (3.2) is shown in Fig. 3.2.

The form of f is not specified, but it is known that it belongs to some type of functions $F_{[0,k]}$, $F_{[0,k)}$, $F_{[k_1,k_2)}$ or F_∞. Here,

$$F_{[0,k]} := \{f | f(0) = 0,\ 0 < \sigma f(\sigma) \le k\sigma^2,\ \sigma \neq 0,\ f \text{ continues}\};$$

$$F_{[0,k)} := \{f | f(0) = 0,\ 0 < \sigma f(\sigma) < k\sigma^2,\ \sigma \neq 0,\ f \text{ continues}\};$$

$$F_{[k_1,k_2]} := \{f | f(0) = 0,\ k_1\sigma^2 \le \sigma f(\sigma) \le k_2\sigma^2,\ \sigma \neq 0,\ f \text{ continues}\};$$

$$F_\infty := \{f | f(0) = 0,\ \sigma f(\sigma) > 0,\ \sigma \neq 0,\ f \text{ continues}\}.$$

The above functions are demonstrated in Fig. 3.3.

Many practical nonlinear feedback control problems can be described by system (3.2), but the form of f is usually not known. Partial information about f may be obtained from experiments. However, experiments can only be carried out under specific loads, and thus f depends upon the leads. Usually one only knows that f belongs to $F_{[0,k]}$, $F_{[0,k]}$, or $F_{[k_1,k_2)}$. Any other information is not available in practice. For example, for the above centrifugal governor, the control signal σ of the governor is proportional to the change of the sliding valve's position, Δs. $f(\sigma)$ measures the amount of the gasoline entering into the server, which is a nonlinear function of the open degree of the sliding valve. Besides knowing that when $\sigma > 0$, $f(\sigma) > 0$ (which means a increase of the gasoline in the upper part of the server cylinder), when $\sigma < 0$, $f(\sigma) < 0$ (implying the reduction of the gasoline in the upper part of the server cylinder), and $f(0) = 0$, very little is known about the function $f(\sigma)$.

Therefore, the Lurie control system (3.2) is actually a multivalued differential equations, also called differential involving or nondeterministic system.

In general, Lurie systems can be divided into three types [2, 159]:

1. Direct control system when A of system (3.2) is a Hurwitz matrix;
2. Indirect control system when A of system (3.2) has an eigenvalues with zero real part and remains have negative real parts;

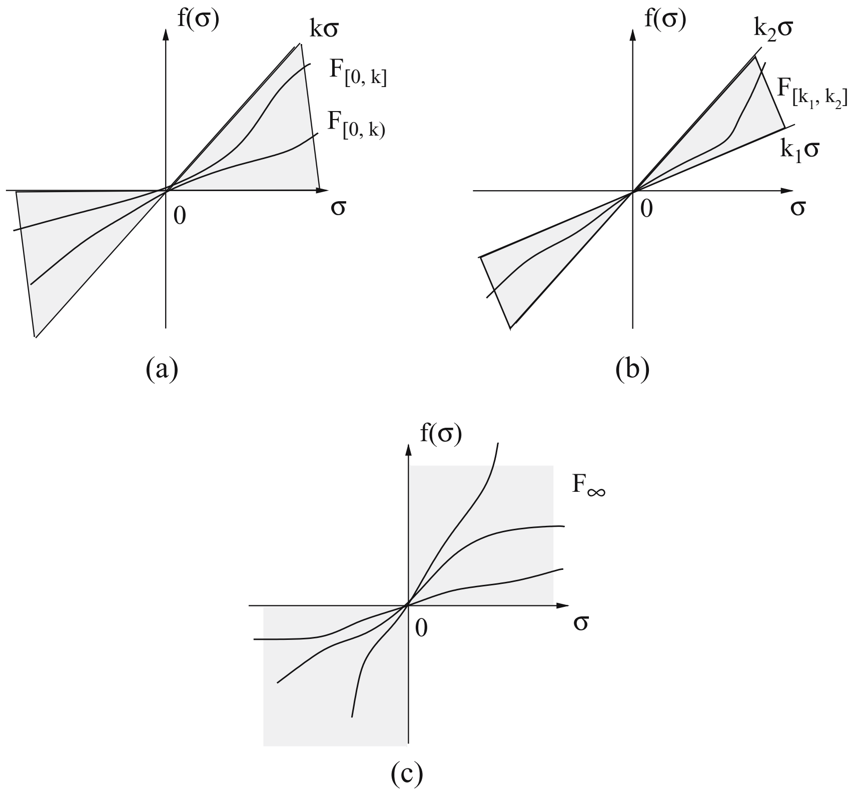

Fig. 3.3 Different types of $f(\sigma)$: (**a**) $F_{[0,k]}$ and $F_{(0,k)}$; (**b**) $F_{[k_1,,k_2]}$; and (**c**) F_∞

3. Critical control system when no eigenvalues of A of system (3.2) have positive real parts.

Definition 3.1. *The zero solution of system (3.2) is absolutely stable if* $\forall f(\sigma) \in F_\infty$*, the zero solution of system (3.2) is globally stable. The zero solution of system (3.2) is absolutely stable in the Hurwitz angle region* $[0,k]$ *(*$[0,k), [k_1,k_2]$*) if* $\forall f(\sigma) \in F_{[0,k]}$ *(*$\forall f(\sigma) \in F_{[0,k)}$*,* $\forall f(\sigma) \in F_{[k_1,k_2]}$*), the zero solution of system (3.2) is globally stable.*

The well-known American mathematician Lefschetz gave the precise mathematical description for the Lurie problem [62]: The Lurie problem is to find the sufficient and necessary conditions of absolute stability for the equilibrium points of system (3.2). Therefore, as long as a sufficient and necessary condition is obtained, it is considered as a solution to the Lurie problem. The particular form of the condition is not a concern here. However, certainly, the simpler the condition, the better it is.

We now first discuss the necessary conditions for the absolute stability of system (3.2) when $f \in F_\infty$.

Theorem 3.2. *The following conditions are all the necessary conditions for the absolute stability of system (3.2) [68, 159]:*

(1) $\forall \varepsilon > 0$, *(*$0 < \varepsilon \le k$*), the matrix* $A + \varepsilon b c^{\mathrm{T}}$ *is a Hurwitz matrix;*
(2) $\mathrm{Re}\lambda(A) \le 0$, *i.e., all eigenvalues of* A *are located in the closed left half of the complex plane;*
(3) $c^{\mathrm{T}} b \le 0$.

Proof. 1. Let $f(\sigma) = \varepsilon \sigma = \varepsilon c^{\mathrm{T}} x$. Then system (3.2) becomes

$$\dot{x} = (A + \varepsilon b c^{\mathrm{T}}) x.$$

Since the zero solution of (3.2) is asymptotically stable, $A + \varepsilon b c^{\mathrm{T}}$ is a Hurwitz matrix.

2. Suppose at least one of eigenvalues of A has positive real part, say, $\mathrm{Re}\lambda_0(A) > 0$. Because of the the continuous dependence of the eigenvalue on the elements of the matrix, we can always choose $0 < \varepsilon \ll 1$ such that at least one of the eigenvalues of $A + \varepsilon b c^{\mathrm{T}}$ has $\mathrm{Re}\lambda_0(A + \varepsilon b c^{r}) > 0$, leading to a contradiction with absolute stability. Thus $\mathrm{Re}(A) \le 0$.
3. Suppose $c^{\mathrm{T}} b > 0$. Let $f(\sigma) = h\sigma$. Then system (3.2) can be written as

$$\dot{x} = (A + h b c^{\mathrm{T}}) x.$$

Since $\mathrm{tr}(b c^{\mathrm{T}}) = c^{\mathrm{T}} b$, when $c^{\mathrm{T}} b > 0$, $h \gg 1$,

$$\mathrm{tr}(A + h b c^{\mathrm{T}}) = \mathrm{tr}A + h\,\mathrm{tr} b c^{\mathrm{T}} = \mathrm{tr}A + h c^{\mathrm{T}} b > 0.$$

Thus the zero solution of system (3.2) is unstable, a contradiction, implying that $c^{\mathrm{T}} b \le 0$. □

3.2 Lyapunov–Lurie Function Method

We first consider system (3.2) being a direct control system, that is, A is a Hurwitz matrix, and introduce the sufficient condition that was first derived by Lurie [99, 100] for the absolute stability of the zero solution of system (3.2). Since A is a Hurwitz matrix, for a given arbitrary symmetric, negative definite matrix $-B$, the Lyapunov matrix equation,

$$A^{\mathrm{T}} P + P A = -B, \tag{3.3}$$

has symmetric positive definite matrix solution.

Theorem 3.3. [99, 100] *If there exists a symmetric, positive definite matrix* P *such that the derivative of the radially unbounded, positive definite Lyapunov function*

$$V(x) = x^{\mathrm{T}} P x + \int_0^{\sigma} f(\sigma)\, \mathrm{d}\sigma, \tag{3.4}$$

evaluated along the trajectories of system (3.2), given by

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)} = -x^{\mathrm{T}}Bx + (c^{\mathrm{T}}A + 2b^{\mathrm{T}}P)xf(\sigma) + c^{\mathrm{T}}bf^2(\sigma), \tag{3.5}$$

is negative definite. Then the zero solution of the system (3.2) is absolutely stable.

The conclusion of Theorem 3.3 is correct, which was the earliest original method used to study the absolute stability of Lurie controls systems. However, it is difficult to verify the negativity of (3.5). Researchers once considered equation (3.5) as a quadratic form with respect to (w.r.t.) $(x, f(\sigma))$, and hoped to find the condition of $(x, f(\sigma))$ satisfying Sylvester condition to overcome the difficulty. However, this failed. This idea is equivalent to add a differential equation to system (3.2) to obtain

$$\begin{aligned} \dot{x} &= Ax + bf(\sigma), \\ \dot{\sigma} &= c^{\mathrm{T}}Ax + c^{\mathrm{T}}bf(\sigma). \end{aligned} \tag{3.6}$$

We only need to consider a special case by choosing $f(\sigma) = \sigma$ from which the failure can be seen immediately. Thus, (3.6) becomes

$$\begin{aligned} \dot{x} &= Ax + b\sigma, \\ \dot{\sigma} &= c^{\mathrm{T}}Ax + c^{\mathrm{T}}b\sigma. \end{aligned} \tag{3.7}$$

The last equation of (3.7) is actually not independent, but a linear combination of the previous n equations. If formally considering this equation as an independent equation, we may construct the following Lyapunov function:

$$V = x^{\mathrm{T}}Px + \frac{1}{2}\sigma^2.$$

Then if $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)}$ is negative definite, it implies that system (3.7) treats σ as an independent variable and so the zero solution of the system is also asymptotically stable. Thus the matrix

$$\begin{bmatrix} A & b \\ c^{\mathrm{T}}A & c^{\mathrm{T}}b \end{bmatrix}$$

is Hurwitz stable. However,

$$\det \begin{bmatrix} A & b \\ c^{\mathrm{T}}A & c^{\mathrm{T}}b \end{bmatrix} = 0$$

indicates that

$$\begin{bmatrix} A & b \\ c^{\mathrm{T}}A & c^{\mathrm{T}}b \end{bmatrix}$$

is not a Hurwitz matrix, a contradiction, showing the failure of this method.

However, the Lyapunov function given in the form of

$$V(x) = c^{\mathrm{T}}Px + \int_0^{\sigma} f(\sigma)\,d\sigma$$

can still be used as a valid V function to study the absolute stability. To obtain the absolute stability, one only needs to show that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)}$ is negative w.r.t. x, not necessarily for $(x, f(\sigma))$. Nevertheless, even considering the negativity of V w.r.t. x is still difficult. This is the main source causing the mistakes made by the early researchers who tried to get rid of the difficulty. This is also the motivation for the latter researchers to develop new methodology and new techniques to overcome the difficulty of verifying $\frac{dV}{dt}\big|_{(3.2)}$ being negative definite w.r.t. x.

3.3 Lyapunov–Lurie Type *V*-Function Method Plus *S*-Program

In this section, we introduce the S-program, which was developed by Lurie to verify the negativity of $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(3.2)}$ w.r.t. to x. This is a very efficient classical method to verify $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(3.2)}$ being negative definite w.r.t. x, and widely used.

We first point out that the use of the following two Lurie types of V functions are equivalent:

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma \quad (\beta > 0,\ P = P^{\mathrm{T}},\ P > 0) \tag{3.8}$$

and

$$W(x) = x^{\mathrm{T}}Qx + \int_0^{\sigma} f(\sigma) \quad (Q = Q^{\mathrm{T}},\ Q > 0). \tag{3.9}$$

That is, the result of application is same. Obviously, the latter is a special case of the former. In fact, let $Q = \frac{P}{\beta}$. Then $W(x) = \frac{V(x)}{\beta}$, and the latter becomes the former. Therefore, we use the latter to reduce a variable, and assume $\beta = 1$ hereafter.

If $f(\sigma) \in F_\infty$, we can add and subtract the same term $2\tau c^{\mathrm{T}}x$ (where $\tau > 0$ to be determined) to rewrite system (3.5) as

$$\begin{aligned}
-\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(3.2)} &= x^{\mathrm{T}}Bx - 2\Big(\frac{1}{2}c^{\mathrm{T}}A + b^{\mathrm{T}}P + \tau c^{\mathrm{T}}\Big)xf(\sigma) - c^{\mathrm{T}}bf^2(\sigma) + 2\tau c^{\mathrm{T}}xf(\sigma)\\
&:= S(x,f) + 2\tau\sigma f(\sigma).
\end{aligned} \tag{3.10}$$

Obviously, because of $2\tau\sigma f(\sigma) \geq 0$, if

$$S(x,f) := x^{\mathrm{T}}Bx - 2\Big(\frac{1}{2}c^{\mathrm{T}}A + b^{\mathrm{T}}P + \tau c^{\mathrm{T}}\Big)xf(\sigma) - c^{\mathrm{T}}bf^2(\sigma)$$

is positive definite w.r.t. $x, f(\sigma)$, then $\frac{dV}{dt}\big|_{(3.2)}$ is negative definite w.r.t. x. The Sylvester condition for $S(x,f)$ being positive definite w.r.t. $(x, f(\sigma))$ can usually be realized by choosing appropriate $\tau > 0$.

Since B is positive definite, $\forall x \neq 0$, let $y = Bx$. We then have $y^{\mathrm{T}}B^{-1}y = x^{\mathrm{T}}B^{\mathrm{T}}B^{-1}Bx = x^{\mathrm{T}}BB^{-1}Bx = x^{\mathrm{T}}Bx > 0$ $(x \neq 0)$, and thus B^{-1} is positive definite.

Because of B being positive definite, the condition for $S(x,f)$ being positive definite w.r.t. $(x, f(\sigma))$ is

$$\det\begin{bmatrix} B & -\Big(\frac{1}{2}A^{\mathrm{T}}c + Pb + \tau c\Big) \\ -\Big(\frac{1}{2}A^{\mathrm{T}}c + Pb + \tau c\Big)^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix} > 0,$$

which is equivalent to

$$\det\begin{bmatrix} B^{-1} & 0 \\ 0 & 1 \end{bmatrix} \det\begin{bmatrix} B & -\left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right) \\ -\left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right)^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix}$$

$$= \det\begin{bmatrix} I & -B^{-1}\left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right) \\ -\left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right)^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix}$$

$$= -c^{\mathrm{T}}b - \left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right)^{\mathrm{T}} B^{-1}\left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right) > 0.$$

Hence, the following Lurie theorem (also called Lurie S-program) is obtained.

Theorem 3.4. *If there exist a symmetric positive definite matrix B and a positive number $\tau > 0$ such that*

$$-c^{\mathrm{T}}b > \left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right)^{\mathrm{T}} B^{-1}\left(\frac{1}{2}A^{\mathrm{T}}c+Pb+\tau c\right),$$

then the zero solution of (3.2) is absolutely stable, where P is the solution of the matrix equation (3.3).

Corollary 3.5. *If there exists a symmetric positive definite matrix B such that*

$$-c^{\mathrm{T}}b > \left(\frac{1}{2}A^{\mathrm{T}}c+Pb+c\right)^{\mathrm{T}} B^{-1}\left(\frac{1}{2}A^{\mathrm{T}}c+Pb+c\right),$$

then the zero solution of (3.2) is absolutely stable.

Corollary 3.5 is special case of Theorem 3.4 when $\tau = 1$.

If $f(\sigma) \in F_{[0,k)}$, in (3.5) add and subtract the same term

$$2f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right) \quad (\tau > 0 \text{ to be determined}),$$

then (3.5) becomes

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)} &= -x^{\mathrm{T}}Bx + 2\left(\frac{1}{2}c^{\mathrm{T}}A + b^{\mathrm{T}}P + \tau c^{\mathrm{T}}\right)xf(\sigma) \\ &\quad + \left(c^{\mathrm{T}}b - \frac{2\tau}{k}\right)f^2(\sigma) - 2\tau f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right) \\ &:= -S_1(x, f(\sigma)) - 2\tau f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right). \end{aligned} \tag{3.11}$$

Since $f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right) > 0 \ \ (\forall \sigma \neq 0)$,

$$S(x,f) := x^{\mathrm{T}}Bx - 2\left(\frac{1}{2}c^{\mathrm{T}}A + b^{\mathrm{T}}P + \tau c^{\mathrm{T}}\right)xf(\sigma) - \left(c^{\mathrm{T}}b - \frac{2\tau}{k}\right)f^2(\sigma)$$

is positive definite w.r.t. $(x, f(\sigma))$, then $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)}$ is negative definite w.r.t. x. It leads to a similar theorem.

Theorem 3.6. *If there exist a symmetric positive definite matrix B and a positive number $\varepsilon > 0$ such that*

$$-\left(c^{\mathrm{T}}b - \frac{2\tau}{k}\right) > \left(\frac{1}{2}A^{\mathrm{T}}c + Pb + \tau c\right)^{\mathrm{T}} B^{-1}\left(\frac{1}{2}A^{\mathrm{T}}c + Pb + \tau c\right),$$

then the zero solution of (3.2) is absolutely stable in the Hurwitz region $[0,k)$.

The above method is called S-program [159].

It should be noted that $S(x,f)$ being positive definite w.r.t. (x,f) is only a sufficient condition for $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(3.2)}$ being negative definite w.r.t. x, not a necessary condition. Does there exist a system such that $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(3.2)}$ is negative definite w.r.t. x, but no $\tau > 0$ exists for $S(x,f)$ being positive definite. If this is true, then S-program is called defective, otherwise, it is called nondefective. A counter example has been found by Zhao [182] indicating that S-program is defective. This shows that the S-method gives only a sufficient condition for the derivative of the Lyapunov–Lurie type V function being negative, not a necessary condition. However, S-method first solved the difficulty in verifying the negativity of $\dot{V}$. It is convenient in applications and widely used. Furthermore, we show that the S-method is equivalent to the well-known Popov method.

3.4 Several Equivalent SANC for Negative Definite Derivatives

Now we turn to introduce several sufficient and necessary conditions (SANC) for determining the negativity of the derivative of the Lurie type V function, $\frac{\mathrm{d}V}{\mathrm{d}t}$, given in (3.5) along the trajectories of (3.2).

Zhao [180] obtained the following result.

Theorem 3.7. [180] *Assume that system (3.2) is a direct control system, then the sufficient and necessary conditions for V given by (3.4) such that its derivative*

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)} = -x^{\mathrm{T}}Bx + (c^{\mathrm{T}}A + 2b^{\mathrm{T}}P)xf(\sigma) + c^{\mathrm{T}}bf^2(\sigma)$$

is negative definite with respect to x are the following:

(1) $U(x) := x^{\mathrm{T}}Bx - (c^{\mathrm{T}}A + 2b^{\mathrm{T}}P)x - c^{\mathrm{T}}b \geq 0$ (when $c^{\mathrm{T}}x = 0$, i.e., $U(x)$ is positive semi-definite on the hyperplane $\sigma = 0$);

(2) $c^{\mathrm{T}}HH^{\mathrm{T}}(A^{\mathrm{T}}c + 2P^{\mathrm{T}}b) \leq 0$ (where $H^{\mathrm{T}}BH = I$).

The proof of Theorem 3.7 is quite lengthy (see [180]), yet the condition (2) is still not easy to verify. If one can determine the lowest bound of $U(x)$ on $\sigma = c^{\mathrm{T}}x = 0$, then one can use $\inf_{x\in\{x|\sigma=0\}} U(x) \geq 0$ to replace the condition (1).

Based on this idea with the principle of Theorem 3.7, Zhu [184] proved the following theorem.

Theorem 3.8. [184] *Suppose A is a Hurwitz matrix. For a given symmetric positive definite matrix B, $\forall f(\sigma) \subset F_{[0,k]}$, choose (3.4),*

$$V(x) = x^{\mathrm{T}} P x + \int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma,$$

then the sufficient and necessary conditions for

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)} = -x^{\mathrm{T}} B x + 2x^{\mathrm{T}}\left(Pb + \frac{1}{2}A^{\mathrm{T}}c\right) f(\sigma) + c^{\mathrm{T}} b f^2(\sigma) \tag{3.12}$$

being negative definite w.r.t. x are

$$\begin{aligned} &\frac{1}{k} - c^{\mathrm{T}} B^{-1} d > 0, \\ &\frac{1}{c^{\mathrm{T}} B^{-1} c}\left(\frac{1}{k} - c^{\mathrm{T}} B^{-1} d\right)^2 - d^{\mathrm{T}} B^{-1} d - c^{\mathrm{T}} b > 0, \end{aligned} \tag{3.13}$$

where $d = Pb + \frac{1}{2}A^{\mathrm{T}}c$.

Proof. Since B is a symmetric positive definite, so is B^{-1}, and there exists matrix H such that

$$B = H^{\mathrm{T}} H, \quad B^{-1} = H^{-1}(H^{-1})^{\mathrm{T}}.$$

Rewrite (3.12) as

$$\begin{aligned} -\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)} &= x^{\mathrm{T}} B x - 2x^{\mathrm{T}}\left(Pb + \frac{1}{2}A^{\mathrm{T}}c\right) f(\sigma) - c^{\mathrm{T}} b f^2(\sigma) \\ &= x^{\mathrm{T}} H^{\mathrm{T}} H x - 2x^{\mathrm{T}} d f(\sigma) - c^{\mathrm{T}} b f^2(\sigma) \\ &= (Hx)^{\mathrm{T}}(Hx) - 2(Hx)^{\mathrm{T}}(H^{-1})^{\mathrm{T}} d f(\sigma) - c^{\mathrm{T}} d f^2(\sigma) \\ &= \left[Hx - (H^{-1})^{\mathrm{T}} d f(\sigma)\right]^{\mathrm{T}}\left[Hx - (H^{-1})^{\mathrm{T}} d f(\sigma)\right] - (d^{\mathrm{T}} B^{-1} d + c^{\mathrm{T}} b) f^2(\sigma) \\ &= \begin{cases} 0 & \text{when } x = 0, \\ (Hx)^{\mathrm{T}}(Hx) = x^{\mathrm{T}} B x > 0 & \text{when } f(\sigma) > 0,\ x \neq 0, \\ U f^2(\sigma) & \text{when } f(\sigma) \neq 0, \end{cases} \end{aligned} \tag{3.14}$$

where

$$U = \left[H\frac{x}{f(\sigma)} - (H^{-1})d\right]^{\mathrm{T}}\left[H\frac{x}{f(\sigma)} - (H^{-1})d\right] - (d^{\mathrm{T}} B^{-1} d + c^{\mathrm{T}} b). \tag{3.15}$$

Since

$$\frac{c^{\mathrm{T}} x}{f(\sigma)} = \frac{\sigma}{f(\sigma)} \geq \frac{1}{k},$$

we only need to prove that for any $c^{\mathrm{T}}x \geq \dfrac{1}{k}$, we have $U > 0$. To achieve this, introduce the nonsingular transform

$$y = Hx - (H^{-1})^{\mathrm{T}}d$$

under which U becomes

$$U = y^{\mathrm{T}}y - \rho,$$

where

$$\rho = d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b.$$

Then the condition $c^{\mathrm{T}}x \geq \frac{1}{k}$ is now equivalent to

$$c^{\mathrm{T}}H^{-1}\left[y + (H^{-1})^{\mathrm{T}}d\right] \geq \frac{1}{k}. \tag{3.16}$$

Therefore, we need and only need to prove that $U > 0$ on the half closed space R^n_y, defined by (3.16).

It is easy to show that $\rho \geq 0$. So when $y = 0$, $U \leq 0$. Thus $y = 0$ is not on the half closed space (3.16), implying that

$$c^{\mathrm{T}}H^{-1}\left[0 + (H^{-1})^{\mathrm{T}}d\right] = c^{\mathrm{T}}H^{-1}(H^{-1})^{\mathrm{T}}d = c^{\mathrm{T}}B^{-1}d < \frac{1}{k}, \tag{3.17}$$

which is the first equation of (3.13).

It is easy to see that on the half closed space R^n_y, U reaches its minimum at the intersection point, $y = y^*$, of the hyperplane:

$$c^{\mathrm{T}}H^{-1}\left[y + (H^{-1})^{\mathrm{T}}d\right] = \frac{1}{k}$$

and its normal line $\mathbf{n}$ passing through $y = 0$, as shown in Fig. 3.4.

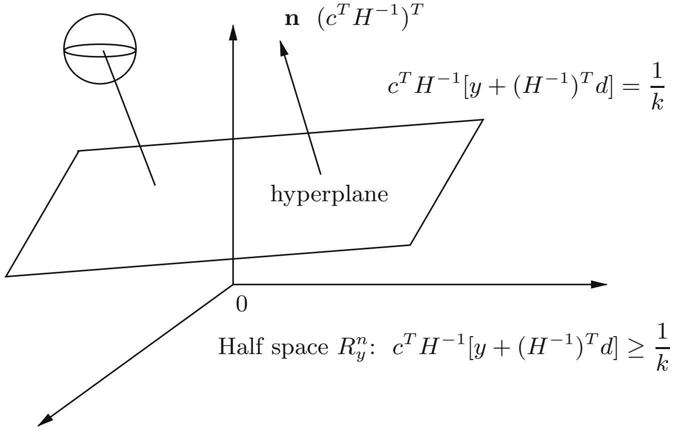

Fig. 3.4 Hyperplane and the half space R^n_y

Let

$$y^* = \lambda\,(c^{\mathrm{T}}H^{-1})^{\mathrm{T}} \quad (\lambda \text{ to be determined}).$$

Then solve for λ from the equation

$$c^{\mathrm{T}}H^{-1}\left[\lambda\,(c^{\mathrm{T}}H^{-1})^{\mathrm{T}} + (H^{-1})^{\mathrm{T}}d\right] = \frac{1}{k}$$

to obtain

$$\lambda = \frac{\frac{1}{k} - c^{\mathrm{T}}B^{-1}d}{c^{\mathrm{T}}B^{-1}c}.$$

Hence,

$$U(Y^*) = (y^{*T})y^* - \rho > 0$$

is equivalent to $U > 0$ on the half closed plane R_y^n (3.16). That is,

$$\begin{aligned} U(y^*) &= \left[\lambda\,(c^{\mathrm{T}}H^{-1})^{\mathrm{T}}\right]^{\mathrm{T}}\left[\lambda\,(c^{\mathrm{T}}H^{-1})^{\mathrm{T}}\right] - \rho, \\ &= \lambda^2 c^{\mathrm{T}}B^{-1}c - (d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b), \\ &= \frac{1}{c^{\mathrm{T}}B^{-1}c}\left(\frac{1}{k} - c^{\mathrm{T}}B^{-1}d\right)^2 - d^{\mathrm{T}}B^{-1}d - c^{\mathrm{T}}b > 0. \end{aligned} \tag{3.18}$$

(3.18) is the condition (3.13) and thus the proof is complete. □

Following Theorem 3.7, we can prove the following results [184].

Corollary 3.9. *Take the V function in the form (3.4), where $f(\sigma) \in F_{[0,k)}$, then the sufficient and necessary conditions for $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(3.2)}$ being negative definite w.r.t. x are*

$$\begin{cases} \dfrac{1}{k} - c^{\mathrm{T}}B^{-1}d \geq 0, \\ \dfrac{1}{c^{\mathrm{T}}B^{-1}c}\left(\dfrac{1}{k} - c^{\mathrm{T}}B^{-1}d\right)^2 - d^{\mathrm{T}}B^{-1}d - c^{\mathrm{T}}b \geq 0. \end{cases}$$

Letting $k \to \infty$ in the above condition yields

Corollary 3.10. *Choose the V function in the form (3.4), where $f(\sigma) \in F_\infty$, then the sufficient and necessary conditions for $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(3.2)}$ being negative definite w.r.t. x are*

$$\begin{cases} c^{\mathrm{T}}B^{-1}d \leq 0, \\ \dfrac{1}{c^{\mathrm{T}}B^{-1}c}(c^{\mathrm{T}}B^{-1}d)^2 - d^{\mathrm{T}}B^{-1}d - c^{\mathrm{T}}b \geq 0. \end{cases}$$

Peng [119] used the V function given in (3.4) to prove the following equivalent result.

Theorem 3.11. [119] *The following three conditions are equivalent, any of them gives a sufficient and necessary condition for $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(3.2)}$ being negative definite w.r.t. x.*

1. $\begin{cases} (1)\ U(x) = x^{\mathrm{T}}Bx - 2d^{\mathrm{T}}x - c^{\mathrm{T}}b \geq 0 \ \ (c^{\mathrm{T}}x = 0), \\ (2)\ c^{\mathrm{T}}B^{-1}d \leq 0, \end{cases}$

which is the condition given in Theorem 3.7;

2. $c^{\mathrm{T}}B^{-1}d + \sqrt{c^{\mathrm{T}}B^{-1}c\,(d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b)} \leq 0$*;*

3. $\begin{cases} (1)\ c^{\mathrm{T}}b + \eta^{\mathrm{T}}B^{-1}d - \dfrac{(c^{\mathrm{T}}B^{-1}d)^2}{c^{\mathrm{T}}B^{-1}c} \leq 0, \\ (2)\ c^{\mathrm{T}}B^{-1}d \leq 0, \end{cases}$ *where* $d = \frac{1}{2}(A^{\mathrm{T}}c + 2Pb)$.

To prove Theorem 3.11, we need the following lemma.

Lemma 3.12. *For any symmetric matrix B, we have*

$$c^{\mathrm{T}}b + d^{\mathrm{T}}B^{-1}d \geq 0,$$

where

$$d = \frac{1}{2}(A^{\mathrm{T}}c + 2Pb), \quad PA + A^{\mathrm{T}}P = -B.$$

Proof. Let $V(x) = x^{\mathrm{T}}Px + \int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma$. Then we have

$$-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.2)} = \begin{pmatrix} x \\ f(\sigma) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} B & -d \\ -d^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix} \begin{pmatrix} x \\ f(\sigma) \end{pmatrix}. \tag{3.19}$$

Since B is positive definite, $\det B > 0$. Then,

$$\det \begin{bmatrix} B^{-1} & 0 \\ 0^{\mathrm{T}} & 1 \end{bmatrix} = \det B^{-1} = (\det B)^{-1} > 0,$$

and

$$\det\left\{ \begin{bmatrix} B^{-1} & 0 \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} B & -d \\ -d^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix} \right\} = \det \begin{bmatrix} I & -B^{-1}d \\ -d^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix} = -c^{\mathrm{T}}b - d^{\mathrm{T}}B^{-1}d.$$

Thus,

$$\det \begin{bmatrix} B & -d \\ -d^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix} = (-c^{\mathrm{T}}b - d^{\mathrm{T}}B^{-1}d)\det B. \tag{3.20}$$

(3.20) is an important formula and has many applications.

If $c^{\mathrm{T}}b + d^{\mathrm{T}}B^{-1}d < 0$, then (3.19) is positive definite w.r.t. $(x, f(\sigma))$, which is, however, impossible as shown above. Therefore,

$$c^{\mathrm{T}}b + d^{\mathrm{T}}B^{-1}d \geq 0. \qquad \square$$

Now we prove Theorem 3.11 following the order (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).

Proof. Suppose condition (1) of Theorem 3.11 holds. It readily follows from (3.15) that $\forall x_0 \neq 0$, $U(x_0) > 0$ when $c^{\mathrm{T}}x_0 > 0$. Let

$$\bar{x} = B^{-1}d - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c,$$

where m is a constant. Thus,

$$c^{\mathrm{T}}\bar{x} = c^{\mathrm{T}}B^{-1}d - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}c^{\mathrm{T}}B^{-1}c = m.$$

So for all $m \in (0, +\infty)$, $U(\bar{x}) > 0$. However, on the other hand,

$$\begin{aligned} U(\bar{x}) &= \bar{x}^{\mathrm{T}}B\bar{x} - 2d^{\mathrm{T}}\bar{x} - c^{\mathrm{T}}b, \\ &= \left[d^{\mathrm{T}}(B^{-1})^{\mathrm{T}} - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}c^{\mathrm{T}}(B^{-1})^{\mathrm{T}}\right] \\ &\quad \times B\left[B^{-1}d - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right] \\ &\quad - 2d^{\mathrm{T}}\left[(B^{-1})^{\mathrm{T}}d - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right] - c^{\mathrm{T}}b. \end{aligned} \tag{3.21}$$

Because B is a symmetric positive definite matrix, so is B^{-1}, and $(B^{-1})^{\mathrm{T}} = B^{-1}$. Hence,

$$\begin{aligned} U(\bar{x}) &= \left[d^{\mathrm{T}} - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}c^{\mathrm{T}}\right]\left[B^{-1}d - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right] \\ &\quad - 2d^{\mathrm{T}}\left[B^{-1}d - \frac{c^{\mathrm{T}}B^{-1}d - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right] - c^{\mathrm{T}}b, \\ &= \frac{(c^{\mathrm{T}}B^{-1}d - m)^2}{c^{\mathrm{T}}B^{-1}c} - d^{\mathrm{T}}B^{-1}d - c^{\mathrm{T}}b, \\ &= \frac{1}{c^{\mathrm{T}}B^{-1}c}m^2 - 2\frac{c^{\mathrm{T}}B^{-1}d}{c^{\mathrm{T}}B^{-1}c}m + \frac{(c^{\mathrm{T}}B^{-1}d)^2}{c^{\mathrm{T}}B^{-1}c} - d^{\mathrm{T}}B^{-1}d - c^{\mathrm{T}}b, \end{aligned}$$

which is quadratic polynomial of m, and the coefficient of m^2 is

$$\frac{1}{c^{\mathrm{T}}B^{-1}c} > 0.$$

The discriminant of $U(\bar{x}) = 0$ is

$$\begin{aligned} \Delta &= 4\frac{(c^{\mathrm{T}}B^{-1}c)^2}{(c^{\mathrm{T}}B^{-1}c)^2} - 4\frac{1}{c^{\mathrm{T}}B^{-1}c}\left[\frac{(c^{\mathrm{T}}B^{-1}d)^2}{c^{\mathrm{T}}B^{-1}c} - d^{\mathrm{T}}B^{-1}d - c^{\mathrm{T}}b\right] \\ &= \frac{4}{c^{\mathrm{T}}B^{-1}c}(d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b). \end{aligned}$$

By Lemma 3.12, we know that $\Delta \geq 0$, so the equation $U(\bar{x}) = 0$ has two real roots:

$$m_{\pm} = c^{\mathrm{T}}B^{-1}d \pm \sqrt{c^{\mathrm{T}}B^{-1}c(d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b)}.$$

From the second condition of condition (1), we know $m_- \leq 0$. If $m_+ > 0$, then $\forall m \in (0, m_+)$, we obtain $U(\bar{x}) < 0$, a contradiction to the fact that $U(\bar{x}) > 0$ for $m \in (0, +\infty)$. Thus,

$$m_+ = c^{\mathrm{T}}B^{-1}d + \sqrt{c^{\mathrm{T}}B^{-1}c(d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b)} \le 0,$$

which is the condition (2), and the proof from (1) to (2) is finished.

Next proof (2) $\Longrightarrow$ (3). Suppose condition (2) holds, that is,

$$c^{\mathrm{T}}B^{-1}d + \sqrt{c^{\mathrm{T}}B^{-1}c(d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b)} \le 0,$$

so $c^{\mathrm{T}}B^{-1}d \le 0$, which is the second condition of (3). Thus we have

$$-c^{\mathrm{T}}B^{-1}d \ge \sqrt{c^{\mathrm{T}}B^{-1}c(d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b)},$$

that is,

$$(c^{\mathrm{T}}B^{-1}d)^2 \ge c^{\mathrm{T}}B^{-1}c(d^{\mathrm{T}}B^{-1}d + c^{\mathrm{T}}b),$$

or

$$c^{\mathrm{T}}b + d^{\mathrm{T}}B^{-1}d - \frac{(c^{\mathrm{T}}B^{-1}d)^2}{c^{\mathrm{T}}B^{-1}c} \le 0,$$

which is the first condition of (3).

Finally we want to prove (3) $\Longrightarrow$ (1). We only need to prove that the first condition of (1) is true under the condition (3). Since B is positive definite, there exists a nonsingular matrix H such that $H^{\mathrm{T}}BH = I$, and so $B^{-1} = HH^{\mathrm{T}}$. Take the nonsingular transform

$$x = H\left[y + \frac{1}{2}H^{\mathrm{T}}(A^{\mathrm{T}}c + 2Pb)\right] = H(y + H^{\mathrm{T}}d).$$

Let the vector y_0 correspond to x_0, then from $c^{\mathrm{T}}x_0 = 0$, we have

$$c^{\mathrm{T}}Hy_0 + c^{\mathrm{T}}HH^{\mathrm{T}}d = c^{\mathrm{T}}Hy_0 + c^{\mathrm{T}}B^{-1}d = 0,$$

and thus

$$(c^{\mathrm{T}}B^{-1}d)^2 = (c^{\mathrm{T}}Hy_0)^2.$$

From the Cauchy inequality:

$$(c^{\mathrm{T}}B^{-1}d)^2 \le \|c^{\mathrm{T}}H\|^2\,\|y_0\|^2,$$

we obtain

$$y_0^{\mathrm{T}}y_0 \ge \frac{(c^{\mathrm{T}}B^{-1}d)^2}{(c^{\mathrm{T}}H)(H^{\mathrm{T}}c)} = \frac{(c^{\mathrm{T}}B^{-1}d)^2}{c^{\mathrm{T}}B^{-1}c}.$$

Therefore, from the first condition of (3), we know that when $c^{\mathrm{T}}x_0 = 0$,

$$U(x_0) = y_0^{\mathrm{T}}y_0 - d^{\mathrm{T}}B^{-1}d - b^{\mathrm{T}}c \ge \frac{(c^{\mathrm{T}}B^{-1}d)^2}{c^{\mathrm{T}}B^{-1}c} - d^{\mathrm{T}}B^{-1}d - b^{\mathrm{T}}c \ge 0.$$

This indicates that the first condition of (1) is true. The proof of Theorem 3.11 is complete. □

3.5 Popov Frequency Criterion

In this section, we briefly introduce the Popov frequency method and the simplified Popov criterion.

3.5.1 The Classical Popov Criterion

In the late 1950s, Popov developed the popov frequency criterion [120–123]. First, we give three Lemmas.

Lemma 3.13. *Suppose the function $f(t)$ is defined for $t \geq 0$, and is continuous and piecewise continuously differentiable. Further assume that $f(t)$ and $f'(t)$ are bounded. The function $G(x)$ is defined for all x, and is continuous, positive definite for all x. If*

$$\int_0^\infty G(f(t))\,\mathrm{d}t = g < \infty, \tag{3.22}$$

then we have

$$\lim_{t\to\infty} f(t) = 0. \tag{3.23}$$

Proof. Let $|f(t)| \leq \alpha$ $(t \geq 0)$, $|f'(t)| \leq \alpha'$, where α and α' are positive constants.

Suppose (3.23) is not true. Then there exist an $\varepsilon > 0$ and a monotone increasing infinite divergent series $\{t_k\}$ such that

$$|f(t_k)| \geq 2\varepsilon.$$

Without loss of generality, assume $t_1 > \delta$, and $t_{k+1} - t_k \geq 2\delta$, where δ is an appropriately chosen positive number.

Since when $t_k - \delta \leq t_k + \delta$,

$$|f(t) - f(t_k)| \leq |f'(\xi)|\,|t - t_k| \leq \alpha'\,\delta.$$

From

$$\int_0^\infty G(f(t))\,\mathrm{d}t \geq \sum_{k=1}^{\infty} \int_{t_k-\delta}^{t_k+\delta} G(f(t))\,\mathrm{d}t,$$

we can derive

$$|f(t)| \geq |f(t_k)| - |f(t) - f(t_k)| \geq 2\varepsilon - \delta\,\alpha' = \varepsilon$$

by taking $\delta = \frac{\varepsilon}{\alpha'}$. Let $\eta = \inf_{\varepsilon \leq |x| \leq \alpha} G(x) > 0$. Then

$$g = \int_0^\infty G(f(t))\,\mathrm{d}t > k\,\delta\,\eta \quad (k = 1, 2, \dots) \quad \to \infty \quad \text{as} \quad k \to \infty,$$

a contradiction. This finishes the proof of Lemma 3.13. □

Lemma 3.14. *Let the functions f_k $(k = 1,2,3)$ be defined for $t \geq 0$ and continuous. Further suppose that there exist constants a_k such that*

$$|f_k(t)|\,\mathrm{e}^{\alpha_k t} \leq c_k \quad (\text{for } g \geq 0),$$

then their Fourier transforms

$$F_k(i\,\omega) = \int_0^{\infty} f_k(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t \quad (k = 1,2,3) \tag{3.24}$$

exist. Let

$$F_1(i\omega) = H(i\omega)\,F_3(i\omega) + F_2(i\omega),$$

where

$$H(i\omega) = \int_0^{\infty} h(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t, \quad \omega \geq 0, \quad \mathrm{Re}H(i\omega) \geq \delta > 0.$$

Then we have

$$\int_0^{\infty} f_1(t)\,f_3(t)\,\mathrm{d}t \geq -\gamma := -\frac{1}{8\pi\delta}\int_{-\infty}^{+\infty} |F_2(i\omega)|^2\,\mathrm{d}\omega = \frac{1}{4\delta}\int_0^{\infty} f_2^2(t)\,\mathrm{d}t. \tag{3.25}$$

Proof. The existence of the Fourier transforms is obvious. To prove (3.25), we use the Parseval formula, the fact that $|F_3(i\omega)|^2$ is even function, and that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_0^{\infty} h(t)\,i\sin(\omega t)\,\mathrm{d}t\right)\mathrm{d}\omega = 0$$

(due to $\displaystyle\int_0^{\infty} h(t)\,i\sin(\omega t)\,\mathrm{d}t$ being odd) to obtain

$$\begin{aligned}
\int_0^{\infty} f_1(t)\,f_3(t)\,\mathrm{d}t &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{F}_1(i\omega)\,F_3(i\omega)\,\mathrm{d}\omega \quad (\text{Parseval formula}),\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\bar{H}(i\omega)\,\bar{F}_3(i\omega) + \bar{F}_2(i\omega)\right\}F_3(i\omega)\,\mathrm{d}\omega,\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\mathrm{Re}\bar{H}(i\omega)\,|F_3(i\omega)|^2\right.\\
&\quad\left. + \frac{1}{2}\left[\bar{F}_2(i\omega)\,\bar{F}_3(i\omega) + F_2(i\omega)\,\bar{F}_3(i\omega)\right]\right\}\mathrm{d}\omega,\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|(\mathrm{Re}\bar{H}(i\omega))^{1/2}F_3(i\omega) + \frac{F_2(i\omega)}{2(\mathrm{Re}\bar{H}(i\omega))^{1/2}}\right|^2\mathrm{d}\omega\\
&\quad - \frac{1}{8\pi}\int_{-\infty}^{+\infty}\frac{|F_2(i\omega)|^2}{\mathrm{Re}\bar{H}}\,\mathrm{d}\omega,\\
&\geq -\frac{1}{8\pi}\int_{-\infty}^{+\infty}|F_2(i\omega)|^2\,\mathrm{d}\omega.
\end{aligned}$$

This shows that (3.25) is true. □

Lemma 3.15. *If $A = (a_{ij})_{n\times n}$ is Hurwitz stable, and $f(t)$ is a continuous function defined for $t \geq t_0$, and $\lim_{t\to+\infty} f(t) = 0$, then every solution of the nonhomogeneous equation*

$$\dot{x} = Ax + bf(t) \tag{3.26}$$

holds and $x(t) \to 0$ as $t \to +\infty$, where

$$x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}, \quad b = (b_1, b_2, \ldots, b_n)^{\mathrm{T}}.$$

Proof. By the method of constant variation, the solution of system (3.26) can be expressed as

$$x(t) = \mathrm{e}^{A(t-t_0)}x_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}bf(\tau)\,\mathrm{d}\tau.$$

Since A is Hurwitz stable, there exist $M > 1$ and $\alpha > 0$ such that

$$\|\mathrm{e}^{A(t-t_0)}\| \leq M\mathrm{e}^{-\alpha(t-t_0)}.$$

so the first term of $x(t)$ goes to zero as $t \to +\infty$. For the second term of x, we have

$$\begin{aligned}\left\|\int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}bf(\tau)\,\mathrm{d}\tau\right\| &\leq M\int_{t_0}^{t} \mathrm{e}^{-\alpha(t-t_0)}\,\|b\|\,|f(\tau)|\,\mathrm{d}\tau \quad (t \geq t_0)\\ &= M\,\|b\|\,\frac{\int_{t_0}^{t} \mathrm{e}^{\alpha\tau}|f(\tau)|\,\mathrm{d}\tau}{\mathrm{e}^{\alpha t}}.\end{aligned}$$

Then by L'Hospital Rule we obtain

$$\lim_{t\to+\infty} \frac{\int_{t_0}^{t} \mathrm{e}^{\alpha\tau}|f(\tau)|\,\mathrm{d}\tau}{\mathrm{e}^{\alpha t}} = \lim_{t\to+\infty} \frac{\mathrm{e}^{\alpha t}\,|f(t)|}{\alpha\,\mathrm{e}^{\alpha t}} = \lim_{t\to+\infty} \frac{|f(t)|}{\alpha} = 0,$$

which implies that

$$\lim_{t\to+\infty} x(t) = 0.$$

This proves Lemma 3.15. □

Theorem 3.16. [120, 122] *If there exists a real number q such that*

$$\mathrm{Re}\Big[(1 + i\,q\,\omega)W(i\omega)\Big] + \frac{1}{k} > 0 \quad \text{for} \quad \omega \geq 0, \tag{3.27}$$

where

$$W(i\omega) = \frac{K(i\omega)}{D(i\omega)} = \frac{\det\begin{bmatrix} i\omega I - A & b \\ c^{\mathrm{T}} & 0 \end{bmatrix}}{\det(i\omega I - A)},$$

then the zero solution of (3.2) is absolutely stable in the Hurwitz angle region $[0,k]$.

Because the proof is very lengthy and many people are familiar with the result of this classical method, we will not give the detailed proof here (interested readers can find the complete proof in [127]), but give some geometrical explanation, and outline the proof.

Let

$$X(\omega) = \mathrm{Re} W(i\omega) = \frac{\mathrm{Re}\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2} \tag{3.28}$$

and

$$Y(\omega) = \omega \mathrm{Im} W(i\omega) = \omega \frac{\mathrm{Im}\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2}. \tag{3.29}$$

Then the condition of theorem (3.27) is equivalent to

$$X(\omega) - qY(\omega) + \frac{1}{k} > 0 \quad (\omega \geq 0). \tag{3.30}$$

Thus, on the complex plane $X + iY$, for all ω, the graph of the characteristic curve

$$W^*(\omega) = X(\omega) + iY(\omega) \qquad (W^*(-\omega) = W^*(\omega)) \tag{3.31}$$

is located on the right side of the line l^*, which passes the point $(-1/k, 0)$ with the slope $1/q$ (see Figs. 3.5–3.7).

The following is the outline of the proof for Popov theorem.

(1) First prove that the solution of (3.2), $x(t)$, is defined for $t \geq 0$. Consider an arbitrary solution of (3.2), $x(t, t_0, x_0) := x(t)$, which can be written as

$$x(t) = \mathrm{e}^{At} x_0 + \int_0^{\mathrm{T}} f(c^{\mathrm{T}} x(\tau))\, \mathrm{e}^{A(t-\tau)} b\, \mathrm{d}\tau. \tag{3.32}$$

Since A is stable, there exists $\alpha > 1$, $M > 0$ such that

$$\|x(t)\| \leq M \mathrm{e}^{-\alpha t} \|x_0\| + \int_0^t M K \|b\| \, \|c\| \, \|x(\tau)\| \, \mathrm{e}^{-\alpha(t-\tau)} \, \mathrm{d}\tau.$$

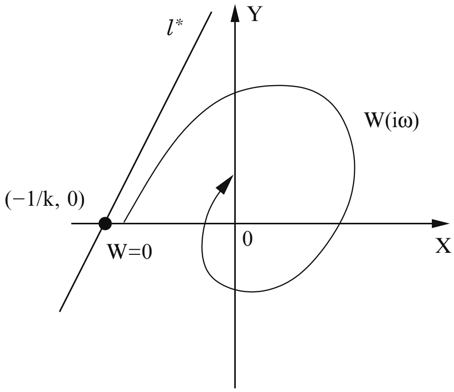

Fig. 3.5 The characteristic frequency curve $W(i\omega)$ with slope $1/q > 0$

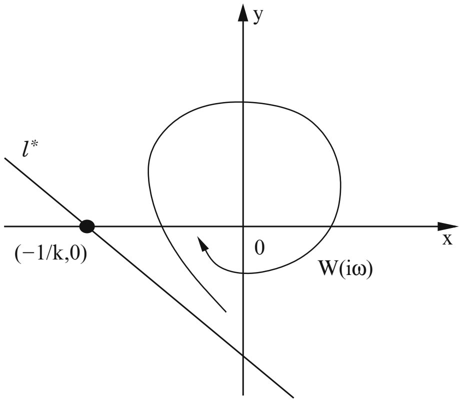

Fig. 3.6 The characteristic frequency curve $W(i\omega)$ with slope $1/q < 0$

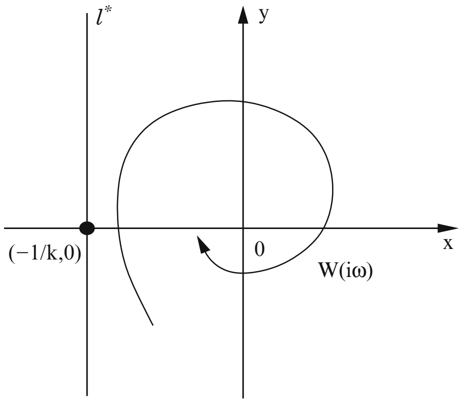

Fig. 3.7 The characteristic frequency curve $W(i\omega)$ with slope $1/q = +\infty$

Then from the Gronwall–Bellman inequality we obtain

$$\|x(t)\| \leq M\,\|x_0\|\,\mathrm{e}^{(MK\,\|b\|\,\|c\|-\alpha)t} \tag{3.33}$$

from which we know that $x(t)$ globally exists for all $t \geq 0$.

(2) Suppose in (3.27) $q > 0$. Take $0 < h \ll 1$ and rewrite (3.2) as

$$\dot{x} = Ax + (f(\sigma) - h\sigma)\,b. \tag{3.34}$$

Further let

$$\bar{f}(\sigma) = h\sigma + f(\sigma) \qquad \left(h \leq \frac{f(\sigma)}{\sigma} \leq k+h,\ \sigma \neq 0\right).$$

Then (3.34) can be rewritten as

$$\dot{x} = A_{-h}x + \bar{f}(\sigma)\,b \qquad (A_{-h} = A - h\,b\,c^{\mathrm{T}}). \tag{3.35}$$

For any $L > 0$, we have

$$\xi_T(t) = \begin{cases} \xi(t) = \bar{f}(\sigma(t)) = \bar{f}(c^{\mathrm{T}}x(t,t_0,x_0)) & (0 \le t \le L), \\ 0 \quad (t > L). \end{cases}$$

Repeatedly apply Lemma 3.14 to prove $\int_0^{+\infty} \bar{f}(\sigma(t))\,\mathrm{d}t < \infty$. Then apply Lemma 3.13 to show that $\sigma(t) \to 0$ as $t \to 0$. Finally use Lemma 3.15 to obtain $x(t) \to 0$ as $t \to +\infty$. This shows that the zero solution of (3.2) is absolutely stable in the Hurwitz angle region $[0,k]$.

Definition 3.17. *Suppose A has eigenvalues with zero real parts, but no eigenvalues with positive real parts. Then $\forall \varepsilon > 0$, when $f(\sigma) = \varepsilon\,\sigma$, the zero solution of (3.2) is globally stable, and we say that the zero solution of (3.2) is limit (or ε) stable.*

Theorem 3.18. *Suppose the following system*

$$\begin{aligned} \dot{x} &= Ax + bf(\sigma), \\ \sigma &= c^{\mathrm{T}}x \end{aligned} \tag{3.36}$$

satisfies the limit stability in the singular situation, and

$$\mathrm{Re}\,[\,(1 + iq\,\omega)W(i\omega)\,] + \frac{1}{k} > 0, \quad \omega \ge 0 \quad (q \text{ is a constant}), \tag{3.37}$$

then for each continuous function satisfying

$$0 < \varepsilon \le \frac{f(\sigma)}{\sigma} \le k \quad (\sigma \ne 0),$$

the zero solution of (3.36) is globally asymptotically stable, that is, the zero solution of (3.36) is absolutely stable in the Hurwitz angle region $[\varepsilon,k]$.

The proof can be found in [127] and omitted here.

3.5.2 The Simplified Popov Criterion

Popov frequency criterion is actually a generalization of the Nyquist frequency criterion. Since the domain of the characteristic curve can be infinite, it is not convenient in applications. In the following, we introduce a simplified Popov criterion, due to Zhang [178]. The result obtained in [178] applied the idea of Wang [152, 153] to change the domain of frequency characteristic curve from an infinite interval $[0,\infty]$ to a finite interval $[0,\rho]$ ($\rho > 0$), which improves the Popov criterion from both theoretical and application aspects.

Again consider system (3.2). Let

$$W(i\omega) = -\,c^{\mathrm{T}}(i\omega I - A)^{-1}b = \frac{K(i\omega)}{D(i\omega)},$$

where $D(i\omega) = \det(i\omega I - A)$, and

$$X(\omega) = \mathrm{Re}W(i\omega) = \frac{\mathrm{Re}\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2},$$

$$Y(\omega) = \omega\ \mathrm{Im}W(i\omega) = \omega\ \frac{\mathrm{Im}\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2} := \frac{H(\omega)}{|D(i\omega)|^2}.$$

Suppose the real polynomial $H(\omega)$ is in the form of

$$H(\omega) = h_{2n} + h_{2n-1}\,\omega + \cdots + h_1\,\omega^{2n-1}. \tag{3.38}$$

Further let

$$\rho = 1 + \max_{2\le i\le 2n}\left\{\left|\frac{h_i}{h_1}\right|\right\}. \tag{3.39}$$

Then we have

Lemma 3.19. *In system (3.2) let* $f(\sigma) = \varepsilon\sigma$ $(0 \le \varepsilon \le k)$. *Then the sufficient and necessary condition for the asymptotic stability of the corresponding linear system is that the frequency characteristic curve* $W(i\omega)$ $(0 \le \omega \le \rho)$ *does not intersect the part of the real axis* $(-\infty, -1/k)$.

Proof. When $0 \le \varepsilon \le k$, the sufficient and necessary condition for the asymptotic stability of the corresponding linear system is that the characteristic frequency curve $W(i\omega)$ $(-\infty \le \omega \le +\infty)$ does not intersect the part of the real axis $(-\infty, -1/k)$. So the *necessity* of Lemma 3.19 is obvious.

For *sufficiency*, since $W(i\omega) = X(\omega) + iY(\omega)$, from the bounded value theorem of zero point for polynomials, we know that all zeros of $H(\omega) = 0$ satisfy $|\omega_j| < \rho$ $(1 \le j \le 2n-1)$. Thus, $H(\omega) \ne 0$ for $|\omega| > \rho$, that is, $Y(\omega) \ne 0$ when $|\omega| > \rho$, for which $W(i\omega)$ does not intersect the real axis. From

$$X(\omega) = \frac{K(i\omega)D(-i\omega) + K(-i\omega)D(i\omega)}{2D(i\omega)D(-i\omega)}$$

we obtain

$$X(-\omega) = X(\omega).$$

Hence, if $W(i\omega)$ $(0 \le \omega \le \rho)$ does not intersect the part of the real axis $(-\infty, -1/k)$, then $W(i\omega)$ $(-\rho \le \omega \le 0)$ also does not intersect the part of the real axis $(-\infty, -1/k)$. Therefore, $W(i\omega)$ $(-\infty \le \omega \le +\infty)$ does not intersect the part of the real axis $(-\infty, -1/k)$. The sufficiency is proved. □

Lemma 3.19 actually provides the necessary condition for system (3.2) being absolutely stable in the Hurwitz angle region $[0, k]$.

Before discussing the absolute stability of system (3.2), we introduce the following notations:

$$X^*(\omega) = \mathrm{Re}W(i\omega) = \frac{\mathrm{Re}\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2} := \frac{A(\omega)}{E(\omega)},$$

$$Y^*(\omega) = \mathrm{Im}W(i\omega) = \frac{\mathrm{Im}\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2} := \frac{B(\omega)}{E(\omega)}.$$

Theorem 3.20. [178] *If there exists a real number, q, such that*

$$\mathrm{Re}\left[(1+iq\omega)W(i\omega)\right]+\frac{1}{k}>0, \quad \omega\in[0,\rho], \tag{3.40}$$

then the zero solution of system (3.2) is absolutely stable in the Hurwitz angle region $[0,k]$*, where*

$$\rho = 1+\max_{2\le i\le 2n+1}\left|\frac{\rho_i}{\rho_1}\right|,$$

in which ρ_i $(i=1,2,\dots,2n+1)$ *are the coefficients of the polynomial:*

$$\rho(\omega)=A(\omega)-qB(\omega)+\frac{1}{k}E(\omega).$$

Proof. The condition (3.40) is equivalent to

$$X^*(\omega)-qY^*(\omega)+\frac{1}{k}>0, \quad \omega\in[0,\rho], \tag{3.41}$$

that is,

$$\frac{A(\omega)}{E(\omega)}-q\frac{B(\omega)}{E(\omega)}+\frac{1}{k}>0, \quad \omega\in[0,\rho],$$

or

$$\frac{A(\omega)-qB(\omega)+\frac{1}{k}E(\omega)}{E(\omega)}=\frac{\rho(\omega)}{E(\omega)}>0, \quad \omega\in[0,\rho].$$

From the definition of ρ and the intermediate value theorem, we have $\rho(\omega)\neq 0$. When $\omega>\rho$, $\rho(\omega)>0$. Thus when $\omega>\rho$, $\rho(\omega)>0$, that is,

$$X^*(\omega)-qY^*(\omega)+\frac{1}{k}=\frac{\rho(\omega)}{E(\omega)}>0, \quad (\omega>\rho).$$

Combining the above result with (3.41) yields

$$\mathrm{Re}\left[(1+iq\omega)W(i\omega)\right]+\frac{1}{k}>0, \quad (\omega\ge 0).$$

Therefore, the zero solution of (3.2) is absolutely stable in the Hurwitz angle region $[0,k]$. □

By Theorem 3.20 it makes possible to use a finite part of the characteristic frequency curve to determine the absolute stability of nonlinear control systems. Comparing with the Popov criterion, it greatly reduces computational demanding. From the geometrical point of view, the zero solution of (3.2) is absolutely stable provided that for $\omega\in[0,\rho)$, the graph of the characteristic curve

$$W^*(\omega)=X(\omega)+iY(\omega) \tag{3.42}$$

is located on the right side of the line that passes through the point $[-1/k,0]$ having the slope $1/q$.

However, the parameter ρ in Theorem 3.20 still depends upon q and k, where q is an existence parameter. Thus, it is still difficult to determine ρ, and needs to further overcome this obstacle.

For a given $k > 0$, let

$$\begin{aligned} G(\omega) = A(\omega) + \frac{1}{k} E(\omega) &= c_{2n} + c_{2n-1}\,\omega + \cdots + c_0\,\omega^{2n}, \\ B(\omega) &= b_{2n} + b_{2n-1}\,\omega + \cdots + b_0\,\omega^{2n}, \\ \rho_1 &= 1 + \max_{1\le i\le 2n} \left|\frac{c_i}{c_0}\right|, \\ \rho_2 &= 1 + \max_{1\le i\le 2n} \left|\frac{b_i}{c_0}\right|. \end{aligned}$$

Theorem 3.21. [178] *If one of the following two conditions holds, then the zero solution of system (3.2) is absolutely stable in the Hurwitz angle region* $[0,k]$*:*

(1) $b_0 < 0$ *(or* $b_0 = 0,\ b_1 < 0$*), and there exits* $q \ge 0$ *such that*

$$\mathrm{Re}\,[(1 + i\,q\,\omega)W(i\omega)] + \frac{1}{k} > 0, \quad \text{for } \omega \in [0,\rho]; \tag{3.43}$$

(2) $b_0 > 0$ *(or* $b_0 = 0,\ b_1 > 0$*), and there exits* $q \le 0$ *such that*

$$\mathrm{Re}\,[(1 + i\,q\,\omega)W(i\omega)] + \frac{1}{k} > 0, \quad \text{for } \omega \in [0,\rho]. \tag{3.44}$$

Here $\rho = \max(\rho_1, \rho_2)$.

Proof. Since (3.41) is equivalent to

$$\frac{A(\omega) - qB(\omega) + \frac{1}{k}E(\omega)}{E(\omega)} > 0, \quad \omega \in [0,\rho],$$

we only need to prove

$$A(\omega) - qB(\omega) + \frac{1}{k}E(\omega) = G(\omega) - qB(\omega) > 0 \quad (\omega > \rho). \tag{3.45}$$

If condition (1) holds, then from the definition of ρ, we know that when $\omega > \rho$, $G(\omega) \ne 0$ and $B(\omega) \ne 0$. Further from $b_0 > 0$ (or $b_0 = 0,\ b_1 < 0$), $B(\omega) < 0$ for enough large ω. Noticing that the degree of $A(\omega)$ is at least one less than that of $E(\omega)$, the coefficient of the highest order term is $1/k$. Therefore, when $\omega \gg 1$, $G(\omega) > 0$. This implies that when $\omega > \rho$, $G(\omega) > 0$ and $B(\omega) < 0$, and so (3.43) holds.

A similar proof can be given to condition (2). □

Finally, to end this section, we establish a criterion that ρ is independent of q and k. Let

$$A(\omega) = a_{2n-1} + a_{2n-2}\,\omega + \cdots + a_1\,\omega^{2n-2} + a_0\,\omega^{2n-1},$$

where at least one of a_0 and a_1 is nonzero, and $\rho_3 = 1 + \max_{0\le i\le 2n-1} |\frac{a_i}{a_0}|$.

Theorem 3.22. [178] *If one of the following two conditions holds, then the zero solution of system (3.2) is absolutely stable in the Hurwitz angle region* $[0,k]$*:*

(1) $a_0 > 0$ *(or* $a_0 = 0,\ a_1 < 0$*),* $b_0 < 0$ *(or* $b_0 = 0,\ b_1 > 0$*), and there exists* $q \geq 0$ *such that*

$$\mathrm{Re}\left[(1+iq\omega)W(i\omega)\right]+\frac{1}{k}>0, \quad \text{for } \omega\in[0,\rho]; \tag{3.46}$$

(2) $a_0 < 0$ *(or* $a_0 = 0,\ a_1 < 0$*),* $b_0 > 0$ *(or* $b_0 = 0,\ b_1 > 0$*), and there exists* $q \leq 0$ *such that*

$$\mathrm{Re}\left[(1+iq\omega)W(i\omega)\right]+\frac{1}{k}>0, \quad \text{for } \omega\in[0,\rho]. \tag{3.47}$$

Here $\rho = \max(\rho_2, \rho_3)$.

Proof. If condition (1) holds, (3.46) is equivalent to

$$X^*(\omega)-qY^*(\omega)+\frac{1}{k}=\frac{A(\omega)}{E(\omega)}-q\frac{B(\omega)}{E(\omega)}+\frac{1}{k}>0 \quad \text{for } \omega\in[0,\rho]. \tag{3.48}$$

From the definition of ρ, we know that when $\omega > \rho$, $A(\omega) \neq 0$ and $B(\omega) \neq 0$. Further it follows from condition (1) that when $\omega > \rho$, $A(\omega) > 0$ and $B(\omega) < 0$. Thus when $\omega > \rho$,

$$A(\omega)-qB(\omega)>0,$$

which implies that

$$\frac{A(\omega)}{E(\omega)}-q\frac{B(\omega)}{E(\omega)}+\frac{1}{k}>0.$$

Combining this with (3.46) results in

$$\mathrm{Re}\left[(1+iq\omega)W(i\omega)\right]+\frac{1}{k}>0, \quad \text{for } \omega\geq 0.$$

Hence, the zero solution of system (3.2) is absolutely stable. Similarly, we can prove condition (2). □

By the improved Popov criterion, one only needs to obtain the characteristic frequency curve for $\omega \in [0,\rho]$, and the verification of Theorem 3.21 can be easily carried out by a digital computer. The condition given in Theorem 3.22 is a little bit stronger than that of the standard Popov criterion, but much convenient in applications.

In this chapter, we have discussed three classical methods: Lurie-type V-function method, Lurie-type V-function combined with the S-program, and the Popov frequency approach. Naturally a question is raised: How are the three methods related? This has been discussed in detail in [159]. A simple relationship is given below.

$$
\begin{array}{ccccccc}
\text{Lurie Method} & \supset & S\text{-program} & \Longleftrightarrow & \text{Popov Criterion} & \Longleftrightarrow & \text{Simplified Popov Criterion} \\
\text{Thm 3.3} & \supset & \text{Thm 3.4} & & \text{Thm 3.16} & & \text{Thm 3.20} \\
\Updownarrow & & \text{Thm 3.6} & & & & \\
\text{Thm 3.7} & & & & & & \\
\Updownarrow & & & & & & \\
\text{Thm 3.8} & & & & & & \\
\Updownarrow & & & & & & \\
\text{Thm 3.11} & & & & & &
\end{array}
$$

The condition of Popov method is merely sufficient condition, which guarantees $\sigma(t) \to 0$ as $t \to +\infty$. Are there any other conditions different from that of Popov's method that also guarantees $\sigma(t) \to 0$ as $t \to +\infty$? Is the condition that $\sigma(t) \to 0$ as $t \to +\infty$ still a necessary condition for absolute stability? These questions motivate us to find sufficient and necessary conditions for the absolute stability, which will be discussed in the next chapter.

4

Necessary and Sufficient Conditions for Absolute Stability

In this chapter, we discuss the necessary and sufficient conditions for absolute stability of various Lurie control systems described by ordinary differential equations. The absolute stability for all the system's variables will be equivalently transformed into that of a single variable or partial variables, and that of the Hurwitz stability for matrix, which is easy to be verified. Based on obtained theoretical results, some practically useful algebraic sufficient conditions will be derived, which provide guidelines for designers and engineers. The material given in 4.1 and 4.3 is chosen from [78, 80] and that presented in 4.2 and 4.4 is based on [67, 76, 77]. The results given in 4.5 are mainly taken from [89].

4.1 Necessary and Sufficient Conditions for Absolute Stability

Consider the general Lurie control system:

$$\begin{aligned} \dot{x} &= Ax + bf(\sigma), \\ \sigma &= c^{\mathrm{T}}x = \sum_{i=1}^{n} c_i x_i, \end{aligned} \tag{4.1}$$

where $f(\sigma) \in F_\infty$, $f(\sigma) \in F_{[0,k)}$ $f(\sigma) \in F_{[0,k]}$, or $f(\sigma) \in F_{[k_1,k_2]}$.

Definition 4.1. *The zero solution of* (4.1) *is said to be absolutely stable for the set* $\Omega = \{x : \sigma = 0\}$ *(absolutely stable for* Ω *in* $[0,k)$*, or* $[0,k], [k_1,k_2]$*) if for every* $\varepsilon > 0$*, there exists* $\delta(\varepsilon) > 0$ *such that for any* $f(\sigma) \in F_\infty$ *(for any* $f(\sigma) \in F_{[0,k)}$ *or* $F_{[0,k]}$*,* $F_{[k_1,k_2]}$*), the solution* $x(t) = x(t,t_0,x_0)$ *of* (4.1) *satisfies*

$$|\sigma(t,t_0,x_0)| = |c^{\mathrm{T}}x(t,t_0,x_0)| < \varepsilon \quad \textit{for all} \quad t \geq t_0$$

if $\|x_0\| < \delta(\varepsilon)$*, and for any* $x_0 \in R^n$*,*

$$\lim_{t\to+\infty} \sigma(t,t_0,x_0) = \lim_{t\to+\infty} c^{\mathrm{T}}x(t,t_0,x_0) = 0.$$

Definition 4.2. *A function* $V(x) \in C[R^n, R^1]$ *is said to be positive definite for the set* Ω *if*

$$V(x) \begin{cases} =0 & \textit{when} \quad x \in \Omega, \\ >0 & \textit{when} \quad x \notin \Omega. \end{cases}$$

$V(x) \in C[R^n, R^1]$ is negative definite for Ω if $-V(x)$ is positive definite for Ω. $V(x) \in C[R^n, R^1]$ is radially unbounded, positive definite for Ω if $V(x)$ is positive definite for Ω and $V(x) \to +\infty$ as $|\sigma| = |c^{\mathrm{T}} x| \to +\infty$.

Theorem 4.3. *The necessary and sufficient conditions for the zero solution of (4.1) to be absolutely stable (absolutely stable in $[0,k)$, $[0,k]$, $[k_1,k_2]$) are the following:*

(1) The matrix $A + bc^{\mathrm{T}}\theta := B$ is Hurwitz stable, where $\theta = 0$ or $\theta = 1$ ($\theta = 0$ or $\theta = \frac{1}{2}k$, or $\theta = \frac{1}{2}(k_2 - k_1)$);
(2) The zero solution of (4.1) is absolutely stable for Ω (absolutely stable in $[0,k)$ or $[0,k]$ or $F_{[k_1,k_2]}$ for Ω).

Proof. *Necessity*. (1) When $\mathrm{Re}\lambda(A) < 0$, we choose $\theta = 0$; when $\mathrm{Re}\lambda(A) \le 0$, we take $\theta = 1$ $\left(\theta = \frac{1}{2}k \text{ or } \theta = \frac{1}{2}(k_2 - k_1)\right)$. By putting $f(\sigma) = \sigma = c^{\mathrm{T}}x$ ($f(\sigma) = \frac{1}{2}k\sigma = \frac{1}{2}kc^{\mathrm{T}}x$) in (4.1) it follows that the matrix $B = A + bc^{\mathrm{T}}\theta$ is Hurwitz stable.

(2) For any $\varepsilon > 0$, we take $\tilde{\varepsilon} = \varepsilon \max_{1\le i\le n} |c_i|$. Then there exists $\delta(\varepsilon) > 0$ such that for any $f \in F_\infty$ (for any $f \in F_{[0,k)}$, $F_{[0,k]}$ or $F_{[k_1,k_2]}$), $\|x_0\| < \delta(\varepsilon)$ implies that

$$\|x(t,t_0,x_0)\| := \|x(t)\| = \sum_{i=1}^{n} |x_i(t)| < \tilde{\varepsilon} \quad \text{for all} \quad t \ge t_0,$$

and further

$$\begin{aligned} |\sigma(t,t_0,x_0) := |\sigma(t)| &= \sum_{i=1}^{n} |c_i x_i(t)| \\ &\le \max_{1\le i\le n} |c_i| \sum_{i=1}^{n} |x_i(t)| < \max_{1\le i\le n} |c_i|\, \tilde{\varepsilon} = \varepsilon. \end{aligned}$$

Clearly, we have

$$\lim_{t\to+\infty} |\sigma(t)| \le \lim_{t\to+\infty} \max_{1\le i\le n} |c_i| \sum_{i=1}^{n} |x_i(t)| = 0$$

for any $x_0 \in \mathrm{R}^n$.

Sufficiency. For any $f \in F_\infty$ $\left(\text{for any } f \in F_{[0,k)},\ F_{[0,k]} \text{ or } F_{[k_1,k_2]}\right)$, the solution of (4.1) can be expressed as

$$x(t) = \mathrm{e}^{B(t-t_0)} x_0 + \int_{t_0}^{t} \mathrm{e}^{B(t-\tau)} [bf(\sigma(\tau)) - \theta b \sigma(\tau)]\, \mathrm{d}\tau.$$

Since B is Hurwitz stable, there exist constants $M \ge 1$ and $\alpha > 0$ such that

$$\|\mathrm{e}^{B(t-t_0)}\| \le M \mathrm{e}^{-\alpha(t-t_0)} \quad \text{for all} \quad t \ge t_0.$$

Since $\sigma(t) \to 0$ as $t \to +\infty$, $\sigma(t)$ continuously depends on x_0, and $f(\sigma(t))$ is a continuous function of x_0 and $f(\sigma(t)) \to 0$ as $t \to +\infty$, for any $\varepsilon > 0$, there exit $\delta_1(\varepsilon) > 0$ and $t_1 > t_0$ such that

$$\int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)}\Big[\|bf(\sigma(\tau))\|+\|b\theta\sigma(\tau)\|\Big]\mathrm{d}\tau<\frac{\varepsilon}{3}\quad \text{for } t_1\geq t_0,$$
$$\int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)}\Big[\|bf(\sigma(\tau))\|+\|b\theta\sigma(\tau)\|\Big]\mathrm{d}\tau<\frac{\varepsilon}{3}\quad \text{for all}\quad t\geq t_1,$$
$$\|\mathrm{e}^{B(t-t_0)}x_0\|\leq M\mathrm{e}^{-\alpha(t-\tau)}\|x_0\|<\frac{\varepsilon}{3}\quad \text{for all}\quad t\geq t_0$$

if

$$\|x_0\|<\delta_1(\varepsilon).$$

Thus, it follows

$$\begin{aligned}\|x(t)\| &\leq \|\mathrm{e}^{B(t-t_0)}x_0\|+\int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)}\Big[\|bf(\sigma(\tau))\|+\|\theta b\sigma(\tau)\|\Big]\mathrm{d}\tau\\ &\quad+\int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)}[\|bf(\sigma(\tau))\|+\|\theta b\sigma(\tau)\|]\mathrm{d}\tau\\ &<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\quad \text{for all}\quad t\geq t_0.\end{aligned}$$

$\forall x_0\in \mathrm{R}^n$, by using the L'Hospital rule we have

$$\begin{aligned}0&\leq \lim_{t\to+\infty}\|x(t)\|\\ &\leq \lim_{t\to+\infty} M\mathrm{e}^{-\alpha(t-t_0)}\|x_0\|\\ &\quad+\lim_{t\to+\infty}\int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-\tau)}\Big[\|bf(\sigma(\tau))\|+\|\theta b\sigma(\tau)\|\Big]\mathrm{d}\tau=0.\end{aligned}$$

Therefore, the zero solution of (4.1) is absolutely stable (absolutely stable in $[0,k)$ $[0,k]$, or $k_1,k_2]$). □

Theorem 4.4. *The necessary and sufficient conditions for the zero solution of (4.1) to be absolutely stable (absolutely stable in* $[0,k)$*,* $[0,k]$ *or* $[k_1,k_2]$*) are the following:*

(1) The condition (1) of Theorem 4.3 is true;
(2) There exists a differentiable function $V_f(x)\in C[R^n,R^1]$ *such that* $V_f(0)=0$*, and*

$$V_f(x)\geq\varphi_f(|\sigma|),\quad \varphi_f\in KR, \tag{4.2}$$

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(4.1)}\leq-\tilde{\psi}_f(|\sigma|),\quad \tilde{\psi}_f\in K. \tag{4.3}$$

Proof. *Sufficiency*. It suffices to prove that condition (2) implies that the zero solution of (4.1) is absolutely stable for Ω.

Since $V_f(0)=0$ $(0\in\Omega)$ and $V_f(x)$ is continuous, for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that

$$V_f(x_0)<\varphi_f(\varepsilon)\quad \text{for}\quad \|x_0\|<\delta(\varepsilon).$$

From (4.2) and (4.3), it follows

$$\varphi_f(|\sigma(t)|) \le V_f(x(t)) \le V_f(x_0) < \varphi_f(\varepsilon).$$

Hence, we deduce that $|\sigma(t)| < \varepsilon$, that is, the zero solution of (4.1) is Hurwitz stable for Ω.

Next, we prove that $\lim\limits_{t\to+\infty} \sigma(t) = 0$.

For any $x_0 \in \mathrm{R}^n$ it follows from (4.3) that $V_f(x(t))$ is monotone decreasing and has a lower bound. Thus there exists

$$\inf_{t\ge t_0} V_f(x(t)) = \lim_{t\to+\infty} V_f(x(t)) := \alpha \ge 0,$$

and α can only be reached on Ω. If α is reached outside Ω, then there must exist a constant $\beta > 0$ such that $|\sigma(t)| \ge \beta > 0$ for all $t \ge t_0$. Otherwise, there exists a sequence $\{t_k\}$ with $t_k \to +\infty$ as $k \to +\infty$ such that $\lim\limits_{t_k\to+\infty} \sigma(t_k) = 0$. Thus

$$\alpha = \inf_{t\ge t_0} V_f(x(t)) = \lim_{t\to+\infty} V_f(x(t)) = \lim_{t_k\to+\infty} V_f(x(t_k)) = \lim_{\sigma(t_k)\to 0} V_f(x(t_k)),$$

namely, α is reached on Ω, a contradiction with that α is reached outside Ω.

For any $x_0 \in \mathrm{R}^n$, from (4.3), it follows that

$$|\sigma(t)| \le |c^{\mathrm{T}} x(t_0)| := h < +\infty.$$

If $\lim\limits_{t\to+\infty} \sigma(t) \ne 0$, then according to the uniform continuity of $\sigma(t)$, there exist constants $\beta > 0$, $\eta > 0$ and a sequence $\{t_j\}$ such that

$$|\sigma(t)| \ge \beta \quad \text{for} \quad t \in [t_j - \eta, t_j + \eta].$$

We fix $r_f = \inf\limits_{\beta\le|\sigma|\le h} \tilde{\psi}_f(|\sigma|) > 0$, then

$$\begin{aligned} V_f(x(t)) &= V_f(x(t_0)) + \int_{t_0}^{t} \frac{\mathrm{d}V_f}{\mathrm{d}t}\,\mathrm{d}t \\ &\le V_f(x(t_0)) - \int_{t_0}^{t} \tilde{\psi}_f(|\sigma(\tau)|)\,\mathrm{d}\tau \\ &\le V_f(x(t_0)) - \sum_{j=1}^{n} \int_{t_j-\eta}^{t_j+\eta} \tilde{\psi}_f(|\sigma(\tau)|)\,\mathrm{d}\tau \\ &\le V_f(x(t_0)) - 2\eta n r_f \to -\infty \quad \text{as} \quad n \to +\infty. \end{aligned}$$

This is in contradiction with $V_f(x(t)) \ge 0$, and so $\lim\limits_{t\to+\infty} \sigma(t) = 0$. Then the zero solution of (4.1) is absolutely stable for Ω (absolutely stable in $[0,k)$, $[0,k]$ or $[k_1,k_2]$ for Ω). The sufficiency follows directly from Theorem 4.3.

Necessity. Since the zero solution of (4.1) is absolutely stable, R^n is a region of attraction. For any $x_0 \in \mathrm{R}^n$, let

$$\varphi_f(x) = \sup_{t\ge 0} \|x(t,0,x)\|^2.$$

Obviously, $\varphi_f(x)$ possesses the following properties:

1. $\varphi_f(x) \geq 0$, where the equality holds if and only if $x = 0$, and $\varphi(x)$ is radially unbounded, positive definite;
2. $\varphi_f(x(\eta)) = \sup\limits_{t \geq \eta} \|x(t)\|^2$ is monotone decreasing;
3. $\varphi_f(x)$ is continuous on R^n.

Again we define

$$V_f(x) = \int_0^{+\infty} \varphi_f(x(\eta, 0, x))\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta,$$

then we get

$$V_f(x(t)) = \int_0^{+\infty} \varphi(x(t+\eta))\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta.$$

Denoting

$$\Phi(t+\eta) = \int_0^{t+\eta} \varphi_f(x(\xi))\, \mathrm{d}\xi,$$

we obtain

$$\Phi'_\eta = \Phi'_t = \varphi_f(x(t+\eta)).$$

Using integration by parts, we obtain

$$\begin{aligned}
V_f(x(t)) &= \int_0^{+\infty} \mathrm{e}^{-\eta}\, \mathrm{d}\Phi \\
&= \mathrm{e}^{-\eta} \int_0^{t+\eta} \varphi_f(x(\xi))\, \mathrm{d}\xi \Big|_0^{+\infty} + \int_0^{+\infty} \Phi(t+\eta)\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta \\
&= -\int_0^{t} \varphi_f(x(\xi))\, \mathrm{d}\xi + \int_0^{+\infty} \Phi(t+\eta)\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta.
\end{aligned}$$

Since $\varphi(x(\xi))$ is monotone decreasing, it is bounded. Thus

$$\lim_{t \to +\infty} \mathrm{e}^{-\eta} \int_0^{t+\eta} \varphi_f(x(\xi))\, \mathrm{d}\xi = 0.$$

Now we take the derivative of V_f along the solution of (4.1). Clearly,

$$\begin{aligned}
\frac{\mathrm{d}V_f(x(t))}{\mathrm{d}t}\Big|_{(4.1)} &= -\varphi_f(x(t)) + \int_0^{+\infty} \Phi'_t\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta \\
&= -\varphi_f(x(t)) + \int_0^{+\infty} \varphi_f(x(t+\eta))\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta \\
&= \int_0^{+\infty} [\varphi_f(x(t+\eta)) - \varphi_f(x(t))]\, \mathrm{e}^{-\eta} \mathrm{d}\eta.
\end{aligned}$$

Since $\varphi_f(x(t))$ is monotone decreasing, we have

$$\varphi_f(x(t)) \geq \varphi_f(x(t+\eta)) \quad \text{for} \quad \eta \geq 0.$$

In particular, if $x(t)$ is not the zero solution, then there exists

$$\varphi_f(x(t)) \not\equiv \varphi_f(x(t+\eta)).$$

Otherwise, we have

$$\varphi_f(x(t)) \equiv \varphi_f(x(t+\eta)) \to 0 \quad \text{as} \quad \eta \to +\infty,$$

that is, $\varphi_f(x(t)) \equiv 0$, which is a contradiction. Thus if $x(t) \neq 0$, we have

$$\int_0^{+\infty} [\varphi(x(t+\eta)) - \varphi(x(t))]\,\mathrm{e}^{-\eta}\,\mathrm{d}\eta < 0,$$

that is,

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(4.1)} < 0 \quad \text{for} \quad x \neq 0.$$

Therefore, we see that

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(4.1)} \leq -W_f(x),$$

where $W_f(x)$ is a positive definite function. From the equivalence relation between the positive definite function and the K-class function, it follows that there exist $\tilde{\varphi}_f(r) \in KR$ and $\tilde{\psi}(r) \in K$ such that

$$\tilde{\varphi}(\|x\|) \leq V_f(x) \quad \text{and} \quad -W_f(x) \leq -\tilde{\psi}_f(\|x\|).$$

Therefore, we write

$$\begin{aligned} V_f(x) \geq \tilde{\varphi}_f(\|x\|) &:= \tilde{\varphi}_f\Big(\sum_{i=1}^n |x_i|\Big) \\ &\geq \tilde{\varphi}_f\Big(\frac{1}{\max\limits_{1\leq i\leq n} |c_i|} \sum_{i=1}^n |c_i x_i|\Big) \\ &= \tilde{\varphi}_f\Big(\frac{1}{\max\limits_{1\leq i\leq n} |c_i|} |\sigma|\Big) := \varphi_f(|\sigma|) \in KR, \end{aligned}$$

and confirm that $V_f(x)$ is positive definite for Ω. Moreover, we have

$$\begin{aligned} \left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(4.1)} &\leq -W_f(x) \leq -\tilde{\psi}_f(\|x\|) \\ &\leq -\tilde{\psi}_f\Big(\frac{1}{\max\limits_{1\leq i\leq n} |c_i|} \sum_{i=1}^n |c_i x_i|\Big) \\ &= -\tilde{\psi}_f\Big(\frac{1}{\max\limits_{1\leq i\leq n} |c_i|} |\sigma|\Big) := -\varphi_f(|\sigma|) \in K. \end{aligned}$$

The condition (2) of Theorem 4.4 is satisfied.

The proof of condition (1) of Theorem 4.4 is trivial. □

By imitating Theorem 4.3 and 4.4 we formulate

Theorem 4.5. *The necessary and sufficient conditions for the zero solution of* (4.1) *to be absolutely stable (absolutely stable in* $[0,k)$, $[0,k]$ *or* $[k_1,k_2]$*) are given below:*

(1) There exists $\tilde{b} \in R^n$ *such that* $A+\tilde{b}c^{\mathrm{T}}$ *is Hurwitz stable;*
(2) The condition (2) *of Theorem* 4.3 *is satisfied.*

Theorem 4.6. *The necessary and sufficient conditions for the zero solution of* (4.1) *to be absolutely stable (absolutely stable in* $[0,k)$, $[0,k]$, *or* $[k_1,k_2]$*) are given by the following:*

(1) The condition (1) *of Theorem* 4.5 *is satisfied;*
(2) The condition (2) *of Theorem* 4.4 *is satisfied.*

Theorems 4.5 and 4.6 are useful because sometime it is more convenient to verify the stability of $A+\tilde{b}c^{\mathrm{T}}$ than that of $A+bc^{\mathrm{T}}\theta$.

Theorem 4.7. *Suppose the following conditions are satisfied:*

(1) $A+\theta bc^{\mathrm{T}}$ *is Hurwitz stable, where* $\theta=0$ *or* $\theta=1$*;*
(2) There exists a symmetric matrix $P_{n\times n}$ *such that*

$$\begin{cases} x^{\mathrm{T}}Px \geq \alpha\sigma^2, \quad \alpha>0, \\ x^{\mathrm{T}}(PA+A^{\mathrm{T}}P)x+(2Pb+\beta A^{\mathrm{T}}c)^{\mathrm{T}}xf(\sigma)+\beta c^{\mathrm{T}}bf^2(\sigma) \leq -\varepsilon\tau, \quad \beta \geq 0, \end{cases}$$

or

$$\begin{cases} x^{\mathrm{T}}Px \geq 0, \quad \int_0^{\pm\infty} f(\sigma)d\sigma=+\infty, \\ x^{\mathrm{T}}(PA+A^{\mathrm{T}}P)x+(2Pb+\beta A^{\mathrm{T}}c)^{\mathrm{T}}xf(\sigma)+\beta c^{\mathrm{T}}bf^2(\sigma) \leq -\varepsilon\tau, \quad \beta>0, \end{cases}$$

where $\tau \in \{\sigma^2, \sigma f(\sigma), f^2(\sigma)\}$, $0<\varepsilon\ll 1$, α, β, ε *are constants.*
Then the zero solution of (4.1) is absolutely stable.

Proof. It suffices to prove that the condition (2) of Theorem 4.7 implies the condition (2) of Theorem 4.4.

In fact, we can construct the Lyapunov function

$$V(x)=x^{\mathrm{T}}Px+\beta\int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma.$$

By condition (2), it bears

$$V(x) \geq \alpha\sigma^2 \geq \varphi(|\sigma|), \quad \varphi \in KR$$

or

$$V(x) \geq \beta\int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma \geq \varphi(|\sigma|), \quad \varphi \in KR,$$

and

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(4.1)} &= x^{\mathrm{T}}(PA+A^{\mathrm{T}}P)x+(2Pb+\beta A^{\mathrm{T}}c)^{\mathrm{T}}xf(\sigma)+\beta c^{\mathrm{T}}bf^2(\sigma)\\ &\leq -\varepsilon\tau \leq -\psi(|\sigma|), \quad \varphi\in K.\end{aligned}$$

Thus, all the conditions of Theorem 4.4 are satisfied, and the desired conclusion of the theorem holds. □

As a special case, Theorem 4.7 can contain all the criteria of absolute stability obtained by the Lyapunov function

$$V(x)=x^{\mathrm{T}}Px+\beta\int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma,$$

which makes $\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(4.1)}$ negative definite for positive definite P.

Corollary 4.8. *If A is a Hurwitz matrix and there exists a constant $\beta>0$ and a symmetric positive definite matrix P such that*

$$x^{\mathrm{T}}(PA+A^{\mathrm{T}}P)x+2(Pb+\frac{1}{2}\beta A^{\mathrm{T}}c)^{\mathrm{T}}xf(\sigma)+\beta c^{\mathrm{T}}bf^2(\sigma)$$

is negative definite, then the zero solution of (4.1) is absolutely stable.

Proof. It suffices to prove that the conditions of Theorem 4.7 are satisfied. In fact, since A is Hurwitz stable, the condition (1) of Theorem 4.7 holds.

Now we construct the positive definite and radially unbounded Lyapunov function

$$V(x)=x^{\mathrm{T}}Px+\beta\int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma.$$

Obviously, $V(0)=0$ and

$$\begin{aligned}V(x)\geq\lambda_1 x^{\mathrm{T}}x \geq \lambda_1\frac{n\sum\limits_{i=1}^{n}|c_ix_i|^2}{n\max\limits_{1\leq i\leq n}|c_i|^2}\geq\lambda_1\frac{\left(\sum\limits_{i=1}^{n}c_ix_i\right)^2}{n\max\limits_{1\leq i\leq n}|c_i|^2}\\ =\lambda_1\frac{\sigma^2}{n\max\limits_{1\leq i\leq n}|c_i|^2}:=\varphi(|\sigma|)\in KR.\end{aligned}$$

Here, λ_1 refers to the smallest eigenvalue of P. Thus, $V(x)$ is radially unbounded, positive definite for Ω. Again, we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(4.1)}&\leq-\varphi(\|x\|)\leq-\varphi\Big(\frac{1}{\max\limits_{1\leq i\leq n}|c_i|}\sum_{i=1}^{n}|c_ix_i|\Big)\\ &=-\varphi\Big(\frac{1}{\max\limits_{1\leq i\leq n}|c_i|}|\sigma|\Big):=-\varphi_1(|\sigma|),\end{aligned}$$

where $\varphi_1,\varphi\in K$. Therefore, all the conditions of Theorem 4.7 are satisfied and the corollary follows. □

Now we turn to the Lurie indirect control system:

$$\begin{aligned} \dot{x} &= Ax + bf(\sigma), \\ \dot{\sigma} &= c^{\mathrm{T}}x - \rho f(\sigma), \end{aligned} \tag{4.4}$$

where $A \in \mathrm{R}^{n\times n}$, $b \in \mathrm{R}^n$, $c \in \mathrm{R}^n$, $f(\sigma) \in F$.

Corollary 4.9. *If A and $\begin{bmatrix} A & b \\ c^{\mathrm{T}} & -\rho \end{bmatrix}$ are Hurwitz stable, and there exist a symmetric positive semi-definite matrix G and a positive constant p such that*

$$W(x) = -x^{\mathrm{T}}Gx + f(\sigma)\{2u^{\mathrm{T}}x + pc^{\mathrm{T}}A^{-1}x\} - \rho f^2(\sigma)$$

is negative semi-definite, where $u = Pb + \frac{c}{2}$, P is the solution of the Lyapunov matrix equation

$$PA + A^{\mathrm{T}}P = -G,$$

then the zero solution of (4.4) is absolutely stable.

Proof. We construct the Lyapunov function

$$V(x,\sigma) = x^{\mathrm{T}}Px + \int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma + \frac{p}{2(\rho + c^{\mathrm{T}}A^{-1}b)}(c^{\mathrm{T}}A^{-1}x - \sigma)^2.$$

It is obvious that $V(x,\sigma)$ is radially unbounded, positive definite with respect to σ. Then,

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.4)} &= -x^{\mathrm{T}}Gx + 2f(\sigma)\left(Pb + \frac{1}{2}c\right)^{\mathrm{T}} x - \rho f^2(\sigma) \\ &\quad + \frac{p}{(\rho + c^{\mathrm{T}}A^{-1}b)}(c^{\mathrm{T}}A^{-1}x - \sigma)(\rho + c^{\mathrm{T}}A^{-1}b)f(\sigma) \\ &= -x^{\mathrm{T}}Gx + f(\sigma)\{2u^{\mathrm{T}}x + pc^{\mathrm{T}}A^{-1}x\} \\ &\quad - \rho f^2(\sigma) - p\sigma f(\sigma) \\ &= W(x) - p\sigma f(\sigma) \\ &\leq -p\sigma f(\sigma) < 0 \quad \text{if} \quad \sigma \neq 0. \end{aligned}$$

Thus, the zero solution of (4.4) is absolutely stable w.r.t. the single variable σ. □

Example 4.10. Discuss the absolute stability of the zero solution of the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 - f(x_1 - x_2), \\ \dot{x}_2 &= -x_1 + f(x_1 - x_2), \end{aligned} \tag{4.5}$$

where $f \in F_\infty$, and the coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has a pair of purely imaginary eigenvalues. Thus, Example 4.10 is neither a Lurie direct control system, nor a Lurie indirect control system, but a more complicated critical case. Therefore, the Lyapunov matrix equation

$$AP + A^{\mathrm{T}}P = -G$$

has no solution P of symmetric positive definite matrix for any positive definite matrix G. Thus, the traditional Lurie method, that is, the method of using the Lyapunov function

$$V(x) = x^{\mathrm{T}}Px + \beta \int_0^{\sigma} f(\sigma)d\sigma \quad (P \text{ being positive definite}),$$

which makes $\frac{dV}{dt}\Big|_{(4.5)}$ negative definite, cannot be applied. The traditional Popov method cannot be applied either. Instead we can use Theorem 4.7.

(i) Let $f(x_1 - x_2) = x_1 - x_2$. Then system (4.5) changes to

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2, \\ \dot{x}_2 &= -x_2. \end{aligned} \tag{4.6}$$

Obviously, the coefficient matrix $B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ is Hurwitz stable.

(ii) Construct the Lyapunov function:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2).$$

Then,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(4.5)} &= -x_1 f(x_1 - x_2) + x_2 f(x_1 - x_2) \\ &= -(x_1 - x_2) f(x_1 - x_2) \end{aligned}$$

is negative definite for $\Omega = \{x : \sigma = x_1 - x_2 = 0\}$. Thus, all the conditions of Theorem 4.7 are satisfied. The zero solution of (4.5) is absolutely stable.

Example 4.11. Consider the three-dimensional system:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 + x_3 - f(x_1 + 2x_2 + x_3), \\ \dot{x}_2 &= x_1 - 2x_2 + x_3 + f(x_1 + 2x_2 + x_3), \\ \dot{x}_3 &= x_1 + 3x_2 - 3x_3 - 2f(x_1 + 2x_2 + x_3). \end{aligned} \tag{4.7}$$

It is easy to verify that

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & -3 \end{bmatrix}$$

is not Hurwitz stable matrix. We construct the positive semi-definite Lyapunov function

$$V = (x_1 + 2x_2 + x_3)^2.$$

Then,

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.7)} &= 2(x_1 + 2x_2 + x_3)\,(\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3)\\ &= -2(x_1 + 2x_2 + x_3) f(x_1 + 2x_2 + x_3).\end{aligned}$$

Consequently, $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.7)}$ is negative definite for $\Omega = \{x : \sigma = x_1 + 2x_2 + x_3 = 0\}$. Clearly, V is radially unbounded, positive definite for $\Omega = \{x : \sigma = x_1 + 2x_2 + x_3 = 0\}$. We conclude that the zero solution of (4.7) is absolutely stable.

Example 4.12. Consider the indirect control system:

$$\begin{aligned}\dot{x}_1 &= x_2 - f(\sigma),\\ \dot{x}_2 &= -x_1 + f(\sigma),\\ \dot{\sigma} &= x_1 - x_2 - \rho f(\sigma),\end{aligned} \tag{4.8}$$

where $\sigma = x_1 - x_2$, $\rho > 0$, $f(\sigma) \in F_\infty$, and $\int_0^{\pm\infty} f(\sigma)\,\mathrm{d}\sigma = +\infty$.

Since $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is not Hurwitz stable, the Lyapunov matrix equation

$$A^{\mathrm{T}}P + PA = -G$$

has no positive definite matrix solution for any positive definite matrix G. So the traditional Lurie method and Popov method cannot be applied. Instead, we can use Theorem 4.7.

(i) Let $f(\sigma) = \sigma$. Then (4.8) changes into

$$\begin{aligned}\dot{x}_1 &= x_2 - \sigma,\\ \dot{x}_2 &= -x_1 + \sigma,\\ \dot{\sigma} &= x_1 - x_2 - \rho\sigma.\end{aligned} \tag{4.9}$$

Since

$$\det|\lambda I_3 - B| = \begin{vmatrix} \lambda & -1 & 1 \\ 1 & \lambda & -1 \\ -1 & 1 & \lambda + \rho \end{vmatrix} = \lambda^3 + \rho\lambda^2 + 3\lambda + \rho,$$

the necessary and sufficient conditions for the characteristic polynomial to be Hurwitz are given by

$$\rho > 0, \quad \Delta_1 := 3 > 0, \quad \Delta_1 := \begin{vmatrix} 3 & \rho \\ 1 & \rho \end{vmatrix} = 2\rho > 0.$$

Thus

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & -\rho \end{bmatrix}$$

is Hurwitz stable.

(ii) Construct the Lyapunov function:

$$V(x,\sigma) = \frac{1}{2}(x_1^2 + x_2^2) + \int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma,$$

then $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.8)} = -\rho f^2(\sigma) < 0$ when $\sigma \neq 0$. Therefore, the zero solution of (4.8) is absolutely stable.

4.2 Lurie Direct Control Systems

In this section, we first transform the n-dimensional direct control system to a variable-separated nonlinear system with full rank linear transformation. The feedback control variable σ is changed into a state variable. It will be shown that the absolute stability of the Lurie direct control system is equivalent to the absolute stability of this new state variable. Also, some algebraic sufficient conditions for absolute stability will be obtained.

The conditions for all the results in this section are based on the systems' parameters, which are independent of the unknown function V, or the solutions of the Lyapunov matrix equation. Thus, the conditions are easily verified.

Consider the n-dimensional Lurie direct control system [76, 77]:

$$\begin{aligned} \dot{\xi}_i &= \sum_{j=1}^{n} \tilde{a}_{ij}\xi_j + h_i f(\sigma) \quad i = 1,\ldots,n, \\ \sigma &= \sum_{i=1}^{n} c_i \xi_i, \end{aligned} \tag{4.10}$$

where $\tilde{a}_{ij}, h_i, c_i$ $(i,j = 1,\ldots,n)$ are all constants. $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ is a Hurwitz matrix and $f \in F_\infty$. The description of the control system (4.10) is shown in Fig. 4.1.

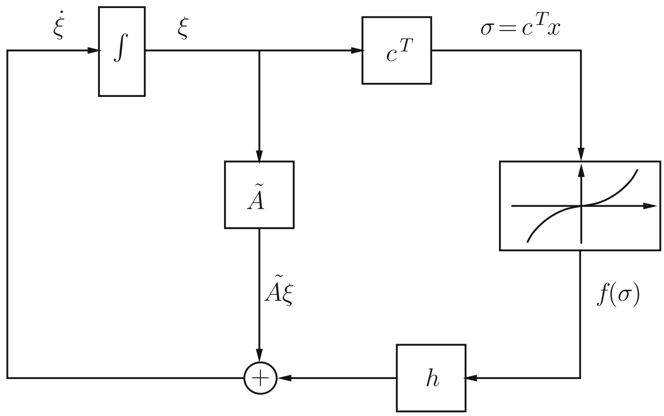

Fig. 4.1 Lurie direct control system

Without loss of generality, we assume $c_n \neq 0$ (otherwise, we can adjust the orders of the state variables and the equations to make $c_n \neq 0$). With the full rank linear transformation $x = \Omega\xi$, that is,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \\ c_1 & \cdots & c_{n-1} & c_n \end{bmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{pmatrix}. \tag{4.11}$$

Equation (4.10) then changes into

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^{n} a_{ij}x_j + b_i f(x_n) \quad (i = 1,\ldots,n-1), \\ \dot{x}_n &= \sum_{j=1}^{n} a_{nj}x_j + b_n f(x_n), \end{aligned} \tag{4.12}$$

where $x_n = \sigma$ is an independent state variable, and

$$\begin{aligned} a_{ij} &= \left(\tilde{a}_{ij} - \frac{\tilde{a}_{ij}}{c_n}c_j\right) \quad (i,j = 1,\ldots,n-1), \\ a_{in} &= \frac{\tilde{a}_{in}}{c_n} \quad (i = 1,\ldots,n-1), \\ a_{nj} &= \sum_{i=1}^{n} c_i\tilde{a}_{ij} - \sum_{i=1}^{n} c_i\frac{\tilde{a}_{in}}{c_n}c_j \quad (j = 1,\ldots,n-1), \\ a_{nn} &= \frac{1}{c_n}\sum_{i=1}^{n} c_i\tilde{a}_{in}, \\ b_i &= h_i \quad (i = 1,\ldots,n-1), \\ b_n &= \sum_{i=1}^{n} c_i h_i. \end{aligned}$$

It is easy to prove that if $f \in F_\infty$, the necessary and sufficient condition for the zero solution of (4.12) to be absolutely stable is $h_n \leq 0$. It is obvious that the absolute stability of (4.10) and that of (4.12) are equivalent, and $A(a_{ij})_{n\times n}$ in (4.12) is a Hurwitz matrix.

Definition 4.13. *The zero solution of (4.12) is absolutely stable w.r.t. the partial variable x_n if $\forall f \in F_\infty$, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, when $\sum_{i=1}^{n} x_i^2(t_0) < \delta(\varepsilon)$, the component of the solution $x(t,t_0,x(t_0))$ of equation (4.12) satisfies*

$$x_n^2(t,t_0,x(t_0)) < \varepsilon, \quad \text{if} \quad t \geq t_0$$

and

$$\lim_{t\to+\infty} x_n(t,t_0,x(t_0)) = 0 \quad \forall x(t_0) \in R^n.$$

Theorem 4.14. *The necessary and sufficient condition for the zero solution of the direct control system (4.10) to be absolutely stable is that the zero solution of (4.12) is absolutely stable w.r.t. the single variable* x_n.

Proof. *Necessity*. If the zero solution of (4.10) is absolutely stable, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\|\xi(t,t_0,\xi_0)\| \leq \frac{\varepsilon}{\|c\|} \quad \text{if} \quad \|\xi_0\| < \delta. \tag{4.13}$$

Also, we have the condition

$$x_0 = \Omega\xi_0. \tag{4.14}$$

Let R^n_ξ and R^n_x be the n-dimensional linear spaces having the components ξ and x, respectively, $\Omega : \mathrm{R}^n_\xi \to \mathrm{R}^n_x$ is a nonsingular linear transformation. From the uniqueness of the solution, we can establish the one to one map between the solution of (4.10) and that of (4.12) using (4.14). Assume the initial conditions of (4.10) and (4.12) satisfy (4.14), we have

$$\|\xi_0\| \leq \|\Omega^{-1}\| \cdot \|x_0\| < \delta, \quad \text{if} \quad \|x_0\| < \frac{\delta}{\|\Omega^{-1}\|}.$$

Thus,

$$\|x_n(t,t_0,x_0)\| = \|c^{\mathrm{T}}\xi(t,t_0,\xi_0)\| \leq \|c^{\mathrm{T}}\| \cdot \|\xi(t,t_0,\xi_0)\| \leq \frac{\varepsilon}{\|c\|} \cdot \|c\| = \varepsilon.$$

For all $x_0 \in \mathrm{R}^n$, we have

$$\lim_{t\to+\infty} x_n(t,t_0,x_0) = \lim_{t\to+\infty} c^{\mathrm{T}}\xi(t,t_0,\xi_0) = 0,$$

as $\lim_{t\to+\infty} \xi(t,t_0,\xi_0) = 0$. The necessity is proved.

Sufficiency. Express the solution of (4.12) $x(t) := x(t,t_0,x_0)$ as

$$x(t) = \mathrm{e}^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} bf(x_n(t,t_0,x_0))\,\mathrm{d}\tau. \tag{4.15}$$

Using the Hurwitz stability of matrix A, we can complete the proof by imitating Theorem 4.3. □

Definition 4.15. *The zero solution of (4.12) is absolutely stable w.r.t. the partial variables* $x_j, x_{j+1}, \ldots, x_n$ *if* $\forall f(x_n) \in F_\infty$, $\forall\varepsilon > 0$, $\exists\delta(\varepsilon) > 0$, *when* $\sum_{i=1}^{n} x_i^2(t_0) < \delta(\varepsilon)$, *we have*

$$\sum_{i=j}^{n} x_i^2(t,t_0,x(t_0)) < \varepsilon \quad \textit{if} \quad t \geq t_0$$

and

$$\lim_{t\to+\infty} \sum_{i=j}^{n} x_i^2(t,t_0,x(t_0)) = 0 \quad \textit{for} \quad x_0 \in R^n.$$

Theorem 4.16. *The zero solution of (4.10) is absolutely stable if and only if the zero solution of (4.12) is absolutely stable w.r.t.* $x_j, \ldots, x_n$.

Proof. If the zero solution of (4.12) is absolutely stable w.r.t. $x_j, \ldots, x_n$ $(1 < j \le n)$, especially it is absolutely stable w.r.t. x_n, the conditions of Theorem 4.14 are satisfied. The sufficiency holds.

On the other hand, assume that the zero solution of (4.10) is absolutely stable, especially it is absolutely stable w.r.t. $\xi_j, \ldots, \xi_n$. We have the equation $x = \Omega\xi$ (i.e., $x_i = \xi_i$, $(i = j, j+1, \ldots, n-1)$, $x_n = \sum_{i=1}^{n} c_i \xi_i$) between (4.10) and (4.12). Thus, the zero solution of (4.12) is absolutely stable w.r.t. $x_j, \ldots, x_n$. □

In the above proof, we have changed the feedback control variable σ to a state variable, which not only avoids the difficulty in discussing the absolute stability, but also shows that the absolute stability with respect to all variables is equivalent to that with respect to a single variable in Lurie direct control system.

The conditions of Theorem 4.16 seem stronger than that of Theorem 4.14, but we can see from the following theorem that the requirement for the conditions of Theorem 4.16 can be reduced.

Theorem 4.17. *For a given symmetric positive definite matrix* $G_{n\times n}$*, there exists* $\varepsilon > 0$ *such that the following matrix*

$$\begin{bmatrix} -G_{n\times n} & Pb + \frac{1}{2}A_n + \varepsilon e_n \\ \left(Pb + \frac{1}{2}A_n + \varepsilon e_n\right)^{\mathrm{T}} & b_n \end{bmatrix}$$

is negative semi-definite, where $A_n^{\mathrm{T}} = (a_{n1}, \ldots, a_{nn})$, $e_n = (\overbrace{0, \ldots, 0}^{n-1}, 1)$*, P is the symmetric positive definite solution of the Lyapunov matrix equation:*

$$PA + A^{\mathrm{T}}P = -G_{n\times n},$$

then the zero solution of (4.12) is absolutely stable.

Proof. Construct the radially unbounded, positive definite Lyapunov function:

$$V(x) = x^{\mathrm{T}}Px + \int_0^{x_n} f(x_n)\,\mathrm{d}x_n. \tag{4.16}$$

Then,

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.12)} &= \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} + [A_n^{\mathrm{T}}x + b_n f(x_n)]f(x_n), \\ &= [Ax + bf(x_n)]^{\mathrm{T}}Px + x^{\mathrm{T}}P[Ax + bf(x_n)] + \left[A_n^{\mathrm{T}}x + b_n f(x_n)\right]f(x_n), \\ &= x^{\mathrm{T}}A^{\mathrm{T}}Px + x^{\mathrm{T}}PAx + \left[b^{\mathrm{T}}Px + x^{\mathrm{T}}Pb + \left(A_n^{\mathrm{T}}x + b_n f(x_n)\right)\right]f(x_n), \end{aligned}$$

$$
\begin{aligned}
&= x^{\mathrm{T}} A^{\mathrm{T}} P x + x^{\mathrm{T}} P A x + \left[b^{\mathrm{T}} P x + x^{\mathrm{T}} P b + A_n^{\mathrm{T}} x\right] f(x_n) + b_n f^2(x_n), \\
&= \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -G_{n\times n} & Pb + \frac{1}{2}A_n + \varepsilon e_n \\ \left(Pb + \frac{1}{2}A_n + \varepsilon e_n\right)^{\mathrm{T}} & b_n \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix} \\
&\quad - \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \varepsilon e_n \\ (\varepsilon e_n)^{\mathrm{T}} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix}, \\
&\leq -2\varepsilon x_n f(x_n) < 0 \quad \text{for} \quad x_n \neq 0.
\end{aligned}
$$

Thus, we know that the zero solution of (4.12) is absolutely stable from Theorem 4.14. □

If $f \in F_\infty$, the necessary and sufficient condition for the absolute stability of the zero solution of (4.12) is $b_n \leq 0$. In the following, we assume $b_n < 0$. With the full rank linear transformation

$$
y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{bmatrix} 1 & \cdots & & -\frac{b_1}{b_n} \\ 0 & 1 & \cdots & \vdots \\ \vdots & & \ddots & -\frac{b_{n-1}}{b_n} \\ 0 & \cdots & & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} := Hx, \tag{4.17}
$$

(4.12) changes into the following system having the same absolute stability,

$$
\begin{aligned}
\dot{y}_i &= \sum_{j=1}^{n} r_{ij} y_j \quad i = 1, \ldots, n-1, \\
\dot{y}_n &= \sum_{j=1}^{n} r_{nj} y_j + b_n f(y_n),
\end{aligned} \tag{4.18}
$$

where $R(r_{ij})_{n\times n} = HAH^{-1}$ and $h = Hb = (\overbrace{0, \ldots, 0}^{n-1}, b_n)^{\mathrm{T}}$.

Theorem 4.18. *There exists a symmetric positive semi-definite matrix given by*

$$
P = \begin{bmatrix} p_{11} & \cdots & p_{1,n-1} & 0 \\ \vdots & & & \vdots \\ p_{n-1,1} & \cdots & p_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & p_{nn} \end{bmatrix}, \quad p_{nn} > 0,
$$

such that

$$
PR + R^{\mathrm{T}} P
$$

is negative semi-definite, then the zero solution of (4.12) is absolutely stable.

Proof. Construct $V = y^{\mathrm{T}}Py$, then V is a radially unbounded, positive definite Lyapunov function w.r.t. y_n.

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.18)} &= \dot{y}^{\mathrm{T}}Py + y^{\mathrm{T}}P\dot{y} = (Ry + hf(y_n))^{\mathrm{T}}Py + y^{\mathrm{T}}P(Ry + bf(y_n)),\\ &= y^{\mathrm{T}}(R^{\mathrm{T}}P + PR)y + 2p_{nn}b_n y_n f(y_n),\\ &< 0 \quad \text{for} \quad y_n \neq 0.\end{aligned}$$

Thus, the zero solution of (4.18) is absolutely stable w.r.t. y_n. The zero solution of (4.12) is absolutely stable as $x_n = y_n$. From Theorem 4.14, the conclusion holds. □

Theorem 4.19. *There exist constants $c_j \geq 0$ $(j = 1, \ldots, n-1)$, $c_n > 0$ such that one of the following inequalities*

$$\begin{aligned} -c_j a_{jj} &\geq \sum_{i=1, i\neq j}^{n} c_i |a_{ij}|, \quad j = 1, \ldots, n-1,\\ -c_n a_{nn} &\geq \sum_{i=1}^{n-1} c_i |a_{in}|,\\ -c_n b_n &\geq \sum_{i=1}^{n-1} c_i |b_i|,\end{aligned}$$

holds, and at least one of the last two inequalities is a strict inequality, then the zero solution of (4.12) is absolutely stable.

Proof. Construct a radially unbounded, positive definite Lyapunov function w.r.t. x_n:

$$V = \sum_{i=1}^{n} c_i |x_i|.$$

Then we have

$$\begin{aligned} D^{+}V|_{(4.12)} \leq & \sum_{j=1}^{n-1} \left[c_j a_{jj} + \sum_{i=1, i\neq j}^{n} c_i |a_{ij}| \right] |x_j| \\ & + \left[c_n a_{nn} + \sum_{i=1}^{n-1} c_i |a_{in}| \right] |x_n| + \left[c_n b_n + \sum_{i=1}^{n-1} c_i |b_i| \right] |f(x_n)|,\\ < & \; 0 \quad \text{if} \quad x_n \neq 0.\end{aligned}$$

From Theorems 2.33, the conclusion holds. □

According to Theorem 2.33 and (4.11), some useful algebraic sufficient conditions for absolute stability are given in the following.

Corollary 4.20. *If one of the following conditions is satisfied,*

(1) $\sum_{j=1}^{n-1} a_{nj}^2 = 0$, *and* $\begin{cases} a_{nn} \le 0, \\ b_n < 0, \end{cases}$

(2) $\sum_{j=1}^{n-1} a_{nj}^2 = 0$, *and* $\begin{cases} a_{nn} < 0, \\ b_n \le 0, \end{cases}$

the zero solution of (4.12) is absolutely stable.

Proof. Construct a radially unbounded, positive definite Lyapunov function $V(x) = x_n^2$ w.r.t. x_n. Then,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.12)} = 2a_{nn}x_n^2 + 2b_n x_n f(x_n) < 0 \quad \text{if} \quad x_n \neq 0.$$

The condition of Theorem 4.14 is satisfied. The conclusion holds. □

Next, we generalize the results of Corollary 4.20 to more general cases. To do this, let the variables and matrices in system (4.12) be

$$x^{(m)} = (x_1, \ldots, x_m)^{\mathrm{T}}, \quad x^{(n-m)} = (x_{m+1}, \ldots, x_n)^{\mathrm{T}},$$
$$b^{(m)} = (b_1, \ldots, b_m)^{\mathrm{T}}, \quad b^{(n-m)} = (b_{m+1}, \ldots, b_n)^{\mathrm{T}}$$

$$A_{n\times n} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \text{where}$$

$$A_{11} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ldots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}_{m\times m} \qquad A_{12} = \begin{bmatrix} a_{1\,(m+1)} & \cdots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{m\,(m+1)} & \cdots & a_{mn} \end{bmatrix}_{m\times(n-m)}$$

$$A_{21} = [O]_{(n-m)\times m} \qquad A_{22} = \begin{bmatrix} a_{(m+1)(m+1)} & \cdots & a_{(m+1)n} \\ \vdots & \ldots & \vdots \\ a_{n(m+1)} & \cdots & a_{nn} \end{bmatrix}_{(n-m)\times(n-m)}.$$

Then system (4.12) can be rewritten as

$$\begin{aligned} \dot{x}^{(m)} &= A_{11}x^{(m)} + A_{12}x^{(n-m)} + b^{(m)} f(x_n), \\ \dot{x}^{(n-m)} &= O x^{(m)} + A_{22}\, x^{(n-m)} + b^{(n-m)} f(x_n). \end{aligned} \tag{4.12$'$}$$

For system (4.12)$'$, we have the following result.

Corollary 4.21. *If* A_{11} *is a Hurwitz matrix, then the zero solution of the following* $(n-m)$*-dimensional system*

$$\dot{x}^{(n-m)} = A_{22}x^{(n-m)} + b^{(n-m)} f(x_n) \tag{4.19}$$

is absolutely stable, implying that the zero solution of system (4.12)$'$ is absolutely stable.

Proof. Since the zero solution of system (4.19) is abosultely stable, the zero solution of system $(4.12)'$ is absolutely stable with respect to the partial variable $x^{(n-m)}$. The solution for the first m variables of system $(4.12)'$ can be expressed as

$$x^{(m)} = \mathrm{e}^{A_{11}(t-t_0)} x^{(m)}(t_0) + \int_{t_0}^{t} \mathrm{e}^{A_{11}(t-\tau)} \left[A_{12} x^{(n-m)}(\tau) + b^{(m)} f(x_n(\tau)) \right] \tau.$$

The rest can follow the part of the proof for the sufficient condition of Theorem 4.3, and thus omitted here. □

In the following, we discuss the absolute stability in the Hurwitz angle $[0,k]$ when $f \in F_{[0,k]}$ in (4.12).

Theorem 4.22. *Suppose that the following conditions hold:*

(1) A has stability degree at least $\alpha > 0$, that is, there exists $M \geq 1$ such that

$$\|\mathrm{e}^{A(t-t_0)}\| \leq M\,\mathrm{e}^{-\alpha(t-t_0)};$$

(2) $0 \leq f(\sigma) \leq F_{[0,k]}$.

Then the zero solution of (4.12) is absolutely stable in the Hurwitz angle $[0,k]$ if $Mk\|b\| < \alpha$.

Proof. Expressing the solution of (4.12) as

$$x(t) := x(t,t_0,x_0) = \mathrm{e}^{A(t-t_0)} x_0 + \int_{t_0}^{t} b f(x_n(\tau))\,\mathrm{d}\tau \tag{4.20}$$

yields

$$|x_n(t)| \leq \|x(t)\| \leq M\|x_0\|\mathrm{e}^{-\alpha(t-t_0)} + \int_{t_0}^{t} M\,\mathrm{e}^{-\alpha(t-\tau)} k\|b\| \cdot |x_n(\tau)|\,\mathrm{d}\tau,$$

that is,

$$\mathrm{e}^{\alpha(t-t_0)}|x_n(t)| \leq M\|x_0\| + \int_{t_0}^{t} Mk\|b\|\mathrm{e}^{\alpha(\tau-t_0)}|x_n(\tau)|\,\mathrm{d}\tau.$$

Applying the Gronwall–Bellman inequality yields

$$\mathrm{e}^{\alpha(t-t_0)}|x_n(t)| \leq M\|x_0\|\mathrm{e}^{Mk\|b\|(t-t_0)},$$

that is,

$$|x_n(t)| \leq M\|x_0\|\mathrm{e}^{(-\alpha+Mk\|b\|)(t-t_0)} \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.21}$$

Thus, the conditions of Theorem 4.14 are satisfied, the zero solution of (4.12) is absolutely stable in the Hurwitz angle $[0,k]$. □

Rewrite (4.12) as

$$\dot{x}_i = \sum_{j=1}^{n-1} a_{ij}x_j + [a_{in} + b_i g(x_n)]x_n \quad (i = 1,\ldots,m-1), \tag{4.22}$$

where

$$g(x_n) = \begin{cases} 0 & \text{if } x_n = 0, \\ \dfrac{f(x_n)}{x_n} & \text{if } x_n \neq 0, \end{cases} \qquad (0 \leq g(x_n) \leq k).$$

It is obvious that if $m = n$, (4.22) is exactly (4.12), and if $m < n$, (4.22) has the same form as (4.19).

Note that $f(0) \in F_{[0,k]}$, $b_m \leq 0$ may not be the necessary condition for the absolute stability of the zero solution of (4.22).

We denote

$$q_{ij} = \begin{cases} a_{ij} & 1 \leq i,j \leq n-1, \\ |a_{in} + b_i k| & \text{if} \quad b_i a_{in} \geq 0,\ \ i = 1,\ldots,n-1,\ j = n, \\ \max\{|a_{in}|, k|b_i|\} & \text{if} \quad b_i a_{in} \leq 0,\ \ i = 1,\ldots,n-1,\ j = n, \\ |a_{nj}| & i = n,\ 1 \leq j \leq n-1, \\ a_{nn} & \text{if} \quad b_n \leq 0,\ \ i,j = n, \\ a_{n-1} - b_n k & \text{if} \quad b_n > 0,\ \ i,j = n. \end{cases}$$

Theorem 4.23. *If the following conditions are satisfied:*

(1) $q_{ii} < 0,\ i = 1,\ldots,m;$
(2) $-Q(q_{ij})_{m\times m}$ *is an M-matrix;*

then the zero solution of (4.22) is absolutely stable in the Hurwitz angle $[0,k]$.

Proof. If $-Q(q_{ij})_{m\times m}$ is an M-matrix, there exists $\eta_i > 0$ $(i = 1,\ldots,m)$ such that

$$q_{jj}\eta_j + \sum_{i=1, i\neq j}^{n} \eta_i q_{ij} < 0.$$

Construct a radially unbounded, positive definite Lyapunov function for (4.22):

$$V = \sum_{i=1}^{n} \eta_i |x_i|.$$

Then

$$\begin{aligned} D^+V_{(4.22)} \leq& \sum_{i=1}^{n-1} \eta_i \Big[a_{ij}x_j + (a_{in} + b_i g(x_n))x_n \Big] \operatorname{sign} x_i \\ &+ \eta_n \Big[\sum_{i=1}^{n-1} a_{nj}x_j + (a_{nn} + b_n g(x_n))x_n \Big] \operatorname{sign} x_n, \\ \leq& \sum_{j=1}^{n} \Big[\eta_j q_{jj} + \sum_{i=1, i\neq j}^{n} \eta_i q_{ij} \Big] |x_j| < 0 \quad \text{when} \quad x \neq 0. \end{aligned}$$

Thus, the conclusion is true. □

4.3 The *S*-Method and Modified *S*-Method

If the system (4.10) is a direct control system, the Lyapunov function consists of an integral term and a quadratic form:

$$V(x) = x^{\mathrm{T}}Px + \beta \int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma,$$

where P denotes the solution of the Lyapunov matrix equation

$$PA + A^{\mathrm{T}}P = -R,$$

R stands for a given $n \times n$ symmetric positive definite matrix and $\beta > 0$ is a constant.

Provided that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)} = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} x f(\sigma) + \beta c^{\mathrm{T}} b f^2(\sigma) \tag{4.23}$$

is negative definite, the zero solution of (4.10) is absolutely stable. One can consider (4.23) as a quadratic form in x and $f(\sigma)$, and estimate its sign by means of the Sylvester condition. Thus, (4.23) is negative definite in x and $f(\sigma)$ if and only if

$$\det \begin{bmatrix} R & -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right) \\ -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right)^{\mathrm{T}} & -\beta c^{\mathrm{T}} b \end{bmatrix} > 0. \tag{4.24}$$

But the condition (4.24) can never be satisfied. It can be shown that

$$\det \begin{bmatrix} R & -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right) \\ -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right)^{\mathrm{T}} & -\beta c^{\mathrm{T}} b \end{bmatrix} \le 0. \tag{4.25}$$

Hence, it is impossible to use Sylvester condition to find the sign of (4.23) regarded as a quadratic form in x and $f(\sigma)$. This was pointed out by Xie [159].

4.3.1 The S-Method

To overcome this difficulty, consider $f(\cdot) \in F_{[0,k])}$. A new method called S-method or S-process was developed.

By adding and subtracting $\alpha f(\sigma)\left(\sigma - \frac{1}{k} f(\sigma)\right)$ in (4.23) with constant $\alpha > 0$, we deduce

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)} &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} x f(\sigma) \\ &\quad + \beta c^{\mathrm{T}} b f^2(\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k} f(\sigma)\right) - \alpha f(\sigma)\left(\sigma - \frac{1}{k} f(\sigma)\right) \\ &:= -S(x,\sigma) - \alpha f(\sigma)\left(\sigma - \frac{1}{k} f(\sigma)\right), \end{aligned}$$

where

$$\begin{aligned} S(x,\sigma) &= x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma), \\ R &= -PA - A^{\mathrm{T}}P, \\ d &= -\left[Pb + \frac{1}{2}(\alpha c + \beta A^{\mathrm{T}}c)\right], \\ r &= -\beta c^{\mathrm{T}}b + \frac{\alpha}{k}. \end{aligned}$$

If $S(x,\sigma)$ is positive definite in x and σ, then $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)}$ is negative definite. The Sylvester condition for $S(x,\sigma)$ to be positive definite in x and σ is usually satisfied. Then the following result is valid.

Theorem 4.24. *(S-process) Let A be stable and suppose there exist constants $\alpha > 0, \beta > 0$, and a real symmetric positive definite matrix P such that*

$$r > 0, \quad R - \frac{1}{r}dd^{\mathrm{T}} > 0$$

or

$$R > 0, \quad r - d^{\mathrm{T}}R^{-1}d > 0.$$

Then (4.23) is negative definite, and the zero solution of (4.10) is absolutely stable in the Hurwitz angle $[0,k]$.

Proof. It suffices to prove that

$$S(x,\sigma) = x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma)$$

is positive definite, that is,

$$\det\begin{bmatrix} R & d \\ d^{\mathrm{T}} & r \end{bmatrix} > 0.$$

Using linear algebra, we reach the conclusion from the following two relations:

$$\begin{bmatrix} I & -d/r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R & d \\ d^{\mathrm{T}} & r \end{bmatrix}\begin{bmatrix} I & 0 \\ -d^{\mathrm{T}}/r & 1 \end{bmatrix} = \begin{bmatrix} R - dd^{\mathrm{T}}/r & 0 \\ 0 & r \end{bmatrix},$$

$$\begin{bmatrix} I & 0 \\ -d^{\mathrm{T}}R^{-1} & 1 \end{bmatrix}\begin{bmatrix} R & d \\ d^{\mathrm{T}} & r \end{bmatrix}\begin{bmatrix} I & -R^{-1}d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & r - d^{\mathrm{T}}R^{-1}d \end{bmatrix}. \qquad \square$$

Naturally, readers can easily see that positive definiteness of $S(x,\sigma)$ is only the sufficient condition for $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)}$ to be negative definite. If this condition is not necessary, we say that the S-method is defect. Aizeman and Gantmather [1] (p.119) presented two problems. The second one is whether there is an example in which the S-method cannot be applied but one can judge if $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)}$ is negative definite by other methods.

In [182] Zhao presented the following example.

Example 4.25. Consider the system

$$\begin{aligned} \dot{x}_1 &= -2x_1 + f(x_2), \\ \dot{x}_2 &= x_1 - x_2 - \frac{1}{2}f(x_2), \end{aligned} \qquad k = +\infty, \tag{4.26}$$

where $A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$ is stable; $b = (1, -1/2)^{\mathrm{T}}$; $c = (0, 1)^{\mathrm{T}}$, and take

$$V = \frac{1}{2}x_1^2 + x_2^2 + \int_0^{x_2} f(x_2)\,\mathrm{d}x_2.$$

Then we arrive at

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.26)} &= -2x_1^2 + 2x_1x_2 - 2x_2^2 + 2x_1 f(x_2) - 2x_2 f(x_2) - \frac{1}{2}f^2(x_2), \\ &= \begin{pmatrix} x_1 \\ x_2 \\ f(x_2) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1+\alpha \\ 1 & -1+\alpha & -1/2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ f(x_2) \end{pmatrix} - 2\,\alpha x_2 f(x_2). \end{aligned}$$

It is not difficult to verify that there is no $\alpha > 0$ such that

$$\det \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1+\alpha \\ 1 & -1+\alpha & -1/2 \end{bmatrix} < 0.$$

However, Zhao [182] proved that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.26)}$ is negative definite by another method.

4.3.2 The Modified S-Method

When we verify if the derivative, $\dot{V}$, of the function V with the Lurie form is negative definite, the original S-method is the most useful one and is used extensively. This method is also widely used for nonautonomous systems, multiple adjusted systems, etc. However, the original S-method requires that $S(x, \sigma)$ is negative definite in x and σ. If it requires only negative semi-definite, one can apply Barbashin–Krasovskii's theorem [5, 53] or LaSalle's invariance principle [61]. But it is rather troublesome. Our goal is to improve the S-method such that it can be widely used [78].

Theorem 4.26. *Assume that A is stable. If there exists constants $\alpha > 0$, $\beta \geq 0$, and a real symmetric positive definite matrix P such that*

$$r > 0, \quad R - \frac{1}{r}dd^{\mathrm{T}} \geq 0 \tag{4.27}$$

or

$$R > 0, \quad r - d^{\mathrm{T}}R^{-1}d \geq 0, \tag{4.28}$$

then the zero solution of (4.10) is absolutely stable in the Hurwitz angle $[0,k]$, *where*

$$
\begin{aligned}
R &= -PA - A^{\mathrm{T}}P,\\
d &= -\left[Pb + \frac{1}{2}(\alpha c + \beta A^{\mathrm{T}}c)\right],\\
r &= -\beta c^{\mathrm{T}}b + \frac{\alpha}{k},\\
0 &\le \frac{f(\sigma)}{\sigma} < k \le +\infty.
\end{aligned}
$$

Proof. We construct the Lyapunov function

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma.$$

Then we have

$$
\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)} &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} xf(\sigma)\\
&\quad + \beta c^{\mathrm{T}}bf^2(\sigma).
\end{aligned}
$$

By using the S-method, we rearrange $-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)}$ as follows

$$
\begin{aligned}
-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)} &= x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right)\\
&:= S(x,\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right),
\end{aligned}
$$

where

$$S(x,\sigma) = x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma).$$

When $k \to +\infty$, the condition (4.27) or (4.28) implies that $S(x,\sigma) \ge 0$. Thus,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.10)} \le -\alpha f(\sigma)\sigma$$

is negative definite for Ω. Therefore, from Theorem 4.14 it follows that the conclusion is true. □

Corollary 4.27. *Suppose* $k \to +\infty$ *and one of the following conditions holds:*

(1) $\left(\frac{\beta A^{\mathrm{T}}c}{2} + Pb\right)^{\mathrm{T}} R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2} + Pb\right) + \beta c^{\mathrm{T}}b < 0$;

(2) $\left(\frac{\beta A^{\mathrm{T}}c}{2} + Pb\right)^{\mathrm{T}} R^{-1}c < 0$ *and*

$$\left[\left(\frac{\beta A^{\mathrm{T}}c}{2} + Pb\right)^{T} R^{-1}c\right]^2 - (c^{\mathrm{T}}R^{-1}c)\left[\left(\frac{\beta A^{\mathrm{T}}c}{2} + Pb\right)^{T} R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2} + Pb\right) + \beta c^{\mathrm{T}}b\right] \ge 0;$$

(3) $\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)^{\mathrm{T}} R^{-1} c \leq 0$ *and*

$$\left[\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)^{T} R^{-1} c\right]^{2}-(c^{\mathrm{T}} R^{-1} c)\left[\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)^{T} R^{-1}\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)+\beta c^{\mathrm{T}} b\right]>0.$$

Then there exists $\alpha>0$ such that $S(x,\sigma)\geq 0$. Therefore, the zero solution of (4.10) is absolutely stable.

Proof. We have $R>0$. Let $d^{\mathrm{T}} R^{-1} d-r=0$, that is,

$$\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb+\frac{\alpha c}{2}\right)^{\mathrm{T}} R^{-1}\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb+\frac{\alpha c}{2}\right)+\beta c^{\mathrm{T}} b=0,$$

or

$$\begin{aligned}\frac{c^{\mathrm{T}}}{2} R^{-1} \frac{c}{2} \alpha^{2}&+\left[\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)^{\mathrm{T}} R^{-1} \frac{c}{2}+\frac{c^{\mathrm{T}}}{2} R^{-1}\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)\right] \alpha \\ &+\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)^{\mathrm{T}} R^{-1}\left(\frac{\beta A^{\mathrm{T}} c}{2}+Pb\right)+\beta c^{\mathrm{T}} b=0. \end{aligned} \tag{4.29}$$

Equation (4.29) has the positive solution α if and only if one of the conditions (1), (2) and (3) holds. In this case, the condition of Theorem 4.26 is satisfied, so the conclusion is clear. □

Corollary 4.28. *If $k<+\infty$ and the following conditions hold:*

$$\begin{aligned}&\frac{1}{k}-c^{\mathrm{T}} R^{-1} d>0, \\ &\left(\frac{1}{k}-c^{\mathrm{T}} R^{-1} d\right)^{2}-c^{\mathrm{T}} R^{-1} c(d^{\mathrm{T}} R^{-1} d+\beta c^{\mathrm{T}} b)>0,\end{aligned}$$

then the derivative of $V(x)=x^{\mathrm{T}} P x+\beta \int_{0}^{\sigma} f(\sigma) \mathrm{d} \sigma$ along the solution of (4.10) is negative definite for Ω. Thus, the zero solution of (4.10) is absolutely stable in the Hurwitz angle $[0,k]$.

Proof. The conditions imply that there exists ε with $0<\varepsilon \ll 1$ such that

$$\frac{1}{k+\varepsilon}-c^{\mathrm{T}} R^{-1} d>0, \tag{4.30}$$

$$\left(\frac{1}{k+\varepsilon}-c^{\mathrm{T}} R^{-1} d\right)^{2}-c^{\mathrm{T}} R^{-1} c(d^{\mathrm{T}} R^{-1} d+\beta c^{\mathrm{T}} b)>0. \tag{4.31}$$

Consequently, we derive

$$\begin{aligned}-\left.\frac{\mathrm{d} V}{\mathrm{d} t}\right|_{(4.10)} &= x^{\mathrm{T}} R x+2 d^{\mathrm{T}} x f(\sigma)+\tilde{r} f^{2}(\sigma)+\alpha f(\sigma)\left(\sigma-\frac{1}{k+\varepsilon} f(\sigma)\right) \\ &:=\tilde{S}(x,\sigma)+\alpha f(\sigma)\left(\sigma-\frac{1}{k+\varepsilon} f(\sigma)\right),\end{aligned}$$

where

$$d = -\left[Pb + \frac{1}{2}(\alpha c + \beta A^{\mathrm{T}} c)\right],$$
$$\tilde{r} = -\left(\beta c^{\mathrm{T}} b - \frac{\alpha}{k+\varepsilon}\right).$$

Obviously, R is positive definite. Thus, the conditions (4.30) and (4.31) guarantee that the equation

$$\det \begin{vmatrix} R & d \\ d^{\mathrm{T}} & \tilde{r} \end{vmatrix} = 0$$

has positive solution for α, implying that $\tilde{S}(x,\alpha) \geq 0$. Therefore, the zero solution of (4.10) is absolutely stable in the Hurwitz angle $[0,k]$. □

Example 4.25 indicates that the original S-method is not efficient. In the following, we again adopt this example to illustrate that the modified S-method can be used to easily verify the absolute stability.

Taking

$$V = \frac{1}{2}x_1^2 + x_2^2 + \int_0^{x_2} f(x_2)\,\mathrm{d}x_2,$$

we derive

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.26)} &= -2x_1^2 + 2x_1x_2 - 2x_2^2 + 2x_1 f(x_2) - 2x_2 f(x_2) - \frac{1}{2}f^2(x_2), \\ &= \begin{pmatrix} x_1 \\ x_2 \\ f(x_2) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1+\alpha \\ 1 & -1+\alpha & -1/2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ f(x_2) \end{pmatrix} - 2\,\alpha x_2 f(x_2).\end{aligned}$$

Choosing $\alpha = 1/2$, we get

$$\det \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1+\alpha \\ 1 & -1+\alpha & -1/2 \end{bmatrix} = 0.$$

Thus, $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.26)} \leq -x_2 f(x_2)$. That is, $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.20)}$ is negative definite for $\Omega = \{x : \sigma = x_2 = 0\}$. Therefore, the zero solution of (4.26) is absolutely stable. Obviously, the modified S-method is more general than the standard one.

4.4 Lurie Indirect Control System

Consider the Lurie indirect control system

$$\begin{aligned} \dot{y} &= Dy + s\xi, \\ \dot{\xi} &= f(\sigma), \\ \sigma &= c^{\mathrm{T}} y - \rho\xi, \end{aligned} \tag{4.32}$$

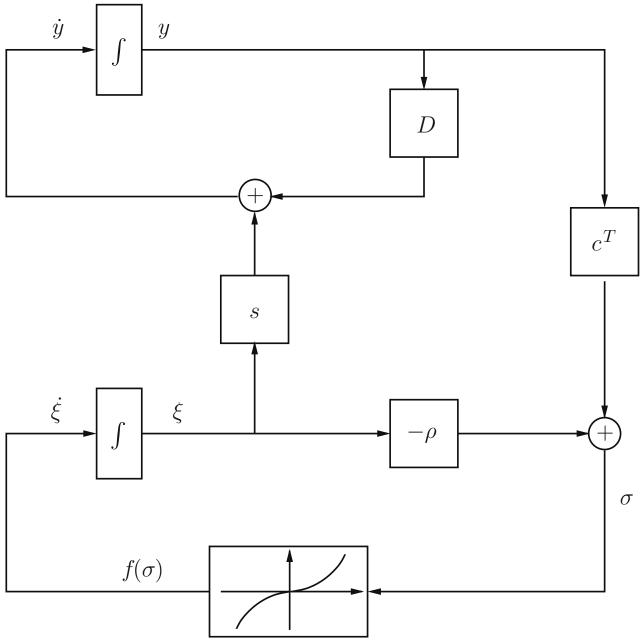

Fig. 4.2 Lurie indirect control system

where $D = (d_{ij})_{n\times n} \in \mathrm{R}^{n\times n}$, $s \in \mathrm{R}^n$, $y \in \mathrm{R}^n$, $\xi \in \mathrm{R}$, $\rho \in \mathrm{R}$, and $f(\sigma) \in F_\infty$. The indirect Lurie control system descried by (4.32) is depicted in Fig. 4.2.

It can be easily proved that the necessary conditions for absolute stability of the zero solution of (4.32) are that $\rho \geq 0$ and $\begin{bmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{bmatrix}$ is Hurwitz stable.

If $\rho \neq 0$, introduce the n-dimensional full-rank linear transformation:

$$\begin{aligned} x_i &= y_i, \quad i = 1,\ldots,n, \\ x_{n+1} &= \sigma = \sum_{i=1}^{n} c_i y_i - \rho\xi. \end{aligned} \tag{4.33}$$

Then (4.33) is transformed into the following system with separable variables

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^{n+1} a_{ij}x_j, \quad i = 1,\ldots,n, \\ \dot{x}_{n+1} &= \sum_{j=1}^{n+1} a_{n+1,j}x_j - \rho f(x_{n+1}), \end{aligned} \tag{4.34}$$

where

$$\begin{aligned} a_{ij} &= d_{ij} + \frac{s_i}{\rho}c_j, \quad i,j = 1,\ldots,n, \\ a_{i,n+1} &= -\frac{s_i}{\rho}, \quad i = 1,\ldots,n, \end{aligned}$$

$$a_{n+1,j} = \sum_{i=1}^{n} c_i a_{ij} = \sum_{i=1}^{n} c_i \left(d_{ij} + \frac{s_i c_j}{\rho} \right), \quad j = 1, \ldots, n,$$
$$a_{n+1,n+1} = \sum_{i=1}^{n} c_i a_{i,n+1} = \sum_{i=1}^{n} c_i \left(-\frac{s_i}{\rho} \right).$$

Since the necessary condition for absolute stability of the zero solution of (4.32) is $\begin{bmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{bmatrix}$ being stable, we take $\begin{bmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{bmatrix}$ nonsingular.

Again, by the nonsingular linear transformation

$$\begin{aligned} z &= Dy + s\xi, \\ z_{n+1} &= c^{\mathrm{T}} y - \rho\xi, \end{aligned} \qquad z, y \in \mathrm{R}^n, \tag{4.35}$$

(4.34) can be rewritten as

$$\begin{aligned} \dot{z} &= Bz + hf(z_{n+1}), \\ \dot{z}_{n+1} &= c^{\mathrm{T}} z - \tilde{\rho} f(z_{n+1}), \end{aligned} \tag{4.36}$$

where

$$\begin{bmatrix} B & h \\ c^{\mathrm{T}} & -\tilde{\rho} \end{bmatrix} = \begin{bmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{bmatrix} \begin{bmatrix} D & s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{bmatrix}^{-1},$$

$$\begin{pmatrix} h \\ -\tilde{\rho} \end{pmatrix} = \begin{bmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{bmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix}.$$

Obviously, under the condition that $\rho \neq 0$ and $\begin{bmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{bmatrix}$ is nonsingular, the stabilities of (4.32), (4.34), and (4.35) are equivalent. In analogy with Theorems 4.3–4.4, we formulate [75] the following:

Theorem 4.29. *The zero solution of (4.34) is absolutely stable if and only if*

(1) The zero solution of (4.34) is absolutely stable for x_{n+1};
(2) The matrix $G(g_{ij})_{(n+1)\times(n+1)}$ is Hurwitz stable, where

$$g_{ij} = \begin{cases} a_{n+1,n+1} - \rho, & i = j = n+1, \\ a_{ij}, & \textit{otherwise}. \end{cases}$$

Theorem 4.30. *The zero solution of (4.34) is absolutely stable if and only if*

(1) The zero solution of (4.34) is absolutely stable for the partial variables $x_j, \ldots,$ x_{n+1} $(i < j \leq n+1)$;
(2) The condition (2) of Theorem 4.29 holds.

Theorem 4.31. *The zero solution of (4.36) is absolutely stable if and only if*

(1) The zero solution of (4.36) is absolutely stable for z_{n+1};

(2) $\begin{bmatrix} B & h \\ c^{\mathrm{T}} & -\tilde{\rho} \end{bmatrix}$ is Hurwitz stable.

Theorem 4.32. *The zero solution of (4.36) is absolutely stable if and only if*

(1) The zero solution of (4.36) is absolutely stable for the partial variables $z_j, \ldots, z_{n+1}$ $(1 < j \leq n+1)$*;*
(2) $\begin{bmatrix} B & h \\ c^{\mathrm{T}} & -\tilde{\rho} \end{bmatrix}$ *is Hurwitz stable.*

The proofs for Theorems 4.29–4.32 can be completed similarly to that of Theorem 4.3, and are omitted.

In the following, we derive a series of practical sufficient conditions from the above theorems. Henceforth, we assume $\rho > 0$ $(\tilde{\rho} > 0)$.

Theorem 4.33. *If* $a_{ii} < 0, i = 1, \ldots, n+1$*, and*

$$G_a := -((-1)^{\delta_{ij}} |a_{ij}|)_{(n+1)\times(n+1)} \text{ is an } M\text{-matrix},$$

the zero solution of (4.34) is absolutely stable.

Proof. Since G_a is an M-matrix and $a_{ii} < 0$ $(i = 1, \ldots, n+1)$, there exists $n+1$ positive constants $r_i > 0$ $(i = 1, \ldots, n+1)$ such that

$$-r_j a_{jj} > \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}|, \quad j = 1, \ldots, n+1.$$

We construct the radially unbounded, positive definite Lyapunov function

$$V(x) = \sum_{i=1}^{n+1} r_i |x_i|.$$

Then, we obtain

$$\begin{aligned} D^+V(x)|_{(4.34)} &\leq \sum_{j=1}^{n+1} \left[r_j a_{jj} + \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}| \right] |x_j| - \rho r_{n+1} |f(x_{n+1})| \\ &< 0 \quad \text{for} \quad x \neq 0. \end{aligned}$$

Consequently, the zero solution of (4.34) is absolutely stable. □

Theorem 4.34. *Suppose that*

(1) The matrix A or the matrix $A + \begin{bmatrix} 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & -\rho \end{bmatrix}$ *is Hurwitz stable;*
(2) There exists constants $r_i \geq 0$ $(i = 1, \ldots, n)$, $r_{n+1} > 0$ *such that*

$$-r_j a_{jj} \geq \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}|, \quad j = 1, \ldots, n+1.$$

Then the zero solution of (4.34) is absolutely stable.

Proof. We construct the radially unbounded, positive definite Lyapunov function for x_n:

$$V(x) = \sum_{i=1}^{n+1} r_i |x_i|.$$

The argument used in the proof of Theorem 4.33 works, and

$$\begin{aligned} D^+V(x)|_{(4.34)} &\leq \sum_{j=1}^{n+1} \Big[r_j a_{jj} + \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}| \Big] |x_j| - \rho r_{n+1} |f(x_{n+1})| \\ &\leq -\rho\, r_{n+1} |\, f(x_{n+1})| < 0 \quad \text{for} \quad x_{n+1} \neq 0. \end{aligned}$$

Thus, the zero solution of (4.34) is absolutely stable for x_{n+1}. According to Theorem 4.29 the assertion holds. □

Theorem 4.35. *If* $b_{ii} < 0$, $i = 1, \ldots, n$, $\tilde{\rho} > 0$, *and*

$$\Omega := \begin{bmatrix} |b_{11}| & -|b_{12}| & \cdots & -|b_{1n}| & -|h_1| \\ -|b_{21}| & |b_{22}| & \cdots & -|b_{2n}| & -|h_2| \\ \vdots & \vdots & & \vdots & \vdots \\ -|b_{n1}| & -|b_{n2}| & \cdots & |b_{nn}| & -|h_n| \\ -|c_1| & -|c_2| & \cdots & -|c_n| & \tilde{\rho} \end{bmatrix} \text{ is an } M\text{-matrix},$$

then the zero solution of (4.36) is absolutely stable.

Proof. Since Ω is an M-matrix, there exist constants $r_i > 0$ $(i = 1, \ldots, n+1)$ such that

$$\begin{aligned} r_j |b_{jj}| &> \sum_{i=1, i\neq j}^{n} r_i |b_{ij}| + r_{n+1} |c_j|, \quad j = 1, \ldots, n, \\ r_{n+1} \tilde{\rho} &> \sum_{i=1}^{n} r_i |h_i|. \end{aligned}$$

We construct the radially unbounded and positive definite Lyapunov function

$$V(z) = \sum_{i=1}^{n+1} r_i |z_i|.$$

As in Theorem 4.33, we obtain

$$\begin{aligned} D^+V(z)|_{(4.36)} \leq & \sum_{j=1}^{n} \Big[r_j b_{jj} + \sum_{i=1, i\neq j}^{n} r_i |b_{ij}| + r_{n+1} |c_j| \Big] |z_j| \\ & + \Big[-\tilde{\rho} r_{n+1} + \sum_{i=1}^{n} r_i |h_i| \Big] |f(z_{n+1})| \\ & < 0 \quad \text{for} \quad z \neq 0. \end{aligned}$$

Thus, the zero solution of (4.36) is absolutely stable. □

Theorem 4.36. *(1) Let the matrix* $\begin{bmatrix} B & h \\ c^{\mathrm{T}} & -\tilde{\rho} \end{bmatrix}$ *be Hurwitz stable;*
(2) Suppose that there exist constants $r_i \geq 0\ (i=1,\ldots,n),\ r_{n+1} > 0$ *such that*

$$r_j|b_{jj}| \geq \sum_{i=1,i\neq j}^{n} r_i|b_{ij}| + r_{n+1}|c_j|, \quad j=1,\ldots,n,$$
$$r_{n+1}\tilde{\rho} > \sum_{i=1}^{n} r_i|h_i|.$$

Then the zero solution of (4.36) is absolutely stable.

Proof. We construct the radially unbounded, positive definite Lyapunov function for z_{n+1}:

$$V(z) = \sum_{i=1}^{n+1} r_i|z_i|.$$

Then,

$$\begin{aligned} D^+V(z)|_{(4.36)} &\leq \sum_{j=1}^{n}\Big[r_j b_{jj} + \sum_{i=1,i\neq j}^{n} r_i|b_{ij}| + r_{n+1}|c_j|\Big]|z_j| \\ &\quad + \Big[-\tilde{\rho}r_{n+1} + \sum_{i=1}^{n} r_i|h_i|\Big]|f(z_{n+1})| \\ &\leq \Big[-\tilde{\rho}r_{n+1} + \sum_{i=1}^{n} r_i|h_i|\Big]|f(z_{n+1})| \\ &< 0 \quad \text{for} \quad z_{n+1} \neq 0. \end{aligned}$$

Accordingly, $D^+V(z)|_{(4.36)}$ is negative definite for z_{n+1}, and it follows from condition (1) that the conditions of Theorem 4.31 are satisfied. Hence, the conclusion of this theorem holds. □

In the following, we take

$$A_{(j_0)} = \begin{bmatrix} a_{11} & \cdots & a_{1j_0} \\ \vdots & & \vdots \\ a_{j_01} & \cdots & a_{j_0j_0} \end{bmatrix}, \qquad A^{(n+1-j_0)} = \begin{bmatrix} a_{1,j_0+1} & \cdots & a_{1,n+1} \\ \vdots & & \vdots \\ a_{j_0,j_0+1} & \cdots & a_{j_0,n+1} \end{bmatrix},$$

$$B_{(j_0)} = \begin{bmatrix} b_{11} & \cdots & b_{1j_0} \\ \vdots & & \vdots \\ b_{j_01} & \cdots & b_{j_0j_0} \end{bmatrix}, \qquad B^{(n+1-j_0)} = \begin{bmatrix} b_{1,j_0+1} & \cdots & b_{1n} & h_1 \\ \vdots & & \vdots & \vdots \\ b_{j_0,j_0+1} & \cdots & b_{j_0n} & h_{j_0} \end{bmatrix},$$

$$x_{(j_0)} = (x_1,\ldots,x_{j_0})^{\mathrm{T}}, \qquad x^{(n+1-j_0)} = (x_{j_0+1},\ldots,x_{n+1})^{\mathrm{T}},$$
$$z_{(j_0)} = (z_1,\ldots,z_{j_0})^{\mathrm{T}}, \qquad z^{(n+1-j_0)} = (z_{j_0+1},\ldots,z_n,f(z_{n+1}))^{\mathrm{T}},$$
$$f^{(n+1-j_0)} = (\overbrace{0,\ldots,0}^{n+1-j_0}, -\rho f(x_{n+1})).$$

Theorem 4.37. *Suppose that*

(1) The matrix $A_{(j_0)}$ *is Hurwitz stable;*
(2) There exist constants $r_i \geq 0\ (i = 1, \ldots, j_0)$, $r_j > 0\ (j = j_0 + 1, \ldots, n+1)$ *such that*

$$-r_j a_{jj} \geq \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}|, \quad j = 1, \ldots, j_0,$$
$$-r_j a_{jj} > \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}|, \quad j = j_0 + 1, \ldots, n+1.$$

Then the zero solution of (4.34) is absolutely stable.

Proof. We construct the radially unbounded, positive definite Lyapunov function w.r.t. the partial variables $x_{j_0+1}, \ldots, x_{n+1}$:

$$V(z) = \sum_{i=1}^{n+1} r_i |x_i|.$$

Then,

$$\begin{aligned} D^+V|_{(4.34)} &\leq \sum_{j=1}^{j_0} \left[r_j a_{jj} + \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}| \right] |x_j| - \rho r_{n+1} |f(x_{n+1})|, \\ &\leq \sum_{j=j_0+1}^{n+1} \left[r_j a_{jj} + \sum_{i=1, i\neq j}^{n+1} r_i |a_{ij}| \right] |x_j| - \rho\, r_{n+1} |f(x_{n+1})|, \\ &< 0 \qquad \text{for} \sum_{j=j_0+1}^{n+1} x_j^2 \neq 0. \end{aligned}$$

Thus, the zero solution of (4.34) is absolutely stable w.r.t. the partial variables $x_{j_0+1}, \ldots, x_{n+1}$.

The first j_0 components of the solution of (4.34) can be expressed as

$$\begin{aligned} x_{(j_0)}(t, t_0, x_0) &= \mathrm{e}^{A_{(j_0)}(t-t_0)} x_{(j_0)}(t_0) \\ &\quad + \int_{t_0}^{t} \left[\mathrm{e}^{A_{(j_0)}(t-\tau)} A^{(n+1-j_0)} x^{(n+1-j_0)}(\tau) + f^{(n+1-j_0)}(x_{nn}(\tau)) \right] \mathrm{d}\tau. \end{aligned}$$

Following the proof of the sufficiency in Theorem 4.3, we can complete the rest of the proof. □

Similarly, we have the following theorems.

Theorem 4.38. *Suppose that*

(1) The matrix $B_{(j_0)}$ *is Hurwitz stable;*

(2) There exist constants $r_i \geq 0\ (i = 1, \dots, j_0)$, $r_j > 0\ (j = j_0 + 1, \dots, n+1)$ *such that*

$$\begin{aligned}
-r_j b_{jj} &\geq \sum_{i=1, i\neq j}^{n+1} r_i |b_{ij}| + r_{n+1}|c_j|, \quad j = 1, \dots, j_0,\\
-r_j b_{jj} &> \sum_{i=1, i\neq j}^{n+1} r_i |b_{ij}| + r_{n+1}|c_j|, \quad j = j_0 + 1, \dots, n+1,\\
r_{n+1}\tilde{\rho} &> \sum_{i=1}^{n} r_i |h_i|.
\end{aligned}$$

Then the zero solution of (4.36) is absolutely stable.

Theorem 4.39. *Assume that*

(1) $A(a_{ij})_{(n+1)\times(n+1)}$ *or* $A + \begin{bmatrix} 0 & 0 \\ 0 & -\rho \end{bmatrix}$ *is Hurwitz stable;*
(2) There exist a symmetric positive semi-definite matrix of the form

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ p_{n1} & \cdots & p_{nn} & 0 \\ 0 & \cdots & 0 & p_{n+1,n+1} \end{bmatrix} \quad (p_{n+1,n+1} > 0)$$

such that $A^{\mathrm{T}}P + PA$ *is negative semi-definite.*

Then the zero solution of (4.34) is absolutely stable.

Proof. Obviously, the condition (1) is the same as the condition (2) in Theorem 4.29.

We construct the radially unbounded, positive definite Lyapunov function w.r.t. x_{n+1}:

$$V(x) = x^{\mathrm{T}} P x.$$

We fix $l = (\overbrace{0, \dots, 0}^{n}, -\rho)^{\mathrm{T}}$, then

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.34)} &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + (l^{\mathrm{T}}Px + x^{\mathrm{T}}Pl) f(x_{n+1}),\\
&= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - 2\rho\, p_{n+1,n+1} x_{n+1} f(x_{n+1}),\\
&\leq -2\rho\, p_{n+1,n+1} x_{n+1} f(x_{n+1}),\\
&< 0 \quad \text{for} \quad x_{n+1} \neq 0.
\end{aligned}$$

Thus, the condition (1) of Theorem 4.29 is satisfied and the conclusion of this theorem is true. □

Theorem 4.40. *Assume that*

(1) The condition (1) of Theorem 4.39 holds;

(2) There exist a constant $\varepsilon > 0$ and an $(n+1) \times (n+1)$ symmetric, positive semi-definite matrix P such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & Pl + \frac{1}{2}A_{n+1} + \varepsilon e_{n+1} \\ (Pl + \frac{1}{2}A_{n+1} + \varepsilon e_{n+1})^{\mathrm{T}} & -\rho \end{bmatrix}$$

is negative semi-definite, where

$$A_{n+1} = (a_{n+1,1}, \ldots, a_{n+1,n+1})^{\mathrm{T}}, \quad \ell = (\overbrace{0, \ldots, 0}^{n}, -\rho)^{\mathrm{T}},$$
$$e_{n+1} = (\overbrace{0, \ldots, 0}^{n}, 1)^{\mathrm{T}};$$

(3) $\int_0^{\pm\infty} f(x_{n+1})\,\mathrm{d}x_{n+1} = +\infty$.

Then the zero solution of (4.34) is absolutely stable.

Proof. We construct the Lyapunov function

$$V(x) = x^{\mathrm{T}}Px + \int_0^{x_{n+1}} f(x_{n+1})\,\mathrm{d}x_{n+1}.$$

Obviously,

$$V(x) \geq \int_0^{x_{n+1}} f(x_{n+1})\,\mathrm{d}x_{n+1} > 0 \quad \text{for} \quad x_{n+1} \neq 0,$$

and $V(x) \to +\infty$ as $|x_{n+1}| \to +\infty$. Moreover, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.34)} &= \dot{x}Px + x^{\mathrm{T}}P\dot{x} + [A_{n+1}^{\mathrm{T}}x - \rho f(x_{n+1})]f(x_{n+1}), \\
&= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + [l^{\mathrm{T}}Px + x^{\mathrm{T}}Pl + A_{n+1}^{\mathrm{T}}x]f(x_{n+1}) - \rho f^2(x_{n+1}), \\
&= \begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}}P + PA & Pl + \frac{A_{n+1}}{2} + \varepsilon e_{n+1} \\ (Pl + \frac{A_{n+1}}{2} + \varepsilon e_{n+1})^{\mathrm{T}} & -\rho \end{bmatrix} \begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix} \\
&\quad - \begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 0_{(n+1)\times(n+1)} & \varepsilon e_{n+1} \\ (\varepsilon e_{n+1})^{\mathrm{T}} & 0 \end{bmatrix} \begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix}, \\
&\leq -2\varepsilon x_{n+1} f(x_{n+1}) < 0 \quad \text{as} \quad x_{n+1} \neq 0.
\end{aligned}$$

Thus, the zero solution of (4.34) is absolutely stable. □

Remark 4.41. Suppose that the condition for the matrix P being positive semi-definite is replaced by the condition for the matrix P being positive definite, or that there exists a constant $\varepsilon > 0$ satisfying

$$x^{\mathrm{T}}Px \geq \varepsilon x_{n+1}^2.$$

Then the condition (3) of Theorem 4.40 can be dropped.

Theorem 4.42. *(1) Let the condition (1) of Theorem 4.40 be satisfied.*
(2) Suppose that there exists an $n \times n$ symmetric, positive semi-definite matrix P such that $B^{\mathrm{T}}P + PB := -Q$ is negative semi-definite, and that there exists a constant $\varepsilon > 0$ such that

$$\det \begin{bmatrix} Q & -(Ph + \frac{c}{2}) \\ -(Ph + \frac{c}{2})^{\mathrm{T}} & \tilde{\rho} - \varepsilon \end{bmatrix} \geq 0$$

and

$$\int_0^{\pm\infty} f(z_{n+1})\,\mathrm{d}z_{n+1} = +\infty.$$

Then the zero solution of (4.36) is absolutely stable.

Proof. We construct the Lyapunov function

$$V(z) = z^{\mathrm{T}}Pz + \int_0^{z_{n+1}} f(z_{n+1})\,\mathrm{d}z_{n+1}$$

with $z = (z_1, \ldots, z_n)^{\mathrm{T}}$. Obviously, $V(z)$ is radially unbounded, positive definite for z_{n+1}, and

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.36)} &= z^{\mathrm{T}}(B^{\mathrm{T}}P + PB)z + (h^{\mathrm{T}}Pz + z^{\mathrm{T}}Ph + c^{\mathrm{T}}z)f(z_{n+1}) - \tilde{\rho}f^2(z_{n+1}), \\ &= -z^{\mathrm{T}}Qz + 2f(z_{n+1})(Ph + \frac{c}{2})^{\mathrm{T}}z - \tilde{\rho}f^2(z_{n+1}), \\ &= \begin{pmatrix} z \\ f(z_{n+1}) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -Q & Ph + \frac{c}{2} \\ (Ph + \frac{c}{2})^{\mathrm{T}} & -\tilde{\rho} + \varepsilon \end{bmatrix} \begin{pmatrix} z \\ f(z_{n+1}) \end{pmatrix} - \varepsilon f^2(z_{n+1}), \\ &\leq -\varepsilon f^2(z_{n+1}) < 0 \quad \text{for} \quad z_{n+1} \neq 0. \end{aligned}$$

In this case, the zero solution of (4.36) is absolutely stable for z_{n+1}. Hence, all the conditions of Theorem 4.31 are satisfied; thus the conclusion of this theorem is valid. □

Theorem 4.43. *Suppose the following conditions are satisfied:*

(1) $A_{(j_0)}$ is stable;
(2) There exist an $n \times n$ matrix P and a constant $\varepsilon > 0$ $(\varepsilon \ll 1)$ such that

$$x^{\mathrm{T}}Px \geq \varepsilon \sum_{j=j_0+1}^{n+1} x_j^2$$

and

$$\begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}}P + PA & Ph + \frac{1}{2}A_{n+1} \\ (Ph + \frac{1}{2}A_{n+1})^{\mathrm{T}} & -\rho \end{bmatrix} \begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix}$$

$$\leq \begin{cases} -\varepsilon \sum\limits_{i=j_0+1}^{n} x_i^2 - \varepsilon f^2(x_{n+1}), & \textit{or} \\ -\varepsilon \sum\limits_{i=j_0+1}^{n+1} x_i^2, & \textit{or} \\ -\varepsilon \Big[\sum\limits_{i=j_0+1}^{n} x_i^2 + x_{n+1} f(x_{n+1}) \Big], \end{cases}$$

where $A_{n+1} = (a_{n+1,1}, \ldots, a_{n+1,n+1})^{\mathrm{T}}$.

Then the zero solution of (4.34) is absolutely stable.

Proof. We construct the Lyapunov function

$$V = x^{\mathrm{T}} P x + \int_0^{x_{n+1}} f(x_{n+1})\, \mathrm{d}x_{n+1}.$$

The condition (2) asserts that $V(x)$ is radially unbounded and positive definite w.r.t. the partial variables $x_{j_0+1}, \ldots, x_{n+1}$, and that $\left.\dfrac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.34)}$ is negative definite w.r.t. $x_{j_0+1}, \ldots, x_{n+1}$.

In addition, the first j_0 components of the solution of (4.34) can be expressed as

$$\begin{aligned} x_{(j_0)}(t, t_0, x_0) &= \mathrm{e}^{A_{(j_0)}(t-t_0)} x_{(j_0)}(t_0) \\ &\quad + \int_{t_0}^{t} \Big[\mathrm{e}^{A_{(j_0)}(t-\tau)} A^{(n+1-j_0)} x^{(n+1-j_0)}(\tau) + f^{n+1-j_0}(\tau) \Big]\, \mathrm{d}\tau. \end{aligned}$$

The rest of the proof can be completed as in Theorem 4.3. □

Similarly, we formulate

Theorem 4.44. *(1) Let* $B_{(j_0)}$ *be stable;*
(2) Suppose that there exist an $n \times n$ *matrix* P *and a constant* $\varepsilon > 0$ *such that*

$$z^{\mathrm{T}} B z \geq \varepsilon \sum_{i=j_0+1}^{n+1} z_i^2$$

and

$$\begin{pmatrix} z \\ f(z_{n+1}) \end{pmatrix}^T \begin{bmatrix} B^{\mathrm{T}}P + PB & Ph + \dfrac{c}{2} \\ (Ph + \dfrac{c}{2})^{\mathrm{T}} & -\tilde{\rho} \end{bmatrix} \begin{pmatrix} z \\ f(z_{n+1}) \end{pmatrix} \leq -\varepsilon \sum_{i=j_0+1}^{n+1} z_i^2 - \varepsilon f^2(z_{n+1});$$

(3) $\int_0^{\pm\infty} f(z_{n+1})\, \mathrm{d}z_{n+1} = +\infty$.
Then the zero solution of (4.36) is absolutely stable.

4.5 Lurie Systems with Loop Feedbacks

The result presented in this section is new and is in press [89].

Consider the more general Lurie control system with loop feedbacks:

$$\begin{aligned} \dot{x} &= Ax + b\xi + df(\sigma), \\ \dot{\xi} &= f(\sigma), \\ \sigma &= c^{\mathrm{T}}x - r\xi - Nf(\sigma), \end{aligned} \tag{4.37}$$

where $x \in \mathrm{R}^n$, $A \in \mathrm{R}^{n\times n}$, $b,c,d \in \mathrm{R}^n$, $N,r \in \mathrm{R}^1$, $f \in F_k$. When $N = 0$, $d = 0$, (4.37) is a standard Lurie indirect control system. If the second scalar equation does not exist, and $N = \xi = 0$, $b = 0$, A is a Hurwitz matrix. Then (4.37) is a standard Lurie direct control system. If the second scalar equation does not exist, and $N = \xi = 0$, $b = 0$, $\mathrm{Re}\lambda(A) \le 0$, A has only one zero eigenvalue, and all other eigenvalues have negative real parts, then (4.37) is a Lurie critical control system. The system (4.37) describing the Lurie system with loop feedback is shown in Fig. 4.3.

In the above three control systems, we have $N = 0$, that is, the feedback control variable σ is a linear combination of state variables. When $N \neq 0$ in (4.37), the feedback variable σ and the state variables are in an implicit relation. To consider the effect of delay, we can further see that the feedback control variable and the feedback function $f(\sigma)$ have a recursive relation. This is more complicated, but better describes practical control process.

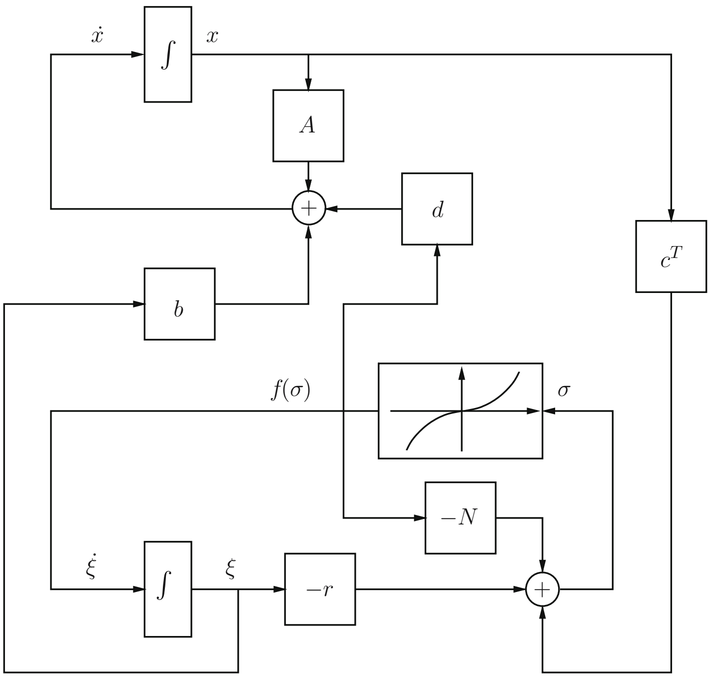

Fig. 4.3 Lurie system with loop feedback

Let $\bar{\Omega} = \{\|c^{\mathrm{T}}x - r\xi\| = 0\}$. Similar to Definition 4.13, we can define the absolute stability of the zero solution of (4.37) w.r.t. $\bar{\Omega}$. Similar to Definition 4.15, we can also define the Lyapunov function $V(x,\xi) \in C[\mathrm{R}^{n+1}, \mathrm{R}^1]$, which is radially unbounded and positive definite w.r.t. the set $\bar{\Omega}$.

Definition 4.45. *$V(x,\xi)$ is positive definite (negative definite) w.r.t. $\sigma = 0$ if*

$$V(x,\xi) = \begin{cases} > 0 & \text{if} \quad \sigma \neq 0, \\ = 0 & \text{if} \quad \sigma = 0. \end{cases} \qquad \left(V(x,\xi) = \begin{cases} < 0 & \text{if} \quad \sigma \neq 0, \\ = 0 & \text{if} \quad \sigma = 0. \end{cases} \right)$$

$V(x,\xi)$ is radially unbounded w.r.t. σ if $V(x,\xi) \to +\infty$ as $\sigma \to \infty$.

Note that σ is not a state variable. Its relation with x and ξ is not explicit. Thus, it is difficult to check if the above conditions are satisfied.

In the following, we assume that $|\sigma + Nf(\sigma)|$ is positive definite. $|\sigma + Nf(\sigma)|$ is obviously positive definite as $N \geq 0$. If $N < 0$, and $|f(\sigma)| < \frac{1}{N}|\sigma|$ or $|f(\sigma)| > \frac{1}{N}|\sigma|$, we can also make $|\sigma + Nf(\sigma)|$ positive definite.

Lemma 4.46. *The zero solution of (4.37) is absolutely stable with respect to the set $\bar{\Omega}$ if and only if it is absolutely stable with respect to $\sigma = 0$.*

Proof. The conclusion is obvious if $N = 0$. In this case, σ is explicitly expressed by the linear combination of the state variables as $\sigma = c^{\mathrm{T}}x - r\xi = 0$.

Next, consider the case $N \neq 0$.

Sufficiency. Assume that the zero solution of (4.37) is absolutely stable w.r.t. $\sigma = 0$. Since $f(0) = 0$ and $f(\sigma)$ is continuous, $\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $|\sigma(t,t_0,\sigma_0)| < \frac{\varepsilon}{2}$ and $|Nf(\sigma(t,t_0,\sigma_0))| < \frac{\varepsilon}{2}$ when $|\sigma_0| < \delta$. Thus,

$$|c^{\mathrm{T}}x(t) - r\xi(t)| \leq |\sigma(t,t_0,\sigma_0)| + |Nf(\sigma(t,t_0,\sigma_0))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For all $(x_0, \xi_0) \in \mathrm{R}^{n+1}$,

$$\lim_{t\to+\infty} |c^{\mathrm{T}}x(t) - r\xi(t)| \leq \lim_{t\to+\infty} |\sigma(t)| + \lim_{t\to+\infty} |Nf(\sigma(t))| = 0.$$

Thus, the zero solution of (4.37) is absolutely stable with respect to $\bar{\Omega}$.

Necessity. Assume that the zero solution of (4.37) is absolutely stable w.r.t. $\bar{\Omega}$, and $|\sigma + Nf(\sigma)|$ is positive definite w.r.t. σ. From Lemmas 2.5 and 2.6, we know that there exists kth order function $\varphi(\sigma)$ such that

$$\varphi(|\sigma|) \leq |\sigma(t) + Nf(\sigma(t))| = |c^{\mathrm{T}}x(t) - r\xi(t)|.$$

From the characteristics of the K-class function, we have

$$|\sigma| \leq \varphi^{-1}|c^{\mathrm{T}}x(t) - r\xi(t)|.$$

$\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$|\sigma(t)| \leq \varphi^{-1}(|c^{\mathrm{T}}x(t) - r\xi(t)|) < \varepsilon,$$

if $\|x_0\| + |\xi_0| < \delta$. Also, $\forall (x_0, \xi_0) \in \mathrm{R}^{n+1}$,

$$|\sigma(t) \leq \varphi^{-1}(|c^{\mathrm{T}}x(t) - r\xi(t)|) \to 0, \quad \text{as} \quad t \to +\infty.$$

Thus, the zero solution of (4.37) is absolutely stable w.r.t. $\sigma = 0$. The proof is complete. □

Theorem 4.47. *The zero solution of (4.37) is absolutely stable if and only if the following two conditions are satisfied:*

(1) B_1, B_2, and B_3 are all Hurwitz matrices, where

$$B_1 = \begin{bmatrix} A + dc^{\mathrm{T}} & b - dr \\ c^{\mathrm{T}} & -r \end{bmatrix}, \quad \text{if} \quad N = 0,$$

$$B_2 = \begin{bmatrix} A + \dfrac{dc^{\mathrm{T}}}{1+N} & b - \dfrac{dr}{1+N} \\ \dfrac{c^{\mathrm{T}}}{1+N} & -\dfrac{r}{1+N} \end{bmatrix}, \quad \text{if} \quad N > 0,$$

$$B_3 = \begin{bmatrix} A - \dfrac{dc^{\mathrm{T}}}{N} & b + \dfrac{dr}{N} \\ -\dfrac{c^{\mathrm{T}}}{N} & \dfrac{r}{N} \end{bmatrix}, \quad \text{if} \quad N < 0;$$

(2) The zero solution of (4.37) is absolutely stable w.r.t. the set $\bar{\Omega}$.

Proof. *Necessity*. If $N = 0$, $f(\sigma) = c^{\mathrm{T}}x - r\xi = \sigma$. Then (4.37) becomes

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{pmatrix} = \begin{bmatrix} A + dc^{\mathrm{T}} & b - dr \\ c^{\mathrm{T}} & -r \end{bmatrix} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = B_1 \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}. \tag{4.38}$$

If $N \neq 0$, we have $f(\sigma) = \frac{c^{\mathrm{T}}x - r\xi - \sigma}{N}$. Then (4.37) can be rewritten as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{pmatrix} = \begin{bmatrix} A & b \\ \dfrac{c^{\mathrm{T}}}{N} & -\dfrac{r}{N} \end{bmatrix} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} + \begin{pmatrix} d f(\sigma) \\ -\dfrac{\sigma}{N} \end{pmatrix}. \tag{4.39}$$

If $N > 0$, let $f(\sigma) = \sigma$. It is obvious that $|\sigma + Nf(\sigma)|$ is positive definite. Substituting $\sigma = \frac{c^{\mathrm{T}}x - r\xi}{1+N}$ in (4.39) yields

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{pmatrix} &= \begin{bmatrix} A & b \\ \dfrac{c^{\mathrm{T}}}{N} & -\dfrac{r}{N} \end{bmatrix} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} + \begin{pmatrix} \dfrac{d(c^{\mathrm{T}}x - r\xi)}{1+N} \\ -\dfrac{c^{\mathrm{T}}x - r\xi}{N(1+N)} \end{pmatrix}, \\ &= \begin{bmatrix} A + \dfrac{dc^{\mathrm{T}}}{1+N} & b - \dfrac{dr}{1+N} \\ \dfrac{c^{\mathrm{T}}}{1+N} & -\dfrac{r}{1+N} \end{bmatrix} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = B_2 \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}. \end{aligned} \tag{4.40}$$

If $N < 0$, let $f(\sigma) = -\frac{\sigma}{2N}$. It is clear that $|\sigma + Nf(\sigma)| = \frac{1}{2}|\sigma|$ is positive definite. Substituting $\sigma = 2(c^{\mathrm{T}}x - r\xi)$ in (4.39) results in

$$\begin{aligned}\begin{pmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{pmatrix} &= \begin{bmatrix}A & b\\ \dfrac{c^{\mathrm{T}}}{N} & -\dfrac{r}{N}\end{bmatrix}\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix} + \begin{pmatrix}-\dfrac{d(c^{\mathrm{T}}x - r\xi)}{N}\\ \dfrac{-2(c^{\mathrm{T}}x - r\xi)}{N}\end{pmatrix},\\ &= \begin{bmatrix}A - \dfrac{dc^{\mathrm{T}}}{N} & b + \dfrac{dr}{N}\\ -\dfrac{c^{\mathrm{T}}}{N} & \dfrac{r}{N}\end{bmatrix}\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix} = B_3\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix}.\end{aligned} \tag{4.41}$$

Thus, B_1, B_2, and B_3 are all Hurwitz matrices, and the condition (1) holds.

$\forall f(\cdot) \in F_{[0,k]}$, let $\max[\max_{1\le i\le n}(|c_i|), |r|] = r$. $\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\sum_{i=1}^{n} |x_i(t)| + |\xi(t)| < \frac{\varepsilon}{r}.$$

Thus,

$$\sum_{i=1}^{n} |c_i x_i(t)| + |r\xi(t)| \le r \cdot \frac{\varepsilon}{r} = \varepsilon.$$

$\forall (x_0, \xi_0) \in \mathrm{R}^{n+1}$, we have

$$0 \le \lim_{t\to+\infty} |c^{\mathrm{T}}x(t) - r\xi(t)| \le \lim_{t\to+\infty}\left[\sum_{i=1}^{n}|c_i x_i(t)| + |r\xi(t)|\right] = 0.$$

Thus, the zero solution of (4.37) is absolutely stable w.r.t. $\bar{\Omega}$. The condition (2) holds, and the necessity is proved.

Sufficiency. Using (4.38), (4.40), and (4.41), we can rewrite (4.37) as the following three equations:

When $N = 0$,

$$\begin{pmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{pmatrix} = B_1\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix} + \begin{pmatrix}d(f(\sigma(t))) - \sigma(t)\\ f(\sigma(t)) - \sigma(t)\end{pmatrix}; \tag{4.42}$$

When $N > 0$,

$$\begin{pmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{pmatrix} = B_2\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix} + \begin{pmatrix}d(f(\sigma(t))) - \sigma(t)\\ f(\sigma(t)) - \sigma(t)\end{pmatrix}; \tag{4.43}$$

When $N < 0$,

$$\begin{pmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{pmatrix} = B_3\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix} + \begin{pmatrix}d(f(\sigma(t))) - \sigma(t)\\ f(\sigma(t)) + \dfrac{\sigma(t)}{2N}\end{pmatrix}. \tag{4.44}$$

In the following, we first prove the absolute stability of the zero solution of (4.42). With the formula of variation of constants, the solution of (4.42) can be expressed as

$$\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \mathrm{e}^{B_1(t-t_0)} \begin{pmatrix} x(t_0) \\ \xi(t_0) \end{pmatrix} + \int_{t_0}^{t} \mathrm{e}^{B_1(t-\tau)} \begin{pmatrix} d\,(f(\sigma(\tau)) - \sigma(\tau)) \\ f(\sigma(\tau)) - \sigma(\tau) \end{pmatrix} \mathrm{d}\tau. \quad (4.45)$$

Since B_1 is a Hurwitz matrix, there exist $\xi > 0$ and $M \geq 1$ such that

$$\left\| \mathrm{e}^{B_1(t-t_0)} \right\| \leq M\,\mathrm{e}^{-\xi(t-t_0)}.$$

Further, because of that continuity of $f(\sigma) - \sigma$, $\forall \varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(\sigma) - \sigma| < \frac{\xi \varepsilon}{2 \left\| \begin{matrix} d \\ 1 \end{matrix} \right\|} M$$

when $|\sigma| < \delta_1$. The absolute stability of the zero solution of (4.42) w.r.t. the set $\bar{\Omega}$ is equivalent to that w.r.t. $\sigma = 0$. Thus, for $\delta_1 > 0$, there exists $\delta > 0$ such that (4.42) satisfies

$$\begin{aligned}
&\left\| \int_{t_0}^{t} \mathrm{e}^{B_1(t-\tau)} \begin{bmatrix} d\,(f(\sigma(\tau)) - \sigma(\tau)) \\ (f(\sigma(\tau)) - \sigma(\tau) \end{bmatrix} \mathrm{d}\tau \right\| \\
&\leq \int_{t_0}^{t} \left\| \mathrm{e}^{B_1(t-\tau)} \right\| \left\| \begin{matrix} d\,(f(\sigma(\tau)) - \sigma(\tau)) \\ f(\sigma(\tau)) - \sigma(\tau) \end{matrix} \right\| \mathrm{d}\tau \\
&\leq \int_{t_0}^{t} M\,\mathrm{e}^{-\xi(t-\tau)} \left\| \begin{matrix} d \\ 1 \end{matrix} \right\| \|f(\sigma(\tau)) - \sigma(\tau)\| \,\mathrm{d}\tau \\
&\leq \xi \varepsilon \int_{t_0}^{t} \mathrm{e}^{-\xi(t-\tau)}\,\mathrm{d}\tau \leq \frac{\varepsilon}{2}
\end{aligned}$$

when $\left\| \begin{matrix} x(t_0) \\ \xi(t_0) \end{matrix} \right\| < \delta$ and $\|\sigma(t)\| \leq \delta_1$.

Let $\delta_2 = \min\left\{ \frac{\varepsilon}{2M}, \delta \right\}$. For the above ε, we have

$$\begin{aligned}
\left\| \begin{matrix} x(t) \\ \xi(t) \end{matrix} \right\| &\leq \|\mathrm{e}^{B_1(t-t_0)}\| \left\| \begin{matrix} x(t_0) \\ \xi(t_0) \end{matrix} \right\| + \int_{t_0}^{t} \left\| \mathrm{e}^{B_1(t-\tau)} \right\| \left\| \begin{matrix} d\,(f(\sigma(\tau)) - \sigma(\tau)) \\ f(\sigma(\tau)) - \sigma(\tau) \end{matrix} \right\| \mathrm{d}\tau \\
&\leq M\,\mathrm{e}^{-\xi(t-t_0)} \left\| \begin{matrix} x(t_0) \\ \xi(t_0) \end{matrix} \right\| + \left\| \begin{matrix} d \\ 1 \end{matrix} \right\| M \int_{t_0}^{t} \mathrm{e}^{-\xi(t-\tau)} \|f(\sigma(\tau) - \sigma(\tau)\| \,\mathrm{d}\tau \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned} \quad (4.46)$$

when $\left\| \begin{matrix} e(t_0) \\ \xi(t_0) \end{matrix} \right\| < \delta_2$. Thus, the zero solution of (4.42) is stable.

The first term in (4.45) $\to 0$ as $t \to +\infty$ due to $\mathrm{Re}[\lambda(B_1)] < 0$. Since $\lim\limits_{t\to+\infty} \sigma(t) = 0$, $f(\sigma) - \sigma$ is continuous, and $f(0) = 0$, we have

$$\lim_{t\to+\infty} |f(\sigma(t)) - \sigma(t)| = 0.$$

Now we derive

$$\begin{aligned}\int_{t_0}^{t} \mathrm{e}^{B_1(t-\tau)} \begin{pmatrix} d(f(\sigma(\tau)) - \sigma(\tau)) \\ f(\sigma(\tau)) - \sigma(\tau) \end{pmatrix} \mathrm{d}\tau &\le M \int_{t_0}^{t} \mathrm{e}^{-\xi(t-\tau)} \left\| \begin{matrix} d(f(\sigma(\tau)) - \sigma(\tau)) \\ f(\sigma(\tau)) - \sigma(\tau) \end{matrix} \right\| \mathrm{d}\tau \\ &\le M \frac{\int_{t_0}^{t} \mathrm{e}^{\xi\tau} \left\| \begin{matrix} d(f(\sigma(\tau)) - \sigma(\tau)) \\ f(\sigma(\tau)) - \sigma(\tau) \end{matrix} \right\| \mathrm{d}\tau}{e^{\xi t}}.\end{aligned}$$

Using the L'Hospital rule, we have

$$\lim_{t\to+\infty} \frac{\int_{t_0}^{t} \left\| \begin{matrix} d(f(\sigma(\tau)) - \sigma(\tau)) \\ f(\sigma(\tau)) - \sigma(\tau) \end{matrix} \right\| \mathrm{d}\tau}{\mathrm{e}^{\xi t}} = \frac{1}{\xi} \lim_{t\to+\infty} \left\| \begin{matrix} d(f(\sigma(t)) - \sigma(t)) \\ f(\sigma(t)) - \sigma(t) \end{matrix} \right\| = 0.$$

Thus, $\lim\limits_{t\to+\infty} \left\| \begin{matrix} x(t) \\ \xi(t) \end{matrix} \right\| = 0$. The zero solution of (4.42) is absolutely stable.

Since (4.43) has the same nonlinear terms as that of (4.42), B_1, B_2 have the same Hurwitz's characteristics. Thus, the proof of the absolute stability of the zero solution of (4.43) is same as that of (4.42).

Although the nonlinear terms of (4.44) are different from that of (4.42), we have the following estimation:

$$\left\| \begin{matrix} df(\sigma) - \sigma \\ \frac{\sigma}{2N} + \sigma \end{matrix} \right\| \le \left\| \begin{matrix} d \\ \max[\frac{1}{2N}, 1] \end{matrix} \right\| (|f(\sigma)| + |\sigma|). \tag{4.47}$$

From Lemma 4.46, we know that $\forall \varepsilon > 0$, there exists $\delta_3 > 0$ such that

$$|f(\sigma)| + |\sigma| < \frac{\eta \varepsilon}{2 \left\| \begin{matrix} d \\ \max\left[\frac{1}{2N}, 1\right] \end{matrix} \right\|} \tag{4.48}$$

when $|\sigma| < \delta_3$. Also, $\lim\limits_{t\to+\infty} \sigma(t) = 0$ implies

$$\lim_{t\to+\infty} |\, f(\sigma(t)) + \sigma(t)\,| = 0. \tag{4.49}$$

With (4.47), (4.48), and (4.49), the proof of the absolute stability of the zero solution of (4.44) is similar to that of (4.42). The proof is completed. □

Theorem 4.48. *The zero solution of (4.37) is absolutely stable if and only if*

(1) The condition (1) of Theorem 4.47 holds;

(2) There exists radially unbounded positive definite Lyapunov function $V(x,\xi) \in C[R^{n+1}, R^1]$ w.r.t. $\bar{\Omega}$ such that $D^+V(x,\xi)|_{(4.37)}$ is negative definite w.r.t. $\bar{\Omega}$.

Proof. *Sufficiency*. From condition (2), we know that there exist $\varphi(|c^{\mathrm{T}}x - r\xi|) \in KR$ and $\psi(|c^{\mathrm{T}}x - r\xi|) \in K$ such that

$$\varphi\Big(|c^{\mathrm{T}}x - r\xi|\Big) \leq V(x,\xi) \tag{4.50}$$

and

$$D^{+}V(x,\xi)\big|_{(4.37)} \leq -\psi\Big(|c^{\mathrm{T}}x - r\xi|\Big). \tag{4.51}$$

Thus the zero solution of (4.37) is absolutely stable w.r.t. $\bar{\Omega}$.

Necessity. (4.37) is an autonomous system. The globally and uniformly asymptotic stability is equivalent to the globally and asymptotic stability. The Lyapunov theorem for globally and uniformly asymptotic stability is invertible, that is, the necessary and sufficient condition for the zero solution of a system to be globally and uniformly asymptotically stable is that there exists a radially unbounded, positive definite Lyapunov function, whose derivative evaluated on the solution of the system is negative definite. Thus, $\forall f(\sigma) \in F_{[0,k]}$, there exists a radially unbounded, positive definite Lyapunov function $V(x,\xi) \in C[R^{n+1}, R^1]$, $\varphi(\sum_{i=1}^{n}|x_i| + |\xi|) \in KR$, and $\psi(\sum_{i=1}^{n}|x_i| + |\xi|) \in K$ such that

$$\varphi\Big(\sum_{i=1}^{n}|x_i| + |\xi|\Big) \leq V(x,\xi) \tag{4.52}$$

and

$$D^{+}V(x,\xi) \leq -\psi\Big(\sum_{i=1}^{n}|x_i| + |\xi|\Big). \tag{4.53}$$

Let $\mu = \max[\max\limits_{1\leq i\leq n}|c_i|, |r_i|]$, we have

$$\begin{aligned}
\tilde{\varphi}\Big(|c^{\mathrm{T}}x - r\xi|\Big) &:= \varphi\left(\frac{1}{\mu}|c^{\mathrm{T}}x - r\xi|\right) \leq \varphi\left(\frac{\sum_{i=1}^{n}|c_i x_i| + |r\xi|}{\mu}\right) \\
&\leq \varphi\Big(\sum_{i=1}^{n}|x_i| + |\xi|\Big) \leq V(x,\xi)
\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}
D^{+}V(x,\xi)|_{(4.37)} &\leq -\psi\Big(\sum_{i=1}^{n}|x_i| + |\xi|\Big) \leq -\psi\left(\frac{\sum_{i=1}^{n}|c_i x_i| + |r\xi|}{\mu}\right) \\
&\leq -\psi\left(\frac{\|c^{\mathrm{T}}x - r\xi\|}{\mu}\right) := -\tilde{\psi}\left(\|c^{\mathrm{T}}x - r\xi\|\right),
\end{aligned}$$

where $\tilde{\varphi} \in KR$ and $\tilde{\psi} \in K$. Thus, the condition (2) holds. The proof is complete. □

In the following, we further discuss how to change the absolute stability of the zero solution of (4.37) to the Hurwitz stability of a matrix, and the absolute stability with respect to $\bar{\Omega}$ or $\sigma = 0$.

For $N = 0$, system (4.37) is the Lurie indirect control system, which has been discussed earlier in detail. Thus, we only consider the case $N \neq 0$.

Assume $f \in C^1$, and $\omega(\sigma) = 1 + N\frac{\mathrm{d}f}{\mathrm{d}\sigma} > 0$. From $\sigma = c^{\mathrm{T}}x - r\xi - Nf(\sigma)$, we have $\xi = r^{-1}(c^{\mathrm{T}}x - \sigma - Nf(\sigma))$. Thus, system (4.37) can be written as

$$\begin{aligned} \dot{x} &= \tilde{A}x + \tilde{b}\sigma + \tilde{d}f(\sigma), \\ \omega(\sigma)\dot{\sigma} &= \tilde{c}^{\mathrm{T}}x - \tilde{r}\sigma - \tilde{N}f(\sigma), \end{aligned} \tag{4.55}$$

where

$$\begin{aligned} &\tilde{A} = A + br^{-1}c^{\mathrm{T}}, \ \tilde{b} = -r^{-1}b, \ \tilde{d} = d - r^{-1}bN, \\ &\tilde{c}^{\mathrm{T}} = c\tilde{A}, \qquad \tilde{r} = -c^{\mathrm{T}}\tilde{b}, \quad \tilde{N} = -c^{\mathrm{T}}\tilde{d} + r. \end{aligned}$$

In (4.55), σ is a state variable and ξ is replaced by σ, but the derivative of $f(\sigma)$ appears on the left-hand side of (4.55).

System (4.37) can be further rewritten as

$$\begin{aligned} \dot{x} &= Ax + b\xi + df(\sigma), \\ \dot{\xi} &= \frac{c^{\mathrm{T}}}{N}x - \frac{r\xi}{N} - \frac{\sigma}{N}. \end{aligned} \tag{4.56}$$

Therefore, we use (4.55) to obtain the absolute stability of the zero solution of (4.37) with respect to σ, and use the fact that the coefficient matrix of the linear part of (4.56) $W := \begin{bmatrix} A & b \\ \frac{c^{\mathrm{T}}}{N} & -\frac{r}{N} \end{bmatrix}$ is a Hurwitz matrix to show that the zero solution of (4.37) is absolutely stable.

Theorem 4.49. *Suppose that*

(1) $\omega(\sigma) > 0$ *and* $\int_0^{\pm\infty}\omega(\sigma)\,\mathrm{d}\sigma = \pm\infty$*;*

(2) $W := \begin{bmatrix} A & b \\ \frac{c^{\mathrm{T}}}{N} & -\frac{r}{N} \end{bmatrix}$ *is a Hurwitz matrix;*

(3) There exist constants $\eta_i \geq 0,\ (i = 1,\ldots,n)$, $\eta_{n+1} > 0$ *such that*

$$-\eta_j\tilde{a}_{jj} \geq \sum_{i=1, i\neq j}^{n} \eta_i|\tilde{a}_{ij}| + \eta_{n+1}|\tilde{c}_j| \quad (j = 1,\ldots,n),$$

and

$$\begin{aligned} &\eta_{n+1}\tilde{r} \geq \sum_{i=1}^{n} |\tilde{b}_i|\eta_i, \\ &\eta_{n+1}\tilde{N} \geq \sum_{i=1}^{n} \eta_i|\tilde{d}_i|, \end{aligned}$$

where at least one of the last two inequalities strictly holds. Then the zero solution of (4.37) is absolutely stable.

Proof. Construct the radially unbounded, positive definite Lyapunov function w.r.t. σ:

$$V(x,\sigma)=\sum_{i=1}^{n}\eta_i|x_i|+\eta_{n+1}\int_0^{\sigma}(\operatorname{sign}\sigma)\omega(\sigma)\,\mathrm{d}\sigma\geq\eta_{n+1}\int_0^{\sigma}(\operatorname{sign}\sigma)\omega(\sigma)\,\mathrm{d}\sigma\in KR.$$

Then we obtain

$$\begin{aligned}
D^+V(x,\sigma)|_{(4.55)} &\leq \sum_{j=1}^{n}\Big[\eta_j\tilde{a}_{jj}+\sum_{i=1,i\neq j}^{n}\eta_i|\tilde{a}_{ij}|+\eta_{n+1}|\tilde{c}_j|\Big]|x_j| \\
&\quad+\Big[-\eta_{n+1}\tilde{r}+\sum_{i=1}^{n}|\tilde{b}_i|\eta_i\Big]|\sigma(t)| \\
&\quad+\Big[-\eta_{n+1}\tilde{N}+\sum_{i=1}^{n}\eta_i|\tilde{d}_i|\Big]|f(\sigma(t))| \\
&\leq\begin{cases}\big[-\eta_{n+1}\tilde{N}+\sum_{i=1}^{n}\eta_i|\tilde{d}_i|\big]|f(\sigma(t))|<0, & \sigma\neq 0,\\ \quad\text{or} & \\ \big[-\eta_{n+1}\tilde{r}+\sum_{i=1}^{n}|\tilde{b}_i|\eta_i\big]|\sigma(t)|, & \sigma\neq 0.\end{cases}
\end{aligned}\tag{4.57}$$

Thus, the zero solution of (4.55) is absolutely stable w.r.t. $\sigma=0$.

The general solution of (4.56) can be expressed as

$$\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix}=\mathrm{e}^{W(t-t_0)}\begin{pmatrix}x(t_0)\\ \xi(t_0)\end{pmatrix}+\int_{t_0}^{t}\mathrm{e}^{W(t-\tau)}\begin{pmatrix}d\,f(\sigma(\tau))\\ -\frac{\sigma(\tau)}{N}\end{pmatrix}\mathrm{d}\tau,$$

and the remaining proof is similar to that of Theorem 4.47. □

Theorem 4.50. *Suppose that the following conditions are satisfied:*

(1) $\omega(\sigma)>0$ *and* $\int_0^{\pm\infty}\omega(\sigma)\sigma\,\mathrm{d}\sigma=+\infty$*;*
(2) $\tilde{a}_{jj}<0,\ \tilde{r}>0,\ \tilde{N}>0\ (j=1,\ldots,n)$*;*

(3) (a) $$G_1:=\begin{bmatrix}|\tilde{a}_{11}| & -|\tilde{a}_{12}| & \cdots & -|\tilde{a}_{1n}| & -|\tilde{b}_1|\\ -|\tilde{a}_{21}| & |\tilde{a}_{22}| & \cdots & -|\tilde{a}_{2n}| & -|\tilde{b}_2|\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -|\tilde{a}_{n1}| & -|\tilde{a}_{n2}| & \cdots & |\tilde{a}_{nn}| & -|\tilde{b}_n|\\ -|\tilde{c}_1| & -|\tilde{c}_2| & \cdots & -|\tilde{c}_n| & \tilde{r}\end{bmatrix}$$

is an M-matrix and $|\tilde{b}_i|\geq|\tilde{d}_i|\ (i=1,\ldots,n),\ \tilde{r}\leq\tilde{N}$*, or*

(b) $$G_2:=\begin{bmatrix}|\tilde{a}_{11}| & -|\tilde{a}_{12}| & \cdots & -|\tilde{a}_{1n}| & -|\tilde{d}_1|\\ -|\tilde{a}_{21}| & |\tilde{a}_{22}| & \cdots & -|\tilde{a}_{2n}| & -|\tilde{d}_2|\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -|\tilde{a}_{n1}| & -|\tilde{a}_{n2}| & \cdots & |\tilde{a}_{nn}| & -|\tilde{d}_n|\\ -|\tilde{c}_1| & -|\tilde{c}_2| & \cdots & -|\tilde{c}_n| & \tilde{N}\end{bmatrix}$$

is an M-matrix and $\tilde{N}\leq\tilde{r}$.

Then the zero solution of (4.55) is absolutely stable.

Proof. If conditions (1), (2), and (3)(a) are all satisfied, there exist constants $\eta_i>0,\ (i=1,\ldots,n-1)$ such that

$$\eta_j \tilde{a}_{jj} + \sum_{i=1, i \neq j}^{n} \eta_i |\tilde{a}_{ij}| + \eta_{n+1} |\tilde{c}_j| < 0,$$

$$-\eta_{n+1} \tilde{r} + \sum_{i=1, i \neq j}^{n} \eta_i |\tilde{b}_i| < 0.$$

Construct the radially unbounded, positive definite Lyapunov function

$$V(x, \sigma) = \sum_{i=1}^{n} \eta_i |x_i| + \eta_{n+1} \int_0^{\sigma} \omega(\sigma) \operatorname{sign} \sigma \, \mathrm{d}\sigma. \tag{4.58}$$

Using $|\tilde{b}_i| \geq |\tilde{d}_i|, \ (i = 1, \ldots, n)$, $\tilde{r} \leq \tilde{N}$, and

$$-\eta_{n+1} \tilde{r} + \sum_{i=1, i \neq j}^{n} \eta_i |\tilde{b}_i| < 0,$$

we have

$$-\eta_{n+1} \tilde{N} + \sum_{i=1}^{n} \eta_i |\tilde{d}_i| < 0.$$

Thus,

$$\begin{aligned} D^+ V(x, \sigma)|_{(4.55)} \leq & \sum_{j=1}^{n} \left[\eta_j \tilde{a}_{jj} + \sum_{i=1, i \neq j}^{n} \eta_i |\tilde{a}_{ij}| + \eta_{n+1} |\tilde{c}_j| \right] |x_j| \\ & + \left[-\eta_{n+1} \tilde{r} + \sum_{i=1}^{n} \eta_i |\tilde{b}_i| \right] |\sigma| \\ & + \left[-\eta_{n+1} \tilde{N} + \sum_{i=1}^{n} \eta_i |\tilde{d}_i| \right] |f(\sigma)| \\ & < 0, \quad \text{when} \quad |x| + |\sigma| \neq 0. \end{aligned}$$

Thus, the zero solution of (4.55) is absolutely stable. Similarly, we can prove that the conditions (1), (2), and (3)(b) imply that the zero solution of (4.55) is absolutely stable. The theorem is proved. $\square$

Theorem 4.51. *Suppose that*

$$p_i = \left(\tilde{b}_i + \tilde{d}_i \frac{f(\sigma)}{\sigma} \right), \ (i = 1, \ldots, n), \quad p_{n+1} = \left(-\tilde{r} - \tilde{N} \frac{f(\sigma)}{\sigma} \right),$$

and the following conditions are satisfied

(1) $\omega(\sigma) > 0$, $\int_0^{\pm\infty} \omega(\sigma) \mathrm{d}\sigma = \pm\infty$;

(2) There exist constants $\eta_i > 0$, $(i = 1, \ldots, n+1)$ *such that the matrix* $H = (h_{ij})_{(n+1) \times (n+1)}$ *is negative definite, where*

$$h_{ij} = h_{ji} = \begin{cases} \eta_j \tilde{a}_{ij} + \eta_j \tilde{a}_{ii} & 1 \leq i, j \leq n, \\ \eta_i p_i + \eta_{n+1} c_i & 1 \leq i \leq n, \ j = n+1, \\ 2\eta_{n+1} p_{n+1} & i = j = n+1. \end{cases}$$

Then the zero solution of (4.55) is absolutely stable.

Proof. Construct the radially unbounded, positive definite Lyapunov function

$$V(x,\sigma)=\sum_{i=1}^{n}\eta_i x_i^2+2\eta_{n+1}\int_0^{\sigma}\omega(\sigma)\sigma\,\mathrm{d}\sigma. \tag{4.59}$$

Then,

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(4.55)}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\\\sigma\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}h_{11}&\cdots&h_{1,n+1}\\\vdots&\ddots&\vdots\\h_{n+1,1}&\cdots&h_{n+1,n+1}\end{bmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\\\sigma\end{pmatrix}<0\quad\text{when}\quad |x|+|\sigma|\neq 0.$$

Thus, the zero solution of (4.55) is absolutely stable. □

Theorem 4.52. *Suppose that the following conditions are satisfied:*

(1) $\omega(\sigma)>0$, $\int_0^{\pm\infty}\omega(\sigma)\sigma\,\mathrm{d}\sigma=+\infty$, *and* $\int_0^{\pm\infty}\omega(\sigma)f(\sigma)\,\mathrm{d}\sigma=+\infty$;
(2) There exist a symmetric positive definite matrix $P=(p_{ij})_{n\times n}$ *and* $\alpha>0,\beta>0$ *such that*

$$G=\begin{bmatrix}P\tilde{A}+\tilde{A}^{\mathrm{T}}P & P\tilde{b}+\alpha\tilde{c} & p\tilde{d}+\beta\tilde{c}\\(P\tilde{b}+\alpha\tilde{c})^{\mathrm{T}} & -2\alpha\tilde{r} & 0\\(p\tilde{d}+\beta\tilde{c})^{\mathrm{T}} & 0 & -2\beta\tilde{N}\end{bmatrix}$$

is negative definite.

Then the zero solution of (4.55) is absolutely stable.

Proof. Construct the radially unbounded, positive definite Lyapunov function

$$V(x,\sigma)=x^{\mathrm{T}}Px+2\alpha\int_0^{\sigma}\omega(\sigma)\sigma\,\mathrm{d}\sigma+2\beta\int_0^{\sigma}\omega(\sigma)f(\sigma)\,\mathrm{d}\sigma. \tag{4.60}$$

Then we obtain

$$\begin{aligned}D^{+}V(x,\sigma)|_{(4.55)}&=\dot{x}^{\mathrm{T}}Px+x^{\mathrm{T}}P\dot{x}+2\alpha\omega(\sigma)\sigma\dot{\sigma}+2\beta\omega(\sigma)f(\sigma)\dot{\sigma},\\&=x^{\mathrm{T}}(P\tilde{A}+\tilde{A}^{\mathrm{T}}P)x+2x^{\mathrm{T}}P\tilde{b}\sigma+2x^{\mathrm{T}}P\tilde{d}f(\sigma)\\&\quad+2\alpha\tilde{c}^{\mathrm{T}}x\sigma-2\alpha\tilde{r}\sigma^2-2\alpha\tilde{N}\sigma f(\sigma)\\&\quad+2\beta\tilde{c}^{\mathrm{T}}xf(\sigma)-2\beta\tilde{r}\sigma f(\sigma)-2\beta\tilde{N}f^2(\sigma),\\&=\begin{pmatrix}x\\\sigma\\f(\sigma)\end{pmatrix}^{T}\begin{bmatrix}P\tilde{A}+\tilde{A}^{\mathrm{T}}P & P\tilde{b}+\alpha\tilde{c} & p\tilde{d}+\beta\tilde{c}\\(P\tilde{b}+\alpha\tilde{c})^{\mathrm{T}} & -2\alpha\tilde{r} & 0\\(p\tilde{d}+\beta\tilde{c})^{\mathrm{T}} & 0 & -2\beta\tilde{N}\end{bmatrix}\begin{pmatrix}x\\\sigma\\f(\sigma)\end{pmatrix}\\&\quad-(2\alpha\tilde{N}+2\beta\tilde{r})\sigma f(\sigma)\\&<0,\quad\text{when}\quad |x|+|\sigma|\neq 0,\end{aligned} \tag{4.61}$$

indicating that the conclusion of the theorem is true. □

Theorem 4.53. *Suppose that*

(1) The condition (1) of Theorem 4.52 is satisfied;
(2) G is negative definite in the condition (2) of Theorem 4.52,
(3) $\tilde{A}$ is a Hurwitz matrix.

Then the zero solution of (4.55) is absolutely stable.

Proof. Use the same Lyapunov function as that in Theorem 4.53. Applying (4.61) and condition (2) yields

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.55)} \leq -(2\alpha\tilde{N}+2\beta\tilde{r})\sigma f(\sigma) < 0 \quad \text{when} \quad \sigma \neq 0.$$

Thus, the zero solution of (4.55) is absolutely stable w.r.t. σ.

Then, using

$$x(t) = \mathrm{e}^{\tilde{A}(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{\tilde{A}(t-\tau)}(\tilde{b}\sigma(\tau)+\tilde{d}f(\sigma(\tau))\,\mathrm{d}\tau,$$

we can prove that the zero solution of (4.55) is absolutely stable w.r.t. $x(t)$. The proof is complete. □

Example 4.54. Consider a system in the form of (4.45):

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \omega(\sigma)\dot{\sigma} \end{pmatrix} &= \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{b}_1 & \tilde{d}_1 \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & \tilde{b}_2 & \tilde{d}_2 \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & \tilde{b}_3 & \tilde{d}_3 \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 & -\tilde{r} & -\tilde{N} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ f(\sigma) \end{pmatrix}, \\
&= \begin{bmatrix} -4 & 1 & 1 & 2 & -\frac{3}{2} \\ -1 & -4 & 1 & 1 & -1 \\ 1 & 2 & -3 & -1 & 1 \\ 1 & 2 & 1 & -4 & -4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ f(\sigma) \end{pmatrix}.
\end{aligned} \tag{4.62}$$

Assume that $\omega(\sigma) > 0$, $f(\sigma) \in F_{[0,k]}$, and $\int_0^{\pm\infty} \omega(\sigma)\sigma\,\mathrm{d}\sigma = +\infty$. It is obvious that $|\tilde{b}_i| \geq |d_i|$, $(i = 1,2,3)$, and $r = N = 4$.

It is easy to show that

$$G_1 = \begin{bmatrix} |\tilde{a}_{11}| & -|\tilde{a}_{12}| & -|\tilde{a}_{13}| & -|\tilde{b}_1| \\ -|\tilde{a}_{21}| & |\tilde{a}_{22}| & -|\tilde{a}_{23}| & -|\tilde{b}_2| \\ -|\tilde{a}_{31}| & -|\tilde{a}_{32}| & |\tilde{a}_{33}| & -|\tilde{b}_3| \\ -|\tilde{c}_1| & -|\tilde{c}_2| & -|\tilde{c}_3| & |\tilde{r}| \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 & -2 \\ -1 & 4 & -1 & -1 \\ -1 & -2 & 3 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}$$

is an M-matrix. Thus, the conditions of Theorem 4.50 are all satisfied. The zero solution of (4.62) is absolutely stable.

5

Special Lurie-Type Control Systems

In this chapter, we present some necessary and sufficient algebraic conditions for the absolute stability of several special classes of Lurie-type control systems. Moreover, the algebraic sufficient conditions for absolute stability of other systems are obtained. All conditions are convenient in applications, in particular, for designing absolute stable control systems, or for stabilizing nonabsolute stable control systems.

Part of this chapter is based on the results of Ye [163], Xie [158], and Zhang [177] (Sect. 5.1); Liao [72, 78] (Sects. 5.3–5.5); Letov [63] (Sects. 5.2 and 5.4); and Shu et al. [136] (Sect. 5.6).

5.1 Three Special Order Control Systems

In this section, we present results for second-order direct control system, a class of third-order control system, and special nth-order direct control systems.

5.1.1 The Second-Order Direct Control Systems

We consider the second-order Lurie-type direct control system:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b_1 f(c_1x_1 + c_2x_2), \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_2 f(c_1x_1 + c_2x_2), \end{aligned} \tag{5.1}$$

where $f(\sigma) \in F_\infty$, $\sigma = c^{\mathrm{T}}x = c_1x_1 + c_2x_2$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \ c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{5.2}$$

Ye [163] obtained the necessary and sufficient conditions for the absolute stability of the zero solution of system (5.1).

Theorem 5.1. *If the matrix A is a Hurwitz matrix, then the necessary and sufficient conditions for the absolute stability of the zero solution of (5.1) are*

$$c^{\mathrm{T}}b \le 0 \quad and \quad c^{\mathrm{T}}A^{-1}b \ge 0.$$

To prove the theorem, a theorem by Krasovaskii [53] must be used and the following lemma is needed.

Lemma 5.2. *For the second order nonlinear system,*

$$\begin{aligned} \dot{x} &= f_1(x) + by, \\ \dot{y} &= f_2(x) + dy, \end{aligned} \qquad f_1(0) = f_2(0) = 0, \tag{5.3}$$

if the following conditions are satisfied:

(1) $[bf_2(x) - df_1(x)]x < 0$ *for* $x \neq 0$*;*

(2) $\frac{f_1(x)}{x} + d < 0$ *for* $x \neq 0$*;*

(3) $\lim_{|x|\to+\infty}\left\{(f_1(x)+dx)\mathrm{sgn}x - \int_0^x [df_1(x) - bf_2(x)]\,\mathrm{d}x\right\} = -\infty.$

Then the zero solution of system (5.3) is globally stable.

Now we turn to prove Theorem 5.1.

Proof. *Necessity*. This is a special case of Theorem 3.2 proved in Chap. 3.

Sufficiency. Without loss of generality, let $c_1 \neq 0$, and we introduce the transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{c_1} & -\frac{c_2}{c_1} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{5.4}$$

which reduces the system (5.1) to

$$\begin{cases} \dot{y}_1 = f_1(y_1) + by_2, \\ \dot{y}_2 = f_2(y_1) + dy_2, \end{cases} \tag{5.5}$$

where

$$\begin{aligned} f_1(y_1) &= \left(a_{11} + \frac{c_2}{c_1}a_{21}\right)y_1 + (c_1b_1 + c_2b_2)f(y_1), \\ f_2(y_1) &= \frac{a_{21}}{c_1}y_1 + b_2f(y_1), \\ b &= -c_2a_{11} - \frac{c_2^2}{c_1}a_{21} + c_1a_{12} + c_2a_{22}, \\ d &= -\frac{c_2}{c_1}a_{21} + a_{22}, \\ & y_1f(y_1) > 0 \quad \text{if} \quad y_1 \neq 0, \quad f(0) = 0. \end{aligned}$$

Now we prove that the conditions of Theorem 5.1 imply the condition of Lemma 5.2. We observe that

$$\begin{aligned} \Big[bf_2(y_1) - df_1(y_1)\Big]y_1 &= -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} y_1^2 - \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} y_1f(y_1) \\ &< 0 \quad \text{if} \quad y_1 \neq 0, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \frac{f_1(y_1)}{y_1} + d &= (a_{11} + a_{22}) + (c_1b_1 + c_2b_2)\frac{f(y_1)}{y_1} \\ &< 0 \quad \text{if} \quad y_1 \neq 0. \end{aligned} \tag{5.7}$$

Obviously, (5.6) yields

$$-\int_0^{y_1}\Big[df_1(y_1)-bf_2(y_1)\Big]\,\mathrm{d}y_1 \to -\infty \quad \text{as} \quad |y_1|\to+\infty,$$

and (5.7) gives

$$\lim_{|y_1|\to+\infty}(f_1(y_1)+\mathrm{d}y_1)\,\mathrm{sgn}\,y_1=-\infty.$$

So, the condition (3) of Lemma 5.2 is satisfied.

As a result, the zero solution of the system (5.1) is absolutely stable. □

The conclusion can be proved in the case of $c_2\neq 0$ as well.

Theorem 5.3. *If the two eigenvalues of the matrix* $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ *are pure imaginary numbers, then the zero solution of system (5.1) is absolutely stable if and only if*

$$c^{\mathrm{T}}b<0 \quad \textit{and} \quad c^{\mathrm{T}}A^{-1}b\geq 0.$$

Theorem 5.4. *If* $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ *has two real eigenvalues, namely, one equals zero and the other is negative, then the zero solution of the system (5.1) is absolutely stable if and only if*

$$c^{\mathrm{T}}b\leq 0 \quad \textit{and} \quad c^{\mathrm{T}}(adj\,A)b\geq 0,$$

where adj A is the adjoint matrix of A, that is,

$$adj\,A=\begin{bmatrix}a_{22} & -a_{21}\\ -a_{12} & a_{11}\end{bmatrix}.$$

We can prove Theorems 5.3 and 5.4 similar to that for Theorem 5.1, and therefore, omit the proofs.

Example 5.5. Discuss the absolute stability of the zero solution of the system:

$$\begin{aligned}\dot{x}_1 &= -2x_1+x_2-2f(x_1-x_2),\\ \dot{x}_2 &= -x_1-x_2+f(x_1-x_2),\end{aligned} \tag{5.8}$$

where

$$A=\begin{bmatrix}-2 & 1\\ -1 & -1\end{bmatrix},\quad b=\begin{pmatrix}-2\\ 1\end{pmatrix},\quad c=\begin{pmatrix}1\\ -1\end{pmatrix}.$$

Since

$$c^{\mathrm{T}}b=(1\ -1)\begin{pmatrix}-2\\ 1\end{pmatrix}=-3<0,$$

$$c^{\mathrm{T}}A^{-1}b=\begin{pmatrix}1\\ -1\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}-1 & 1\\ -1 & -2\end{bmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix}=3>0.$$

A has two eigenvalues with negative real parts, and thus, the zero solution of (5.8) is absolutely stable on the basis of Theorem 5.1.

5.1.2 A Class of the Third-Order Control Systems

Consider the third-order control system:

$$\begin{aligned} \dot{x} &= Ax + bf(\sigma), \\ \sigma &= c^{\mathrm{T}}x, \end{aligned} \tag{5.9}$$

where $x \in R^3$, $A \in R^{3\times 3}$, $c \in R^3$, $f(0) = 0$, and

$$0 < f(\sigma)\sigma < k\sigma^2 \ (k \leq +\infty) \quad \text{for all} \quad \sigma \neq 0.$$

Xie [159] has proved that for a class of the third-order control systems including the third-order indirect control systems, the conditions of Popov's frequency criterion are not only sufficient but also necessary for absolute stability. Then the Lurie problem of this class of control systems has been solved completely.

Theorem 5.6. *In system (5.9), suppose that A has at least one eigenvalue with zero real part and no eigenvalues with positive real parts. If there exist two constants* $p \geq 0$ *and q, both of which are not zero, such that*

$$\mathrm{Re}\{(p + i\omega q)c^{\mathrm{T}}(A - i\omega T)^{-1}b\} + \frac{p}{k} \geq 0 \quad \textit{for} \quad \omega \in (-\infty, +\infty), \tag{5.10}$$

then (5.10) is the necessary and sufficient condition for the absolute stability of the zero solution to the system (5.9) in the Hurwitz angle $[0, k]$.

Popov's criterion implies sufficiency, while the proof of necessity is too lengthy to be quoted in detail; the reader is referred to Xie [159].

5.1.3 Special nth-Order Direct Control Systems

Consider a special Lurie direct control system [177]:

$$\begin{aligned} \dot{x} &= Ax + bf(\sigma), \\ \sigma &= c^{\mathrm{T}}x, \end{aligned} \tag{5.11}$$

where $A \in R^{n\times n}$, $c, b \in R^n$ and $f(\sigma) \in F_{[0,k]}$.

Theorem 5.7. *Let the matrix A of the system (5.11) be of the form*

$$A = \begin{bmatrix} -\lambda & 1 & \cdots & 0 \\ & -\lambda & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & & \cdots & -\lambda \end{bmatrix}, \quad \lambda > 0.$$

Then the necessary and sufficient conditions for the absolute stability of the zero solution of system (5.11) are

$$c^{\mathrm{T}}b \leq 0 \quad \textit{and} \quad c^{\mathrm{T}}A^{-1}b \geq 0.$$

Proof. *Necessity*. It has been proved in Chap. 3 (see Theorem 3.2).

Sufficiency. By virtue of Popov's criterion, if there exists a real number $q \geq 0$ such that

$$\mathrm{Re}\{(1+i\omega q)W(i\omega)\} \geq 0 \quad \text{for all} \quad \omega \geq 0, \tag{5.12}$$

where $W(z) = -c^{\mathrm{T}}(zI-A)^{-1}b$, then the zero solution of system (5.11) is absolutely stable.

The condition (5.12) can be equivalently rewritten as

$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\} \leq 0 \quad \text{for} \quad \omega \geq 0,$$

where $A_{i\omega} = i\omega I - A$. In this case, we have

$$A_{i\omega} = \begin{bmatrix} i\omega+\lambda & -1 & 0\cdots & 0 \\ 0 & i\omega+\lambda & \cdots & \vdots \\ \vdots & & \ddots & -1 \\ 0 & & \cdots & i\omega+\lambda \end{bmatrix},$$

$$A_{i\omega}^{-1} = \begin{bmatrix} \dfrac{1}{i\omega+\lambda} & \dfrac{1}{(i\omega+\lambda)^2} & \cdots & 0 \\ 0 & \dfrac{1}{i\omega+\lambda} & \cdots & \vdots \\ \vdots & & \ddots & \dfrac{1}{(i\omega+\lambda)^2} \\ 0 & & \cdots & \dfrac{1}{i\omega+\lambda} \end{bmatrix},$$

thus

$$\begin{aligned} c^{\mathrm{T}}A_{i\omega}^{-1}b &= (c_1,c_2,\ldots,c_n)A_{i\omega}^{-1}(b_1,b_2,\ldots,b_n)^{\mathrm{T}} \\ &= \sum_{j=1}^{n}\frac{b_jc_j}{i\omega+\lambda}+\frac{c_1b_2}{(i\omega+\lambda)^2}, \\ &= \frac{c^{\mathrm{T}}b(\lambda-i\omega)}{\lambda^2+\omega^2}+\frac{c_1b_2(\lambda^2-\omega^2)-c_1b_2\cdot 2\omega_i\lambda}{(\lambda^2-\omega^2)^2+4\omega^2\lambda^2}. \end{aligned}$$

We conclude

$$\begin{aligned} &\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\} \\ &= \frac{c^{\mathrm{T}}b\lambda}{\lambda^2+\omega^2}+\frac{c_1b_2(\lambda^2-\omega^2)}{(\lambda^2-\omega^2)^2+4\omega^2\lambda^2}+\frac{q\omega^2(c^{\mathrm{T}}b)}{\lambda^2+\omega^2} \\ &\quad +\frac{2q\omega^2c_1b_2\lambda}{(\lambda^2-\omega^2)^2+4\omega^2\lambda^2} \\ &:= \frac{F(\omega^2)}{(\lambda^2+\omega^2)[(\lambda^2-\omega^2)^2+4\lambda^2\omega^2]}, \end{aligned}$$

where

$$\begin{aligned}F(\omega^2) &= \left[(\lambda^2-\omega^2)+4\omega^2\lambda^2\right]\left[(c^{\mathrm{T}}b)\lambda+q\omega^2(c^{\mathrm{T}}b)\right]\\&\quad+\left[\lambda^2+\omega^2\right]\left[c_1b_2(\lambda^2-\omega^2)+2q\omega^2c_1b_2\lambda\right],\\&=(c^{\mathrm{T}}b)\left[\lambda^4+2\omega^2\lambda^2+\omega^4\right]\left[\lambda+q\omega^2\right]\\&\quad+c_1b_2(\lambda^4-\omega^4)+2q\lambda^3\omega^2c_1b_2+2q\omega^4c_1b_2\lambda,\\&=q(c^{\mathrm{T}}b)\omega^6+\left[(c^{\mathrm{T}}b)\lambda+2(c^{\mathrm{T}}b)\lambda^2q-c_1b_2+2qc_1b_2\lambda\right]\omega^4\\&\quad+\left[(c^{\mathrm{T}}b)\lambda^4q+2(c^{\mathrm{T}}b)\lambda^3+2q\lambda^3c_1b_2\right]\omega^2\\&\quad+\left[(c^{\mathrm{T}}b)\lambda^5+c_1b_2\lambda^4\right].\end{aligned}$$

The conditions

$$c^{\mathrm{T}}b\le 0 \quad \text{and} \quad -c^{\mathrm{T}}A^{-1}b=\frac{1}{\lambda^2}\left[(c^{\mathrm{T}}b)\lambda+c_1b_2\right]\le 0$$

indicate that the coefficient of the ω^6 term and the constant term of $F(\omega^2)$ are not positive. Now we discuss the coefficients of ω^4 and ω^2 terms.

1. If $c_1b_2\le 0$, then, obviously, for any $q\ge 0$

$$(c^{\mathrm{T}}b)\lambda^4q+2(c^{\mathrm{T}}b)\lambda^3+2q\lambda^3c_1b_2\le 0.$$

   Choosing $q>\frac{1}{2\lambda}$, it follows that

$$\begin{aligned}&(c^{\mathrm{T}}b)\lambda+2(c^{\mathrm{T}}b)\lambda^2q-c_1b_2+2qc_1c_2\\&=(c^{\mathrm{T}}b)\lambda+2(c^{\mathrm{T}}b)\lambda^2q+c_1b_2(2q\lambda-1)\le 0.\end{aligned}$$

   Therefore, the coefficients of ω^4 and ω^2 terms are not positive.
2. If $c_1b_2>0$, we choose $q=0$. Then the coefficient of ω^4 term is $(c^{\mathrm{T}}b)\lambda-c_1b_2\le 0$ and the coefficient of ω^2 term is $2(c^{\mathrm{T}}b)\lambda^3\le 0$.

In any case, we can choose $q\ge 0$ such that $F(\omega^2)\le 0$; therefore, the zero solution of system (5.11) is absolutely stable by Popov's criterion (5.12). □

Corollary 5.8. *If there exists a real similarity transformation that transforms the matrix A of the system (5.11) into the form presented in Theorem 5.7, then the necessary and sufficient conditions for the absolute stability of the zero solution of system (5.11) are* $c^{\mathrm{T}}b\le 0$ *and* $c^{\mathrm{T}}A^{-1}b\ge 0$.

Proof. It is suffices to prove that $c^{\mathrm{T}}b$ and $c^{\mathrm{T}}A^{-1}b$ are not changed by similarity transformation.

By the nonsingular transformation $x=By, B\in R^{n\times n}$, system (5.11) is transformed into

$$\dot{y}=B^{-1}ABy+B^{-1}bf(c^{\mathrm{T}}By):=\tilde{A}y+\tilde{b}f(\tilde{c}^{\mathrm{T}}y),$$

where $\tilde{A}=B^{-1}AB$, $\tilde{b}=B^{-1}b$, $\tilde{c}=B^{\mathrm{T}}c$. Thus,

$$\tilde{c}^{\mathrm{T}}\tilde{b}=c^{\mathrm{T}}BB^{-1}b=c^{\mathrm{T}}b,$$

$$\tilde{c}^{\mathrm{T}}\tilde{A}^{-1}\tilde{b}=c^{\mathrm{T}}BB^{-1}A^{-1}BB^{-1}b=c^{\mathrm{T}}A^{-1}b.$$

The corollary follows. □

Theorem 5.9. *In system (5.11), we assume that*

$$A=\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}=\begin{bmatrix}-\lambda I_1 & 0\\ 0 & -\rho I_2\end{bmatrix},$$

where $\lambda>0,\rho>0,I_1\in R^{n_1\times n_1}$, $I_2\in R^{n_2\times n_2}$ are identity matrices and $n_1+n_2=n$. Then the necessary and sufficient conditions for the absolute stability of the zero solution of system (5.11) amount to $c^{\mathrm{T}}b\le 0$, $c^{\mathrm{T}}A^{-1}b\ge 0$.

Proof. The conditions are obviously necessary.

Now we prove that they are also sufficient. From

$$A_{i\omega}=\begin{bmatrix}(i\omega+\lambda)I_1 & 0\\ 0 & (i\omega+\rho)I_2\end{bmatrix},\quad A_{i\omega}^{-1}=\begin{bmatrix}\dfrac{1}{i\omega+\lambda}I_1 & 0\\ 0 & \dfrac{1}{i\omega+\rho}I_2\end{bmatrix},$$

taking

$$\tilde{c_1}=\begin{bmatrix}I_1 & 0\\ 0 & 0\end{bmatrix}_{n\times n}c,\quad \tilde{c_2}=\begin{bmatrix}0 & 0\\ 0 & I_2\end{bmatrix}_{n\times n}c,$$

$$\tilde{b_1}=\begin{bmatrix}I_1 & 0\\ 0 & 0\end{bmatrix}b,\quad \tilde{b_2}=\begin{bmatrix}0 & 0\\ 0 & I_2\end{bmatrix}b,$$

we observe

$$\begin{aligned}c^{\mathrm{T}}A_{i\omega}^{-1}b&=\frac{\tilde{c}_1^{\mathrm{T}}\tilde{b}_1}{i\omega+\lambda}+\frac{\tilde{c}_2^{\mathrm{T}}\tilde{b}_2}{i\omega+\rho},\\ &=\frac{\tilde{c}_1^{\mathrm{T}}\tilde{b}_1\lambda-i\tilde{c}_1^{\mathrm{T}}\tilde{b}_1\omega}{\omega^2+\lambda^2}+\frac{\tilde{c}_2^{\mathrm{T}}\tilde{b}_2\rho-i\tilde{c}_2^{\mathrm{T}}\tilde{b}_2\omega}{\omega^2+\rho^2}.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\}&=\frac{\tilde{c}_1^{\mathrm{T}}\tilde{b}_1\lambda+q\omega^2\tilde{c}_1^{\mathrm{T}}\tilde{b}_1}{\omega^2+\lambda^2}+\frac{\tilde{c}_2^{\mathrm{T}}\tilde{b}_2\rho+q\omega^2\tilde{c}_2^{\mathrm{T}}\tilde{b}_2}{\omega^2+\rho^2},\\ &=\frac{F(\omega^2)}{(\omega^2+\lambda^2)(\omega^2+\rho^2)},\end{aligned}$$

where

$$F(\omega^2) = q(\tilde{c}_1^{\mathrm{T}}\tilde{b}_1 + \tilde{c}_2^{\mathrm{T}}\tilde{b}_2)\omega^4 + [\tilde{c}_1^{\mathrm{T}}\tilde{b}_1\lambda + \tilde{c}_2^{\mathrm{T}}\tilde{b}_2\rho \\ + q(\tilde{c}_1^{\mathrm{T}}\tilde{b}_1\rho^2 + \tilde{c}_2^{\mathrm{T}}\tilde{b}_2\lambda^2)]\omega^2 + \Big(\frac{\tilde{c}_1^{\mathrm{T}}\tilde{b}_1}{\lambda} + \frac{\tilde{c}_2^{\mathrm{T}}\tilde{b}_2}{\rho}\Big)\lambda^2\rho^2.$$

The conditions of Theorem 5.9 give

$$c^{\mathrm{T}}b = \tilde{c}_1^{\mathrm{T}}\tilde{b}_1 + \tilde{c}_2^{\mathrm{T}}\tilde{b}_2 \le 0,$$

$$-c^{\mathrm{T}}A^{-1}b = \frac{\tilde{c}_1^{\mathrm{T}}\tilde{b}_1}{\lambda} + \frac{\tilde{c}_2^{\mathrm{T}}\tilde{b}_2}{\rho} \le 0.$$

It is easy to prove that there exists $q \ge 0$ such that

$$\tilde{c}_1^{\mathrm{T}}\tilde{b}_1\lambda + \tilde{c}_2^{\mathrm{T}}\tilde{b}_2\rho + q(\tilde{c}_1^{\mathrm{T}}\tilde{b}_1\rho^2 + \tilde{c}_2^{\mathrm{T}}\tilde{b}_2\lambda^2) \le 0,$$

that is, there exists $q \ge 0$ satisfying $F(\omega^2) \le 0$. Then, the zero solution of system (5.11) is absolutely stable. □

Corollary 5.10. *Assume that in system (5.11) $A = A^{\mathrm{T}}$ and that A has at most two different real eigenvalues and simple elementary divisor. Then the necessary and sufficient condition for the absolute stability of the zero solution of system (5.11) are $c^{\mathrm{T}}b \le 0$ and $c^{\mathrm{T}}A^{-1}b \ge 0$.*

Corollary 5.11. *Suppose that in system (5.11) $A = A^{\mathrm{T}}$, while A has only one real eigenvalue and corresponds to the simple elementary divisor. Then the necessary and sufficient condition for the absolute stability of the zero solution of system (5.11) is $c^{\mathrm{T}}b \le 0$.*

Theorem 5.12. *Suppose that in system (5.11) the vector b is an eigenvector of A, or the vector c is an eigenvector of A^{T}. Then the zero solution of the system (5.11) is absolutely stable if and only if $c^{\mathrm{T}}b \le 0$.*

Proof. The necessity is trivial. We prove the sufficiency only. First, suppose $Ab = -\lambda b$ $(\lambda > 0)$. Then,

$$(i\omega I - A)b = [\lambda + i\omega]b, \quad \omega \ge 0, \quad (i\omega I - A)^{-1} = \frac{1}{\lambda + i\omega}b.$$

Thus,

$$c^{\mathrm{T}}(i\omega I - A)^{-1}b = \frac{c^{\mathrm{T}}b}{i\omega + \lambda}.$$

It follows from the condition $c^{\mathrm{T}}b \le 0$ that

$$\mathrm{Re}\{c^{\mathrm{T}}(i\omega I - A)^{-1}b\} = \frac{\lambda c^{\mathrm{T}}b}{\lambda^2 + \omega^2} \le 0.$$

This shows that the zero solution of system (5.11) is absolutely stable.

Next, suppose that $A^{\mathrm{T}}c = -\lambda c\,(\lambda > 0)$. Then we obtain

$$(i\omega I - A^{\mathrm{T}})^{-1}c = \frac{1}{i\omega + \lambda}c.$$

Thus,

$$c^{\mathrm{T}}(i\omega I - A)^{-1}b = \frac{c^{\mathrm{T}}b}{i\omega + \lambda}.$$

We also have

$$\mathrm{Re}\{c^{\mathrm{T}}(i\omega I - A)^{-1}b\} \le 0.$$

Consequently, the zero solution of system (5.11) is absolutely stable. □

Theorem 5.13. *Suppose that in system (5.11) A is a quasi-diagonal matrix:*

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_m),$$

where A_r $(r = 1, \ldots, m)$ are square submatrices. If $\tilde{b}_r$ is an eigenvector of A_r for $r = 1, \ldots, m$, or $\tilde{c}_r$ is an eigenvector of A_r^{T}, then $\tilde{c}_r^{\mathrm{T}}\tilde{b}_r \le 0\,(r = 1, \ldots, m)$ imply the absolute stability of the zero solution of system (5.11), where $\tilde{c}_r, \tilde{b}_r$ refer to column vectors corresponding to A_r, $b = (\tilde{b}_1, \ldots, \tilde{b}_m)^{\mathrm{T}}$, $c = (\tilde{c}_1, \ldots, \tilde{c}_m)^{\mathrm{T}}$, and I_r is the identity matrix corresponding to A_r.

Proof. If $A_r\tilde{b}_r = -\lambda\tilde{b}_r$, since

$$\tilde{c}_r^{\mathrm{T}}(i\omega I_r - A_r)^{-1}\tilde{b}_r = \frac{\tilde{c}_r^{\mathrm{T}}\tilde{b}_r}{\lambda + i\omega},$$

we get

$$\mathrm{Re}\{\tilde{c}_r^{\mathrm{T}}(i\omega I_r - A_r)^{-1}\tilde{b}_r\} = \frac{\lambda\tilde{c}_r^{\mathrm{T}}\tilde{b}_r}{\lambda^2 + \omega^2} \le 0, \quad r = 1, \ldots, m.$$

When $A_r^{\mathrm{T}}\tilde{c}_r = -\lambda\tilde{c}_r$, it gives rise to

$$\tilde{c}_r^{\mathrm{T}}(i\omega I_r - A_r)^{-1}\tilde{b}_r = \frac{\lambda\tilde{c}_r^{\mathrm{T}}\tilde{b}_r}{i\omega + \lambda},$$

$$\mathrm{Re}\{\tilde{c}_r^{\mathrm{T}}(i\omega I_r - A_r)^{-1}\tilde{b}_r\} \le 0, \quad r = 1, \ldots, m.$$

Now we proceed to prove that the expressions $\mathrm{Re}\{\tilde{c}_r^{\mathrm{T}}(i\omega I_r - A_r)^{-1}\tilde{b}_r\} \le 0$ $(r = 1, \ldots, m)$ imply the absolute stability of the zero solution of system (5.11). Since

$$\begin{aligned}
A &= \mathrm{diag}(A_1, A_2, \ldots, A_m),\\
A_{1\omega} &= \mathrm{diag}(i\omega I_1 - A_1, \ldots, i\omega I_m - A_m),\\
c^{\mathrm{T}}A_{1\omega}^{-1}b &= (\tilde{c}_1, \ldots, \tilde{c}_m)\,\mathrm{diag}((i\omega I_1 - A_1)^{-1}, \ldots, (i\omega I_m - A_m)^{-1})\,(\tilde{b}_1^{\mathrm{T}}, \ldots, \tilde{b}_m^{\mathrm{T}})^{\mathrm{T}}\\
&= \sum_{r=1}^{m} \tilde{c}_r^{\mathrm{T}}(i\omega I_r - A_r)^{-1}\tilde{b}_r,
\end{aligned}$$

it follows that

$$\mathrm{Re}(c^{\mathrm{T}}A_{1\omega}^{-1}b) = \sum_{r=1}^{m} \mathrm{Re}\,\tilde{c}_r^{\mathrm{T}}(i\omega I_r - A_r)^{-1}\tilde{b}_r \le 0.$$

Therefore, the zero solution of system (5.11) is absolutely stable. □

Example 5.14. Consider the three-dimensional control system:

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 - x_3 + f(x_1 - x_2 - x_3),\\
\dot{x}_2 &= \ x_1 - x_2 - x_3 + f(x_1 - x_2 - x_3),\\
\dot{x}_3 &= x_1 + x_2 - 3x_3 + f(x_1 - x_2 - x_3),
\end{aligned}
$$

where

$$
A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad f \in F_{[0,k]}.
$$

It is clear that A is a stable matrix and b is an eigenvector of A associated with the eigenvalue $\lambda = -1$. Moreover

$$
c^{\mathrm{T}} b = -1 < 0.
$$

Thus, the system is absolutely stable by Theorem 5.12.

5.2 The First Canonical Form of Control Systems

Letov [63] has considered a class of control systems called the first canonical form in which A is diagonal:

$$
\begin{aligned}
\dot{x}_j &= -\lambda_j x_j + f(\sigma), \quad \lambda_j > 0, \quad j = 1, \ldots, n,\\
\sigma &= c^{\mathrm{T}} x = \sum_{i=1}^{n} c_i x_i.
\end{aligned}
\tag{5.13}
$$

We discuss the absolute stability of system (5.13).

For a real number $q \geq 0$, let

$$
a_j = \begin{cases} c_j q & \text{if } c_j(q\lambda_j - 1) > 0, \\ c_j/\lambda_j & \text{if } c_j(q\lambda_j - 1) < 0, \\ c_j/\lambda_j & \text{if } c_j(q\lambda_j - 1) = 0, c_j \neq 0, \\ 0 & \text{if } c_j(q\lambda_j - 1) = 0, c_j = 0. \end{cases}
$$

Theorem 5.15. [177] *If there exists $q \geq 0$ such that $\sum_{j=1}^{n} a_j < \frac{1}{k}$, then the zero solution of system (5.13) is absolutely stable in the Hurwitz angle $[0,k]$.*

Proof. Since

$$
A = \begin{bmatrix} -\lambda_1 & \cdots & & 0 \\ \vdots & -\lambda_2 & & \\ & & \ddots & \vdots \\ 0 & & \cdots & -\lambda_n \end{bmatrix}, \quad A_{i\omega}^{-1} = \begin{bmatrix} \dfrac{1}{i\omega + \lambda_1} & 0 & \cdots & 0 \\ 0 & \dfrac{1}{i\omega + \lambda_2} & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \dfrac{1}{i\omega + \lambda_n} \end{bmatrix},
$$

it follows that

$$c^{\mathrm{T}}A_{i\omega}^{-1}b = \sum_{j=1}^{n}\frac{c_j}{i\omega+\lambda_j},$$
$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\} = \sum_{j=1}^{n}\frac{c_j(\lambda_j+q\omega^2)}{\omega^2+\lambda_j^2}.$$

Let $f_j(\omega)=\frac{c_j(\lambda_j+q\omega^2)}{\omega^2+\lambda_j^2}$. We compute the derivative of $f_j(\omega)$:

$$f_j'(\omega)=\frac{2c_jq\omega(\omega^2+\lambda_j^2)-(c_j\lambda_j+c_jq\omega^2)2\omega}{(\omega^2+\lambda_j^2)^2}=\frac{2c_j\lambda_j(q\lambda_j-1)\omega}{(\omega^2+\lambda_j^2)^2}.$$

We see that $f_j'(\omega)=0$ only if $\omega=0$. The function $f_j(\omega)$ is monotone increasing on $[0,\infty)$ when $c_j(q\lambda_j-1)>0$, thus $f_j(\omega)\le f_j(\infty)=c_jq$; $f_j(\omega)$ is monotone decreasing on $[0,\infty)$ when $c_j(q\lambda_j-1)<0$, thus $f_j(\omega)\le f_j(0)=c_j/\lambda_j$. Since $f_j'(\omega)=0$, $q=1/\lambda_j$, when $c_j(q\lambda_j-1)=0$, $c_j\neq 0$, we have

$$f_j(\omega)=\frac{c_j(\lambda_j+\omega^2/\lambda_j)}{\omega^2+\lambda_j^2}=\frac{c_j}{\lambda_j}.$$

When $c_j(q\lambda_j-1)=0$, $c_j=0$, it obviously follows that $f_j(\omega)=0$. Choosing appropriate a_j, we always have $f_j(\omega)\le a_j$. From the hypothesis of the theorem, we can deduce that

$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\}=\sum_{j=1}^{n}\frac{c_j(\lambda_j+q\omega^2)}{\omega^2+\lambda_j^2}=\sum_{j=1}^{n}f_j(\omega)\le\sum_{j=1}^{n}a_j<\frac{1}{k},$$

that is,

$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\}-\frac{1}{k}<0.$$

Thus, the zero solution of system (5.13) is absolutely stable in the Hurwitz angle $[0,k]$ by Popov's criterion. □

We can always rearrange the equations in (5.13) and the unknown functions x_j such that

$$c_j\begin{cases}<0, & j=1,\dots,j_1,\\ =0, & j=j_1+1,\dots,j_2,\\ >0, & j=j_2+1,\dots,n.\end{cases}$$

Corollary 5.16. *If $\sum_{j=j_2+1}^{n}\frac{c_j}{\lambda_j}<\frac{1}{k}$, then the zero solution of system (5.13) is absolutely stable in the Hurwitz angle $[0,k]$.*

The conclusion may be easily derived by choosing $q=0$ in Theorem 4.4.1.

Corollary 5.17. *If $\lambda_j > 2k(n-j_2)c_j\,(j = j_2+1,\dots,n)$, then the zero solution of system (5.13) is absolutely stable in the Hurwitz angle* $[0,k]$.

Proof. From $\lambda_j > 2k(n-j_2)c_j\,(j = j_2+1,\dots,n)$,

$$\sum_{j=j_2+1}^{n} \frac{c_j}{\lambda_j} < \sum_{j=j_2+1}^{n} \frac{1}{2k(n-j_2)} = \frac{1}{2k} < \frac{1}{k}.$$

The condition of Corollary 5.16 is satisfied, and so Corollary 5.17 is proved. □

Theorem 5.18. *If there exists $q \geq 0$ such that $\sum_{j=1}^{n} a_j \leq 0$, then the zero solution of system (5.13) is absolutely stable.*

Proof. Repeating the proof of Theorem 5.15, we have $f_j(\omega) \leq a_j$. From

$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\} = \sum_{j=1}^{n} f_j(\omega) \leq \sum_{j=1}^{n} a_j \leq 0,$$

the conclusion is valid by Popov's criterion. □

5.3 Critical Systems

Theorem 5.19. [72] *In system (5.13), assume that $\lambda_i > 0$ $(i = 1,\dots,n-1)$, $\lambda_n = 0$, $c_n < 0$, $c_i < \frac{\lambda_i}{2k(n-1)}$ $(i = 1,\dots,n-1)$. Then the zero solution of system (5.13) is absolutely stable in* $[0,k]$.

Proof. It is clear that the quadratic inequalities for ξ_i

$$k(n-1)\xi_i^2 + 2\Big[k(n-1)c_i - \lambda_i\Big]\xi_i + k(n-1)c_i^2 < 0,\ i = 1,\dots,n-1 \tag{5.14}$$

have positive solutions if and only if

$$c_i < \frac{\lambda_i}{2k(n-1)},\ i = 1,\dots,n-1.$$

Suppose that $\xi_i = r_i$ $(i = 1,\dots,n-1)$ is a positive solution of (5.14). Let

$$\varphi_i(x_i) = \begin{cases} 2r_i x_i, & i = 1,\dots,n-1, \\ -2c_i x_i, & i = n. \end{cases}$$

Choose the radially unbounded, positive definite Lyapunov function

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} \varphi_i(x_i)\,\mathrm{d}x_i.$$

Then,

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.13)} &:= G(x) = \sum_{i=1}^{n} \varphi_i(x_i)\frac{\mathrm{d}x_i}{\mathrm{d}t},\\
&= -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + 2\sum_{i=1}^{n-1} r_i x_i f(\sigma) - 2c_n x_n f(\sigma),\\
&= -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + 2\sum_{i=1}^{n-1} (r_i+c_i) x_i f(\sigma) - 2\sum_{i=1}^{n} c_i x_i f(\sigma),\\
&\le -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + 2\left|\sqrt{k}\sum_{i=1}^{n-1}(r_i+c_i)x_i\right|\left|\frac{f(\sigma)}{\sqrt{k}}\right| - 2\sigma f(\sigma),\\
&\le -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + k(n-1)\sum_{i=1}^{n-1}(r_i+c_i)^2 x_i^2 + \sigma f(\sigma) - 2\sigma f(\sigma),\\
&\le -\sum_{i=1}^{n-1}[k(n-1)(r_i+c_i)^2 - 2r_i\lambda_i]x_i^2 - \sigma f(\sigma),\\
&= -\sum_{i=1}^{n-1}\{k(n-1)r_i^2+2\,[k(n-1)c_i-\lambda_i]\,r_i+k(n-1)c_i^2\}x_i^2-\sigma f(\sigma),\\
&:= W(x).
\end{aligned}$$

Now we have

$$G(x) \le W(x) \le 0.$$

If $W(\tilde{x}) = 0$, since $\sigma f(\sigma) > 0$ for $\sigma \neq 0$, we have $\sum_{i=1}^{n-1}\tilde{x}_i^2 = 0$, and therefore, $c_n\tilde{x}_n f(c_n\tilde{x}_n) = 0$. This implies $c_n\tilde{x}_n = 0$. From $c_n < 0$, we have $\tilde{x}_n = 0$; thus $\tilde{x} = 0$. It follows that $W(x)$ is negative definite. Therefore, the zero solution of system (5.13) is absolutely stable in the Hurwitz angle $[0,k]$. □

Corollary 5.20. *If* $\lambda_i > 0\,(i = 1,\dots,n-1)$, $\lambda_n = 0$, $c_n < 0$, $c_i \le 0$, $(i = 1,\dots,n-1)$. *Then the zero solution of system (5.13) is absolutely stable in the Hurwitz angle* $[0,k]$.

Corollary 5.21. *Provided that* $\lambda_i > 0\,(i = 1,\dots,n-1)$, $\lambda_n = 0$, $c_i < 0\,(i = 1,\dots,n)$, *the zero solution of system (5.13) is absolutely stable.*

Proof. Let $\varphi_i(x_i) = -c_i x_i\,(i = 1,\dots,n)$. We still use the same Lyapunov function $V(x) = \sum_{i=1}^{n}\int_0^{x_i}\varphi_i(x_i)\,\mathrm{d}x_i$. Then

$$\begin{aligned}
G(x) := \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.13)} &= \sum_{i=1}^{n}\varphi_i(x_i)\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{i=1}^{n-1} c_i\lambda_i x_i^2 - \sum_{i=1}^{n} c_i x_i f(\sigma)\\
&= \sum_{i=1}^{n-1} c_i\lambda_i x_i^2 - \sigma f(\sigma).
\end{aligned}$$

Thus, $G(x)$ is negative definite and the zero solution of system (5.13) is absolutely stable. □

Theorem 5.22. *If the following conditions are satisfied:*

(1) $\lambda_i > 0\,(i = 1,\ldots,r)$, $\lambda_j = 0\,(j = r+1,\ldots,n)$, $c_i \le \frac{\lambda_i}{2kr}\,(i = 1,\ldots,r)$, $c_j < 0\,(j = r+1,\ldots,n)$*;*

(2) the matrix $\mathrm{diag}(-\lambda_1,\ldots,-\lambda_n) + bc^{\mathrm{T}}$ *is Hurwitz stable with* $b = (1,\ldots,1)^{\mathrm{T}}$, *then the zero solution of system (5.13) is absolutely stable.*

Proof. The condition $c_i \le \frac{\lambda_i}{2kr}$ implies that the equations for η_i

$$kr\eta_i^2 + 2(krc_i - \lambda_i)\eta_i + krc_i^2 = 0, \quad i = 1,\ldots,r$$

have positive solutions $\eta_i = \xi_i > 0$, $i = 1,\ldots,r$. Let

$$\varphi_i(x_i) = \begin{cases} 2\,\xi_i x_i, & i = 1,\ldots,r, \\ -2\,c_j x_j, & j = r+1,\ldots,n. \end{cases}$$

We again employ the same Lyapunov function $V(x) = \sum_{i=1}^n \int_0^{x_i} \varphi_i(x_i)\,\mathrm{d}x_i$ to obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.13)} &:= G(x) = \sum_{i=1}^n \varphi_i(x_i)\,\frac{\mathrm{d}x_i}{\mathrm{d}t}, \\
&= -2\sum_{i=1}^r \xi_i\lambda_i x_i^2 + 2\sum_{i=1}^r \xi_i x_i f(\sigma) - 2\sum_{j=r+1}^n c_j x_j f(\sigma), \\
&= -2\sum_{i=1}^r \xi_i\lambda_i x_i^2 + 2\sum_{i=1}^r (\xi_i + c_i)x_i f(\sigma) - 2\sum_{i=1}^n c_i x_i f(\sigma), \\
&\le -2\sum_{i=1}^r \xi_i\lambda_i x_i^2 + 2\sqrt{k}\sum_{i=1}^r (\xi_i + c_i)\,|x_i|\,\frac{f(\sigma)}{|k|} - 2\sigma f(\sigma), \\
&\le -2\sum_{i=1}^r \xi_i\lambda_i x_i^2 + kr\sum_{i=1}^r (\xi_i + c_i)^2 x_i^2 + \sigma f(\sigma) - 2\sigma f(\sigma), \\
&\le -\sum_{i=1}^r [kr(\xi_i + c_i)^2 - 2\xi_i\lambda_i]x_i^2 - \sigma f(\sigma), \\
&\le -\sigma f(\sigma).
\end{aligned}$$

Hence, $G(x)$ is negative definite w.r.t. σ. The conclusion follows. □

Corollary 5.23. *If the condition (2) in Theorem 5.22 is satisfied, and*

$$\begin{aligned} &\lambda_i > 0\ (i = 1,\ldots,r), && c_i \le 0\ (i = 1,\ldots,r), \\ &\lambda_j = 0\ (j = r+1,\ldots,n), && c_j < 0\ (j = r+1,\ldots,n), \end{aligned}$$

then the zero solution of system (5.13) is absolutely stable in the Hurwitz angle $[0,k]$.

Corollary 5.24. *If the condition (1) in Theorem 5.22 is satisfied, and*

$$\begin{aligned} &\lambda_i > 0\ (i = 1,\ldots,r), \quad \lambda_j = 0\ (j = r+1,\ldots,n), \\ &c_i < 0\ (i = 1,\ldots,n), \end{aligned}$$

then the zero solution of the system (5.13) is absolutely stable.

5.4 The Second Canonical Form of Control Systems

Consider the second canonical form of the control system [63]:

$$\begin{aligned} \dot{x}_i &= -\rho_i x_i + \sigma, \quad i = 1, \ldots, n, \\ \dot{\sigma} &= \sum_{i=1}^{n} \beta_i x_i - p\sigma - rf(\sigma), \end{aligned} \tag{5.15}$$

where $p > 0$, $r > 0$, $\rho_i > 0$ are constants.

Theorem 5.25. [78] *Suppose*

$$p \geq \sum_{i=1}^{n} \left(\frac{1 + \operatorname{sign} \beta_i}{2} \right) \frac{\beta_i}{\rho_i}.$$

Then the zero solution of system (5.15) is absolutely stable.

Proof. We construct the Lyapunov function:

$$V(x, \sigma) = \sum_{i=1}^{n} c_i x_i^2 + \sigma^2.$$

Obviously, $V(x, \sigma)$ is radially unbounded and positive definite for

$$c_i = \begin{cases} -\beta_i & \text{if } \beta_i < 0, \\ \varepsilon_i \ (0 < \varepsilon_i \ll 1) & \text{if } \beta_i = 0, \\ \beta_i & \text{if } \beta_i > 0. \end{cases}$$

Then,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.15)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1 + \beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2 + \beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_n\rho_n & c_n + \beta_n \\ c_1 + \beta_1 & c_2 + \beta_2 & \cdots & c_n + \beta_n & -2p \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix} - 2r\sigma f(\sigma).$$

Now we prove

$$(-1)^{n+1} D_{n+1} := \frac{(-1)^{n+1}}{2^{n-1}} \begin{vmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1 + \beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2 + \beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_n\rho_n & c_n + \beta_n \\ c_1 + \beta_1 & c_2 + \beta_2 & \cdots & c_n + \beta_n & -2p \end{vmatrix} \geq 0.$$

By induction it can be verified that

$$D_{n+1} = (-1)^{n+1} 4 \prod_{i=1}^{n} c_i \rho_i p + (-1)^n \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \neq j}}^{n} c_i \rho_i (c_j + \beta_j)^2. \tag{5.16}$$

For $n+1 = 2$,

$$D_2 = \begin{vmatrix} -2c_1\rho_1 & c_1+\beta_1 \\ c_1+\beta_1 & -2p \end{vmatrix} = 4c_1\rho_1 p(-1)^2 + (-1)(c_1+\beta_1)^2.$$

Assume that when $n+1 = k$

$$\begin{aligned} D_k &= \frac{1}{2^{k-2}} \begin{vmatrix} -2c_2\rho_2 & 0 & \cdots & 0 & c_1+\beta_1 \\ 0 & -2c_3\rho_3 & \cdots & 0 & c_2+\beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_k\rho_k & c_k+\beta_k \\ c_2+\beta_2 & c_3+\beta_3 & \cdots & c_k+\beta_k & -2p \end{vmatrix}, \\ &= (-1)^k 4 \prod_{i=2}^{k} c_i\rho_i p + (-1)^{k-1} \sum_{j=2}^{k} \prod_{\substack{i=2 \\ i\neq j}}^{k} c_i\rho_i (c_j+\beta_j)^2. \end{aligned}$$

Then, for $n+1 = k+1$, we have

$$\begin{aligned} D_{k+1} &= \frac{1}{2^{k-1}} \begin{vmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1+\beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2+\beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_k\rho_k & c_k+\beta_k \\ c_1+\beta_1 & c_2+\beta_2 & \cdots & c_k+\beta_k & -2p \end{vmatrix}, \\ &= \frac{1}{2^{k-1}} \left[-2c_1\rho_1 2^{k-2} D_k + (-1)^{k+2}(c_1+\beta_1)^2 (-1)^{k+1} \prod_{i=2}^{k} (-2c_i\rho_i) \right], \\ &= (-1)^{k+1} 4 \prod_{i=1}^{k} c_i\rho_i p + (-1)^k \sum_{j=2}^{n} \prod_{\substack{i=1 \\ i\neq j}}^{k} c_i\rho_i (c_j+\beta_j)^2 \\ &\quad + (-1)^{k+2}(-1)^{k+1}(-1)^{k-1} \prod_{i=2}^{k} c_i\rho_i (c_1+\beta_1)^2, \\ &= (-1)^{k+1} 4 \prod_{i=1}^{k} c_i\rho_i p + (-1)^k \sum_{j=1}^{k} \prod_{\substack{i=1 \\ i\neq j}}^{k} c_i\rho_i (c_j+\beta_j)^2. \end{aligned}$$

Therefore, for any natural number n, the expression (5.16) holds. Since,

$$p - \sum_{i=1}^{n} \left(\frac{1+\operatorname{sign}\beta_i}{2} \right) \frac{\beta_i}{\rho_i} \geq 0$$

we write

$$-4p - \sum_{i=1}^{n} \frac{(c_i + \beta_i)^2}{c_i \rho_i} = \sum_{i=1}^{n} \left(\frac{4p}{n} - \frac{(c_i + \beta_i)^2}{c_i \rho_i} \right) \geq 0.$$

Consequently,

$$(-1)^{n+1} D_{n+1} \geq 0$$

and

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.15)} \leq -2r\sigma f(\sigma) < 0 \quad \text{when} \quad \sigma \neq 0.$$

Thus, the zero solution of system (5.15) is absolutely stable w.r.t. σ.

On the other hand, take $f(\sigma) = \sigma$ in (5.15). Then system (5.15) is turned to be a linear system

$$\begin{aligned} \dot{x}_i &= -\rho_i x_i + \sigma, \\ \dot{\sigma} &= \sum_{i=1}^{n} \beta_i x_i - (p + r)\sigma. \end{aligned} \tag{5.17}$$

For system (5.17), using the Lyapunov function discussed above, we can prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.17)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1 + \beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2 + \beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_n\rho_n & c_n + \beta_n \\ c_1 + \beta_1 & c_2 + \beta_2 & \cdots & c_n + \beta_n & -2p - 2r \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}$$

is negative definite. Then the zero solution of system (5.17) is globally stable and thus the coefficient matrix is stable. This completes the proof. □

Example 5.26. Consider the equations of longitudinal motion of a plane

$$\begin{aligned} \dot{x}_i &= -\rho_i x_i + \sigma, \quad i = 1, 2, 3, 4, \\ \dot{\sigma} &= \sum_{j=1}^{4} \beta_j x_j - r p_2 \sigma - f(\sigma), \end{aligned} \tag{5.18}$$

where $rp_2 > 0$, $\rho_i > 0$ $(i = 1, 2, 3, 4)$, $f(\sigma) \in F_\infty$.

System (5.18) is clearly a particular case of (5.15) for $n = 4$.

For the above system (5.18), we have the following result.

Corollary 5.27. [78] *If* $rp_2 \geq \sum_{i=1}^{4} \frac{1+\mathrm{sign}\beta_i}{2} \frac{\beta_i}{\rho_i}$, *then the zero solution of system (5.18) is absolutely stable.*

It is of great interest in estimating the parameter values for the stability of aircrafts in the vertical direction. The larger the stable parameter regimes, the better

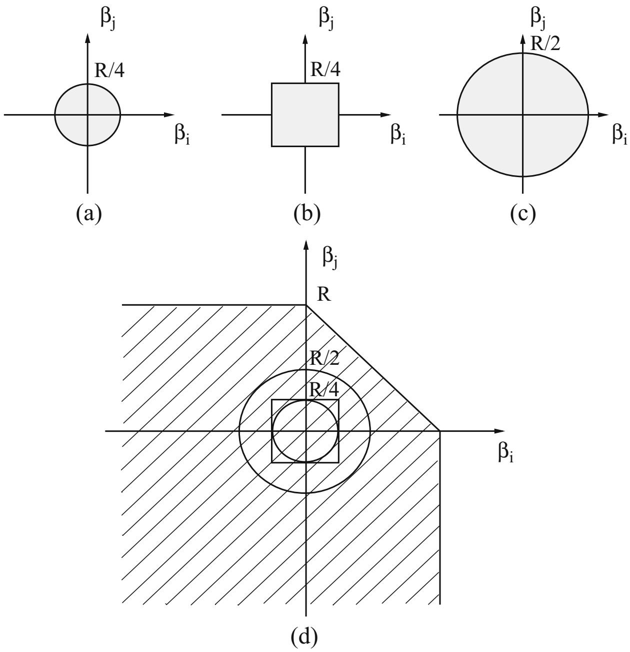

Fig. 5.1 Stable parameter regimes for (**a**) $\sum_{i=1}^{4}\beta_i^2 < \frac{R}{4}$; (**b**) $\max_{1\le i\le 4}|\beta_i| < \frac{R}{4}$; (**c**) $\max_{1\le i\le 4}|\beta_i| < \frac{R}{2}$; and (**d**) $rp_2 \ge \sum_{i=1}^{4}\frac{1+\mathrm{sign}\beta_i}{2}\frac{\beta_i}{\rho_i}$

the designing technical characteristics. The typical stable parameter regimes in the literature are given below:

(a) $\min\limits_{1\le i\le 4}\rho_i^2 r^2 p_2^2 - 16\sum\limits_{i=1}^{4}\beta_i^2 > 0$;

(b) $\min\limits_{1\le i\le 4}\rho_i r p_2 - 4\max\limits_{1\le i\le 4}|\beta_i| > 0$;

(c) $\min\limits_{1\le i\le 4}\rho_i^2 r^2 p_2^2 - 4\sum\limits_{i=1}^{4}\beta_i^2 > 0$.

Let $R = \min_{1\le i\le 4}\rho_i r p_2$. Then the geometrical meaning of the above cases are demonstrated in Fig. 5.1a–c. The result given in Corollary 5.27 substantially increases the stable parameter regimes, as shown in Fig. 5.1d. Note that Fig. 5.1a–c are all compact set, but Fig. 5.1d is unbounded set.

5.5 Two Special Systems

Consider the following special Lurie system [78]:

$$\dot{x} = Ax + hf(x_n), \tag{5.19}$$

where $f \in F_\infty, A \in R^{n\times n}, h = (h_1,\ldots,h_n)^{\mathrm{T}} \in R^n$.

Theorem 5.28. *If* $a_{ii} < 0$ $(i = 1,2,\dots,n)$, $a_{ij} \geq 0$ $(i \neq j,\, i,j = 1,\dots,n)$, $h_n < 0$, $h_i \geq 0$ $(i = 1,2,\dots,n-1)$, *and*

$$(a_{1n}, a_{2n}, \dots, a_{nn}) = \lambda(h_1, \dots, h_n), \quad \lambda > 0,$$

then the zero solution of system (5.19) is absolutely stable if and only if A is a Hurwitz matrix.

Proof. *Necessity*. Let $f(x_n) = x_n$, system (5.19) becomes

$$\dot{x} = Ax + hx_n = \left[A + (O_{n\times(n-1)}, h)\right] x, \tag{5.20}$$

where

$$(O_{n\times(n-1)}, h) = \begin{bmatrix} 0 & \dots & 0 & h_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & h_n \end{bmatrix}_{n\times n}.$$

Since the zero solution of $(S - Z_o)$ is asymptotically stable.

Thus, $[A + (O_{n\times(n-1)}, h)]$ is a Hurwitz matrix, which is equivalent to that $-[A + (O_{n\times(n-1)}, h)]$ is an M matrix because $a_{ii} < 0$ $(i = 1,2,\dots,n)$, $a_{ij} \geq 0$ $(i \neq j,\ i,j = 1,\dots,n)$, $h_n < 0$, $h_i \geq 0$ $(i = 1,2,\dots,n-1)$. Thus, there exist constants $r_i > 0$ $(i = 1,\dots,n)$ such that

$$\begin{aligned} -r_j a_{jj} &> \sum_{i=1, i\neq j}^{n} r_i a_{ij}, \quad j = 1,\dots,n-1, \\ -r_n[a_{nn} + h_n] &> \sum_{i=1}^{n-1} r_i\,(a_{in} + h_i). \end{aligned} \tag{5.21}$$

Rewriting the last inequality in equation (5.21) yields

$$-r_n \left(1 + \frac{1}{\lambda}\right) a_{nn} > \sum_{i=1}^{n-1} r_i \left(1 + \frac{1}{\lambda}\right) a_{in},$$

that is,

$$-r_n a_{nn} > \sum_{i=1}^{n-1} r_i a_{in}. \tag{5.22}$$

It is obvious that $-A$ is an M matrix, that is, A is a Hurwitz matrix, followed from the first $n-1$ inequalities in equation (5.21) and equation (5.22).

Sufficiency. Since the condition implies that there exist constants $r_i > 0$, $i = 1,2,\dots,n$ such that

$$r_j a_{jj} + \sum_{i=1, i\neq j}^{n} r_k |a_{ii}| < 0, \quad i = 1, 2 \dots, n,$$

we construct the radially unbounded, a positive definite Lyapunov function

$$V(x) = \sum_{i=1}^{n} r_i |x_i|.$$

From the conditions

$$\lambda h_i = a_{in}\,(i = 1,2,\dots,n), \quad \text{and} \quad -r_n a_{nn} > \sum_{i=1, i\neq j}^{n-1} r_i |a_{in}|,$$

we have

$$-r_n \frac{a_{nn}}{\lambda} > \sum_{i=1}^{n-1} r_i \frac{|a_{in}|}{\lambda},$$

that is,

$$-r_n h_n > \sum_{i=1}^{n-1} r_i |h_i|.$$

Thus,

$$\begin{aligned} D^+V(x)\big|_{(5.19)} &\le \sum_{j=1}^{n} \left[r_j a_{jj} + \sum_{i=1, i\neq j}^{n} r_i |a_{ij}| \right] |x_j| + \left[r_n h_n + \sum_{i=1}^{n-1} r_i \,|h_i| \,\right] |f(x_n)| \\ &< 0, \quad \text{if} \quad x \neq 0, \end{aligned}$$

which indicates that the zero solution of (5.19) is absolutely stable. □

Equation (5.19) can be transformed into

$$\dot{\xi} = HAH^{-1}\xi + Hhf(x_n) := P\xi + qf(\xi_n), \tag{5.23}$$

with a nonsingular linear transformation

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{bmatrix} 1 & 0 & \cdots & -\frac{h_1}{h} \\ 0 & 1 & \cdots & \vdots \\ \vdots & & \ddots & -\frac{h_{n-1}}{h} \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := Hx.$$

Rewrite equation (5.23) as

$$\begin{aligned} \dot{\xi}_{(n-1)} &= P_{(n-1)}\xi_{(n-1)} + K\xi_n, \\ \dot{\xi}_n &= L\xi_{(n-1)} + p_{nn}\xi_n + h_n f(\xi_n), \end{aligned} \tag{5.24}$$

where

$$\xi_{(n-1)} = (\xi_1, \dots, \xi_{n-1})^{\mathrm{T}}, \quad P_{(n-1)} = \begin{bmatrix} p_{11} & \cdots & p_{1,n-1} \\ \vdots & \ddots & \vdots \\ p_{n-11} & \cdots & p_{n-1,n-1} \end{bmatrix},$$

$$K = (p_{1n}, \dots, p_{n-1,n}), \qquad L = (p_{n,1}, \dots, p_{n,n-1}).$$

Theorem 5.29. *(1) If $L = 0$, the zero solution of system (5.24) is absolutely stable if and only if (i) $p_{nn} < 0$ and (ii) $P_{(n-1)}$ is a Hurwitz matrix.*
(2) If $K = 0$ and $\int_0^{\pm\infty} f(\xi_n)\,\mathrm{d}\xi_n = +\infty$, the zero solution of system (5.24) is absolutely stable if and only if $p_{nn} \le 0$ and $P_{(n-1)}$ is a Hurwitz matrix.
(3) If $K = 0$ and $p_{nn} < 0$, the zero solution of system (5.24) is absolutely stable if and only if $P_{(n-1)}$ is a Hurwitz matrix.

Proof. *Sufficiency.*

(1) If $L = 0$, $p_{nn} < 0$, P_{n-1} is a Hurwitz matrix. Choosing the radially unbounded, positive definite Lyapunov function w.r.t. ξ_n: $V(\xi) = \xi_n^2$, we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.24)} = 2p_{nn}\xi_n^2 + 2h_n\xi_n f(\xi_n) < 0, \quad \text{if} \quad \xi_n \ne 0. \tag{5.25}$$

Thus, the zero solution of system (5.24) is absolutely stable w.r.t. ξ_n.
$\xi_{(n-1)}(t)$ can be expressed as

$$\xi_{(n-1)}(t) = \mathrm{e}^{P_{(n-1)}(t-t_0)}\xi_{(n-1)}(t_0) + \int_{t_0}^{t} \mathrm{e}^{P_{(n-1)}(t-\tau)}K\xi_n(\tau)\,\mathrm{d}\tau. \tag{5.26}$$

It follows from equation (5.26) that the zero solution of system (5.24) is also absolutely stable w.r.t. $\xi_{(n-1)}$.

(2) Suppose $K = 0$ and $\int_0^{\pm\infty} f(\xi_n)\,\mathrm{d}\xi_n = +\infty$, then $p_{nn} \le 0$, $P_{(n-1)}$ is a Hurwitz matrix.

Choose the radially unbounded, positive definite Lyapunov function:

$$V(\xi) = \xi_{(n-1)}^{\mathrm{T}} B_{(n-1)}\xi_{(n-1)} + \varepsilon\int_0^{\xi_n} f(\xi_n)\,\mathrm{d}\xi_n, \quad \varepsilon = \text{const.},$$

where $B_{(n-1)}$ is the solution of the Lyapunov matrix equation

$$B_{(n-1)}P_{(n-1)} + P_{(n-1)}^{\mathrm{T}}B_{(n-1)} = -I_{(n-1)}. \tag{5.27}$$

Then we have

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.24)} &= \dot{\xi}_{(n-1)}^{\mathrm{T}}B_{(n-1)}\xi_{(n-1)} + \xi_{(n-1)}^{\mathrm{T}}B_{(n-1)}\dot{\xi}_{(n-1)} + \varepsilon\dot{\xi}_{(n)}f(\xi_n) \\
&\le \begin{pmatrix} \xi_{(n-1)}^{\mathrm{T}} \\ f(\xi_n) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -E_{(n-1)} & \frac{1}{2}\varepsilon L^{\mathrm{T}} \\ \frac{1}{2}\varepsilon L & \varepsilon h_n \end{bmatrix} \begin{pmatrix} \xi_{(n-1)}^{\mathrm{T}} \\ f(\xi_n) \end{pmatrix} \\
&< 0, \quad \text{if} \quad \xi \ne 0,\ 0 < \varepsilon \ll 1.
\end{aligned}$$

Thus, the zero solution of system (5.24) is absolutely stable.

(3) If $K = 0, p_{nn} < 0$, we construct the radially unbounded, positive definite Lyapunov function:

$$V(\xi) = \xi^{\mathrm{T}}_{(n-1)} B_{(n-1)} \xi_{(n-1)} + \varepsilon \xi_n^2,$$

where $\varepsilon = \text{const.}$, $0 < \varepsilon \ll 1$, and $B_{(n-1)}$ satisfies the Lyapunov matrix equation (5.27). Thus,

$$\left.\frac{\mathrm{d}V(\xi)}{\mathrm{d}t}\right|_{(5.24)} \le \xi^{\mathrm{T}} \begin{bmatrix} -E & \varepsilon K^{\mathrm{T}} \\ \varepsilon K & 2\varepsilon p_{nn} \end{bmatrix} \xi < 0, \quad \text{if} \quad \xi \neq 0 \ \ (0 < \varepsilon \ll 1),$$

from which it is easy to see that the zero solution of system (5.24) is absolutely stable.

Necessity. Let $f(\xi_n) = \varepsilon \xi_n \, (0 < \varepsilon \ll 1)$. Then (5.24) becomes

$$\begin{aligned} \dot{\xi}_{(n-1)} &= P_{(n-1)} \xi_{(n-1)} + K \xi_n, \\ \dot{\xi}_n &= L \xi_{(n-1)} + (p_{nn} + \varepsilon h_n) \xi_n. \end{aligned} \tag{5.28}$$

When $K = 0$ or $L = 0$, the Hurwitz stability of the matrix

$$\begin{bmatrix} P_{(n-1)} & K^{\mathrm{T}} \\ L & p_{nn} + \varepsilon h_n \end{bmatrix}$$

implies that the matrix $P_{(n-1)}$ is Hurwitz stable and $p_{nn} + \varepsilon h_n < 0$. Thus, $p_{nn} \le 0$ due to $0 < \varepsilon \ll 1$ and $h_n < 0$. □

5.6 The Systems with $A^{\mathrm{T}}A = AA^{\mathrm{T}}, A^{\mathrm{T}} = A$, or $A + A^{\mathrm{T}} = -2\rho E$

In this section, we consider the absolute stability for the following nonlinear direct control system [136]:

$$\begin{aligned} \dot{x} &= Ax + bf(\sigma), \\ \sigma &= c^{\mathrm{T}} x, \end{aligned} \tag{5.29}$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

$f(\sigma)$ is an arbitrary continuous function of σ, satisfying the condition $f(0) = 0$, $\sigma f(\sigma) > 0 \ (\sigma \neq 0)$.

In [102], Malkin gave a criterion for the absolute stability of system (5.29), but this criterion can not be realized [179]. Later in 1979, using different methods, Zhang [179] and Zhao [183] established a criterion for the absolute stability of system (5.29), in which A is stable, that is, A has only characteristic roots with negative real parts. However, this criterion is not an explicit criterion depending only on the system parameters A, b, c, and so it is inconvenient to use. Now, we propose some explicit criteria for the absolute stability of several classes of systems (5.29) in which A is stable and $A^{\mathrm{T}}A = AA^{\mathrm{T}}$, especially $A^{\mathrm{T}} = A$ or $A + A^{\mathrm{T}} = -2\rho E$.

Theorem 5.30. *Assume in system (5.29) that $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ and A is stable. If there exist two integers ℓ and m such that*

$$\begin{cases} \Delta_3 = 0, & \Delta_0 \le 0 & \textit{when} \quad \Delta_2 < 0, \\ \Delta_3 = 0, & \Delta_0 \Delta_1 \ge 0 & \textit{when} \quad \Delta_2 = 0, \\ \Delta_2^2 - \Delta_1 \Delta_3 \ge 0, & \Delta_2 + \sqrt{\Delta_2^2 - \Delta_1 \Delta_3} \ge -\Delta_0 \Delta_1 & \textit{when} \quad \Delta_2 > 0, \end{cases}$$

where

$$\begin{aligned} \Delta_0 &= 2(-1)^m c^{\mathrm{T}}(A + A^{\mathrm{T}})^{-1} b / c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell}(A + A^{\mathrm{T}})^{-m+1} c, \\ \Delta_1 &= \frac{1}{4}\left[c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell}(A + A^{\mathrm{T}})^{-m+1} c\right]^2 \\ &\quad - c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell}(A + A^{\mathrm{T}})^{-m} c\, c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell+1}(A + A^{\mathrm{T}})^{-m} c, \\ \Delta_2 &= \frac{1}{2}(-1)^{m+1} c^{\mathrm{T}}(A + A^{\mathrm{T}})^{-1} b\, c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell+1}(A + A^{\mathrm{T}})^{-m+1} c \\ &\quad + (-1)^m c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell}(A + A^{\mathrm{T}})^{-m} c \left[b^{\mathrm{T}}(A + A^{\mathrm{T}})^{-\ell} A^{\mathrm{T}} c - b^{\mathrm{T}} c\right], \\ \Delta_3 &= \left[c^{\mathrm{T}}(A + A^{\mathrm{T}})^{-1} b\right]^2 - c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell}(A + A^{\mathrm{T}})^{-m} c\, b^{\mathrm{T}}(AA^{\mathrm{T}})^{\ell}(A + A^{\mathrm{T}})^{m-2} b, \end{aligned} \tag{5.30}$$

then the zero solution of (5.29) is absolutely stable.

Proof. We give a brief proof. Let $A^o = E$ and $A^{-k} = (A^{-1})^k$ $(k > 0)$. If A is stable and $A^{\mathrm{T}}A = AA^{\mathrm{T}}$, then $A + A^{\mathrm{T}}$ is negative definite and $(-1)^m(A + A^{\mathrm{T}})^m$ is positive definite for any integer m, and $A^{\ell}(A + A^{\mathrm{T}})^m = (A + A^{\mathrm{T}})^m A^{\ell}$ and $(-1)^m(AA^{\mathrm{T}})^{\ell}(A + A^{\mathrm{T}})^m$ are positive definite for any integers ℓ and m. Using these facts, we obtain $\Delta_1 \le 0$ and $\Delta_3 \le 0$ in Theorem 5.30. From the results in [183] we know that when A is stable, the zero solution of (5.29) is absolutely stable if and only if there exist a positive definite matrix B and a nonnegative number β (noting that in [183], $\beta > 0$ is assumed, but the relevant proof in [183] also holds for $\beta = 0$) such that

$$\text{(i)} \quad c^{\mathrm{T}} B^{-1} d \le 0, \tag{5.31}$$

$$\text{(ii)} \quad (c^{\mathrm{T}} B^{-1} d)^2 - (c^{\mathrm{T}} B^{-1} c)(d^{\mathrm{T}} B^{-1} d + 2\beta c^{\mathrm{T}} b) \ge 0, \tag{5.32}$$

where $d = Gb + \beta A^{\mathrm{T}} c$ and G is determined by equation $A^{\mathrm{T}} G + GA = -B$. Now we take $B = (-1)^m (AA^{\mathrm{T}})^{\ell}(A + A^{\mathrm{T}})^m$, then B is positive definite. Next, we take $G = (-1)^{m-1}(AA^{\mathrm{T}})\ell(A + A^{\mathrm{T}})^{m-1}$, then B and G satisfy the equation $A^{\mathrm{T}} G + GA = -B$. Substituting the expressions of B, G, and d in the left-hand sides of the inequalities (5.31) and (5.32) and writing them in the form of a polynomial of β, we have

$$c^{\mathrm{T}} B^{-1} d = \frac{1}{2}(-1)^{m+1} c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell}(A + A^{\mathrm{T}})^{-m+1} c(\Delta_0 - \beta),$$

$$(c^{\mathrm{T}} B^{-1} d)^2 - (c^{\mathrm{T}} B^{-1} c)(d^{\mathrm{T}} B^{-1} d + 2\beta c^{\mathrm{T}} d) = \Delta_1 \beta^2 + 2\Delta_2 \beta + \Delta_3.$$

Since

$$(-1)^{m+1}(AA^{\mathrm{T}})^{-\ell}(A+A^{\mathrm{T}})^{-m+1} = (-1)^{-m+1}(AA^{\mathrm{T}})^{\ell}(A+A^{\mathrm{T}})^{-m+1},$$

which is positive definite, we know that

$$\frac{1}{2}(-1)^{m+1}c^{\mathrm{T}}(AA^{\mathrm{T}})^{-\ell}(A+A^{\mathrm{T}})^{-m+1}c > 0.$$

Hence, the inequalities (5.31) and (5.32) are equivalent to

$$\Delta_0 - \beta \le 0, \quad \Delta_1\beta^2 + 2\Delta_2\beta + \Delta_3 \ge 0. \tag{5.33}$$

Because $\Delta_1 \le 0$ and $\Delta_3 \le 0$, we know that the inequalities given in (5.33) have nonnegative solution of β if and only if the conditions of Theorem 5.30 hold. The proof is complete. □

Theorem 5.31. *Assume in system (5.29) that $A^{\mathrm{T}} = A$, A is stable, $c^{\mathrm{T}}b \le 0$, and $c^{\mathrm{T}}A^{-1}b \ge 0$. If there exists an integer m such that*

$$\begin{cases} \delta_3 = 0 & \text{when} \quad \delta_2 = 0, \\ \delta_2^2 - \delta_1\delta_3 \ge 0 & \text{when} \quad \delta_2 > 0, \end{cases}$$

where

$$\begin{aligned} \delta_1 &= (c^{\mathrm{T}}A^{-m+1}c)^2 - c^{\mathrm{T}}A^{-m}c\,c^{\mathrm{T}}A^{-m+2}c, \\ \delta_2 &= (-1)^{m+1}c^{\mathrm{T}}A^{-1}b\,c^{\mathrm{T}}A^{-m+1}c + (-1)^{m+1}c^{\mathrm{T}}A^{-m}c\,c^{\mathrm{T}}b, \\ \delta_3 &= (c^{\mathrm{T}}A^{-1}b)^2 - c^{\mathrm{T}}A^{-m}c\,b^{\mathrm{T}}A^{m-2}b, \end{aligned} \tag{5.34}$$

then the zero solution of system (5.29) is absolutely stable.

When $A^{\mathrm{T}} + A = -2\rho E$, we still have $A^{\mathrm{T}}A = AA^{\mathrm{T}}$. Let $\ell = m = 0$ and $-2\rho R$ take the place of $A + A^{\mathrm{T}}$ in Theorem 5.30, we obtain the following theorem.

Theorem 5.32. *Assume in (5.29) that $A^{\mathrm{T}} + A = -2\rho R$, A is stable, and $c^{\mathrm{T}}b \le 0$. If*

$$\begin{cases} (c^{\mathrm{T}}b)^2 - c^{\mathrm{T}}c\,b^{\mathrm{T}}b = 0 & \text{when} \;\; b^{\mathrm{T}}A^{\mathrm{T}}c + 3c^{\mathrm{T}}b\rho \ge 0, \\ c^{\mathrm{T}}c(b^{\mathrm{T}}A^{\mathrm{T}}c + 3c^{\mathrm{T}}b\rho)^2 & \\ -(c^{\mathrm{T}}c\rho^2 - c^{\mathrm{T}}A^{\mathrm{T}}Ac)\left[(c^{\mathrm{T}}b)^2 - c^{\mathrm{T}}c\,b^{\mathrm{T}}b\right] \ge 0 & \text{when} \;\; b^{\mathrm{T}}A^{\mathrm{T}}c + 3c^{\mathrm{T}}b\rho < 0, \end{cases}$$

then the zero solution of system (5.29) is absolutely stable.

Theorems 5.31 and 5.32 include the corresponding results given in [148, 150].

Applying Theorem 5.31 to the following second-order first canonical form of control system:

$$\begin{aligned} x_i &= -\rho_i x_i + f(\sigma), \quad \rho_i > 0, \;\; i = 1, 2, \\ \sigma &= r_1x_1 + r_2x_2 \end{aligned} \tag{5.35}$$

yields the necessary and sufficient conditions for the absolute stability of the system as

$$r_1 + r_2 \leq 0 \quad \text{and} \quad \frac{r_1}{\rho_1} + \frac{r_2}{\rho_2} \leq 0.$$

Note that employing the criteria given in [148, 150] to system (5.35) can give only sufficient conditions. This shows that Theorems 5.30 and 5.31 are better than the results in [148, 150].

6

Nonautonomous Systems

In this chapter, we consider nonautonomous systems. Lurie control systems are mainly autonomous systems. Thus, the Lurie method or Popov method was developed for single-variable autonomous systems, which are very difficult to be used to study nonautonomous systems. However, many practical systems contain time-variant parameters though they usually vary slowly. Stability of time-variant systems has been a relatively difficult problem in control systems and dynamical systems, and has less results compared to that of autonomous systems. The results presented in this chapter are mainly taken from [67, 86] for Sects. 6.1–6.4, and from [67, 105] for Sect. 6.5.

6.1 Nonautonomous Systems

Consider the nonlinear nonautonomous control system:

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + bf(\sigma,t), \\
\sigma &= c^{\mathrm{T}}x = \sum_{i=1}^{n} c_i x_i,
\end{aligned}
\tag{6.1}
$$

where $A(t) \in C[\,[0,+\infty), R^{n\times n}]$, $b \in R^n$, $c \in R^n$, $x \in R^n$, and

$$
\begin{aligned}
f \in F_{[0,k]} := \{f : f(t,0) \equiv 0,\, 0 \le f(\sigma,t)/\sigma \le k < +\infty,\\
f \in C[\,[0,+\infty) \times R^n, R^1]\}.
\end{aligned}
$$

If for any $f \in F_{[0,k]}$, the zero solution of (6.1) is globally stable, we say that the zero solution of (6.1) to be *absolutely stable in* $[0,k]$.

Definition 6.1. *We say that the zero solution of (6.1) is absolutely stable w.r.t. the set* $\Omega = \{x : \sigma = c^{\mathrm{T}}x = 0\}$ *in the Hurwitz angle* $[0,k]$*, if for any* $f \in F_{[0,k]}$ *and any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that the solution* $x(t,t_0,x_0)$ *of (6.1) satisfies*

$$
\left| c^{\mathrm{T}}x(t,t_0,x_0) \right| = \left| \sum_{i=1}^{n} c_i x_i(t,t_0,x_0) \right| < \varepsilon \quad \textit{for all} \quad t \ge t_0
$$

if $\|x_0\| < \delta(\varepsilon)$*, and for any* $x_0 \in R^n$*,*

$$
\lim_{t\to+\infty} \sigma(t,t_0,x_0) = \lim_{t\to+\infty} c^{\mathrm{T}}x(t,t_0,x_0) = 0.
$$

Definition 6.2. *A function $V(x) \in C[R^n, R^1]$ is said to be positive definite (negative definite) w.r.t. the set $\Omega := \{x | \sigma = 0\}$ if*

$$V(x) \begin{cases} =0 & \text{for } x \in \Omega, \\ >0 & \text{for } x \overline{\in} \Omega. \end{cases} \qquad \left(V(x) \begin{cases} =0 & \text{for } x \in \Omega, \\ <0 & \text{for } x \overline{\in} \Omega. \end{cases} \right)$$

The function $V(x)$ is said to be radially unbounded positive definite for Ω if $V(x)$ is positive definite for Ω, and $V(x) \to +\infty$ as $|\sigma| \to +\infty$.

Theorem 6.3. *Suppose that the following conditions are satisfied:*

(1) The zero solution of the following linear system

$$\dot{x} = A(t)x \tag{6.2}$$

is uniformly asymptotically stable;

(2) The zero solution of (6.1) is absolutely stable for the set Ω in $[0,k]$.

Then the zero solution of (6.1) is absolutely stable in $[0,k]$.

Proof. According to the formula of variation of constants, the solutions of (6.1) can be expressed as

$$x(t) := x(t,t_0,x_0) = K(t,t_0)x_0 + \int_{t_0}^{t} K(t,\tau) b f(\sigma(\tau),\tau)\,\mathrm{d}\tau, \tag{6.3}$$

where $K(t,t_0)$ is the Cauchy matrix solution of (6.2), that is,

$$\begin{aligned} \frac{\mathrm{d}K(t,t_0)}{\mathrm{d}t} &= A(t)K(t,t_0), \\ K(t_0,t_0) &= I_n. \end{aligned}$$

Since the condition (1) is satisfied if and only if the zero solution of (6.2) is exponentially stable, there exist constants $\alpha > 0$ and $M \geq 1$ such that

$$\|K(t,t_0)\| \leq M\mathrm{e}^{-\alpha(t-t_0)}.$$

Since $\sigma(t,t_0,x_0) \to 0$ as $t \to +\infty$, and $\sigma(t,t_0,x_0)$ continuously depends on x_0, $f(\sigma(t,t_0,x_0),t)$ is a continuous function of x_0, and $|f(\sigma(t,t_0,\sigma_0),t)| \leq k|\sigma(t,t_0,\sigma_0)|$, which implies that

$$f(\sigma(t,t_0,x_0),t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Thus, $\forall \varepsilon > 0$, there exist $\delta_1(\varepsilon) > 0$ and $t_1 > t_0$ such that

$$M\mathrm{e}^{-\alpha(t-t_0)}\|x_0\| < \frac{\varepsilon}{3} \quad \text{for all} \quad t \geq t_0, \tag{6.4}$$

$$\int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)}\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau < \frac{\varepsilon}{3}, \tag{6.5}$$

$$\int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)}\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau < \frac{\varepsilon}{3} \quad \text{for all} \quad t \geq t_1 \tag{6.6}$$

if $\|x_0\| < \delta_1(\varepsilon)$.

Let us take $\delta_2(\varepsilon) = \frac{\varepsilon}{3M}$, $\delta(\varepsilon) = \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$. From (6.3), (6.4), (6.5), and (6.6), we have

$$\begin{aligned}
\|x(t,t_0,x_0)\| &\le \|K(t,t_0)\|\,\|x_0\| + \int_{t_0}^{t_1} \|K(t,\tau)\|\,\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau \\
&\quad + \int_{t_1}^{t} \|K(t,\tau)\|\,\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau, \\
&\le M\mathrm{e}^{-\alpha(t-t_0)}\|x_0\| + \int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)}\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau \\
&\quad + \int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)}\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon \quad \text{for all} \quad t \ge t_0 \ \text{ if } \ \|x_0\| < \delta(\varepsilon).
\end{aligned}$$

Therefore, the zero solution of (6.1) is stable.

For any $x_0 \in R^n$, by using the L'Hospital rule, we deduce

$$\begin{aligned}
0 &\le \lim_{t\to+\infty} \|x(t)\| \le \lim_{t\to+\infty} M\mathrm{e}^{-\alpha(t-t_0)} + \lim_{t\to+\infty} \int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-\tau)}\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau \\
&= 0 + \lim_{t\to+\infty} \frac{1}{\mathrm{e}^{\alpha t}} \int_{t_0}^{t} M\mathrm{e}^{\alpha\tau}\|bf(\sigma(\tau),\tau)\|\,\mathrm{d}\tau = 0.
\end{aligned}$$

Therefore, the zero solution of (6.1) is absolutely stable in $[0,k]$. The proof of the theorem is completed. □

Theorem 6.4. *If there exists a function $V(x) \in [R^n, R^1]$ such that*

$$\begin{aligned}
V(0) = 0, \quad V(x) &\ge \varphi(|\sigma|), \qquad \varphi \in KR, \\
D^+V(x)|_{(6.1)} &\le -\psi(|\sigma|), \qquad \psi \in K,
\end{aligned} \tag{6.7}$$

then the zero solution of (6.1) is absolutely stable for the set Ω in $[0,k]$.

Proof. Since $V(0) = 0$ $(0 \in \Omega)$ and $V(x) \in C[R^n, R^1]$, $\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $V(x_0) < \varphi(\varepsilon)$ for $\|x_0\| < \delta(\varepsilon)$. According to (6.7), it yields

$$\varphi(|\sigma(t,t_0,x_0)|) \le V(x(t,t_0,x_0)) \le V(x_0) \le \varphi(\varepsilon) \quad \text{for all} \quad t \ge t_0,$$

which implies

$$|\sigma(t,t_0,x_0)| < \varepsilon \quad \text{for all} \quad t \ge t_0.$$

Now, we prove that

$$\lim_{t\to+\infty} \sigma(t,t_0,x_0) = 0, \quad \text{for all} \quad x_0 \in R^n.$$

Expression (6.7) gives

$$\inf_{t\geq t_0} V(x(t)) = \lim_{t\to+\infty} V(x(t)) := \alpha \geq 0.$$

We can easily check that the function $V(x(t))$ can reach its inferior limit only in Ω.

For any $x_0 \in R^n$, it follows from (6.7) that

$$|\sigma(t)| \leq |c^{\mathrm{T}} x(t_0)| := h < H < +\infty.$$

Assuming that $\lim_{t\to\infty}\sigma(t) \neq 0$, from the uniform continuity of $\sigma(t)$, there exist constants $\beta > 0$, $\eta > 0$ and a sequence $\{t_j\}$ such that

$$|\sigma(t)| \geq \beta \quad \text{for} \quad t \in [t_j - \eta, t_j + \eta].$$

Let $r = \inf_{\beta\leq|\sigma|\leq h}\varphi(|\sigma|) > 0$. We have

$$\begin{aligned} 0 \leq V(t) &\leq V(t_0) + \int_{t_0}^{t} D^+V(t)\,\mathrm{d}t, \\ &\leq V(t_0) - \int_{t_0}^{t} \psi(|\sigma(\tau)|)\,\mathrm{d}\tau, \\ &\leq V(t_0) - \sum_{j=1}^{n}\int_{t_j-\eta}^{t_j+\eta} \psi(|\sigma(\tau)|)\,\mathrm{d}\tau, \\ &\leq V(t_0) - 2n\eta r \to -\infty \quad \text{as} \quad n \to +\infty, \end{aligned}$$

leading to a contradiction. Thus

$$\lim_{t\to+\infty} \sigma(t, t_0, x_0) = 0.$$

The theorem is proved. □

Theorem 6.5. *(1) Let the condition (1) of Theorem 6.3 be satisfied, and* $|f(\sigma,t)| \geq \varphi(|\sigma|) \in K$*;*

(2) Suppose that there exist a symmetric matrix $B(t)_{n\times n}$ *and constants* $\alpha > 0$, $\beta > 0$, $\varepsilon > 0$ *such that* $x^{\mathrm{T}}B(t)x \geq \beta|\sigma|^2$*, and the matrix*

$$\begin{bmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/(k+\varepsilon) \end{bmatrix},$$

is negative semi-definite, where

$$\begin{aligned} G(t) &= A^{\mathrm{T}}(t)B(t) + B(t)A(t) + \dot{B}(t), \\ g(t) &= B(t)b + \frac{\alpha}{2}c. \end{aligned} \tag{6.8}$$

Then the zero solution of (6.1) is absolutely stable in $[0,k]$.

Proof. We choose the Lyapunov function:

$$V(t,x) = x^{\mathrm{T}} B(t) x.$$

Then $V(t,x)$ is radially unbounded, positive definite for the set Ω. Using the fact that

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.1)} &= x^{\mathrm{T}}(A^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)A(t))x + 2x^{\mathrm{T}}B(t)bf(\sigma,t), \\
&= x^{\mathrm{T}}(A^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)A(t))x + 2x^{\mathrm{T}}B(t)bf(\sigma,t) \\
&\quad + \alpha c^{\mathrm{T}} x f(\sigma,t) - \frac{\alpha}{k+\varepsilon} f^2(\sigma,t) \\
&\quad - \alpha\Big(\sigma - \frac{f(\sigma,t)}{k+\varepsilon}\Big) f(\sigma,t), \\
&= \begin{pmatrix} x \\ f(\sigma,t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\frac{\alpha}{(k+\varepsilon)} \end{bmatrix} \begin{pmatrix} x \\ f(\sigma,t) \end{pmatrix}, \\
&\le -\alpha\left(\sigma - \frac{f(\sigma,t)}{k+\varepsilon}\right) f(\sigma,t), \\
&\le -\frac{\alpha}{k+\varepsilon}\varepsilon\sigma f(\sigma,t), \\
&\le -\frac{\alpha\varepsilon}{k+\varepsilon}|\sigma|\,\varphi(|\sigma|).
\end{aligned}$$

We see that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.1)}$ is negative definite for the set Ω. The conditions of Theorem 6.4 are satisfied, and therefore the zero solution of (6.1) is absolutely stable in $[0,k]$. □

6.2 Systems with Separable Variables

In this section, we study the absolute stability for the set Ω by turning this stability into the absolute stability for one state variable.

Without loss of generality, we assume that $c_n \neq 0$. Let

$$y_i = x_i, \quad i = 1,\dots,n-1,$$
$$y_n = c^{\mathrm{T}} x = \sum_{i=1}^{n} c_i x_i.$$

System (6.1) is then transformed into

$$\dot{y}_i = \sum_{j=1}^{n} \tilde{a}_{ij}(t) y_j + \tilde{b}_i f(y_n,t), \quad i = 1,\dots,n, \tag{6.9}$$

where

$$\begin{aligned}
\tilde{a}_{ij}(t) &= a_{ij}(t) - \frac{a_{ij}c_i}{c_n}, \quad i,j = 1,\dots,n-1, \\
\tilde{a}_{in}(t) &= \frac{a_{ij}(t)}{c_n}, \quad i = 1,\dots,n-1,
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_{nn}(t) &= \frac{1}{c_n}\sum_{i=1}^{n} c_i\, a_{in}(t),\\
\tilde{a}_{nj}(t) &= \sum_{i=1}^{n} c_i\, a_{ij}(t) - \sum_{i=1}^{n} c_i\, a_{in}(t)\,\frac{c_j}{c_n}, \quad j=1,\dots,n-1,\\
\tilde{b}_i &= b_i, \quad i=1,\dots,n-1,\\
\tilde{b}_n &= \sum_{i=1}^{n} c_i b_i.
\end{aligned}$$

Obviously, the stability of the zero solution of (6.1) is equivalent to that of (6.9).

Definition 6.6. *The zero solution of (6.9) is said to be absolutely stable in* $[0,k]$ *w.r.t. partial variables* $y_j,\dots,y_n$ *if for any* $f\in F_{[0,k]}$ *and any* $\varepsilon>0$*, there exists* $\delta(\varepsilon)>0$ *such that*

$$\|y_j(t,t_0,y_0)\cdots y_n(t,t_0,y_0)\|<\varepsilon \quad \textit{for all} \quad t\geq t_0$$

when $\|y_0\|<\delta(\varepsilon)$*, and for any* $y_0\in R^n$*,*

$$\lim_{t\to+\infty}\|y_j(t,t_0,y_0)\cdots y_n(t,t_0,y_0)\|=0.$$

Theorem 6.7. *Suppose that*

(1) The zero solution of the following linear system

$$\dot{y}=\tilde{A}(t)y$$

be uniformly asymptotically stable;

(2) There exist constants $r_i\geq 0$ $(i=1,\dots,n-1)$, $r_n>0$*, and* $\delta>0$ *such that*

$$-r_j\tilde{a}_{jj}(t) \geq \sum_{i=1,i\neq j}^{n} r_i|\tilde{a}_{ij}(t)|, \quad j=1,\dots,n-1, \tag{6.10}$$

$$-r_n\tilde{a}_{nn}(t) \geq \sum_{i=1}^{n-1} r_i|\tilde{a}_{in}(t)|+\delta, \tag{6.11}$$

$$-r_n\tilde{b}_n \geq \sum_{i=1}^{n-1} r_i|\tilde{b}_i|, \tag{6.12}$$

or

$$-r_j\tilde{a}_{jj}(t) \geq \sum_{i=1,i\neq j}^{n} r_i|\tilde{a}_{ij}(t)|, \quad j=1,\dots,n-1,$$

$$-r_n\tilde{a}_{nn}(t) \geq \sum_{i=1}^{n-1} r_i|\tilde{a}_{in}(t)|, \tag{6.13}$$

$$-r_n\tilde{b}_n > \sum_{i=1}^{n-1} r_i|\tilde{b}_i|. \tag{6.14}$$

(3) $|f(\sigma,t)| \geq \varphi(|\sigma|) \in K$.
In other words, either (6.10), (6.11), and (6.12), or (6.10), (6.13), and (6.14) are simultaneously valid.

Then the zero solution of (6.9) is absolutely stable in $[0,k]$.

Proof. We construct the Lyapunov function:

$$V(y) = \sum_{i=1}^{n} r_i |y_i|.$$

$V(y) \geq r_n|y_n| \to +\infty$ as $y_n \to +\infty$ and, as a consequence, $V(y)$ is radially unbounded, positive definite for y_n. In addition, there exists

$$\begin{aligned} D^+V(y)|_{(6.9)} &\leq \sum_{j=1}^{n} \left[r_j\tilde{a}_{jj}(t) + \sum_{i=1, i\neq j}^{n-1} r_i|\tilde{a}_{ij}(t)| \right] |y_j| + \left[r_n b_n + \sum_{j=1}^{n-1} r_j b_j \right] |f(y_n,t)|, \\ &\leq \left[r_n\tilde{a}_{nn}(t) + \sum_{i=1}^{n-1} r_i|a_{in}(t)| \right] |y_n| + \left[r_n\tilde{b}_n + \sum_{i=1}^{n-1} r_i|\tilde{b}_i| \right] |f(y_n,t)|, \\ &\leq \left[r_n\tilde{a}_{nn}(t) + \sum_{i=1}^{n-1} r_i|a_{in}(t)| \right] |y_n| + \left[r_n\tilde{b}_n + \sum_{j=1}^{n-1} r_j|\tilde{b}_{jj}| \right] |\varphi(y_n)|, \end{aligned}$$

thus $D^+V(y)|_{(6.9)}$ is negative definite for y_n. According to Theorem 6.5, the conclusion of this theorem is valid. □

Theorem 6.8. *(1) Let the condition (1) of Theorem 6.7 hold;*
(2) Suppose there exist a symmetric matrix $B(t)_{n\times n}$ *and constants* $\alpha > 0$, $\beta > 0$, $\varepsilon > 0$ *such that*

$$y^{\mathrm{T}}B(t)y \geq \beta y_n^2,$$

and either

$$\begin{pmatrix} y \\ f(y_n(t),t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/(k+\varepsilon) \end{bmatrix} \begin{pmatrix} y \\ f(y_n(t),t) \end{pmatrix} \leq -\varepsilon y_n^2,$$

or the matrix $\begin{bmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\frac{\alpha}{(k+\varepsilon)} \end{bmatrix}$ *is negative semi-definite, where*

$$\begin{aligned} G(t) &= A^{\mathrm{T}}(t)B(t) + B(t)A(t) + \dot{B}(t), \\ g(t) &= B(t)\tilde{b} + \frac{\alpha}{2}c, \end{aligned}$$

and $|f(y_n,t)| \geq \varphi(|y_n|) \in K$.
Then the zero solution of (6.9) is absolutely stable in $[0,k]$.

Proof. We choose the Lyapunov function:

$$V(t,y) = y^{\mathrm{T}} B(t) y.$$

According to the hypothesis, we know that $V(t,y)$ is radially unbounded and positive definite, and

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.9)} &= y^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)\tilde{A}(t))y + 2y^{\mathrm{T}}B(t)\tilde{b}f(y_n(t),t),\\
&= y^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)\tilde{A}(t))y + 2y^{\mathrm{T}}B(t)\tilde{b}f(y_n(t),t)\\
&\quad + \alpha y_n(t) f(y_n(t),t) - \frac{\alpha}{k+\varepsilon} f^2(y_n(t),t)\\
&\quad - \left(\alpha y_n(t) - \frac{\alpha}{k+\varepsilon} f(y_n(t),t)\right) f(y_n(t),t),\\
&\le \begin{cases} -\dfrac{\alpha\varepsilon}{k+\varepsilon} |y_n(t)| \cdot |\varphi(y_n(t))|, & \text{or}\\ -\varepsilon y_n^2(t), \end{cases}\\
&< 0 \quad \text{for} \quad y_n^2 \neq 0.
\end{aligned}$$

As a result, the zero solution of (6.9) is absolutely stable in $[0,k]$. □

Assume $b_n < 0$. By the topological transformation

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -b_1/b_n \\ 0 & 1 & \cdots & 0 & -b_2/b_n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{n-1}/b_n \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} := Hy,$$

the system (6.9) becomes

$$\begin{aligned}
\dot{z}_i &= \sum_{j=1}^{n} p_{ij}(t) z_j, \quad i = 1, \ldots, n-1,\\
\dot{z}_n &= \sum_{j=1}^{n} p_{nj}(t) z_j + \tilde{h}_n f(z_n(t),t),
\end{aligned} \tag{6.15}$$

where $P(t) = (p_{ij}(t))_{n\times n} = H\tilde{A}(t)H^{-1}$.

Theorem 6.9. *Assume that*

(1) The condition (1) of Theorem 6.7 be satisfied;
(2) There exists a symmetric positive semi-definite matrix

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,n-1} & 0 \\ 0 & \vdots & 0 & b_{nn} \end{bmatrix} \quad (b_{nn} > 0),$$

such that either

$$z^{\mathrm{T}}(P^{\mathrm{T}}(t)B + BP(t))z \le -\varepsilon z_n$$

or

$$z^{\mathrm{T}}(P^{\mathrm{T}}(t)B + BP(t))z \le 0 \quad \textit{and} \quad z_n f(z_n,t) \textit{ is positive definite}.$$

Then the zero solution of (6.15) is absolutely stable in $[0,k]$.

Proof. We choose the Lyapunov function:

$$V(z) = z^{\mathrm{T}}Bz + b_{nn}z_n^2,$$

which is radially bounded, positive definite w.r.t. z_n. Then,

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.15)} &= z^{\mathrm{T}}(P^{\mathrm{T}}(t)B + BP(t))z + 2\tilde{h}_n z_n f_n(z_n,t),\\ &\le \begin{cases} -\varepsilon z_n^2, & \text{or} \\ 2\tilde{h}_n z_n f(z_n,t). \end{cases} < 0 \quad \text{when} \quad Z_n \ne 0\end{aligned}$$

Consequently, the zero solution of (6.15) is absolutely stable in $[0,k]$. □

6.3 Direct Control Systems

Consider the general nonautonomous direct control system

$$\dot{x} = A(t)x + b(t)f(\sigma,t), \quad \sigma = c^{\mathrm{T}}x, \tag{6.16}$$

where $x \in R^n$, $A(t)$ is an $n \times n$ continuous matrix, $c(t)$ and $b(t)$ are n-dimensional continuous vectors, which are bounded and differentiable, and

$$f \in F_{[0,k]} := \{f : f(0,t) \equiv 0, 0 \le \sigma f(\sigma,t) \le k\sigma^2, 0 < k < +\infty,$$

$$f \in C[[0,+\infty) \times R^n, R^1]\}.$$

Theorem 6.10. *Suppose that there exists a symmetric, differentiable, and bounded $n \times n$ matrix $B(t)$ such that $x^{\mathrm{T}}B(t)x$ is radially unbounded, positive definite, and there exists a constant $\alpha > 0$ such that*

$$\begin{pmatrix} x \\ \xi \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/k \end{bmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}$$

is negative definite, where

$$\begin{aligned} -G(t) &= A^{\mathrm{T}}(t)B(t) + B(t)A(t) + \dot{B}(t),\\ g(t) &= B(t)b(t) + \frac{\alpha}{2}c(t). \end{aligned}$$

Then the zero solution of (6.16) is absolutely stable in $[0,k]$.

Proof. We construct the radially unbounded, positive definite Lyapunov function $V = x^{\mathrm{T}}B(t)x$. Using the S-method, we deduce

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.16)} &= x^{\mathrm{T}}(A^{\mathrm{T}}(t)B(t)+B(t)A(t)+\dot{B}(t))x+2b^{\mathrm{T}}(t)B(t)x(t)f(\sigma(t),t),\\ &= -x^{\mathrm{T}}G(t)x+2b^{\mathrm{T}}(t)B(t)x(t)f(\sigma(t),t)+\alpha\sigma(t)f(\sigma(t),t)\\ &\quad -\alpha\frac{1}{k}f^2(\sigma(t),t)-\alpha\left(\sigma(t)-\frac{1}{k}f(\sigma(t),t)\right)f(\sigma(t),t),\\ &\le \begin{pmatrix} x\\ f(\sigma(t),t)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}-G(t) & g(t)\\ g^{\mathrm{T}}(t) & -\alpha/k\end{bmatrix}\begin{pmatrix} x\\ f(\sigma(t),t)\end{pmatrix},\\ &< 0 \quad \text{for} \quad x\neq 0.\end{aligned}$$

Thus, the zero solution of (6.16) is absolutely stable in $[0,k]$. □

6.4 Indirect Control Systems

Consider the nonautonomous indirect control system

$$\begin{aligned}\dot{x} &= A(t)x+b(t)\xi+d(t)f(\sigma,t),\\ \dot{\xi} &= f(\sigma,t),\ \sigma = c^{\mathrm{T}}(t)x-r(t)\xi,\end{aligned} \tag{6.17}$$

where $x\in R^n$, $b(t), d(t), c(t)$ are n-dimensional continuous differentiable vectors; ξ,σ are scalars; and $r(t)\neq 0$ is a scalar continuously differentiable function. Suppose that the coefficients in system (6.17) are continuously differentiable, and

$$\Delta(t) := \det\begin{vmatrix} A(t) & b(t)\\ -c^{\mathrm{T}}(t) & r(t)\end{vmatrix}\neq 0 \quad \text{for all} \quad t\in[0,+\infty),$$

$$\begin{aligned}f\in F_{[k_1,k_2]} = \{f : f(0,t)\equiv 0, k_1\sigma^2\le \sigma f(\sigma,t)<k_2\sigma^2 \text{ for } \sigma\neq 0,\\ f\in C[[0,+\infty)\times R^n, R]\}.\end{aligned}$$

We set

$$\sigma = c^{\mathrm{T}}(t)x-r(t)\xi, \quad \text{that is,} \quad \xi(t) = r^{-1}(t)(c^{\mathrm{T}}(t)x-\sigma).$$

Then (6.17) can be reduced to

$$\begin{aligned}\dot{x} &= \tilde{A}(t)x+\tilde{b}(t)\sigma+d(t)f(\sigma,t),\\ \dot{\sigma} &= \tilde{c}^{\mathrm{T}}(t)x-\rho(t)\sigma-\gamma(t)f(\sigma,t),\end{aligned} \tag{6.18}$$

where

$$\begin{aligned}\tilde{A}(t) &= A(t)+b(t)r^{-1}(t)c^{\mathrm{T}}(t),\\ \tilde{b}(t) &= -b(t)r^{-1}(t),\end{aligned}$$

$$\begin{aligned}\tilde{c}^{\mathrm{T}}(t) &= c^{\mathrm{T}}(t)\tilde{A}(t) + \dot{c}^{\mathrm{T}}(t) - \dot{r}(t)r^{-1}(t)c^{\mathrm{T}}(t),\\ \rho(t) &= -c^{\mathrm{T}}(t)\tilde{b}(t) + \dot{r}(t)r^{-1}(t),\\ \gamma(t) &= -c^{\mathrm{T}}(t)d(t) + r(t).\end{aligned}$$

Let

$$P(t) = \begin{bmatrix} \tilde{A}(t) & \tilde{b}(t) \\ \tilde{c}^{\mathrm{T}}(t) & -\rho(t) \end{bmatrix}, \quad z = \begin{pmatrix} x \\ \sigma \end{pmatrix}, \quad l(t) = \begin{pmatrix} d(t) \\ -\gamma(t) \end{pmatrix}.$$

Then system (6.18) can be written as

$$\dot{z} = P(t)z + l(t)f(\sigma,t). \tag{6.19}$$

Theorem 6.11. *Suppose that there exists an $n \times n$ symmetric differentiable matrix $H(t)$ such that $x^{\mathrm{T}}H(t)x$ is radially unbounded, positive definite, and $x^{\mathrm{T}}(G(t) + k_1S(t)S^{\mathrm{T}}(t))x$ is positive definite, where*

$$\begin{aligned}G(t) &= P^{\mathrm{T}}(t)H(t) + H(t)P(t) + \dot{H}(t),\\ \sigma &= S^{\mathrm{T}}(t)z,\\ S(t) &= 2H(t)l(t).\end{aligned}$$

Then the solution of (6.19) is absolutely stable in $[k_1, k_2]$.

Proof. We construct the radially unbounded, positive definite Lyapunov function $V(t,x) = x^{\mathrm{T}}H(t)x$. It follows that

$$-\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(6.19)} = z^{\mathrm{T}}G(t)z + \sigma f(\sigma,t).$$

Since $k_1\sigma^2 < \sigma f(\sigma) < k_2\sigma^2$ for $\sigma \neq 0$, we deduce

$$z^{\mathrm{T}}(G(t) + k_1S(t)S^{\mathrm{T}}(t))z \leq -\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(6.19)} \leq z^{\mathrm{T}}(G(t) + k_2S(t)S^{\mathrm{T}}(t))z.$$

Furthermore, there exists

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(6.19)} \leq -z^{\mathrm{T}}(G(t) + k_1S(t)S^{\mathrm{T}}(t))z < 0 \quad \text{for} \quad z \neq 0.$$

This completes the proof. □

In the following, we use the method of absolute stability for partial variables to determine the absolute stability of the zero solution of (6.18).

Suppose that all the coefficients in (6.19) are bounded, and that

$$f \in F_{[0,k]} := \{f : f(0,t) \equiv 0, 0 < \sigma f(\sigma,t) \leq k_2\sigma^2 (k_2 < +\infty) \text{ for } \sigma \neq 0,$$

$$f \in C[[0,+\infty) \times R^n, R^1]\}.$$

For the coefficients of (6.19), we adopt the following notations:

$$P(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1,n+1}(t) \\ \vdots & & \vdots \\ p_{n+1,1}(t) & \cdots & p_{n+1,n+1}(t) \end{bmatrix},$$

$$P_{(j_0)}(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1j_0}(t) \\ \vdots & & \vdots \\ p_{j_01}(t) & \cdots & p_{j_0j_0}(t) \end{bmatrix},$$

$$P^{(n+1-j_0)} = \begin{bmatrix} p_{1,j_0+1}(t) & \cdots & p_{1,n+1}(t) \\ \vdots & & \vdots \\ p_{j_0,j_0+1}(t) & \ldots & p_{j_0,n+1}(t) \end{bmatrix},$$

$$\begin{aligned} l(t) &= (l_1(t),\ldots,l_{n+1}(t))^{\mathrm{T}}, \\ l_{(j_0)}(t) &= (l_1(t),\ldots,l_{j_0}(t))^{\mathrm{T}}, \\ z_{(j_0)}(t) &= (x_1(t),\ldots,x_{j_0}(t))^{\mathrm{T}}, \\ z^{(n-j_0)}(t) &= (x_{j_0+1}(t),\ldots,x_n(t),\sigma)^{\mathrm{T}}. \end{aligned}$$

Theorem 6.12. *Assume that the following conditions are satisfied:*

(1) The zero solution of the system

$$\dot{z}_{(j_0)} = A_{j_0}(t) z_{j_0} \tag{6.20}$$

is uniformly asymptotically stable;

(2) There exist constants $r_i \geq 0$ $(i = 1,\ldots,j_0)$, $r_j > 0$ $(j = j_0+1,\ldots,n+1)$, $\varepsilon > 0$ such that

$$\begin{aligned} -r_j p_{jj}(t) &\geq \sum_{i=1,i\neq j}^{n+1} r_i |p_{ij}(t)|, \quad j = 1,\ldots,j_0, \\ -r_j a_{jj}(t) &\geq \sum_{i=1,i\neq j}^{n+1} r_i |p_{ij}(t)| + \varepsilon, \quad j = j_0+1,\ldots,n+1, \\ -l_{n+1}(t) r_{n+1} &\geq \sum_{i=1}^{n} r_i l_i(t). \end{aligned}$$

Then the zero solution of (6.19) is absolutely stable in $[0,k_2]$.

Proof. We construct the radially unbounded, positive definite Lyapunov function for $z_{j_0+1},\ldots,z_{n+1}$:

$$V(z) = \sum_{i=1}^{n+1} r_i |z_i|.$$

Then,

$$
\begin{aligned}
D^+V(z)|_{(6.19)} &\le \sum_{j=1}^{n+1}\Big[r_j p_{jj}(t) + \sum_{i=1, i\neq j}^{n+1} r_i|p_{ij}(t)|\Big]|z_j| \\
&\quad + \Big[r_{n+1}\, l_{n+1}(t) + \sum_{i=1}^{n} r_i|l_i(t)|\Big]|f(\sigma,t)| \\
&\le \sum_{j=j_0+1}^{n+1}\Big[r_j p_{jj}(t) + \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i|p_{ij}(t)|\Big]|z_j| \\
&\le -\varepsilon \sum_{j=j_0+1}^{n+1} |z_j| \\
&< 0 \qquad \text{for} \ \sum_{j=j_0+1}^{n+1} z_j^2 \neq 0.
\end{aligned}
$$

As a result, the zero solution of (6.19) is absolutely stable w.r.t. the variables $z_{j_0+1}, \dots, z_{n+1}$.

Assume that the Cauchy matrix solution of (6.20) is $K(t,t_0)$. The condition (1) indicates that there exist two constants $M \ge 1$ and $\alpha > 0$ such that

$$\|K(t,t_0)\| \le M\,\mathrm{e}^{-\alpha(t-t_0)}. \tag{6.21}$$

The first j_0 components of the solution of (6.19) can be expressed as

$$
\begin{aligned}
z_{(j_0)}(t,t_0,z_0) = K(t,t_0)z_{(j_0)}(t_0) &+ \int_{t_0}^{t} K(t,\tau)P^{(n+1-j_0)}(\tau)z^{(n+1-j_0)}(\tau)\,\mathrm{d}\tau \\
&+ \int_{t_0}^{t} K(t,\tau)l_{(j_0)}(\tau)f(\sigma(\tau),\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Using (6.21) and the method used in proving Theorem 4.3, we can easily prove that the zero solution of (6.19) is absolutely stable w.r.t $z_{j_0+1}, \dots, z_n$, as well. The proof is complete. □

We can also prove the following corollary, along the same line.

Corollary 6.13. *(1) Let the zero solution of the system* $\frac{\mathrm{d}z}{\mathrm{d}t} = P(t)z$ *be uniformly asymptotically stable;*
(2) Suppose that there exist constants $r_i \ge 0$ $(i = 1, \dots, n)$, $r_{n+1} > 0$, $\varepsilon > 0$ *such that*

$$
\begin{aligned}
-r_j p_{jj}(t) &\ge \sum_{i=1, i\neq j}^{n+1} r_i|p_{ij}(t)|, \quad j = 1, \dots, n+1, \\
-l_{n+1}(t)r_{n+1} &\ge \sum_{i=1}^{n} r_i|l_i(t)| + \varepsilon, \quad 0 < \varepsilon \ll 1.
\end{aligned}
$$

Then the zero solution of (6.19) is absolutely stable in $[0, k_2]$.

Theorem 6.14. *(1) Let the condition (1) of Corollary 6.13 be satisfied;*
(2) Suppose that there exists a symmetric differentiable bounded $(n+1)\times(n+1)$ *matrix* $B(t)$ *such that*

$$z^{\mathrm{T}}B(t)z \geq \varepsilon z^{\mathrm{T}}z, \quad 0<\varepsilon \ll 1;$$

(3) Suppose that there exists a constant $\alpha>0$ *such that*

$$\begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -\tilde{G}(t) & \tilde{g}(t) \\ \tilde{g}^{\mathrm{T}}(t) & -\alpha/k \end{bmatrix} \begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix} \leq -\delta\tau, \quad 0<\delta\ll 1,$$

where

$$\begin{aligned} -\tilde{G}(t) &= P^{\mathrm{T}}(t)B(t)+B(t)P(t)+\dot{B}(t), \\ \tilde{g}(t) &= B(t)l(t)+\frac{\alpha}{2}\tilde{c}(t), \\ \tau &\in \{\sigma^2, \sigma f(\sigma,t), f^2(\sigma,t)\}. \end{aligned}$$

Then the zero solution of (6.19) is absolutely stable in $[0,k_2]$.

Proof. We construct the radially unbounded positive definite Lyapunov function $V(z)=z^{\mathrm{T}}B(t)z$. Differentiating V w.r.t. time t along the solution of (6.19), making use of the S-method and the proof of Theorem 6.10, we get

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.19)} &\leq \begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -\tilde{G}(t) & \tilde{g}(t) \\ \tilde{g}^{\mathrm{T}}(t) & -\alpha/k \end{bmatrix} \begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix} \\ &\quad -\alpha\Big(\sigma(t)-\frac{1}{k}f(\sigma(t),t)\Big)f(\sigma(t),t), \\ &\leq \begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -\tilde{G}(t) & \tilde{g}(t) \\ \tilde{g}^{\mathrm{T}}(t) & -\alpha/k \end{bmatrix} \begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix} \leq -\delta\tau. \end{aligned}$$

Therefore, the zero solution of (6.19) is absolutely stable for σ in $[0,k]$.

Suppose that the Cauchy matrix solution of the system $\dot{z}=P(t)z$ is $K(t,t_0)$. Using the method of variation of constants, we can express the solution of (6.19) as

$$z(t,t_0,z_0)=K(t,t_0)z_0+\int_{t_0}^{t}K(t,\tau)l(\tau)f(\sigma(\tau),\tau)\,\mathrm{d}\tau.$$

A similar reasoning to that used in the proof of Theorem 4.3 can be applied here. □

6.5 Systems with Loop Revolving Feedbacks

Consider the control systems with loop revolving feedbacks [67, 105]:

$$\begin{aligned} \dot{x} &= A(t)x+b(t)\xi+d(t)f(\sigma), \\ \dot{\xi} &= f(\sigma), \quad \sigma=c^{\mathrm{T}}(t)x-r(t)\xi-Nf(\sigma). \end{aligned} \tag{6.22}$$

Again we adopt the notations used in (6.17). However, note that here the control function $f(\sigma)$ is assumed to be differentiable and satisfy

$$-v_1 \le \frac{\partial f(\sigma)}{\partial \sigma} \le v_2,$$

where v_1, v_2 are some positive constants.

Suppose that $r(t) \neq 0$. Using (6.22), we can write

$$\xi(t) = r^{-1}(t)(c^{\mathrm{T}}(t)x - \sigma - Nf(\sigma)). \tag{6.23}$$

Then substituting (6.23) into (6.22) yields

$$\begin{aligned} \dot{x} &= \tilde{A}(t)x + \tilde{b}(t)\sigma + \tilde{d}(t)f(\sigma), \\ w(\sigma)\dot{\sigma} &= \tilde{c}^{\mathrm{T}}x - \rho(t)\sigma - \tilde{N}f(\sigma), \end{aligned} \tag{6.24}$$

where

$$\begin{aligned} \tilde{A}(t) &= A(t) + b(t)r^{-1}(t)c^{\mathrm{T}}(t), \\ \tilde{b}(t) &= b(t)r^{-1}(t), \\ \tilde{d}(t) &= d(t) - r^{-1}(t)b(t)N, \\ \rho(t) &= -c^{\mathrm{T}}(t)\tilde{b}(t) - \dot{r}(t)r^{-1}(t), \\ \tilde{c}^{\mathrm{T}}(t) &= c^{\mathrm{T}}(t)\tilde{A}(t) + \dot{c}^{\mathrm{T}}(t) - \dot{r}(t)r^{-1}(t)c^{\mathrm{T}}(t), \\ \omega(\sigma) &= 1 + N\frac{\partial f}{\partial \sigma}, \\ \tilde{N} &= r(t) - \dot{r}(t)r^{-1}(t)N - c^{\mathrm{T}}(t)d(t). \end{aligned}$$

Theorem 6.15. *Suppose that*

(1) $\omega(\sigma)$ does not change its sign for any $f \in F_{[k_1,k_2]}$;

(2) There exists an $(n+1)\times(n+1)$ symmetric matrix $H(t)$ such that $x^{\mathrm{T}}H(t)x$ is radially unbounded, positive definite;

(3)

$$\begin{pmatrix} x \\ \sigma \\ f \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} G(t) & -H(t)\tilde{b}(t) & -\alpha\tilde{c}(t) - H(t)\tilde{d}(t) \\ -\tilde{b}^{\mathrm{T}}(t)H(t) & 2\alpha\rho(t)k_1 & \alpha\rho(t) \\ -\alpha\tilde{c}^{\mathrm{T}}(t) - (H(t)\tilde{d}(t))^{\mathrm{T}} & \alpha\rho^{\mathrm{T}}(t) & 2\alpha\tilde{N} \end{bmatrix} \begin{pmatrix} x \\ \sigma \\ f \end{pmatrix}$$

is positive definite, where α is a constant with the same sign as $\omega(\sigma)$, and $H(t)$ is solution of the Lyapunov matrix equation:

$$\tilde{A}^{\mathrm{T}}(t)H(t) + H(t)\tilde{A}(t) + \dot{H}(t) = -G(t),$$

in which $G(t)$ is a given positive definite matrix.

Then the zero solution of (6.22) is absolutely stable in $[k_1,k_2]$.

Proof. We take the Lyapunov function

$$V(x,\sigma) = x^{\mathrm{T}}H(t)x + 2\alpha\int_0^{\sigma} f(\sigma)\omega(\sigma)\,\mathrm{d}\sigma,$$

where α is a constant having the same sign as $\omega(\sigma)$. Obviously, $V(x,\sigma)$ is radially unbounded, positive definite. Differentiating V along the solution of (6.22), we obtain

$$\begin{aligned}-\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(6.22)} &= x^{\mathrm{T}}G(t)x - 2x^{\mathrm{T}}H(t)\tilde{b}(t)\sigma - 2x^{\mathrm{T}}H(t)\tilde{d}(t)f(\sigma)\\ &\quad -2\alpha x^{\mathrm{T}}c(t)f(\sigma) + 2\alpha\rho(t)\sigma f(\sigma) + 2\alpha\tilde{N}f^2(\sigma),\end{aligned} \tag{6.25}$$

where $-G(t) = \tilde{A}^{\mathrm{T}}(t)H(t) + H(t)\tilde{A}(t) + \dot{H}(t)$.

We introduce the following notations: $y = (x,\,\sigma,\,t)^{\mathrm{T}}$ and

$$G_i = \begin{bmatrix} G(t) & -H(t)\tilde{b}(t) & -\alpha\tilde{c}(t) - H(t)\tilde{d}(t)\\ -\tilde{b}^{\mathrm{T}}(t)H(t) & 2\alpha\rho(t)k_i & \alpha\rho(t)\\ -\alpha\tilde{c}^{\mathrm{T}}(t) - (H(t)\tilde{d}(t))^{\mathrm{T}} & \alpha\rho^{\mathrm{T}}(t) & 2\alpha\tilde{N}\end{bmatrix}, \quad i = 1,2.$$

Using the facts that $f(0) = 0$ and $k_1\sigma^2 < \sigma f(\sigma) < k_2\sigma^2$, the expression (6.25) becomes

$$y^{\mathrm{T}}G_1(t)y \le -\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(6.24)} \le y^{\mathrm{T}}G_2(t)y.$$

From (6.25) we find that $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(6.24)}$ is negative definite. Thus, the zero solution of (6.24) is absolutely stable in $[k_1,k_2]$. □

Theorem 6.16. *Assume that*

(1) The condition (1) of Theorem 6.15 is satisfied;
(2) The zero solution of the system $\dot{x} = \tilde{A}(t)x$ is uniformly asymptotically stable;
(3) There exist constants $r_i \ge 0$ $(i = 1,\ldots,n)$, $r_{n+1} > 0$ such that

$$-r_j\tilde{a}_{jj}(t) \ge \sum_{i=1,i\neq j}^{n} r_i|\tilde{a}_{ij}(t)| + r_j|\tilde{c}_j(t)|, \quad j = 1,\ldots,n, \tag{6.26}$$

$$r_{n+1}\rho(t) \ge \sum_{i=1}^{n} |\tilde{b}_i(t)|r_i + \varepsilon, \tag{6.27}$$

$$r_{n+1}\tilde{N} \ge \sum_{i=1}^{n} r_i|\tilde{d}_i(t)|, \tag{6.28}$$

where ε is a constant with $0 < \varepsilon \ll 1$;
(4) $\int_0^{+\infty}|\omega(\sigma)|\,\mathrm{d}\sigma = +\infty$.

Then the zero solution of (6.24) is absolutely stable in $[k_1,k_2]$.

Proof. We construct the Lyapunov function

$$V = \sum_{i=1}^{n} r_i|x_i| + r_{n+1}\int_0^{\sigma} \mathrm{sign}\sigma|\omega(\sigma)|\,\mathrm{d}\sigma.$$

V is radially unbounded, positive definite for σ, and thus

$$
\begin{aligned}
D^+V|_{(6.24)} \le & \sum_{j=1}^{n}\Big[r_j\tilde{a}_{jj}(t)+\sum_{i=1,i\neq j}^{n} r_i|\tilde{a}_{ij}(t)|+r_j|\tilde{c}_j(t)|\Big]|x_j| \\
& +\Big[-r_{n+1}\rho(t)+\sum_{i=1}^{n}|\tilde{b}_i(t)|r_i\Big]|\sigma| \\
& +\Big[-\tilde{N}r_{n+1}+\sum_{i=1}^{n} r_i|\tilde{d}_i(t)|\Big]|f(\sigma)|, \\
\le & -\varepsilon|f(\sigma)|, \\
< & \ 0 \quad \text{for} \quad \sigma \neq 0.
\end{aligned}
$$

In this case, $D^+V|_{(6.24)}$ is negative definite in σ. Therefore, the zero solution of (6.24) is absolutely stable in $[k_1,k_2]$ for partial variabel σ.

Let the Cauchy matrix solution of the following system be $K(t,t_0)$:

$$\dot{x}=\tilde{A}(t)x.$$

From the condition (2), we know that there exist constants $M \ge 1$, $\alpha > 0$ such that

$$\|K(t,t_0)\| \le M\,\mathrm{e}^{-\alpha(t-t_0)}.$$

The first equations in (6.24) gives

$$x(t,t_0,x_0)=K(t,t_0)x_0+\int_{t_0}^{t}K(t,\tau)[\tilde{b}(\tau)\sigma(\tau)+\tilde{d}(\tau)f(\sigma(\tau))]\,\mathrm{d}\tau.$$

The proof can be completed along the lines of the proof for the sufficiency of Theorem 4.3. This proves the theorem. □

Theorem 6.17. *If the conditions* (1) *and* (2) *of Theorem 6.16 hold, and (6.26) holds as well, while (6.27) and (6.28) are replaced by*

$$
\begin{aligned}
r_{n+1}\rho(t) &\ge \sum_{i=1}^{n} r_i|\tilde{b}_i(t)|, \\
r_{n+1}\tilde{N} &\ge \sum_{i=1}^{n} r_i|\tilde{d}_i(t)|+\varepsilon, \quad 0<\varepsilon \ll 1,
\end{aligned}
$$

then the zero solution of (6.24) is absolutely stable in $[k_1,k_2]$.

Proof. We construct the radially unbounded, positive definite Lyapunov function:

$$V=\sum_{i=1}^{n} r_i|x_i|+r_{n+1}\int_0^{\sigma}\operatorname{sign}\sigma|\omega(\sigma)|\,\mathrm{d}\sigma,$$

then we have

$$
\begin{aligned}
D^+V|_{(6.24)} \leq \sum_{j=1}^{n} &\left[r_j \tilde{a}_{jj}(t) + \sum_{i=1, i\neq j}^{n} r_i |\tilde{a}_{ij}(t)| + r_j |\tilde{c}_j(t)| \right] |x_j| \\
&+ \left[-r_{n+1}\rho(t) + \sum_{i=1}^{n} |\tilde{b}_i(t)| r_i \right] |\sigma| \\
&+ \left[-\tilde{N} r_{n+1} + \sum_{i=1}^{n} r_i |\tilde{d}_i(t)| \right] |f(\sigma)|, \\
\leq\ & -\varepsilon |f(\sigma)|, \\
<\ & 0 \quad \text{for} \quad \sigma \neq 0.
\end{aligned}
$$

Thus, the zero solution of (6.24) is absolutely stable for σ in $[k_1, k_2]$. The rest of the proof is identical to that of Theorem 6.16. □

7

Systems with Multiple Nonlinear Feedback Controls

As a result of the fast development of science and technology, control systems have become more and more complex. Usually, single feedback control is not enough to finish a complicated task and needs multiple feedbacks. In 1988, SIAM published a research report "The Future Development of Control Theory – Mathematical Prospect." It was indicated in this report that though the stability of nonlinear control systems had been paid much attention and many mathematical results had been found, the results are still mainly on single-variable nonlinear control systems such as Popov's principal and Lyapunov method. Multivariable nonlinear control systems are not yet well generalized. It is still difficult to extend the single-variable case to the multivariable case.

In this chapter, we will discuss the absolute stability of control systems with multiple nonlinear control terms. The results given in Sects. 7.1–7.4 are taken from Liao et al. [91, 92], and that presented in Sect. 7.5 are from Gan and Ge [29].

7.1 Necessary and Sufficient Conditions for Absolute Stability

Consider the following control system with m nonlinear control terms [91]:

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^{m} b_j f_j(\sigma_j), \\ \sigma_j &= c_j^{\mathrm{T}} x = \sum_{i=1}^{n} c_{ij} x_i, \ j = 1, \ldots, m, \end{aligned} \tag{7.1}$$

where $A \in R^{n\times n}$, $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, $b_j = (b_{1j}, \ldots, b_{nj})^{\mathrm{T}}$, $c_j = (c_{1j}, \ldots, c_{nj})^{\mathrm{T}}$,

$$f_j \in F_\infty := \Big\{ f : f(0) = 0, f(\sigma)\sigma > 0, \ \sigma \neq 0, \ f(\sigma) \in C[(-\infty, +\infty), \ R^1] \Big\},$$
$$j = 1, \ldots, m,$$

$\mathrm{Re}\lambda(A) \leq 0$. Let

$$\begin{aligned} \Omega_j &= \Big\{ x : \sigma_j = c_j^{\mathrm{T}} x = 0 \Big\}, \quad j = 1, \ldots, m, \\ \Omega &= \left\{ x : \|\sigma\| = \sum_{j=1}^{m} |\sigma_j| = \sum_{j=1}^{m} |c_j^{\mathrm{T}} x| = 0 \right\}. \end{aligned}$$

Definition 7.1. *The zero solution of (7.1) is said to be absolutely stable for the set $\Omega[\Omega_i]$ if for any $f_j(\sigma_j) \in F_\infty (j = 1,\dots,m)$ and any $\varepsilon > 0$, there exits $\delta(\varepsilon) > 0$ such that for $\|x_0\| < \delta(\varepsilon)$ the distance from the solution $x(t) := x(t,t_0,x_0)$ to the set $\Omega[\Omega_j]$ satisfies*

$$\rho(x,\Omega) := \sum_{j=1}^{m} |c_j^{\mathrm{T}} x(t)| < \varepsilon \qquad \Big(\rho(x,\Omega_j) := |c_j^{\mathrm{T}} x(t)| < \varepsilon\Big),$$

and

$$\lim_{t\to+\infty} \sum_{j=1}^{m} |c_j^{\mathrm{T}} x(t)| = 0 \qquad \Big(\lim_{t\to+\infty} |c_j^{\mathrm{T}} x(t)| = 0\Big)$$

for every $x_0 \in \mathrm{R}^n$.

Definition 7.2. *A function $V(x) \in C[\mathrm{R}^n,\mathrm{R}^1]$ is said to be positive definite w.r.t. the set $\Omega[\Omega_j]$ if*

$$V(x)\begin{cases} =0 & \text{for } x \in \Omega, \\ >0 & \text{for } x \overline{\in} \Omega. \end{cases} \qquad \left(V(x)\begin{cases} =0 & \text{for } x \in \Omega_j, \\ >0 & \text{for } x \overline{\in} \Omega_j. \end{cases}\right)$$

The function $V(x) \in C[\mathrm{R}^n,\mathrm{R}^1]$ is said to be negative definite w.r.t. the set Ω (Ω_j) if $-V(x)$ is positive definite for Ω (Ω_j).

Definition 7.3. *A function $V(x) \in C[\mathrm{R}^n,\mathrm{R}^1]$ is said to be radially unbounded, positive definite for Ω (Ω_j) if $V(x)$ is positive definite for Ω (Ω_j) and $V(x) \to +\infty$ as $\sum_{j=1}^{m} |\sigma_j| \to +\infty$ $(|\sigma_j| \to +\infty)$.*

Theorem 7.4. *The necessary and sufficient conditions for the absolute stability of the zero solution of (7.1) are given by*

1. $B := A + \sum_{j=1}^{m} \theta_j b_j c_j^{\mathrm{T}}$ is Hurwitz stable with $\theta_j = 1$ or $\theta_j = 0, j = 1,\dots,m;$
2. The zero solution of (7.1) is absolutely stable for Ω.

Proof. *Necessity.* (1) In the case $\mathrm{Re}\lambda(A) < 0$, we take $\theta_j = 0,\ j = 1,\dots,m$, and then $B = A$. B is obviously Hurwitz stable. In the case $Re\lambda(A) \le 0$, we take some $\theta_j = 1$. Let in (7.1) $f_j(\sigma_j) = \sigma_j = c_j^{\mathrm{T}} x (j = 1,\dots,m)$. Then (7.1) can be transformed into

$$\dot{x} = \left(A + \sum_{j=1}^{m} \theta_j b_j c_j^{\mathrm{T}}\right) x.$$

Therefore, $B = A + \sum_{j=1}^{m} \theta_j b_j c_j^{\mathrm{T}}$ is Hurwitz stable.

(2) $\forall \varepsilon > 0$, taking

$$\tilde{\varepsilon} = \frac{\varepsilon}{\sum_{j=1}^{m} \|c_j^{\mathrm{T}}\|},$$

there exits $\delta(\tilde{\varepsilon}) > 0$ such that for $\|x_0\| < \delta(\tilde{\varepsilon})$,

$$\|x(t)\| := \|x(t,t_0,x_0)\| := \sum_{j=1}^{n} |x_j(t)| < \tilde{\varepsilon} \quad \text{for all } t \geq t_0.$$

This implies that

$$\sum_{j=1}^{m} \|c_j^{\mathrm{T}} x(t)\| \leq \sum_{j=1}^{m} \|c_j^{\mathrm{T}}\| \, \|x(t)\| \leq \sum_{j=1}^{m} \|c_j^{\mathrm{T}}\| \tilde{\varepsilon} = \varepsilon$$

for all $t \geq t_0$.

Furthermore, since $\lim\limits_{t\to+\infty} \|x(t)\| = 0$ for every $x_0 \in \mathrm{R}^n$, we have

$$0 \leq \lim_{t\to+\infty} \sum_{j=1}^{m} \|c_j^{\mathrm{T}} x(t)\| \leq \sum_{j=1}^{m} \|c_j^{\mathrm{T}}\| \lim_{t\to+\infty} \|x(t)\| = 0.$$

Consequently, the zero solution of (7.1) is absolutely stable for Ω. The necessity is proved.

Sufficiency. In accordance with the formula of variation of constants, the solution $x(t) := x(t,t_0,x_0)$ of (7.1) satisfies

$$x(t) = \mathrm{e}^{B(t-t_0)} x_0 + \int_{t_0}^{t} \mathrm{e}^{B(t-\tau)} \sum_{j=1}^{m} [b_j f_j(\sigma_j(\tau)) - \theta_j b_j \sigma_j(\tau)] \, \mathrm{d}\tau.$$

Since B is Hurwitz stable, there exist constants $M \geq 1$ and $\alpha > 0$ such that

$$\left\| \mathrm{e}^{B(t-t_0)} \right\| \leq M \mathrm{e}^{-\alpha(t-t_0)}, \quad t \geq t_0.$$

We define $\sigma_j(t) = \sigma_j(t,t_0,x_0)$. Since $\sigma = \sum_{j=1}^{m} |\sigma_j(t)| \to 0$ as $t \to +\infty$, we have $\lim_{t\to+\infty} \sigma_j(t) = 0$. Because $\sigma_j(t)$ continuously depends on the initial value x_0, and $f_j(\sigma_j(t))$ is a composite continuous function of x_0 and $f_j(\sigma_j(t)) \to +\infty$ as $t \to +\infty$ $\forall \varepsilon > 0$ there exits $\delta(\varepsilon) > 0$ and $t_1 > t_0$ such that $\|x_0\| < \delta(\varepsilon)$ implies

$$\left\| \mathrm{e}^{B(t-t_0)} x_0 \right\| \leq \left\| \mathrm{e}^{B(t-t_0)} \right\| \|x_0\| < \frac{\varepsilon}{3} \qquad \text{for } t \geq t_0,$$

$$\int_{t_0}^{t_1} M \mathrm{e}^{-\alpha(t-\tau)} \left[\sum_{j=1}^{m} \|b_j f_j(\sigma_j(\tau))\| + \sum_{j=1}^{m} \theta_j \|b_j \sigma_j(\tau)\| \right] \mathrm{d}\tau < \frac{\varepsilon}{3} \quad \text{for } t_0 \leq t_1 < t,$$

$$\int_{t_1}^{t} M \mathrm{e}^{-\alpha(t-\tau)} \left[\sum_{j=1}^{m} \|b_j f_j(\sigma_j(\tau))\| + \sum_{j=1}^{m} \theta_j \|b_j \sigma_j(\tau)\| \right] \mathrm{d}\tau < \frac{\varepsilon}{3} \quad \text{for } t \geq t_1.$$

Thus, we have

$$\begin{aligned} \|x(t)\| \leq{} & \|\mathrm{e}^{B(t-t)} x_0\| + \int_{t_0}^{t_1} M \mathrm{e}^{-\alpha(t-\tau)} \left[\sum_{j=1}^{m} \|b_j f_j(\sigma_j(\tau))\| + \sum_{j=1}^{m} \theta_j \|b_j \sigma_j(\tau)\| \right] \mathrm{d}\tau \\ & + \int_{t_1}^{t} M \mathrm{e}^{-\alpha(t-\tau)} \left[\sum_{j=1}^{m} \|b_j f_j(\sigma_j(\tau))\| + \sum_{j=1}^{m} \theta_j \|b_j \sigma_j(\tau)\| \right] \mathrm{d}\tau \\ <{} & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

For any $x_0 \in \mathrm{R}^n$, by the L'Hospital rule, we get

$$
\begin{aligned}
0 &\leq \lim_{t\to+\infty} \|x(t)\| \\
&\leq \lim_{t\to+\infty} M\mathrm{e}^{-\alpha(t-t_0)} \\
&\quad + \lim_{t\to+\infty} \int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-\tau)} \left[\sum_{j=1}^{m} \|b_j f_j(\sigma_j(\tau))\| + \sum_{j=1}^{m} \theta_j \|b_j \sigma_j(\tau)\| \right] \mathrm{d}\tau \\
&= 0.
\end{aligned}
$$

Thus the zero solution of (7.1) is absolutely stable. The proof of the theorem is complete. □

Theorem 7.5. *The zero solution of (7.1) is absolutely stable if and only if*

1. $A + \sum_{j=1}^{m} \theta_j b_j c_j^{\mathrm{T}} := B$ *is a Hurwitz matrix, where* $\theta_j = 0$ *or* $\theta_j = 1, j = 1, \ldots, m$
2. There exists a differential Lyapunov function $V_f \in [\mathrm{R}^n, \mathrm{R}^1]$, *where* $V_f(x)$ *is radially unbounded, positive definite for* Ω, *i.e., there exist* $\varphi_f \in KR$ *and* $\psi_f \in K$ *such that*

$$
V_f(x) \geq \varphi_f(|\sigma|),
$$

$$
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7.1)} \leq -\psi_f(|\sigma|). \tag{7.2}
$$

Proof. *Sufficiency*. It is suffice to prove that the condition (2) implies that the zero solution of (7.1) is absolutely stable for Ω.

Since $V_f(0) = 0,\ 0 \in \Omega$, and $V_f(x)$ is a continuous function of x $\forall \varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$
V_f(x_0) < \varphi_f(\varepsilon) \quad \text{for} \quad \|x_0\| < \delta(\varepsilon).
$$

If follows from (7.2) that:

$$
\varphi_f(|\sigma|) \leq V_f(x(t)) \leq V_f(x_0) \leq \varphi_f(\varepsilon),
$$

and therefore, $|\sigma(t)| < \varepsilon$. Thus, the zero solution of (7.1) is stable for Ω. Now we prove that $\lim_{t\to+\infty} \sigma(t, t_0, x_0) = 0$ for any $x_0 \in \mathrm{R}^n$.

Since $V_f(x(t)) := V_f(t)$ is a monotone decreasing and bounded function,

$$
\inf_{t \geq t_0} V_f(x(t)) := \lim_{t\to+\infty} V_f(x(t)) := \alpha \geq 0.
$$

We want to show that α can be reached only in Ω. If it can be reached outside Ω, then there must exist a sequence t_k with $t_k \to +\infty$ as $k \to +\infty$ such that $\lim_{t_k\to+\infty} \sigma(t_k) = 0$. As a result,

$$
\alpha = \lim_{t_k\to+\infty} V_f(t_k) = \lim_{\substack{t_k\to+\infty \\ \sigma(t_k)\to 0}} V_f(t_k).
$$

In other words, α can be reached in Ω, a contradiction with the pre-assumption that α can be reached outside Ω.

For any $x_0 \in \mathrm{R}^n$, the expression (7.1) gives

$$|\sigma(t)| \le |\sigma(t_0)| := h < H < +\infty.$$

Now, let us prove that $\lim_{t\to+\infty} \sigma(t) = 0$. Assume that $\lim_{t\to+\infty} \sigma(t) \neq 0$. Since $\sigma(t)$ is uniformly continuous, there exist constants $\beta > 0,\ \eta > 0$ and point sequence t_j such that $|\sigma(t)| \ge \beta$ for $t \in [t_j - \eta, t_j + \eta]$, $j = 1, 2, \ldots$.

Setting $\gamma_f = \lim_{\beta\le\sigma\le h} \psi_f(\sigma)$, we deduce

$$\begin{aligned}
0 \le V_f(t) &\le V_f(t_0) + \int_{t_0}^{t} \frac{\mathrm{d}V_f}{\mathrm{d}t}\mathrm{d}t \\
&\le V_f(t_0) - \int_{t_0}^{t} \psi_f(|\sigma(\tau)|)\mathrm{d}\tau \\
&\le V_f(t_0) - \sum_{j=1}^{n} \int_{t_j-\eta}^{t_j+\eta} \psi_f(|\sigma(\tau)|)\mathrm{d}\tau \\
&\le V_f(t_0) - 2n\eta\gamma_f \to -\infty \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This yields a contradiction; thus $\lim_{t\to+\infty} \sigma(t) = 0$. This proves that the zero solution of (7.1) is absolutely stable for Ω. The sufficiency is proved.

Necessity. Since the zero solution of (7.1) is absolutely stable, R^n is an attractive space. for any $f_i \in F(j = 1, \ldots, m)$ and any $x \in \mathrm{R}^n$, let

$$W_f(x) := \sup \|x_f(t, 0, x)\|^2, \quad t \ge 0,$$

where $x_f(t)$ denotes a solution of (7.1). From Theorem 3.2 of Bhatia and Szegö [7], we know that $W_f(x)$ has the following properties:

1. $W_f(x) \ge 0$ and $W_f(x) = 0$ if and only if $x = 0$, $W_f(x)$ is radially unbounded, positive definite
2. $W_f(x)$ is monotone decreasing function
3. $W_f(x)$ is continuous in R^n

Furthermore, we define

$$V_f(x) := \int_0^{+\infty} W_f(x_f(\eta, 0, x))\mathrm{e}^{-\eta}\mathrm{d}\eta.$$

Obviously, $V_f(x)$ is radially unbounded, positive definite. Thus, there exists $\tilde{\varphi}_f \in KR$ such that

$$V_f(x) \ge \tilde{\varphi}_f(\|x\|).$$

Let

$$\Phi = \int_0^{t+\eta} W_f(x_f(\xi))\mathrm{d}\xi.$$

If follows that:

$$\Phi'_\eta = \Phi'_t = W_f(x_f(t+\eta)).$$

Integrating by parts yields

$$\begin{aligned} V_f(x_f(t)) &= \int_0^{+\infty} \mathrm{e}^{-\eta}\mathrm{d}\Phi = \mathrm{e}^{-\eta}\int_0^{t+\eta} W_f(x_f(\xi))\mathrm{d}\xi\,|_0^{+\infty} + \int_0^{+\infty} \Phi(t+\eta)\mathrm{e}^{-\eta}\mathrm{d}\eta \\ &= -\int_0^t W_f(x_f(\xi))\mathrm{d}\xi + \int_0^{+\infty} \Phi(t+\eta)\mathrm{e}^{-\eta}\mathrm{d}\eta. \end{aligned}$$

Since $W_f(x_f(t))$ is a monotone nonincreasing function, $W_f(x_f(t))$ is bounded. Furthermore, we note that

$$\lim_{\eta\to+\infty} \mathrm{e}^{-\eta}\int_0^{t+\eta} W_f(x_f(\xi))\mathrm{d}\xi = 0$$

and

$$\begin{aligned} \frac{\mathrm{d}V_f}{\mathrm{d}t}|_{(7.1)} &= -W_f(x_f(t)) + \int_0^{+\infty} \Phi'_t \mathrm{e}^{-\eta}\mathrm{d}\eta \\ &= -W_f(x_f(t)) + \int_0^{+\infty} W_f(x_f(t+\eta))\mathrm{e}^{-\eta}\mathrm{d}\eta \\ &= \int_0^{+\infty} \left[W_f(x_f(t+\eta)) - W_f(x_f(t))\right] \mathrm{d}\eta. \end{aligned}$$

Since $W_f(x_f(t))$ is a monotone nonincreasing function, we obtain

$$W_f(x_f(t+\eta)) \geq W_f(x_f(t)), \quad \eta \geq 0.$$

In particular, if $x(t)$ is a nonzero solution of (7.1),

$$W_f(x_f(t+\eta)) \neq W_f(x_f(t))$$

or

$$W_f(x_f(t+\eta)) \equiv W_f(x_f(t)) \to 0 \quad \text{as} \quad \eta \to +\infty.$$

Thus, $W_f(x_f(t+\eta)) \equiv 0$ that fact

$$W_f(x) = \sup\left\{\|x_f(t,0,x)\|^2, t \geq 0\right\} \not\equiv 0.$$

Therefore, if $x_f(t) \not\equiv 0$, then

$$\int_0^{+\infty} [W_f(x_f(t+\eta)) - W_f(x_f(t))]\mathrm{e}^{-\eta}\mathrm{d}\eta,$$

i.e.

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(7.1)} < 0 \quad \text{for} \ \ x \neq 0.$$

Therefore, we conclude

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(7.1)} \leq -u_f(x) \quad \text{for } x \neq 0$$

with $u_f(x)$ being a positive definite function. Thus, we have

$$\begin{aligned}
u_f(x) \;&\geq\; \tilde{\varphi}_f(\|x\|) := \tilde{\varphi}_f\left(\sum_{i=1}^{n}|x_i|\right)\\
&\geq\; \tilde{\varphi}_f\left(\frac{1}{m}\sum_{i=1}^{n}\frac{1}{\max\limits_{1\leq i\leq n,1\leq j\leq m}|c_{ij}|}\sum_{i,j=1}|c_{ij}x_i|\right)\\
&\geq\; \tilde{\varphi}_f\left(\frac{1}{m}\frac{1}{\max\limits_{1\leq i\leq n,1\leq j\leq m}|c_{ij}|}\sum_{j=1}^{m}|c_{ij}x_i|\right)\\
&:= \varphi_f(\sigma) \in KR.
\end{aligned}$$

Hence, $u_f(x)$ is radially unbounded, positive definite for Ω. Further, we have

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(7.1)} \leq -u_f(x) \leq -\tilde{\varphi}_f(\|x\|) \leq -\varphi_f(\sigma), \quad \varphi_f \in K.$$

The condition (2) of Theorem 7.5 is satisfied; the condition (2) of this theorem is trivial. The necessity is proved. □

Theorem 7.6. *The zero solution of (7.1) is absolutely stable if and only if*

1. *The condition (1) of Theorem 7.5 is satisfied*
2. *For any $f_j \in F$ $(j = 1,\ldots,m)$, there exist m Lyapunov fictions $V_f^{(j)}(x) \in [R^n, R^1]$ $(j = 1,\ldots,m)$ such that*

$$\begin{aligned}
&V_f^{(j)}(x) \geq \varphi_f^{(j)}(|\sigma_j|), \quad \varphi_f^{(j)} \in KR, \quad j = 1,\ldots,m,\\
&\left.\frac{\mathrm{d}V_f^{(j)}}{\mathrm{d}t}\right|_{(7.1)} \leq -\psi_f^{(j)}(|\sigma_j|), \quad \psi_f^{(j)} \in K, \quad j = 1,\ldots,m.
\end{aligned}$$

Proof. *Necessity*. Theorem 7.5 guarantees that the condition (1) is satisfied, and that there exists $V_f(x) \geq \varphi_f(\sigma) \in KR$ such that

$$\left.\frac{\mathrm{d}V_f(x)}{\mathrm{d}t}\right|_{(7.1)} \leq -\psi_f(\sigma), \quad \varphi_f \in K.$$

Take $V_f^{(j)} = V_f(x),\ j = 1,\ldots,m$. Due to

$$\begin{aligned}
V_f^{(j)} &= V_f(x) \geq \varphi_f(\sigma) = \varphi_f\Big(\sum_{j=1}^{m}|c_j^{\mathrm{T}}x|\Big) \geq \varphi_f(|c_j^{\mathrm{T}}x|)\\
&= \varphi_f(|\sigma_j|) \in KR, \quad j = 1,\ldots,m,
\end{aligned}$$

we have

$$\left.\frac{\mathrm{d}V_f^{(j)}}{\mathrm{d}t}\right|_{(7.1)} = \left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(7.1)} \leq -\psi_f(\sigma) = -\psi_f\left(\sum_{j=1}^{m} |c_j^{\mathrm{T}}x|\right)$$
$$\leq -\psi_f\left(|c_j^{\mathrm{T}}x|\right) = \psi_j(\sigma_j),\ \psi_j \in K,\ j = 1,\dots,m.$$

This verifies the necessity.

Sufficiency. Similar to the proof of sufficiency for Theorem 7.5, the condition (2) can be proved to imply that the zero solution of (7.1) is absolutely stable for $\Omega_j,\ j = 1,\dots,m$. Thus, as in Theorem 7.4 one can show that the zero solution of (7.1) is absolutely stable. This verifies the sufficiency. □

Theorem 7.7. *Let the following conditions be satisfied:*

1. *The condition (1) of Theorem 7.4 holds.*
2. *There exist an $n \times n$ real symmetric matrix B and constants $\beta_i \geq 0$ $(i = 1,\dots,m)$, $\alpha > 0$ such that*

$$V(x) = x^{\mathrm{T}}Bx + \sum_{j=1}^{m} \beta_i \int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j$$

with

$$x^{\mathrm{T}}Bx \geq \alpha \sum_{i=1}^{m} \sigma_i^2$$

or

$$V(x) = \sum_{j=1}^{m} \beta_i \int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j,\ \beta_j > 0,\ \int_0^{+\infty} f(\sigma_j)\mathrm{d}\sigma_j = +\infty,\ j = 1,\dots,m.$$

3. $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(7.1)} \leq -\varepsilon\tau,\ \tau \in \left\{\sigma^2, \sum_{j=1}^{m} \sigma_j f_j(\sigma_j), \sum_{j=1}^{m} f_j^2(\sigma_j)\right\}.$

Then the zero solution of (7.1) is absolutely stable.

Proof. It is suffice to prove that the conditions (2) and (3) of this theorem imply Theorem 7.5.

In fact, by the Lyapunov function

$$V(x) = x^{\mathrm{T}}Bx + \sum_{j=1}^{m} \beta_i \int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j,$$

the condition (2) implies that

$$V(x) \geq \alpha \sum_{j=1}^{m} \sigma_j^2 + \sum_{j=1}^{m} \beta_i \int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j = \varphi(\sigma) \in KR$$

or

$$V(x) \geq \sum_{j=1}^{m} \beta_i \int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j := \varphi(\sigma) \in KR,$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}t}|_{(7.1)} = -\varepsilon\tau := -\psi(\sigma),\ \psi \in K.$$

Therefore the conditions of Theorem 7.5 are satisfied. Theorem 7.7 is proved. □

Corollary 7.8. *Suppose that there exit constant $\beta_j \geq 0$ $(j = 1,\dots,m)$ and a symmetric positive matrix P such that the function:*

$$V(x) = x^{\mathrm{T}}Px + \sum_{j=1}^{n} \beta_j \int_0^{\sigma_i} f_j(\sigma_j)\,\mathrm{d}\sigma_j$$

satisfies $\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(7.1)} < 0$ *for* $x \neq 0$. *Then the zero solution of (7.1) is absolutely stable.*

Proof. It is suffice to prove that $V(x)$ is radially unbounded, positive definite for Ω, while $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(7.1)}$ is negative definite for Ω.

Let $\bar{c} = \max\limits_{1\leq i\leq n, 1\leq i\leq n} |c_{ij}|$, $\underline{\lambda} = \min\limits_{1\leq i\leq n} \lambda_i(P)$, and λ_i be eigenvalues of P. Then

$$V(x) \geq x^{\mathrm{T}}Px \geq \underline{\lambda}x^{\mathrm{T}}x \geq \underline{\lambda}\,\frac{\sum_{j=1}^{m} n\sum_{i=1}^{n} |c_{ij}x_i|}{mn\bar{c}} \geq \underline{\lambda}\frac{\sum_{j=1}^{m} \sigma_i^2}{mn\bar{c}} := \varphi(|\sigma|) \in KR.$$

Therefore, $V(x)$ is radially unbounded, positive definite for Ω, and

$$\frac{\mathrm{d}V}{\mathrm{d}t}|_{(7.1)} \leq -\psi(\|x\|) \leq -\psi\Big(\frac{1}{m\bar{c}}\sum_{j=1}^{m}|\sigma_j|\Big) := -\psi_1(\sigma),\ \psi_1 \in K.$$

Thus $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(7.1)}$ is negative definite for Ω. The conditions of Theorem 7.7 are satisfied. The proof of Corollary 7.8 is complete. □

7.2 Some Simple Sufficient Conditions for Absolute Stability

Without loss of generality, we assume that $c_i = (c_{i1},\dots,c_{im})^{\mathrm{T}}$ $(i = 1,\dots,m)$ are linearly independent. by an n-dimensional full-rank linear transformation, (7.1) can be transformed into the following form:

$$\dot{x} = Ax + \sum_{j=m+1}^{n} b_j f_j(x_j), \tag{7.3}$$

or into the vector component form:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=m+1}^{n} b_j f_j(x_j). \tag{7.4}$$

Theorem 7.9. *Suppose that*

1. $A = (a_{ij})_{n\times n}$ *is a Hurwitz matrix,*
2. There exist constant $r_j \geq 0$ $(j = 1,\dots,m)$, $r_j > 0$ $(j = m+1,\dots,n)$ *such that*

$$\begin{cases} -r_j a_{jj} + \sum\limits_{i=1,i\neq j}^{n} r_i |a_{ij}| \leq 0, & j = 1,\dots,m, \\ -r_j a_{jj} + \sum\limits_{i=1,i\neq j}^{n} r_i |a_{ij}| < 0, & j = m+1,\dots,n, \\ -r_j b_{jj} + \sum\limits_{i=1,i\neq j}^{n} r_i |b_{ij}| \leq 0, & j = m+1,\dots,n; \end{cases}$$

or

$$\begin{cases} -r_j a_{jj} + \sum\limits_{i=1,i\neq j}^{n} r_i |a_{ij}| \leq 0, & j = 1,\dots,m, \\ -r_j a_{jj} + \sum\limits_{i=1,i\neq j}^{n} r_i |a_{ij}| \leq 0, & j = m+1,\dots,n, \\ -r_j b_{jj} + \sum\limits_{i=1,i\neq j}^{n} r_i |b_{ij}| < 0, & j = m+1,\dots,n. \end{cases}$$

Then the zero solution of the system (7.3) is absolutely stable.

Proof. We construct the Lyapunov function:

$$V = \sum_{i=1}^{n} r_i |x_i|.$$

Obviously, we have

$$V = \sum_{i=1}^{n} r_i |x_i| \geq \sum_{i=m+1}^{n} r_i |x_i| \to +\infty \text{ as } \quad \sum_{i=m+1}^{n} |x_i| \to +\infty.$$

Thus, V is radially unbounded, positive definite for $x_{n-m+1},\dots,x_n$. Since

$$\begin{aligned} D^+V(x)|_{(7.3)} \leq & \sum_{j=1}^{n} \left[r_j a_{jj} + \sum_{i=1,i\neq j}^{n} r_i |a_{ij}| \right] |x_j| \\ & + \sum_{l=m+1}^{n} \left[r_l b_{ll} + \sum_{i=1,i\neq l}^{n} |r_i b_{il}| \right] |f_l(x_l)| \\ & < 0 \quad \text{for} \quad \sum_{j=m+1}^{n} |x_j| \neq 0, \end{aligned}$$

the zero solution of (7.3) is absolutely stable w.r.t. $x_{m+1},\dots,x_n$. Since the matrix A is Hurwitz stable, there exit $M \geq 1$ and $\alpha > 0$ such that

$$\|e^{A(t-t_0)}\| \leq M e^{-\alpha(t-t_0)}.$$

The solution of (7.3) can be expressed in the following form:

$$x(t,t_0,x_0) = \mathrm{e}^{A(t-t_0)x_0} + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} \sum_{j=m+1}^{n} b_j f_j(x_j(\tau))\mathrm{d}\tau.$$

Following the proof of sufficiency for Theorem 7.4, we can prove that the zero solution of (7.3) is absolutely stable. □

Let us denote

$$\tilde{a}_{ij} = \begin{cases} a_{ij},\ i=1,\dots,n, & j=1,\dots,m,\\ b_{ij},\ i=1,\dots,n, & j=m+1,\dots,n. \end{cases}$$

Theorem 7.10. *The zero solution of (7.3) is absolutely stable if either of the following two sets of conditions is satisfied:*

1. $a_{ii} < 0, i=1,\dots,n,$ *the matrix*$((-1)^{\delta_{ij}}|a_{ij}|)_{n\times n}$ *is Hurwitz stable*
2. There exits a constant $k>0$ *such that*

$$\begin{cases} kb_{ll} < a_{ll}, & l=m+1,\dots,n,\\ k|b_{il}| \le a_{il}, & i=1,\dots,n,\ i\ne l,\quad l=m+1,\dots,n. \end{cases}$$

Proof. Since the condition (1) implies that $-((-1)^{\delta_{ij}}|a_{ij}|)_{n\times n}$ is an M-matrix, there exist constant $r_i>0$, $(j=1,\dots,n)$ such that

$$r_j a_{jj} + \sum_{i=1,i\ne j}^{n} |r_i a_{ij}| < 0, \quad j=1,\dots,n.$$

Condition 2 implies that

$$r_l b_{ll} + \sum_{i=1,i\ne l}^{n} |r_i b_{il}| \le \frac{1}{k}\left(r_l a_{ll} + \sum_{i=1}^{n} r_i |a_{il}|\right) < 0 \quad l=m+1,\dots,n.$$

We construct the radially unbounded, positive definite Lyapunov function:

$$V = \sum_{i=1}^{n} r_i |x_i|$$

and we deduce that

$$\begin{aligned} D^+V|_{(7.3)} \le & \sum_{j=1}^{n}\left[r_j a_{jj} + \sum_{i=1,i\ne j}^{n} r_i|a_{ij}|\right]|x_j| \\ & + \sum_{l=m+1}^{n}\left[r_l b_{ll} + \sum_{i=1,i\ne l}^{n} |r_i b_{il}|\right]|f_l(x_l)| \\ < & \ 0 \qquad \text{for } x \ne 0, \end{aligned}$$

Consequently, the zero solution of (7.3) is absolutely stable. □

Example 7.11. Consider the absolute stability of the following system:

$$\begin{aligned} \dot{x}_1 &= -x_1 - 2f_1(x_1) + 2f_2(x_2), \\ \dot{x}_2 &= -x_2 + 2f_1(x_1) - 2f_2(x_2), \end{aligned} \qquad f_1, f_2, \in F_\infty. \tag{7.5}$$

We choose the radially unbounded, positive definite Lyapunov function:

$$V = |x_1| + |x_2|.$$

Thus,

$$\begin{aligned} D^+V|_{(7.5)} &\leq -|x_1| - |x_2| + [-2+2]|f_1(x_1)| \\ &\quad + (-2+2)\,|f_2(x_2)| \\ &\leq -|x_1| - |x_2| < 0 \quad \text{for } x \neq 0 \end{aligned}$$

and the zero solution of (7.5) is absolutely stable.

Example 7.12. Consider the absolute stability of the system:

$$\begin{aligned} \dot{x}_1 &= -4x_1 + 2x_2 + x_3 + 2f_2(x_2) + 2f_3(x_3), \\ \dot{x}_2 &= 2x_1 - 3x_2 + x_3 - 4f_2(x_2) - 2f_3(x_3), \\ \dot{x}_3 &= -2x_1 + x_2 - 3x_3 + f_2(x_2) - 7f_3(x_3), \end{aligned} \tag{7.6}$$

where $f_1, f_2 \in F_\infty$, and

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -4 & 2 & 1 \\ 2 & -3 & 1 \\ -2 & 1 & -3 \end{bmatrix},$$

$$\begin{bmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -2 \\ 1 & -7 \end{bmatrix}.$$

It is easy to prove that the matrix $((-1)^{\delta_{ij}}|a_{ij}|)$ is stable. Furthermore, notice that

$$0 > a_{22} > b_{22}, \quad |a_{i2}| = |b_{i2}|, \quad i = 1, 3,$$

$$|a_{i3}| = \frac{1}{2}|b_{i3}|, \quad i = 1, 2, \qquad \text{by taking } k = \frac{1}{2}.$$

Thus, the condition of Theorem 7.10 are satisfied. The zero solution of (7.6) is absolutely stable.

Let

$$\tilde{f}_{ij}(x_j) = \begin{cases} a_{ij}x_j + b_{ij}f(x_j), & i = 1, \ldots, n, \quad j = m+1, \ldots, n \\ a_{ij}x_j, & i = 1, \ldots, n, \quad j = 1, \ldots, m \end{cases}$$

System (7.4) can be written as

$$\dot{x}_i = \sum_{j=1}^{n} \tilde{f}_{ij}(x_j), \quad i = 1, \ldots, n. \tag{7.7}$$

Theorem 7.13. *Let the following conditions be satisfied:*

1. $A = (a_{ij})_{n\times n}$ *is Hurwitz stable*
2. $\int_0^{\pm\infty} \tilde{f}_{ii}(x_i)dx_i = -\infty,\ \tilde{f}_{ii}(x_i)x_i < 0 \ \ for\ \ x_i \neq 0,\ \ i = 1,\dots,n$
3. *There exist constants* $c_i \geq 0$ $(i = 1,\dots,m)$, $c_i > 0$ $(i = m+1,\dots,n)$, $\varepsilon > 0$, *such that the matrix*

$$G(g_{ij})_{n\times n} + \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon E_{(n-m)\times(n-m)} \end{bmatrix}$$

is negative semi-definite, where

$$g_{ij}(x) = \begin{cases} -\dfrac{1}{2}\Big(\dfrac{c_{ij}\tilde{f}_{ij}(x_j)}{\tilde{f}_{jj}(x_j)} + \dfrac{c_j\tilde{f}_{ji}(x_i)}{\tilde{f}_{ii}(x_i)}\Big), & x_ix_j \neq 0, \\ 0, & x_ix_j = 0, \end{cases} \qquad i,\, j = 1,\dots,n$$

Then the zero solution of (7.7) is absolutely stable.

Proof. We construct the radially unbounded, positive definite Lyapunov function

$$V(x) = -\sum_{i=1}^{n} \int_0^{x_i} c_i\tilde{f}_{ii}(x_i)\mathrm{d}x_i.$$

Following the proof of Theorem 7.4, we can show that $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(7.7)}$ is negative definite for $x_{m+1},\dots,x_n$. Thus, the zero solution of (7.7) is absolutely stable w.r.t. $x_{m+1},\dots,x_n$, i.e., the zero solution of (7.3) is absolutely stable for $x_{m+1},\dots,x_n$.

Using the fact that the matrix A is Hurwitz stable, the solution of (7.3) can be expressed as

$$x(t,t_0,x_0) = \mathrm{e}^{A(t-t_0)}x_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} \sum_{j=m+1}^{n} b_jf_j(x_j(\tau))\mathrm{d}\tau.$$

As in the proof of Theorem 7.4, we can prove that the zero solution of (7.3) is absolutely stable. Consequently, the zero solution of (7.7) is absolutely stable. □

Theorem 7.14. *Let the following conditions be satisfied:*

1. *The conditions (1) and (2) of Theorem 7.5 are satisfied*
2. $\left|\frac{\tilde{f}_{ij}(x_j)}{\tilde{f}_{jj}(x_j)}\right| \leq b_{ij}$
3.

$$\tilde{B} = \begin{bmatrix} 1 & -b_{21} & \cdots & -b_{n1} \\ -b_{12} & 1 & \cdots & -b_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ -b_{1n} & -b_{2n} & \cdots & 1 \end{bmatrix} := \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}$$

with $\tilde{B}_{11}$, $\tilde{B}_{22}$, *and* $I - \tilde{B}_{11}^{-1}\tilde{B}_{12}\tilde{B}_{22}^{-1}\tilde{B}_{21}$ *being M-matrices, where* $\tilde{B}_{11}$, $\tilde{B}_{12}$, $\tilde{B}_{21}$, *and* $\tilde{B}_{22}$ *are* $m\times m$, $m\times(n-m)$, $(n-m)\times m$ *and* $(n-m)\times(n-m)$ *matrices, respectively.*

Then the zero solution of (7.7) is absolutely stable.

Proof. For any $\xi=(\xi_1,\dots,\xi_m)^{\mathrm T}\geq 0$ and any $\eta=(\eta_1,\dots,\eta_{n-m})^{\mathrm T}>0$, we consider the following equations:

$$\begin{aligned}\tilde B_{11}c+\tilde B_{12}\tilde c&=\xi,\\ \tilde B_{21}c+\tilde B_{22}\tilde c&=\eta.\end{aligned}\tag{7.8}$$

Obviously, (7.8) is equivalent to

$$\tilde c=-\tilde B_{22}^{-1}\tilde B_{21}c+\tilde B_{22}^{-1}\eta,\tag{7.9}$$

$$c=\tilde B_{11}^{-1}\xi+\tilde B_{11}^{-1}\tilde B_{12}\tilde B_{22}^{-1}\tilde B_{21}c-\tilde B_{11}^{-1}\tilde B_{12}\tilde B_{22}^{-1}\eta.\tag{7.10}$$

Since $\eta\geq 0$, $\tilde B_{11}$, and $\tilde B_{22}$ are M-matrices, we have $\tilde B_{11}^{-1}\geq 0$, $\tilde B_{22}^{-1}\geq 0$, $\tilde B_{12}\leq 0$, and

$$-\tilde B_{11}^{-1}\tilde B_{12}\tilde B_{22}^{-1}\eta\geq 0,\quad \tilde B_{11}^{-1}\xi>0.$$

Recalling that $I-\tilde B_{11}^{-1}\tilde B_{12}\tilde B_{22}^{-1}\tilde B_{21}$ is an M-matrix, we know that there exist a positive solution of (7.10) for $\tilde c$ and a nonnegative solution of (7.9) for c.

We choose the Lyapunov function

$$V=\sum_{i=1}^{n}c_i|x_i|\geq\sum_{i=m+1}^{n}c_i|x_i|.$$

Then

$$\begin{aligned}D^+V|_{(7.7)}&\leq\sum_{j=1}^{n}\left[-c_j|\tilde f_{jj}(x_j)|+\sum_{i=1,i\neq j}^{n}c_i|\tilde f_{ij}|\right]|x_j|\\ &\leq\sum_{j=m+1}^{n}\left[-c_j|\tilde f_{jj}(x_j)|+\sum_{i=1,i\neq j}^{n}c_i|\tilde f_{ij}|\right]|x_j|\\ &\leq\sum_{j=m+1}^{n}\left[-c_j+\sum_{i=1,i\neq j}^{n}c_ib_{ij}]|x_j|\right]|\tilde f_{jj}(x_j)|\\ &<0\qquad\text{for}\sum_{j=n-m+1}^{n}|f_{jj}(x_j)|\neq 0.\end{aligned}$$

Therefore, the zero solution of system (7.7) is absolutely stale w.r.t. x_{n-m+j} $(j=1,\dots,m)$, i.e., the zero solution of system (7.3) is absolutely stable w.r.t. x_{n-m+j} $(j=1,\dots,m)$. Following the proof of Theorem 7.5, we can complete the proof of the remaining part of this theorem. □

Example 7.15. We examine the absolute stability of the following system:

$$\begin{aligned}\dot x_1&=-x_1-\frac{1}{2}x_2-f_1(x_1)-\frac{1}{2}f_2(x_2),\\ \dot x_2&=\frac{1}{2}x_1-x_2+\frac{1}{2}f_1(x_1)-f_2(x_2),\end{aligned}\tag{7.11}$$

where $f_1,f_2\in F_\infty$.

Let

$$\tilde{f}_{11}(x_1) = -x_1 - f_1(x_1), \qquad f_{12}(x_2) = -\frac{1}{2}x_2 - \frac{1}{2}f_2(x_2),$$
$$\tilde{f}_{21}(x_1) = \frac{1}{2}(x_1 + f_1(x_1)), \quad f_{22}(x_2) = -x_2 - f_2(x_2).$$

We choose the positive definite and radially unbounded Lyapunov function

$$V(x) = -\int_0^{x_1} \tilde{f}_{11}(x_1)\mathrm{d}x_1 - \int_0^{x_2} \tilde{f}_{22}(x_2)\mathrm{d}x_2.$$

It follows that:

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7.11)} = -\tilde{f}_{11}^2(x_1) - \tilde{f}_{22}^2(x_2) < 0 \qquad \text{for } x_1^2 + x_2^2 \neq 0.$$

Therefore, the zero solution of (7.11) is absolutely stable.

7.3 Special Systems

Consider the following special system:

$$\begin{aligned} \dot{x} &= Ax + Bf(\sigma), \\ \sigma &= C^{\mathrm{T}}x, \end{aligned} \tag{7.12}$$

where $A \in \mathrm{R}^{n\times n}, x \in \mathrm{R}^n$, B and C are $n \times m$ constant matrices, σ and f are m-dimensional vectors, and $f(\sigma)$ satisfies:

$$f(0) = 0 \quad \text{and} \quad \sigma^{\mathrm{T}}K_1\sigma \leq \sigma^{\mathrm{T}}K_3 f(\sigma) \leq \sigma^{\mathrm{T}}K_2\sigma, \tag{7.13}$$

where K_1, K_2, and K_3 are $m \times m$ symmetric positive definite constant matrices.

Remark 7.16.

1. Let $\varphi(\sigma) = K_3 f(\sigma) - K_1\sigma$, $K = K_2 - K_1$, then (7.13) becomes

$$\varphi(0) = 0, \quad 0 < \sigma^{\mathrm{T}}\varphi(\sigma) \leq \sigma^{\mathrm{T}}K\sigma.$$

2. If K_3 is a nonsingular matrix, then using the transformation, we have

$$f(\sigma) = K_3^{-1}K_1\sigma + K_3^{-1}\varphi(\sigma).$$

System (7.12) is then transformed into

$$\begin{aligned} \dot{x} &= \tilde{A}x + \tilde{B}f(\sigma), \\ \sigma &= C^{\mathrm{T}}x, \end{aligned} \tag{7.14}$$

where $\tilde{A} = A + BK_3^{-1}K_1C^{\mathrm{T}}$ and $\tilde{B} = BK_3^{-1}$.

Theorem 7.17. *Assume that*

1. *The matrix $\tilde{A}$ is stable;*
2. *There exists a positive definite matrix $\tilde{G}$ such that the solution of the Lyapunov matrix equation:*
$$\tilde{A}^{\mathrm{T}}\tilde{H}+\tilde{H}\tilde{A}=-2\tilde{G} \tag{7.15}$$
satisfies
$$C=-\tilde{H}\tilde{B} \tag{7.16}$$

Then the zero solution of (7.14) is absolutely stable.

Proof. Since $\tilde{A}$ is stable, the Lyapunov matrix equation (7.15) has only a symmetric and positive definite solution $\tilde{H}$.

Constructing the Lyapunov function:

$$V(x)=x^{\mathrm{T}}\tilde{H}x$$

which is radically unbounded, positive definite. We deduce

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7.14)}=-2x^{\mathrm{T}}\tilde{G}x+2x^{\mathrm{T}}\tilde{H}\tilde{B}\varphi(\sigma)=-2x^{\mathrm{T}}\tilde{G}x-2\sigma^{\mathrm{T}}\varphi(\sigma)<0 \quad \text{for } x\neq 0.$$

Therefore, the zero solution of (7.14) is absolutely stable. □

The significance of Theorem 7.4 lies in the fact that when $\tilde{A}$ is Hurwitz stable, we can first construct a positive definite matrix $\tilde{H}$, by choosing the control matrix C such that $\tilde{H}$ satisfies (7.16). This makes an easy design for stable systems.

Theorem 7.18. *Assume that*

1. *The matrix $\tilde{A}$ is stable*
2. $\varphi^{\mathrm{T}}K^{-1}\varphi\leq\varphi^{\mathrm{T}}\sigma$
3. *There exists a constant $\tau>0$ such that*
$$\begin{bmatrix} 2\tilde{G} & -Q \\ -Q^{\mathrm{T}} & \tau K^{-1} \end{bmatrix}$$
is positive definite, where $\tilde{G}$ is a positive definite matrix defined by (7.15), and $Q=\tilde{H}\tilde{B}+\frac{1}{2}\tau C$.

Then the zero solution of system (7.15) is absolutely stable.

Proof. We construct the Lyapunov function in the positive definite quadratic form:

$$V(x)=x^{\mathrm{T}}\tilde{H}x,$$

where $\tilde{H}$ denotes a solution of (7.15). Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7.14)}=-2x^{\mathrm{T}}\tilde{G}x+2x^{\mathrm{T}}\tilde{H}\tilde{B}\varphi(\sigma).$$

Using the S-process, we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7.14)} &= -2x^{\mathrm{T}}\tilde{G}x + 2x^{\mathrm{T}}Q\varphi - \tau\varphi^{\mathrm{T}}K^{-1}\varphi \\ &= -\begin{pmatrix} x \\ \varphi \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2\tilde{G} & -Q \\ -Q^{\mathrm{T}} & \tau K^{-1} \end{bmatrix} \begin{pmatrix} x \\ \varphi \end{pmatrix} \\ &< 0 \qquad \text{for } x \neq 0.\end{aligned}$$

This completes the proof of Theorem 6.3.2. □

7.4 Nonautonomous Systems

Consider the m-dimensional nonautonomous and nonlinear control system:

$$\begin{aligned}\dot{y} &= \tilde{A}(t)y + \sum_{j=m+1}^{n} \tilde{b}_j(t) f_j(\sigma_j, t),\ 1 \leq m < n, \\ \sigma_j &= c_j^{\mathrm{T}} y,\end{aligned} \tag{7.17}$$

where

$$\begin{aligned} f_j \in F_{[0,k_j]} := \Big\{ f \,\Big|\, & f(0) = 0, 0 \leq \sigma_j f(\sigma_j, t) \leq k_j \sigma_j^2,\ k_j < +\infty, \\ & f \in C[(-\infty, +\infty) \times [0, +\infty), \mathrm{R}] \Big\},\ j = m+1, \ldots, n, \end{aligned}$$

$A(t) = (a_{ij}(t))_{n\times n}$ is continuous matrix function matrix on $[0, +\infty)$, and

$$b_j(t) = (b_{j1}(t), \ldots, b_{jn}(t)) \in C[0, +\infty)^{\mathrm{T}}, \mathrm{R}^n],$$

$C_j(t) = (c_{j1}(t), \ldots, c_{jn}(t))^{\mathrm{T}}$ is a constant vector. Suppose that $c_j\,(j = m+1, \ldots, n)$ are linearly independent. Without loss of generality, we assume that

$$\det \begin{bmatrix} c_{m+1,m+1} & \cdots & c_{m+1,n} \\ \vdots & & \vdots \\ c_{m,m+1} & \cdots & c_{nn} \end{bmatrix} \neq 0.$$

By the following nonsingular linear transformation:

$$x = Qy,$$

i.e.,

$$\begin{aligned} x_i &= y_i, \quad i = 1, \ldots, m, \\ x_i &= \sigma_i, \quad i = m+1, \ldots, n. \end{aligned}$$

System (7.17) can be transformed into

$$\dot{x} = A(t)x + \sum_{j=m+1}^{n} b_j(t) f_j(x_j, t),$$

where

$$A(t) = Q^{-1}\tilde{A}(t)Q, \quad B_j(t) = Q^{-1}\tilde{b}_j(t),$$

or, in the vector component form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j + \sum_{j=m+1}^{n} b_{ij}(t) f_j(x_j, t), \quad i = 1, \dots, n \tag{7.18}$$

Theorem 7.19. *Assume that there exist constant $r_i \geq 0$ $(i = 1, \dots, m)$ and $r_i > 0$ $(i = m+1, \dots, n)$, $\delta > 0$ such that*

$$\begin{cases} -r_j a_{jj}(t) + \sum_{i=1,\, j\neq i}^{n} r_i |a_{ij}(t)| \leq 0, & j = 1, \dots, m, \\ -r_j a_{jj}(t) + \sum_{i=1,\, j\neq j}^{n} r_i |a_{ij}(t)| \leq -\delta < 0, & j = m+1, \dots, n, \\ -r_j b_{jj}(t) + \sum_{i=1,\, j\neq i}^{n} r_i |b_{ij}(t)| \leq 0, & j = m+1, \dots, n. \end{cases}$$

Let the zero solution of the linear system

$$\dot{x}_i = \sum_{j=1}^{m} a_{ij}(t)x_j, \quad i = 1, \dots, m \tag{7.19}$$

be uniformly asymptotically stable.

Then the zero solution of system (7.18) is absolutely stable in

$$K = \operatorname{diag}\Big([0, k_1], \dots, [0, k_m]\Big).$$

Proof. We choose the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} c_i |x_i|,$$

which is radially unbounded, positive definite w.r.t. $x_{m+1}, \dots, x_n$. Then

$$\begin{aligned} D^+V(x)|_{(7.18)} &\leq \sum_{j=1}^{n} \left[r_j a_{jj}(t) + \sum_{i=1, i\neq j}^{n} r_i |a_{ij}(t)| \right] |x_j| \\ &\quad + \sum_{j=m+1}^{n} \left[r_j b_{jj}(t) + \sum_{i=1, i\neq j}^{n} r_i |b_{ij}(t)| \right] |f_j(x_j, t)| \\ &\leq \sum_{j=m+1}^{n} \left[r_j a_{jj}(t) + \sum_{i=1, i\neq j}^{n} r_i |a_{ij}(t)| \right] |x_j| \\ &\leq -\delta \sum_{j=m+1}^{n} |x_j|. \end{aligned}$$

Thus, $D^+V|_{(7.18)}$ is negative definite w.r.t. $x_{m+1},\dots,x_n$, and the zero solution of (7.18) is absolutely stable w.r.t. $x_{m+1},\dots,x_n$ in $K = \mathrm{diag}([0,k_1],\dots,[0,k_m])$.

The Cauchy matrix solution $K(t,t_0)$ of (7.19) satisfies

$$\|K(t,t_0)\| \le M\mathrm{e}^{-\alpha(t-t_0)}.$$

Let

$$\begin{aligned} x^{(n-m)} &= (x_{m+1},\dots,x_n)^{\mathrm{T}}, \\ A^{(n-m)}(t) &= (a_{ij}(t))_{(m)\times(n-m)}, \\ b_j^{(n-m)}(t) &= (b_{m+1,j}(t),\dots,b_{n,j}(t))^{\mathrm{T}}. \end{aligned}$$

Then, the first $n-m$ components of the solution of (7.18) can be written as

$$\begin{aligned} x^{(n-m)}(t,t_0,x_0) = \ & K(t,t_0)x_0^{(n-m)} + \int_{t_0}^{t} K(t,\tau)A^{(n-m)}(\tau)x^{(n-m)}(\tau)\mathrm{d}\tau \\ & + \int_{t_0}^{t} K(t,\tau)\sum_{j=m+1}^{n} b_j^{(n-m)}(\tau)f_j(x_j(\tau),\tau)\mathrm{d}\tau. \end{aligned}$$

Since $0 \le f_j(x_j,t)x_j \le kx^2$, we have $f_j(x_j(t),t) \to 0$ as $x_j \to 0$ uniformly in t.

Following the proof of Theorem 4.3, we can prove that the zero solution of (7.18) is absolutely stable w.r.t. $x^{(n-m)}$ in $K = \mathrm{diag}([0,k_1],\dots,[0,k_m])$. Theorem 7.19 is proved. $\square$

Theorem 7.20. *1. Suppose that there exist constants $r_i \ge (i = 1,\dots,m)$, $r_i > 0$ $(i = m+1,\dots,n)$ $\delta > 0$ such that*

$$\begin{cases} -r_j a_{jj}(t) + \displaystyle\sum_{i=1,i\neq j}^{n} r_i|a_{ij}(t)| \le 0, & j = 1,\dots,m, \\ -r_j b_{jj}(t) + \displaystyle\sum_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}(t)| \le -\delta < 0, & j = m+1,\dots,n, \end{cases} \qquad f_1, f_2 \in F_\infty$$

2. Let the condition (2) of Theorem 7.19 be satisfied

3. Let $|f_i(x_i,t)|$ be positive definite, $j = 1,\dots,m$.

Then the zero solution of the system (7.18) is absolutely stable in

$$K = \mathrm{diag}\Big([0,k_1],\dots,[0,k_m]\Big).$$

Proof. We choose the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} r_i|x_i|,$$

which is radially unbounded, positive definite for $x_{n-m+1},\dots,x_n$. We find

$$\begin{aligned} D^+V(x)|_{(7.18)} &\le \sum_{j=m+1}^{n}\left[r_j b_{jj}(t)+\sum_{i=1,i\neq j}^{n} r_i|b_{ij}(t)|\right]|f_i(\sigma_j,t)| \\ &< 0 \quad \text{for} \quad x^{(n-m)}\neq 0. \end{aligned}$$

Thus, the zero solution of (7.18) is absolutely stable w.r.t. x^{n-m}. The rest of the proof is exactly the same as in Theorem 7.19 and we do not repeat it here. □

Example 7.21. Examine the absolute stability of the following system:

$$\begin{aligned} \dot{x}_1 &= (-4+\sin t)x_1+(\sin t)x_2+(\frac{1}{2}\cos t)x_3+f_2(x_2)+(1+\sin t)f_3(x_3), \\ \dot{x}_2 &= (2\cos t)x_1-3x_2-\frac{t}{1+t^2}x_3-2f_2(x_2)+2f_3(x_3), \\ \dot{x}_3 &= \frac{t}{1+t^2}x_1-(\cos t)x_2-2x_3-f_2(x_2)-4f_3(x_3), \end{aligned} \tag{7.20}$$

where $f_2\in F_{[0,k_2]}$, $f_3\in F_{[0,k_3]}$.

Proof. Choosing the radially unbounded and positive definite Lyapunov function:

$$V=|x_1|+|x_2|+|x_3|,$$

we deduce that

$$\begin{aligned} D^+V(x)|_{(7.20)} \le & \left[(-4+\sin t)+|2\cos t|+\frac{1}{1+t^2}\right]|x_1| \\ &+[-3+|\sin t|+|\cos t|]\,|x_2| \\ &+\left[-2+\frac{1}{1+t^2}+\frac{1}{2}|\cos t|\right]|x_3|+(-2+1+1)\,|f_2(x_2)| \\ &+(-4+2+1+|\sin t|)\,|f_3(x_3)| \\ \le & -\frac{1}{2}|x_1|-|x_2|-|x_3| \qquad \text{for } t^2\ge 1, \end{aligned}$$

implying that the zero solution of system (7.20) is absolutely stable in

$$K=\operatorname{diag}\Big([0,k_1],[0,k_2]\Big).$$ □

Next, take

$$\tilde{f}_{ij}(x_j,t)=\begin{cases} a_{ij}(t)x_j+b_{ij}(t)f_j(t,x_j), & i=1,\dots,n, \quad j=n-m+1,\dots,n, \\ a_{ij}(t)x_j, & i=1,\dots,n, \quad j=1,\dots,n-m. \end{cases}$$

Then system (7.18) can be rewritten as

$$\dot{x}_i=\sum_{j=1}^{n}\tilde{f}_{ij}(x_j,t), \tag{7.21}$$

where $f_j\in F_{k_j}$, $j=1,\dots,n$.

Theorem 7.22. *Let the following conditions be satisfied:*

1. $\tilde{f}_{jj}(x_j,t)x_j < 0, j = 1,\ldots,n, x_j \neq 0$
2. *There exist function $F_{jj}(x_j)$ defined on $(-\infty,+\infty)$, $j = 1,\ldots,m$, which are continuous or have only finite discontinuous points of the first or third kind such that*
$$F_{jj}(x_j)x_j < 0 \quad for\ \ x_j \neq 0, \quad |F_{jj}(x_j)| \leq |f_{jj}(x_j,t)|,$$
$$\int_0^{\pm\infty} F_{jj}(x_j)\mathrm{d}x_j = -\infty, \quad j = 1,\ldots,m;$$
3. *The matrix $G(g_{ij})_{n\times n}$ is negative definite, where*

$$g_{ij}\begin{cases} = -1, & i = 1,\ldots,n, \\ \geq \dfrac{1}{2}\left|\dfrac{\tilde{f}_{ij}(x_j,t)}{F_{jj}(x_j)} + \dfrac{\tilde{f}_{ji}(x_i,t)}{F_{ii}(x_i)}\right|, & i \neq j,\ x_ix_j \neq 0,\ i,j, = 1,\ldots,n, \\ =0, & i \neq j,\ x_ix_j = 0,\ i,j, = 1,\ldots,n. \end{cases}$$

Then the zero solution of system (7.21) is absolutely stable in

$$K = \mathrm{diag}\left([0,k_1],\ldots,[0,k_m]\right).$$

Proof. We choose the Lyapunov function

$$V(x) = -\sum_{i=1}^{n}\int_0^{x_i} F_{ii}(x_i)\mathrm{d}x_i.$$

Obviously, it is radically unbounded, positive definite.

Following the proof of Theorem 7.19, we can prove that $D^+V|_{(7.21)}$ is negative definite. Therefore, the zero solution of (7.21) is absolutely stable in $K = \mathrm{diag}([1,k_1],\ldots,[0,k_m])$. □

Theorem 7.23. *1. The condition (1) of Theorem 7.22 is satisfied*
2. $|\tilde{f}_{jj}(x_j,t)|$ *is positive definite,* $j = 1,\ldots,m$
3. $h\left|\frac{\tilde{f}_{ij}(x_j,t)}{\tilde{f}_{jj}(x_j,t)}\right| \leq \tilde{g}_{ij},\ i \neq j,\ i,\ j = 1,\ldots,n$ *and*

$$G = \begin{bmatrix} 1 & -\tilde{g}_{12} & \cdots & -\tilde{g}_{1n} \\ -\tilde{g}_{21} & 1 & \cdots & -\tilde{g}_{2n} \\ \vdots & \vdots & & \vdots \\ -\tilde{g}_{n1} & -\tilde{g}_{n2} & \cdots & 1 \end{bmatrix}$$

is an M-matrix.

Then the zero solution of the system (7.21) is absolutely stable in

$$K = \mathrm{diag}([0,k_1],\ldots,[0,k_m]).$$

Proof. Since G is an M-matrix, for any $\xi = (\xi_1, \ldots, \xi_n)^{\mathrm{T}} > 0$, the algebraic equation $G^{\mathrm{T}}\eta = \xi$ has a positive solution:

$$r = \eta = (G^{\mathrm{T}})^{-1}\xi > 0.$$

Constructing the Lyapunov function

$$V(x) = -\sum_{1}^{n} \left[\operatorname{sgn}\tilde{f}_{ii}(x_i,t)\right] r_i|x_i|,$$

we find

$$\begin{aligned} D^+V(x)|_{(7.21)} &\le \sum_{j=1}^{n} \left[-r_j|\tilde{f}_{jj}(x_j,t)| + \sum_{i=1,i\neq j}^{n} r_i\,|\tilde{f}_{ij}(x_j,t)| \right] \\ &\le \sum_{j=1}^{n} \left[-r_j + \sum_{i=1,i\neq j}^{n} r_i\tilde{g}_{ij} \right] |\tilde{f}_{jj}(x_j,t)| \\ &\le 0 \qquad \text{for } \ x \neq 0. \end{aligned}$$

Therefore, the zero solution of the system (7.21) is absolutely stable in $K = \operatorname{diag}([1,k_1],\ldots,[0,k_m])$. The theorem is proved. □

7.5 Lurie Systems with Multiple Nonlinear Loop Feedbacks

Consider a Lurie system with multiple nonlinear loop feedbacks:

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^{m} b_j f_j(\sigma_j), \\ \sigma_j &= c_j^{\mathrm{T}}x + d_j f_i(\sigma_j), \end{aligned} \tag{7.22}$$

where $A \in \mathrm{R}^{n\times n}$, $\mathrm{Re}\lambda(A) \le 0$, x, b_j, $c_j \in \mathrm{R}^n$, $d_j \le 0$, $j = 1,\ldots m$, and $f_j(\cdot) \in F_\infty$.

Similar to Sect. 7.1, we can define $\Omega_1,\ldots,\Omega_j$, $(j = 1,\ldots,m)$ and its corresponding Lyapunov functions. $\Omega_1,\ldots,\Omega_j$ are radially unbounded, positive definite.

Similarly, as in Sect. 7.1, we can define the absolute stability of the zero solution of (7.22) w.r.t. Ω, Ω_j. Here, we will give the absolute stability for the zero solution of (7.22) w.r.t. σ.

Definition 7.24. *The zero solution of (7.22) is said to be absolutely stable if* $\forall\, f_j(\cdot) \in F_\infty\,(j = 1,\ldots,m)$ $\forall \varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that when* $\|x_0\| < \delta(\varepsilon)$, $\|\sigma(\delta(\varepsilon))\| < \varepsilon$, *and* $\forall x_0 \in \mathrm{R}^n$, $\lim_{t\to\infty} \delta(t,t_0,x_0) = 0$, *where* σ_0 *depends on* x_0.

Since in (7.22), σ_j's are no longer given in explicit linear combinations of the state variables, but implicitly depend on the state variables, the positive definite, negative definite, and radially unboundedness for the corresponding Lyapunov functions can only be formally defined. They are not easy to verify. Noticing the property of $f_j(\sigma)$ and $d_i \le 0$, we have the following important property:

σ_j and $c_j^{\mathrm{T}}x$ have the same sign, and $\sigma_j \to 0$ if and only if $c_j^{\mathrm{T}}x \to 0$. Thus, do not need to distinguish the absolute stabilities w.r.t. the set Ω_j, $\sigma = 0$, or $\sigma_j = 0$. They can be employed and replaced with each other.

When we determine the sign and radial unboundedness of a Lyapunov function, we need the explicit of the state variables Ω, Ω_j. We also need the property that $\sigma \to 0$ as $t \to +\infty$ in proofs.

Theorem 7.25. *The necessary and sufficient conditions for the zero solution of system (7.22) being absolutely stable are given by*

1. $B = A + \sum_{j=1}^{m} \frac{\theta_j b_j c_j^{\mathrm{T}}}{1-d_j}$ *is a Hurwitz matrix, where* $\theta_j = 1$ *or* $\theta_j = 0$*,* $j = 1, 2, \ldots, m$*;*
2. *The zero solution of (7.22) is absolutely stable w.r.t.* Ω*.*

Theorem 7.26. *The necessary and sufficient conditions for the zero solution of the system (7.22) being absolutely stable are:*

1. *The condition (1) of Theorem 7.25 is satisfied*
2. *There exits a differentiable function* $V \in [\mathrm{R}^n, \mathrm{R}^1]$ *which is positive definite and radially unbounded w.r.t.* Ω *satisfying*

$$\begin{aligned} V &\geq \varphi(\|c^{\mathrm{T}}x\|), \quad \varphi \in KR, \\ \frac{\mathrm{d}V}{\mathrm{d}t} &\leq -\psi(\|c^{\mathrm{T}}x\|), \ \psi \in K. \end{aligned}$$

Theorem 7.27. *The necessary and sufficient conditions for system (7.22) being absolutely stable are:*

1. *The condition (1) of Theorem 7.25 is satisfied*
2. *The zero solution of (7.22) is absolutely stable w.r.t.* σ*.*

Theorem 7.28. *The necessary and sufficient condition for system (7.22) being absolutely stable are:*

1. *The condition (1) of Theorem 7.25 is satisfied*
2. *There exits a differentiable function* $V \in [\mathrm{R}^n, \mathrm{R}^1]$ *satisfies:*

$$\begin{aligned} V &\geq \varphi\left(\|c^{\mathrm{T}}x\|\right), \quad \varphi \in KR, \\ \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7.22)} &\leq -\psi\left(\|c^{\mathrm{T}}x\|\right), \quad \psi \in K. \end{aligned}$$

The proofs for the above four theorems are almost exactly same as those in proving the theorems in Sects. 7.1 and 7.2, and thus omitted here. Note that the relationship between the variable σ and the state variable x is not clear in (7.22). Although the first condition in Theorems 7.25 and 7.28 is algebraic and easy to verify, verifying second condition is quite difficult. However, we may, in addition, assume that f is continuously differentiable, then we have the following practically useful corollary.

Corollary 7.29. *Assume the following conditions are satisfied:*

1. *The condition (1) in Theorem 7.25 is satisfied*
2. *There exists an $n \times n$ symmetric matrix B and constant $\beta_j > 0,\ j = 1, \ldots, n$ such that B is positive definite or positive semi-definite and $\int_0^{\pm\infty} f_i(\sigma_i)\mathrm{d}\sigma_i = +\infty$ or $\lim\limits_{\sigma \to \infty} f_j^2(\sigma) = +\infty$*
3. *$f_j \in C^1$ and*

$$V(x,\sigma) = x^{\mathrm{T}}Bx + \sum_{j=1}^{m} \beta_i \int_0^{\sigma_j} f(\sigma_j)\mathrm{d}j - \frac{1}{2}\sum_{j=1}^{m} \beta_j d_j f_j^2(\sigma_j)$$

satisfying

$$\frac{dV(x,\sigma)}{\mathrm{d}t}|_{(7.22)} \leq -\varepsilon\tau,$$

where

$$\tau \in \left\{\sigma^{\mathrm{T}}\sigma,\ \sum_{j=1}^{m} \sigma_j f_j(\sigma_j),\ \sum_{j=1}^{n} f_j^2(\sigma_j),\ x^{\mathrm{T}}x\right\}.$$

Then the zero solution of system (7.22) is absolutely stable.

Proof. We only need to prove that the zero solution of (7.22) is absolutely stable w.r.t. σ under the conditions (1) and (2).

1. When B is positive definite, if $\tau = x^{\mathrm{T}}x$, $\frac{dV(x,\sigma)}{\mathrm{d}t}|_{(7.22)} \leq -\varepsilon x^{\mathrm{T}}x$. Thus, the zero solution of (7.22) is absolutely stable.
   If
   $$\tau \in \{\sigma^{\mathrm{T}}\sigma,\ \sum_{j=1}^{n} \sigma_j f_j(\sigma_j),\ \sum_{j=1}^{n} f_j^2(\sigma_j)\},$$
   then the zero solution of (7.22) is absolutely stable w.r.t. σ.
2. When B is positive semi-definite, if
   $$\tau \in \left\{\sigma^{\mathrm{T}}\sigma,\ \sum_{j=1}^{n} \sigma_j f_j(\sigma_j),\ \sum_{j=1}^{n} f_j^2(\sigma_j)\right\},$$
   $\frac{\mathrm{d}V(x,\sigma)}{\mathrm{d}t}|_{(7.22)}$ is negative definite w.r.t. σ.

If $\tau = x^{\mathrm{T}}x$, since $\frac{\|cx\|^2}{\|c\|^2} \leq \frac{\|c\|^2\|x\|^2}{\|c\|^2} = x^{\mathrm{T}}x$, $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(7.22)} \leq -\tau x^{\mathrm{T}}x \leq -\tau\frac{\|cx\|^2}{\|c\|^2} < 0$. When $x \bar{\in} \Omega$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ is negative w.r.t. Ω. Thus, the zero solution of (7.22) is absolutely stable w.r.t. σ.

Example 7.30. Consider a two-dimensional system with two loop feedback nonlinear terms:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} f(\sigma_1) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} f_2(\sigma_2), \\
\sigma_1 &= (1, 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 0.5 f_1(\sigma_1), \\
\sigma_2 &= (0, 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 0.3 f_2(\sigma_2).
\end{aligned} \tag{7.23}$$

(1) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a Hurwitz matrix.

(2) Assume $f_i \in F_\infty$ where f_i is differentiable, and $\int_0^{\pm\infty} f(\sigma_j) \mathrm{d}j = +\infty$ or $f_j^2(\sigma_j) \to +\infty$ when $\sigma_j \to \infty$. Let

$$V(\sigma) = \int_0^{\sigma_1} f_1(\sigma_1) \mathrm{d}\sigma_1 + \int_0^{\sigma_2} f_2(\sigma_2) \mathrm{d}\sigma_2 + \frac{1}{4} f_1^2(\sigma_1) + \frac{3}{20} f_2^2(\sigma_2),$$

and thus,

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7.23)} &= f_1(\sigma_1) \frac{\mathrm{d}\sigma_1}{\mathrm{d}t} + f_2(\sigma_2) \frac{\mathrm{d}\sigma_2}{\mathrm{d}t} + \frac{1}{2} f_1(\sigma_1) \frac{\mathrm{d}f}{\mathrm{d}\sigma_1} \frac{\mathrm{d}\sigma_1}{\mathrm{d}t} + \frac{3}{10} f_2(\sigma_2) \frac{\mathrm{d}f}{\mathrm{d}\sigma_2} \frac{\mathrm{d}\sigma_2}{\mathrm{d}t} \\
&= f_1(\sigma_1) \left[1 + \frac{1}{2} \frac{\mathrm{d}f_1}{\mathrm{d}\sigma_1} \right] \frac{\mathrm{d}\sigma_1}{\mathrm{d}t} + f_2(\sigma_2) \left[1 + \frac{3}{10} \frac{\mathrm{d}f_2}{\mathrm{d}\sigma_2} \right] \frac{\mathrm{d}\sigma_2}{\mathrm{d}t} \\
&= f_1(\sigma_1) \frac{dx_1}{\mathrm{d}t} + f_2(\sigma_2) \frac{dx_2}{\mathrm{d}t} \\
&= f_1(\sigma_1) \left[-x_1 - f_1(\sigma_1) \right] + f_2(\sigma_2) \left[-x_2 - f_2(\sigma_2) \right] \\
&= f_1(\sigma_1) \left[-\sigma_1 - \frac{3}{2} f_1(\sigma_1) \right] + f_2(\sigma_2) \left[-\sigma_2 - \frac{13}{10} f_2(\sigma_2) \right] \\
&= -\sigma_1 f_1(\sigma_1) - \frac{3}{2} f_1^2(\sigma_1) - \sigma_1 f_2(\sigma_2) - \frac{13}{10} f_2^2(\sigma_2) \\
&< 0 \qquad \text{when } \ \sigma_1^2 + \sigma_2^2 \neq 0.
\end{aligned}$$

This indicates that the conditions in Corollary 7.29 are satisfied, and the proof is complete. □

8

Robust Absolute Stability of Interval Control Systems

Strictly speaking, a mathematical model is only an approximate description of a real system since the information of the system coefficients are usually the upper and lower bounds, not the exact values [44]. In the past two decades, the stability study for linear control systems with parameters varied in a finite closed interval has been a hot topic in control society. However, not many results have been obtained for stability of nonlinear control systems with varied parameters in an interval. In this chapter, we will systematically introduce robust stability of control systems with interval varied parameters. In fact, such idea and methodology can be generalized to consider other Lurie control systems. The materials presented in this chapter are chosen from Liao et al. [85] (Sects. 8.1–8.4), and from Yu and Liao [172] (Sects. 8.5–8.9) and Liao [79].

8.1 Interval Lurie Control Systems

In Sect. 2.4, we have used a linear transform to change a general Lurie control system (including direct, indirect, and critical control systems) into two types of nonlinear control systems with separable variables in which feedback states become part of the state variables. Therefore, without loss of generality, here we assume the system is given in the transformed standard form.

Consider the following Lurie interval control system:

$$\dot{x} = A_I x + h_I\, f(x_n), \tag{8.1}$$

and a simpler system:

$$\dot{y} = B_I y + r_I f(y_n), \tag{8.2}$$

where

$$\begin{aligned}
f(\cdot) \in F &:= \{x_n | 0 < x_n f(x_n) \le +\infty,\ \ x_n \neq 0,\ \ f(x_n) \in C[(-\infty, +\infty), R^1],\\
A_I &:= \{A(a_{ij})_{n\times n}:\ \underline{A} \le A \le \overline{A},\ i.e., \underline{a_{ij}} \le a_{ij} \le \overline{a_{ij}},\ i,\ j = 1, 2, \ldots, n\},\\
h_I &:= \{h:\ \underline{h} \le h \le \overline{h},\ \ i.e., \underline{h_i} \le h_i \le \overline{h_i},\ i = 1, 2, \ldots, n\},\\
B_I &:= \{B(b_{ij})_{n\times n}:\ \underline{B} \le B \le \overline{B},\ i.e., \underline{b_{ij}} \le b_{ij} \le \overline{b_{ij}},\ i,\ j = 1, 2, \ldots, n\},\\
r_I &:= \{r: \underline{r} \le r \le \overline{r}, i.e., \underline{r_i} \le r_i \le \overline{r_i},\ \underline{r_i} = \overline{r_i} = 0, i = 1, \ldots, n-1, \underline{r_n} < \overline{r_n}\}
\end{aligned}$$

in which $\underline{A}, \overline{A}, \underline{B}, \overline{B}$ are known $n \times n$ matrices, $\underline{h}, \overline{h}, \underline{r}, \overline{r}$ are known n-dimensional vectors, while A, B, h and r are not precisely known.

$\forall A \in A_I\ \forall h \in h_I$ ($\forall B \in B_I$, $\forall r \in r_I$), the corresponding Lurie systems of (8.1) and (8.2) are given, respectively, by

$$\dot{x} = Ax + hf(x_n), \tag{8.1$'$}$$

$$\dot{y} = Ay + rf(x_n). \tag{8.2$'$}$$

Definition 8.1. *If $\forall A \in A_I$, $\forall h \in h_I$ ($\forall B \in B_I$, $\forall r \in r_I$), the zero solutions of the corresponding systems (8.1)$'$ and (8.2)$'$ are absolutely stable, then it is said the the zero solutions of the Lurie interval control systems (8.1) and (8.2) are robustly absolutely stable.*

Definition 8.2. *If $\forall A \in A_I$, $\forall h \in h_I$ ($\forall B \in B_I$, $\forall r \in r_I$), the zero solutions of the corresponding systems (8.1)$'$ and (8.2)$'$ are absolutely stable w.r.t. the partial variables $x_{j+1}, x_{j+2}, \dots, x_n$ ($y_{j+1}, y_{j+2}, \dots, y_n$), then it is said the the zero solutions of the Lurie interval control systems (8.1) and (8.2) are robustly absolutely stable w.r.t. the partial variables $x_{j+1}, x_{j+2}, \dots, x_n$, ($y_{j+1}, y_{j+2}, \dots, y_n$).*

Definition 8.3. *If $\forall A \in A_I$, ($\forall B \in B_I$), A (B) is a Hurwitz matrix, then A_I (B_I) is called an interval Hurwitz matrix.*

8.2 Sufficient and Necessary Conditions for Robust Absolute Stability

Since system (8.2) is a special case of system (8.1), we only discuss system (8.1), and the results for system (8.2) can be directly obtained from the results of system (8.1), as corollaries of the theorems obtained for system (8.1).

Theorem 8.4. *The sufficient and necessary conditions for the robust absolute stability of the zero solution of the Lurie interval control system (8.1) are:*

1. *$A_I + (O_{n\times(n-1)}\, h_I\theta)$ is an interval Hurwitz matrix, where $\theta = 0$ or 1, and $O_{n\times(n-1)}$ is an $n \times (n-1)$ zero matrix, i.e.,*

$$(O_{n\times(n-1)}, h_I\theta) = \begin{bmatrix} 0 & \cdots & 0 & h_1\theta \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & h_n\theta \end{bmatrix}_{n\times n}, \qquad h_i \in [\underline{h_i}, \overline{h_i}], \quad i = 1, 2, \dots, n.$$

2. *The zero solution of system (8.1) is robustly absolutely stable w.r.t. the partial variable x_n.*

Proof. *Necessity.* When the Lurie interval control system (8.1) is a direct control system, i.e., A_I is a Hurwitz matrix, then take $\theta = 0$; otherwise, choosing $f(x_n) = x_n$, $\theta = 1$, we know that $A_I + (O_{n\times(n-1)}\, h_I\theta)$ is a Hurwitz matrix. Thus condition (1) holds. Condition 2 is obvious since the robust absolute stability of the zero solution of system (8.1) implies that the zero solution is robustly absolutely stable with respect to (w.r.t.) the partial variable x_n.

Sufficiency. Let $W = A + (O_{n\times(n-1)}, h\theta)$. Then the zero solution of $(8.1)'$ can be expressed as

$$x(t,t_0;x_0) = \mathrm{e}^{W(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{W(t-\tau)}h\left[f(x_n(\tau)) - \theta x_n(\tau)\right]\mathrm{d}\tau. \tag{8.3}$$

Then, we can follow Theorem 4.3 to prove that $\forall\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that when $\|x_0\| < \delta$, we have $\|x(t,t_0;x_0)\| < \varepsilon$, and $\forall x_0 \in R^n$, $\lim\limits_{t\to+\infty} x(t,t_0;x_0) = 0$. This completes the proof. □

Corollary 8.5. *The sufficient and necessary conditions for the robust absolute stability of the zero solution of the Lurie interval control system (8.2) are:*

1. $B_I + (O_{n\times(n-1)}r_I\theta)$ *is a interval Hurwitz matrix, where* $\theta = 0$ *or* 1
2. *The zero solution of system (8.2) is robustly absolutely stable w.r.t. the partial variable* y_n.

Since in (8.2), $\underline{r}_i = \overline{r}_i = 0$ $(i = 1,\ldots,n-1)$, which is a special case of (8.1).

Theorem 8.6. *The sufficient and necessary conditions for the robust absolute stability of the zero solution of the Lurie interval control system (8.1) are:*

1. *There exists an n-dimensional interval vector* η_I *such that* $A_I + (O_{n\times(n-1)}, \eta_I)$ *is an interval Hurwitz matrix.*
2. *The zero solution of system (8.1) is robustly absolutely stable w.r.t. the partial variable* x_n.

Proof. *Necessity*. The existence of condition (1) is obvious. When A_I is a Hurwitz matrix, we can choose $\eta_I = (0,0,\ldots,0)^{\mathrm{T}}$; otherwise, take $\eta_I = h_I$, $f(x_n) = x_n$. It is easy to verify under these choices, condition (1) holds. Condition 2 is obviously true.

Sufficiency. $\forall A \in A_I$, let $\tilde{W} = A + (O_{n\times(n-1)}, \eta)$. Then rewrite (8.1) as

$$\dot{x} = \tilde{W}x + hf(x_n) - \eta x_n. \tag{8.4}$$

Now for system (8.4), applying the method of constant variation yields

$$x(t,t_0;x_0) = \mathrm{e}^{\tilde{W}(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{\tilde{W}(t-\tau)}\left[hf(x_n(t)) - \eta x_n(\tau)\right]\mathrm{d}\tau.$$

The remaining proof can follow Theorem 8.4. This completes the proof. □

Corollary 8.7. *The zero solution of the Lurie interval control system (8.2) is robustly absolutely stable if and only if the following conditions are satisfied:*

1. *There exists an n-dimensional interval vector* η_I *such that* $B_I + (O_{n\times(n-1)}\eta_I)$ *is an interval Hurwitz matrix*

2. *The zero solution of system (8.2) is robustly absolutely stable w.r.t. the partial variable* y_n.

Remark 8.8. Compared to the constructive conditions given in Theorem 8.4 and Corollary 8.5, the existence conditions in Theorem 8.6 and Corollary 8.7 are not so convenient. However, if they are chosen properly, sometimes the verification of the conditions can be simplified.

Similar to Theorems 8.4 and 8.6, we can prove the following theorem and corollary.

Theorem 8.9. *The zero solution of the Lurie interval control system (8.1) is robustly absolutely stable if and only if*

1. *The condition (1) in Theorem 8.4 or condition (1) in Theorem 8.6 holds.*
2. *The zero solution of system (8.1) is robustly absolutely stable w.r.t. the partial variables* $x_{j+1}, x_{j+2}, \ldots, x_n$.

Corollary 8.10. *The zero solution of the Lurie interval control system (8.2) is robustly absolutely stable if and only if*

1. *The condition (1) in Corollary 8.5 or condition (1) in Corollary 8.7 holds.*
2. *The zero solution of system (8.2) is robustly absolutely stable w.r.t. partial variables* $y_{j+1}, y_{j+2}, \ldots, y_n$.

8.3 Sufficient Conditions for Robust Absolute Stability

First we introduce the following notations:

$$A_I^{(j_0)} = \begin{bmatrix} a_{11} & \cdots & a_{1j_0} \\ \vdots & & \vdots \\ a_{j_01} & \cdots & a_{j_0j_0} \end{bmatrix}, \quad A_I^{(j_0)C} = \begin{bmatrix} a_{1(j_0+1)} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{j_0(j_0+1)} & \cdots & a_{j_0n} \end{bmatrix}, \quad i \le j_0 \le n.$$

Similarly, we define $B_I^{(j_0)}$ and $B_I^{(j_0)C}$, $\underline{a_{ij}} \le a_{ij} \le \overline{a_{ij}}$, $1 \le i \le j_0$, $1 \le j \le n$.

Theorem 8.11. *If the following conditions are satisfied:*

1. $A_I^{(j_0)}$ *is an interval Hurwitz matrix.*
2. *The zero solution of system (8.1) is robustly absolutely stable w.r.t. the partial variables* $x_{j_0+1}, x_{j_0+2}, \ldots, x_n$.

Then the zero solution of system (8.1) is robustly absolutely stable w.r.t. all the state variables.

Proof. $\forall A \in A_I, h \in h_I$, let $x^{(j_0)}(t) := x^{(j_0)}(t, t_0; x_0) = (x_1(t, t_0; x_0), \ldots x_{j_0}(t, t_0; x_0))^{\mathrm{T}}$, $x^{(j_0)C}(t) := (x_{j_0+1}(t, t_0; x_0), \ldots, x_n(t, t_0; x_0))^{\mathrm{T}}$, $h^{(j_0)}(t) := (h_1(t), \ldots, h_{j_0}(t))^{\mathrm{T}}$. Thus, the first j_0 solutions, $x^{(j_0)}(t)$, of system (8.1) can be expressed as

$$x^{(j_0)}(t) = \mathrm{e}^{A^{(j_0)}(t-t_0)} x^{(j_0)}(t_0) + \int_{t_0}^{t} \mathrm{e}^{A^{(j_0)}(t-\tau)} A^{(j_0)C} x^{(j_0)C}(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t} \mathrm{e}^{A^{(j_0)}(t-\tau)} h^{(j_0)} f(x_n(\tau))\,\mathrm{d}\tau. \tag{8.5}$$

Since $\|x^{(j_0)}(t_0)\| \le \|x(t_0)\|$, the zero solution of (8.1)' is robustly absolutely stable w.r.t. $x^{(j_0)}$. Then we may follow the proof of the sufficiency of Theorem 8.4 to show that $\forall \varepsilon > 0$, there exists $\delta(\varepsilon)$ such that when $\|x^{(j_0)}(t_0)\| < \|x(t_0)\| < \delta$, we have $\|x^{(j_0)}(t)\| < \varepsilon$, and $\forall x_0 \in R^n$, $\lim\limits_{t\to+\infty} x^{(j_0)}(t) = 0$. Thus the zero solution of system (8.1) is robustly absolutely stable w.r.t. $x^{(j_0)}(t)$, and thus also robustly absolutely stable w.r.t. all the state variables. $\square$

Corollary 8.12. *If the following conditions hold:*

1. $B_I^{(j_0)}$ *is an interval Hurwitz matrix*
2. *The zero solution of system (8.2) is robustly absolutely stable w.r.t. partial variables* $y_{j_0+1}, y_{j_0+2}, \ldots, y_n$

the zero solution of system (8.2) is robustly absolutely stable w.r.t. all the state variables.

Theorem 8.13. *If there exist constants* $c_i > 0$ $(i = 1, 2, \ldots, n)$ *such that*

$$\begin{cases} -c_j \overline{a_{jj}} > \sum\limits_{i=1, i\ne j}^{n} c_i a_{jj}^{(m)} \quad j = 1, 2, \ldots, n-1, \\ -c_n \overline{a_{nn}} \ge \sum\limits_{i=1}^{n-1} c_i a_{in}^{(m)}, \\ -c_n \overline{h_n} \ge \sum\limits_{i=1}^{n-1} c_i h_i^{(m)}, \end{cases} \tag{8.6}$$

and at least one of the last two inequalities in (8.6) is a strict inequality. Then the zero solution of system (8.1) is robustly absolutely stable. Here, $a_{ij}^{(m)} := \max_{i,j=1,2,\ldots,n}\left\{|\underline{a_{ij}}|, |\overline{a_{ij}}|\right\}$, *and* $h_i^{(m)} := \max_{i=1,2,\ldots,n-1}\left\{|\underline{h_i}|, |\overline{h_i}|\right\}$.

Proof. $\forall A \in A_I, h \in h_I$, construct the radially unbounded, positive definite Lyapunov function:

$$V = \sum_{i=1}^{n} c_i |x_i|.$$

Thus,

$$
\begin{aligned}
D^+V\big|_{(8.1)} &\leq \sum_{j=1}^{n}\left[c_j\,a_{jj}+\sum_{i=1,i\neq j}^{n}c_i\,|a_{ij}|\right]|x_j(t)|+\left[c_n\,h_n+\sum_{j=1}^{n-1}c_i\,|h_i|\right]|f(x_n(t))| \\
&\leq \sum_{j=1}^{n}\left[c_j\,\overline{a_{jj}}+\sum_{i=1,i\neq j}^{n}c_i\,a_{ij}^{(m)}\right]|x_j(t)|+\left[c_n\,\overline{h_n}+\sum_{j=1}^{n-1}c_i\,h_i^{(m)}\right]|f(x_n(t))| \\
&< 0 \qquad \text{when } x\neq 0.
\end{aligned}
$$

Therefore, the zero solution of system $(8.1)'$ is robustly absolutely stable. □

Corollary 8.14. *When $\overline{r_n}<0$, if there exist constants $c_i>0$ $(i=1,2,\ldots,n)$ such that*

$$
\begin{aligned}
&-c_j\,\overline{b_{jj}} > \sum_{i=1,i\neq j}^{n}c_i\,b_{ij}^{(m)} \;\; j=1,2,\ldots,n-1, \\
&-c_n\,\overline{b_{nn}} > \sum_{i=1}^{n-1}c_i\,b_{in}^{(m)},
\end{aligned}
\tag{8.7}
$$

while when $\overline{r_n}\leq 0$, the the last inequality in (8.7) is a strict inequality, then the zero solution of system (8.2) is robustly absolutely stable.

Example 8.15. Consider the robust absolute stability of the zero solution of the following Lurie interval control system:

$$
\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}[-1.2,-1] & [-\frac{1}{2},\frac{3}{2}]\\ [-2,2] & [-5,-4]\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}[-2,2]\\ [-4.5,-4.2]\end{pmatrix}f(x_2), \tag{8.8}
$$

where $f(\cdot)\in F_\infty$.

It is easy to verify that the conditions in Theorem 8.13 are satisfied: $\overline{a_{11}}=-1$, $\overline{a_{22}}=-4$, $a_{12}^{(m)}=\frac{3}{2}$, $a_{21}^{(m)}=2$, $b_1^{(m)}=2$, $\overline{b_2}=-4.2$. Take $c_1=2.1$ and $c_2=1$. Then construct the radially unbounded, positive definite Lyapunov function:

$$
V=c_1\,|x_1|+c_2\,|x_2|
$$

and find that

$$
\begin{aligned}
D^+V\big|_{(8.8)} &\leq \left(c_1\,\overline{a_{11}}+c_2\,a_{21}^{(m)}\right)|x_1|+\left(c_2\,\overline{a_{22}}+c_1\,a_{12}^{(m)}\right)|x_2| \\
&\quad+\left(c_2\,\overline{b_2}+c_1\,b_1^{(m)}\right)|f(x_2)| \\
&=(-2.1+2)\,|x_1|+\left(-4+2.1\times\tfrac{3}{2}\right)|x_2|+(-4.2+4.2)\,|f(x_2)| \\
&\leq -0.1\times|x_1|-0.85\times|x_2| \\
&<0, \qquad \text{when} \quad |x_1|+|x_2|\neq 0.
\end{aligned}
$$

Thus, all the conditions in Theorem 8.13 are satisfied. Hence, the zero solution of system (8.8) is robustly absolutely stable.

Theorem 8.16. *If the following conditions are satisfied:*

1. $A_I^{(j_0)}$ *is an interval Hurwitz matrix*
2. *If there exist constants* $c_i \geq 0$ $(i = 1, 2, \ldots, j_0)$, $c_j > 0$, $j = j_0 + 1, \ldots, n$, $\varepsilon > 0$ *such that*

$$\begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{a_{11}} & m_{12} & \cdots & m_{1(n+1)} \\ m_{21} & 2c_2\overline{a_{22}} & \cdots & m_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & 2c_n\overline{a}_{nn} & m_{n(n+1)} \\ m_{(n+1)1} & \cdots & m_{(n+1)n} & 2\overline{h_n} \end{bmatrix} \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix} \leq \begin{cases} -\varepsilon \sum\limits_{i=j_0+1}^{n} x_i^2 \quad \text{or} \\ -\varepsilon \sum\limits_{i=j_0+1}^{n-1} x_i^2 - \varepsilon f(x_n) \text{ or} \\ -\varepsilon \sum\limits_{i=j_0+1}^{n-1} x_i^2 - \varepsilon x_n f^2(x_n), \end{cases}$$

where

$$m_{ij} = m_{ji} = \max_{\underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}} [|c_i a_{ij} + c_j a_{ji}|], \quad i \neq j, \ 1 \leq i, j \leq n,$$

$$m_{(n+1)i} = m_{i(n+1)} = \max_{\substack{\underline{a_{ni}} \leq a_{ni} \leq \overline{a_{ni}} \\ \underline{h_i} \leq h_i \leq \overline{h_i}}} [|c_i h_i + a_{ni}|], \quad 1 \leq i \leq n,$$

then the zero solution of the Lurie interval control system (8.1) is robustly absolutely stable.

Proof. For the variables $x_{j_0+1}, \ldots, x_n$, construct the radially unbounded, positive definite Lyapunov function:

$$V(x) = \sum_{i=1}^{n} c_i x_i^2 + 2\int_0^{x_n} f(x_n)\,\mathrm{d}x_n.$$

Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(8.1)'} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1a_{11} & (c_1a_{12}+c_2a_{21}) & \cdots (c_1b_1+a_{n1}) \\ (c_1a_{12}+c_2a_{21}) & 2c_2a_{22} & \cdots (c_1b_2+a_{n2}) \\ \vdots & & \vdots \\ (c_1b_1+a_{n1}) & \cdots & \cdots \ \ 2h_n \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix}$$

$$\leq \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{a_{11}} & m_{12} & \cdots & m_{1(n+1)} \\ m_{21} & 2c_2\overline{a_{22}} & \cdots & m_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & 2c_n\overline{a}_{nn} & m_{n(n+1)} \\ m_{(n+1)1} & \cdots & m_{(n+1)n} & 2\overline{h_n} \end{bmatrix} \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix}$$

$$\leq \begin{cases} -\varepsilon \sum_{i=j_0+1}^{n} x_i^2 & \text{or} \\ -\varepsilon \sum_{i=j_0+1}^{n-1} x_i^2 - \varepsilon f^2(x_n) & \text{or} \\ -\varepsilon \sum_{i=j_0+1}^{n-1} x_i^2 - \varepsilon x_n f(x_n). \end{cases} \tag{8.9}$$

Therefore, the zero solution of $(8.1)'$ is robustly absolutely stable w.r.t. the partial variables $x_{j_0+1}, \ldots, x_n$. Further, due to the condition (1) and Theorem 8.13, we know that the conclusion of Theorem 8.16 is true. □

Theorem 8.17. *If the following conditions are satisfied: when $\overline{r_n} \leq 0$ and*

1. *$B_I^{(j_0)}$ is an interval Hurwitz matrix*
2. *If there exist constants $c_i \geq 0\ i = 1, 2, \ldots, j_0$, $c_j > 0,\ j = j_0 + 1, \ldots, n$, and $\varepsilon > 0$ such that*

$$\begin{pmatrix} |y_1| \\ \vdots \\ |y_n| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n} \\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n(n-1)} & 2c_n\overline{b_{nn}} \end{bmatrix} \begin{pmatrix} |y_1| \\ \vdots \\ |y_n| \end{pmatrix} \leq -\varepsilon \sum_{i=j_0+1}^{n} y_i^2,$$

or when $\overline{r_n} < 0$ and

1. *B_I is an interval Hurwitz matrix*
2. *If there exist constants $c_i \geq 0\ i = 1, 2, \ldots, n-1$, and $c_n > 0$ such that*

$$\begin{bmatrix} 2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n} \\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n(n-1)} & 2c_n\overline{b_{nn}} \end{bmatrix} \leq 0,$$

then the zero solution of system (8.2) is robustly absolutely stable. Here $m_{ij} = m_{ji} = \max_{\underline{b_{ij}} \leq b_{ij} \leq \overline{b_{ij}}} [|c_i b_{ij} + c_j b_{ji}|]$, $i \neq j$, $1 \leq i, j \leq n$.

Proof. When $\overline{r_n} \leq 0$, for the variables $y_{j_0+1}, \ldots, y_n$, construct the radially unbounded, positive definite Lyapunov function:

$$V(y) = \sum_{i=1}^{n} c_i y_i^2.$$

Then,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(8.2)'} \leq \begin{pmatrix} |y_1| \\ |y_2| \\ \vdots \\ |y_n| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n} \\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n(n-1)} & 2c_n\overline{b_{nn}} \end{bmatrix} \begin{pmatrix} |y_1| \\ |y_2| \\ \vdots \\ |y_n| \end{pmatrix}$$

$$+ 2c_n\overline{r_n}y_n f(y_n)$$

$$\leq -\varepsilon \sum_{i=j_0+1}^{n} y_i^2. \tag{8.10}$$

Equation (8.10) indicates that the zero solution of $(8.2)'$ is robustly absolutely stable w.r.t. the partial variables $y_{j_0+1}, \ldots, y_n$. Further, following the proof of Theorem 8.13 for the robust absolute stability w.r.t. $x^{(j_0)}$, we can show that the zero solution of $(8.2)'$ is also absolutely stable w.r.t. the partial variables $y_1, \ldots, y_{j_0}$.

Next, consider $\overline{r_n} < 0$. For the variable y_n, construct the radially unbounded, positive definite Lyapunov function:

$$V(y) = \sum_{i=1}^{n} c_i y_i^2.$$

Then differentiating V w.r.t. time t along the trajectory of system $(8.2)'$ yields

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(8.2)'} \leq \begin{pmatrix} |y_1| \\ |y_2| \\ \vdots \\ |y_n| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n} \\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n(n-1)} & 2c_n\overline{b_{nn}} \end{bmatrix} \begin{pmatrix} |y_1| \\ |y_2| \\ \vdots \\ |y_n| \end{pmatrix} + 2c_n\overline{r_n}y_n f(y_n)$$

$$\leq 2c_n\overline{r_n}y_n f(y_n) < 0 \qquad \text{when} \quad y_n \neq 0. \tag{8.11}$$

Further, use the method of constant variation to express $y(t)$ as

$$y(t,t_0;y_0) = \mathrm{e}^{B(t-t_0)}y_0 + \int_{t_0}^{t} \mathrm{e}^{B(t-\tau)} r f(y_n(\tau))\,\mathrm{d}\tau,$$

and follow Theorem 8.4 to finish the proof. □

8.4 Algebraic Sufficient and Necessary Conditions

For an interval matrix, it is difficult to verify if it is a Hurwitz matrix. Although we have applied finite cover theorem to show that the Hurwitz stability of an infinite number of interval matrices can be found from the Hurwitz stability of its finite number of interval matrices. However, to find these finite number of matrices are very difficult.

In this section, we will consider some special Lurie interval control systems and derive very simple algebraic sufficient and necessary conditions for the robust absolute stability of these systems.

Again consider system (8.1), but now assume that $-\underline{a_{ij}} = \overline{a_{ij}},\ i \neq j,\ i,j = 1,2,\ldots,n,\ \overline{a_{ii}} < 0,\ i = 1,2,\ldots,n;\ \overline{h_n} < 0,\ -\underline{h_i} = \overline{h_i},\ i = 1,2,\ldots,n-1$ and $\lambda\,\overline{h_i} = \overline{a_{in}},\ i = 1,2,\ldots,n,\ \lambda > 0$. Then we have the following theorem.

Theorem 8.18. *The sufficient and necessary condition for the zero solution of the Lurie interval control system (8.1) being robustly absolutely stable is that $-\overline{A}$ is an M matrix.*

Proof. *Necessity.* Take $\overline{A} \in A_I,\ \overline{h} \in h_I,\ f(x_n) = x_n$. Substituting these expressions into system $(8.1)'$ yields

$$\dot{x} = \overline{A}x + \overline{h}x_n = (\overline{A} + O_{n\times(n-1)}, \overline{h})x. \tag{8.12}$$

So $(\overline{A} + O_{n\times(n-1)}, \overline{h})$ must be a Hurwitz matrix. The diagonal elements of this matrix are negative, and non-negative elements are non-negative. Thus, the matrix $-(\overline{A} + O_{n\times(n-1)}, \overline{h})$ is an M matrix. Hence, there exist constants $c_i > 0,\ i = 1,2,\ldots,n$ such that

$$-c_j\overline{a_{ij}} > \sum_{i=1,i\neq j}^{n} c_i\overline{a_{ij}} \qquad j = 1,2,\ldots,n-1 \tag{8.13}$$

and

$$-c_n(\overline{a_{nn}} + \overline{h_n}) > \sum_{i=1}^{n-1} c_i(\overline{a_{in}} + \overline{h_i}). \tag{8.14}$$

Equation (8.14) can be rewritten as $-c_n\left(1 + \frac{1}{\lambda}\right)\overline{a_{nn}} > \sum_{i=1}^{n-1} c_i\left(1 + \frac{1}{\lambda}\right)\overline{a_{in}}$, i.e.,

$$-c_n\overline{a_{nn}} > \sum_{i=1}^{n-1} c_i\overline{a_{in}}. \tag{8.15}$$

Equations (8.14) and (8.15) imply that $-\overline{A}$ is an M matrix.

Sufficiency. For system $(8.1)'$ choose the radially unbounded, positive definite Lyapunov function:

$$V(x) = \sum_{i=1}^{n} c_i|x_i|,$$

where c_i are determined by (8.13) and (8.14). It follows from $\lambda\,\overline{h_i} = \overline{a_{in}}$ $(i = 1,2,\ldots,n)$ and $-c_n\overline{a_{nn}} > \lim_{j=1}^{n-1} c_i\overline{a_{in}}$ that $-c_n\lambda\,\overline{h_n} > \sum_{i=1}^{n-1}\lambda c_i\overline{h_i}$, i.e., $-c_n\overline{h_n} > \sum_{i=1}^{n-1} c_i\overline{h_i}$. Thus, by (8.13) and (8.14) we have

$$
\begin{aligned}
D^{+}V(x)\big|_{(8.1)'} &\le \sum_{j=1}^{n}\left[c_j a_{jj} + \sum_{i=1,i\neq j}^{n} c_i\,|a_{ij}|\right]|x_j| + \left[c_n h_n + \sum_{i=1}^{n-1} c_i\,|h_i|\right]|f(x_n)| \\
&\le \sum_{j=1}^{n}\left[c_j \overline{a_{jj}} + \sum_{i=1,i\neq j}^{n} c_i\,\overline{a_{ij}}\right]|x_j| + \left[c_n \overline{h_n} + \sum_{i=1}^{n-1} c_i\,\overline{h_i}\right]|f(x_n)| \\
&< 0 \qquad \text{when} \quad x \neq 0.
\end{aligned}
\tag{8.16}
$$

Therefore, the zero solution of system (8.1) is robustly absolutely stable. The proof is complete. □

Next, consider system (8.2). Assume that $\overline{b_{ii}} < 0,\ i = 1, 2, \ldots, n,\ -\underline{b_{ij}} = \overline{b_{ij}},\ i \neq j,\ i, j = 1, 2, \ldots, n;\ -\underline{r_i} = \overline{r_i},\ i = 1, 2, \ldots, n-1$ and $\overline{r_n} < 0$.

Theorem 8.19. *The sufficient and necessary conditions for the zero solution of the Lurie interval control system (8.2) being robustly absolutely stable are:*

1. The zero solution of system (8.2) is robustly absolutely stable w.r.t. y_n.

2. $\overline{B_{(n-1)}} := \begin{bmatrix} \overline{b_{11}} & \cdots & \overline{b_{1(n-1)}} \\ \vdots & \cdots & \vdots \\ \overline{b_{(n-1)1}} & \cdots & \overline{b_{(n-1)(n-1)}} \end{bmatrix}$ *is a Hurwitz matrix.*

Proof. *Necessity*. (1) The robust absolute stability w.r.t. y_n is obvious. For 2, substituting $f(y_n) = y_n$ into (8.2) results in an interval system:

$$
\dot{y} = \Big(B_I + (O_{n\times(n-1)}\ r_I\Big)\,y.
$$

So $(B_I + (O_{n\times(n-1)}\ r_I))$ is an interval Hurwitz matrix. Thus, $(\overline{B} + (O_{n\times(n-1)}\ \overline{r}))$ is a Hurwitz matrix. In particular, $\overline{B}_{(n-1)}$ is a Hurwitz matrix.

Sufficiency. Let

$$
B^{(n-1)} := \begin{bmatrix} b_{11} & \cdots & b_{1(n-1)} \\ \vdots & & \vdots \\ b_{(n-1)1} & \cdots & b_{(n-1)(n-1)} \end{bmatrix}, \qquad B^{(n-1)C} := (b_{1n}, b_{2n}, \ldots, b_{(n-1)n})^{\mathrm{T}},
$$

and $y^{(n-1)} := (y_1(t), y_2(t), \ldots, y_{n-1}(t))^{\mathrm{T}}$. Then, the first $n-1$ solutions of system $(8.2)'$ can be expressed as

$$
\begin{gathered}
y^{(n-1)}(t) = \mathrm{e}^{B^{(n-1)}(t-t_0)}y^{(n-1)}(t_0) + \int_{t_0}^{t} \mathrm{e}^{B^{(n-1)}(t-\tau)}B^{(n-1)C}y_n(\tau)\,\mathrm{d}\tau \\
\max \operatorname{Re}\lambda(B^{(n-1)}) \le \max \operatorname{Re}\lambda(\overline{B}^{(n-1)}).
\end{gathered}
\tag{8.17}
$$

There exist constants $M \ge 1$ and $\alpha > 0$ such that

$$
\|\mathrm{e}^{B^{(n-1)}(t-t_0)}\| \le \|\mathrm{e}^{(\overline{B}^{(n-1)})(t-t_0)}\| \le M\,\mathrm{e}^{\max \operatorname{Re}\lambda(\overline{B}^{(n-1)})(t-t_0)} \le M\,\mathrm{e}^{-\alpha(t-t_0)}.
$$

Hence,

$$\begin{aligned}\|y^{(n-1)}(t)\| &\leq M\mathrm{e}^{\max \mathrm{Re}\lambda(\overline{B}^{(n-1)})(t-t_0)}\|y^{(n-1)}(t_0)\| \\ &\quad + \int_{t_0}^{t} M\mathrm{e}^{\max \mathrm{Re}\lambda(\overline{B}^{(n-1)})(t-\tau)}|y_n(\tau)|\mathrm{d}\tau \\ &\leq M\mathrm{e}^{-\alpha(t-t_0)}\|y^{n-1}(t_0)\| \\ &\quad + \int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-\tau)}\|B^{-(n-1)C}\|\,|y_n(\tau)|\mathrm{d}\tau,\end{aligned}$$

where $M \gg 1$. Due to $y_n(t) \to 0$ as $t \to +\infty$, we can follow the proof of Theorem 8.4 to prove that the zero solution of (8.2) is robustly absolutely stable w.r.t. the partial variable $y^{(n-1)}(t)$. □

Remark 8.20. Although the conditions given in Theorems 8.18 and 8.19 are obtained for special cases, they are quite useful in realizing robust absolute stability via feedback controls.

Example 8.21. Analyze the stability of the zero solution of the following Lurie interval control system:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}[-5,-4] & [-3,3]\\ [-2,2] & [-4,-3]\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}[-2,2]\\ [-3,-2]\end{pmatrix} f(x_2), \tag{8.18}$$

where $f(x_2) \in F$.

Since $\overline{A} = \begin{bmatrix}-4 & 3\\ 2 & -3\end{bmatrix}$, it is obvious to see that $\overline{A}$ is a Hurwitz matrix. Further, from $\overline{h_1} = 2$, $\overline{h_2} = -2$, $\overline{a_{12}} = 3$, $\overline{a_{22}} = -3$, and taking $\lambda = \frac{3}{2}$ yields $\lambda\,\overline{h_i} = \overline{a_{i2}}$, $i = 1, 2$, all the conditions in Theorem 8.18 are satisfied. Thus, the zero solution of system (8.18) is robustly absolutely stable.

8.5 Interval Yocubovich Control Systems

In this section, we consider the robust absolute stability of interval Yocubovich control systems [172], including the famous Lurie indirect control system [96], the well-known Popov indirect control system [119] and the Yocubovich indirect control system [163] as particular cases.

Now, we consider the following interval Yocubovich control system:

$$(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, \dot{\sigma})^{\mathrm{T}} = A_I(a_{ij})_{(n+1)\times(n+1)}(x_1, x_2, \ldots, x_n, f(\sigma))^{\mathrm{T}}, \tag{8.19}$$

which can be written in the vector form:

$$\begin{pmatrix}\dot{x}\\ \dot{\sigma}\end{pmatrix} = A_I(a_{ij})_{(n+1)\times(n+1)}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix}. \tag{8.19$'$}$$

Here, $x \in R^n$, $\sigma \in R^1$, $\sigma = c^{\mathrm{T}}x - \rho\xi$, $\rho, \xi \in R^1$, $\rho > 0$, and

$$\begin{aligned}&A_I(a_{ij})_{(n+1)\times(n+1)}\\ &:= \left\{A \,\middle|\, \underline{A} \le A \le \overline{A}, \quad \text{i.e.,} \quad \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, \;\; i,j = 1,2,\ldots,(n+1)\right\}\end{aligned}$$

in which $\underline{A}$ and $\overline{A}$ are known, but A is not known.

$$f(\cdot) \in F := \left\{f(\cdot)\,|\,0 < \sigma f(\sigma) \le +\infty,\; \sigma \neq 0,\; f(0) = 0,\; f(\cdot) \in C[(-\infty,+\infty),R^1]\right\}$$

Therefore, system (8.19) (or $(8.19)'$) is an uncertain system.

$\forall A \in A_I$, the corresponding certain Yocubovich control system is given by

$$\begin{pmatrix}\dot{x}\\ \dot{\sigma}\end{pmatrix} = A(a_{ij})_{(n+1)\times(n+1)}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix}. \tag{8.20}$$

Now, we show that the Yocubovich control system (8.20) includes the Lurie indirect control system [96], and the Popov control system [119], as special cases.

1. The famous Lurie indirect control system [96] is given by

$$\begin{cases}\dot{y} = Dy + S\xi,\\ \dot{\xi} = f(\sigma),\end{cases} \tag{8.21}$$

where $y \in R^n$, $\xi \in R^1$, $D = D(d_{ij})_{n\times n} \in R^{n\times n}$, $c \in R^n$, $S \in R^n$, $\sigma = c^{\mathrm{T}}y - \rho\xi$, $\rho \in R^1$, $\rho > 0$. When the necessary condition of absolute stability for the zero solution of system (8.21) is satisfied, the matrix

$$\begin{bmatrix}D & S\\ c^{\mathrm{T}} & -\rho\end{bmatrix}$$

must be Hurwitz matrix, implying that $\det\begin{bmatrix}D & S\\ c^{\mathrm{T}} & -\rho\end{bmatrix} \neq 0$. Then, introducing the following full-rank transformation:

$$\begin{cases}x = Dy + S\xi,\\ \sigma = c^{\mathrm{T}}y - \rho\xi,\end{cases}$$

we obtain $\frac{\mathrm{d}y}{\mathrm{d}t} = x$, and

$$\begin{pmatrix}\dot{x}\\ \dot{\sigma}\end{pmatrix} = \begin{bmatrix}D & S\\ c^{\mathrm{T}} & -\rho\end{bmatrix}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix} := A^{(1)}(a_{ij})_{(n+1)\times(n+1)}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix}, \tag{8.22}$$

which is in the form of (8.20).

2. The well-known Popov indirect control system [119] is described by

$$\begin{cases}\dot{x} = Dx + Sf(\sigma),\\ \dot{\xi} = f(\sigma),\\ \sigma = c^{\mathrm{T}}x - \rho\xi,\end{cases} \tag{8.23}$$

where $x \in R^n,\ \xi \in R^1$, and $D \in R^{n\times n},\ S \in R^n,\ c \in R^n,\ \rho > 0,\ \rho \in R^1$ are fixed coefficients. Since

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = c^{\mathrm{T}}\frac{\mathrm{d}x}{\mathrm{d}t} - \rho\frac{\mathrm{d}\xi}{\mathrm{d}t} = c^{\mathrm{T}}(Dx + Sf(\sigma)) - \rho f(\sigma) = c^{\mathrm{T}}Dx - (\rho - c^{\mathrm{T}}S)f(\sigma),$$

we have

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{\sigma}\end{pmatrix} &= \begin{bmatrix} D & S \\ c^{\mathrm{T}}D & -(\rho - c^{\mathrm{T}}S)\end{bmatrix}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix}\\ &:= A^{(2)}(a_{ij})_{(n+1)\times(n+1)}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix},\end{aligned} \tag{8.24}$$

which is also in the form of (8.20).

3. Finally, the Yocubovich indirect control system [163] is

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{\sigma}\end{pmatrix} &= \begin{pmatrix}Dx - Sf(\sigma)\\ c^{\mathrm{T}}x - \rho f(\sigma)\end{pmatrix} = \begin{bmatrix} D & -S \\ c^{\mathrm{T}} & -\rho\end{bmatrix}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix}\\ &:= A^{(3)}(a_{ij})_{(n+1)\times(n+1)}\begin{pmatrix}x\\ f(\sigma)\end{pmatrix},\end{aligned} \tag{8.25}$$

where $x \in R^n$, $\sigma \in R^1$, $c, S \in R^n$, $\rho \in R^1$, $\rho > 0$. Equation (8.25) is again in the form of (8.20).

Therefore, we only need to study control system (8.20).

Following Definitions 8.1–8.3, it is not difficulty for readers to define the robust absolute stability as well as the robust absolute stability with respect to partial variables for the zero solution of system (8.19).

8.6 SANC for the Robust Absolute Stability of the Interval Yocubovich System (8.19)

In this section, we present several sufficient and necessary conditions (SANC) for the robust absolute stability of the zero solution of the interval Yocubovich control system (8.19) [165–168].

Theorem 8.22. *The SANC for the robust absolute stability of the zero solution of system (8.19) are:*

1. *$A_I(a_{ij})_{(n+1)\times(n+1)}$ is a interval Hurwitz matrix*
2. *The zero solution of system (8.19) is robustly absolutely stable w.r.t. one variable σ.*

Proof. *Necessity*. Let $f(\sigma) = \sigma$. Then system (8.19) becomes an interval linear systems:

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} \end{pmatrix} = A_I(a_{ij})_{(n+1)\times(n+1)} \begin{pmatrix} x \\ \sigma \end{pmatrix}. \tag{8.26}$$

Since the zero solution of system (8.26) is robustly globally asymptotically stable, $A_I(a_{ij})_{(n+1)\times(n+1)}$ is an interval Hurwitz matrix, i.e., condition (1) of Theorem 8.22 holds. Condition 2 is obvious, since the robustly absolute stability of the zero solution of (8.19) w.r.t. all variables $x_1, x_2, \ldots, \sigma$, and particularly, w.r.t. one variable σ.

Sufficiency. $\forall A \in A_I$, let $f(\sigma) = \sigma + [f(\sigma) - \sigma]$. Simply noting that $A = A_{(n+1)\times(n+1)}$, $A_{(n+1)\times 1} = (a_{1(n+1)}, a_{2(n+1)}, \ldots, a_{(n+1)(n+1)})^{\mathrm{T}}$, any solution of (8.20) can be expressed by

$$\begin{pmatrix} x(t) \\ \sigma(t) \end{pmatrix} = \mathrm{e}^{A(t-t_0)} \begin{pmatrix} x(t_0) \\ \sigma(t_0) \end{pmatrix} + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} A_{(n+1)\times 1} [f(\sigma(\tau)) - \sigma(\tau)]\,\mathrm{d}\tau. \tag{8.27}$$

Since A is a Hurwitz matrix, there exist $M \geq 1$ and $\alpha > 0$ such that

$$\|\mathrm{e}^{A(t-t_0)}\| \leq M\mathrm{e}^{-\alpha(t-t_0)}.$$

In addition, $f(\sigma(t)) \to 0$ as $t \to +\infty$, $\sigma(t)$ continuously depends on (x_0, σ_0) and $f(\sigma(t))$ is a continuous function of x_0. Thus, $\forall \varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $t_1 > t_0$ such that for all $t \geq t_1 > t_0$,

$$\int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)} \left[\|A_{(n+1)\times 1} f(\sigma(\tau))\| + \|A_{(n+1)\times 1}\sigma(\tau)\|\right] \mathrm{d}\tau < \frac{\varepsilon}{3} \quad \text{for } t_0, \leqslant t_1 < t,$$

$$\int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)} \left[\|A_{(n+1)\times 1} f(\sigma(\tau))\| + A_{(n+1)\times 1}\sigma(\tau)\|\right] \mathrm{d}\tau < \frac{\varepsilon}{3} \quad \text{for } t \geq t_1,$$

$$\|\mathrm{e}^{A(t-t_0)}\| \leq M\mathrm{e}^{-\alpha(t-t_0)} < \frac{\varepsilon}{3\delta(\varepsilon)}.$$

Thus, when $\left\| \begin{matrix} x_0 \\ \sigma_0 \end{matrix} \right\| \leq \delta(\varepsilon)$, we have

$$\begin{aligned} \left\| \begin{matrix} x(t) \\ \sigma(t) \end{matrix} \right\| &\leq \|\mathrm{e}^{A(t-t_0)}\| \left\| \begin{matrix} x_0 \\ \sigma_0 \end{matrix} \right\| + \int_{t_0}^{t} \|\mathrm{e}^{A(t-\tau)} A_{(n+1)\times 1} [f(\sigma(\tau)) - \sigma(\tau)]\|\,\mathrm{d}\tau \\ &\leq M\mathrm{e}^{-\alpha(t-t_0)} \left\| \begin{matrix} x_0 \\ \sigma_0 \end{matrix} \right\| \\ &\quad + \int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)} \left[\|A_{(n+1)\times 1} f(\sigma(\tau))\| + \|A_{(n+1)\times 1}\sigma(\tau)\|\right] \mathrm{d}\tau \\ &\quad + \int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)} \left[\|A_{(n+1)\times 1} f(\sigma(\tau))\| + \|A_{(n+1)\times 1}\sigma(\tau)\|\right] \mathrm{d}\tau \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \text{for all } t \geq t_0. \end{aligned} \tag{8.28}$$

Further, for any $(x_0, \sigma_0) \in R^{n+1}$, applying the L'Hospital rule to the above inequality yields:

$$
\begin{aligned}
0 &\le \lim_{t\to+\infty} \left\| \begin{matrix} x(t) \\ \sigma(t) \end{matrix} \right\| \\
&\le \lim_{t\to+\infty} M\|\phi\| \mathrm{e}^{-\alpha(t-t_0)} \left\| \begin{matrix} x_0 \\ \sigma_0 \end{matrix} \right\| \\
&\quad + \lim_{t\to+\infty} \int_{t_0}^{t} M \mathrm{e}^{-\alpha(t-\tau)} \left[\|A_{(n+1)\times 1} f(\sigma(\tau))\| + \|A_{(n+1)\times 1}\sigma(\tau)\| \right] \mathrm{d}\tau \\
&= 0 + M \lim_{t\to+\infty} \frac{\int_{t_0}^{t} \mathrm{e}^{\alpha\tau} [\|A_{(n+1)\times 1} f(\sigma(\tau))\| + \|A_{(n+1)\times 1}\sigma(\tau)\|] \mathrm{d}\tau}{\mathrm{e}^{\alpha t}} \\
&= \frac{M}{\alpha} \lim_{t\to+\infty} \left[\|A_{(n+1)\times 1} f(\sigma(t))\| + \|A_{(n+1)\times 1}\sigma(t)\| \right] \\
&= 0.
\end{aligned} \tag{8.29}
$$

This implies that the zero solution of system (8.20) is globally asymptotically stable. Thus, due to arbitrary $f(\sigma) \in F_\infty$, the zero solution of system (8.19) is robustly absolutely stable.

The proof is complete. □

Let

$$
A_I(a_{ij})_{(n+1)\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{(n+1)1} & \cdots & a_{(n+1)n} \end{bmatrix},
$$

$$
\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, \ \ 1 \le i \le n+1, \ 1 \le j \le n,
$$

$$
\begin{aligned}
\eta_{(n+1)\times 1} &= (\eta_1, \ldots, \eta_{n+1})^{\mathrm{T}}, \\
A_I(a_{ij})_{(n+1)\times 1} &= (a_{1(n+1)}, a_{2(n+1)}, \ldots, a_{(n+1)(n+1)})^{\mathrm{T}}, \\
&\qquad \underline{a}_{i(n+1)} \le a_{i(n+1)} \le \overline{a}_{i(n+1)}, \ 1 \le i \le n+1.
\end{aligned}
$$

Theorem 8.23. *The zero solution of the interval Yocubovich control system (8.19) is robustly absolutely stable if and only if:*

1. *The zero solution of (8.19) is robustly absolutely stable w.r.t. one variable σ*
2. *There exists a constant vector $\eta_{(n+1)\times 1}$ such that*

$$
\left(A_I(a_{ij})_{(n+1)\times n}, \eta_{(n+1)\times 1} \right) := \begin{bmatrix} a_{11} & \cdots & a_{1n} & \eta_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n+1)1} & \cdots & a_{(n+1)n} & \eta_{n+1} \end{bmatrix}
$$

is an interval Hurwitz matrix.

Proof. *Necessity*. Condition 1 of Theorem 8.23 is obvious. Condition 2 holds, implying the existence of $\eta_{(n+1)\times 1}$. For example, we can take $\eta_{(n+1)\times 1} = (a_{1(n+1)}, \dots, a_{(n+1)(n+1)})^{\mathrm{T}}$. When we choose $\eta_{(n+1)\times 1} = (0,\dots,0,-1)^{\mathrm{T}}$, the Hurwitz property of

$$\left[A_I(a_{ij})_{(n+1)\times n},\ \eta_{(n+1)\times 1}\right]$$

is equivalent to the Hurwitz property of

$$A_I(a_{ij})_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

which has 1 less dimension and so is more convenient to verify.

Sufficiency. Since any solution of (8.20) can be expressed by

$$\begin{aligned} \begin{pmatrix} x(t) \\ \sigma(t) \end{pmatrix} &= \mathrm{e}^{(A_I(a_{ij})_{(n+1)\times n},\eta_{(n+1)\times 1})(t-t_0)} \\ &\quad + \int_{t_0}^{t} \mathrm{e}^{(A_I(a_{ij})_{(n+1)\times n},\eta_{(n+1)\times 1})(t-\tau)} A_I(a_{ij})_{(n+1)\times 1} && (8.30) \\ &\quad \times \left[f(\sigma(\tau)) - \eta_{(n+1)\times 1}\sigma(\tau)\right] \mathrm{d}\tau, && (8.31) \end{aligned}$$

we can complete the proof by following the proof of Theorem 8.22. □

Based on Theorems 8.22 and 8.23, we can prove the following theorem.

Theorem 8.24. *The zero solution of the interval indirect control system (8.19) is robustly absolutely stable if and only if*

1. *Condition 1 in Theorem 8.22 (or the condition (2) in Theorem 8.23) holds.*
2. *The zero solution of system (8.19) is robustly absolutely stable w.r.t. the partial variable* $x_{j+1}, \dots, x_n, \sigma$.

8.7 Sufficient Conditions for the Robust Absolute Stability of System (8.19)

For the convenience of verifying the conditions in the results given in Sect. 8.6, in this section, we give some sufficient conditions for the robust absolute stability of the interval Yocubovich control system (8.19). First, we introduce the following notation:

$$a_{ij}^{(m)} = \max\left\{|\underline{a}_{ij}|,\ |\overline{a}_{ij}|\right\},\ 1 \le i,j \le n+1,\ i \ne j;\ a_{ii}^{(m)} = \overline{a}_{ii},\ i = 1,2,\dots,n+1.$$

Let

$$A_I^{(j_0)} := \begin{bmatrix} a_{11} & \cdots & a_{1j_0} \\ \vdots & \vdots & \vdots \\ a_{j_01} & \cdots & a_{j_0j_0} \end{bmatrix}_{j_0\times j_0}$$

$$A_I^{(j_0)C} := \begin{bmatrix} a_{1(j_0+1)} & \cdots & a_{1(n+1)} \\ \vdots & \vdots & \vdots \\ a_{j_0(j_0+1)} & \cdots & a_{j_0(n+1)} \end{bmatrix}_{(j_0+1)\times(n+1-j_0)}$$

$$x^{(j_0)} := (x_1, \dots, x_{j_0})^{\mathrm{T}}.$$

Theorem 8.25. *If the following conditions are satisfied:*

1. $A_I^{(j_0)}$ is an interval Hurwitz matrix
2. There exist constants $c_i \geq 0,\ i = 1,2,\dots,j_0,\ c_i > 0,\ i = j_0+1,\dots,n$ such that

$$-c_j\overline{a}_{jj} \geq \sum_{i=1,i\neq j}^{n+1} c_i a_{ij}^{(m)}, \quad j = 1,2,\dots,j_0;$$
$$-c_j\overline{a}_{jj} > \sum_{i=1,i\neq j}^{n+1} c_i a_{ij}^{(m)}, \quad j = j_0+1, j_0+2,\dots,n+1,$$

Then the zero solution of system (8.19) is robustly absolutely stable.

Proof. $\forall A \in A_I$, construct the positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n+1} c_i|x_i|.$$

Then the derivation of $V(x)$ along the solution of (8.20) is given by

$$\begin{aligned} D^+V(t)\big|_{(8.20)} &\leq \sum_{j=1}^{n}\left[c_j a_{jj} + \sum_{j=1,j\neq i}^{n} c_i a_{ij}^{(m)}\right]|x_j| \\ &\quad + \left[c_{n+1}a_{(n+1)(n+1)} + \sum_{i=1}^{n} c_i\,|a_{i(n+1)}|\right]|f(x_{n+1})| \\ &\leq \sum_{j=j_0+1}^{n}\left[c_j\overline{a}_{jj} + \sum_{i=1,i\neq j}^{n+1} c_i a_{ij}^{(m)}\right]|x_j| \\ &\quad + \left[c_{n+1}\overline{a}_{(n+1)(n+1)} + \sum_{j=1}^{n} c_i\, a_{i(n+1)}^{(m)}\right]|f(x_{n+1})| \\ &< 0 \qquad \text{when} \sum_{j=j_0+1}^{n+1} x_j^2 \neq 0. \end{aligned}$$

So the zero solution of (8.19) is robustly absolutely stable w.r.t. $x_{j_0+1},\dots,x_n,\sigma$. Then the first j_0 solutions: $x^{(j_0)}(t) := (x_1(t),\dots,x_{j_0}(t))^{\mathrm{T}}$ of (8.20) can be expressed as

$$\begin{aligned} x^{(j_0)}(t) &= \mathrm{e}^{A^{(j_0)}(t-t_0)}x^{(j_0)}(t_0) \\ &\quad + \int_{t_0}^{t} \mathrm{e}^{A^{(j_0)}(t-\tau)}A^{(j_0)C}\Big(x_{j_0+1}(\tau),\dots,x_n(\tau),f(\sigma(\tau)\Big)^{\mathrm{T}}\mathrm{d}\tau. \end{aligned} \tag{8.32}$$

Then we can follow part of proving the sufficiency of Theorem 8.22 to show that the zero solution of (8.19) is robustly absolutely stable for the partial variables $x_1,\dots,x_{j_0}$. With the result of robust absolute stability w.r.t. $x_{j_0+1},\dots,x_{n+1}$, we know that the robust absolute stability is w.r.t. all stable variables. The proof of Theorem 8.25 is complete. □

Example 8.26. Consider the absolute stability of the zero solution of the following four-dimensional deterministic Yocubovich control system (i.e., $\underline{a_{ij}} = \overline{a_{ij}} = a_{ij}$):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix}, \tag{8.33}$$

where

$$A = (a_{ij})_{4\times 4} = \begin{bmatrix} -1 & -1 & 4 & 7 \\ 1 & -1 & -5 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \qquad f(x_4) =\in F_\infty.$$

It is obvious that

$$-\overline{A} = \begin{bmatrix} 1 & 1 & -4 & -7 \\ -1 & 1 & 5 & -2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

is not an M matrix. So there does not exist $c_i > 0,\ i = 1, 2, 3, 4$ such that

$$c_j\overline{a_{jj}} + \sum_{i=1, i\neq j}^{4} c_j\overline{a_{ij}} < 0, \qquad i = 1, 2, 3, 4;$$

but there exist $c_i \geq 0,\ i = 1, 2,\ c_i > 0,\ i = 3, 4$ such that

$$\begin{aligned} c_j\overline{a_{jj}} + \sum_{i=1}^{4} c_j\overline{a_{ij}} &\leq 0, \quad j = 1, 2 \\ c_j\overline{a_{jj}} + \sum_{i=1}^{4} c_j\overline{a_{ij}} &< 0, \quad j = 3, 4. \end{aligned}$$

In fact, choosing $c_1 = c_2 = 0$, $c_3 = c_4 = 1$, we have

$$
\begin{aligned}
&c_1 \overline{a_{11}} + c_2 \overline{a_{21}} + c_3 \overline{a_{31}} + c_4 \overline{a_{41}} = 0 - 0 = 0, \\
&c_2 \overline{a_{22}} + c_1 \overline{a_{12}} + c_3 \overline{a_{32}} + c_4 \overline{a_{42}} = 0 - 0 = 0, \\
&c_3 \overline{a_{33}} + c_1 \overline{a_{13}} + c_2 \overline{a_{23}} + c_4 \overline{a_{43}} = -2 + 1 = -1 < 0, \\
&c_4 \overline{a_{44}} + c_1 \overline{a_{41}} + c_2 \overline{a_{24}} + c_3 \overline{a_{34}} = -2 + 1 = -1 < 0.
\end{aligned}
$$

Thus, for this example, we do not require $\int_0^{\pm\infty} f(x_4)\,\mathrm{d}x_4 = +\infty$, but $\int_0^{\pm\infty} f(x_4)\,\mathrm{d}x_4 < +\infty$.

Now, for the variables x_3 and x_4, we construct the positive definite and radially unbounded Lyapunov function:

$$
V = \sum_i^4 c_i\, |x_i| = \frac{1}{5}\,(|x_1| + |x_2|) + |x_3| + |x_4|,
$$

and thus

$$
D^+V\Big|_{(8.33)} \le -\frac{4}{5}\,(|x_3| + |x_4|) \quad \forall\ x_3^2 + x_4^2 \neq 0.
$$

Therefore, the zero solution of (8.33) is absolutely stable with respect to the partial variables x_3 and x_4.

Let

$$
A_{11} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 7 \\ -5 & 2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}.
$$

It is easy to see that A_{11} is a Hurwitz matrix. The solution for the first two variables of (8.33) can be expressed as

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \mathrm{e}^{A_{11}(t-t_0)} \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} + \int_{t_0}^{t} \mathrm{e}^{A_{11}(t-\tau)} A_{12} \begin{pmatrix} x_3(\tau) \\ f(x_4(\tau)) \end{pmatrix} \mathrm{d}\tau
$$

from which we know that the zero solution of (8.33) is also absolutely stable with respect to the variables x_1 and x_2. Summarizing the above results we have shown that the zero solution of (8.33) is absolutely stable for all its variables.

Corollary 8.27. *Assume that*

1. There exist constants $\tilde{c}_i > 0,\ i = 1, 2, \ldots, j_0$ *such that*

$$
-\tilde{c}_j\, \bar{a}_{jj} > \sum_{i=1, i\neq j}^{j_0} c_i\, a_{ij}^{(m)},\ j = 1, 2, \ldots, j_0;
$$

2. The condition (2) of Theorem 8.25 is satisfied

then the zero solution of (8.19) is robustly absolutely stable.

Proof. Since condition (1) implies that $A_I(a_{ij})_{j_0\times j_0}$ is a Hurwitz interval matrix, the conditions of Theorem 8.25 hold. This implies that the conclusion of Corollary is true. □

Theorem 8.28. *Suppose that the following conditions are satisfied:*

1. $A_I^{(j_0)}$ *is a Hurwitz interval matrix, and* $\int_0^{+\infty} f(\sigma)\mathrm{d}\sigma = +\infty$;
2. *There exists constants* $c_i > 0,\ i = 1,2,\ldots,n+1$ *and* $0 < \varepsilon \ll 1$ *such that the matrix:*

$$H := \begin{bmatrix} 2c_1\overline{a}_{11} & m_{12} & \cdots & m_{1j_0} & m_{1(j_0+1)} & \cdots & m_{1(n+1)} \\ m_{12} & 2c_2\overline{a}_{22} & \cdots & m_{2j_0} & m_{2(j_0+1)} & \cdots & m_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ m_{1j_0} & m_{2j_0} & \cdots & 2c_{j_0}\overline{a}_{j_0j_0} & m_{j_0(j_0+1)} & \cdots & m_{j_0(n+1)} \\ m_{1(j_0+1)} & m_{2(j_0+1)} & \cdots & m_{j_0(j_0+1)} & 2c_{j_0+1}\overline{a}_{(j_0+1)(j_0+1)}+\varepsilon & \cdots & m_{(j_0+1)(n+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{1(n+1)} & m_{2(n+1)} & \cdots & m_{j_0(n+1)} & m_{(j_0+1)(n+1)} & \cdots & 2c_{n+1}\overline{a}_{(n+1)(n+1)}+\varepsilon \end{bmatrix} \le 0, \tag{8.34}$$

where

$$m_{ij} = \max_{\underline{a}_{ij}\le a_{ij}\le\overline{a}_{ij}} \left[|c_i a_{ij} + c_j a_{ji}|\right], \quad i \ne j, \quad 1 \le i, \quad j \le n+1$$

then the zero solution of system (8.19) is robustly absolutely stable.

Proof. First, we construct the positive definite and radially unbounded Lyapunov function:

$$V = \sum_{i=1}^{n} c_i x_i^2 + 2c_{n+1}\int_0^{\sigma} f(s)\,\mathrm{d}s$$

from which we obtain

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(8.20)} = 2\sum_{i=1}^{n} c_i x_i \dot{x}_i + 2c_{n+1} f(\sigma)\,\dot{\sigma}$$

$$\le \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ f(\sigma) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{a}_{11} & m_{12} & \cdots & m_{1j_0} & \cdots & m_{1(n+1)} \\ m_{12} & 2c_2\overline{a}_{22} & \cdots & m_{2j_0} & \cdots & m_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ m_{1j_0} & m_{2j_0} & \cdots & 2c_{j_0}\overline{a}_{j_0j_0} & \cdots & m_{j_0(n+1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{1(n+1)} & m_{2(n+1)} & \cdots & m_{j_0(n+1)} & \cdots & 2c_n\overline{a}_{(n+1)(n+1)} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ f(\sigma) \end{pmatrix}$$

$$\leq \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ f(\sigma) \end{pmatrix}^{\mathrm{T}} H \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ f(\sigma) \end{pmatrix} - \sum_{j=j_0+1}^{n} \varepsilon x_i^2 - \varepsilon f^2(\sigma)$$

$$\leq -\varepsilon \left[\sum_{j=j_0+1}^{n} x_i^2 + f^2(\sigma) \right] < 0 \quad \text{when} \quad \sum_{j=j_0+1}^{n} x_i^2 + f^2(\sigma) \neq 0. \tag{8.35}$$

Thus, the zero solution of system (8.19) is robustly absolutely stable w.r.t. the partial variables $x_{j_0+1}, \ldots, x_n, \sigma$.

Next, note that the first j_0 component solutions $x^{(j_0)} := (x_1(t), \ldots, x_{j_0}(t))^{\mathrm{T}}$ of (8.20) can be expressed by (8.32). This shows that the zero solution of (8.19) is robustly absolutely stable w.r.t. $x^{(j_0)}$. The proof is complete. □

Corollary 8.29. *If*

1. There exist constants $\tilde{c}_i > 0$, $i = 1, 2, \ldots, j_0$ such that

$$\begin{bmatrix} 2\tilde{c}_1 \overline{a}_{11} & \tilde{m}_{12} & \cdots & \tilde{m}_{1j_0} \\ \tilde{m}_{12} & 2\tilde{c}_2 \tilde{a}_{22} & \cdots & 2\tilde{j}_0 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{m}_{1j_0} & \tilde{m}_{2j_0} & \cdots & 2\tilde{c}_{j_0} \overline{a}_{j_0 j_0} \end{bmatrix} < 0,$$

where

$$\tilde{m}_{ij} = \max_{\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}} [|\tilde{c}_i a_{ij} + \tilde{c}_j a_{ji}|], \quad i \neq j, \quad 1 \leq i, \quad j \leq j_0 + 1.$$

2. Condition (2) of Theorem 8.28 holds.

Then the zero solution of (8.19) is robustly absolutely stable.

Proof. Because condition (1) implies that $A_I(a_{ij})_{j_0 \times j_0}$ is a Hurwitz interval matrix, so the conclusion is true. □

Example 8.30. Consider the absolute stability of the zero solution of the following four-dimensional deterministic Yocubovich control system (i.e., $\underline{a_{ij}} = \overline{a_{ij}} = a_{ij}$):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} 0 & 3 & -1 & 0 \\ -6 & -1 & 4 & -5 \\ 2 & -4 & -2 & 1 \\ 0 & 5 & -1 & -3 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} \tag{8.36}$$

where $f(x_4) \in F_\infty$ satisfying $\int_0^{\pm\infty} f(x_4)\, dx_4 = +\infty$.

With the positive definite and radially unbounded Lyapunov function

$$V(x) = x_1^2 + \frac{1}{2}(x_2^2 + x_3^2) + \int_0^{x_4} f(x_4)\, \mathrm{d}x_4,$$

we obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}V(x)}{\mathrm{d}t}\right|_{(8.36)} &= 2x_1(3x_2 - x_3) + x_2[-6x_1 - x_2 + 4x_3 - 5f(x_4)] \\
&\quad + x_3[2x_1 - 4x_2 - 2x_3 + f(x_4)] + f(x_4)[5x_2 - x_3 - 3f(x_4)] \\
&\quad - x_2^2 - 2x_3^2 - 3f^2(x_4) \\
&< 0 \qquad \text{when } x_2^2 + x_3^2 + x_4^2 \neq 0.
\end{aligned} \tag{8.37}$$

Thus, the zero solution of (8.36) is absolutely stable about the variables x_2, x_3, and x_4, and in particular, with respect to the variables x_3, and x_4.

Now let

$$A_{11} = \begin{bmatrix} 0 & 3 \\ -6 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -1 & 0 \\ 4 & -5 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix}.$$

Obviously, A_{11} is a Hurwitz matrix. Expressing the solution of the first two equations of (8.36) as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \mathrm{e}^{A_{11}(t-t_0)} \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} + \int_{t_0}^{t} \mathrm{e}^{A_{11}(t-\tau)} A_{12} \begin{pmatrix} x_3(\tau) \\ f(x_4(\tau)) \end{pmatrix} \mathrm{d}\tau,$$

we know that the zero solution of (8.36) is also absolutely stable about variables x_1 and x_2.

Alternatively, let

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 3 & -1 \\ -6 & -1 & 4 \\ 2 & -4 & -2 \end{bmatrix}, \quad \tilde{A}_{12} = \begin{pmatrix} 0 \\ -5 \\ 1 \end{pmatrix}, \quad \tilde{A}_{21} = (0\ 5\ 1), \quad \tilde{A}_{22} = (-3).$$

It is easy to verify that $\tilde{A}_{11}$ is a Hurwitz matrix. The solution of the first three equations of (8.36) can be written as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \mathrm{e}^{\tilde{A}_{11}(t-t_0)} \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{pmatrix} + \int_{t_0}^{t} \mathrm{e}^{\tilde{A}_{11}(t-\tau)} \tilde{A}_{12} f(x_4(\tau))\, \mathrm{d}\tau$$

from which we can conclude that the zero solution of (8.36) is also stable with respect to the variables x_1, x_2 and x_3.

In a summary for this example, the zero solution of (8.36) is absolutely stable about all of its variables.

Remark 8.31. It is easy to see that for system (8.36) one cannot use the approach of the so-called diagonal stability, i.e., the following positive definite and radially unbounded Lyapunov function:

$$V = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 + c_4 \int_0^{x_4} f(x_4)\, dx_4 \quad (c_i > 0,\ i = 1, 2, 3, 4)$$

cannot be applied just in one step to prove the absolute stability of the zero solution of system (8.36) about all of its variables. This is because the necessary condition for applying the above Lyapunov function to get negative definite of $\frac{\mathrm{d}V}{\mathrm{d}t}$ is $a_{ii} < 0$. However, for this example, $a_{11} = 0$.

For a similar reason, one cannot use the following form of Lyapunov function:

$$V = \sum_{i=1}^{4} c_i |x_i|$$

to prove the absolute stability of the zero solution of (8.36) about all of its variables.

However, for such a system, with the theory and method of absolute stability about partial variables, we have very high flexibility to choose Lyapunov functions, and use different methods to deal with the absolute stability of partial variables, and thus obtain the absolute stability for all variables. For example, in Example 8.30, to obtain the absolute stability of the zero solution of system (8.36) about all its variables, we first used the direct Lyapunov method to prove that the zero solution is absolutely stable about the partial variables x_3 and x_4, then we apply the Lagrange constant variation approach and L'Hospital rule to that the zero solution is also absolutely stable with respect to the variables x_1 and x_2.

In the following, we show that there exists a full-rank linear transformation between systems (8.19) and (8.20) [93].

Since $\overline{a}_{(n+1)(n+1)} \le 0$ is a necessary condition for the robust absolute stability of the zero solution of (8.19), we only consider the case of $\overline{a}_{(n+1)(n+1)} < 0$. To achieve this, we introduce the following nonsingular linear transformation:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \end{pmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 \; 0 \; \cdots \; 0 & -\dfrac{a_{1(n+1)}}{a_{(n+1)(n+1)}} \\ 0 \; 1 \; \cdots \; 0 & -\dfrac{a_{2(n+1)}}{a_{(n+1)(n+1)}} \\ \vdots \; \vdots \; \ddots \; \vdots & \vdots \\ 0 \; 0 \; \cdots \; 1 & -\dfrac{a_{n(n+1)}}{a_{(n+1)(n+1)}} \\ 0 \; 0 \; \cdots \; 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix} := H \begin{pmatrix} x^{(n)} \\ \sigma \end{pmatrix} \tag{8.38}$$

into (8.20) yields

$$\begin{aligned} \dot{y} = H\dot{x} = HA \begin{pmatrix} x \\ f(\sigma) \end{pmatrix} &= HA \left[\begin{pmatrix} x \\ \sigma \end{pmatrix} + \begin{pmatrix} 0 \\ f(\sigma) - \sigma \end{pmatrix} \right] \\ &= HA \begin{pmatrix} x \\ \sigma \end{pmatrix} + HA \begin{pmatrix} 0 \\ f(\sigma) \end{pmatrix} - HA \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \\ &= HAH^{-1}y - HA \begin{pmatrix} 0 \\ y_{n+1} \end{pmatrix} + HA \begin{pmatrix} 0 \\ f(\sigma) \end{pmatrix} \\ &\coloneqq By + hf(\sigma), \end{aligned} \tag{8.39}$$

where $x^{(n)} = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $x = (x^{(n)}, \sigma)$. The corresponding interval indirect control system of (8.39) is given by

$$\dot{y} = B_I y + h_I f(y_{n+1}), \tag{8.40}$$

where

$$\begin{aligned}
&B_I = \left\{B : \ \underline{B} \leq B \leq \overline{B}, \text{ i.e., } \underline{b}_{ij} \leq b_{ij} \leq \overline{b}_{ij},\ i,j = 1,2,\ldots,n+1\right\},\\
&h_I = \left\{h : \ \underline{h} \leq h \leq \overline{h}, \text{ i.e., } \underline{h}_i \leq h_i \leq \overline{h}_i,\ i = 1,2,\ldots,n\right\},\\
&\underline{a}_{(n+1)(n+1)} = \underline{h}_{n+1} \leq h_{n+1} \leq \overline{h}_{n+1} = \overline{a}_{(n+1)(n+1)} < 0.
\end{aligned}$$

System (8.40) is the same as system (2) of [93], except for that (8.40) here has $n+1$ dimension, while that in [93] has n dimension. Therefore, the conclusions obtained for system (2) of [93] can be directly applied here, and the details are not repeated here.

Thus, in the following, we will only present two new results.

Theorem 8.32. *If the zero solution of system (8.40) is robustly absolutely stable w.r.t. $y_1, y_2, \ldots, y_n$, and $\overline{b}_{(n+1)(n+1)} < 0$, then the zero solution of (8.40) is robustly absolutely stable w.r.t. all state variables.*

Proof. We employ the following positive definite and radially unbounded Lyapunov function:

$$V = y_{n+1}^2.$$

Take ε satisfying $0 < n\varepsilon \leq -\overline{b}_{(n+1)(n+1)}$, and let

$$\frac{(\max\{|\underline{b}_{(n+1)j}|, |\overline{b}_{(n+1)j}|\})^2}{\varepsilon} = \xi_j.$$

Then differentiating V w.r.t. time t along the solution trajectory of system (8.40) results in

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(8.40)} &= 2\sum_{j=1}^{n} b_{n+1,j}\, y_j\, y_{n+1} + 2\, b_{(n+1)(n+1)} y_{n+1}^2 + 2\, h_{(n+1)}\, y_{n+1}\, f(y_{n+1})\\
&\leq 2\sum_{j=1}^{n} \frac{(\max\{|\underline{b}_{(n+1)j}|, |\overline{b}_{(n+1)j}|\})}{\sqrt{\varepsilon}}\ |y_j|\sqrt{\varepsilon}\,|y_{n+1}| + 2\overline{b}_{(n+1)(n+1)} y_{n+1}^2\\
&\leq \overline{b}_{(n+1)(n+1)} y_{n+1}^2 + \sum_{j=1}^{n} \xi_j y_j^2\\
&= \overline{b}_{(n+1)(n+1)}\, V + \sum_{j=1}^{n} \xi_j y_j^2.
\end{aligned}$$

Thus,

$$y_{n+1}^2(t) \le \mathrm{e}^{\overline{b}_{(n+1)(n+1)}(t-t_0)} y_{n+1}^2(t_0) + \int_{t_0}^{t} \mathrm{e}^{\overline{b}_{(n+1)(n+1)}(t-\tau)} \sum_{j=1}^{n} \xi_j y_j^2(\tau)\,\mathrm{d}\tau. \tag{8.41}$$

Now following the proof for the sufficiency of Theorem 8.22, we can show that the zero solution of system (8.40) is robustly absolutely stable w.r.t. y_{n+1}, and therefore, Theorem 8.32 is proved. □

Theorem 8.33. *If the interval indirect control system (8.40) satisfies one of the following conditions:*

1. B_I is an interval Hurwitz matrix, and there exist constants $c_j > 0,\ j = 1,2,\ldots,$ $n+1$ such that

$$c_j \overline{b}_{jj} + \sum_{i=1, i\neq j}^{n+1} c_i \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\} \le 0, \quad j = 1,2,\ldots,n+1 \tag{8.42}$$

2. There exist constants $c_j > 0,\ j = 1,2,\ldots,n+1$ such that (8.42) holds and

$$\max\{|\underline{b}_{(n+1)j}|, |\overline{b}_{(n+1)j}|\} \neq 0, \quad j = 1,2,\ldots,n;$$

then the zero solution of system (8.40) is robustly absolutely stable.

Proof. First consider when condition (1) is satisfied. For this case, we can construct the positive definite and radially unbounded Lyapunov function:

$$V = \sum_{i=1}^{n+1} c_i |y_i|,$$

and thus obtain

$$\begin{aligned} D^+V\big|_{(8.40)} &\le \sum_{i=1}^{n+1} c_i \dot{y}_i \operatorname{sign}(y_i) \\ &\le \sum_{j=1}^{n+1} \left[c_j \overline{b}_{jj} + \sum_{i=1, i\neq j}^{n+1} c_i \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\} \right] |y_j| + c_{n+1}\overline{h}_{n+1}|f(y_{n+1})| \\ &\le c_{n+1}\overline{h}_{n+1}|f(y_{n+1})| < 0 \qquad \text{when} \quad y_{n+1} \neq 0. \end{aligned} \tag{8.43}$$

Therefore, the zero solution of (8.40) is robustly absolutely stable w.r.t. y_{n+1}.

Since any solution $y(t)$ of (8.40) can be expressed as

$$y(t) = \mathrm{e}^{B_I(t-t_0)} y(t_0) + \int_{t_0}^{t} \mathrm{e}^{B_I(t-\tau)} h_I f(y_{n+1}(\tau))\,\mathrm{d}\tau, \tag{8.44}$$

it is easy to follow the proof of Theorem 8.22 to show that the zero solution of (8.40) is robustly absolutely stable w.r.t. all state variables.

Now, consider condition (2). We rewrite (8.40) as

$$\dot{y} = B_I y + h_I y_{n+1} + h_I\Big(f(y_{n+1}) - y_{n+1}\Big). \tag{8.45}$$

Let

$$\tilde{c}_{n+1} = \frac{c_{n+1}(b_{(n+1)(n+1)} + \frac{1}{2}h_{n+1})}{b_{(n+1)(n+1)} + h_{n+1}}.$$

Then $\tilde{c}_{n+1} < c_{n+1}$. We construct the positive definite and radially unbounded Lyapunov function:

$$V = \sum_{i=1}^{n} c_i\,|y_i| + \tilde{c}_{n+1}|y_{n+1}|.$$

Due to $\max\{|\underline{b}_{(n+1)j}|, |\overline{b}_{(n+1)j}|\} \neq 0$, we consider the linear part of (8.45):

$$\dot{y} = B_I y + h_I y_{n+1}, \tag{8.46}$$

and then obtain

$$\begin{aligned}
D^+V\big|_{(8.46)} \leq \sum_{j=1}^{n} \Bigg[& c_j\overline{b}_{jj} + \sum_{i=1, i\neq j}^{n} c_i \max\left(|\underline{b}_{ij}|, |\overline{b}_{ij}|\right) \\
& + \tilde{c}_{n+1}\max\Big(|\underline{b}_{(n+1)j}|, |\overline{b}_{(n+1)j}|\Big)\Bigg]\,|y_j(t)| \\
& + \Bigg[\tilde{c}_{n+1}(\overline{b}_{(n+1)(n+1)} + \overline{h}_{n+1}) \\
& + \sum_{i=1}^{n} c_i \max\Big(|\underline{b}_{i(n+1)}|, |\overline{b}_{i(n+1)}|\Big)\Bigg]\,|y_{n+1}(t)| \\
:= \sum_{j=1}^{n} -\xi_j|y_j| + \Bigg[& c_{n+1}\overline{b}_{(n+1)(n+1)} \\
& + \sum_{i=1}^{n} c_i \max\Big(|\underline{b}_{i(n+1)}|, |\overline{b}_{i(n+1)}|\Big) + \frac{1}{2}c_{n+1}h_{n+1}\Bigg]\,|y_{n+1}(t)| \\
\leq \sum_{j=1}^{n} -\xi_j|y_i| + \frac{1}{2}c_{n+1}\overline{h}_{n+1}|y_{n+1}| < 0 & \qquad \text{when} \quad \sum_{i=1}^{n+1}|y_i| \neq 0.
\end{aligned}$$

So the coefficient matrix of (8.46) is an interval Hurwitz matrix.

The general solution of (8.45) can be expressed as

$$\begin{aligned}
y(t) &= \mathrm{e}^{[B_I + (O_{(n+1)\times n} h_I)](t-t_0)} \\
&\quad + \int_{t_0}^{t} \mathrm{e}^{[B_I + (O_{(n+1)\times n} h_I)](t-\tau)} \left[h_I\Big(f(y_{n+1}(\tau)) - y_{n+1}(\tau)\Big)\right] \mathrm{d}\tau.
\end{aligned} \tag{8.47}$$

Then, similarly following the proof of Theorem 8.22, we can show that the zero solution of (8.45) (i.e., (8.40)) is robustly absolutely stable.

The proof of Theorem 8.33 is complete. □

To end this section, we give an example to demonstrate the applicability of the theoretical results obtained in this section.

8.8 Numerical Examples and Simulation Results

In this section, we present several numerical examples to demonstrate the applicability of theorems given in the previous two sections. Numerical simulation is employed to verify the analytical predictions.

Example 8.34. Consider the robust absolute stability of the zero solution of the following interval indirect control system:

$$\begin{pmatrix} \dot{x} \\ \dot{\sigma} \end{pmatrix} = \begin{bmatrix} [-2,0] & [-2,-1] \\ [3,4] & [-2,-1] \end{bmatrix} \begin{pmatrix} x \\ f(\sigma) \end{pmatrix} := A_I \begin{pmatrix} x \\ f(\sigma) \end{pmatrix}, \quad f(\sigma) \in F. \tag{8.48}$$

$\forall a_{11} \in [-2,0],\ a_{12} \in [-2,-1],\ a_{21} \in [3,4],\ a_{22} \in [-2,-1]$, the corresponding system can be written as

$$\begin{aligned} \dot{x} &= a_{11} x + a_{12} f(\sigma), \\ \dot{\sigma} &= a_{21} x + a_{22} f(\sigma). \end{aligned} \tag{8.49}$$

Construct the positive definite and radially unbounded Lyapunov function:

$$V = \frac{x^2}{|a_{12}|} + \frac{2\int_0^{\sigma} f(s)\,ds}{a_{21}}.$$

Thus,

$$\begin{aligned} \left.\frac{dV}{\mathrm{d}t}\right|_{(8.49)} &= \frac{2a_{11}}{|a_{12}|} x^2 - 2x f(\sigma) + 2x f(\sigma) + \frac{2a_{22}}{a_{21}} f^2(\sigma) \\ &\le \frac{2a_{22}}{a_{21}} f^2(\sigma) \le \frac{-2f^2(\sigma)}{a_{21}} < 0 \qquad \text{when } \sigma \neq 0. \end{aligned} \tag{8.50}$$

Thus, the zero solution of (8.48) is robustly absolutely stable w.r.t. σ.

Now, we verify that A_I is an interval Hurwitz matrix. In fact, $\forall a_{11} \in [-2,0]$, $a_{12} \in [-2,-1]$, $a_{21} \in [3,4]$, $a_{22} \in [-2,-1]$, since $a_{11} \le 0$, $a_{22} < 0$, $a_{12} < 0$, $a_{21} > 0$, $a_{11} + a_{22} < 0$, $a_{11}a_{22} + a_{12}a_{21} \ge 0 + a_{12}a_{21} > 3 > 0$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a Hurwitz matrix. Due to arbitrary of $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in A_I$, so A_I is an interval Hurwitz matrix. Thus, according to Theorem 8.22, the zero solution of (8.48) is robustly absolutely stable.

Example 8.35. Consider the robust absolute stability of the zero solution of a three-dimensional interval indirect control system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\sigma} \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ f(\sigma) \end{pmatrix} = \begin{bmatrix} [-4,-3] & [3,4] & [-4,-3] \\ [-3,-2] & [-3,-2] & [-1,1] \\ [1,2] & [-1,1] & [-4,-3] \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ f(\sigma) \end{pmatrix} \tag{8.51}$$

Construct the positive definite and radially unbounded Lyapunov function:

$$V = \frac{x_1^2}{2a_{12}} - \frac{x_2^2}{2a_{21}} + \int_0^{\sigma} f(s)\,ds.$$

Then,

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(8.51)} &= \frac{a_{11}}{a_{12}}x_1^2 + x_1x_2 + \frac{a_{13}}{a_{12}}x_1 f(\sigma) - x_1x_2 - \frac{a_{22}}{a_{21}}x_2^2 - \frac{a_{23}}{a_{21}}x_2 f(\sigma) \\
&\quad + a_{31}x_1 f(\sigma) + a_{32}x_2 f(\sigma) + a_{33} f^2(\sigma) \\
&\le -\frac{3}{4}x_1^2 + \left(\frac{a_{13}}{a_{12}} + a_{31}\right)x_1 f(\sigma) - \frac{2}{3}x_2^2 + \left|-\frac{a_{23}}{a_{21}} + a_{32}\right| |x_2 f(\sigma)| - 3f^2(\sigma) \\
&\le -\frac{3}{4}x_1^2 - \frac{2}{3}x_2^2 + \left|\frac{a_{13}}{a_{12}} + a_{31}\right| |x_1 f(\sigma)| + \frac{3}{2}|x_2 f(\sigma)| - 3f^2(\sigma) \\
&\le \begin{pmatrix} |x_1| \\ |x_2| \\ |f(\sigma)| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -\frac{3}{4} & 0 & \frac{1}{2}\left|\frac{a_{13}}{a_{12}} + a_{31}\right| \\ 0 & -\frac{2}{3} & \frac{1}{2}\left|-\frac{a_{23}}{a_{21}} + a_{32}\right| \\ \frac{1}{2}\left|\frac{a_{13}}{a_{12}} + a_{31}\right| & \frac{1}{2}\left|-\frac{a_{23}}{a_{21}} + a_{32}\right| & -3 \end{bmatrix} \begin{pmatrix} |x_1| \\ |x_2| \\ |f(\sigma)| \end{pmatrix} \\
&< 0 \quad \text{for } x_1^2 + x_2^2 + f^2(\sigma) \neq 0.
\end{aligned} \tag{8.52}$$

Thus, the zero solution of (8.48) is robustly absolutely stable.

For simulation of this example, we take the upper bounds of the system coefficients and $f(\sigma) = \sigma^3$ to obtain the system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\sigma} \end{pmatrix} = \begin{bmatrix} -3 & 4 & -3 \\ -2 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \sigma^3 \end{pmatrix}. \tag{8.53}$$

The simulation result is shown in Fig. 8.1, where the initial point is taken as $(x_1, x_2, \sigma) = (8, -6, 4)$. It is seen that all the three components converge to the origin – the equilibrium point.

Example 8.36. Analyze the robust absolute stability of the zero solution of the following interval indirect control system:

$$\begin{aligned}
\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\sigma} \end{pmatrix} &= \begin{bmatrix} [\underline{b}_{11}, \overline{b}_{11}] & [\underline{b}_{12}, \overline{b}_{12}] & [\underline{b}_{13}, \overline{b}_{13}] \\ [\underline{b}_{21}, \overline{b}_{21}] & [\underline{b}_{22}, \overline{b}_{22}] & [\underline{b}_{23}, \overline{b}_{23}] \\ [\underline{b}_{31}, \overline{b}_{31}] & [\underline{b}_{32}, \overline{b}_{32}] & [\underline{b}_{33}, \overline{b}_{33}] \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \sigma \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ [\underline{h}_3, \overline{h}_3] \end{bmatrix} f(\sigma) \\
&= \begin{bmatrix} [-1.1, -1] & [-0.5, 0.5] & [-0.5, 0.5] \\ [-1, 0] & [-3.5, -3] & [-1, 1] \\ [0, 1] & [-2, 2] & [-2.5, -2] \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \sigma \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ [-1.1, -1] \end{bmatrix} f(\sigma).
\end{aligned} \tag{8.54}$$

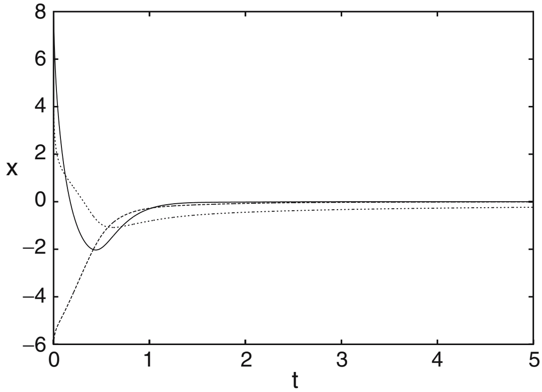

Fig. 8.1 Simulated solution of system (8.53) for Example 8.35 converging to the origin, from the initial point: $(x_1, x_2, \sigma) = (8, -6, 4)$

Taking $c_1 = 2$, $c_2 = c_3 = 1$, we obtain

$$\begin{aligned}
&c_1\overline{b}_{11} + c_2 \max[|\underline{b}_{21}|, |\overline{b}_{21}|] + c_3 \max[|\underline{b}_{31}|, |\overline{b}_{31}|] = -2+1+1 = 0,\\
&c_2\overline{b}_{22} + c_1 \max[|\underline{b}_{12}|, |\overline{b}_{12}|] + c_3 \max[|\underline{b}_{32}|, |\overline{b}_{32}|] = -3+1+2 = 0,\\
&c_3\overline{b}_{33} + c_1 \max[|\underline{b}_{13}|, |\overline{b}_{13}|] + c_2 \max[|\underline{b}_{23}|, |\overline{b}_{23}|] = -2+1+1 = 0,\\
&\overline{h}_3 = -1,
\end{aligned}$$

and $\overline{b}_{13}\overline{b}_{32}\overline{b}_{33} \neq 0$. Thus, condition (2) Theorem 8.33 is satisfied. Therefore, the zero solution of (8.54) is robustly absolutely stable.

For this example, we take $\underline{b}_{ij} \leq b_{ij} \leq \overline{b}_{ij}$, $h_3 = \overline{h}_3$ and $f(\sigma) = \sigma^5$, giving the following system:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\sigma} \end{pmatrix} = \begin{bmatrix} -1 & 0.5 & 0.5 \\ -1 & -3 & 1 \\ 1 & 2 & -2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \sigma \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \sigma^5. \tag{8.55}$$

The simulated result is depicted in Fig. 8.2, which again shows that all solution components converge to the origin, as expected. The initial point is taken as the same as that for Example 8.35: $(y_1, y_2, \sigma) = (8, -6, 4)$.

Example 8.37. The final example is to consider the robust absolute stability of the zero solution of a four-dimensional interval indirect control system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{\sigma} \end{pmatrix} = \begin{bmatrix} [\underline{a}_{11}, \overline{a}_{11}] & [\underline{a}_{12}, \overline{a}_{12}] & [\underline{a}_{13}, \overline{a}_{13}] & [\underline{a}_{14}, \overline{a}_{14}] \\ [\underline{a}_{21}, \overline{a}_{21}] & [\underline{a}_{22}, \overline{a}_{22}] & [\underline{a}_{23}, \overline{a}_{23}] & [\underline{a}_{24}, \overline{a}_{24}] \\ [\underline{a}_{31}, \overline{a}_{31}] & [\underline{a}_{32}, \overline{a}_{32}] & [\underline{a}_{33}, \overline{a}_{33}] & [\underline{a}_{34}, \overline{a}_{34}] \\ [\underline{a}_{41}, \overline{a}_{41}] & [\underline{a}_{42}, \overline{a}_{42}] & [\underline{a}_{43}, \overline{a}_{43}] & [\underline{a}_{44}, \overline{a}_{44}] \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ f(\sigma) \end{pmatrix}$$

$$= \begin{bmatrix} [-3.1, -3] & [-2, 2] & [0, 1] & [-0.9, 0.9] \\ [-1, 1] & [-4.5, -4] & [-1, 1] & [0, 1] \\ [-1, 0] & [-1, 1] & [-3.5, -3] & [-1, 1] \\ [0, 1] & [0, 1] & [-1, 0] & [-4, -3] \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ f(\sigma) \end{pmatrix}. \tag{8.56}$$

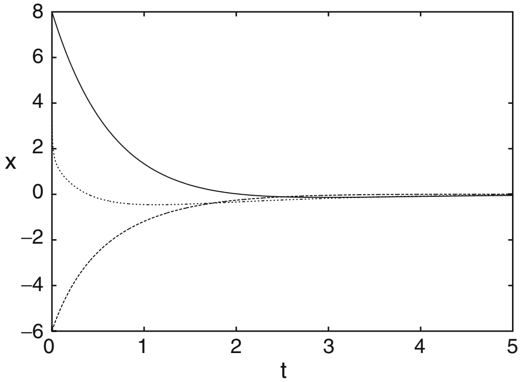

Fig. 8.2 Simulated solution of system (8.55) for Example 8.36 converging to the origin, from the initial point: $(y_1, y_2, \sigma) = (8, -6, 4)$

We take $c_1 = c_2 = c_3 = c_4 = 1$. Then

$$-c_j\overline{a}_{jj} = \sum_{i=1, i\neq j}^{4} c_i a_{ij}^{(m)}, \quad j = 1,2,3, \quad \text{and} \quad -c_4\overline{a}_{44} > \sum_{i=1}^{3} c_i a_{ij}^{(m)}, \tag{8.57}$$

where $a_{ij}^{(m)} = \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$. So the zero solution of (8.56) is robustly absolutely stable w.r.t. σ. But since

$$-c_j\overline{a}_{jj} > \sum_{i=1, i\neq j}^{3} c_i a_{ij}^{(m)}, \quad j = 1,2,3, \tag{8.58}$$

implies that $A^{(j_0)}(a_{ij})_{j_0\times j_0} = A^{(3)}(a_{ij})_{3\times 3}$ is an interval Hurwitz matrix, by Theorem 8.25 we know that the zero solution of (8.56) is robustly absolutely stable.

To simulate this example, we similarly choose a definite system as the follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{\sigma} \end{pmatrix} = \begin{bmatrix} -3 & 2 & 1 & 0.9 \\ 1 & -4 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & -1 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \sigma^7 \end{pmatrix}. \tag{8.59}$$

where $f(\sigma)$ has been taken as $f(\sigma) = \sigma^7$.

The simulation result, as shown in Fig. 8.3, again confirms the analytical prediction: all the four state components converge to the origin. Here the initial point is chosen as $(x_1, x_2, x_3, \sigma) = (8, -6, 6, 1)$. It is seen that all the three components converge to the origin. However, note that the convergence of the variable σ is slow due to the nonlinear term σ^7 and the relatively large initial condition.

8.9 More General Interval Systems [171, 172]

In this section, we consider more general systems, which include interval Yocubovich control system as a special case, described by

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} f_{ij}(x_j), \qquad j = 1, 2, \ldots, n, \tag{8.60}$$

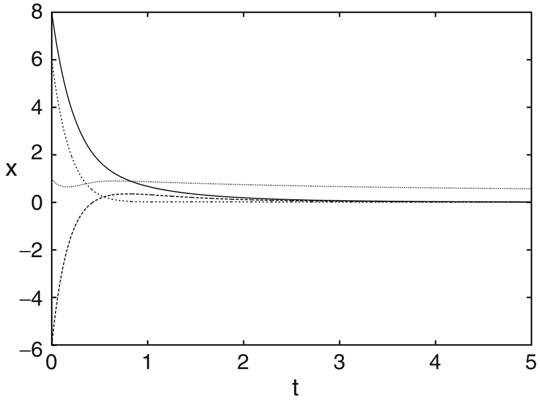

Fig. 8.3 Simulated solution of system (8.53) for Example converging to the origin, from the initial point: $(x_1, x_2, x_3, \sigma) = (8, -6, 6, 1)$

where $a_{ij} \in [\underline{a_{ij}}, \overline{a_{ij}}]$, $\overline{a_{ii}} < 0$, $i = 1, 2, \ldots, n$.

Without loss of generality, we may assume that $-\underline{a_{ij}} = \overline{a_{ij}}$, $i \neq j$, $i, j = 1, 2, \ldots, n$, since if $-\underline{a_{ij}} \neq \overline{a_{ij}}$, we can consider $\overline{a}_{ij} = \max[|\underline{a_{ij}}|, |\overline{a_{ij}}|]$, $-\underline{a}_{ij} = \overline{a}_{ij}$. The larger interval $[-\underline{a}_{ij}, \overline{a}_{ij}]$ is symmetric. Further, assume that $|f_{ij}(x_j)| \leq |f_{jj}(x_j)|$ $f_{ij}(x_j) \in F_\infty$, $i, j = 1, 2, \ldots, n$.

Theorem 8.38. *The sufficient condition for the zero solution of the interval nonlinear control system (8.60) being absolutely stable is* $-\overline{A} = (-\overline{a_{ij}})_{n\times n}$ *is an* M *matrix (i.e.,* $\overline{A} = (\overline{a_{ij}})_{n\times n}$ *is a Hurwitz matrix).*

Proof. *Necessity*. Take $f_{ij}(x_j) = x_j$, then system (8.60) becomes an interval linear system:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, 2, \ldots, n. \tag{8.61}$$

Since the zero solution of (8.60) is robust absolutely stable, the zero solution of (8.61) is robust globally and asymptotically stable. In particular, choose $\overline{A} \in A_I = \{A,\ \underline{A} \leq A \leq \overline{A}\}$. Then the zero solution of the system

$$\dot{x}_i = \sum_{j=1}^{n} \overline{a_{ij}} x_j, \qquad i = 1, 2, \ldots, n$$

is globally and asymptotically stable. From the assumption: $\overline{a_{ii}} < 0$, $\overline{a_{ij}} \geq 0$, we know that $\overline{A} = (\overline{a_{ij}})_{n\times n}$ is a Hurwitz matrix (or $-\overline{A}$ is an M matrix). The necessity is proved.

Sufficiency. Since $\overline{A} = (\overline{a_{ij}})_{n\times n}$ is an M matrix, based on the definition of M matrix, there exist positive constants $c_i > 0$, $i = 1, 2, \ldots, n$ such that

$$c_j \overline{a_{jj}} + \sum_{i=1, i\neq j}^{n} c_i \overline{a_{ij}} < 0, \qquad j = 1, 2, \ldots, n. \tag{8.62}$$

Constructing the positive definite and radially unbounded Lyapunov function:

$$V = \sum_{i}^{n} c_i |x_i|,$$

we have

$$\begin{aligned}
D^+V\Big|_{(8.60)} &= \sum_{i=1}^{n} c_i \dot{x}_i \operatorname{sign}(x_i) \\
&= \sum_{i=1}^{n} c_i \Big(\sum_{j}^{n} a_{ij} f_{ij}(x_j) \Big) \operatorname{sign}(x_i) \\
&\leq \sum_{i=1}^{n} c_i \Big(a_{ii} |f_{ii}(x_i)| + \sum_{j=1, j\neq i}^{n} a_{ij} |f_{ij}(x_j)| \Big) \\
&= \sum_{j=1}^{n} c_j \overline{a_{jj}} |f_{jj}(x_j)| + \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i \overline{a_{ij}} |f_{ij}(x_j)| \\
&\leq \sum_{j=1}^{n} c_j \overline{a_{jj}} |f_{jj}(x_j)| + \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i \overline{a_{ij}} |f_{jj}(x_j)| \\
&= \sum_{j=1}^{n} \Big(c_j \overline{a_{jj}} + \sum_{i=1, i\neq j}^{n} c_i \overline{a_{ij}} \Big) |f_{jj}(x_j)| \\
&< 0 \qquad \text{when } \|x\| \neq 0.
\end{aligned}$$

Hence, the zero solution of (8.60) is robust absolutely stable. □

Remark 8.39. Consider the absolute stability of the zero solution of the following deterministic separate variable control system (i.e., not interval system):

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij} f_j(x_j), \qquad i = 1, 2, \ldots, n, \tag{8.63}$$

where $f_j(x_j) \in F_\infty$, and p_{ij} are fixed constants. Consider the absolute stability of the zero solution of the comparison system:

$$\dot{x}_i = \sum_{j=1}^{n} \overline{p_{ij}} f_j(x_j), \qquad i = 1, 2, \ldots, n, \tag{8.64}$$

implies the absolute stability of the zero solution of system (8.62). Here,

$$\overline{p_{ii}} = p_{ii} < 0, \quad \overline{p_{ij}} = |p_{ij}|, \quad i \neq j.$$

It was also proved that the necessary and sufficient condition for the zero solution of system (8.64) being absolute stable is that there exist constants $c_i > 0$ such that

$$\sum_{j}^{n} p_{ij} c_i < 0, \qquad i = 1, 2, \ldots, n, \tag{8.65}$$

i.e., $- - \overline{P} = (-\overline{p_{ij}})_{n\times n}$ is an M matrix.

Obviously, $a_{ij} = \overline{a_{ij}} = a_{ij} = p_{ij}$, $a_{ii} = \overline{a_{ii}} = a_{ii} = p_{ii} < 0$, the condition (8.65) is equivalent to (8.62), and thus the result can be directly obtained from our above theorem as special case.

However, in general, $f_{ij}(x_j) \neq f_j(x_j)$, i.e., for different $1 \leq k, l \leq n$, there may exist $f_{kj}(x_j) \neq f_{lj}(x_j)$. Thus, if we suppose $f_{ij}(x_j) \equiv f_j(x_j)$, then one cannot consider any deterministic system in the form of $(8.1)'$ as a particular case of (8.1). This is because though the last two terms on the right-hand side of $(8.1)'$ contain the same x_n, they can only be written as

$$f_{in}(x_n) = a_{in} x_n + h_i f(x_n), \qquad i = 1, 2, \ldots, n.$$

In general $f_{in}(x_n) \neq f_{jn}(x_n) \neq f_{nn}(x_n)$.

Remark 8.40. For the interval Yocubovich control system (considered in Sects. 8.5–8.8), formally it is a particular case of system (8.60) when $f_{ij}(x_j) \equiv f_j(x_j)$. However, because every term in (8.60) is nonlinear, the necessary and sufficient condition of the absolute stability, (8.62) or (8.65) is very strong. (In fact, the necessary and sufficient condition has restriction, or more precisely, it is a sufficient condition.) In Sects. 8.5–8.8), we made well use of the property of the linear part of the nonlinear systems, and thus we transformed the complex stability problem of a nonlinear system to study the property of interval matrix and the absolute stability of the system about partial variables (in particular one variable). This is the basic idea of reducing the dimension of given system and thus solving the original problem becomes easier. Moreover, there are many existing theories and methods we can use to consider the absolute stability about partial variables. As a matter of fact, some examples we presented in Sects. 8.5–8.7) cannot be considered using the M matrix method described in this section.

Remark 8.41. It is seen from the proof for the sufficiency of Theorem 8.38 that if the condition

$$|f_{ij}(x_j)| \leq |f_{jj}(x_j)|, \quad f_{jj}(x_j) \in F_\infty \quad j = 1, 2, \ldots, n$$

is satisfied and allows $f_{ij}(x_j) \overline{\in} F_\infty$, $i \neq j$, then the sufficiency of Theorem 8.38 still holds. So we have the following result.

Corollary 8.42. *If $|f_{ij}(x_j)| \leq |f_{ij}(x_j)|$, $f_{ij}(x_j) \overline{\in} F_\infty$, $i \neq j$, and $-\overline{A}$ is an M matrix, then the zero solution of system (8.60) is robust absolutely stable.*

Example 8.43. Consider the interval control system:

$$\begin{aligned}
\dot{x}_1 &= a_{11} x_1 + a_{12} \sin(x_2^3) + a_{13} \sin(x_3^5), \\
\dot{x}_2 &= a_{21} \sin(x_1) + a_{22} x_2^3 + a_{23} \sin(x_3^5), \\
\dot{x}_3 &= a_{31} \sin(x_1) + a_{32} \sin(x_2^3) + a_{33} \sin(x_3^5),
\end{aligned} \tag{8.66}$$

where $a_{ij} \in [-\underline{a_{ij}}, \overline{a_{ij}}]$, $-\underline{a_{ij}} = \overline{a_{ij}}$, $i \neq j$, $\overline{a_{ii}} < 0$.

Obviously,

$$
\begin{aligned}
&f_{11}(x_1) = x_1 \in F_\infty, \;\; f_{22}(x_2) = x^3 \in F_\infty, \;\; f_{33}(x_3) = x_3^5 \in F_\infty, \\
&|f_{21}(x_1)| = |f_{31}(x_1)| = |\sin(x_1)| \le |x_1| = |f_{11}(x_1)|, \\
&|f_{12}(x_2)| = |f_{32}(x_2)| = |\sin(x_2^3)| \le |x_2^3| = |f_{22}(x_2)|, \\
&|f_{13}(x_3)| = |f_{23}(x_3)| = |\sin(x_3^5)| \le |x_3^5| = |f_{33}(x_3)|.
\end{aligned}
$$

Thus, if $-(\overline{A}(\overline{a_{ij}})_{n\times n}$ is an M matrix, then the zero solution of the interval system (8.66) is robust stable.

In [68, 71], we constructed a separate variable positive definite and radially unbounded Lyapunov function to study the global stability of the zero solution for the general high dimensional nonlinear autonomous system:

$$
\dot{x} = f(x), \qquad x \in R^n, \;\; f(0) = 0, \;\; f \in C[R^n, R^1] \tag{8.67}
$$

and the separate variable high dimensional nonlinear autonomous system:

$$
\dot{x}_i = \sum_{j=1}^{n} f_{ij}(x_j), \quad j = 1, 2, \ldots, n, \quad f_{ij}(0) = 0, \quad f_{jj}(x_j) \in C[R^1, R^1]. \tag{8.68}
$$

Here, we shall generalize the method and results to consider the robust stability of system (8.60).

Theorem 8.44. *Suppose*

1. $\overline{a_{ii}} < 0$, $f_{ii}(x_i) \in F_\infty$, $i = 1, 2, \ldots, n$, *and* $\int_0^{\pm\infty} f_{ii}(x_i)\,\mathrm{d}x_i = +\infty$.
2. *There exist continuous functions* $c_i(x_i) \ge \delta > 0$ *(in particular,* c_i *are constants),* $i = 1, 2, \ldots, n$ *such that the interval matrix function* $B(b_{ij}(x_j))_{n\times n}$ *is negative definite.*

Then the zero solution of the interval control system (8.60) is robust absolutely stable. Here,

$$
b_{ij}(x_j) = \begin{cases} c_i(x_i)\, a_{ii}, & \text{when } i = j, \; i, j = 1, \ldots, n, \\ \dfrac{1}{2}\left[\dfrac{c_i(x_j) a_{ij} f_{ij}(x_j)}{f_{jj}(x_j)} + \dfrac{c_j(x_j) a_{ji} f_{ji}(x_i)}{f_{ii}(x_i)}\right], & \text{when } i \ne j, \; x_i x_j \ne 0, \\ & \qquad i, j = 1, \ldots, n. \end{cases}
$$

Proof. We construct the following positive definite and radially unbounded Lyapunov function

$$
V(x) = \sum_{i=1}^{n} \int_0^{x_i} c_i(x_i)\, f_{ii}(x_i)\, dx_i, \tag{8.69}
$$

and then evaluate the derivative of $\frac{\mathrm{d}V}{\mathrm{d}t}$ along the solution of (8.60) to obtain

$$\begin{aligned}\left.\frac{\mathrm{d}V(x)}{\mathrm{d}t}\right|_{(8.60)} &= \sum_{i=1}^{n} c_i(x_i)\, f_{ii}(x_i) \left[\sum_{j=1}^{n} a_{ij}\, f_{ij}(x_j)\right] \\ &= \sum_{i=1}^{n} c_i(x_i)\, a_{ii}\, f_{ii}^2(x_i) \\ &\quad + \frac{1}{2} \sum_{i=1, i\neq j}^{n} \sum_{j=1}^{n} \left[c_i(x_i)\, a_{ij}\, f_{ii}(x_i) f_{ij}(x_j) + c_j(x_j) a_{ji}\, f_{jj}(x_j)\, f_{ji}(x_i)\right].\end{aligned}$$

For $x = \xi \in R^n$, without loss of generality, we may assume $\Pi_{i=1}^{k}\xi_i \neq 0$, $\sum_{i=k+1}^{n}\xi_i^2 = 0$ $(0 \leq k < n)$, then

$$\begin{aligned}\left.\frac{\mathrm{d}V(\xi)}{\mathrm{d}t}\right|_{(8.60)} &= \sum_{i=1}^{k} c_i(\xi_i)\, a_{ii}\, f_{ii}^2(\xi_i) \\ &\quad + \frac{1}{2} \sum_{i=1, i\neq j}^{k} \sum_{j=1}^{k} \left[c_i(\xi_i)\, a_{ij}\, f_{ii}(\xi_i) f_{ij}(\xi_j) + c_j(\xi_j) a_{ji}\, f_{jj}(\xi_j)\, f_{ji}(\xi_i)\right] \\ &= \sum_{i=1}^{k} c_i(\xi_i)\, a_{ii}\, f_{ii}^2(\xi_i) \\ &\quad + \frac{1}{2} \sum_{i=1, i\neq j}^{k} \sum_{j=1}^{k} \left[\frac{c_i(\xi_i) a_{ij} f_{ii}(\xi_i)}{f_{jj}(\xi_j)} + \frac{c_j(\xi_j) a_{ji} f_{jj}(\xi_j)}{f_{ii}(\xi_i)}\right] f_{ji}(\xi_i) f_{jj}(x_j) \\ &= \begin{pmatrix} f_{11}(\xi_1) \\ \vdots \\ f_{kk}(\xi_k) \end{pmatrix}^{\mathrm{T}} B(b_{ij}(x_j))_{k\times k} \begin{pmatrix} f_{11}(\xi_1) \\ \vdots \\ f_{kk}(\xi_k) \end{pmatrix} \\ &:= W(f_{11}, \ldots, f_{kk}) \leq 0.\end{aligned} \tag{8.70}$$

From the conditions of the theorem, we know that the generalized quadratic form $W(f_{11}, \ldots, f_{kk})$ is negative definite about $f_{11}, \ldots, f_{kk}$. However, due to $f_{ii}(x_i) \in F_\infty$, we have

$$\begin{aligned} &W(0) = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} f_{ii}^2 = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} \xi_i^2 = 0, \\ &W(f_{11}, \ldots, f_{kk}) < 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} f_{ii}^2 \neq 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} \xi_i^2 \neq 0.\end{aligned}$$

Thus, $\left.\frac{\mathrm{d}V(\xi)}{\mathrm{d}t}\right|_{(8.60)}$ is negative definite about ξ. Due to arbitrary of ξ, we have proved that $\left.\frac{\mathrm{d}V(\xi)}{\mathrm{d}t}\right|_{(8.60)}$ is negative definite about x. Therefore, the zero solution of the interval control system (8.60) is robustly absolutely stable. □

Corollary 8.45. *If there exist constants $c_i > 0,\ i = 1, 2, \ldots, n$ such that the matrix $\overline{B}(\overline{b_{ij}})_{n\times n}$ is negative definite, where*

$$\overline{b_{ij}} \begin{cases} = c_i \overline{a_{ii}}, & \text{when } i = j,\ i, j = 1, \ldots, n, \\ \geq \dfrac{1}{2} \max\limits_{a_{ij}\in[\overline{a_{ii}}, \overline{a_{ij}}]} \left[\dfrac{c_i a_{ij} f_{ji}(x_i)}{f_{ii}(x_i)} + \dfrac{c_j a_{ji} f_{ij}(x_j)}{f_{jj}(x_j)} \right], & \text{when } i \neq j,\ x_i x_j \neq 0, \\ & \quad i, j = 1, \ldots, n, \end{cases}$$

then the zero solution of system (8.60) is robust absolutely stable.

Proof. Obviously, the condition of the corollary implies the condition of Theorem 8.44, so the conclusion is true. □

Corollary 8.46. *Assume condition (1) in Theorem 8.44 holds, and condition is changed to that there exist continuous functions $c_i(x_i) \geq \delta > 0$, and $M_i^{(j)}(x_i, x_j) M_j^{(i)}(x_j, x_i) \geq 0$ satisfying*

$$\begin{aligned} &\frac{1}{2} \left[\frac{c_i a_{ii} f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{c_j a_{ji} f_{ji}(x_i)}{f_{ii}(x_i)} \right] \leq M_i^{(j)}(x_i, x_j) M_j^{(i)}(x_j, x_i), \\ &\text{and} \quad \sum_{j=1, j\neq i}^{n} \left(M_i^{(j)}(x_i, x_j) \right)^2 < -c_i(x_i) \overline{a_{ii}} \quad (i = 1, 2, \ldots, n), \end{aligned} \tag{8.71}$$

then the zero solution of the interval control system (8.60) is robust absolutely stable.

Proof. Again take the same Lyapunov function used in Theorem 8.44. For $x = \xi \in R^n$, $\Pi_{i=1}^k \xi_i \neq 0$, and $\sum_{i=k+1}^n \xi_i^2 = 0$. Then following the derivation in (8.69)–(8.70), we have

$$\begin{aligned} \left.\frac{\mathrm{d}V(\xi)}{\mathrm{d}t}\right|_{(8.60)} &= \sum_{i=1}^{k} c_i a_{ii} f_{ii}^2(\xi_i) \\ &\quad + \frac{1}{2} \sum_{i=1, i\neq j}^{k} \sum_{j=1}^{k} \left[\frac{c_i(\xi_i) a_{ij} f_{ij}(\xi_j)}{f_{jj}(x_j)} + \frac{c_j(\xi_j) a_{ji} f_{ji}(x_i)}{f_{ii}(x_i)} \right] f_{ii}(\xi_i) f_{jj}(x_j) \\ &\leq \sum_{i=1}^{k} c_i a_{ii} f_{ii}^2(\xi_i) \\ &\quad + 2 \sum_{i=1, i\neq j}^{k} \sum_{j=1}^{k} M_i^{(j)}(\xi_i, \xi_j) M_i^{(j)}(\xi_i, \xi_j) \left| f_{ii}(\xi_i) f_{jj}(x_j) \right| \\ &\leq \sum_{i=1}^{k} c_i a_{ii} f_{ii}^2(\xi_i) + \sum_{i=1, i\neq j}^{k} \sum_{j=1}^{k} \left(M_i^{(j)}(\xi_i, \xi_j) \right)^2 f_{ii}^2(\xi_i) \\ &= \sum_{i=1}^{k} \left[c_i a_{ii} + \sum_{i=1, i\neq j}^{k} \left(M_i^{(j)}(\xi_i, \xi_j) \right)^2 \right] f_{ii}^2(\xi_i) \\ &\ll 0. \end{aligned}$$

Thus, the conclusion of the corollary is true. □

Particularly, if $c_i(x_i), M_i^{(j)}(\xi_i, \xi_j)$ and $M_j^{(i)}(\xi_j, \xi_i)$ can be taken as constants, for example,

$$c_i(x_i) \equiv 1, \quad M_i^{(j)}(\xi_i, \xi_j) = M_j^{(i)}(\xi_j, \xi_i) = \left[\frac{1}{2}\left(\frac{a_{ij} f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{a_{ji} f_{ji}(x_i)}{f_{ii}(x_i)}\right)\right]^{1/2},$$

then the conditions in Corollary 8.46 become

$$\overline{a_{ii}} < 0 \quad \text{and} \quad \sum_{j=1, j\neq i}^{n} \left|\frac{a_{ij} f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{a_{ji} f_{ji}(x_i)}{f_{ii}(x_i)}\right| < -2\overline{a_{ii}}, \tag{8.72}$$

which are easy to verify.

Example 8.47. Consider the interval control system:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} f_j(x_j), \quad i = 1, 2, \cdot, n, \tag{8.73}$$

where $a_{ij} \in [\underline{a_{ij}}, \overline{a_{ij}}],\ f_j(x_j) \in F_\infty$.

If $\overline{a_{ii}} < 0$ and

$$\sum_{j=1, j\neq i}^{n} \max_{\substack{a_{ij}\in[\underline{a_{ij}}, \overline{a_{ij}}] \\ a_{ji}\in[\underline{a_{ji}}, \overline{a_{ji}}]}} [|a_{ij} + a_{ji}|] < -2\overline{a_{ii}},$$

then the zero solution of (8.73) is robust absolutely stable.

Remark 8.48.

$$\begin{aligned} &\max_{\substack{a_{ij}\in[\underline{a_{ij}}, \overline{a_{ij}}] \\ a_{ji}\in[\underline{a_{ji}}, \overline{a_{ji}}]}} [|a_{ij} + a_{ji}|] \\ &\quad = \max\left[\left[|\underline{a_{ij}} + \underline{a_{ji}}|\right], \left[|\underline{a_{ij}} + \overline{a_{ji}}|\right], [|\overline{a_{ij}} + \overline{a_{ji}}|], \left[|\overline{a_{ij}} + \underline{a_{ji}}|\right]\right]. \end{aligned}$$

9

Discrete Control Systems

The Lurie control system and the well-known Lurie problem were originally developed for solving the nonlinear systems described by ordinary differential equations (ODE), and the most research interest and results in this area were focused on ODE systems. With the very fast development of computer systems and technology, the dynamics and asymptotic behavior of discrete systems described by difference equations (DE) play more important roles in solving practical problems [115], attracting more and more researchers [47]. The absolute stability of discrete Lurie control systems is naturally raised. However, the results obtained so far for such systems are relatively less than that of continuous Lurie control systems.

In this chapter, we will generalize the recently developed theory and methodology for continuous Lurie control systems to study discrete Lurie control systems described by difference equations. We will mainly discuss the sufficient and necessary conditions of absolute stability, and some practically useful algebraic sufficient conditions of absolute stability. The material of this chapter is mainly chosen from [77, 84].

9.1 Sufficient and Necessary Conditions for the Absolute Stability

We examine the following discrete Lurie control system:

$$\begin{aligned} x(t_{k+1}) &= A\,x(t_k) + h\,f(\sigma(t_k)), \\ \sigma &= c^{\mathrm{T}}x = \sum_{i=1}^{n} c_i\,x_i\,, \end{aligned} \tag{9.1}$$

where $x \in R^n$, $A \in R^{n\times n}$, $b \in R^n$, $c \in R^n$, $f \in F_\infty$, or

$$f \in F_{[k_1,k_2]} := \{f\,|\,f(0)=0,\ 0 \le k_1 \le f(\sigma)/\sigma \le k_2,\ \sigma \ne 0,\ f \in C[(-\infty,+\infty),R^1]\}.$$

We choose $J = \{t_k : t_0 < t_1 < \cdots < t_k < \cdots\}$, $N := \{0, 1, 2, \ldots\}$.

Definition 9.1. *The zero solution of (9.1) is said to be absolutely stable (absolutely stable in Hurwitz angle $[k_1,k_2]$) if for any $f \in F_\infty$ ($f \in F_{[k_1,k_2]}$), the zero solution of (9.1) is globally asymptotically stable (globally asymptotically stable in Hurwitz angle $[k_1,k_2]$).*

Definition 9.2. *The zero solution of (9.1) is said to be absolutely stable for the set* $\Omega = \{x \,|\, c^{\mathrm{T}} x = 0\}$ *(absolutely stable for* Ω *in* $[k_1, k_2]$*) if for any* $f \in F_\infty$ *(*$f \in F_{[k_1,k_2]}$*), the zero solution of (9.1) is globally asymptotically stable w.r.t.* Ω.

Lemma 9.3. *Let* $x(t_{k+1})$ *be the solution of the following system:*

$$\begin{aligned} x(t_{k+1}) &= Ax(t_k) + F(t_k, x(t_k)), \\ x(t_0) &= x_0. \end{aligned}$$

Then for any natural number k, the following formula of variation of constants holds:

$$x(t_{k+1}) = A^{k+1} x_0 + \sum_{l=0}^{k} A^{k-l} F(t_l, x(t_l)).$$

Proof. The lemma can be easily verified by the method of mathematical induction, and the proof is omitted. □

Corollary 9.4. *The solution* $x(t_{k+1}) := x(t_{k+1}, t_0; x_0)$ *of (9.1) can be written as*

$$x(t_{k+1}) = A^{k+1} x_0 + \sum_{l=0}^{k} A^{k-l} h f(\sigma(t_l)).$$

Corollary 9.5. *Suppose that* $f(\sigma) \in F_\infty$ *(*$F_{[k_1,k_2]}$*). For arbitrary fixed m, the solution of (9.1) depends continuously on the initial value* x_0.

Proof. Obviously, when $m = 0$, the following expression:

$$x(t_1) = x_0 + hf(\sigma(t_0)), \qquad \sigma(t_0) = c^{\mathrm{T}} x_0$$

is a continuous function of x_0. Suppose that

$$x(t_k) = A^k x_0 + \sum_{l=0}^{k-1} A^{k-l} h f(\sigma(t_l))$$

depends continuously on x_0. Since

$$x(t_{k+1}) = Ax(t_k) + hf(\sigma(t_k))$$

is a continuous function of $x(t_k)$, thus $x(t_{k+1})$ depends continuously on x_0. □

Theorem 9.6. *The zero solution of (9.1) is absolutely stable (absolutely stable in* $[k_1, k_2]$*) if and only if*

1. $\rho(B) < 1$ $(\rho(B^*) < 1)$, *where* $\rho(B)$ *and* $\rho(B^*)$ *are the respectively, spectral radius of B and* B^*
2. *The zero solution of (9.1) is absolutely stable for the set* $\Omega = \{x : c^{\mathrm{T}} x = 0\}$ *(absolutely stable for* Ω *in* $[k_1, k_2]$*). Here*

$$B = (b_{ij})_{n \times n} = A + hc^{\mathrm{T}} \theta, \qquad \theta = 0 \quad or \quad \theta = 1,$$
$$\Big(B^* = (b^*_{ij})_{n \times n} = A + hc^{\mathrm{T}}(\frac{k_2 - k_1}{2})\Big).$$

Proof. It is suffice to prove the necessary and sufficient conditions (NASC) of the absolute stability because the proof of NASC of absolute stability in $[k_1, k_2]$ is the same.

Necessity. Since the zero solution of (9.1) is absolutely stable, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that the solution of (9.1) satisfies

$$|x(t_k)| < \frac{\varepsilon}{\max_{1 \le i \le n} |c_i|} \qquad \text{for} \quad \|x_0\| < \delta(\varepsilon).$$

Further, we have

$$|\sigma(t_k)| = |\sum_{i=1}^{n} c_i x_i(t_k)| \le \max_{1 \le i \le n} |c_i| \sum_{i=1}^{n} |x_i(t_k)| := \max_{1 \le i \le n} |c_i| \, \|x(t_k)\| < \varepsilon.$$

Obviously, $\lim\limits_{k \to +\infty} x(t_k) = 0$ implies $\lim\limits_{k \to +\infty} \sigma(t_k) = 0$, which leads to the conclusion that the zero solution of (9.1) is absolutely stable w.r.t. the set Ω.

If $f(\sigma) = \sigma$, then (9.1) is transformed into

$$x(t_k) = Bx(t_{k-1}).$$

Since the zero solution of (9.1) is globally asymptotically stable, it can be shown that $\rho(B) < 1$. Necessity is proved.

Sufficiency. The solution $x(t_{k+1})$ of (9.1) can be written as

$$x(t_{k+1}) = B^{k+1} x_0 + \sum_{l=0}^{k} B^{k-l} (h f(\sigma(t_l)) - h \theta \sigma(t_l)).$$

Since $\rho(B) < 1$, B^m is bounded, we can define

$$\|B^{k+1}\| \le M = \text{const.} \qquad \text{for all} \quad k \in N.$$

For any $\varepsilon > 0$, we take $\delta(\varepsilon) = \frac{\varepsilon}{3M}$. Since $\lim\limits_{k \to +\infty} \sigma(t_k) = 0$ and $\lim\limits_{k \to +\infty} f(\sigma(t_k)) = 0$, there exists a constant $k_1 > k_0$ such that the following estimation holds:

$$\left\| \sum_{i=k_1+1}^{k} B^{k-1} (h f(\sigma(t_l)) - h \theta \sigma(t_l)) \right\| < \frac{\varepsilon}{3}.$$

By virtue of the facts that $\lim_{k \to +\infty} \sigma(t_k, t_0; x_0) = 0$, $\sigma(t_k, t_0; x_0)$ depends continuously on the initial value x_0 and $f(\sigma(t_k))$ is continuous, there exists a constant $\delta_2(\varepsilon) > 0$ such that

$$\left\| \sum_{l=0}^{k_1} B^{k-l} (h f(\sigma(t_l)) - h \theta \sigma(t_l)) \right\| < \frac{\varepsilon}{3} \qquad \text{for} \quad \|x_0\| < \delta_2(\varepsilon).$$

Let $\delta(\varepsilon) = \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$. Then, we obtain

$$\begin{aligned}\|x(t_{k+1})\| &\le \|B^{k+1}x(t_0)\| + \left\|\sum_{l=0}^{k_1} B^{k-l}(hf(\sigma(t_l)) - h\theta\sigma(t_l))\right\| \\ &\quad + \left\|\sum_{l=k_1+1}^{k} B^{k-l}(hf(\sigma(t_l)) - h\theta\sigma(t_l))\right\| \\ &< 0 \qquad \text{for} \quad \|x_0\| < \delta(\varepsilon).\end{aligned}$$

Therefore, the zero solution of (9.1) is stable.

Since $\lim_{k\to+\infty} f(\sigma(t_k)) = 0$ and $\lim_{k\to+\infty}\sigma(t_k) = 0$ for any $x_0 \in R^n$, there exists a constant $M_1 > 0$ such that

$$\|hf(\sigma(t_l)) - h\theta\sigma(t_l)\| \le M_1.$$

Taking into account that $\rho(B) < 1$, we have $\sum_{l=1}^{\infty}\|B^l\| < +\infty$. We know that there exists a constant $M_2 > 0$ such that

$$\sum_{l=[\frac{k}{2}]}^{k} \|B^{k-l}\| \le M_2.$$

Therefore,

$$\begin{aligned}0 &\le \lim_{k\to+\infty}\|x(t_0)\| \\ &\le \lim_{k\to+\infty}\|B^k x_0\| + \lim_{k\to+\infty}\sum_{l=0}^{[\frac{k}{2}]}\|B^{k-l}\|\,\|\overline{h}f(\sigma(t_l)) - h\theta\sigma(t_l)\| \\ &\quad + \lim_{k\to+\infty}\sum_{l=[\frac{k}{2}]+1}^{k}\|B^{k-l}\|\,\|hf(\sigma(t_l)) - h\theta\sigma(t_l)\| \\ &= 0 + M_1\lim_{k\to+\infty}\sum_{l=0}^{[\frac{k}{2}]}\|B^{k-l}\| + M_2\lim_{k\to+\infty}\max_{[\frac{k}{2}]\le l\le k}\|hf(\sigma(t_l)) - h\theta\sigma(t_l)\| \\ &= 0.\end{aligned}$$

□

Theorem 9.7. *The zero solution of (9.1) is absolutely stable (absolutely stable in* $[k_1, k_2]$*) if and only if*

1. *The condition (1) of Theorem 9.6 is satisfied*
2. *For any* $f \in F$ *(*$f \in F_{[k_1,k_2]}$*), there exists a Lyapunov function* $V_f(x)$ *which is radially unbounded positive definite w.r.t* Ω *such that*

$$\Delta V_f = V_f(x(t_{k+1})) - V_f(x(t_k))$$

is negative definite for Ω.

Proof. It is suffice to prove the NASC for absolute stability, because the NASC of absolute stability in $[k_1, k_2]$ can be proved along the same lines.

Sufficiency. On the basis of Theorem 9.6, what we need is to prove that the condition (2) implies that the zero solution of (9.1) is absolutely stable w.r.t. σ.

Since $V_f(x(t_{k+1})) - V_f(x(t_k))$ is negative definite w.r.t. σ, we find that

$$V_f(x(t_{k+1})) - V_f(x(t_k)) \le -\psi_f(|\sigma(t_k)|), \quad \psi_f \in K.$$

It can be deduced that there exists $\varphi \in KR$ such that

$$\begin{aligned} 0 \le \varphi_f(|\sigma(t_{k-1})|) \le V_f(x(t_{k+1})) &\le V_f(x(t_k)) - \psi_f(|\sigma(t_k)|) \\ &\le V_f(x(t_{k-1})) - \psi_f(|\sigma(t_k)|) - \psi_f(|\sigma(t_{k-1})|) \\ &\le V_f(x(t_0)) - \psi_f(|\sigma(t_k)|) - \cdots - \psi_f(|\sigma(t_0)|). \end{aligned}$$

Consequently,

$$\begin{aligned} \varphi_f(|\sigma(t_{k-1})|) &\le V_f(x(t_{k+1})) \le V_f(x(t_0)), \\ |\sigma(t_{k+1})| &\le \psi_f^{-1}(V_f(x(t_0))) \ll 1 \qquad \text{for} \quad \|x_0\| \ll 1. \end{aligned}$$

Now we show that

$$\lim_{k\to+\infty} \sigma(t_{k+1}) = 0 \qquad \forall \quad x_0 \in R^n.$$

If there exists some $x_0 \in R^n$ satisfying $\lim_{k\to+\infty} \sigma(t_{k+1}) \ne 0$, then there exist $\varepsilon > 0$ and a sequence $\{k_j\}$ such that

$$\sigma(t_{k_i}) \ge \varepsilon, \qquad i = 1, 2, \ldots$$

Provided $k_1 < k_2 < \cdots < k_j < k+1$, k being a sufficiently large constant, we derive

$$\begin{aligned} 0 \le \varphi_f(|\sigma(t_{k+1})|) &\le V_f(x(t_{k+1})) \\ &\le V_f(x(t_0)) - \psi_f(|\sigma(t_k)|) - \cdots - \psi_f(|\sigma(t_0)|) \\ &\le V_f(x(t_0)) - \varepsilon \cdots - \varepsilon \to -\infty \qquad \text{as} \quad k \to +\infty, \end{aligned}$$

which yields a contradiction. Hence, we have $\lim_{k\to+\infty} \sigma(t_{k+1}) = 0$, i.e., the zero solution of (9.1) is absolutely stable w.r.t. Ω.

Necessity. Since the zero solution of (9.1) is absolutely stable, one can prove that for any $f \in F_\infty$ $(F_{[k_1,k_2]})$, there exists a radially unbounded, positive definite function $V_f(x)$ such that

$$V_f(x(k+1)) - V_f(x(0)) < 0 \qquad \text{(i.e., negative definite)}.$$

Accordingly, there exist two functions, namely $\varphi \in KR$, $\psi \in KR$ such that

$$V_f(x) \ge \varphi(\|x\|)$$

and

$$V_f(x(t_{k+1})) - V_f(x(t_k)) \leq -\psi(\|x(t_k)\|).$$

By virtue of

$$|\sigma| = \left|\sum_{i=1}^{n} c_i x_i\right| \leq \sum_{i=1}^{n} |c_i|\,|x_i| \leq \max_{1\leq i\leq n} |c_i \left|\sum_{i=1}^{n} |x_i\right| := \max_{1\leq i\leq n} |c_i|\,\|x\|,$$

we obtain

$$V_f(x) \geq \varphi_f(\|x\|) \geq \varphi_f\left(\frac{|\sigma|}{\max_{1\leq i\leq n} |c_i|}\right) := \tilde{\varphi}_f(|\sigma|),$$

where $\varphi_1 \in KR$, and

$$V_f(x(t_{k+1})) - V_f(x(t_k)) \leq -\psi_f(\|x(t_k)\|) \leq -\psi_f\left(\frac{|\sigma|}{\max_{1\leq i\leq n} |c_i|}\right) := -\tilde{\psi}_f(|\sigma|).$$

The condition (2) is satisfied, the condition (1) holds trivially and the conclusion follows. □

Theorem 9.8. *The zero solution of (9.1) is absolutely stable (absolutely stable in $[k_1, k_2]$) if and only if*

1. *There exists a constant vector $\eta = (\eta_1, \ldots, \eta_n)^{\mathrm{T}}$ such that $\rho(B_1(b_{ij})) < 1$, where $B_1 = A + \eta\, c^{\mathrm{T}}$.*
2. *The condition* (2) *of Theorem 9.6 holds.*

Proof. *Necessity*. The existence of $\eta = (\eta_1, \ldots, \eta_n)^{\mathrm{T}}$ is obvious. For example, we may take $\eta_i = h_i \left[\eta_i = \frac{k_2 - k_1}{2} h_i\right]$, and the condition (1) is satisfied. The necessity of the condition (2) has been proved in Theorem 9.6.

Sufficiency. For any $f \in F_\infty$ ($f \in F_{[k_1,k_2]}$), we rewrite (9.1) as

$$x(t_{k+1}) = B_1 x(t_k) + h f(x(t_k)) - \eta\, \sigma(t_k).$$

In accordance with Lemma 9.3, we deduce that

$$x(t_{k+1}) = B_1^{k+1} x(t_0) + \sum_{l=0}^{k} B_1^{k-l} (h f(\sigma(t_l)) - \eta\, \sigma(t_k).$$

The proof of the remaining part is similar to that of Theorem 9.7. □

Similarly, we can prove the following theorem.

Theorem 9.9. *The zero solution of (9.1) is absolutely stable (absolutely stable in $[k_1, k_2]$) if and only if both the condition (1) of Theorem 9.7 and the condition (2) of Theorem 9.6 hold.*

9.2 Sufficient Algebraic Conditions for the Absolute Stability

In this section, we present some useful sufficient algebraic conditions for absolute stability.

Without loss of generality, we may assume that $c_n \neq 0$. Let

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}.$$

A full rank linear transformation, given by

$$\xi = Qx,$$

transforms system (9.1) into

$$\xi(t_{k+1}) = QAQ^{-1}\xi(t_k) + Qhf(\xi_n(t_k)) := \tilde{A}\xi(t_k) + \tilde{h}f(\xi_n(t_k)), \tag{9.2}$$

where $\tilde{A} = QAQ^{-1}$, $\tilde{h} = Qh$.

Obviously, the absolute stability of the zero solution of (9.1) and (9.2) are equivalent. However, all the variables of (9.2) are separated. The definition of the absolute stability of the zero solution for the variable ξ_n can be stated as in Definition 9.2.

Let

$$\frac{f(\xi_n)}{\xi_n} = \begin{cases} g(\xi_n), & \text{when } \xi_n \neq 0, \\ 0, & \text{when } \xi_n = 0. \end{cases} \tag{9.3}$$

Accordingly, (9.2) can be rewritten as

$$\begin{aligned} \xi_i(t_{k+1}) &= \sum_{j=1}^{n} \tilde{a}_{ij}\,\xi_j(t_k) + \tilde{h}_i\, g(\xi_n(t_k))\,\xi_n(t_k) \\ &= \sum_{j=1}^{n-1} \tilde{a}_{ij}\,\xi_j(t_k) + (\tilde{a}_{in} + \tilde{b}_i\, g(\xi_n(t_k)))\,\xi_n(t_k). \end{aligned} \tag{9.4}$$

Theorem 9.10. *1. Let the condition (1) of Theorem 9.6 be satisfied.*

2. Suppose that for any $f \in F_\infty$ ($f \in F_{[k_1,k_2]}$), there exist positive constants r_i $(i = 1, 2, \ldots, n)$ which are independent of f such that

$$\max_{1\le j\le n-1} \left\{ \sum_{i=1}^{n} \frac{r_i}{r_j} |\tilde{a}_{ij}| \right\} \le 1,$$

$$\sum_{i=1}^{n} \frac{r_i}{r_n} \left| \tilde{a}_{in} + \tilde{h}_i\, g(\xi_n) \right| \le \rho < 1.$$

Then the zero solution of (9.4) is absolutely stable (absolutely stable in $[k_1,k_2]$).

Proof. We take the radially unbounded, positive definite Lyapunov function

$$V(\xi) = \sum_{i=1}^{n} r_i\,|\xi_i|$$

and obtain

$$\begin{aligned}
V(\xi(t_{k+1})) - V(\xi(t_k)) \;&=\; \sum_{i=1}^{n} r_i \Big| \sum_{j=1}^{n} \tilde{a}_{ij}\,\xi_j(t_k) + \tilde{h}_i\, f(\xi_n(t_k)) \Big| - \sum_{i=1}^{n} r_i\,|\xi_i(t_k)| \\
&\le\; \sum_{j=1}^{n-1} \sum_{i=1}^{n} \frac{r_i}{r_j} |\tilde{a}_{ij}|\;|r_j|\;|\xi_j(t_k)| \\
&\quad + \sum_{i=1}^{n} \frac{r_i}{r_n} |\tilde{a}_{in}\,\xi_n(t_k) + \tilde{h}_i g(\xi_n(t_k))|\, r_n - \sum_{j=1}^{n} r_j |\xi_j(t_k)| \\
&=\; \sum_{j=1}^{n-1} \left[\sum_{i=1}^{n} \frac{r_i}{r_j} |\tilde{a}_{ij}| - 1 \right] r_j\,|\xi_j(t_k)| \\
&\quad + \left[\sum_{i=1}^{n} \frac{r_i}{r_n} |\tilde{a}_{in} + \tilde{h}_i g(\xi_n(t_k))| - 1 \right] r_n |\xi_n(t_k)| \\
&\le\; (\rho - 1)\, r_n\,|\xi_n(t_k)| \\
&:=\; -\,\delta\, r_n\,|\xi_n(t_k)|.
\end{aligned} \tag{9.5}$$

Thus, it follows that:

$$r_n\,|\xi_n(t_{k+1})| \le V(t_{k+1}) < V(t_k) < V(t_{k-1}) < \cdots < V(t_0). \tag{9.6}$$

The expression (9.6) shows that the zero solution of (9.4) is stable w.r.t. the partial variable ξ_n, and (9.5) gives

$$\begin{aligned}
r_n\,|\xi_n(t_{k+1})| &\le V(t_{k+1}) < V(t_k) - \delta\, r_n\,|\xi_n(t_k)| \\
&\le V(t_0) - \delta\, r_n\,|\xi_n(t_k)| - \delta\, r_n\,|\xi_n(t_{k-1})| - \cdots - \delta\, r_n\,|\xi_n(t_0)|.
\end{aligned}$$

Now we will prove that $\lim_{k\to+\infty} \xi_n(t_{k+1}) = 0$. If $\lim_{k\to+\infty} \xi_n(t_{k+1}) \neq 0$, then there exist a constant $\varepsilon > 0$ and a sequence $\{k_i\}$ such that

$$|\xi_n(t_{k_i})| \ge \varepsilon, \qquad i = 1, 2, \ldots$$

Assume that $k_1 < k_2 < \cdots < k_j < L+1$, where the constant L is large enough. In this case,

$$\begin{aligned}
0 \le r_n\,|\xi(t_{k+1})| &\le V(t_0) - \delta\, r_n\,|\xi_n(t_k)| - \delta\, r_n |\xi_n(t_{k-1})| - \cdots - \delta\, r_n\,|\xi_n(t_0)| \\
&\to -\infty \qquad \text{as} \quad k \to +\infty,
\end{aligned}$$

which leads to a contradiction. Thus, we have $\lim_{k\to+\infty} \xi(t_{k+1}) = 0$. Following the same idea used in proving Theorem 9.6 we see that the conclusion of Theorem 9.10 holds. □

Corollary 9.11. *1. Let the condition (1) of Theorem 9.6 be satisfied*
2. Let $\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|\tilde{a}_{ij}|\right\}\le 1$ *and* $\sum_{i=1}^{n}\|\tilde{a}_{in}+\tilde{h}_i g(\xi_n)\|\le\rho<1$ *hold for any* $f(\sigma)\in F_\infty$ *(*$f\in F_{[k_1,k_2]}$*). Then the zero solution of (9.4) is absolutely stable (absolutely stable in* $[k_1,k_2]$*).*

Proof. Taking $r_i=1$ $(i=1,2,\dots,n)$ in Theorem 9.10, we see that all the conditions of Theorem 9.10 are satisfied. The conclusion of Corollary 9.11 is true. □

In the following, we will use another Lyapunov function to consider absolute stability.

For any $\varepsilon>0$, let

$$D=(d_{ij})_{n\times n},\qquad G_\varepsilon:=\begin{bmatrix}0&0&\cdots&0&0\\0&0&\cdots&0&0\\ \vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&0&0\\0&0&\cdots&0&\varepsilon\end{bmatrix}_{n\times n},$$

where

$$d_{ij}:=\begin{cases}\tilde{a}_{ij}, & 1\le i,j\le n-1,\\ \tilde{a}_{in}+\tilde{h}_i\,g(\xi_n), & i=1,2,\dots,n.\end{cases}$$

Theorem 9.12. *1. Let the condition (1) of Theorem 9.6 be satisfied*
2. Suppose that there exists a symmetric, positive definite matrix $B(b_{ij})_{n\times n}$ *such that the matrix* $D^{\mathrm{T}}BD-B+G_\varepsilon$ *is negative semi-definite for any* $f\in F_\infty$ *(*$f\in F_{[k_1,k_2]}$*). Then the zero solution of (9.4) is absolutely stable (absolutely stable in* $[k_1,k_2]$*).*

Proof. Choosing the Lyapunov function $V=\xi^{\mathrm{T}}B\xi$, we get

$$\begin{aligned}V(t_{k+1})-(t_k)&=\xi^{\mathrm{T}}(t_{k+1})B\xi(t_{k+1})-\xi^{\mathrm{T}}(t_k)B\xi(t_k)\\&=(D\xi(t_k)^{\mathrm{T}}BD\xi(t_k))-\xi^{\mathrm{T}}(t_k)B\xi(t_k)\\&=\xi^{\mathrm{T}}(t_k)(D^{\mathrm{T}}BD-B+G_\varepsilon)\xi(t_k)-\varepsilon\xi_n^2(t_k)\\&\le-\varepsilon\xi_n^2(t_k).\end{aligned}$$

Then there exists a constant $\alpha>0$ such that

$$\begin{aligned}\xi_n^2(t_k)\le\alpha V(t_k)&\le\alpha V(t_{k-1})-\alpha\varepsilon\xi_n^2(t_{k-1})\\&\le\alpha V(t_0)-\alpha\varepsilon\xi_n^2(t_{k-1})-\alpha\varepsilon\xi_n^2(t_{k-2})-\cdots-\alpha\varepsilon\xi_n^2(t_0).\end{aligned}$$

The rest of the proof can be completed similarly to the proof of Theorem 9.6. □

In the case $B=I$, we have the following result.

Corollary 9.13. *1. Let the condition (1) of Theorem 9.6 be satisfied*

2. Suppose that there exists a symmetric, positive definite matrix $D^{\mathrm{T}}D - I + G_{\varepsilon}$ *is negative semi-definite for any* $f \in F_{\infty}$ $(f \in F_{[k_1,k_2]})$*. Then the zero solution of (9.4) is absolutely stable (absolutely stable in* $[k_1,k_2]$*).*

Example 9.14. Consider the two-dimensional discrete control system:

$$\begin{aligned} x_1(t_k) &= \frac{2}{5}x_1(t_{k-1}) - \frac{1}{5}x_2(t_{k-1}) + \frac{3}{10}f(x_2(t_{k-1})), \\ x_2(t_k) &= \frac{3}{10}x_1(t_{k-1}) - \frac{3}{10}x_2(t_{k-1}) + \frac{2}{5}f(x_2(t_{k-1})), \end{aligned} \tag{9.7}$$

where $f(x_2) \in F_{[0,2]} = \{f(x_2)|f(0)=0,\ 0 \le f(x_2)/x_2 \le 2,\ f(x_2) \in C[(-\infty,+\infty),R^1]\}$.

1. Let $f(x_2) = x_2$. Then (9.7) is transformed into

$$\begin{aligned} x_1(t_k) &= \frac{2}{5}x_1(t_{k-1}) + \frac{1}{10}x_2(t_{k-1}), \\ x_2(t_k) &= \frac{3}{10}x_1(t_{k-1}) + \frac{1}{10}x_2(t_{k-1}), \end{aligned}$$

   where $B := \begin{bmatrix} \frac{2}{5} & \frac{1}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$ and $\rho(B) \le \|B\| < 1$. The condition (1) of Theorem 9.6 holds true.

2. We choose the radially unbounded, positive definite Lyapunov function $V(x) = |x_1| + |x_2|$. By virtue of

$$\begin{aligned} |\tilde{a}_{11}| + |\tilde{a}_{21}| &= \frac{2}{5} + \frac{3}{10} = \frac{7}{10} < 1, \\ |\tilde{a}_{21} + \tilde{h}_1 g(x_2)| + |\tilde{a}_{22} + \tilde{h}_2 g(x_2)| &= \left| -\frac{1}{5} + \frac{3}{10}\frac{f(x_2)}{x_2} \right| + \left| -\frac{3}{10} + \frac{2}{5}\frac{f(x_2)}{x_2} \right| \\ &\le \left| -\frac{1}{5} + \frac{3}{5} \right| + \left| -\frac{3}{10} + \frac{4}{5} \right| \\ &= \frac{2}{5} + \frac{1}{2} = \frac{9}{10}, \end{aligned}$$

   the condition of Corollary 9.13 are satisfied.

Thus, the zero solution of (9.7) is absolutely stable in the Hurwitz angle $[0, 2]$.

Example 9.15. Consider the system:

$$\begin{aligned} x_1(t_{k+1}) &= \frac{1}{\sqrt{2}}x_1(t_k) - \frac{1}{2}x_2(t_k) + \frac{1}{3}f(x_2(t_k)), \\ x_2(t_{k+1}) &= \frac{1}{\sqrt{2}}x_1(t_k) + \frac{1}{2}x_2(t_k) - \frac{1}{3}f(x_2(t_k)), \end{aligned} \tag{9.8}$$

where $f(x_2) \in F_{[0,7/2]} = \{f(x_2)|f(0) = 0, 0 \le f(x_2)/x_2 \le 7/2, f(x_2) \in C[(-\infty, +\infty),R^1]\}$.

Let us discuss the absolute stability of this system.

1. We fix $f(x_2(t_k)) = x_2(t_k)$. The system (9.8) is changed into

$$x_1(t_{k+1}) = \frac{1}{\sqrt{2}} x_1(t_k) - \frac{1}{6} x_2(t_k),$$
$$x_2(t_{k+1}) = \frac{1}{\sqrt{2}} x_1(t_k) + \frac{1}{6} x_2(t_k),$$

where $B_1 := \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{6} \\ \frac{1}{\sqrt{2}} & \frac{1}{6} \end{bmatrix}$ and $\rho(B_1) \le \|B_1\| = \frac{1}{\sqrt{2}} + \frac{1}{6} < 1$. We see that the condition (1) of Theorem 9.6 is satisfied.

2. We take the radially unbounded, positive definite Lyapunov function

$$V(x) = x_1^2 + x_2^2, \qquad G_\varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}.$$

By virtue of

$$-\frac{1}{2} + \left[\frac{2}{9} g^2(x_2) - \frac{2}{3} g(x_2)\right] + \frac{1}{10} \le -\frac{1}{2} + \left[\frac{2}{9} \times \left(\frac{7}{2}\right)^2 - \frac{2}{3} \times \frac{7}{2}\right] + \frac{1}{10}$$
$$\le -\frac{1}{2} + \frac{2}{5} + \frac{1}{10} < 1,$$

it follows that

$$D^{\mathrm{T}}D - I_2 + G_\varepsilon = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} + \frac{1}{3} g(x_2) & \frac{1}{2} - \frac{1}{3} g(x_2) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} + \frac{1}{3} g(x_2) \\ \frac{1}{\sqrt{2}} & \frac{1}{2} - \frac{1}{3} g(x_2) \end{bmatrix}$$
$$- \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} + \frac{1}{10} + \frac{2}{9} g^2(x_2) - \frac{2}{3} g(x_2) \end{bmatrix}$$

is negative semi-definite.

In accordance with Corollary 9.13, the zero solution of (9.8) is absolutely stable in $[0, \frac{7}{2}]$.

9.3 Discrete Lurie Control Systems with Multiple Loops Feedback

In this section, we will generalize the results obtained in Sect. 9.1 to discrete Lurie control systems with multiple loops feedback [84]. First, we introduce the sufficient and necessary conditions for the explicit systems in which feedback control variables are no longer state variables.

Consider the following more general discrete Lurie control systems with loops feedback:

$$\begin{aligned} x(k+1) &= Ax(k) + Bf(\sigma(k)), \\ \sigma(k) &= C^{\mathrm{T}}x(k) - Df(\sigma(k)), \end{aligned} \tag{9.9}$$

where

$$\begin{aligned} &A \in R^{n\times n}, \quad B,\, C \in R^{n\times m}, \quad D \in R^{m\times m}, \quad m \le n, \\ &\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)^{\mathrm{T}}, \quad \sigma_i(k) = \sigma_i(x(k)), \\ &f = (f_1(\sigma_1)), f_2(\sigma_2)), \ldots, f_m(\sigma_m)))^{\mathrm{T}} \in C[R^m, R^m], \\ &0 \le \frac{f_i(\sigma_i)}{\sigma_i} \le \beta_i < +\infty, \quad j = 1, 2, \ldots, m. \end{aligned}$$

Let set $\Omega = \{x|C^{\mathrm{T}}x = 0\}$, $\Omega_i\{x|C^{\mathrm{T}}x = 0\}$, $i = 1, 2, \ldots, n$, where C_i is the ith-column vector of C $(1 \le i \le m)$.

Similar to Sect. 9.1, we can define the absolute stability of the zero solution of system (9.9) w.r.t. Ω, Ω_i, $\sigma = 0$, $\sigma_i = 0$, and positive definite and radially unbounded Lyapunov function w.r.t. Ω, Ω_i.

Since in (9.9) the feedback control variable $\sigma(k)$ is given implicitly, not in explicit form of state variables, it is not convenient to use σ to measure the positivity or negativity of the Lyapunov function w.r.t. the state variables.

To obtain general conclusion of absolute stability, we need the following basic hypothesis: $\|\sigma + Df(\sigma)\|$ is positive definite w.r.t. σ, i.e.,

$$\|\sigma + Df(\sigma)\| \begin{cases} > 0 & \text{when } \sigma \ne 0, \\ = 0 & \text{when } \sigma = 0. \end{cases}$$

It is obvious that the hypothesis is true when D is a positive definite matrix. When $\|\sigma\| \ne \|Df(\sigma)\|$, if $\sigma \ne 0$, the hypothesis is also true. In fact,

$$\begin{aligned} &\|\sigma + Df(\sigma)\| \ge \|\sigma\| - \|Df(\sigma)\| > 0, \text{ when } \|\sigma\| > \|Df(\sigma)\|, \ \sigma \ne 0, \\ &\|\sigma + Df(\sigma)\| \ge \|Df(\sigma)\| - \|\sigma\| > 0, \text{ when } \|Df(\sigma)\| > \|\sigma\|, \ \sigma \ne 0. \end{aligned}$$

Lemma 9.16. *When $\|\sigma + Df(\sigma)\|$ is positive definite, the zero solution of (9.9) is absolutely stable w.r.t. the set Ω (Ω_i) if and only if the zero solution of (9.9) is absolutely stable w.r.t. $\sigma = 0$ ($\sigma_i = 0$).*

Proof. We only prove for the set Ω and $\sigma = 0$. The cases for the set Ω_i and $\sigma_i = 0$ are similar and omitted.

Sufficiency. Suppose that the zero solution of (9.9) is absolutely stable w.r.t. $\sigma = 0$. Then $\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that when $|\sigma_0| < \delta$, we have $|\sigma(t, t_0; \sigma_0)| < \frac{\varepsilon}{2}$, and $|Df(\sigma(t, t_0; \sigma_0)| < \frac{\varepsilon}{2}$. Further, since $f(\sigma)$ is continuous and $f(0) = 0$, we obtain

$$\|C^{\mathrm{T}}x(t, t_0; x_0)\| \le \|\sigma(t, t_0; \sigma_0)\| + \|Df(\sigma(t, t_0; \sigma_0)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then $\forall x_0 \in R^n$, one has

$$\lim_{t\to+\infty} \|C^{\mathrm{T}}x(t,t_0;x_0)\| \le \lim_{t\to+\infty} \|\sigma(t,t_0;\sigma_0)\| + \lim_{t\to+\infty} \|Df(\sigma(t,t_0;\sigma_0)\| = 0.$$

which indicates that the zero solution of (9.9) is absolute stable w.r.t. the set Ω.

Necessity. From the relation of the positive definite function and the R-type function (see Lemmas 2.5 and 2.6) we know that there exists $\varphi(\|\sigma\|) \in R$ such that

$$\varphi(\|\sigma\|) \le \|\sigma + Df(\sigma)\| = \|C^{\mathrm{T}}x\|.$$

Thus,

$$\|\sigma(t,t_0;\sigma_0)\| \le \varphi^{-1}(\|C^{\mathrm{T}}x(t,t_0;x_0)\|).$$

Now $\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, for $\|x_0\| < \delta(\varepsilon)$, we have

$$\varphi^{-1}(\|C^{\mathrm{T}}x(t,t_0;x_0)\|) < \varepsilon.$$

Therefore, $\forall x_0 \in R^n$,

$$\lim_{t\to+\infty} \|\sigma(t,t_0;\sigma_0)\| \le \lim_{t\to+\infty} \varphi^{-1}(\|C^{\mathrm{T}}x(t,t_0;x_0)\|) = 0.$$

This indicates that the zero solution of (9.9) is absolutely stable w.r.t. $\sigma = 0$. □

Theorem 9.17. *The sufficient and necessary conditions for the zero solution of (9.9) being absolutely stable are:*

1. *The spectral radius of the matrix* $G := A + D\theta C^{\mathrm{T}}$ *is less than* 1, *i.e.,* $\rho(G) < 1$. *Here,*

$$\theta = \begin{cases} I_m & \text{when } D = 0, \\ \dfrac{1}{2\|D\|}\left(I_m + \dfrac{D}{2\|D\|}\right)^{-1} & \text{when } D \ne 0; \end{cases}$$

2. *The zero solution of (9.9) is absolutely stable w.r.t.* Ω.

Proof. *Sufficiency*. When $D = 0$, take $f(\sigma) = C^{\mathrm{T}}x$. When $D \ne 0$, choose $f(\sigma) = \frac{\sigma}{2\|D\|}$. Obviously one can obtain

$$\|\sigma + Df(\sigma)\| \begin{cases} = \|\sigma\| > 0, \qquad \text{when } \sigma \ne 0,\ D = 0, \\ \ge \|\sigma\| - \left\|D\dfrac{\sigma}{2\|D\|}\right\| \ge \|\sigma\| - \|D\|\dfrac{\|\sigma\|}{2\|D\|} = \dfrac{1}{2}\|\sigma\| > 0, \\ \qquad\qquad \text{when } \sigma \ne 0,\ D \ne 0. \end{cases}$$

Thus, $\|\sigma + Df(\sigma)\|$ is positive definite w.r.t. σ.

Rewrite (9.9) as

$$\begin{aligned} x(k+1) &= (A + B\theta C^{\mathrm{T}})x(k) + Bf(\sigma(k)) - B\theta\sigma(k), \\ \sigma(k) &= C^{\mathrm{T}}x(k) - Df(\sigma(k)). \end{aligned} \tag{9.10}$$

Applying the constant variation formulas obtained in Corollaries 9.4 and 9.5 for discrete Lurie control systems into (9.10) yields

$$x(k+1) = G^{k+1}x_0 + \sum_{i=0}^{k} G^{k-i}B\left[f(\sigma(i,0;\sigma_0) - \theta(i,0;\sigma_0))\right],$$

where $x_0 = x(0)$, $\sigma_0 = \sigma(0)$. $\forall \varepsilon > 0$, since $\rho(G) < 1$, G^{k+1} is bounded. Hence, there exists $M > 0$ such that $\|G^{k+1}\| \leq M = \text{const}$. Take $\delta_1(\varepsilon) = \frac{\varepsilon}{3M}$, then when $\|x_0\| < \delta_1$, we obtain $\|G^{k+1}x_0\| \leq M\delta_1 = \frac{\varepsilon}{3}$. Further, from Lemma 9.16 and the condition (2) in Theorem 9.17 we have

$$C^{\mathrm{T}}x(k) \to 0 \iff \sigma(k) \to 0 \quad \text{and} \quad f(\sigma(k)) \to 0 \quad \text{as } k \to +\infty.$$

Now, take enough large $k > 0$ such that

$$\left\| \sum_{i=[k/2]+1}^{k} G^{k-i}B\left[f(\sigma(i,0;\sigma_0) - \theta(i,0;\sigma_0))\right] \right\| < \frac{\varepsilon}{3}.$$

Since the zero solution of (9.9) is absolutely stable w.r.t. Ω, it follows from Lemma 9.16 that it is absolutely stable w.r.t. the set $\sigma = 0$. $x(k,0;x_0)$ depends continuously on x_0, and $f(\sigma)$ is continuous, thus there exists $\delta_2(\varepsilon) > 0$ such that for any $\|x_0\| < \delta_2$, we have

$$\left\| \sum_{i=0}^{[k/2]} G^{[k/2]-i}B\left[f(\sigma(i,0;\sigma_0) - \theta(i,0;\sigma_0))\right] \right\| < \frac{\varepsilon}{3}. \tag{9.11}$$

By taking $\delta = \min(\delta_1, \delta_2)$, it follows from (9.9) and (9.11) that when $\|x_0\| < \delta$,

$$\begin{aligned} \|x(k+1)\| &\leq \left\|G^{k+1}x_0\right\| + \left\| \sum_{i=0}^{[k/2]} G^{[k/2]-i}B\left[f(\sigma(i,0;\sigma_0) - \theta(i,0;\sigma_0))\right] \right\| \\ &\quad + \left\| \sum_{i=[k/2]+1}^{k} G^{k-i}B\left[f(\sigma(i,0;\sigma_0) - \theta(i,0;\sigma_0))\right] \right\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{9.12}$$

Equation (9.12) indicates that the zero solution of (9.9) is stable.

Since $f(\sigma(k)) \to 0$ as $k \to +\infty$, there exists constant $M_1 > 0$ such that

$$\|Bf(\sigma(i,i;\sigma_0) - \theta\,\sigma(i,0;\sigma_0))\| \leq M_1.$$

From $\rho(G) < 1$, there exists constant $M_2 > 0$ such that

$$\sum_{i=[\frac{k}{2}]+1}^{k} \|G^{k-1-i}\|\,\|B\| \leq M_2.$$

Finally, $\forall x_0 \in R^n$, we obtain

$$\begin{aligned}0 \leq \lim_{k\to+\infty} \|x(k+1)\| \leq & \lim_{k\to+\infty} \|G^{k+1}x_0\| \\ & + \lim_{k\to+\infty} \sum_{i=[\frac{k}{2}]+1}^{k} \|G^{k-i}\| \, \|B\| \, \|f(\sigma(i,i;\sigma_0)) - \theta(\sigma(i,0;\sigma_0))\| \\ & + \lim_{k\to+\infty} \sum_{i=0}^{[\frac{k}{2}]} \|G^{k-i}\| \, \|B\| \, \|f(\sigma(i,i;\sigma_0)) - \theta(\sigma(i,0;\sigma_0))\| \\ = & \ 0.\end{aligned}$$

Therefore, the zero solution of (9.9) is absolutely stable.

Necessity. One can apply Lemma 9.16 to prove the necessity. The detail is omitted here. □

Remark 9.18. The condition (1) in Theorem 9.17 is a constructive algebraic condition, which is easy to verify in applications, while the condition (2) of Theorem 9.17 is difficult to verify. Thus, we pay particular attention to the absolute stability of the zero solution of (9.9) w.r.t. to the set Ω.

Theorem 9.19. *If there exists a radially unbounded, positive definite Lyapunov function w.r.t. the set Ω such that $\Delta V|_{(9.9)}$ is negative definite w.r.t. Ω, then the zero solution of (9.9) is absolutely stable w.r.t. the set Ω.*

Proof. The condition of the theorem implies that there exists $\varphi_1(\|C^{\mathrm{T}}x\|) \in KR$, $\varphi_2(\|C^{\mathrm{T}}x\|) \in KR$ and $\psi(\|C^{\mathrm{T}}x\|) \in K$ such that

$$\varphi_1(\|C^{\mathrm{T}}x\|) \leq V(x) \leq \varphi_2(\|C^{\mathrm{T}}x\|) \tag{9.13}$$

and

$$\Delta V|_{(9.9)} = V(x(k+1)) - V(x(k)) \leq -\psi(\|C^{\mathrm{T}}x(k)\|). \tag{9.14}$$

Now $\forall \varepsilon > 0$, take $\delta = \varphi_2^{-1}(\varphi_1(\varepsilon))$. Then when $\|C^{\mathrm{T}}x_0\| < \delta$, we have

$$\varphi_1(\|C^{\mathrm{T}}x(k)\|) \leq V(x(k)) \leq V(x(0)) \leq \varphi_2(\|C^{\mathrm{T}}x_0\|) \leq \varphi_2(\delta) = \varphi_1(\varepsilon),$$

i.e., $\|C^{\mathrm{T}}x(k)\| \leq \varepsilon$, implying that the zero solution of (9.9) is stable.

It follows from (9.14) that:

$$\begin{aligned}& V(x(k+1)) - V(x(0)) \\ & \quad = V(x(k+1)) - V(x(k)) + V(x(k)) - V(x(k-1)) + \cdots - V(x(0)) \\ & \quad \leq -\sum_{j=1}^{k} \psi(\|C^{\mathrm{T}}x(j)\|),\end{aligned} \tag{9.15}$$

which, in turn, yields $\sum_{j=1}^{\infty} \psi(\|C^{\mathrm{T}}x(j)\|) \leq V(x(0))$ and thus $\psi(\|C^{\mathrm{T}}x(j)\|) \to 0$ as $j \to +\infty$. This implies that

$$\|C^{\mathrm{T}}x(j)\| \to 0 \qquad \text{as} \quad j \to +\infty. \tag{9.16}$$

Equation (9.16) clearly shows that the zero solution of (9.9) is absolutely stable w.r.t. the set Ω. □

Corollary 9.20. *If $D = D^{\mathrm{T}}$, and there exists constant $\xi > 0$ such that the following matrix:*

$$\begin{bmatrix} A^{\mathrm{T}}CC^{\mathrm{T}}A - CC^{\mathrm{T}} & A^{\mathrm{T}}CC^{\mathrm{T}}B + \dfrac{1}{2}\xi C \\ B^{\mathrm{T}}CC^{\mathrm{T}}B + \dfrac{1}{2}\xi C^{\mathrm{T}} & B^{\mathrm{T}}CC^{\mathrm{T}}B - \xi D \end{bmatrix}$$

is negative semi-definite, then the zero solution of (9.9) is absolutely stably w.r.t. the set Ω.

Proof. Choose the radially unbounded, positive definite Lyapunov function w.r.t. the set Ω as

$$V(x) = (C^{\mathrm{T}}x)^{\mathrm{T}}(C^{\mathrm{T}}x) = x^{\mathrm{T}}CC^{\mathrm{T}}x.$$

Then we have

$$\begin{aligned}
\Delta V(x(k))|_{(9.9)} &= x^{\mathrm{T}}(k+1)CC^{\mathrm{T}}x(k+1) - x^{\mathrm{T}}(k)CC^{\mathrm{T}}x(k) \\
&= \left[x^{\mathrm{T}}(k)(A^{\mathrm{T}}C + f^{\mathrm{T}}(\sigma(k))B^{\mathrm{T}}C\right]\left[C^{\mathrm{T}}Ax(k) + C^{\mathrm{T}}Bf(\sigma(k))\right] \\
&\quad - x^{\mathrm{T}}(k)CC^{\mathrm{T}}x(k) + \xi f^{\mathrm{T}}(\sigma(k))C^{\mathrm{T}}x(k) \\
&\quad - \xi f^{\mathrm{T}}(\sigma(k))\sigma(x(k)) - \xi f^{\mathrm{T}}(\sigma(k))Df(\sigma(k)) \\
&= \begin{pmatrix} x(k) \\ f(\sigma(k)) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}}CC^{\mathrm{T}}A - CC^{\mathrm{T}} & A^{\mathrm{T}}CC^{\mathrm{T}}B + \dfrac{1}{2}\xi C \\ B^{\mathrm{T}}CC^{\mathrm{T}}B + \dfrac{1}{2}\xi C^{\mathrm{T}} & B^{\mathrm{T}}CC^{\mathrm{T}}B - \xi D \end{bmatrix} \begin{pmatrix} x(k) \\ f(\sigma(k)) \end{pmatrix} \\
&\quad - \xi f(\sigma(k))\sigma(k) \\
&< 0 \qquad \text{when} \;\; \sigma \neq 0.
\end{aligned} \tag{9.17}$$

Thus, the zero solution of (9.9) is absolutely stable w.r.t. the set Ω. □

Corollary 9.21. *If $D = D^{\mathrm{T}}$, and there exist $\xi > 0$ and a symmetric matrix P satisfying $x^{\mathrm{T}}P\,x \geq (C^{\mathrm{T}}x)^2$ such that the matrix*

$$\begin{bmatrix} A^{\mathrm{T}}PA - P & A^{\mathrm{T}}PB + \dfrac{1}{2}\xi C \\ B^{\mathrm{T}}PA + \dfrac{1}{2}\xi C^{\mathrm{T}} & B^{\mathrm{T}}CB - \xi D \end{bmatrix}$$

is negative semi-definite, then the zero solution of (9.9) is absolutely stably w.r.t. the set Ω.

Proof. Construct the radially unbounded, positive definite Lyapunov function w.r.t. Ω as $V(x) = x^{\mathrm{T}} P x$. Then we obtain

$$\begin{aligned}
\Delta V(x(k))|_{(9.9)} &= x^{\mathrm{T}}(k+1)\,PA\,x(k+1) - x^{\mathrm{T}}(k)\,Px(k) \\
&= x^{\mathrm{T}}(k)A^{\mathrm{T}}Px(k) + f^{\mathrm{T}}(\sigma(k))\,B^{\mathrm{T}}PAx(k) \\
&\quad + x^{\mathrm{T}}(k)A^{\mathrm{T}}PBf(\sigma(k)) + f^{\mathrm{T}}(\sigma(k))\,B^{\mathrm{T}}PBf(\sigma(k)) \\
&\quad - x^{\mathrm{T}}(k)\,Px(k) + \xi\, f^{\mathrm{T}}(\sigma(k))\,C^{\mathrm{T}}x(k) \\
&\quad - \xi\, f^{\mathrm{T}}(\sigma(k))\,\sigma(x(k)) - \xi\, f^{\mathrm{T}}(\sigma(k))\,Df(\sigma(k)) \\
&= \begin{pmatrix} x(k) \\ f(\sigma(k)) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}}PA - P & A^{\mathrm{T}}PB + \frac{1}{2}\xi C \\ B^{\mathrm{T}}PA + \frac{1}{2}\xi C^{\mathrm{T}} & B^{\mathrm{T}}PA - \xi D \end{bmatrix} \begin{pmatrix} x(k) \\ f(\sigma(k)) \end{pmatrix} \\
&\quad - \xi\, f(\sigma(k))\,\sigma(k), \\
&\le -\xi\, f(\sigma(k))\,\sigma(k) < 0 \qquad \text{when } \sigma \neq 0.
\end{aligned} \tag{9.18}$$

Thus, $\Delta V(x(k))|_{(9.9)}$ is negative definite w.r.t. Ω. Then it follows from Lemma 9.16 that the zero solution of (9.9) is absolutely stable w.r.t. the set Ω. □

9.4 Sufficient Conditions for Discrete Control Systems with Loops Feedback

In the following, suppose $m = n$. Let $C^{\mathrm{T}}x = \sigma + Df(\sigma) = \sigma + DJ\sigma = (I_n + DJ)\sigma$, where $J = (\frac{\partial f_i}{\partial \sigma_j})_{n\times n}$ is the Jacobian matrix of f, but it is not a constant matrix.

Theorem 9.22. *Suppose that one of the following conditions is satisfied:*

1. C^{T} *is a full-rank matrix, satisfying*

$$\|C^{\mathrm{T}}A(C^{\mathrm{T}})^{-1}\| + \|C^{\mathrm{T}}B\|\;\|J\|\;\|(I_n + DJ)^{-1}\| = h_1 < 1$$

or

2. $C^{\mathrm{T}}A = AC^{\mathrm{T}}$ *is a full-rank matrix, satisfying*

$$\|A\| + \|C^{\mathrm{T}}B\|\;\|J\| + \|(I_n + DJ)^{-1}\| = h_2 < 1.$$

Then the zero solution of (9.9) is absolutely stable.

Proof. If the condition (1) holds, we have

$$
\begin{aligned}
\|C^{\mathrm{T}}x(k+1)\| &\le \|C^{\mathrm{T}}A(C^{\mathrm{T}})^{-1}C^{\mathrm{T}}x(k)\| + \|C^{\mathrm{T}}B\|\ \|J\|\ \|\sigma(k)\| \\
&\le \|C^{\mathrm{T}}A(C^{\mathrm{T}})^{-1}\|\ \|C^{\mathrm{T}}x(k)\| + \|C^{\mathrm{T}}B\|\ \|J\|\ \|\sigma(k)\| \\
&\le \|C^{\mathrm{T}}A(C^{\mathrm{T}})^{-1}\|\ \|C^{\mathrm{T}}x(k)\| + \|C^{\mathrm{T}}B\|\|J\|\|(I_m+DJ)^{-1}\|\ \|C^{\mathrm{T}}x(k)\| \\
&= \left[\|C^{\mathrm{T}}A(C^{\mathrm{T}})^{-1}\| + \|C^{\mathrm{T}}B\|\ \|\ \|J\|\ \|(I_m+DJ)^{-1}\|\right] \|C^{\mathrm{T}}x(k)\| \\
&= h_1\,\|C^{\mathrm{T}}x(k)\| \le h_1^2\|C^{\mathrm{T}}x(k-1)\|\cdots \le h_1^k\|C^{\mathrm{T}}x(0)\| \\
&\to 0 \qquad \text{as} \quad k\to+\infty.
\end{aligned} \tag{9.19}
$$

If the condition (2) is satisfied, we obtain

$$
\begin{aligned}
\|C^{\mathrm{T}}x(k+1)\| &\le \|C^{\mathrm{T}}PAx(k)\| + \|C^{\mathrm{T}}B\|\ \|J\|\ \|(I_m+DJ)^{-1}\|\ \|C^{\mathrm{T}}x(k)\| \\
&\le \|A\|\ \|C^{\mathrm{T}}x(k)\| + \|C^{\mathrm{T}}B\|\ \|J\|\ \|(I_m+DJ)^{-1}\|\ \|C^{\mathrm{T}}x(k)\| \\
&= \Big(\|A\| + \|C^{\mathrm{T}}B\|\ \|\ \|J\|\ \|(I_m+DJ)^{-1}\|\Big)\,\|C^{\mathrm{T}}x(k)\| \\
&= h_2\,\|C^{\mathrm{T}}x(k)\| \le h_2^2\,\|C^{\mathrm{T}}x(k-1)\|\cdots \le h_2^k\,\|C^{\mathrm{T}}x(0)\| \\
&\to 0 \qquad \text{as} \quad k\to+\infty.
\end{aligned} \tag{9.20}
$$

Therefore, the zero solution of (9.9) is absolutely stable w.r.t. the set Ω. □

In the following, we rewrite system (9.9) into a more detailed form:

$$
\begin{aligned}
x(k+1) &= Ax(k) + \sum_{j=1}^{m} b_j f_j(\sigma_j(k)), \\
\sigma_j(k) &= C_j^{\mathrm{T}}x(k) - d_j f_j(\sigma_j(k)), \qquad j=1,2,\ldots,m,
\end{aligned} \tag{9.21}
$$

where $x\in R^n, A\in R^{n\times n}, f_i\in F_{[0,k]}, C_j^{\mathrm{T}}=(C_{1j},C_{2j},\ldots,C_{nj}), b_j^{\mathrm{T}}=(b_{1j},b_{2j},\ldots,b_{nj})$, $j=1,2,\ldots,m,\ m\le n$; C_{ij}, b_{ij} are column vectors, and $C_1,\ldots,C_m$ are linearly independent.

We assume that $|\sigma_j + d_j f_j(\sigma)|$ is positive definite for $j=1,2,\ldots,m$. Without loss of generality, suppose

$$
\det\begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \cdots & \vdots \\ c_{m1} & \cdots & c_{mm} \end{bmatrix} \ne 0.
$$

Introducing the following linear nonsingular transformation:

$$
y = Gx = (g_{ij})_{n\times n}x \quad \text{where} \quad g_{ij} = \begin{cases} c_{ij} & 1\le i\le m,\ \ 1\le j\le n, \\ 1 & i=j=m+1,m+2,\ldots,n, \\ 0 & \text{otherwise} \end{cases} \tag{9.22}
$$

into system (9.21) yields

$$\begin{aligned} y(k+1) &= \tilde{A}y(k) + \sum_{j=1}^{m} \tilde{b}_j f_j(\delta_j(k)), \\ \delta_j(k) &= y_j(k) - d_j f_j(\delta_j(k)). \end{aligned} \tag{9.23}$$

Since G is non-singular, the absolute stabilities of the zero solutions of systems (9.21) and (9.23) are equivalent. More precisely, the zero solution of system (9.21) is absolutely stable w.r.t. the set $\Omega_i = \{x | C_i^{\mathrm{T}} x = 0\}$ $(i = 1, 2, \dots, m)$ is equivalent to the absolute stability of the zero solution of system (9.23) w.r.t. the partial variables $\eta_1, \eta_2, \dots, \eta_m$; and also equivalent to the absolute stability of the zero solution of system (9.23) w.r.t. $\delta_j = 0$ $(j = 1, 2, \dots, m)$. Thus, similar to Theorems 9.19 and 9.22 we have the following results.

Theorem 9.23. *The zero solution of system (9.23) is absolutely stable if and only if the following conditions are satisfied:*

1. $B = \tilde{A} + \sum_{i=1}^{m} \theta_i \tilde{b}_i c_i^{\mathrm{T}}$ *is a Schur matrix, i.e., the spectral radius of* B *is less then* 1, $\rho(B) < 1$, *where*

$$\theta_i = \begin{cases} \dfrac{1}{2}, & \text{when} \quad d_i = 0 \quad i = 1, 2, \dots, m, \\ \dfrac{1}{3 d_i} & \text{when} \quad d_i \neq 0 \quad i = 1, 2, \dots, m; \end{cases}$$

2. The zero solution of system (9.23) is absolutely stable w.r.t. the partial variables $\eta_1, \eta_2, \dots \eta_m$.

Proof. When $d_j = 0$, take $f_j(\sigma_j) = \frac{1}{2}\delta_j$; while if $d_j \neq 0$, choose $f_j(\sigma_j) = \frac{1}{2\delta_j}\delta_j$. Then from $\delta_j = \eta_j - d_j f_j(\sigma_j)$, we have $\delta_j = \frac{2}{3}\eta_j$, and then simple calculations lead to Q_i given above.

Rewrite system (9.23) as

$$\begin{aligned} y(k+1) &= \tilde{A}y(k) + \sum_{j=1}^{m} \theta_j \tilde{b}_j c_j^{\mathrm{T}}(y(k)) + \sum_{j=1}^{m} \tilde{b}_j f_j(\delta_j(k)) - \sum_{j=1}^{m} \theta_j \tilde{b}_j c_j^{\mathrm{T}}(y(k)), \\ \delta_j(k) &= y_j(k) - d_j f_j(\delta_j(k)). \end{aligned}$$

The remaining of the proof can follow the proof of Theorem 9.22. □

Next, we discuss the case $d_i = 0$, $i = 1, 2, \dots, m$. To obtain some applicable criteria, we further rewrite (9.23) as

$$\begin{aligned} y_i(k+1) &= \sum_{j=1}^{n} \tilde{a}_{ij} y_j(k) + \sum_{j=1}^{m} \tilde{b}_{ij} f_j(y_j(k)), \\ &= \sum_{j=1}^{n} \tilde{a}_{ij} y_j(k) + \sum_{j=1}^{m} \tilde{b}_{ij} g_j(y_j(k)) y_j(k), \end{aligned} \tag{9.24}$$

where

$$\frac{f_j(y_j)}{y_j} = \begin{cases} y_j(y_j) & \text{when } y_j \neq 0, \\ 0 & \text{when } y_j = 0. \end{cases}$$

Theorem 9.24. *If system (9.24) satisfies the following conditions:*

1. *The condition* (1) *in Theorem 9.23 holds.*
2. *There exist constants* $c_i > 0,\ i = 1, 2, \ldots, n$ *such that*

$$\max_{1 \le j \le m} \sum_{i=1}^{n} \left[\frac{c_i}{c_j} \left| \tilde{a}_{ij} + \frac{c_i \tilde{b}_{ij}}{c_j} g_j(\eta_j) \right| \right] \le \xi < 1 \ \ \text{and} \ \ \max_{m+1 \le j \le n} \sum_{i=1}^{n} \left[\frac{c_i}{c_j} |\tilde{a}_{ij}| \right] \le 1,$$

where ξ *is a constant.*

Then the zero solution of system (9.24) is absolutely stable.

Proof. Construct the radially unbounded, positive definite Lyapunov function

$$V(y) = \sum_{i=1}^{n} c_i |y_i|.$$

Then we have

$$\begin{aligned}
\Delta(y(k))|_{(9.24)} &= V(y(k+1)) - V(y(k)) \\
&= \sum_{i=1}^{n} c_i \left[\left| \sum_{j=1}^{n} \tilde{a}_{ij} y_j(k) + \sum_{j=1}^{m} \tilde{b}_{ij}(y_j(k)) y_j(k) \right| \right] - \sum_{j=1}^{n} c_j |y_j(k)| \\
&\le \sum_{j=1}^{m} \left| \sum_{i=1}^{n} c_i \big(|\tilde{a}_{ij}| + |\tilde{b}_{ij} g_j(y_j(k))| \big) \right| |y_j(k)| + \sum_{j=m+1}^{n} \sum_{i=1}^{n} c_i \, |\bar{a}_{ij}| \, |y_j(k)| \\
&\quad - \sum_{j=1}^{n} c_j \, |y_{ij}(k)| \\
&= \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \frac{c_i}{c_j} \big(|\tilde{a}_{ij}| + |\tilde{b}_{ij} g_j(y_j(k))| \big) \right] c_j |y_j(k)| \\
&\quad + \sum_{j=m+1}^{n} \sum_{i=1}^{n} \frac{c_i |\tilde{a}_{ij}}{c_j} |c_j| \, |(y_j(k))| - \sum_{j=1}^{n} c_j \, |y_{ij}(k)| \\
&= \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \frac{c_i}{c_j} \Big(|\tilde{a}_{ij}| + |\tilde{b}_{ij} g_j(y_j(k))| \Big) - 1 \right] c_j |y_j(k)| \\
&\quad + \sum_{j=m+1}^{n} \left[\sum_{i=1}^{n} \frac{c_i |\tilde{a}_{ij}}{c_j} - 1 \right] c_j \, |y_j(k)| \\
&\le \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \frac{c_i}{c_j} \Big(|\tilde{a}_{ij}| + |\tilde{b}_{ij} g_j(y_j(k))| \Big) - 1 \right] c_j |y_j(k)| \\
&< 0, \qquad \text{when} \quad \sum_{j=1}^{m} |y_j| \neq 0.
\end{aligned}$$

Therefore, the zero solution of system (9.24) is absolutely stable w.r.t. the partial variables $y_1, y_2, \ldots, y_m$. Then the conclusion of this theorem follows the condition (1) of this theorem and Theorem 9.23. □

In particular, in Theorem 9.24 take $c_i = 1,\ i = 1, 2 \ldots, n$, the condition (2) of Theorem 9.24 becomes a form which is easier to verify:

$$\max_{1\le j\le m} \sum_{i=1}^{n} \left[|\tilde{a}_{ij}| + |\tilde{b}_{ij}|\,|g_j(y_{ij})|\right] \le \xi < 1 \quad \text{and} \quad \max_{m+1\le j\le n} \sum_{i=1}^{n} |\tilde{a}_{ij}| \le 1.$$

In the following, let $U = (u_{ij})_{n\times n},\ \tilde{U} = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$, where

$$u_{ij} = \begin{cases} |\tilde{a}_{ij}| + |\tilde{b}_{ij}|\,|g_j(y_{ij})|, & i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m, \\ |\tilde{a}_{ij}|, & i = 1, 2, \ldots, n,\ j = m+1, m+2, \ldots, n. \end{cases}$$

Theorem 9.25. *If the following conditions are satisfied:*

1. *The condition* (1) *in Theorem 9.23 holds.*
2. *There exist* $n\times n$ *positive definite matrix* $W = W^{\mathrm{T}}$ *and constant* $\varepsilon > 0$ *such that* $U^{\mathrm{T}} W U - W + \varepsilon \tilde{U}$ *is negative semi-definite.*

Then the zero solution of system (9.24) is absolutely stable.

Proof. Choose the positive definite radially unbounded Lyapunov function:

$$V(y) = y^{\mathrm{T}} W y.$$

Thus, we obtain

$$\begin{aligned} \Delta V(y(k))|_{(9.24)} &= V(y(k+1)) - V(y(k)) \\ &= y^{\mathrm{T}}(k+1) W y(k+1) - y^{\mathrm{T}}(k) W y(k) \\ &= (U y(k))^{\mathrm{T}} W (U y(k)) - y^{\mathrm{T}}(k) W y(k) \\ &= y^{\mathrm{T}}(k) U^{\mathrm{T}} W U y(k) - y^{\mathrm{T}}(k) W y(k) \\ &= y^{\mathrm{T}}(k)(U^{\mathrm{T}} W U - W + \varepsilon \tilde{U}) y(k) - y^{\mathrm{T}}(k) W y(k) \\ &\le -\varepsilon \sum_{i=1}^{m} y_i^2(k) < 0 \qquad \text{when} \quad \sum_{i=1}^{m} y_i^2 \ne 0. \end{aligned}$$

The above result shows that the zero solution of system (9.24) is absolutely stable w.r.t. the partial variables $y_1, y_2, \ldots, y_m$. Then it follows from the condition (1) of this theorem and the proof of Theorem 9.23 that the conclusion of this theorem is true. □

In particular, if take $W = I_n$, then the condition (2) of Theorem 9.23 becomes simpler and easier to verify.

10

Time-Delayed and Neutral Lurie Control Systems

In modeling natural and social phenomena, the dynamic behavior of many systems depends upon not only the current state, but also the system's history, which is called time-delayed phenomenon. Mathematical models arising from the areas of engineering, physics, mechanics, control system, chemical reaction, biological, and medical systems always involve time delay. In particular, time delay often appears in control systems. Any system with a feedback control involves unavoidable time delay since time is needed for the system to appropriately react to the input. Therefore, studying the absolute stability of time-delayed Lurie systems is naturally important and necessary [51, 66].

In Chap. 3, we have obtained absolute stability conditions for the direct, indirect, and general controls of Lurie systems without time delay. However, not much attention has been paid to time-delayed Lurie systems (i.e., the Lurie systems described by differential difference equations (DDE)). Although many researchers are, with the aid of the Matlab software LMI, still investigating the stability of Lurie systems with or without time delay, the conditions they obtained are only sufficient.

In this chapter, based on the results we obtained in Chap. 3, we will continue to consider the absolute stability of Lurie systems with time delay. Some materials presented in this chapter are based on the results of [94] (Sects. 10.1 and 10.2), [160, 161] (Sect. 10.3), and Nain [110–112] (Sects. 10.4 and 10.5).

10.1 Lurie Systems with Constant Time Delays

In this section, we consider Lurie system with constant delays. We first derive sufficient and necessary conditions for absolute stability of Lurie systems, and then present some simple and easy-applicable algebraic sufficient conditions for such systems.

10.1.1 Sufficient and Necessary Conditions for Absolute Stability

Consider the following Lurie system with constant time delays [94]:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= \tilde{A}z(t) + \tilde{B}z(t-\tau_1) + \tilde{h}f(\sigma(t-\tau_2)), \quad \tilde{h} = (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)^{\mathrm{T}}, \\ \sigma &= c^{\mathrm{T}}z = \sum_{i=1}^{n} c_i z_i, \end{aligned} \tag{10.1}$$

where $c, \tilde{h} \in R^n$ are constant vectors, $z \in R^n$ is the state vector, and $\tilde{A}, \tilde{B} \in R^{n\times n}$ are real matrices; the time delays $\tau_1 > 0$ and $\tau_2 > 0$ are constants; the function f is defined as

$$f(\cdot) \in F_\infty := \left\{ f \,|\, f(0) = 0,\ \sigma f(\sigma) > 0,\ \sigma \neq 0,\ f \in C[(-\infty, +\infty), R^1] \right\}$$

or

$$f(\cdot) \in F_{[0,k]} := \left\{ f \,|\, f(0) = 0, 0 < \sigma f(\sigma) \leq k\sigma^2, \sigma \neq 0, f \in C[(-\infty, +\infty), R^1] \right\},$$

where k is a positive real number. Let $\tau = \max\{\tau_1, \tau_2\}$, then $C[[-\tau, 0], R^n]$ represents a Banach space with uniformly continuous topological structure.

Definition 10.1. *If $\forall f(\cdot) \in F_\infty$ (or $\forall f(\cdot) \in F_{[0,k]}$), the zero solution of system (10.1) is globally, asymptotically stable for any values of $\tau_1, \tau_2 \geq 0$, then the zero solution of system (10.1) is said to be time-delay independent absolutely stable (or time-delay independent, absolutely stable in the Hurwitz angle $[0, k]$).*

It is easy to show that the necessary condition for system (10.1) to be time-delay independent absolutely stable is $c^{\mathrm{T}}\tilde{h} \leq 0$; and the necessary condition for system (10.1) to be time-delay independent absolutely stable in the Hurwitz angle $[0, k]$ is that $\forall \mu \in [0, k]$, the matrix $\tilde{A} + \tilde{B} + \mu\,\tilde{h}\,c^{\mathrm{T}}$ is a Hurwitz matrix.

In fact, let

$$f(\sigma(t - \tau_2)) = \mu\,\sigma(t - \tau_2) = \sum_{i=1}^{n} \mu\, c_i\, z_i(t - \tau_2).$$

Then system (10.1) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \tilde{A}\,z(t) + \tilde{B}z(t - \tau_1) + \mu\,\tilde{h}\,c^{\mathrm{T}}z(\sigma(t - \tau_2)). \tag{10.2}$$

In particular, when $\tau_1 = \tau_2 = 0$, for an arbitrary $\mu \in [0, +\infty)$, the matrix $\tilde{A} + \tilde{B} + \mu\,\tilde{h}\,c^{\mathrm{T}}$ is a Hurwitz matrix. Thus we have

$$\mathrm{tr}(\tilde{A} + \tilde{B} + \mu\,\tilde{h}\,c^{\mathrm{T}}) = \mathrm{tr}\tilde{A} + \mathrm{tr}\tilde{B} + \mu\,\mathrm{tr}(\tilde{h}\,c^{\mathrm{T}}) = \mathrm{tr}\tilde{A} + \mathrm{tr}\tilde{B} + \mu c^{\mathrm{T}}\tilde{h} < 0.$$

The above inequality holds for $\mu \gg 1$, implying that $c^{\mathrm{T}}\tilde{h} \leq 0$.

Next, take $\mu \in [0, k]$, $f(\sigma(t - \tau_2)) = \mu\,\sigma(t - \tau_2)$, $\tau_1 = \tau_2 = 0$. Then it is easy to see that $\tilde{A} + \tilde{B} + \mu\,\tilde{h}\,c^{\mathrm{T}}$ must be a Hurwitz matrix.

In the following, we use two nonsingular linear transformations to change system (10.1) into a nonlinear system with separable variables. There are two cases.

1. When $c^{\mathrm{T}}\tilde{h} \leq 0$. Without loss of generality, suppose $c_n \neq 0$. Let

$$x = \Omega\, z, \tag{10.3}$$

where

$$\Omega = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}.$$

Then system (10.1) becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \Omega \tilde{A} \Omega^{-1} x(t) + \Omega \tilde{B} \Omega^{-1} x(t-\tau_1) + \Omega \tilde{h} f(x_n(t-\tau_2)) \\ &:= Ax(t) + Bx(t-\tau_1) + hf(x_n(t-\tau_2)), \end{aligned} \tag{10.4}$$

where $A = \Omega \tilde{A} \Omega^{-1}$, $B = \Omega \tilde{B} \Omega^{-1}$, and $h = \Omega \tilde{h}$. Since (10.3) is a nonsingular linear transformation, the time-delay independent absolute stabilities of the zero solutions of systems (10.1) and (10.4) are equivalent.

2. When $c^{\mathrm{T}} \tilde{h} < 0$. Without loss of generality, assume $\tilde{h}_n c_n \neq 0$. Let

$$y = Gz,$$

where

$$G = \begin{bmatrix} \tilde{h}_n & 0 & \cdots & 0 & -\tilde{h}_1 \\ 0 & \tilde{h}_n & \cdots & 0 & -\tilde{h}_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \tilde{h}_n & -\tilde{h}_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}.$$

Then system (10.1) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= G\tilde{A}G^{-1}y(t) + G\tilde{B}G^{-1}y(t-\tau_1) + G\tilde{h}f(y_n(t-\tau_2)) \\ &:= Py(t) + Qy(t-\tau_1) + bf(y_n(t-\tau_2)), \end{aligned} \tag{10.5}$$

where $P = G\tilde{A}G^{-1}$, $Q = G\tilde{B}G^{-1}$, and $b = G\tilde{h} = (\overbrace{0, 0, \ldots, 0}^{n-1},\ c^{\mathrm{T}}\tilde{h})^{\mathrm{T}}$. Similarly, due to the nonsingularity of G, the time-delay independent absolute stabilities of the zero solutions of systems (10.1) and (10.5) are equivalent.

Definition 10.2. *The zero solution of system (10.4) is said to be time-delay independent absolutely stable (or time-delay independent absolutely stable in the Hurwitz angle $[0,k]$) w.r.t. the partial variable x_n of the system, if $\forall f(\cdot) \in F_\infty$ (or $\forall f(\cdot) \in F_{[0,k]}$), the zero solution of system (10.4) is globally asymptotically time-delay independent stable (or globally asymptotically time-delay independent stable in the Hurwitz angle $[0,k]$) w.r.t. the partial variable x_n.*

Similarly, we can define the time-delay independent stability for the zero solution of system (10.5) w.r.t. the partial variable y_n.

Theorem 10.3. *The sufficient and necessary conditions for the zero solution of system (10.4) to be time-delay independent stable are:*

1. *the matrix* $A+B+(O_{n\times(n-1)},h\theta)$ *is a Hurwitz matrix, where* $\theta=0$ *or* $\theta=1$*, and*

$$(O_{n\times(n-1)},h\theta)=\begin{bmatrix}0&\cdots&0&h_1\theta\\ \vdots&\vdots&\vdots&\vdots\\ 0&\cdots&0&h_n\theta\end{bmatrix}_{n\times n}.$$

2. $\det\left(\mathrm{i}\sigma-A-B\mathrm{e}^{-\mathrm{i}\sigma\tau_1}-(O_{n\times(n-1)},h\theta)\mathrm{e}^{-\mathrm{i}\sigma\tau_2}\right)\neq 0 \quad \forall\sigma\in R,\ \forall\tau_1,\tau_2\geq 0.$
3. *The zero solution of system (10.4) is time-delay independent absolutely stable w.r.t. the partial variable* x_n.

Proof. *Necessity*. Suppose that the zero solution of system (10.4) is time-delay independent, absolutely stable. When $A+B$ is a Hurwitz matrix, we can choose $\theta=0$ and thus $A+B+(O_{n\times(n-1)},h\theta)=A+B$ is a Hurwitz matrix. When $A+B$ is not a Hurwitz matrix, we take $f(x_n)=x_n$. Then system (10.4) becomes a linear time-delayed system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=Ax(t)+Bx(t-\tau_1)+hx_n(t-\tau_2). \tag{10.6}$$

From the sufficient and necessary conditions of global time-delay independent absolute stability for constant time-delayed systems with constant coefficients [40], we know that all the eigenvalues of the characteristic equation of system (10.6), given by

$$\det\left(\lambda I-A-B\mathrm{e}^{-\mathrm{i}\lambda\tau_1}-(O_{n\times(n-1)},h\theta)\mathrm{e}^{-\mathrm{i}\lambda\tau_2}\right)=0, \tag{10.7}$$

must have negative real parts. This is equivalent to the conditions (1) and (2) in Theorem 10.3 ($\theta=1$) [40]. The condition (3) of Theorem 10.3 is obvious. The necessity is proved.

Sufficiency. Rewrite system (10.4) as

$$\frac{\mathrm{d}x}{\mathrm{d}t}=Ax(t)+Bx(t-\tau_1)+h\theta x_n(t-\tau_2)+hf(x_n(t-\tau_2)-\theta hx_n(t-\tau_2)). \tag{10.8}$$

Let $x^*(t)=x(t_0,\varphi)(t)$ be the solution of the following system:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=Ax(t)+Bx(t-\tau_1)+h\theta x_n(t-\tau_2),\\ x(t)&=\varphi(t)\quad t_0-\tau\leq t\leq t_0,\quad \tau=\max[\tau_1,\tau_2].\end{aligned} \tag{10.9}$$

Then from the method of constant variation, we know that the solution of (10.8) passing through the initial point (t_0,φ) can be expressed as

$$x(t)=x^*(t)+\int_{t_0}^{t}U(t,s)\,[hf(x_n(s-\tau_2)-\theta hx_n(t-\tau_2))]\,\mathrm{d}s, \tag{10.10}$$

where $U(t,s)$ is the fundamental matrix solution, satisfying

$$\frac{\partial U(t,s)}{\partial t} = AU(t,s) + BU(t-\tau_1,s) + \left(O_{n\times(n-1)}, h\theta\right) U(t-\tau_2,s),$$
$$U(t,s) = \begin{cases} 0 \text{ when } \tau - s \le t \le s, \\ I \text{ when } t = s. \end{cases}$$

From the conditions given in Theorem 10.3, it is known that there exist constants $M \ge 1, N \ge 1$, and $\alpha > 0$ such that

$$\|x^*(t)(t_0,\varphi)(t)\| \le M\|\varphi\|\mathrm{e}^{-\alpha(t-t_0)} \qquad \text{when } t \ge t_0, \tag{10.11}$$
$$\|U(t,s)\| \le N\mathrm{e}^{-\alpha(t-s)} \qquad \text{when } t \ge s. \tag{10.12}$$

Therefore, we have

$$\|x(t)\| \le M\|\varphi\|\mathrm{e}^{-\alpha(t-t_0)} + N\int_{t_0}^{t} \mathrm{e}^{-\alpha(t-s)} \left[\|hf(x_n(t-\tau_1)\| + \|\theta h x_n(s-\tau_2)\|\right] \mathrm{d}s \tag{10.13}$$

for $t \ge t_0$. $\forall \varepsilon > 0$, since $x_n(t) \to 0$ as $t \to +\infty$ and $f(\cdot)$ is a continuous function of x_0, there exists $\delta_1(\varepsilon) > 0$ such that when $\|\varphi\| < \delta_1(\varepsilon)$, the following inequalities hold:

$$\begin{aligned} &N\textstyle\int_{t_0}^{t_1} \mathrm{e}^{-\alpha(t-s)} \left[\|hf(x_n(t-\tau_1)\| + \|\theta h x_n(s-\tau_2)\|\right] \mathrm{d}s < \dfrac{\varepsilon}{3} \text{ when } t_0 < t_1 < t, \\ &N\textstyle\int_{t_1}^{t} \mathrm{e}^{-\alpha(t-s)} \left[\|hf(x_n(t-\tau_1)\| + \|\theta h x_n(s-\tau_2)\|\right] \mathrm{d}s < \dfrac{\varepsilon}{3} \text{ when } t > t_1. \end{aligned} \tag{10.14}$$

Further let $\delta_2 = \frac{\varepsilon}{3M}$, and $M\|\varphi\|\mathrm{e}^{-\alpha(t-t_0)} \le \frac{\varepsilon}{3}$ when $\|\varphi\| \le \delta_2$. Then define

$$\delta(\varepsilon) = \min\left(\delta_1(\varepsilon), \delta_2(\varepsilon)\right). \tag{10.15}$$

Now combining (10.13), (10.14) and (10.15) yields $\|x(t)\| < \varepsilon$ when $t \ge t_0$ and $\|\varphi\| < \delta(\varepsilon)$. Hence, the zero solution of system (10.8) is stable in the sense of Lyapunov.

Further, it can be shown by applying the L'Hospital rule to (10.13) that $\forall x_0 \in R^n$,

$$\begin{aligned} \lim_{t\to+\infty} \|x(t)\| \le\ & \lim_{t\to+\infty} M\|\varphi\|\mathrm{e}^{-\alpha(t-t_0)} \\ & + \lim_{t\to+\infty} \frac{1}{\mathrm{e}^{\alpha t}} \int_{t_0}^{t} \mathrm{e}^{\alpha s} \left[\|hf(x_n(s-\tau_1))\| + \|\theta h x_n(s-\tau_2)\|\right] \mathrm{d}\tau \\ =\ & 0 + \frac{1}{\alpha}\left[\lim_{t\to+\infty} \|hf(x_n(t-\tau_1))\| + \|\theta h x_n(t-\tau_2)\|\right] \\ =\ & 0, \end{aligned}$$

which implies that the zero solution of system (10.8) is globally asymptotically stable. Due to the arbitrary of $f(\cdot) \in F$, the zero solution of system (10.4) is time-delay independent absolutely stable. The sufficiency is also proved. $\square$

Theorem 10.4. *The sufficient and necessary conditions for system (10.4) to be time-delay independent absolutely stable are:*

1. *There exists non-negative vector* $\eta = (\eta_1, \dots, \eta_n)^{\mathrm{T}}$ *such that the matrix* $A + B + (O_{n\times(n-1)}, \eta)$ *is a Hurwitz matrix.*
2. $\det\left(\mathrm{i}\,\sigma - A - B\mathrm{e}^{-\mathrm{i}\sigma\tau_1} - (O_{n\times(n-1)}, \eta)\,\mathrm{e}^{-\mathrm{i}\sigma\tau_2}\right) \neq 0 \quad \forall \sigma \in R.$
3. *The zero solution of system (10.4) is time-delay independent absolutely stable w.r.t. the partial variable* x_n.

The proof of Theorem 10.4 is similar to that of Theorem 10.3, and thus omitted.

Remark 10.5. The existence of the vector $\eta = (\eta_1, \dots, \eta_n)^{\mathrm{T}}$ is obvious. For example, $\eta = \theta h$ is defined in condition (1) of Theorem 10.3, which is a constructive condition, while condition (1) in Theorem 10.4 is an existence condition, which is certainly not as good as condition (1) of Theorem 10.3. The condition (1) of Theorem 10.3 is easy to verify. However, if an appropriate η is chosen, it may simplify the validation of other conditions in the theorem.

Similar to Theorems 10.3 and 10.4, we have the following theorems.

Theorem 10.6. *The sufficient and necessary conditions for system (10.5) to be time-delay independent absolutely stable are:*

1. *the matrix* $P + Q + (O_{n\times(n-1)}, \theta b)$ *is a Hurwitz matrix, where*

$$(O_{n\times(n-1)}, \theta b) = \begin{bmatrix} 0 & \cdots & 0 & \theta b_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \theta b_n \end{bmatrix}_{n\times n} := (\theta_{ij})_{n\times n}, \quad \theta = 0 \text{ or } \theta = 1.$$

2. $\det\left(\mathrm{i}\,\sigma - P - Q\mathrm{e}^{-\mathrm{i}\sigma\tau_1} - (O_{n\times(n-1)}, \theta b)\mathrm{e}^{-\mathrm{i}\sigma\tau_2}\right) \neq 0 \quad \forall \sigma \in R.$
3. *The zero solution of system (10.5) is time-delay independent, absolutely stable w.r.t. the partial variable* y_n.

Theorem 10.7. *The sufficient and necessary conditions for system (10.5) to be time-delay independent absolutely stable are:*

1. *There exists non-negative vector* $\eta = (\eta_1, \cdots, \eta_n)^{\mathrm{T}}$ *such that the matrix* $P + Q + (O_{n\times(n-1)}, \eta)$ *is a Hurwitz matrix*
2. $\det\left(i\sigma - P - Q\mathrm{e}^{-\mathrm{i}\sigma\tau_1} - (O_{n\times(n-1)}, \eta)\,\mathrm{e}^{-\mathrm{i}\sigma\tau_2}\right) \neq 0 \quad \forall \sigma \in R;$
3. *The zero solution of system (10.5) is time-delay independent absolutely stable w.r.t. the partial variable* y_n.

10.1.2 Algebraic Sufficient Conditions

The sufficient and necessary conditions obtained above are sometimes not easy to verify in practice. Therefore, it is necessary to obtain some simple and practically useful, algebraic criteria. Again, we consider time-delay independent absolute stability of constant time-delayed Lurie control systems in the Hurwitz angle $[0, k]$.

Thus, in system (10.4), assume that

$$f(\cdot) \in F_{[0,k]} := \left\{ f \mid f(0) = 0,\ 0 \le x_n f(x_n) \le k x_n^2 \right\}$$

and f is continuous. Then we have the following theorem.

Theorem 10.8. *If system (10.4) satisfies the following conditions:*

1. $a_{ii} < 0,\ i = 1, 2, \ldots, n.$
2. $G = \left[-(-1)^{\delta_{ij}} |a_{ij}| - |b_{ij}| - \theta_{ij} |h_i| k \right]_{n \times n}$ *is an M matrix, where*

$$\theta_{ij} = \begin{cases} 1 \text{ when } i = 1, \ldots, n,\ j = n, \\ 0 \text{ when } i = 1, \ldots, n,\ j = 1, 2, \ldots, n-1, \end{cases} \qquad \delta_{ij} = \begin{cases} 1, \text{ when } i = j, \\ 0, \text{ when } i \ne j. \end{cases}$$

Then the zero solution of system (10.4) is time-delay independent absolute stable in the Hurwitz angle $[0, k]$.

Proof. Since G is an M matrix, it is known from the property of M matrix that $\forall \beta = (\beta_1, \ldots, \beta_n)^{\mathrm{T}} > 0$ (i.e., $\beta_i > 0,\ i = 1, 2, \ldots, n$), there exist constants $c_i > 0,\ i = 1, 2, \ldots, n$ such that $c = (G^{\mathrm{T}})^{-1} \beta,\ c = (c_1, \ldots, c_n)^{\mathrm{T}}$, i.e.,

$$-c_j a_{jj} - \left(\sum_{i=1, i \ne j}^{n} |a_{ij}| c_i + \sum_{i=1}^{n} |b_{ij}| c_i + \sum_{i=1}^{n} c_i \theta_{ij} \|h_i\| k \right) = \beta_j, \quad j = 1, 2, \ldots, n.$$

Consider the radially unbounded, positive definite Lyapunov functional:

$$V(t) = \sum_{i=1}^{n} c_i \left[|x_i(t)| + \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_1}^{t} |x_j(s)|\, \mathrm{d}s + \sum_{i=1}^{n} \theta_{ij} |h_i| k \int_{t-\tau_2}^{t} |x_i(s)|\, \mathrm{d}s \right].$$

Suppose that the initial condition for the solution of system (10.4) is given by $x(t) = \varphi(t),\ -\tau \le t \le 0$. Then we have

$$V(t_0) \le \sum_{i=1}^{n} c_i \left[|x_i(t_0)| + \sum_{j=1}^{n} |b_{ij}| \|\varphi\| \tau + \sum_{j=1}^{n} \theta_{ij} |h_i| k \|\varphi\| \tau_2 \right] := M < +\infty$$

and $V(t) \ge \sum\limits_{i=1}^{n} c_i |x_i(t)|$. Thus, along the trajectory of system (10.4) differentiating V w.r.t. time yields

$$\begin{aligned} D^+ V(t)\big|_{(10.4)} \le \sum_{i=1}^{n} c_i \Bigg[& \frac{\mathrm{d}x_i}{\mathrm{d}t} \operatorname{sign}(x_i) + \sum_{j=1}^{n} |b_{ij}| |x_j(t)| - \sum_{j=1}^{n} |b_{ij}| |x_j(t-\tau_1)| \\ & + \sum_{j=1}^{n} \theta_{ij} |h_i| |x_j(t)| - \sum_{j=1}^{n} \theta_{ij} |h_j| |x_j(t-\tau_2)| \Bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} c_i \Bigg\{ \Bigg[\sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t-\tau_1) \\
&\qquad + \sum_{j=1}^{n} \theta_{ij} h_i f(x_n(t-\tau_2)) \Bigg] \operatorname{sign}(x_i) \\
&\qquad + \sum_{j=1}^{n} |b_{ij}|\,|x_j(t)| - \sum_{j=1}^{n} |b_{ij}|\,|x_j(t-\tau_1)| \\
&\qquad + \sum_{j=1}^{n} \theta_{ij}|h_i|\,|x_j(t)| - \sum_{j=1}^{n} \theta_{ij}|h_i|\,|x_j(t-\tau_2)| \Bigg\} \\
&\leq \sum_{j=1}^{n} \left[c_j a_{jj} + \sum_{i=1, i\neq j}^{n} c_i\,|a_{ij}| + \sum_{i=1}^{n} c_i\,|b_{ij}| + \sum_{i=1}^{n} \theta_{ij}|h_i|\,k \right] |x_j(t)| \\
&\leq - \sum_{j=1}^{n} \beta_j\,|x_j(t)|. \qquad (10.16)
\end{aligned}
$$

Hence,

$$
0 \leq V(t) \leq V(t_0) - \int_{t_0}^{t} \sum_{j=1}^{n} \beta_j x_j(\tau)|\,\mathrm{d}\tau \leq V(t_0). \qquad (10.17)
$$

Equation (10.17) clearly indicates that the zero solution of system (10.4) is time-delay independent stable in the Hurwitz angle $[0,k]$.

Next, we show that the zero solution of system (10.4) is time-delay independent attractive in the Hurwitz angle $[0,k]$.

Because

$$
0 \leq \min_{1\leq i\leq n} c_i \textstyle\sum_{i=1}^{n} |x_i(t)| \leq V(t) \leq V(t_0) < +\infty,
$$

$\sum_{i=1}^{n} |x_i(t)|$ is bounded, and thus $\sum_{i=1}^{n} |\frac{\mathrm{d}x_i}{\mathrm{d}t}|$ is bounded in $[t_0,+\infty)$. This implies that $\sum_{i=1}^{n} |x_i(t)|$ is uniformly continuous in $[t_0,+\infty)$. On the other hand, it follows from (10.17) that $\int_{t_0}^{t} \sum_{j=1}^{n} \beta_j |x_j(t)| \mathrm{d}t \leq V(t_0)$, which, in turn, results in $\sum_{i=1}^{n} |x_i(t)| \in L_1[0,+\infty)$. Therefore, it follows from calculus that $\forall \varphi \in C[[-\tau,0],R^n]$, $\lim_{t\to+\infty} \sum_{i=1}^{n} |x_i(t)| = 0$, which implies that the zero solution of system (10.4) is time-delay independent attractive in the Hurwitz angle $[0,k]$. The proof of Theorem 10.8 is complete. □

Corollary 10.9. *If one of the following conditions is satisfied:*

1. $-a_{jj} > \sum_{i=1, i\neq j}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_{ij}| + \sum_{i=1}^{n} \theta_{ij}\,|h_i|k, \quad j = 1,2,\ldots,n.$
2. $-a_{ii} > \sum_{j=1, j\neq i}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}| + \sum_{j=1}^{n} \theta_{ij}\,|h_i|k, \quad i = 1,2,\ldots,n.$
3. $-a_{ii} > \frac{1}{2} \sum_{j=1, j\neq i}^{n} (|a_{ij}| + |a_{ji}|) + \frac{1}{2} \sum_{j=1}^{n} (|b_{ij}| + |b_{ji}|) + \frac{1}{2} \sum_{j=1}^{n} (\theta_{ij}|h_i|k + \theta_{ji}|h_j|k).$

Then the zero solution of system (10.4) is time-delay independent attractive in the Hurwitz angle $[0,k]$.

This is simply because that any of the above conditions implies that $a_{ii} < 0$ $(i = 1, 2, \ldots, n)$ and G is an M matrix.

Similar to Theorem 10.8 and Corollary 10.9, we have the following theorem and corollary.

Theorem 10.10. *If system (10.5) satisfies the following conditions:*

1. $p_{ii} < 0,\ i = 1, 2, \ldots, n.$
2. $\tilde{G} = \Big[-(-1)^{\delta_{ij}}|p_{ij}| - |q_{ij}| - \theta_{ij}|b_i|k\Big]_{n\times n}$ *is an M matrix, where* θ_{ij} *and* δ_{ij} *are defined in Theorem 10.3.*

Then the zero solution of system (10.4) is time-delay independent absolutely stable in the Hurwitz angle $[0,k]$.

Corollary 10.11. *If one of the following conditions is satisfied:*

1. $-p_{jj} > \sum\limits_{i=1,i\neq j}^{n} |p_{ij}| + \sum\limits_{i=1}^{n} |q_{ij}| + \sum\limits_{i=1}^{n} \theta_{ij}\,|h_i|\,k, \quad j = 1, 2, \ldots, n.$
2. $-p_{ii} > \sum\limits_{j=1,j\neq i}^{n} |p_{ij}| + \sum\limits_{j=1}^{n} |q_{ij}| + \sum\limits_{j=1}^{n} \theta_{ij}\,|h_i|\,k, \quad i = 1, 2, \ldots, n.$
3. $-p_{ii} > \dfrac{1}{2}\sum\limits_{j=1,j\neq i}^{n} (|p_{ij}| + |p_{ji}|) + \dfrac{1}{2}\sum\limits_{j=1}^{n}(|q_{ij}| + |q_{ji}|) + \dfrac{1}{2}\sum\limits_{j=1}^{n}(\theta_{ij}\,|b_i|\,k + \theta_{ji}\,|b_j|\,k).$

Then the zero solution of system (10.5) is time-delay independent attractive in the Hurwitz angle $[0,k]$.

Further, we have the following result.

Theorem 10.12. *If system (10.4) satisfies the following conditions:*

1. *There exist constants* $c_i,\ i = 1, 2, \ldots, n$ *such that*

$$c_j\,a_{jj} + \sum_{i=1,i\neq j}^{n} c_i\,|p_{ij}| + \sum_{i=1,i\neq j}^{n} c_i\,|q_{ij}| \leq 0, \quad j = 1, 2, \ldots, n-1$$

   and

$$c_n\,a_{nn} + \sum_{i=1}^{n-1} c_i\,|a_{in}| + \sum_{i=1}^{n} c_i\,|b_{in}| + \sum_{i=1}^{n} c_i\,\theta_{in}\,|h_i|\,k \leq -\delta < 0.$$

2. *All eigenvalues of* $\det\big(\lambda I_n - A - B\mathrm{e}^{-\mathrm{i}\lambda\tau_1} - (O_{n\times(n-1)}, \theta h)\mathrm{e}^{-\mathrm{i}\lambda\tau_2}\big) = 0$ *have negative real parts.*

Then the zero solution of system (10.4) is time-delay independent absolutely stable in the Hurwitz angle $[0,k]$.

Proof. Construct the radially unbounded, positive definite Lyapunov functional as follows:

$$V(t) = \sum_{i=1}^{n} c_i \left[|x_i(t)| + \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_1}^{t} |x_j(s)|\,\mathrm{d}s + \sum_{j=1}^{n} \theta_{ij} |h_i| k \int_{t-\tau_2}^{t} |x_j(s)|\,\mathrm{d}s \right]. \tag{10.18}$$

Following the proof of Theorem 10.3, we obtain

$$\begin{aligned} D^+V(t)\big|_{(10.4)} &\leq \sum_{j=1}^{n} \left[c_j a_{jj} + \sum_{i=1, i\neq j}^{n} c_i |a_{ij}| + \sum_{i=1}^{n} c_i |b_{ij}| \right] |x_j(t)| \\ &\leq \left[c_n a_{nn} + \sum_{i=1}^{n-1} c_i |a_{in}| + \sum_{i=1}^{n} c_i |b_{in}| + \sum_{j=1}^{n} c_i \theta_{ij} |h_i| k \right] |x_n(t)| \\ &\leq -\,\delta\, |x_n(t)|. \end{aligned} \tag{10.19}$$

Hence, we have

$$0 \leq V(t) \leq V(t_0) - \delta \int_{t_0}^{t} |x_n(s)|\,\mathrm{d}s \leq V(t_0), \tag{10.20}$$

which indicates that the zero solution of system (10.4) is time-delay independent stable.

Again, similar to Theorem 10.3, we can prove that $\int_{t_0}^{t} \delta\, |x_n(s)|\,\mathrm{d}s \leq V(t_0)$ and $|x_n(t)| \in L_1[0, +\infty)$. Therefore, $\lim_{t\to+\infty} |x_n(t)| = 0$, which implies that the zero solution of system (10.4) is time-delay independent attractive w.r.t. the partial variable x_n in the Hurwitz angle $[0, k]$.

Following the proof of Theorem 10.3, we can express the solution of system (10.4) as

$$x(t) = x^*(t) + \int_{t_0}^{t} U(t,s)\, [h f(x_n(s-\tau))]\,\mathrm{d}s,$$

where $U(t,s)$ satisfies the following system:

$$\begin{aligned} \frac{\partial U(t,s)}{\partial t} &= A U(t,s) + B U(t-\tau, s) + \left(O_{n\times(n-1)}, h\theta\right) U(t-\tau_2, s), \\ U(t,s) &= \begin{cases} 0 \text{ when } \tau - s_0 \leq t \leq s_0, \\ I \text{ when } t = s_0, \end{cases} \end{aligned} \tag{10.21}$$

and $x^*(t) = x(t_0, \varphi)$ is the solution of the following equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= A x(t) + B x(t-\tau_1) + h\theta x(t-\tau_2), \\ x(t) &= \varphi(t), \quad -\tau \leq t \leq t_0. \end{aligned} \tag{10.22}$$

The remaining part of the proof can follow the proof of Theorem 10.3 based on (10.11)–(10.15). The details are omitted here. This completes the proof of Theorem 10.12. □

Similar to Theorem 10.12, we have

Theorem 10.13. *If system (10.5) satisfies the following conditions:*

1. There exist constants c_i, $i = 1,2,\dots,n$ *such that*

$$c_j p_{jj} + \sum_{i=1, i\neq j}^{n} c_i |p_{ij}| + \sum_{i=1, i\neq j}^{n} c_i |q_{ij}| \leq 0, \quad j = 1,2,\dots,n-1,$$

and

$$c_n p_{nn} + \sum_{i=1}^{n-1} c_i\,|p_{in}| + \sum_{i=1}^{n} c_i\,|q_{in}| + \sum_{i=1}^{n} c_i\,\theta_{in}\,|h_i|k \leq -\delta < 0.$$

2. All eigenvalues of $\det\left(\lambda I_n - P - Q\mathrm{e}^{-\mathrm{i}\lambda\tau_1}\right) = 0$ *have negative real parts.*

Then the zero solution of system (10.5) is time-delay independent absolutely stable in the Hurwitz angle $[0,k]$.

To end this section, we give an example to demonstrate the applicability of the theoretical results obtained in this section.

Example 10.14. Consider a three-dimensional Lurie control system in the form of (10.4), given by

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} \end{pmatrix} = \begin{bmatrix} -4 & 0 & \frac{3}{4} \\ \frac{3}{2} & -4 & \frac{5}{4} \\ \frac{1}{2} & 1 & -6 \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \\ x_3(t-\tau_1) \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} f(x_3(t-\tau_2)),$$

where $f(\cdot) \in F_{[0,1/2]}$.

It is seen that $a_{11} = -4 < 0$, $a_{22} = -4 < 0$ and $a_{33} = -6 < 0$, and easy to verify that

$$G = \begin{bmatrix} 4-1 & -\frac{1}{2} & -\frac{3}{4} - 0 - 1 \\ -\frac{3}{2} - \frac{1}{2} & 4 - \frac{2}{3} & -\frac{5}{4} - \frac{1}{4} - \frac{3}{2} \\ -\frac{1}{2} - \frac{1}{2} & -1 - \frac{1}{3} & 6 - \frac{1}{4} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{1}{2} & -\frac{7}{4} \\ -2 & \frac{10}{3} & -3 \\ -1 & -\frac{4}{3} & \frac{21}{4} \end{bmatrix}$$

is an M matrix. Thus the conditions given in Theorem 10.8 are satisfied, and the zero solution of this example is time-delay independent absolutely stable in the Hurwitz angel $[0, \frac{1}{2}]$.

10.2 Absolute Stability Based on Partial Variables

In this section, we turn to apply a decomposition method to obtain the absolute stability of the whole system's states, based on a lower dimensional linear system whose all eigenvalues have negative real parts and on that partial variables of the system are stable.

Let

$$A := \begin{bmatrix} A_{m\times m} & A_{m\times(n-m)} \\ A_{(n-m)\times m} & A_{(n-m)\times(n-m)} \end{bmatrix},$$

$$B := \begin{bmatrix} B_{m\times m} & B_{m\times(n-m)} \\ B_{(n-m)\times m} & B_{(n-m)\times(n-m)} \end{bmatrix},$$

$$\left[O_{n\times(n-1)}, h\theta\right] := D = \begin{bmatrix} D_{m\times m} & D_{m\times(n-m)} \\ D_{(n-m)\times m} & D_{(n-m)\times(n-m)} \end{bmatrix},$$

$$\begin{aligned} h_{(m)} &:= (h_1, h_2, \ldots, h_m)^{\mathrm{T}}, \quad h_{(n-m)} := (h_{m+1}, h_{m+2}, \ldots, h_n)^{\mathrm{T}}, \\ x_{(m)} &:= (x_1, x_2, \ldots, x_m)^{\mathrm{T}}, \quad x_{(n-m)} := (x_{m+1}, x_{m+2}, \ldots, x_n)^{\mathrm{T}}. \end{aligned}$$

Then system (10.4) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}x_{(m)}}{\mathrm{d}t} &= A_{m\times m}x_{(m)}(t) + B_{m\times m}x_{(m)}(t-\tau_1) + D_{m\times m}x_{(m)}(t-\tau_2) \\ &\quad + A_{m\times(n-m)}x_{(n-m)}(t) + B_{m\times(n-m)}x_{(n-m)}(t-\tau_1) \\ &\quad + D_{m\times(n-m)}x_{(n-m)}(t-\tau_2) \\ &\quad + k_{(m)}\left[f(x_n(t-\tau_2)) - \theta x_n(t-\tau_2)\right], \end{aligned} \tag{10.23}$$

$$\begin{aligned} \frac{\mathrm{d}x_{(n-m)}}{\mathrm{d}t} &= A_{(n-m)\times m}x_{(m)}(t) + B_{(n-m)\times m}x_{(m)}(t-\tau_1) + D_{(n-m)\times m}x_{(m)}(t-\tau_2) \\ &\quad + A_{(n-m)\times(n-m)}x_{(n-m)}(t) + B_{(n-m)\times(n-m)}x_{(n-m)}(t-\tau_1) \\ &\quad + D_{(n-m)\times(n-m)}x_{(n-m)}(t-\tau_2) \\ &\quad + k_{(n-m)}\left[f(x_n(t-\tau_2) - \theta x_n(t-\tau_2)\right]. \end{aligned} \tag{10.24}$$

It is obvious that the solutions of system (10.4) are equivalent to that of (10.23) and (10.24).

Theorem 10.15. *If the following conditions are satisfied:*

1. *All the eigenvalues of* $\det\left(\lambda I_m - A_{m\times m} - B_{m\times m}\mathrm{e}^{-\mathrm{i}\lambda\tau_1} - D_{m\times m}\mathrm{e}^{-\mathrm{i}\lambda\tau_2}\right) = 0$ *have negative real parts.*
2. *There exist constants* $c_i \geq 0,\ i = 1, 2, \ldots, m$ *and* $c_i > 0,\ j = m+1, m+2, \ldots, n$ *such that*
$$c_j a_{jj} + \left(\sum_{i=1, i\neq j}^{n} c_i |a_{ij}| + \sum_{i=1}^{n} c_i |b_{ij}|\right) \leq 0, \quad j = 1, 2, \ldots, m;$$
$$c_j a_{jj} + \left(\sum_{i=1, i\neq j}^{n} c_i |a_{ij}| + \sum_{i=1}^{n} c_i |b_{ij}|\right) < 0, \quad j = m+1, m+2, \ldots, n-1;$$
$$c_n a_{nn} + \left(\sum_{i=1, i\neq j}^{n} c_i |a_{in}| + \sum_{i=1}^{n} c_i |b_{in}| + \sum_{i=1}^{n} c_i |h_i| k\right) < 0.$$

Then the zero solution of system (10.23) and (10.24) is time-delay independent absolutely stable in the Hurwitz angle $[0,k]$.

Proof. For the partial variables of the system: $x_{m+1}, x_{m+2}, \ldots, x_n$, construct the radially unbounded, positive definite Lyapunov functional:

$$V(x,t) = \sum_{i=1}^{n} c_i\,|x_i| + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_1}^{t} c_i\,|b_{ij}|\,|x_j(s)|\,\mathrm{d}s + \sum_{i=1}^{n} c_i\,|h_i|\int_{t-\tau_2}^{t} k\,|x_n(s)|\,\mathrm{d}s.$$

Then

$$\begin{aligned}
D^+V(x,t)\big|_{(10.4)} &\le \sum_{i=1}^{n} c_i\,\frac{\mathrm{d}x_i}{\mathrm{d}t}\,\mathrm{sign}(x_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\,|b_{ij}|\,|x_j(t)| \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\,|b_{ij}|\,|x_j(t-\tau_1)| \\
&\quad + \sum_{i=1}^{n} c_i\,|h_i|\,k\,|x_n(t)| - \sum_{i=1}^{n} c_i\,|h_i|\,k\,|x_n(t-\tau_2)| \\
&\le \sum_{j=1}^{n-1}\left[c_j\,a_{jj} + \left(\sum_{i=1,i\neq j}^{n} c_i\,|a_{ij}| + \sum_{i=1}^{n} c_i\,|b_{ij}|\right)\right]|x_j(t)| \\
&\quad + \left[c_n a_{nn} + \left(\sum_{i=1}^{n-1} c_i|a_{in}| + \sum_{i=1}^{n} c_i|b_{in}| + \sum_{i=1}^{n} c_i|h_i|k\right)\right]|x_n(t)| \\
&\le \sum_{j=m+1}^{n-1}\left[c_j\,a_{jj} + \left(\sum_{i=1,i\neq j}^{n} c_i\,|a_{ij}| + \sum_{i=1}^{n} c_i\,|b_{ij}|\right)\right] x_j(t)| \\
&\quad + \left[c_n a_{nn} + \left(\sum_{i=1}^{n-1} c_i|a_{in}| + \sum_{i=1}^{n} c_i|b_{in}| + \sum_{i=1}^{n} c_i|h_i|k\right)\right]|x_n(t)| \\
&< 0 \qquad \text{when } \|x_{(n-m)}\| \neq 0.
\end{aligned}$$

Therefore, the complete solution of system (10.23) and (10.24), namely the complete solution of system (10.4), is absolutely stable w.r.t. the partial variables x_{m+1}, $x_{m+2}, \ldots, x_n$.

Now let $x^*_{(m)}(t) = x_m(t_0, \varphi_m)(t)$ be the solution of the homogeneous part of (10.23):

$$\begin{aligned}
\frac{\mathrm{d}x_{(m)}}{\mathrm{d}t} &= A_{m\times m}x_{(m)}(t) + B_{m\times m}x_{(m)}(t-\tau_1) + D_{m\times m}x_{(m)}(t-\tau_2), \\
x_m(t) &= \varphi_m(t), \quad -\tau \le t \le 0.
\end{aligned}$$

Then we may follow the proof of Theorem 10.3 to write the solution of (10.23) as

$$\begin{aligned}
x_m(t) = x^*_{(m)}(t) + \int_{t_0}^{t} U_{(m)}(t,s)\,\big\{&A_{m\times(n-m)}x_{(n-m)}(s) + B_{m\times(n-m)}x_{(n-m)}(s-\tau_1) \\
&+ D_{m\times(n-m)}x_{(n-m)}(s-\tau_2) + k_{(n-m)}\,[f(x_n(s-\tau_2)) - \theta\,x_n(s-\tau_2)]\big\}\,\mathrm{d}s,
\end{aligned}$$

where $U_{(m)}(t,s)$ is the fundamental matrix solution of the system:

$$\frac{\partial U_{(m)}(t,s)}{\partial t} = A_{m\times m}U_{(m)}(t,s) + B_{m\times m}U_{(m)}(t-\tau_1,s) + D_{m\times m}U_{(m)}(t-\tau_2,s),$$
$$U_{(m)}(t,s) = \begin{cases} 0 & \text{when } \tau - s \le t \le s_0, \\ I_m & \text{when } t = s_0. \end{cases}$$

Finally, we can follow the last part of the proof of Theorem 10.3 to show that the zero solution of system (10.4) is also time-delay independent absolutely stable w.r.t. $x_{(m)}$. This completes the proof of Theorem 10.15. □

Similarly, let

$$P := \begin{bmatrix} P_{m\times m} & P_{m\times(n-m)} \\ P_{(n-m)\times m} & P_{(n-m)\times(n-m)} \end{bmatrix},$$

$$Q := \begin{bmatrix} Q_{m\times m} & Q_{m\times(n-m)} \\ Q_{(n-m)\times m} & Q_{(n-m)\times(n-m)} \end{bmatrix},$$

$$\left[O_{n\times(n-1)}\; b\,\theta\right] := \overline{D} = \begin{bmatrix} \overline{D}_{m\times m} & \overline{D}_{m\times(n-m)} \\ \overline{D}_{(n-m)\times m} & \overline{D}_{(n-m)\times(n-m)} \end{bmatrix},$$

$$b_{(m)} := (b_1, b_2, \cdots, b_m)^{\mathrm{T}}, \qquad b_{(n-m)} := (b_{m+1}, b_{m+2}, \ldots, b_n)^{\mathrm{T}},$$
$$y_{(m)} := (y_1, y_2, \cdots, y_m)^{\mathrm{T}}, \qquad y_{(n-m)} := (y_{m+1}, y_{m+2}, \cdots, y_n)^{\mathrm{T}}.$$

Then system (10.5) can be equivalently written as

$$\begin{aligned} \frac{\mathrm{d}y_{(m)}}{\mathrm{d}t} = {} & P_{m\times m}y_{(m)}(t) + P_{m\times(n-m)}y_{(n-m)}(t) \\ & + Q_{m\times m}y_{(m)}(t-\tau_1) + Q_{m\times(n-m)}y_{(n-m)}(t-\tau_1) \\ & + \overline{D}_{m\times m}y_{(m)}(t-\tau_2) + \overline{D}_{m\times(n-m)}y_{(n-m)}(t-\tau_2) \\ & + b_{(m)}\left[f(y_n(t-\tau_2)) - \theta\, y_n(t-\tau_2)\right], \end{aligned} \tag{10.25}$$

$$\begin{aligned} \frac{\mathrm{d}y_{(n-m)}}{\mathrm{d}t} = {} & P_{(n-m)\times m}y_{(m)}(t) + P_{(n-m)\times(n-m)}y_{(n-m)}(t) \\ & + Q_{(n-m)\times m}y_{(m)}(t-\tau_1) + Q_{(n-m)\times(n-m)}y_{(n-m)}(t-\tau_1) \\ & + \overline{D}_{(n-m)\times m}y_{(m)}(t-\tau_2) + \overline{D}_{(n-m)\times(n-m)}y_{(n-m)}(t-\tau_2) \\ & + b_{(n-m)}\left[f(y_n(t-\tau_2)) - \theta\, y_n(t-\tau_2)\right]. \end{aligned} \tag{10.26}$$

Thus we have a similar theorem, as given below.

Theorem 10.16. *If the following conditions are satisfied:*

1. *All the eigenvalues of* $\det\left(\lambda I_m - P_{m\times m} - Q_{m\times m}\,\mathrm{e}^{-\mathrm{i}\lambda\tau_1} - \overline{D}_{m\times m}\,\mathrm{e}^{-\mathrm{i}\lambda\tau_2}\right) = 0$ *have negative real parts.*

2. *There exist constants* $c_i \geq 0,\ i = 1,2,\ldots,m$ *and* $c_i > 0,\ j = m+1, m+2,\ldots,n$ *such that*

$$c_j p_{jj} + \left(\sum_{i=1, i\neq j}^{n} c_i |p_{ij}| + \sum_{i=1}^{n} c_i |q_{ij}| \right) \leq 0, \quad j = 1,2,\ldots,m;$$

$$c_j p_{jj} + \left(\sum_{i=1, i\neq j}^{n} c_i |p_{ij}| + \sum_{i=1}^{n} c_i |q_{ij}| \right) < 0, \ \ j = m+1, m+2,\ldots,n-1;$$

$$c_n p_{nn} + \left(\sum_{i=1, i\neq j}^{n} c_i |p_{in}| + \sum_{i=1}^{n} c_i |q_{in}| + \sum_{j=1}^{n} c_i |h_i| k \right) < 0.$$

Then the zero solution of system (10.25) and (10.26), namely the zero solution of system (10.5), is time-delay independent absolutely stable in the Hurwitz angle $[0,k]$.

The proof of Theorem 10.16 is similar to Theorem 10.15 and thus omitted.

10.3 Lurie Systems with Multiple Time Delays

In this section, we will use modern mathematical tools: linear matrix inequality (LMI) and Lyapunov functional to study the absolute stability of Lurie systems with multiple time delays involved in state variables and feedback controls.

Consider the following Lurie system with multiple time delays involved in state variables and feedback controls:

$$\begin{aligned} \frac{dx}{dt} &= A_0 x(t) + \sum_{i=1}^{l} A_i x(t - h_i(t)) + b_0 f_0(\sigma_0(t)) + \sum_{i=1}^{m} b_i f_i(\sigma_i(t - h_{i+l}(t))), \\ \sigma_i(t) &= c_i^{\mathrm{T}} x(t), \ \ i = 0, 1, \ldots, m, \\ x(t) &= \varphi(t), \ \ t \in [-H, 0], \end{aligned} \tag{10.27}$$

where $x(t) \in R^n$ is the state variable, $A_i \in R^{n\times n}$, $i = 0, 1, \ldots, l$ are the real matrices, $b_i, c_i \in R^n$, $i = 0, 1, \ldots, m$ are real vectors; the time-variant time delays $h_i(t)$ satisfy $0 < h_i(t) \leq h_i < +\infty$ and $\dot{h}_i(t) \leq d_i < 1,\ i = 1, 2, \ldots, m+l$; $H = \max_{1\leq i\leq m+l}\{h_i\}$, and

$$f_i(\cdot) \in F_{[0,k]} = \left\{ f_i(\sigma_i) | f_i(0) = 0 \ 0 < \sigma_i f_i(\sigma_i) \leq k_i \, \sigma_i^2, \ \sigma_i \neq 0 \right\}$$

for $i = 0, 1, \ldots, m$, where $f_i(\cdot)$'s are continuous, and $\varphi(t)$ is the continuous, initial vector function.

Here, we assume use $U < 0$ to denote that U is a negative definite matrix, and $\|\cdot\|$ represents the Euclidean norm in R^n:

$$\|\varphi\|_{C([-H,0])} = \sup_{\theta \in [-H,0]} \|\varphi(\theta)\|.$$

Lemma 10.17. *Let* $a(s) \in R^{n_x}$, $b(s) \in R^{n_y}$, $s \in \Omega$, *then for any positive definite matrix* $X \in R^{n_x \times n_x}$, *and any matrix* $M \in R^{n_y \times n_y}$, *the following inequality holds:*

$$-2\int_{\Omega} b^{\mathrm{T}}(s) a(s) \mathrm{d}s \leq \int_{\Omega} \begin{pmatrix} a(s) \\ b(s) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} X & XM \\ M^{\mathrm{T}} X & (M^{\mathrm{T}} X + I) X^{-1} (XM + I) \end{bmatrix} \begin{pmatrix} a(s) \\ b(s) \end{pmatrix} \mathrm{d}s.$$

Theorem 10.18. *If there exist constants* $\varepsilon > 0,\ \alpha_i > 0,\ i = 1,2,\ldots,l,\ \beta_j > 0,\ j = 1,2,\ldots,m$, *and symmetric matrix* $P > 0$ *such that*

$$\begin{bmatrix} A_0^{\mathrm{T}}P + PA_0 + \left[\sum_{i=1}^{l}\alpha_i + \sum_{i=1}^{m}\beta_i k_i^2 c_i^{\mathrm{T}} c_i + \varepsilon\, b_0^{\mathrm{T}} b_0\, k_0^2 c_0^{\mathrm{T}} c_0\right] I_n & P & L^{\mathrm{T}} \\ P & -\varepsilon I_n & 0 \\ L & 0 & -R \end{bmatrix} < 0, \tag{10.28}$$

then the zero solution of system (10.27) is time-delay independent, absolutely stable. Here,

$$\begin{aligned} L^{\mathrm{T}} &= [PA_1 \ \cdots \ PA_1\ Pb_1 \ \cdots \ Pb_m], \\ R &= \operatorname{diag}(\alpha_1(1-d_1)I_n \ \cdots \ \alpha_l(1-d_l)I_n \ \ S), \\ S &= \operatorname{diag}(\beta_1(1-d_{l+1}) \ \cdots \ \beta_m(1-d_{l+m})). \end{aligned}$$

Proof. Construct the radially unbounded, positive definite Lyapunov functional:

$$V = x^{\mathrm{T}}(t)Px(t) + \sum_{i=1}^{l}\alpha_i\int_{-h_i(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s + \sum_{i=1}^{m}\beta_i\int_{-h_{i+l}(t)}^{t} f_i^2(s)\,\mathrm{d}s.$$

There exist constants $a_1 > 0,\ a_2 > 0$ such that

$$a_1\,\|\varphi(0)\|^2 \le V \le a_2\,\|\varphi(0)\|^2_{C[c+1,0]}.$$

From the vector inequality:

$$2u^{\mathrm{T}}v \le \varepsilon u^{\mathrm{T}}u + \frac{1}{\varepsilon}v^{\mathrm{T}}v, \tag{10.29}$$

where $u \in R^n$, $v \in R^n$ and $\varepsilon > 0$ is an arbitrary real number, we have

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.27)} \le\ & x^{\mathrm{T}}(t)\Bigg\{A_0^{\mathrm{T}}P + PA_0 \\ & + \left[\sum_{i=1}^{l}\alpha_i + \sum_{i=1}^{m}\beta_i k_i^2 c_i^{\mathrm{T}} c_i + \varepsilon\, b_0^{\mathrm{T}} b_0\, k_0^2 c_0^{\mathrm{T}} c_0\right] I_n + \frac{1}{\varepsilon}P^2\Bigg\}x(t) \\ & + 2\sum_{i=1}^{l} x^{\mathrm{T}}(t)PA_i x(t-h_i(t)) + 2\sum_{i=1}^{m} x^{\mathrm{T}}(t)Pb_i f_i(\sigma_i(t-h_{i+l}(t))) \\ & - \sum_{i=1}^{l}\alpha_i(1-d_i)x^{\mathrm{T}}(t-h_i(t))x(t-h_i(t)) \\ & - \sum_{i=1}^{m}\beta_i(1-d_{i+l})f_i^2(\sigma_i(t-h_{i+l}(t))) \\ :=\ & y^{\mathrm{T}}(t)\,\Phi\, y(t), \end{aligned}$$

where

$$
\begin{aligned}
\Phi &:= \begin{bmatrix} B & L^{\mathrm{T}} \\ L & -R \end{bmatrix}, \\
B &:= A_0^{\mathrm{T}} P + P A_0 + \frac{1}{\varepsilon} P^2 + \left[\sum_{i=1}^{l} \alpha_i + \sum_{i=1}^{m} \beta_i k_i^2 c_i^{\mathrm{T}} c_i + \varepsilon b_0^{\mathrm{T}} b_0 k_0^2 c_0^{\mathrm{T}} c_0 \right] I_n, \\
y(t) &:= \left[x^{\mathrm{T}}(t) \;\; x^{\mathrm{T}}(x - h_1(t)) \;\; \cdots \;\; x^{\mathrm{T}}(t - h_l(t)) \;\; f(t) \right], \\
f(t) &:= \left[f_1(\sigma_1(t - h_{l+1}(t))) \;\; \cdots \;\; f_m(\sigma_m(t - h_{l+m}(t))) \right].
\end{aligned}
$$

From Theorems 4.24 and 4.26, it is known that $\Phi < 0$ is equivalent to the LMI (10.28). Thus when the inequality holds, there exists a constant $a > 0$ such that $\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(10.27)} \leq -a\|x(t)\|^2$. Hence, the zero solution of (10.27) is time-delay independent absolutely stable. $\square$

Theorem 10.19. *For a given $\overline{H} > 0$, if there exist constants $\alpha > 0$, $\alpha_i > 0$, $i = 1, 2, \ldots, m$, and $n \times n$ symmetric matrices $P > 0$, $U > 0$, $Q_i > 0$, $i = 1, 2, \ldots, l$, and matrix W such that*

$$
\begin{bmatrix}
B_0 & B_1^{\mathrm{T}} & B_2^{\mathrm{T}} & \cdots & B_l^{\mathrm{T}} \\
B_1 & (d_1 - 1)\Lambda & & & \\
B_2 & & (d_2 - 1)\Lambda & & \\
\vdots & & & \ddots & \\
B_l & & & & (d_l - 1)\Lambda
\end{bmatrix} < 0, \tag{10.30}
$$

then when $H \leq \overline{H}$, the zero solution of system (10.27) is absolutely stable. Here,

$$
B_0 = \begin{bmatrix}
\overline{A} & -W^{\mathrm{T}} A_1 & \cdots & -W^{\mathrm{T}} A_l & P b_0 & P b_1 & \cdots & P b_m \\
-A_1^{\mathrm{T}} W & (d_1 - 1) Q_1 & & & & & & \\
\vdots & & \ddots & & & & & \\
-A_l^{\mathrm{T}} W & & & (d_l - 1) Q_l & & & & \\
b_0^{\mathrm{T}} P & & & & -\alpha & & & \\
b_1^{\mathrm{T}} P & & & & & \alpha_1 (d_{l+1} - 1) & & \\
\vdots & & & & & & \ddots & \\
b_m^{\mathrm{T}} P & & & & & & & \alpha_m (d_{l+m} - 1)
\end{bmatrix},
$$

$$
\begin{aligned}
\overline{A} &= A^{\mathrm{T}} P + PA + \sum_{i=1}^{l} Q_i + W^{\mathrm{T}} \sum_{i=1}^{l} A_i + \sum_{i=1}^{l} A_i W + \sum_{i=1}^{m} \alpha_i k_i^2 c_i^{\mathrm{T}} c_i I_n + \alpha c_0^{\mathrm{T}} c_0 I_n, \\
A &= \sum_{i=0}^{l} A_i, \\
B_i &= \begin{bmatrix} U A_i A_0 & U A_i A_1 & \cdots & U A_i A_l & U A_i b_0 & U A_i b_1 & \cdots & U A_i b_m \\ \theta \overline{H} (W + P) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; i = 1, \cdots, l, \\
\Lambda &= \operatorname{diag}(U \;\; U), \quad \theta = \left[\frac{1}{l} \sum_{i=1}^{l} \frac{1}{1 - d_i} \right]^{-1/2}.
\end{aligned}
$$

Proof. Let $\varphi(t)=\varphi(-H)$, $t\in[-2H,-H]$. From $t\geq h_i(t)$, we have

$$x(t-h_i(t))=x(t)-\int_{-h_i(t)}^{t}\dot{x}(s)\,\mathrm{d}s,\quad i=1,2,\ldots,l.$$

Thus, (10.27) can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t}=Ax(t)-\sum_{i=1}^{l}A_i\int_{-h_i(t)}^{t}\dot{x}(s)\,\mathrm{d}s+b_0f_0(\sigma_0(t))+\sum_{i=1}^{m}b_if_i(\sigma_i(t-h_{i+l}(t))),\quad (10.31a)$$

$$x(t)=\varphi(t),\quad t\in[-2H,0].\qquad (10.31b)$$

Thus, we only need to consider the absolute stability of the above system.

Let $V_1=x^{\mathrm{T}}(t)Px(t)$. Then

$$\begin{aligned}\left.\frac{\mathrm{d}V_1}{\mathrm{d}t}\right|_{(10.31a)}&=x^{\mathrm{T}}(t)\,(A^{\mathrm{T}}P+PA)\,x(t)+2\sum_{i=1}^{m}x^{\mathrm{T}}(t)\,Pb_if_i(\sigma_i(t-h_{i+l}(t)))\\&\quad+2x^{\mathrm{T}}(t)\,Pb_0f_0(\sigma_0(t))-2\sum_{i=1}^{l}x^{\mathrm{T}}(t)\,PA_i\int_{-h_i(t)}^{t}\dot{x}(s)\,\mathrm{d}s.\end{aligned}$$

From Lemma 10.17 we know that

$$\begin{aligned}&-2\sum_{i=1}^{l}x^{\mathrm{T}}(t)\,PA_i\int_{-h_i(t)}^{t}\dot{x}(s)\,\mathrm{d}s\\&\leq\sum_{i=1}^{l}h_ix^{\mathrm{T}}(t)\,P\,(M^{\mathrm{T}}X+I)X^{-1}(XM+I)Px(t)\\&+2\sum_{i=1}^{l}x^{\mathrm{T}}(t)\,PM^{\mathrm{T}}XA_i\int_{-h_i(t)}^{t}\dot{x}(s)\,\mathrm{d}s+\sum_{i=1}^{l}\int_{-h_i(t)}^{t}\dot{x}^{\mathrm{T}}(s)A_i^{\mathrm{T}}xA_i\dot{x}(s)\,\mathrm{d}s.\end{aligned}$$

Let $W=XMP$, $U=HX$. Then $\left.\frac{\mathrm{d}V_1}{\mathrm{d}t}\right|_{(10.31a)}$ becomes

$$\begin{aligned}\left.\frac{\mathrm{d}V_1}{\mathrm{d}t}\right|_{(10.31a)}&\leq x^{\mathrm{T}}(t)\left[A^{\mathrm{T}}P+PA+lH^2(W^{\mathrm{T}}+P)\,U^{-1}(W+P)\right]x(t)\\&\quad+2\sum_{i=1}^{l}x^{\mathrm{T}}(t)\,W^{\mathrm{T}}A_i\int_{-h_i(t)}^{t}\dot{x}(s)\,\mathrm{d}s\\&\quad+\frac{1}{H}\sum_{i=1}^{l}\int_{-h_i(t)}^{t}\dot{x}^{\mathrm{T}}(s)A_i^{\mathrm{T}}UA_i\dot{x}(s)\,\mathrm{d}s\\&\quad+2\sum_{i=1}^{m}x^{\mathrm{T}}(t)\,Pb_if_i(\sigma_i(t-h_{i+l}(t)))+2x^{\mathrm{T}}(t)\,Pb_0f_0(\sigma_0(t)).\end{aligned}$$

Further, let

$$\begin{aligned}V_2&=\frac{1}{H}\sum_{i=1}^{l}\frac{1}{1-d_i}\int_0^{h_i(t)}\int_{-s}^{t}\dot{x}^{\mathrm{T}}(\tau)A_i^{\mathrm{T}}UA_i\dot{x}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,\\V_3&=\sum_{i=1}^{l}\int_{-h_i(t)}^{t}x^{\mathrm{T}}(s)\,Q_ix(s)\,\mathrm{d}s+\sum_{i=1}^{m}\alpha_i\int_{-h_{i+l}(t)}^{t}f_i^2(\sigma_i(s))\,\mathrm{d}s.\end{aligned}$$

Then

$$\left.\frac{\mathrm{d}V_2}{\mathrm{d}t}\right|_{(10.31a)} \le \sum_{i=1}^{l} \frac{1}{1-d_i}\dot{x}^{\mathrm{T}}(t)A_i^{\mathrm{T}}UA_i\dot{x}(t) - \frac{1}{H}\sum_{i=1}^{l}\int_{-h_i(t)}^{t}\dot{x}^{\mathrm{T}}(s)A_i^{\mathrm{T}}UA_i\dot{x}(s)\,\mathrm{d}s,$$

$$\begin{aligned}\left.\frac{\mathrm{d}V_3}{\mathrm{d}t}\right|_{(10.31a)} &\le x^{\mathrm{T}}(s)\sum_{i=1}^{l}Q_i x(t) - \sum_{i=1}^{l}(1-d_i)x^{\mathrm{T}}(t-h_i(t))Q_i x(t-h_i(t))\\ &\quad + \sum_{i=1}^{m}\alpha_i k_i^2 c_i^{\mathrm{T}} c_i x^{\mathrm{T}}(t)x(t) - \sum_{i=1}^{m}\alpha_i(1-d_{i+l})f_i^2(\sigma_i(t-h_{i+l}(t)))\\ &\quad + \alpha c_0^{\mathrm{T}} c_0 x^{\mathrm{T}}(t)x(t) - \alpha f_0^2(\sigma_0(t)).\end{aligned}$$

Now take the functional $V = V_1 + V_2 + V_3$. Then there exist constants $\alpha_3 > 0$, $\alpha_4 > 0$ such that $\alpha_3\|\varphi(0)\|^2 \le V \le \alpha_4\|\varphi\|^2_{C[-2H,0]}$. When $H \le \overline{H}$, we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.31a)} &\le x^{\mathrm{T}}(t)\left[\overline{A} + lH^2(W^{\mathrm{T}}+P)U^{-1}(W+P)\right]x(t)\\ &\quad + 2\sum_{i=1}^{m}x^{\mathrm{T}}(t)Pb_i f_i(\sigma_i(t-h_{i+l}(t)))\\ &\quad + 2x^{\mathrm{T}}(t)Pb_0 f_0(\sigma_0(t)) - 2\sum_{i=1}^{l}x^{\mathrm{T}}(t)W^{\mathrm{T}}A_i x(t-h_i(t))\\ &\quad + \sum_{i=1}^{l}\frac{1}{1-d_i}\dot{x}^{\mathrm{T}}(t)A_i^{\mathrm{T}}UA_i\dot{x}(t)\\ &\quad - \sum_{i=1}^{l}(1-d_i)x^{\mathrm{T}}(t-h_i(t))Q_i x(t-h_i(t))\\ &\quad - \sum_{i=1}^{m}\alpha_i(1-d_{i+l})f_i^2(\sigma_i(t-h_{i+l}(t))) - \alpha f_0^2(\sigma_0(t))\\ &:= y^{\mathrm{T}}(t)\Omega y(t),\end{aligned}$$

where

$$\begin{aligned} y(t) &:= \left[x^{\mathrm{T}}(t)\;\; x^{\mathrm{T}}(t-h_1(t))\;\;\cdots\;\; x^{\mathrm{T}}(t-h_l(t))\;\; \overline{f}(T)\right]^{\mathrm{T}},\\ \overline{f}(t) &:= \left[f_0(\sigma_0(t))\;\; f_1(\sigma_1(t-h_{l+1})))\;\;\cdots\;\; f_m(\sigma_m(t-h_{l+m})))\right],\\ \Omega &:= B_0 + \sum_{i=1}^{l}\frac{1}{1-d_i}B_i^{\mathrm{T}}\Lambda^{-1}B_i.\end{aligned}$$

Thus, when $H \le \overline{H}$, if (10.30) holds, then there exists constant $b > 0$ such that $\frac{\mathrm{d}V}{\mathrm{d}t}\big|_{(10.31a)} \le b\|x(t)\|^2$. Therefore, the zero solution of system (10.27) is time-delay independent absolutely stable. □

Theorem 10.20. *For a given $\overline{H} > 0$, if there exist constants $\varepsilon > 0$, $\lambda_j > 0$, $j = 1, 2, \ldots, m$, $\alpha_i > 0$, $\beta_i > 0$, $\gamma_i > 0$, $\delta_i > 0$, $i = 1, 2, \ldots, l$, and symmetric matrix $P > 0$ such that*

$$\begin{bmatrix} \Sigma & P & \Gamma & \Pi & \Pi & \Pi & \Pi \\ P & -\varepsilon I_n & & & & & \\ \Gamma^{\mathrm{T}} & & -\Theta_1 & & & & \\ \Pi^{\mathrm{T}} & & & -\Theta_2 & & & \\ \Pi^{\mathrm{T}} & & & & -\Theta_3 & & \\ \Pi^{\mathrm{T}} & & & & & -\Theta_4 & \\ \Pi^{\mathrm{T}} & & & & & & -\Theta_5 \end{bmatrix} < 0, \tag{10.32}$$

then when $H \le \overline{H}$, the zero solution of system (10.27) is time-delay independent absolutely stable. Here,

$$\begin{aligned}
\Sigma &= A^{\mathrm{T}}P + PA + \overline{A} + \mu I_n, \qquad A = \sum_{i=0}^{l} A_i, \\
\overline{A} &= \sum_{i=1}^{l} \alpha_i A_0^{\mathrm{T}} A_i^{\mathrm{T}} A_i A_0 + \sum_{i=1}^{l}\sum_{j=1}^{l} \frac{\beta_i}{1-d_j} A_j^{\mathrm{T}} A_i^{\mathrm{T}} A_i A_j, \\
\Gamma &= \begin{bmatrix} P & P & \cdots & P \end{bmatrix}, \qquad \Pi = \begin{bmatrix} \overline{H}P & \overline{H}P & \cdots & \overline{H}P \end{bmatrix}, \\
\Theta_1 &= \operatorname{diag}(\lambda_1 I_n \;\; \lambda_2 I_n \;\; \cdots \;\; \lambda_m I_n), \\
\Theta_2 &= \operatorname{diag}\left((1-d_1)\,\alpha_1 I_n \;\; (1-d_2)\,\alpha_2 I_n \;\; \cdots \;\; (1-d_l)\,\alpha_l I_n\right), \\
\Theta_3 &= \operatorname{diag}\left(\frac{1-d_1}{l}\,\beta_1 I_n \;\; \frac{1-d_2}{l}\,\beta_2 I_n \;\; \cdots \;\; \frac{1-d_l}{l}\,\beta_l I_n\right), \\
\Theta_4 &= \operatorname{diag}\left((1-d_1)\,\gamma_1 I_n \;\; (1-d_2)\,\gamma_2 I_n \;\; \cdots \;\; (1-d_l)\,\gamma_l I_n\right), \\
\Theta_5 &= \operatorname{diag}\left(\frac{1-d_1}{m}\,\delta_1 I_n \;\; \frac{1-d_2}{m}\,\delta_2 I_n \;\; \cdots \;\; \frac{1-d_l}{m}\,\delta_l I_n\right), \\
\mu &= \sum_{i=1}^{m} \frac{\lambda_i}{1-d_{i+l}}\, k_i^2 b_i^{\mathrm{T}} b_i c_i^{\mathrm{T}} c_i + \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{\delta_i}{1-d_{j+l}}\, b_j^2 A_i^{\mathrm{T}} A_i b_j k_j^2 c_j^{\mathrm{T}} c_j \\
&\quad + \sum_{i=1}^{l} \gamma_i k_0^2 b_0^{\mathrm{T}} A_i^{\mathrm{T}} A_i b_0 c_0^{\mathrm{T}} c_0 + \varepsilon\, b_0^{\mathrm{T}} b_0 k_0^2 c_0^{\mathrm{T}} c_0.
\end{aligned}$$

Proof. Let $\varphi(t) = \varphi(-H)$, $t \in [-2H, -H]$, $V_1 = x^{\mathrm{T}}(t)\,P\,x(t)$, and

$$\begin{aligned}
V_2 &= \frac{1}{H}\sum_{i=1}^{l} \alpha_i \int_0^{h_i(t)} \int_{-s}^{t} x^{\mathrm{T}}(\tau) A_0 A_i^{\mathrm{T}} A_i A_0\, x(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \frac{1}{H}\sum_{i=1}^{l}\sum_{j=1}^{l} \beta_i \int_0^{h_i(t)} \int_{-s}^{t} x^{\mathrm{T}}(\tau - h_j(\tau)) A_j^{\mathrm{T}} A_i^{\mathrm{T}} A_i A_j\, x(\tau - h_j(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \frac{1}{H}\sum_{i=1}^{l} \gamma_i k_0^2 b_0^{\mathrm{T}} A_i^{\mathrm{T}} A_i b_0 c_0^{\mathrm{T}} c_0 \int_0^{h_i(t)} \int_{-s}^{t} x^{\mathrm{T}}(\tau)\, x(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \frac{1}{H}\sum_{i=1}^{l}\sum_{j=1}^{m} \delta_i b_j^{\mathrm{T}} A_i^{\mathrm{T}} A_i b_j k_j^2 c_j^{\mathrm{T}} c_j \int_0^{h_i(t)} \int_{-s}^{t} x^{\mathrm{T}}(\tau - h_{j+l}(\tau))\, x(\tau - h_{j+l}(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}$$

$$
\begin{aligned}
V_3 = & \sum_{i=1}^{l}\sum_{j=1}^{l}\frac{\beta_i}{1-d_j}\int_{-h_i(t)}^{t} x^{\mathrm{T}}(s)A_j^{\mathrm{T}}A_i^{\mathrm{T}}A_iA_jx(s)\,\mathrm{d}s \\
& +\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{\delta_i}{1-d_{j+l}}\,b_j^{\mathrm{T}}A_i^{\mathrm{T}}A_ib_jk_j^2c_j^{\mathrm{T}}c_j\int_{-h_{i+l}(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s \\
& +\sum_{j=1}^{m}\frac{\lambda_i}{1-d_{i+l}}k_i^2b_i^{\mathrm{T}}b_ic_i^{\mathrm{T}}c_i\int_{-h_{i+l}(t)}^{t} x^{\mathrm{T}}(s)x(s)\,\mathrm{d}s.
\end{aligned}
$$

Choose the functional $V = V_1 + V_2 + V_3$. Then there exist constants $\alpha_5 > 0$, $\alpha_6 > 0$ such that $\alpha_5\|\varphi(0)\|^2 \le V \le \alpha_6\|\varphi\|^2_{C[-2H,0]}$. Similar to the proof of Theorem 10.19, it is easy to show that

$$
\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.31a)} \le\ & x^{\mathrm{T}}(t)\left[A^{\mathrm{T}}P+PA+\left(\frac{1}{\varepsilon}+\sum_{j=1}^{m}\frac{1}{\lambda_i}\right)P^2\right. \\
& \left.+\sum_{i=1}^{l}\left(\frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}+\frac{m}{\delta_i}\right)\frac{H^2}{1-d_i}P^2\right]x(t) \\
& +x^{\mathrm{T}}(t)\left[\sum_{j=1}^{l}\alpha_iA_0^{\mathrm{T}}A_i^{\mathrm{T}}A_iA_0+\sum_{i=1}^{l}\sum_{j=1}^{l}\frac{\beta_i}{1-d_j}A_j^{\mathrm{T}}A_i^{\mathrm{T}}A_iA_j\right]x(t) \\
& +\left[\sum_{i=1}^{m}\frac{\lambda_i}{1-d_{i+l}}k_i^2b_i^{\mathrm{T}}b_ic_i^{\mathrm{T}}c_i\right. \\
& \left.+\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{\delta_i}{1-d_{j+l}}b_j^{\mathrm{T}}A_i^{\mathrm{T}}A_ib_jk_j^2c_j^{\mathrm{T}}c_j\right]x^{\mathrm{T}}(t)x(t) \\
& +\left[\sum_{i=1}^{l}\gamma_ik_0^2b_0^{\mathrm{T}}A_i^{\mathrm{T}}A_ib_0c_0^{\mathrm{T}}c_0+\varepsilon\,b_0^{\mathrm{T}}b_0k_0^2c_0^{\mathrm{T}}c_0\right]x^{\mathrm{T}}(t)x(t) \\
:=\ & x^{\mathrm{T}}(t)\left[A^{\mathrm{T}}P+PA+\overline{A}+\mu I_n+\xi P^2+H^2\eta P^2\right]x(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\xi &= \frac{1}{\varepsilon}+\sum_{j=1}^{m}\frac{1}{\lambda_i}, \\
\eta &= \sum_{i=1}^{l}\left[\frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}+\frac{m}{\delta_i}\right]\frac{1}{1-d_i}.
\end{aligned}
$$

When $H \le \overline{H}$, we have

$$
\left.\tfrac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.31a)} \le x^{\mathrm{T}}(t)\left[A^{\mathrm{T}}P+PA+\overline{A}+\mu I_n+\xi P^2+H^2\eta P^2\right]x(t).
$$

Thus, we know that $A^{\mathrm{T}}P+PA+\overline{A}+\mu I_n+\xi P^2+H^2\eta P^2<0$ is equivalent to the linear matrix inequality (10.32). Thus, when $H \le \overline{H}$, if (10.32) is satisfied, then there exists constant $\theta > 0$ such that $\left.\tfrac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.31a)} \le -\theta\|x(t)\|^2$. Therefore, the zero solution of system (10.27) is time-delay independent absolutely stable. □

To end this section, we give a numerical example below.

Example 10.21. Consider the following time-delayed system:

$$\begin{aligned}\dot{x}(t) &= A_0x(t)+A_1x(t-\tau)+b_1f(\sigma_1(t-\tau)), \qquad f\in F_{[0,0.5]},\\ \sigma_1(t) &= c_1^{\mathrm{T}}x(t),\\ x(t) &= \varphi(t), \qquad t\in[-H,0],\end{aligned} \tag{10.33}$$

where

$$A_0=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix},\ A_1=\begin{bmatrix}0.0 & 0.5\\ 0.5 & 0.0\end{bmatrix},\ b_1=\begin{pmatrix}0.5\\ 0.5\end{pmatrix},\ c_1=\begin{pmatrix}1\\ -1\end{pmatrix}.$$

There exist $\alpha=\beta=1$, and

$$P=\begin{bmatrix}2.1821 & 0\\ 0 & 1.7813\end{bmatrix}$$

such that the LMI (10.28) holds. Therefore, it follows from Theorem 10.18 that system (10.33) is time-delay independent absolutely stable. One can apply Theorems 10.19 and 10.20 to verify that when $H\le 0.69$, the system (10.33) is time-delay independent absolutely stable.

10.4 Time-Delay Dependent Absolute Stability of Lurie Systems

In the previous sections, we have considered time-delay independent absolute stability of Lurie systems. The conditions obtained for absolute stability are not related to time delay, and so information of time delay is missed. Thus, these conditions are conservative. In this section, we discuss the absolute stability of Lurie systems depending upon time delay, which is called time-delay dependent absolute stability.

Consider the following Lurie system with multiple time delays:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= Ax(t)+\sum_{i=1}^{m}B_ix(t-\tau_i(t))+bf(\sigma(t)),\\ \sigma(t) &= c^{\mathrm{T}}x(t),\end{aligned} \tag{10.34}$$

where $x\in R^n$, $A,B_i\in R^{n\times n}$, $i=1,2,\ldots,m$, $h,c\in R^n$; $0<\tau_i(t)\le\tau$, $i=1,2,\ldots,m$, are time delays; $f(\cdot)\in F_{[0,k]}$. Let $\|A\|=\sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of P, respectively. For any vector y, $\|y\|=\sqrt{y^{\mathrm{T}}y}$.

Theorem 10.22. *If the following conditions are satisfied:*

1. The matrix $A_0=A+\sum_{i=1}^{m}B_i$ is a Hurwitz matrix.

2. There exist positive definite matrices P, Q satisfy the Lyapunov matrix equation:

$$A_0^{\mathrm{T}} P + P A_0 = -Q. \tag{10.35}$$

3. The time delay τ satisfies the inequality:

$$\tau < \frac{\lambda_{\min}(Q - \varepsilon P^2) - \frac{1}{\varepsilon}\|b\|^2\|c\|^2 k^2}{\varepsilon\,(m+2)\sum_{i=1}^{m}\|PB_i\|^2 + \frac{m\alpha^2}{\varepsilon}(\|A\|^2 + \sum_{i=1}^{m}\|B_i\|^2 + \|b\|\|c\|^2 k^2)}, \tag{10.36}$$

where

$$\frac{1-\sqrt{1-\beta}}{2\,\|Q^{-1/2}P\|^2} < \varepsilon < \frac{1+\sqrt{1-\beta}}{2\,\|Q^{-1/2}P\|^2}, \tag{10.37}$$

and

$$\alpha = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \quad \beta = \frac{4\,\|Q^{-1/2}P\|^2\|b\|\|c\|^2 k^2}{\lambda_{\min}(Q)}, \tag{10.38}$$

then the zero solution of system (10.34) is time-delay dependent, absolutely stable.

Proof. Rewrite (10.34) as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A_0 x(t) - \sum_{i=1}^{m} B_i \int_{t-\tau_i(t)}^{t} \left[A\,x(t) + \sum_{j=1}^{m} B_j x(s - \tau_j(s)) + b\,f(\sigma(s)) \right] \mathrm{d}s + b\,f(\sigma(t)). \tag{10.39}$$

Construct the radially unbounded, positive definite Lyapunov function:

$$V(x(t)) = x^{\mathrm{T}}(t)\,P\,x(t). \tag{10.40}$$

Then differentiating V w.r.t. time t along the solution of system (10.34) yields

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.34)} &= -2x^{\mathrm{T}}(t)\,Q\,x(t) + 2x^{\mathrm{T}}(t)\,P\,b\,f(\sigma(t)) \\ &\quad -2\sum_{i=1}^{m}\int_{t-\tau_i(t)}^{t} \Bigg[x^{\mathrm{T}}(t) P B_i A x(s) + \sum_{j=1}^{m} x^{\mathrm{T}}(t) P B_i B_j A x(s - \tau_j(s)) \\ &\quad + x^{\mathrm{T}}(t) P B_i\, b\, f(\sigma(s)) \Bigg] \mathrm{d}s. \end{aligned} \tag{10.41}$$

Based on (10.40) and Razumikhin theorem [126], if for any real number $q > 1$, $V(x(\xi)) < q^2 V(x(t))$, $t - 2\tau \le \xi \le t$, then $\|(x(\xi))\| \le q\,\alpha\,\|x(t)\|$. Then using the vector inequality

$$2u^{\mathrm{T}}v \le \varepsilon\, u^{\mathrm{T}}u + \tfrac{1}{\varepsilon}v^{\mathrm{T}}v, \quad u, v \in R^n \quad (\varepsilon > 0 \text{ is an arbitrary real number}),$$

we can obtain

$$\begin{aligned} 2x^{\mathrm{T}}(t)\,P\,b\,f(\sigma(t)) &\le \varepsilon\, x^{\mathrm{T}}(t)\,P\,P\,x(t) + \frac{1}{\varepsilon} b^{\mathrm{T}} b\, f^2(\sigma(t)) \\ &\le \varepsilon\, x^{\mathrm{T}}(t)\,P^2 x(t) + \frac{1}{\varepsilon}\,\|b\|^2\|c\|^2 k^2 \|x(t)\|^2, \end{aligned} \tag{10.42}$$

and

$$
\begin{aligned}
&-2\sum_{i=1}^{m}\int_{t-\tau_i(t)}^{t}\left[x^T(t)PB_iAx(s)+\sum_{j=1}^{m}x^{\mathrm{T}}(t)PB_iB_jA(x(s-\tau_j(s)))\right.\\
&\qquad\qquad\left.+\,x^T(t)PB_ibf(\sigma(s))\right]\mathrm{d}s\\
&\le\sum_{i=1}^{m}\int_{t-\tau_i(t)}^{t}\left\{2\varepsilon x^{\mathrm{T}}(t)PB_i(PB_i)^{\mathrm{T}}x(t)+\frac{1}{\varepsilon}x^{\mathrm{T}}(s)A^{\mathrm{T}}Ax(s)+\frac{1}{\varepsilon}b^{\mathrm{T}}bf^2(\sigma(t))\right.\\
&\qquad\left.+\sum_{j=1}^{m}\left[\varepsilon x^{\mathrm{T}}(t)PB_i(PB_i)^{\mathrm{T}}x(t)+\frac{1}{\varepsilon}x^{\mathrm{T}}(s-\tau_j(s))B_j^{\mathrm{T}}B_jx(t-\tau_j(s))\right]\right\}\mathrm{d}s\\
&\le\sum_{j=1}^{m}\tau\left[\varepsilon(m+2)\|PB_i\|^2\|x(t)\|^2\right.\\
&\qquad\left.+\frac{q^2\alpha^2}{\varepsilon}\left(\|A\|^2+\|b\|^2\|c\|^2k^2+\sum_{j=1}^{m}\|B_j\|^2\right)\|x(t)\|^2\right]\\
&=\tau\left[\varepsilon(m+2)\sum_{i=1}^{m}\|PB_i\|^2\right.\\
&\qquad\left.+\frac{mq^2\alpha^2}{\varepsilon}\left(\|A\|^2+\|b\|^2\|c\|^2k^2+\sum_{j=1}^{m}\|B_j\|^2\right)\right]\|x(t)\|^2.
\end{aligned}\tag{10.43}
$$

Substituting (10.42) and (10.43) into (10.41) yields

$$
\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.34)}&\le-\left[\lambda_{\min}(Q-\varepsilon P^2)-\frac{1}{\varepsilon}\|b\|^2\|c\|^2k^2\right]\|x(t)\|^2\\
&\quad+\tau\left[\varepsilon(m+2)\sum_{i=1}^{m}\|PB_i\|^2\right.\\
&\quad\left.+\frac{mq^2\alpha^2}{\varepsilon}\left(\|A\|^2+\|b\|^2\|c\|^2k^2+\sum_{j=1}^{m}\|B_j\|^2\right)\right]\|x(t)\|^2\\
&:=-w\|x(t)\|^2,
\end{aligned}\tag{10.44}
$$

where

$$
\begin{aligned}
w:=\;&\lambda_{\min}(Q-\varepsilon P^2)-\frac{1}{\varepsilon}\|b\|^2\|c\|^2k^2\\
&-\tau\left[\varepsilon(m+2)\sum_{i=1}^{m}\|PB_i\|^2+\frac{mq^2\alpha^2}{\varepsilon}\left(\|A\|^2+\|b\|^2\|c\|^2k^2+\sum_{j=1}^{m}\|B_j\|^2\right)\right].
\end{aligned}
$$

If the condition (10.36) is satisfied, then there exists constant $q>1$ such that $w>0$. Therefore, the system (10.34) is time-delay dependent absolutely stable. □

Theorem 10.23. *If the following conditions are satisfied:*

1. *The matrix* $A_0 = A + \sum_{i=1}^{m} B_i$ *is a Hurwitz matrix.*
2. *There exist positive definite matrices* P, Q *satisfy the Lyapunov matrix equation:*

$$A_0^{\mathrm{T}} P + P A_0 = -Q. \tag{10.45}$$

3. *The time delay* τ *satisfies the inequality:*

$$\tau < T := \frac{\lambda_{\min}^2(Q) M\rho - 2\|P\|^2 \left[(LM - N\|P\|^2)(LM - N\|P\|^2 + \rho) + MN\lambda_{\min}^2(Q)\right]}{2M\left[(LM - N\|P\|^2)(LM - N\|P\|^2 + \rho) + MN\lambda_{\min}^2(Q)\right]}, \tag{10.46}$$

where

$$L = \|b\|^2\|c\|^2k^2, \qquad M = (m+2)\sum_{i=1}^{m}\|PB_i\|^2, \tag{10.47}$$

$$N = m\alpha^2\left(\|A\|^2 + \sum_{i=1}^{m}\|B_i\|^2 + \|b\|^2\|c\|^2k^2\right), \tag{10.48}$$

and

$$\rho = \sqrt{(LM - N\|P\|^2 + MN\lambda_{\min}^2(Q))}, \tag{10.49}$$

then the zero solution of system (10.34) is time-delay dependent, absolutely stable.

Proof. Let

$$g(\varepsilon) = \frac{\lambda_{\min}(Q) - \varepsilon\|P\|^2 - \frac{1}{\varepsilon}\|b\|^2\|c\|^2k^2}{\varepsilon(m+2)\sum_{i=1}^{m}\|PB_i\|^2 + \frac{m\alpha^2}{\varepsilon}\left(\|A\|^2 + \sum_{i=1}^{m}\|B_i\|^2 + \|b\|^2\|c\|^2k^2\right)},$$

$$T(\varepsilon) = \frac{\lambda_{\min}(Q - \varepsilon P^2) - \frac{1}{\varepsilon}\|b\|^2\|c\|^2k^2}{\varepsilon(m+2)\sum_{i=1}^{m}\|PB_i\|^2 + \frac{m\alpha^2}{\varepsilon}\left(\|A\|^2 + \sum_{i=1}^{m}\|B_i\|^2 + \|b\|^2\|c\|^2k^2\right)},$$

then $g(\varepsilon) \le T(\varepsilon)$. Thus, it follows Theorem 10.22 that system (10.34) is time-delay dependent, absolutely stable when $\tau < g(\varepsilon)$.

Setting $g'(\varepsilon) = 0$ results in $\varepsilon_0 = \frac{LM - N\|P\|^2 + \rho}{M\lambda_{\min}(Q)}$ and $g''(\varepsilon_0) < 0$, we can conclude that $g(\varepsilon)$ reaches its maximum value: $g(\varepsilon_0) = T$ at $\varepsilon = \varepsilon_0$. The proof is complete. □

Example 10.24. Consider the following time-delayed Lurie system:

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{bmatrix}-2 & 0\\ -1 & -2\end{bmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{bmatrix}-0.2 & -0.5\\ 0.5 & -0.2\end{bmatrix}\begin{pmatrix}x_1(t-\tau)\\ x_2(t-\tau)\end{pmatrix} + \begin{pmatrix}-0.2\\ -0.3\end{pmatrix} f(\sigma(t)),$$

$$\sigma(t) = 0.6x_1(t)_+ 0.8x_2(t), \quad f(\cdot) \in F_{[0,0.5]}.$$

Take $P = I$, then $\alpha = 1$, and $Q = \begin{bmatrix} 4.4 & 1 \\ 1 & 4.4 \end{bmatrix}$, so $\lambda_{\min}(Q) = 3.4$. From Theorem 10.22 we can find $0.0096 < \varepsilon < 3.3904$. Choosing $\varepsilon = 1.33$ yields $\tau = 0.3230$. Thus, it follows from Theorem 10.23 that $\tau < 3.3230$.

If choose $Q = I$, then $\alpha = 1.5880$, and $P = \begin{bmatrix} 0.23965 & -0.05446 \\ -0.05445 & 0.23965 \end{bmatrix}$, so $\lambda_{\min}(Q) = 1$. From Theorem 10.22 one can obtain $0.0326 < \varepsilon < 11.5281$. Choosing $\varepsilon = 4.8$ gives $\tau = 0.2190$. Thus, by Theorem 10.23 we have $\tau < 0.2191$.

10.5 Neutral Lurie Control Systems

Although some results have been achieved for the absolute stability of Lurie control systems described by time-delay difference equations or time-delayed functional differential equations, not much work has been done on Lurie control systems based on neutral differential difference equations.

In this section, we introduce the results obtained for the earliest developed neutral Lurie control systems described by differential difference equations. We first consider the time-delay independent absolute stability, and then the time-delay dependent absolute stability.

10.5.1 Time-Delay Independent Absolute Stability

First, consider the first type of neutral Lurie control systems, given by

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= Ax(t) + Bx(t-\tau) + C\dot{x}(t-\tau) + bf(\sigma(t)), \\ \dot{\sigma}(t) &= c^{\mathrm{T}}x(t) - \rho f(\sigma(t)), \end{aligned} \tag{10.50}$$

where $\tau > 0$ is a constant time delay, $f(\cdot) \in F_\infty$, $A, B, C \in R^{n\times n}$, $x, b, c \in R^n$ and $\rho \in R_+$. Suppose A is a Hurwitz matrix, then for any symmetric positive definite matrix W, the following Lyapunov matrix equation:

$$A^{\mathrm{T}}P + PA = -W \tag{10.51}$$

has the symmetric positive matrix solution A.

Theorem 10.25. *If there exist positive real numbers* α, β *and* γ *such that the following matrix:*

$$U = \begin{bmatrix} W - \alpha I + \beta A^{\mathrm{T}}A & -(PB + \beta A^{\mathrm{T}}B) & -(PC + \beta A^{\mathrm{T}}C) & -(Pb + \beta A^{\mathrm{T}}b + \frac{1}{2}\gamma c) \\ -(PB + \beta A^{\mathrm{T}}B)^{\mathrm{T}} & \alpha I - \beta B^{\mathrm{T}}B & -\beta B^{\mathrm{T}}c & -\beta B^{\mathrm{T}}b \\ -(PC + \beta A^{\mathrm{T}}C)^{\mathrm{T}} & -(\beta B^{\mathrm{T}}c)^{\mathrm{T}} & \beta I - \beta c^{\mathrm{T}}c & -\beta c^{\mathrm{T}}b \\ -(P + \beta A^{\mathrm{T}}b + \frac{1}{2}\gamma c)^{\mathrm{T}} & -(\beta B^{\mathrm{T}}b)^{\mathrm{T}} & -(\beta c^{\mathrm{T}}b)^{\mathrm{T}} & \gamma\rho - \beta b^{\mathrm{T}}b \end{bmatrix}$$

is positive definite, then the zero solution of system (10.50) is time-delay dependent absolutely stable.

Proof. Construct the radially unbounded, positive definite Lyapunov functional:

$$V(t) = x^{\mathrm{T}}(t)Px(t) + \alpha \int_{t-\tau}^{t} x^{\mathrm{T}}(\theta)x(\theta)\,\mathrm{d}\theta + \beta \int_{t-\tau}^{t} \dot{x}^{\mathrm{T}}(\theta)\dot{x}(\theta)\,\mathrm{d}\theta + \gamma \int_{0}^{\sigma(t)} f(\sigma)\,\mathrm{d}\sigma. \tag{10.52}$$

Then

$$\begin{aligned}
\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{(10.50)} &= \dot{x}^{\mathrm{T}}(t)Px(t) + x^{\mathrm{T}}(t)P\dot{x}(t) + \alpha\left[x^{\mathrm{T}}(t)x(t) - x^{\mathrm{T}}(t-\tau)x(t-\tau)\right] \\
&\quad + \beta\left[\dot{x}^{\mathrm{T}}(t)\dot{x}(t) - \dot{x}^{\mathrm{T}}(t-\tau)\dot{x}(t-\tau)\right] + \gamma\dot{\sigma} f(\sigma) \\
&= x^{\mathrm{T}}(t)\left(-W + \alpha I + \beta A^{\mathrm{T}}A\right)x(t) + x^{\mathrm{T}}(t)\left(PB + \beta A^{\mathrm{T}}B\right)x(t-\tau) \\
&\quad + x^{\mathrm{T}}(t)\left(PC + \beta A^{\mathrm{T}}C\right)\dot{x}(t-\tau) + x^{\mathrm{T}}(t)\left(Pb + \beta A^{\mathrm{T}}b + \frac{1}{2}\gamma c\right)f(\sigma) \\
&\quad + x^{\mathrm{T}}(t-\tau)\left(B^{\mathrm{T}}P + \beta B^{\mathrm{T}}A\right)x(t) + x^{\mathrm{T}}(t-\tau)\left(-\alpha I + \beta B^{\mathrm{T}}B\right)x(t-\tau) \\
&\quad + x^{\mathrm{T}}(t-\tau)\beta B^{\mathrm{T}}C\dot{x}(t-\tau) + \dot{x}^{\mathrm{T}}(t-\tau)(\beta B^{\mathrm{T}}b)f(\sigma) \\
&\quad + \dot{x}^{\mathrm{T}}(t-\tau)(c^{\mathrm{T}}P + \beta c^{\mathrm{T}}A)x(t) + \dot{x}^{\mathrm{T}}(t-\tau)\beta c^{\mathrm{T}}Bx(t-\tau) \\
&\quad + \dot{x}^{\mathrm{T}}(t-\tau)(-\beta I + \beta c^{\mathrm{T}}c)\dot{x}(t-\tau) + \dot{x}^{\mathrm{T}}(t-\tau)(\beta c^{\mathrm{T}}b)f(\sigma) \\
&\quad + \left(b^{\mathrm{T}}P + \beta b^{\mathrm{T}}A + \frac{1}{2}\gamma c^{\mathrm{T}}\right)f(\sigma) + (\beta b^{\mathrm{T}}B)x(t-\tau)f(\sigma) \\
&\quad (\beta b^{\mathrm{T}}C)\dot{x}(t-\tau)f(\sigma) + (-\gamma\rho + \beta b^{\mathrm{T}}b)f^{2}(\sigma) \\
&:= -Z^{\mathrm{T}}UZ,
\end{aligned} \tag{10.53}$$

where $Z := (x(t)\ \ x(t-\tau)\ \ \dot{x}(t-\tau)\ \ f(\sigma))^{\mathrm{T}}$. Thus, the condition given in this theorem implies that there exists $u > 0$ such that

$$\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{(10.50)} \leq -ux^{\mathrm{T}}(t)x(t) < 0, \qquad \text{when } \ x \neq 0.$$

This indicates that the zero solution of system (10.50) is time-delay independent absolutely stable. □

Corollary 10.26. *Let A be a Hurwitz matrix, and there exist constants* $\alpha > 0$, $\beta > 0$, *and* $\gamma > 0$ *such that*

$$\overline{U} = \begin{bmatrix} 2-\alpha-\beta\|A^{\mathrm{T}}A\| & \|PB+\beta A^{\mathrm{T}}B\| & \|PC+\beta A^{\mathrm{T}}C\| & \|Pb+\beta A^{\mathrm{T}}b+\frac{1}{2}\gamma c\| \\ \|PB+\beta A^{\mathrm{T}}B\| & \alpha-\beta\|B^{\mathrm{T}}B\| & \beta\|B^{\mathrm{T}}c\| & \beta\|B^{\mathrm{T}}b\| \\ \|PC+\beta A^{\mathrm{T}}C\| & \beta\|B^{\mathrm{T}}c\| & \beta-\beta\|c^{\mathrm{T}}c\| & \beta\|c^{\mathrm{T}}b\| \\ \|Pb+\beta A^{\mathrm{T}}b+\frac{1}{2}\gamma c\| & \beta\|B^{\mathrm{T}}b\| & \beta\|c^{\mathrm{T}}b\| & \gamma\rho-\beta b^{\mathrm{T}}b \end{bmatrix}$$

is positive definite, then the zero solution of system (10.50) is time-delay independent absolutely stable.

Proof. In Theorem 10.25, let $W = 2I$, then the matrix equation

$$PA + A^{\mathrm{T}}P = -2I$$

has the symmetric positive definite matrix solution P. Following Theorem 10.25 one can prove that there exists $u > 0$ such that

$$\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{(10.50)} \leq -\overline{Z}^{\mathrm{T}}(t)\overline{U}\,\overline{Z}(t) \leq -ux^{\mathrm{T}}(t)x(t) < 0 \qquad \text{when } x \neq 0,$$

where $\overline{Z} := (\|x(t)\| \ \|x(t-\tau)\| \ \|\dot{x}(t-\tau)\| \ |f(\sigma)|)^{\mathrm{T}}$. The above result implies that the conclusion of Lemma 10.26 is true. □

Now we turn to the second type of neutral Lurie indirect control system, described by

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= Ax(t) + Bx(t-\tau) + C\dot{x}(t-\tau) + bf(\sigma(t-\tau)), \\ \dot{\sigma}(t) &= c^{\mathrm{T}}x(t) - \rho f(\sigma(t)). \end{aligned} \tag{10.54}$$

The coefficients of (10.54) are exactly the same as that of (10.50). The only difference in (10.54) is that its feedback term involves time delay, but not in (10.50).

Theorem 10.27. *If there exist positive real numbers α, β, γ and η such that the following matrix:*

$$U^* = \begin{bmatrix} -W + \alpha I + \beta A^{\mathrm{T}}A & PB + \beta A^{\mathrm{T}}B & PC + \beta A^{\mathrm{T}}C & \frac{1}{2}\gamma c & Pb + \beta A^{\mathrm{T}}b \\ (PB + \beta A^{\mathrm{T}}B)^{\mathrm{T}} & -\alpha I + \beta B^{\mathrm{T}}B & \beta B^{\mathrm{T}}c & 0 & \beta B^{\mathrm{T}}b \\ (PC + \beta A^{\mathrm{T}}C)^{\mathrm{T}} & (\beta B^{\mathrm{T}}c)^{\mathrm{T}} & -\beta I + \beta c^{\mathrm{T}}c & 0 & \beta c^{\mathrm{T}}b \\ (\frac{1}{2}\gamma c)^{\mathrm{T}} & 0 & 0 & -\gamma\rho + \eta & 0 \\ (Pb + \beta A^{\mathrm{T}}b)^{\mathrm{T}} & (\beta B^{\mathrm{T}}b)^{\mathrm{T}} & (\beta c^{\mathrm{T}}b)^{\mathrm{T}} & 0 & -\eta + \beta b^{\mathrm{T}}b \end{bmatrix}$$

is negative definite, then the zero solution of system (10.54) is time-delay independent absolutely stable. Here, P is the symmetric positive definite matrix solution of (10.51).

Proof. Construct the radially unbounded, positive definite Lyapunov functional as follows:

$$\begin{aligned} V(t) = {} & x^{\mathrm{T}}(t)Px(t) + \alpha\int_{t-\tau}^{t} x^{\mathrm{T}}(\theta)x(\theta)\,\mathrm{d}\theta + \beta\int_{t-\tau}^{t} \dot{x}^{\mathrm{T}}(\theta)\dot{x}(\theta)\,\mathrm{d}\theta \\ & + \gamma\int_0^{\sigma(t)} f(\sigma(\theta))\,\mathrm{d}\theta + \eta\int_{t-\tau}^{t} f^2(\sigma(\theta))\,\mathrm{d}\theta. \end{aligned} \tag{10.55}$$

Then similar to Theorem 10.25 we can show that there exists $u > 0$ such that

$$
\begin{aligned}
\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{(10.54)} &= \dot{x}^{\mathrm{T}}(t)Px(t)+x^{\mathrm{T}}(t)P\dot{x}(t)+\alpha\left[x^{\mathrm{T}}(t)x(t)-x^{\mathrm{T}}(t-\tau)x(t-\tau)\right]\\
&\quad+\beta\left[\dot{x}^{\mathrm{T}}(t)\dot{x}(t)-\dot{x}^{\mathrm{T}}(t-\tau)\dot{x}(t-\tau)\right]\\
&\quad+\gamma\dot{\sigma}f(\sigma)+\eta(f^2(\sigma(t))-f^2(\sigma(t-\tau)))\\
&:= Y^{\mathrm{T}}U^{*}Y\leq -ux^{\mathrm{T}}(t)x(t)<0, \qquad \text{when } x\neq 0,
\end{aligned} \tag{10.56}
$$

where $Y := (x(t)\ x(t-\tau)\ \dot{x}(t-\tau)\ f(\sigma(t))\ f(\sigma(t-\tau)))^{\mathrm{T}}$. Thus, the condition of the Theorem is true, i.e., the zero solution of system (10.54) is time-delay independent absolutely stable. □

Similar to the proof of Lemma 10.26 we can prove the following lemma.

Lemma 10.28. *If there exist positive constants α, β, γ and η such that*

$$
\overline{U}^{*}=\begin{bmatrix}
-2+\alpha+\beta\|A^{\mathrm{T}}A\| & \|PB+\beta A^{\mathrm{T}}B\| & \|PC+\beta A^{\mathrm{T}}C\| & \frac{1}{2}\gamma\|c\| & \|Pb+\beta A^{\mathrm{T}}b\|\\
\|PB+\beta A^{\mathrm{T}}B\| & -\alpha+\beta\|B^{\mathrm{T}}B\| & \beta\|B^{\mathrm{T}}c\| & 0 & \beta\|B^{\mathrm{T}}b\|\\
\|PC+\beta A^{\mathrm{T}}C\| & \beta\|B^{\mathrm{T}}c\| & -\beta+\beta\|c^{\mathrm{T}}c\| & 0 & \beta\|c^{\mathrm{T}}b\|\\
\frac{1}{2}\gamma\|c\| & 0 & 0 & -\gamma\rho+\eta & 0\\
\|P+\beta A^{\mathrm{T}}b\| & \beta\|B^{\mathrm{T}}b\| & \beta\|c^{\mathrm{T}}b\| & 0 & -\eta+\beta\|b^{\mathrm{T}}c\|
\end{bmatrix}
$$

is negative definite, then the zero solution of system (10.54) is time-delay independent, absolutely stable. Here, P is the symmetric positive definite matrix solution of equation $A^{\mathrm{T}}P+PA=-2I$.

Example 10.29. Consider the following neutral Lurie indirect control system:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= Ax(t)+Bx(t-\tau)+C\dot{x}(t-\tau)+bf(\sigma(t)),\\
\dot{\sigma}(t) &= c^{\mathrm{T}}x(t)-\rho f(\sigma(t)),
\end{aligned} \tag{10.57}
$$

where

$$
A=\begin{bmatrix}-1.1 & 0.2\\ 0.1 & -1.0\end{bmatrix},\ B=\begin{bmatrix}0.3 & 0.3\\ 0.3 & 0.2\end{bmatrix},\ C=\begin{bmatrix}0.3 & 0.2\\ 0.2 & 0.1\end{bmatrix},\ b=\begin{pmatrix}1\\1\end{pmatrix},\ c=\begin{pmatrix}-1\\-1\end{pmatrix},\ \rho=10.
$$

Let $W=2I$, $\alpha=\beta=\gamma=0.5$, the a direct calculation shows that:

$$
P=\begin{bmatrix}0.9215 & 0.1368\\ 0.1368 & 1.0273\end{bmatrix},
$$

$$
U=\begin{bmatrix}
1.6015 & 0.0800 & -0.1675 & -0.1488 & -0.1488 & -0.0930 & -0.3082\\
0.0800 & 1.7400 & 0.2292 & -0.1765 & -0.1765 & -0.1001 & -0.5140\\
-0.1675 & -0.0092 & 0.4100 & -0.0750 & -0.0750 & -0.0450 & -0.3000\\
-0.1488 & -0.1765 & -0.0750 & 0.4350 & -0.0650 & -0.0400 & -0.2500\\
-0.1488 & -0.1765 & -0.0750 & -0.0065 & 0.4350 & -0.0400 & -0.2500\\
-0.0930 & -0.1001 & -0.0450 & -0.0400 & -0.0400 & 0.4750 & -0.1500\\
-0.3082 & -0.5140 & -0.3000 & -0.2500 & -0.2500 & -0.1500 & 4.0000
\end{bmatrix}.
$$

Since the smallest eigenvalue of U is $\lambda_{\min}(U) = 0.0331 > 0$, so U is positive definite, indicating that the conditions of Theorem 10.25 are satisfied. Hence, the zero solution of the neutral type of Lurie control system (10.57) is time-delay independent absolutely stable.

10.5.2 Time-Delay Dependent Absolute Stability

Finally, we consider more general neutral Lurie control system, described by

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + Ff(\sigma(t)) + Gf(\sigma(t-\tau)),\\ \sigma(t) &= Cx(t),\\ x(\theta) &= \varphi(\theta), \quad \theta \in [-\tau, 0],\end{aligned} \tag{10.58}$$

where $x \in R^n$, $A, B, D \in R^{n\times n}$, $C \in R^{m\times n}$, $F, G \in R^{n\times m}$. All the matrices are real constant matrices, and

$$\begin{aligned} C &= [C_1^{\mathrm{T}} \ C_2^{\mathrm{T}} \ \cdots \ C_m^{\mathrm{T}}]^{\mathrm{T}},\\ \sigma(t) &= [\sigma_1(t) \ \sigma_2(t) \ \cdots \ \sigma_m(t)]^{\mathrm{T}},\\ f(\sigma(t)) &= [f_1(\sigma_1(t)) \ f_1(\sigma_2(t)) \ \cdots \ f_m(\sigma_m(t))]^{\mathrm{T}},\\ f_i(\cdot) \in F_{R_i} &= \left\{f_i(0) = 0 < \sigma_i f_i(\sigma) \le k_i\sigma_i^2 \ \sigma_i \neq 0\right\}, \ k_i > 0 \ (i = 1, 2, \ldots, n),\end{aligned}$$

and $\tau > 0$ denotes a constant time delay. $\rho(A) = \max_{1\le i\le n}\{|\lambda_i(A)|\}$. If A is a symmetric matrix, then $A > 0$ $(A < 0)$ means that A is positive (negative) definite. $C([-\tau, 0], R^n)$ denotes that the continuous function $\varphi : [-\tau, 0] \to R^n$ has the norm defined by

$$\|\varphi\|_{([-\tau,0],R^n)} = \sup_{\theta\in[-\tau,0]} \|\varphi(\theta)\| \qquad \text{for all } t \ge t_0.$$

$x_t \in ([-\tau, 0], R^n)$ is defined by

$$x_t(\theta) := x(t+\theta) \ (-\tau \le \theta < 0)$$

and

$$\|x_t\|_{C_1} = \sup_{\theta\in[-\tau,0]} \{\|(t+\theta)\|, \|\dot{x}|(t+\theta)\|\}.$$

Theorem 10.30. *If* $\rho(D) < 1$, *and there exist* $n \times n$ *matrices,* $P > 0$, $Q > 0$, $R > 0$, $S > 0$ *and diagonal matrix* $\alpha = \mathrm{diag}(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_m) > 0$, $\beta = \mathrm{diag}(\beta_1 \ \beta_2 \ \cdots \ \beta_m) > 0$ *such that*

$$\Phi = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} & \theta_{16}\\ * & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} & \theta_{26}\\ * & * & \theta_{33} & \theta_{34} & \theta_{35} & \theta_{36}\\ * & * & * & \theta_{44} & \theta_{45} & \theta_{46}\\ * & * & * & * & \theta_{55} & \theta_{56}\\ * & * & * & * & * & \theta_{66}\end{bmatrix} < 0, \tag{10.59}$$

then the zero solution of the neutral Lurie control system is time-delay dependent absolutely stable. Here, $*$ denotes the corresponding symmetric part, and

$$\begin{aligned}
\theta_{11} &= P(A+B)+(A+B)^{\mathrm{T}}P+R+A^{\mathrm{T}}(\tau Q+S)A+C^{\mathrm{T}}\alpha KCA+A^{\mathrm{T}}C^{\mathrm{T}}K\alpha C,\\
\theta_{12} &= -(A+B)^{\mathrm{T}}PD+A^{\mathrm{T}}(\tau Q+S)B+C^{\mathrm{T}}\alpha KCB,\\
\theta_{13} &= A^{\mathrm{T}}(\tau Q+S)D+C^{\mathrm{T}}\alpha KCD,\\
\theta_{14} &= -\tau PB,\\
\theta_{15} &= PF+A^{\mathrm{T}}(\tau Q+S)F+C^{\mathrm{T}}\alpha KCF-A^{\mathrm{T}}C^{\mathrm{T}}\alpha,\\
\theta_{16} &= PG+A^{\mathrm{T}}(\tau Q+S)G+C^{\mathrm{T}}\alpha KCG,\\
\theta_{22} &= -R+B^{\mathrm{T}}(\tau Q+S)B,\\
\theta_{23} &= B^{\mathrm{T}}(\tau Q+S)D,\\
\theta_{24} &= \tau D^{\mathrm{T}}PB,\\
\theta_{25} &= -D^{\mathrm{T}}PF+B^{\mathrm{T}}(\tau Q+S)F-B^{\mathrm{T}}C^{\mathrm{T}}\alpha,\\
\theta_{26} &= -D^{\mathrm{T}}PG+B^{\mathrm{T}}(\tau Q+S)G,\\
\theta_{33} &= D^{\mathrm{T}}(\tau Q+S)D-S,\\
\theta_{34} &= 0,\\
\theta_{35} &= D^{\mathrm{T}}(\tau Q+S)F-D^{\mathrm{T}}C^{\mathrm{T}}\alpha,\\
\theta_{36} &= D^{\mathrm{T}}(\tau Q+S)G,\\
\theta_{44} &= -\tau Q,\\
\theta_{45} &= 0,\\
\theta_{46} &= 0,\\
\theta_{55} &= F^{\mathrm{T}}(\tau Q+S)F+\beta-2\alpha CF,\\
\theta_{56} &= -\alpha CG+F^{\mathrm{T}}(\tau Q+S)G,\\
\theta_{66} &= G^{\mathrm{T}}(\tau Q+S)G-\beta.
\end{aligned}$$

Proof. Let $\varphi(t)=\varphi(t-\tau)$. Then for $t\in[-2\tau,-\tau]$, we can equivalently rewrite system (10.58) as

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t}-D\dot{x}(t-\tau) &= (A+B)x(t)-B\int_{t-\tau}^{t}\dot{x}(\xi)\,\mathrm{d}\xi+Ff(\sigma(t))+Gf(\sigma(t-\tau)),\\
\sigma(t) &= Cx(t),\\
x(\theta) &= \varphi(\theta),\quad \theta\in[-2\tau,0].
\end{aligned} \tag{10.60}$$

We construct the radially unbounded positive definite Lyapunov functional: $V=\sum_{i=1}^{6}V_i$, where

$$\begin{aligned}
V_1 &= [x(t)-Dx(t-\tau)]^{\mathrm{T}}P\,[x(t)-Dx(t-\tau)],\\
V_2 &= \int_{-\tau}^{0}\int_{t+\eta}^{t}\dot{x}^{\mathrm{T}}(\xi)\,Q\dot{x}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta,\\
V_3 &= \int_{t-\tau}^{t}x^{\mathrm{T}}(\xi)Rx(\xi)\,\mathrm{d}\xi,
\end{aligned}$$

$$V_4 = \int_{t-\tau}^{t} \dot{x}^{\mathrm{T}}(\xi)\, S \dot{x}(\xi)\, \mathrm{d}\xi,$$
$$V_5 = 2\sum_{i=1}^{m} \alpha_i \int_0^{c_i x(t)} [k_i \sigma_i - f_i(\sigma_i)]\, \mathrm{d}\sigma_i,$$
$$V_6 = \sum_{i=1}^{m} \beta_i \int_{t-\tau}^{t} f_i^2(\sigma_i(\xi))\mathrm{d}\xi.$$

It is easy to verify that there exists $\delta > 0$ such that

$$\lambda_{\min}(P)\|x(t) - Dx(t-\tau)\|^2 \le V \le \delta\, \|x_t\|_{C_1}^2.$$

It is obvious that the absolute stability of the zero solution of system (10.58) is equivalent to that of the zero solution of system (10.60). Calculating the derivatives w.r.t. time t along the solution of system (10.60) yields

$$\begin{aligned}
\left.\frac{\mathrm{d}V_1}{\mathrm{d}t}\right|_{(10.60)} &= 2\,[x(t) - Dx(t-\tau)]^{\mathrm{T}} P \\
&\quad \times \left[(A+B)x(t) - B\int_{t-\tau}^{t} \dot{x}(\xi)\mathrm{d}\xi + Ff(\sigma(t)) + Gf(\sigma(t-\tau))\right], \\
\left.\frac{\mathrm{d}V_2}{\mathrm{d}t}\right|_{(10.60)} &= \tau \dot{x}^{\mathrm{T}}(t)\, Q x(t) - \int_{t-\tau}^{t} \dot{x}^{\mathrm{T}}(\xi)\, Q \dot{x}(\xi)\, \mathrm{d}\xi \\
\left.\frac{\mathrm{d}V_3}{\mathrm{d}t}\right|_{(10.60)} &= x^{\mathrm{T}}(t)\, R x(t) - x^{\mathrm{T}}(t-\tau)\, R x(t-\tau), \\
\left.\frac{\mathrm{d}V_4}{\mathrm{d}t}\right|_{(10.60)} &= \dot{x}^{\mathrm{T}}(t)\, S \dot{x}(t) - \dot{x}^{\mathrm{T}}(t-\tau)\, S \dot{x}(t-\tau), \\
\left.\frac{\mathrm{d}V_5}{\mathrm{d}t}\right|_{(10.60)} &= 2\sum_{i=1}^{m} \alpha_i\, [k_i C_i x(t) - f_i(\sigma_i(t))\, C_i]\, \dot{x}(t) \\
&= 2\left[x^{\mathrm{T}}(t) C^{\mathrm{T}} \alpha K C - f^{\mathrm{T}}(\sigma(t))\alpha C\right]^{\mathrm{T}} \dot{x}(t) \\
&= \left[x^{\mathrm{T}}(t) C^{\mathrm{T}} \alpha K C - f^{\mathrm{T}}(\sigma(t))\alpha C\right]^{\mathrm{T}} \\
&\quad \times [Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + F f(\sigma(t)) + G f(\sigma(t-\tau))], \\
\left.\frac{\mathrm{d}V_6}{\mathrm{d}t}\right|_{(10.60)} &= \sum_{i=1}^{m} \beta_i \left[f_i^2(\sigma_i(t)) - f_i^2(\sigma_i(t-\tau))\right] \\
&= f^{\mathrm{T}}(\sigma(t))\, \beta\, f(\sigma(t)) - f^{\mathrm{T}}(\sigma(t-\tau))\, \beta\, f(\sigma(t-\tau)).
\end{aligned}$$

In the following, we first use Lemma 10.26 to obtain that for any constant matrix $M \in R^{n\times n}$, scalar $\gamma > 0$, vector function $g:\ [0,\gamma] \to R^m$, the integral inequality

$$\gamma \int_0^{\gamma} g^{\mathrm{T}}(s) M g(s)\, \mathrm{d}s \ge \left(\int_0^{\gamma} g(s)\, \mathrm{d}s\right)^{\mathrm{T}} M \left(\int_0^{\gamma} g(s)\, \mathrm{d}s\right)$$

holds. Then we have

$$\left.\frac{\mathrm{d}V_2}{\mathrm{d}t}\right|_{(10.60)} \leq \tau \dot{x}(t) Q \dot{x}(t) - \left(\frac{1}{\tau}\int_{t-\tau}^{t} \dot{x}(\xi)\,\mathrm{d}\xi\right)^{\mathrm{T}} (\tau Q) \left(\frac{1}{\tau}\int_{t-\tau}^{t} \dot{x}(\xi)\,\mathrm{d}\xi\right),$$

and due to negative definiteness of Φ, we finally obtain that there exists $\varepsilon > 0$ such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.60)} = \sum_{i=1}^{6} \left.\frac{\mathrm{d}V_i}{\mathrm{d}t}\right|_{(10.60)} \leq y^{\mathrm{T}} \Phi y \leq -\varepsilon \|x(t)\|^2 < 0 \qquad \text{when} \ \ \|x\| \neq 0,$$

where

$$y(t) := \left[x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-\tau)\ \dot{x}^{\mathrm{T}}(t-\tau)\ \left(\tfrac{1}{\tau}\int_{t-\tau}^{t} \dot{x}(\xi)\,\mathrm{d}\xi\right)^{\mathrm{T}}\ f^{\mathrm{T}}(\sigma(t))\ f^{\mathrm{T}}(\sigma(t-\tau))\right]^{\mathrm{T}}.$$

Therefore, the zero solution of the Lurie system (10.58) is time-delay dependent absolutely stable. □

Theorem 10.31. *If* $\|D\| + \tau\|B\| < 1$*, and there exist* $n \times n$ *matrices,* $P > 0, Q > 0, R > 0, S > 0$ *and diagonal matrices* $\alpha = \mathrm{diag}(\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_m) > 0,\ \beta = \mathrm{diag}(\beta_1 \ \ \beta_2 \ \ \cdots \ \ \beta_m) > 0$ *such that*

$$\Psi = \begin{bmatrix} \overline{\theta}_{11} & \overline{\theta}_{12} & \overline{\theta}_{13} & \overline{\theta}_{14} & \overline{\theta}_{15} & \overline{\theta}_{16} \\ * & \overline{\theta}_{22} & \overline{\theta}_{23} & \overline{\theta}_{24} & \overline{\theta}_{25} & \overline{\theta}_{26} \\ * & * & \overline{\theta}_{33} & \overline{\theta}_{34} & \overline{\theta}_{35} & \overline{\theta}_{36} \\ * & * & * & \overline{\theta}_{44} & \overline{\theta}_{45} & \overline{\theta}_{46} \\ * & * & * & * & \overline{\theta}_{55} & \overline{\theta}_{56} \\ * & * & * & * & * & \overline{\theta}_{66} \end{bmatrix} < 0, \tag{10.61}$$

then the zero solution of the neutral Lurie control system (10.58) is time-delay dependent absolutely stable. Here,

$$\begin{aligned}
\overline{\theta}_{11} &= P(A+B) + (A+B)^{\mathrm{T}}P + R + \tau Q + A^{\mathrm{T}}SA + C^{\mathrm{T}}\alpha KCA + A^{\mathrm{T}}C^{\mathrm{T}}K\alpha C,\\
\overline{\theta}_{12} &= -(A+B)^{\mathrm{T}}PD + A^{\mathrm{T}}SB + C^{\mathrm{T}}\alpha KCB,\\
\overline{\theta}_{13} &= A^{\mathrm{T}}SD + C^{\mathrm{T}}\alpha KCD,\\
\overline{\theta}_{14} &= \tau(A+B)PB,\\
\overline{\theta}_{15} &= PF + A^{\mathrm{T}}SF + C^{\mathrm{T}}\alpha KCF - A^{\mathrm{T}}C^{\mathrm{T}}\alpha,\\
\overline{\theta}_{16} &= PG + A^{\mathrm{T}}SG + C^{\mathrm{T}}\alpha KCG,\\
\overline{\theta}_{22} &= -R + B^{\mathrm{T}}SB,\\
\overline{\theta}_{23} &= B^{\mathrm{T}}SD,\\
\overline{\theta}_{24} &= 0,\\
\overline{\theta}_{25} &= -D^{\mathrm{T}}PF + B^{\mathrm{T}}SF - B^{\mathrm{T}}C^{\mathrm{T}}\alpha,\\
\overline{\theta}_{26} &= -D^{\mathrm{T}}PG + B^{\mathrm{T}}SG,\\
\overline{\theta}_{33} &= D^{\mathrm{T}}SD - S,
\end{aligned}$$

$$\begin{aligned}
\overline{\theta}_{34} &= 0,\\
\overline{\theta}_{35} &= D^{\mathrm{T}}SF - D^{\mathrm{T}}C^{\mathrm{T}}\alpha,\\
\overline{\theta}_{36} &= D^{\mathrm{T}}SG,\\
\overline{\theta}_{44} &= -\tau Q,\\
\overline{\theta}_{45} &= \tau B^{\mathrm{T}}PF,\\
\overline{\theta}_{46} &= \tau B^{\mathrm{T}}PG,\\
\overline{\theta}_{55} &= F^{\mathrm{T}}SF + \beta - 2\alpha CF,\\
\overline{\theta}_{56} &= -\alpha CG + F^{\mathrm{T}}SG,\\
\overline{\theta}_{66} &= G^{\mathrm{T}}SG - \beta.
\end{aligned}$$

Proof. Rewrite system (10.58) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - D\dot{x}(t-\tau) + B\int_{t-\tau}^{t} x(\xi)\,\mathrm{d}\xi\right] &= (A+B)x(t) + Ff(\sigma(t)) + Gf(\sigma(t-\tau)),\\
\sigma(t) &= Cx(t),\\
x(\theta) &= \varphi(\theta), \quad \theta \in [-r, 0].
\end{aligned} \tag{10.62}$$

Choose the radially unbounded positive definite Lyapunov functional: $V = \sum_{i=1}^{6} V_i$, where

$$\begin{aligned}
V_1 &= \left[x(t) - Dx(t-\tau) + B\int_{t-\tau}^{t} x(\xi)\mathrm{d}\xi\right]^{\mathrm{T}} P \left[x(t) - Dx(t-\tau) + B\int_{t-\tau}^{t} x(\xi)\,\mathrm{d}\xi\right],\\
V_2 &= \int_{-\tau}^{0}\int_{t+\eta}^{t} x^{\mathrm{T}}(\xi)\,Qx^{\mathrm{T}}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned}$$

and the remaining V_i's ($i = 3,4,5,6$) are the same as that defined in Theorem 10.30. It is easy to verify that there exists $\gamma > 0$ such that

$$\lambda_{\min}(P)\|x(t) - Dx(t-\tau) + B\int_{t-\tau}^{t} x(\xi)\,\mathrm{d}\xi\,\|^2 \le V \le \gamma\|x_t\|_{C_1}^2.$$

Since

$$\begin{aligned}
\left.\frac{\mathrm{d}V_1}{\mathrm{d}t}\right|_{(10.62)} &= 2\left[x(t) - Dx(t-\tau) + B\int_{t-\tau}^{t} x(\xi)\,\mathrm{d}\xi\right]^{\mathrm{T}} P\\
&\quad \times \left[(A+B)x(t) + Ff(\sigma(t)) + Gf(\sigma(t-\tau))\right],
\end{aligned}$$

and by Lemma 10.28, we obtain that for any constant matrix $M \in R^{n\times n}$, scalar $\gamma > 0$, vector function $g: [0,\gamma] \to R^m$, the integral inequality

$$\begin{aligned}
\left.\frac{\mathrm{d}V_2}{\mathrm{d}t}\right|_{(10.62)} &= \tau x(t)Qx(t) - \int_{t-\tau}^{t} x^{\mathrm{T}}(\xi)\,Qx(\xi)\,\mathrm{d}\xi\\
&\le \tau x(t)Qx(t) - \left(\frac{1}{\tau}\int_{t-\tau}^{t} x(\xi)\,\mathrm{d}\xi\right)^{\mathrm{T}} (\tau Q) \left(\frac{1}{\tau}\int_{t-\tau}^{t} x(\xi)\,\mathrm{d}\xi\right).
\end{aligned}$$

Then, we can similarly prove that there exists $0 < u \ll 0$ such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(10.62)} \leq y^{\mathrm{T}} \Psi y \leq -u \|x(t)\|^2 < 0 \qquad \text{when} \ \ \|x\| \neq 0.$$

Hence, the zero solution of the Lurie system (10.58) is time-delay dependent absolutely stable. □

Remark 10.32. The conditions given in Theorems 10.30 and 10.31 look quite involved, but can be easily implemented on a computer system using existing software such as Matlab.

11

Control Systems Described by Functional Differential Equations

Control systems described by ordinary differential equations have been thoroughly studied, and the stability theory of such systems has been developed very rapidly [52]. In practice, in particular, for any automatic control problems with feedbacks, time-delay always appears in such systems. This is because the system needs time to process the information and make decision to react. Such time-delays are usually ignored in classical control theory. However, modern control theory has been paid attention to the effect of the time-delay in control systems, and some results have been obtained. Thus, in this second edition, we add the study of Lurie control systems described by differential and difference equations. From the development of mathematical theory, since differential and difference equations are special case of functional differential equations, it is natural to consider Lurie control systems described by functional differential equations [129–131]. In this chapter, we will present the results concerning such systems.

This chapter is mainly due to Somolinos [139] (Sect. 11.1), Zhao [181] (Sect. 11.2), Ruan and Wu [132] (Sect. 11.3), Chukwu [14] (for Sects. 11.4 and 11.5), and He [42] (Sect. 11.6).

11.1 The Systems Described by RFDE

Consider the Lurie indirect control system described by retarded functional differential equations (RFDE):

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= g(t,x_t) + b f(\sigma),\\ \frac{\mathrm{d}\sigma}{\mathrm{d}t} &= q(t,x_t) - \rho f(\sigma),\end{aligned} \tag{11.1}$$

where $g(t,\psi) \in C[[0,+\infty] \times C_n[-r,0],R^n]$, r is a positive constant, $C_n[-r,0]$ is an n-dimensional vector space of continuous functions defined on $[-r,0]$, $g(t,0) \equiv 0$, $q(t,\psi) \in C[[0,+\infty] \times C_n[-r,0],R^1]$, $x \in R^n$, $b \in R^n$, $\sigma \in R$, ρ is a constant and $\rho > 0$. We denote

$|\psi(t)| := \left[\sum_{i=1}^n \psi_i^2(t)\right]^{1/2}$ (the norm of an n-dimensional vector $\psi(t)$),

$\|\psi(t)\| := \sup\limits_{t\in[-r,0]} |\psi(t)|$ (the norm of $C_n[-r,0]$, where $\psi \in C_n[-r,0]$),

$g(t,\psi)$ is Lipschitzian for ψ, i.e., for given $\psi_1, \psi_2 \in C_n[-r,0]$, we have

$$|g(t,\psi_1) - g(t,\psi_2)| \le L\|\psi_1 - \psi_2\|, \qquad L = \text{constant}. \tag{11.2}$$

We also let $x_t \in C_n[-r,0]$, $x_t(\theta) := x(t+\theta)$, $-r \le \theta \le 0$.

$$|g(t,\psi)| \le c\,\|\psi\|, \qquad c = \text{constant}, \tag{11.3}$$

$$f \in F_{[i,k]} := \Big\{ f | f(0) = 0,\ 0 < \sigma f(\sigma) \le k\sigma^2, \\ \sigma \ne 0,\ k > 0,\ f \in C[(-\infty,+\infty),R^1] \Big\}.$$

Somolinos [139] was the first person who discussed the absolutely stability of the system (11.1).

Concerning the absolute stability of the zero solution, one can establish that for any given $f \in F_{[0,k]}$, the zero solution of (11.1) is globally asymptotically stable. We know that the phase equations of system (11.1) can be written as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= g(t,x_t), \\ x(t) &= \psi(t), \qquad t \in [-r,0]. \end{aligned} \tag{11.4}$$

Assume that the solution of its Cauchy problem satisfies:

$$\|x(t,t_0,;x_0)\| \le De^{-\alpha(t-t_0)}\|\psi\|, \qquad \alpha > 0,\ \ D > 0. \tag{11.5}$$

Using Lemma 2.1 in [40], we obtain that for system (11.1) with the condition (11.3) given any $q \in (0,1)$, there exists a functional $V(t,\psi) \in C[[0,+\infty] \times C_n[-r,0],R^1]$ satisfying

$$\|\psi\| \le V(t,\psi) \le D\,\|\psi\|, \tag{11.6}$$

$$\|V(t,\psi_1) - V(t,\psi_2)\| \le M\,\|\psi_1 - \psi_2\|, \tag{11.7}$$

and

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.4)} \le -\gamma^2 V(t,\psi), \tag{11.8}$$

where $\gamma^2 = (1-q)\,\alpha$, and $M = D^{\frac{1}{q\alpha}}\,(L + (1-q)\,\alpha)$ are constants.

Theorem 11.1. *Suppose that (11.2), (11.3), and (11.4) hold and if*

$$\int_0^{\pm\infty} f(s)\,\mathrm{d}s = +\infty, \qquad 4\rho\gamma^2 > (M|b| + c)^2. \tag{11.9}$$

Then the zero solution of system (11.1) is absolutely stable in the Hurwitz angle $[0,k]$.

Proof. Assume that $V(t,\psi)$ is a functional, which satisfies (11.5)–(11.9). Then $V(t,\psi)$ is positive definite and radially unbounded, with infinitesimal upper bound. Along the solution of (11.1) the derivative of V satisfies

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.1)} \leq -\gamma^2 V + M\,|b\,f(\sigma)|,$$

and thus

$$V\,\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.1)} \leq -\gamma^2 V^2 + MV\,|b\,f(\sigma)|.$$

Let

$$W(t,t_0;\sigma) = \frac{1}{2}V^2(t,\psi) + \int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma.$$

It is easy to prove that there are two increasing continuous functions h_1 and h_2 such that

$$h_1(\|\psi,\sigma\|) \leq W(t,\psi;\sigma) \leq h_2(\|\psi,\sigma\|),$$

and $\|\psi,\sigma\| \to +\infty$, as $h_1 \to +\infty$.

If (11.9) holds, then we deduce that

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.1)} \leq -\gamma^2 V^2 + V\,(M\,|b| + c)\,|f(\sigma)| - \rho\,|f(\sigma)|^2,$$

and there is $\eta > 0$ such that

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.1)} \leq -\eta\,(V^2 + |f(\sigma)|^2) \leq -\eta(\|x_t\|^2 + f(\sigma)|^2) := -h_3(\|x_t,\sigma\|),$$

where h_3 refers to a positive definite continuous function in the norm $\|x_t,\sigma\|$ of (x_t,σ).

From Theorem 11.1 of [40], we know that the conclusion of Theorem 11.1 is true. □

Somolinos [139] also considered the following direct control system:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= g(t,x_t) + b\,f(\sigma), \\ \sigma &= c^{\mathrm{T}}x, \end{aligned} \tag{11.10}$$

where $c^{\mathrm{T}}b = -\rho < 0$, and the meaning of $g(t,x_t)$ and $f(\sigma)$ is similar to that of $g(t,x_t)$, $f(\sigma)$ in (11.1).

Theorem 11.2. *If the conditions (11.2) and (11.3) hold, and*

$$\frac{f(\sigma)}{\sigma} < \frac{\gamma^2}{M\,|b|\,|c|},$$

then the zero solution of the system (11.1) is absolutely stable in Hurwitz angle $[0,k]$.

Proof. Suppose that V is a Lyapunov functional satisfying (11.6)–(11.8). Similar to the proof of Theorem 11.1, we find

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.10)} \le -\gamma^2 V + M|b|\,|f(\sigma)| \le -\gamma^2\|x_t\| + M|b|\,|f(\sigma)|. \tag{11.11}$$

For $\sigma = 0$, we deduce

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.10)} \le -\gamma^2\|x_t\|;$$

for $\sigma \ne 0$, the last term on the right-hand side in (11.11) is multiplied by $\frac{|c|\,\|x_t\|}{c^{\mathrm{T}}x_t}$. From $\sigma = c^{\mathrm{T}}x$, it follows that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.10)} \le -\|x_t\|\left(\gamma^2 - M|b|\,|c|\,\frac{f(\sigma)}{\sigma}\right).$$

Hence, if $\frac{f(\sigma)}{\sigma} < \frac{\gamma^2}{M|b|\,|c|}$, then $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.10)}$ is negative definite. The proof is complete. □

Zhu [185] also studied the absolute stability of the zero solution of system (11.10) in $[0,k]$.

Theorem 11.3. [185] *Suppose that the conditions (11.2), (11.3), (11.6)–(11.8) hold and that there is a real number β such that*

$$k|c|\,(M|b| + |\beta|L|c|) < \gamma, \tag{11.12}$$

$$2k|c|\left[M|b| + |c|\,(|\beta|L + \beta c^{\mathrm{T}}b)\right] < \gamma. \tag{11.13}$$

Moreover, let

$$1 + \beta k|c|^2 > 0 \qquad \mathit{for} \quad \beta < 0.$$

Then the zero solution of system (11.10) is absolutely stable in the Hurwitz angle $[0,k]$.

Proof. Let us set

$$u(t,\psi) = \frac{1}{2}V^2(t,\psi) + \beta\int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma,$$

where V is a functional satisfying (11.6)–(11.8). Thus, V is positive definite, radially unbounded and has infinitesimal upper bound. It follows from $0 < \sigma f(\sigma) \le k\sigma^2$ that

$$0 \le \int_0^{\sigma} f(\sigma)\,\mathrm{d}\sigma \le \frac{1}{2}k\sigma^2,$$

hence, by (11.6) we have

$$\left[\frac{1}{2} + (1-\mathrm{sign}\beta)\,\frac{1}{4}k\beta|c|^2\right]\|\psi\|^2 \le u(t,\psi) \le \left[\frac{D^2}{2} + (1+\mathrm{sign}\beta)\,\frac{1}{2}k\beta|c|^2\right]\|\psi\|^2. \tag{11.14}$$

It then follows form the conditions that $u(t,\psi)$ is a radially unbounded positive definite functional of ψ.

In contrast, we have

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{(11.10)} \leq -\gamma^2 V^2 + (M\,|b| + L|\beta|\,|c|)\,V\,|f(\sigma)| + \beta\,c^{\mathrm{T}} b\,f^2(\sigma).$$

Choosing a constant $\tau > 0$, and taking

$$N = \frac{1}{2\gamma}\Big(M\,|b| + L|\beta|\,|c| + \tau\,|c|\Big),$$

we deduce that

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{(11.10)} \leq -(\gamma V - N|f(\sigma)|)^2 + \Big[N^2 - \Big(\frac{\tau}{k} - \beta c^{\mathrm{T}} b\Big)\Big]\,|f(\sigma)|^2 + \tau\left[\frac{f(\sigma)}{k} - \sigma\right] f(\sigma). \tag{11.15}$$

Clearly, when $f = 0$, the above form is reduced to

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{(11.10)} \leq -\gamma^2 V^2 \leq -\gamma^2 \|x_t\|^2.$$

However, when $f \neq 0$, it follows from $q(t,\psi) \in C[\,[0,+\infty] \times C_n[-r,0], R^1]$ that

$$\left[\frac{f(\sigma)}{k} - \sigma\right] f(\sigma) = f^2(\sigma)\left[\frac{1}{k} - \frac{\sigma}{f(\sigma)}\right] \leq 0.$$

Therefore, only if

$$N^2 < \frac{\tau}{k} - \beta\,c^{\mathrm{T}} b, \tag{11.16}$$

can we obtain from (11.15) that $\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{(11.10)}$ is a negative definite functional of ψ. To decide the conditions satisfying (11.16), we substitute the representation of N to (11.16), which can be further reduced to

$$|c|^2\left[\tau^2 + 2\Big(\frac{M\,|b|}{|c|} - \frac{\gamma}{k\,|c|^2} + |\beta|L\Big)\,\tau + \Big(\frac{M\,|b|}{|c|} + |\beta|L\Big)^2 - 2\gamma\beta\,\frac{c^{\mathrm{T}} b}{|c|^2}\right] < 0.$$

Let

$$\lambda = \frac{M\,|b|}{|c|} - \frac{\gamma}{k\,|c|^2} + |\beta|L,$$

$$p = \Big(\frac{M\,|b|}{|c|} + |\beta|L\Big)^2 - 2\gamma\beta\,\frac{c^{\mathrm{T}} b}{|c|^2}.$$

Then the above formula is reduced to

$$|c|^2(\tau^2 + 2\lambda\tau + p) = |c|^2(\tau + \lambda - \sqrt{\lambda^2 - p^2})(\tau + \lambda + \sqrt{\lambda^2 - p^2}) < 0.$$

Clearly, when $\lambda < 0$, $\lambda^2 > p$, we can find a range of τ such that the above inequality is satisfied, viz.

$$0 < -\lambda - \sqrt{\lambda^2 - p^2} < \tau < -\lambda + \sqrt{\lambda^2 - p^2}.$$

From $\lambda < 0$ it follows that $kL|\beta|\,|k|^2 < \gamma - kM\,|b|\,|c|$. This is the condition (11.12). Further, from $\lambda^2 > \rho$, we obtain

$$-2\left(\frac{M\,|b|}{|c|} + |\beta|\,L\right)\frac{\gamma}{k\,|c|^2} + \frac{\gamma^2}{k^2\,|c|^4} > -2\beta\,\frac{c^{\mathrm{T}}b}{|c|^2}.$$

Using the above expression we obtain the condition (11.13).

Till now we proved that when (11.13) is satisfied, $\frac{\mathrm{d}u}{\mathrm{d}t}\Big|_{(11.10)}$ is a negative definite functional.

Finally, it follows from the inequality (11.14) that $u(t,\psi)$ is a positive definite functional with an infinitesimal small upper bound and radially unbounded.

Therefore, the zero solution of system (11.10) is absolutely stable in the Hurwitz angle $[0,k]$. □

The consequences of Theorems 11.2 and 11.3 exclude each other. Ruan [130] generalized Theorem 11.2 to obtain the following result.

Theorem 11.4. [130] *Suppose that the conditions (11.2), (11.3), and (11.3) hold. Let*

$$\alpha^* = \frac{kM\,|b|\,|c|}{\gamma^2}, \quad \beta^* = L - k\rho, \quad \rho = -c^{\mathrm{T}}b.$$

If one of the following four conditions holds:

1. $0 < \alpha^* < 1$
2. $\alpha^* = 1$ *and* $\|\rho| > \dfrac{L}{k}$
3. $2 - \dfrac{L}{k\rho} < \alpha^* < 2$ *and* $|\rho| > \dfrac{L}{k} > 0$
4. $1 < \alpha^* < 2 + \dfrac{L}{k\rho}$ *and* $\rho < -\dfrac{L}{k} < 0$

then the zero solution of system (11.10) is absolutely stable in the Hurwitz angle $[0,k]$.

Proof. First, for the system (11.1), the criterion of absolute stability is that there exists a real number β such that

$$\gamma^2 > \frac{1}{2}k\,|c|\,(M\,|b| + L|\beta|\,|c|), \tag{11.17}$$

$$\gamma^2 > k\,|c|\,(M\,|b| + L|\beta|\,|c| + k\,|c|\,\beta\,c^{\mathrm{T}}b), \tag{11.18}$$

and such that in case of $\beta < 0$, there exists

$$1 + \beta\,k\,|c|^2 > 0 \tag{11.19}$$

or

$$4\gamma^2\rho\beta > (M|b| + L|\beta|\,|c|). \tag{11.20}$$

However, the condition (11.20) makes G a positive definite quadratic form (G will be given below.) This leads to a contradiction.

Second, we prove that when the conditions of Theorem 11.3 are satisfied there exists a real number β satisfying (11.17) and (11.20), and thus the theorem is proved.

For the system (11.10), let us set

$$W(t,\psi) = \frac{1}{2}V^2(t,\psi) + \beta\int_0^{c^{\mathrm{T}}\psi(0)} f(s)\,\mathrm{d}s,$$

where V stands for a functional satisfying (11.6)–(11.8). Then we deduce

$$\left[\frac{1}{2} + (1-\mathrm{sign}\beta)\frac{1}{4}k\beta|c|^2\right]\|\psi\|^2 \le W(t,\psi) \le \left[\frac{D^2}{2} + (1+\mathrm{sign}\beta)\frac{1}{2}k\beta|c|^2\right]\|\psi\|^2.$$

From the hypothesis of the theorem, we find that W is radially unbounded and have an infinitesimal small upper bound. Below we only find a criterion which W satisfies:

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.10)} \le -h_3(|\varphi(0)|)$$

with h_3 being a positive definite continuous nondecreasing function.

Following this line of reasoning, we have

$$\begin{aligned}\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.10)} &\le -\gamma^2V^2 + MV|bf(\sigma)| + \beta f(\sigma)[c^{\mathrm{T}}g(t,x_t) - \rho f(\sigma)]\\ &\le -\gamma^2V^2 + M|b|V|f(\sigma)| + |\beta|L|c|V|f(\sigma)| - \beta\rho|f(\sigma)|^2\\ &= -\gamma^2[V^2 - 2pV|f(\sigma)| + q|f(\sigma)|^2],\end{aligned}$$

where $\rho = -c^{\mathrm{T}}b > 0$, $2p = \frac{M|b|+L|\beta|\,|c|}{\gamma^2}$, and $q = \frac{\beta\rho}{\gamma^2}$. Furthermore, it follows from $q(t,\varphi) \in C[[0,+\infty)\times C_n[-r,0],R^1]$ that

$$\frac{V(t,x_t)}{|f(c^{\mathrm{T}}x_t(0)|} \ge \frac{\|x_t\|}{f(c^{\mathrm{T}}x(t))} \ge \frac{1}{k|c|}$$

and

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.10)} \le -\gamma^2G,$$

where

$$G = [V - (p+\tau)|f(\sigma)|]^2 + N|f(\sigma)|^2 + 2\tau\left[V - \frac{1}{k|c|}|f(\sigma)|\right]|f(\sigma)|,$$

and $\tau > 0$ is yet undetermined constant such that $N > 0$,

$$N = \frac{2\tau}{k|c|} - p^2 - 2p\tau - \tau^2 + q.$$

If $\tau > 0$ exists, it follows from the condition $f(0) = 0$, $0 < \sigma f(\sigma) \leq k\sigma^2$, $\sigma \neq 0$ that

$$\begin{aligned} G &\geq [V - (p+\tau)|f(\sigma)|]^2 + N|f(\sigma)|^2 \\ &\geq \eta\,[V^2(t,x_t) + f^2(c^{\mathrm{T}}x(t))] \\ &\geq \eta\,(\|x_t\|^2 + |f(c^{\mathrm{T}}x(t))|^2) \\ &\geq \eta\,\|x_t\|^2 \geq \eta\,|x(t)|^2, \end{aligned}$$

where $[V-(p+\tau)|f(\sigma)|]^2 + N|f(\sigma)|^2$ is a positive definite quadratic form in V, $|f(\sigma)$, viz. $h_3(|\varphi(0)|) = \gamma^2\eta\,|\varphi(0)|^2$ $(\eta > 0)$ can hold. Therefore, from Theorem 2.1 in [40], we conclude that the zero solution of system (11.10) is absolutely stable in the Hurwitz angle $[0,k]$.

Now we are in position to verify that τ $(\tau > 0)$ exist. We note that

$$f(\tau) = \tau^2 + 2\lambda^*\tau + \rho^2 = -N,$$

where $\lambda^* = p - \frac{1}{k|c|}$ and $p^* = p^2 - q$. The conditions of existence of τ with $\tau > 0$ and $f(\tau) < 0$ are $p^* < 0$ or $\lambda^{*2} > p^* > 0$ and $\lambda^* < 0$. Expanding them, we obtain (11.17) or (11.18) and (11.19). The first part is completed.

Note the criterion condition for $G > 0$. It can be directly obtained from $x_t > \frac{1}{k|c|}|f(\sigma)|$, instead of the existence of $\tau > 0$. If $\psi = 0$, then $G = 0$; if $\psi \neq 0$ but $c^{\mathrm{T}}\psi(0) = 0$, then G is a positive definite function $\|\psi\|^2$. Furthermore, if $\psi \neq 0$ and $c^{\mathrm{T}}\psi(0) \neq 0$, then the sufficient conditions for $G > 0$ are

$$\frac{1}{k|c|} > p + \sqrt{p^2 - q^2}, \quad p^2 - q > 0.$$

Using these conditions, we obtain (11.17) and (11.18) which in combination with (11.19) give rise to the following: $\frac{\mathrm{d}W}{\mathrm{d}t}\Big|_{(11.10)}$ is a criterion for a negative definite function to be smaller than $|\varphi(0)|$.

Let us turn to the second part of the proof.

First, we analyse the necessary and sufficient conditions for existence of a real number β, which satisfies (11.17), (11.18), and (11.19).

1. The case of $\beta > 0$ satisfying (11.17) and (11.18).

   It follows from (11.17) that

$$0 < \beta < \frac{2\gamma^2 - kM|c|\,b}{kL|c|^2}.$$

From (11.18), we deduce

$$\gamma^2 - kM|c|\,|b| > \beta k|c|^2(L - k\rho).$$

Hence it can be shown that if

$$\gamma^2 - k|c|M|b| \tag{11.21}$$

or if

$$\gamma^2 = k|c|Mb, \quad L - k\rho < 0 \tag{11.22}$$

there exists β with $\beta > 0$, which satisfies (11.17) and (11.18). However, if

$$2 - \frac{L}{k\rho} = 1 - \frac{\beta^*}{k\beta} < \alpha^* < 2, \tag{11.23}$$

the existence of β (> 0) implies

$$L - k\rho < 0 \tag{11.24}$$

and

$$\beta > \frac{\gamma^2 - k|c|M|b|}{k|c|^2(L - k\rho)} > 0,$$

$$0 < \beta < \frac{2\gamma^2 - k|c|M|b|}{kL|c|^2}.$$

Since

$$\frac{2\gamma^2 - k|c|M|b|}{kL|c|^2} > \frac{\gamma^2 - k|c|M|b|}{k|c|^2(L - k\rho)},$$

we know that there exists a positive number β satisfying (11.17) and (11.18). If (11.17) and (11.18) are satisfied, (11.21) and (11.22) or (11.23) and (11.24) holds definitely.

2. The case when β with $\beta > 0$ satisfies (11.17), (11.18), and (11.19).

   (11.19) is just $0 > \beta > -\dfrac{1}{k|c|^2}$. The form (11.17) implies

$$0 > \beta > \frac{2\gamma^2 - k|c|M|b|}{-kL|c|^2},$$

and (11.18) implies

$$\gamma^2 - k|c|M|b| > -\beta k|c|^2(\rho k + L).$$

In the case of $\alpha^* = 1$, there exists β with $\beta > 0$ satisfying (11.17), (11.18), and (11.19) only if $k\rho + L < 0$. Under the condition (4) of the theorem, we can also verify the existence of β with $\beta < 0$ by the same argument.

3. The case with $\beta = 0$ satisfies (11.17) and (11.18).
   From (11.17) and (11.18), we have the following independent inequalities

$$\gamma^2 > \frac{1}{2} k|c|M|b|$$

and

$$\gamma^2 > k|c|M|b|.$$

Hence, if $\gamma^2 > k|c|M|b|$, there exists $\beta = 0$, which satisfies (11.17) and (11.18), and vice versa.

To summarize the above three cases, we know that if one of the four conditions is satisfied, the zero solution of the system (11.10) is absolutely stable in the Hurwitz angle $[0,k]$. □

11.2 FDE Lurie Systems with Multiple Feedback Controls

In 1988, SIAM published a research report – "The Future Development of Control Theory: Mathematical Prospect." It was indicated in the report that though the stability of nonlinear control systems had been paid much attention to, and many mathematical results had been found, the results are still mainly on single-variable nonlinear control systems, such as Popov's principal and Lyapunov method. For multivariable nonlinear control systems, they are not well understood. It is still difficult to extend the single-variable case to multivariable case.

In this section, we will introduce a sufficient condition for the absolutely stability of Lurie systems described by the FDE with multiple feedback controls.

Consider the Lurie functional system with multiple feedback controls:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= g(t,x_t) + \textstyle\sum_{j=1}^m b_j f_j(\sigma_j), \\ \sigma_j &= c_j^{\mathrm{T}} x, \end{aligned} \tag{11.25}$$

where

$$\begin{aligned} & x, b_j, c_j \in \mathrm{R}^n \ (j = 1, \cdots, m \le n) \quad \rho_j = c_j^{\mathrm{T}} b_j \ge 0, \\ & \varphi_j(0) = 0,\ 0 \le f_j(\sigma_j) \le k_j \sigma_j^2, \quad k_j > 0 \ (j = 1, \cdots, m), \end{aligned} \tag{11.26}$$

$g(x,x_t)$ satisfies the condition given in Sect. 10.1.

Assume the phase equation of (11.25), given by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= g(t,x_t), \\ x(t) &= \psi(t), \end{aligned} \tag{11.27}$$

satisfies the conditions (11.4)–(11.8) in Sect. 10.1.

From condition (11.26), there exists a functional $V(t,x_t)$, which satisfies the conditions (11.6)–(11.8), satisfying:

$$\frac{V(t,x_t)}{|f_i(\sigma_i)|} \geq \frac{\|x_t\|}{|f_i(\sigma_i)|} \geq \frac{\|x_t\|}{k_i|c_i|\|x_t\|} = \frac{1}{k_i\|c_i\|},$$

i.e.

$$\frac{|f_i(\sigma_i)|}{V(t,x_t)} \leq k_i\|c_i\|. \tag{11.28}$$

Thus, $\xi_i = \frac{f_i(\sigma_i)}{V} \in [0, k_i\|c_i\|]$, $|f_i(\sigma_i)| = \xi V(t,x_t)$, $\xi_i \in [0, k_i|c_i|]$, ξ_i is not a constant.

Theorem 11.5. *If there exist constants β_i $i = 1, \cdots, m$ ($1 + \beta_i k_i\|c_i\|^2 > 0$ when $\beta_i < 0$), satisfying that $\forall \xi \in [0, k_i|c_i|]$, the following condition holds*

$$r^2 > \sum_{i=1}^{m}(M|b_i|\xi_i + |\beta_i|L|c_i|\xi_i + \beta_i c_i^{\mathrm{T}} b_i \xi_i^2) + \sum_{\substack{i=1\\ i\neq j}}^{m} |\beta_i c_i^{\mathrm{T}} b_j|\xi_i\xi_j.$$

Then the zero solution of (11.25) is absolutely stable in the Hurwitz angle $[0, k_i]$.

Proof. Use the following Lyapunov functional for (11.25):

$$W(t,\psi) = \frac{1}{2}V^2(t,\psi) + \sum_{i=1}^{m}\beta_i \int_0^{\sigma_i} f_i(\sigma_i)\mathrm{d}\sigma_i. \tag{11.29}$$

From the condition (11.26), the following inequality holds:

$$\begin{aligned}
&\left[\frac{1}{2} + \sum_{i=1}^{m}(1 - \operatorname{sgn}\beta_i)\frac{1}{4}\beta_i k_i\|c_i\|^2\right]\|\psi\|^2 \\
&\quad \leq W(t,\psi) \leq \left[\frac{D^2}{2} + \sum_{i=1}^{m}(1 + \operatorname{sgn}\beta_i)\frac{1}{2}\beta_i k_i\|c_i\|^2\right]\|\psi\|^2.
\end{aligned} \tag{11.30}$$

We know from (11.30) that the Lyapunov functional (11.29) has infinitely small upper bound, and it is radially unbounded and positive definite.

From the proof of Theorem 11.1, we have

$$\begin{aligned}
D^+V|_{(11.25)} &\leq -\gamma^2 V + M\|bf(\sigma)\| = -\gamma^2 V + M\sum_{i=1}^{m}\|b_i\||f_i(\sigma_i)\|, \\
VD^+V|_{(11.25)} &\leq -\gamma^2 V^2 + MV\|bf(\sigma)\| = -\gamma^2 V^2 + MV\sum_{i=1}^{m}\|b_i\||f_i(\sigma_i)\|.
\end{aligned}$$

Thus,

$$D^+W|_{(11.25)} \leq V(x,\psi)V_\psi\psi' + \sum_{i=1}^{m}\beta_i f_i(\sigma_i)c_i^{\mathrm{T}}\left[g(t,x_t) + \sum_{j=1}^{m} b_j f_j(\sigma_j)\right]$$

$$
\begin{aligned}
&\leq -\gamma^2 V^2 + MV\Big(\sum_{i=1}^{m} \|b_i\||f_i(\sigma_i)|\Big) \\
&\quad + \sum_{i=1}^{m} \Bigg[|\beta_i|L\|c_i\|V|f_i(\sigma_i)| + \sum_{j=1}^{m} \beta_i c_i^{\mathrm{T}} b_j f_j(\sigma_j) f_i(\sigma_i)\Bigg] \\
&\leq -\gamma^2 V^2 + \sum_{i=1}^{m} \Big[MV\|b_i\||f_i(\sigma_i)| + |\beta_i|L\|c_i\|V|f_i(\sigma_i)| \\
&\quad + \beta_i c_i^{\mathrm{T}} b_i f_i^2(\sigma_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} |\beta_i c_i^{\mathrm{T}} b_j\|f_i(\sigma_i) f_j(\sigma_j)|\Big] \\
&\leq -V^2\Big[\gamma^2 - \sum_{i=1}^{m}\Big(M\|b_i\|\frac{f_i(\sigma_i)}{V(t,\psi)} + |\beta_i\||c_i\|L\frac{|f_i(\sigma_i)|}{V(t,\psi)} \\
&\quad + \beta_i c_i^{\mathrm{T}} b_i \left(\frac{f_i(\sigma_i)}{V(t,\psi)}\right)^2 + \sum_{j=1, j\neq i}^{n} |\beta_i c_i^{\mathrm{T}} b_j| \frac{|f_i(\sigma_i)|}{V(t,\psi)}\frac{|f_j(\sigma_j)|}{V(t,\psi)}\Big)\Big] \\
&\leq -V^2\Big[\gamma^2 - \sum_{i=1}^{m}\Big(M\|b_i\|\xi_i + |\beta_i|L\|c_i\|\xi_i \\
&\qquad + \beta_i c_i^{\mathrm{T}} b_i \xi_i^2 + \sum_{j=1, j\neq i}^{n} |\beta_i c_i^{\mathrm{T}} b_j|\xi_i\xi_j\Big)\Big] \\
&< 0, \quad \text{if} \quad \xi \neq 0.
\end{aligned} \tag{11.31}
$$

Thus, the zero solution of system (11.25) is absolutely stable in the Hurwitz angles $[0,k_1],\cdots,[0,k_m]$. □

In the following, on the basis of Theorem 11.5, we give two two corollaries, which are convenient to use in applications.

Corollary 11.6. *If β_i in Theorem 11.5 satisfies that the matrix*

$$
G = \begin{bmatrix}
\beta_1 c_1^{\mathrm{T}} b_1 & \frac{1}{2}|\beta_1 c_1^{\mathrm{T}} b_2| & \cdots & \frac{1}{2}|\beta_1 c_1^{\mathrm{T}} b_m| \\
\frac{1}{2}|\beta_1 c_1^{\mathrm{T}} b_2| & \beta_2 c_2^{\mathrm{T}} b_2 & \cdots & \frac{1}{2}|\beta_2 c_2^{\mathrm{T}} b_m| \\
\vdots & & \ddots & \vdots \\
\frac{1}{2}|\beta_1 c_1^{\mathrm{T}} b_m| & \cdots & & \beta_m c_m^{\mathrm{T}} b_m
\end{bmatrix}
$$

is seminegative definite and

$$
\sum_{i=1}^{n}(M\|b_i\|k_i\|c_i\| + |\beta_i|L\|c_i\|^2 k_i) < \gamma^2,
$$

then the zero solution of (11.25) is absolutely stable in the above Hurwitz angles $[0,k_1],\cdots,[0,k_m]$.

Proof. The following conditions are satisfied:

$$\begin{aligned}
\gamma^2 &> \sum_{i=1}^{n}(Mk_i\|b_i\|\,\|c_i\| + Lk_i|\beta_i|\,\|c_i\|^2) \\
&\geq \sum_{i=1}^{n}(M\|b_i\|\xi_i + L|\beta_i|\,\|c_i\|\xi_i) \\
&\geq \sum_{i=1}^{n}(M\|b_i\|\xi_i + L|\beta_i|\,\|c_i\|\xi_i) + \begin{pmatrix}\xi_1\\ \vdots \\ \xi_m\end{pmatrix}^{\mathrm{T}} G \begin{pmatrix}\xi_1\\ \vdots \\ \xi_m\end{pmatrix} \\
&= \sum_{i=1}^{n}(M\|b_i\|\xi_i + L|\beta_i|\,\|c_i\|\xi_i + \beta_i c_i^{\mathrm{T}} b_i \xi_i^2 + \sum_{j=1, j\neq i}^{n} |\beta_i|c_i^{\mathrm{T}} b_j|\xi_i\xi_j|.
\end{aligned}$$

Thus, the conditions in Theorem 11.5 are satisfied. The proof is complete. □

Let $\lambda_{\max}(G)$ be the largest eigenvalue of matrix G, and $l_i = M\|b_i\| + L|\beta_i|\,\|c_i\|$.

Corollary 11.7. *Assume β_i in Theorem 11.5 satisfies the following conditions:*

1. If $\lambda_{\max}(G) > 0,\ \ \gamma > \sum_{i=1}^{m} \lambda_{\max}(G)\left[k_i\|c_i\| + \dfrac{l_i}{2\lambda_{\max}(G)}\right]^2$

2. If $\lambda_{\max}(G) < 0,\ \ \gamma > -\sum_{i=1}^{m} \dfrac{l_i^2}{4\lambda_{\max}(G)}$

Then, the conclusion in Theorem 11.25 holds.

Proof.

$$\begin{aligned}
&\sum_{i=1}^{n}(M\|b_i\|\xi_i + L|\beta_i|\,\|c_i\|\xi_i + \beta_i c_i^{\mathrm{T}} b_i \xi_i^2 + \sum_{j=1, j\neq i}^{n} |\beta_i|c_i^{\mathrm{T}} b_j|\xi_i\xi_j| \\
&\quad = \sum_{i=1}^{m} l_i\xi_i + \begin{pmatrix}\xi_1\\ \vdots \\ \xi_m\end{pmatrix}^{\mathrm{T}} G \begin{pmatrix}\xi_1\\ \vdots \\ \xi_m\end{pmatrix} \leq \sum_{i=1}^{m} l_i\xi_i + \lambda_{\max}(G)\sum_{i=1}^{m}\xi_i^2 \\
&\quad = \sum_{i=1}^{m} \lambda_{\max}(G)\left[\xi_i + \frac{l_i}{2\lambda_{\max}(G)}\right]^2 - \sum_{i=1}^{m}\frac{l_i^2}{4\lambda_{\max}(G)} \\
&\quad \leq \begin{cases} \displaystyle\sum_{i=1}^{m} \lambda_{\max}(G)\left[k_i\|c_i\| + \frac{l_i}{2\lambda_{\max}(G)}\right]^2 < \gamma, & \lambda_{\max}(G) > 0, \\ \displaystyle -\sum_{i=1}^{m}\frac{l_i^2}{4\lambda_{\max}(G)} < \gamma, & \lambda_{\max}(G) < 0. \end{cases}
\end{aligned}$$

Thus, the conditions in Theorem 11.5 are satisfied. The proof is finished. □

11.3 Large-Scale Control Systems Described by RFDE

In this section, we will introduce a notion of the absolute stability of Lurie large-scale systems described by retarded functional differential equations.

Let $x^{\mathrm{T}} = \big((x^{(1)})^{\mathrm{T}}, \cdots, (x^{(m)})^{\mathrm{T}} \big) \in R^n$, $x^{(i)} \in R^{n_i}$, $\sum_{i=1}^m n_i = n$, $J = [0, +\infty)$.

Assume that $r > 0$, $H_i > 0$ are constants. Let $C_{n_i} := C[[-r,0], R^{n_i}]$. For $\varphi^{(i)} \in C_{n_i}$, we define

$$\|\varphi^{(i)}\| = \sup_{-r\le\theta\le 0} |\varphi^{(i)}(\theta)|.$$

Let $C_{n_i}^{H_i} = \{\varphi^{(i)} \in C_{n_i} : \|\varphi^{(i)}\| < H_i\}$. Then it follows $C_n^H \subset C_{n_1}^{H_1} \times \cdots \times C_{n_m}^{H_m}$.

Consider a Lurie direct control large-scale system, which is described by retarded functional differential equations:

$$\begin{aligned} \frac{\mathrm{d}x^{(i)}}{\mathrm{d}t} &= g_i(t, x_t^{(i)}) - b_i f_i(\sigma_i), \\ y^{(i)} &= c_i^{\mathrm{T}} x^{(i)}, \\ \sigma_i &:= d_i^{\mathrm{T}} y = \textstyle\sum_{j=1}^m d_{ij} y^{(j)}, \quad i = 1, \cdots, m, \end{aligned} \tag{11.32}$$

where $y \in R^m$, $x^{(i)} \in R^{n_i}$, $b_i \in R^{n_i}$, $c_i \in R^{n_i}$, $d_i^{\mathrm{T}} \in R^m$, $\sigma_i \in R$, $g_i \in C[J \times C_{n_i}^{H_i}, R^{n_i}]$, $g_i(t,0) \equiv 0$. Moreover, $g_i(t, \varphi^{(i)})$ is Lipschitzian in $\varphi^{(i)}$, namely for give $\varphi_1^{(i)}, \varphi_2^{(i)} \in C_{n_i}^{H_i}$, we have

$$|g_i(t, \varphi_1^{(i)}) - g_i(t, \varphi_2^{(i)})| \le L_i \|\varphi_1^{(i)} - \varphi_2^{(i)}\|. \tag{11.33}$$

$f_i(\sigma_i)$ denotes a scalar continuous function satisfying

$$f_i(0) = 0,\ 0 < f_i(\sigma_i)\sigma_i \le k_i \sigma_i^2\ (\sigma_i \ne 0), \quad k_i > 0,\ i = 1, \cdots, m.$$

Suppose that the phase equations are

$$\begin{aligned} \frac{\mathrm{d}x^{(i)}}{\mathrm{d}t} &= g_i(t, x_t^{(i)}), \\ x^{(i)}(t) &= \varphi^{(i)}, \quad \tau \le t \le 0 \end{aligned}. \tag{11.34}$$

We assume that the solution of (11.34) satisfies

$$|x^{(i)}(t, t_0, \varphi^{(i)})| \le D_i e^{-\lambda_i (t-t_0)} \|\varphi^{(i)}\|, \quad i = 1, \cdots, m, \tag{11.35}$$

where λ_i is a positive constant. For any i there exists a functional $V_i(t, \varphi^{(i)})$ such that

$$\begin{aligned} &\|\varphi^{(i)}\| \le V_i(t, \varphi^{(i)}) \le D_i \|\varphi^{(i)}\|, \\ &|V_1(t, \varphi_1^{(i)}) - V_2(t, \varphi_2^{(i)})| \le M_i \|\varphi_1^{(i)} - \varphi_2^{(i)}\|, \\ &D^+ V_i(t, \varphi^{(i)})|_{(11.34)} \le -\gamma_i^2 \|\varphi^{(i)}\|, \end{aligned} \tag{11.36}$$

where D_i, M_i are positive constants, and $\gamma_i^2 := (1 - q_i)\lambda_i$,
$M_i := D_i^{[\lambda_i + (1-q_i)\lambda_i]/q_i \lambda_i}$, $0 \le q_i < 1$, $i = 1, \cdots, m$.

Theorem 11.8. *Suppose that the system (11.32) satisfies (11.33), the system (11.34) satisfies (11.35), and there are m constants β_i $i = 1, \cdots, m$, such that the matrix $V_{m\times m}$ is positive definite and $\begin{pmatrix} C & S \\ S^{\mathrm{T}} & R \end{pmatrix}_{2m\times 2m}$ is negative definite. Here, $V = (v_{jl})_{m\times m}$, $C = (c_{ij})_{m\times m}$, $S = (s_{ij})_{m\times m}$, $R = (r_{ij})_{m\times m}$, where*

$$v_{jl} = \begin{cases} \dfrac{1}{2}\alpha_j + \displaystyle\sum_{i=1}^{m} \dfrac{1}{4}(1 - sgn\beta_i)\beta_i\alpha_i k_i |c_j d_{ij}^{\mathrm{T}} c_l^{\mathrm{T}}|, & j = l, \\ \displaystyle\sum_{i=1}^{m} \dfrac{1}{4}(1 - sgn\beta_i)\beta_i\alpha_i k_i |c_i d_{ij}^{\mathrm{T}} d_{il} c_l^{\mathrm{T}}|, & j \neq l; \end{cases}$$

$$c_{ij} = \begin{cases} -\alpha_i\gamma_i^2, & i = j, \\ 0, & i \neq j; \end{cases}$$

$$s_{ij} = \begin{cases} \dfrac{1}{2}L_i c_i d_{ii}|\beta_i|\alpha_i + \dfrac{1}{2}c_i d_{ii}\alpha_i + \dfrac{1}{2}M_i\alpha_i|b_i|D_i, & i = j, \\ \dfrac{1}{2}L_i|c_i d_{ji}|\,\alpha_i\,|\beta_j| + \dfrac{1}{2}|c_i d_{ji}|\alpha_i, & i \neq j; \end{cases}$$

$$r_{ij} = \begin{cases} \alpha_i|\beta_i|\,|d_{ii}c_i^{\mathrm{T}} b_i| - \alpha_i/k_i, & i = j, \\ \dfrac{1}{2}(\alpha_i|\beta_i|\,|d_{ij}c_j^{\mathrm{T}} b_j| + \alpha_j|\beta_j|\,|d_{ji}c_i^{\mathrm{T}} b_i|), & i \neq j. \end{cases}$$

Then the zero solution of the Lurie-type system (11.32) is absolutely stable.

Proof. Let us choose radially unbounded, positive definite Lyapunov functional

$$u(t,\varphi) = \sum_{i=1}^{m} \alpha_i u_i(t, \varphi^{(i)}),$$

where

$$u_i(t,\varphi^{(i)}) = \frac{1}{2}V_i^{\mathrm{T}}(t,\varphi^{(i)})V_i(t,\varphi^{(i)}) + \beta_i\int_0^{\sigma_i} f_i(s)\mathrm{d}s, \quad i = 1, \cdots, m, \tag{11.37}$$

$V_i(t,\varphi^{(i)})$ being given by (11.36). It follows from the properties of $f_i(s)$ that

$$0 \le \int_0^{\sigma_i} f_i(s)\mathrm{d}s \le \frac{1}{2}k_i\sigma_i^2.$$

Combining the first form of (11.36) with (11.37), we deduce

$$\begin{aligned} 0 \le \int_0^{\sigma_i} f_i(s)\mathrm{d}s &\le \frac{1}{2}k_i\Big[\sum_{j=1}^{m}(y^{(i)})^{\mathrm{T}} d_{ij}^{\mathrm{T}}\Big]\Big[\sum_{j=1}^{m} d_{ij}y^{(i)}\Big] \\ &= \frac{1}{2}k_i\sum_{j,l=1}^{m}(y^{(j)})d_{ij}^{\mathrm{T}}d_{il}^{\mathrm{T}}(y^{(l)}) = \frac{1}{2}k_i\sum_{j,l=1}^{m}(x^{(j)})^{\mathrm{T}}c_i d_{ij}^{\mathrm{T}}d_{il}c_l^{\mathrm{T}}(x^{(l)}), \end{aligned}$$

hence

$$
\begin{aligned}
u(t,\varphi) \le \sum_{i=1}^{m} \alpha_i D_i^2 \|\varphi^{(i)}\|^2 + \sum_{i=1}^{m} \frac{1}{4}(1+\operatorname{sgn}\beta_i)\alpha_i\beta_i k_i \\
\times \sum_{j,l=1}^{m} |c_j d_{ij}^{\mathrm{T}} d_{il} c_l^{\mathrm{T}}| \, \|\varphi^{(j)}\| \, \|\varphi^{(l)}\|,
\end{aligned}
$$

$$
\begin{aligned}
u(t,\varphi) &\ge \sum_{i=1}^{m} \frac{1}{2}\alpha_i \|\varphi^{(i)}\|^2 + \sum_{i=1}^{m} \frac{1}{4}(1-\operatorname{sgn}\beta_i)\alpha_i\beta_i k_i \\
&\quad \times \sum_{j,l=1}^{m} |c_j d_{ij}^{\mathrm{T}} d_{il} c_l^{\mathrm{T}}| \, \|\varphi^{(j)}\| \, \|\varphi^{(l)}\| \\
&= (\|\varphi^{(1)}\|, \cdots, \|\varphi^{(m)}\|) V (\|\varphi^{(1)}\|, \cdots, \|\varphi^{(m)}\|)^{\mathrm{T}} \\
&\ge \lambda_{\min}(V) \sum_{i=1}^{m} \|\varphi^{(i)}\|^2,
\end{aligned}
$$

where $\lambda_{\min}$ is the minimum eigenvalue of V.

In addition, along the solution of the system (11.32) we have

$$
\begin{aligned}
D^+u|_{(11.32)} &\le \sum_{i=1}^{m} [-\alpha_i \gamma_i^2 \|\varphi^{(i)}\|^2 + \alpha_i M_i |b_i| D_i \|\varphi^{(i)}\| \, |f_i(\sigma_i)|] \\
&\quad + \sum_{i=1}^{m} \alpha_i |\beta_i| \, |f_i(\sigma_i)| \sum_{j=1}^{m} |d_{ij} c_j^{\mathrm{T}} b_j| \, |f_j(\sigma_j)| \\
&\quad + \sum_{i=1}^{m} \alpha_i |\beta_i| \, |f_i(\sigma_i)| \sum_{j=1}^{m} |d_{ij} c_j^{\mathrm{T}} L_j| \, \|\varphi^{(j)}\| \\
&\quad + \sum_{i=1}^{m} \alpha_i \left(\sigma_i - \frac{f_i(\sigma_i)}{\sigma_i} \right) f_i(\sigma_i) \\
&\quad - \sum_{i=1}^{m} \alpha_i \left(\sigma_i - \frac{f_i(\sigma_i)}{k_i} \right) f_i(\sigma_i) \\
&\le w^{\mathrm{T}} \begin{pmatrix} C & S \\ S^{\mathrm{T}} & R \end{pmatrix} w - \sum_{i=1}^{m} \alpha_i \left(\sigma_i - \frac{f_i(\sigma_i)}{k_i} \right) f_i(\sigma_i),
\end{aligned}
$$

where $w^{\mathrm{T}} = (\|\varphi^{(1)}\|, \cdots, \|\varphi^{(m)}\|)$, $C = (c_{ij})_{m\times m}$, $S = (s_{ij})_{m\times m}$ and $R = (r_{ij})_{m\times m}$ are the matrices given in the theorem. Since the matrix $\begin{pmatrix} C & S \\ S^{\mathrm{T}} & R \end{pmatrix}$ is negative definite, we conclude that the zero solution of (11.32) is absolutely stable in the Hurwitz angles $[0,k_1], \ldots, [0,k_m]$. □

11.4 Systems Described by NFDE

Consider a Lurie indirect control system described by neutral functional differential equations:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(D(t)x_t) = A(t,x_t) + bf(\sigma),\\
&\frac{\mathrm{d}\xi}{\mathrm{d}t} = f(\sigma),\\
&\sigma(t) = B(t,x_t) - r\xi,\\
&x_{t_0} = \varphi, \quad t_0 \in I,\ \varphi \in C_n[-h,0],
\end{aligned} \tag{11.38}$$

where $I = [t_0, +\infty)$, $C_n[-h,0]$ is a set of continuous functions mapping $[-h,0]$ to R^n, $D(\cdot) : [t_0,+\infty) \times C_n[-h,0] \to \mathrm{R}^n$, and

$$D(t)\varphi = \varphi(0) - g(t,\varphi).$$

Let $\|\varphi\| := \sup_{-h\le\theta\le 0} |\varphi(\theta)|$, $x_t \in C_n[-h,0]$ and $x_t(\theta) := x(t+\theta)$, $-h \le \theta \le 0$.

Suppose that $g(t,\varphi)$ is linear in φ. Using Stieltjes integral, we get

$$g(t,\varphi) = \int_{-h}^{0} [d_s u(t,s)]\,\varphi(s),$$

where $u(t,s)$ is an $n\times n$ matrix, $t \in I$, $s \in [-h,0]$, satisfying

$$\left|\int_{-\theta}^{0} [d_s u(t,s)]\varphi(s)\right| \le l(\theta) \sup_{-\theta\le\xi\le 0} |\varphi(\xi)|,$$

and $l(\theta)$ is a nondecreasing continuous function with $\theta \in [0,h]$ and $l(0) = 0$.

We assume that $A : I \times C_n[-h,0] \to \mathrm{R}^n$ is continuous, $b, c \in \mathrm{R}^n$ and $f(\sigma)$ is a continuous function.

Consider the phase equations:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(D(t)x(t)) = A(t,x_t),\\
&x_{t_0} = \varphi, \quad t_0 \in I, \quad \varphi \in C_n[-h,0].
\end{aligned} \tag{11.39}$$

Definition 11.9. *An operator D is said to be uniformly stable if there exist two constants $\alpha > 0$ and $\beta > 0$ such that the solution of difference equations $D(t)x_t = 0$, $x_{t_0} = \varphi$, $D(t_0)\varphi = 0$ satisfies*

$$\|x_t\| \le \beta e^{-\alpha(t-t_0)}\|\varphi\|, \quad t \ge t_0.$$

In the following, we use the existence theorem of neutral functional differential equations, which is uniformly asymptotic stable. This theorem is due to [19].

Lemma 11.10. *Suppose that $D(t)$ and $A(t,\cdot)$ in system (11.39) are linear bounded operators mapping $C_n[-h,0]$ to R^n and there exists $L_1 > 0$ such that for $t \geq t_0$, $\varphi \in C_n[-h,0]$,*

$$|D(t)\varphi| \leq L_1\|\varphi\|.$$

If the zero solution of system (11.39) is uniformly asymptotically stable, then there exist constants M, γ^2, K, and a continuous scalar function $V(t,\varphi)$ on $I \times C_n[-h,0]$ such that for $t \geq t_0$, φ, $\varphi \in C_n[-h,0]$

$$|D(t)\varphi| \leq V(t,\varphi) \leq M\|\varphi\|,$$

$$\left.\frac{\mathrm{d}^+V}{\mathrm{d}t}\right|_{(11.39)} \leq -\gamma^2 V(t,\varphi),$$

$$|V(t,\varphi) - V(t,\varphi)| \leq K(\varphi - \varphi),$$

where $\left.\frac{\mathrm{d}^+V}{\mathrm{d}t}\right|_{(11.39)}$ represents the upper right derivative of V along the solution of (11.39).

Lemma 11.11. *Suppose that $A(t,0) = 0$, and $A(t,\varphi)$ in system (11.39) is locally Lipschitzian in φ (the Lipschitz constant is N, which is uniform in t.) Assume that*

$$|D(t)\varphi| \leq L_1\|\varphi\|, \qquad t \geq t_0,$$

$$|g(t,\varphi)| \leq l(h)\|\varphi\|, \qquad t \geq t_0.$$

If the zero solution of (11.39) is uniformly asymptotically stable, then there exist $\delta_0 > 0$, $M = M(\delta_0) > 0$ together with two positive definite function $b(u)$, $c(u)$ $(0 \leq u \leq \delta_0)$ and a continuous scalar function $V(t,\varphi)$ $(t \in I,\ \varphi \in C_n[-h,0])$ such that for $t \geq t_0$, φ_1, $\varphi_2 \in C_n[-h,0]$ with $\|\varphi_i\| \leq \delta_0$ $(i = 1,2)$

$$|D(t)\varphi| \leq V(t,\varphi) \leq b(\|\varphi\|),$$

$$\left.\frac{\mathrm{d}^+V(t,\varphi)}{\mathrm{d}t}\right|_{(11.39)} \leq -c(|D(t)\varphi|),$$

$$|V(t,\varphi_1) - V(t,\varphi_2)| \leq M(\varphi_1 - \varphi_2).$$

Lemma 11.12. *If $D(t)$ in system (11.39) satisfies the conditions of Lemma 11.11, then for arbitrary $r_0 > 0$, there exists a constant $L = L(r_0)$ such that for $\varphi_1, \varphi_2 \in C_n[-h,0]$ with $\|x_t(t_0,\varphi_1)\| < r_0$ and $\|x_t(t_0,\varphi_2)\| < r_0$ and for all $t \geq t_0$*

$$\|x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)\| \leq e^{L(t-t_0)}\|\varphi_1 - \varphi_2\|.$$

Chukwu [14] generalized the result of [139] to the system of neutral functional differential equations.

Theorem 11.13. *Suppose that the zero solution of system (11.39) is uniformly asymptotically stable. Let γ^2 and K be defined as in Lemma (11.39). Further, let us assume that*

1. $D(t), A(t,\cdot)$, and $B(t,\cdot)$ are bounded linear operators mapping $C_n[-h,0]$ to R^n and

$$|D(t)\varphi| \le K\|\varphi\|, \quad t \ge t_0,\ \varphi \in C_n[-h,0],$$

$$|A(t,\varphi)| \le L|D(t)\varphi|,\ t \ge t_0,\ \varphi \in C_n[-h,0],$$

$$|B(t,\varphi)| \le c|D(t)\varphi|,\ t \in I;$$

2. $f \in F_k = \{f : f(0) = 0, 0 < \sigma f(\sigma) \le k\sigma^2, f \in C(-\infty,+\infty)\}$ and

$$\int_0^{\pm\infty} f(s)\mathrm{d}s = +\infty;$$

3. For $\theta \in [0,h]$ and a function $l(\theta)$, there exists

$$4\gamma^2 r > \left(c + \frac{K|b|}{1-l(\theta)}\right)^2;$$

4. The operator D is uniformly stable

Then the zero solution of the system (11.38) is absolutely stable in the Hurwitz angle $[0,k]$.

Proof.

1. From the hypothesis, the conditions of Lemma 11.10 are satisfied. Hence, we use the same Lyapunov functional $V(t,\varphi)$ as in Lemma 11.10. Suppose that $x = x(t_0,\varphi)$ is the solution of (11.39) and that $y = y(t_0,\varphi)$ is the solution of (11.38). We write $D^+V|_{(11.38)}$ as the upper right derivative of $V(f,\varphi)$ along the solution of (11.38). Then we deduce

$$\begin{aligned} D^+V|_{(11.38)} &= \overline{\lim_{r\to 0^+}} \frac{1}{\tau}[V(t+\tau, y_{t+h}(t,\varphi)) - V(t,\varphi)] \\ &\le \overline{\lim_{r\to 0^+}} \frac{1}{\tau}[V(t+\tau, y_{t+h}(t,\varphi)) - V(t+\tau, x_{t+\tau}(t,\varphi))] \\ &\quad + \overline{\lim_{r\to 0^+}} \frac{1}{\tau}[V(t+\tau, x_{t+\tau}(t,\varphi)) - V(t,\varphi)], \end{aligned}$$

where $x(t,\varphi)$ is a solution of (11.39) through (t,φ), i.e., $x_t = \varphi$.
Clearly, the second part of above inequality is equal to $D^+V|_{(11.39)}$.
In the following, we estimate the first part. Noting that

$$\begin{aligned} &D(t+\tau)(y_{t+\tau}(t,\varphi) - x_{t+\tau}(t,\varphi)) \\ &= D(t+\tau)(y_{t+\tau}(t,\varphi) - D(t)y_t(t,\varphi) + D(t)x_t(t,\varphi) - D(t+\tau)(x_{t+\tau}(t,\varphi) \\ &= \int_t^{t+\tau} [A(\xi, y_\xi) + bf(\sigma(\xi))]\mathrm{d}\xi - \int_t^{t+\tau} A(\xi, x_\xi)\mathrm{d}\xi \\ &= \int_t^{t+\tau} [A(\xi, y_\xi) - A(\xi, x_\xi)]\mathrm{d}\xi + \int_t^{t+\tau} bf(\sigma(\xi))\mathrm{d}\xi, \end{aligned}$$

and that

$$\begin{aligned}&D(t+\tau)(y_{t+\tau}(t,\varphi)-x_{t+\tau}(t,\varphi))\\&\quad=[y(t+\tau)-x(t+\tau)]-[g(t+\tau,y_{t+\tau})-g(t+\tau,x_{t+\tau})],\end{aligned}$$

we find

$$\begin{aligned}|y(t+\tau)-x(t+\tau)| &\le |D(t+\tau)(y_{t+\tau}-x_{t+\tau})|+l(h)\|y_{t+\tau}-x_{t+\tau}\|\\&\le KL\int_t^{t+\tau}\|y_\xi-x_\xi\|\mathrm{d}\xi+\int_t^{t+\tau}|bf(\sigma(\xi))|\mathrm{d}\xi\\&\quad+l(h)\|y_{t+\tau}-x_{t+\tau})\|.\end{aligned}$$

In fact, there exists

$$\begin{aligned}&\sup_{-r\le\theta\le 0}|y(t+\tau+\theta)-x(t+\tau+\theta)|=\|y_{t+\tau}-x_{t+\tau}\|\\&\quad\le\sup_{-r\le\theta\le 0}\left[KL\int_t^{t+\tau+\theta}\|y_\xi-x_\xi\|\mathrm{d}\xi+l(h)\|y_{t+\tau+\theta}-x_{t+\tau+\theta}\|\right.\\&\qquad\left.+\int_t^{t+\tau+\theta}|bf(\sigma(\xi))|\mathrm{d}\xi\right]\\&\quad\le KL\int_t^{t+\tau+\theta}\|y_\xi-x_\xi\|\mathrm{d}\xi+l(h)\|y_{t+\tau}-x_{t+\tau}\|\\&\qquad+\int_t^{t+\tau+\theta^*}|bf(\sigma(\xi))|\mathrm{d}\xi,\quad -\tau\le\theta^*\le 0.\end{aligned}$$

Hence,

$$\|y_{t+\tau}-x_{t+\tau}\|\le\frac{1}{1-l(h)}\left[KL\int_t^{t+\tau}\|y_\xi-x_\xi\|\mathrm{d}\xi+\int_t^{t+\tau}|bf(\sigma(\xi))|\mathrm{d}\xi\right]$$

and

$$\overline{\lim_{\tau\to 0^+}}\frac{1}{\tau}\|y_{t+\tau}-x_{t+\tau}\|\le\frac{|bf(\sigma(t))|}{1-l(h)}.$$

Therefore,

$$D^+V|_{(11.38)}\le-\gamma^2V+\frac{K}{1-l(h)}|bf(\sigma(t))|.$$

2. Defining

$$W=\frac{1}{2}V^2+\int_0^\sigma f(s)\mathrm{d}s,$$

we have

$$D^+W|_{(11.38)}\le-\gamma^2V^2-r|f(\sigma)|^2+V\left(\frac{K|b|}{1-l(h)}+c\right)|f(\sigma)|.$$

The right-hand side of the above expression is a quadratic form with respect to V and $|f(\sigma)|$. From the condition (3), we know that it is negative definite.

Then, there exists a constant m with $m > 0$ such that

$$D^+W|_{(11.38)} \le -m(V^2 + |f(\sigma)|^2) \le -m|D(t)\varphi|^2.$$

Finally, using Theorem 4.1 from [19], we conclude that Theorem 11.13 is true. □

Ruan [131] studied the absolute stability of the Lurie direct control system of neutral type:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[D(t)x_t] = A(t,x_t) + bf(\sigma(t)), \quad t \ge t_0,\\ &\sigma(t) = c^{\mathrm{T}}D(t)x_t,\\ &x_{t_0} = \varphi, \quad t_0 \in I, \quad \varphi \in C_n[-h,0]. \end{aligned} \tag{11.40}$$

Theorem 11.14. *Let the following conditions hold:*

1. $D(t)$ and $A(t,\cdot)$ are linear bounded operator mapping $C_n[-h,0]$ to R^n such that

$$|D(t)\varphi| \le L_1\|\varphi\|, \quad |A(t,\varphi)| \le L|D(t)\varphi|$$

for all $t \ge t_0$.
2. The zero solution of the phase equation (11.39) is globally and uniformly asymptotically stable;
3. $f \in F_k = \{f : f(0) = 0, 0 < \sigma f(\sigma) \le k\sigma^2, f \in C(-\infty,+\infty)\}$,

$$\int_0^{\pm\infty} f(s)\mathrm{d}s = +\infty;$$

4. The operator D is uniformly stable and it is atomic at 0.
In addition, let $l(h) < 1$. Then the zero solution of system (11.40) is absolutely stable in $[0,k]$ if one of the following four conditions is satisfied:
1) $0 < \alpha^* < 1$
2) $\alpha^* = 1$ and $|\rho| > \dfrac{L}{k} > 0$
3) $2 - \dfrac{L}{k\rho} < \alpha^* < 2$ and $\rho > \dfrac{L}{k} > 0$
4) $1 < \alpha^* < 2 + \dfrac{L}{k\rho}$ and $\rho < -\dfrac{L}{k}$ (< 0),
where $\alpha^* = kM|b|\,|c|/[1-l(h)]\gamma^2$, γ^2 and M are the same as in Lemma 11.10, and $\rho = -c^{\mathrm{T}}b$.

Proof.

1. As in Theorem 11.13, we obtain

$$D^+V|_{(11.40)} \le -\gamma^2 V + \frac{M}{1-l(h)}|bf(\sigma(t))|.$$

2. Let

$$W = \frac{1}{2}V^2 + \beta\int_0^{\sigma} f(s)\mathrm{d}s,$$

where β is the undetermined constant. We get

$$\begin{aligned} D^+W|_{(11.40)} &\leq -\gamma^2 V^2 + V\frac{M|b|\,|f(\sigma)|}{1-l(h)} + \beta f(\sigma)\sigma \\ &\leq -\gamma^2 V^2 - \rho\beta|f(\sigma)|^2 + \left[\frac{M|b|}{1-l(h)} + |\beta|L|c|\right] V|f(\sigma)| \\ &= -\gamma^2[V^2 - 2pV|f(\sigma)| + q|f(\sigma)|^2], \end{aligned}$$

where

$$2p = \left[\frac{M|b|}{1-l(h)} + |\beta|\,|c|L\right]/\gamma^2, \quad q = \frac{\beta\rho}{\gamma^2}.$$

In the remainder of this proof, we proceed along the line of Theorem 11.4 with M replaced by $M/[1-l(h)]$. We obtain that if one of the four conditions is satisfied, then we can choose β such that

$$D^+W|_{(11.40)} \leq -m|D(t)\varphi|^2, \quad m > 0.$$

The proof of the theorem is completed. □

11.5 Control Systems in Hilbert Spaces

In this section, we introduce a notion of the absolute stability of the Lurie control system in Hilbert space [15].

Consider the Lurie indirect control system:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= Ax + bu, \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= \varphi(\sigma), \\ \sigma &= (c,x) - \rho u, \end{aligned} \tag{11.41}$$

where the operator A is either bounded, or is assumed to generate C_0 strongly continuous group $T(t), t \in (-\infty,+\infty) = R$. On a real Hilbert space X, we denote an inner product by $(\cdot,\cdot)$ and a norm by $|\cdot|$. Above, $b,c \in X$, $u,\rho \in \mathrm{R}$, and $\varphi : \mathrm{R} \to \mathrm{R}$ is a continuously uniformly Lipschitzan nonlinear function which satisfies the following properties:

$$\sigma\varphi(\sigma) > 0 \quad \text{for} \quad \sigma \neq 0, \quad \varphi(0) = 0,$$

$$|\varphi(\sigma)| \leq K(|\sigma|) \quad \text{for all} \quad \sigma \in \mathrm{R},$$

where $K(s)$ are some monotonically nondecreasing function, $s \in \mathrm{R}_+ = (0,+\infty)$.

Assume that the linear phase equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax \tag{11.42}$$

is exponentially stable, i.e., that there exist constants $M \geq 1$ and $\alpha > 0$ such that

$$|T(t)|_{L(x)} \leq Me^{-\alpha t}, \quad t \geq 0, \tag{11.43}$$

where $L(x)$ is the Banach space of bounded linear operator from X to X. Because of the condition (11.43), it follows from Theorem 2.1 and Theorem 2.2 in [114] that there is an unique symmetric positive definite bounded operator P on X such that

$$(PAx,x) + (x,PAx) = -(x,x), \tag{11.44}$$

where $T(t)$ is a strongly continuous group satisfying

$$(A^*Px,x) + (PAx,x) \leq -\lambda \|x\|^2 \tag{11.45}$$

for any λ with $0 < \lambda < 1$. If $T(t)$ is a strongly continuous semigroup and A is bounded, a similar result is given by Walker [147]. When A satisfies

$$(x,(A-wI)x)_X \leq 0$$

for real $w \in \mathrm{R}$ and for all x in the domain of A, it is clear that (11.41) can be regarded as equations in the Hilbert space $H = X \times \mathrm{R}$ with the inner product $(\cdot,\cdot)$ defined by

$$((x_1,r_1),(x_2,r_2)) = (x_1,x_2) + r_1 r_2.$$

Theorem 11.15. [15]

1. *Let the origin be the only singular point of (11.41)*
2. *Let P be a unique symmetric positive definite bounded operator on H given by (11.45)*
3. *Let λ in (11.45) satisfy*
$$\lambda\rho > \left|Pb + \frac{c}{2}\right|^2$$
4. $\int_0^\sigma \varphi(s)\mathrm{d}s \to +\infty$ *as* $\sigma \to +\infty$

Then the zero solution of (11.41) is absolutely stable.

Proof. Since (11.42) is uniformly exponentially stable, there exists an unique symmetric positive definite bounded operator on H such that

$$(A^*Px,x) + (PAx,x) \leq -\lambda |x|^2$$

for some λ $(0 < \lambda \leq 1)$. We use P to define the functional on H:

$$V(x,\mu) = (Ax + b\mu, P(Ax + b\mu)).$$

Let

$$U(x,\mu) = \int_0^\sigma \varphi(s)\mathrm{d}s, \quad \sigma = (c,x) - \rho\mu,$$

and

$$W = V + U.$$

It is easy to prove that W is positive definite. Since P is symmetric and positive definite if and only if

$$\delta |x|^2 \le |(x, Px)| \le l|x|^2,$$

where $|P| \le l$, we have the estimate

$$\begin{aligned}
&\delta |Ax + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s \\
&\quad \le W(x,\mu) \le l|Ax + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s,
\end{aligned}$$

so that

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.41)} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(Ax+b\mu), P(Ax+b\mu)\right) + \left(Ax+b\mu, \frac{\mathrm{d}}{\mathrm{d}t}(Ax+b\mu)\right) \\
&= (A(Ax+b\mu) + b\varphi(\sigma), P(Ax+b\mu)) \\
&\quad + (Ax+b\mu, P(A(Ax+b\mu) + b\varphi(\sigma))) \\
&= (Ax+b\mu, PA(Ax+b\mu)) + (PA(Ax+b\mu), Ax+b\mu) \\
&\quad + (Ax+b\mu, Pb\varphi(\sigma)) + (Pb\varphi(\sigma), Ax+b\mu),
\end{aligned} \tag{11.46}$$

$$\begin{aligned}
\left.\frac{\mathrm{d}U}{\mathrm{d}t}\right|_{(11.41)} &= \varphi(\sigma)\dot{\sigma} = [(c, Ax+b\mu) - \rho\varphi(\sigma)]\varphi(\sigma) \\
&= (Ax+b\mu, c)\varphi(\sigma) - \rho\varphi^2(\sigma).
\end{aligned} \tag{11.47}$$

Hence, using (11.46), (11.47), (11.44), we deduce

$$\begin{aligned}
\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.41)} &= (A^*P(Ax+b\mu), Ax+b\mu) + (PA(Ax+b\mu), Ax+b\mu) \\
&\quad + (Ax+b\mu, Pb\varphi(\sigma)) + (Pb\varphi(\sigma), Ax+b\mu) \\
&\quad + (Ax+b\mu, c)\varphi(\sigma) - \rho\varphi^2(\sigma) \\
&\le -\lambda|Ax+b\mu|^2 + 2\left(Ax+b\mu, Pb+\frac{c}{2}\right)\varphi(\sigma) - \rho\varphi^2(\sigma).
\end{aligned}$$

Therefore,the condition $\lambda\rho > \left|Pb + \frac{c}{2}\right|^2$ leads to

$$\begin{aligned}
\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.41)} &\le -\lambda\left[|Ax+b\mu|^2 + 2\left|Ax + \frac{c}{2}\right|\varphi(\sigma) - \rho\varphi^2(\sigma)\right] \\
&\le -k[|Ax+b\mu|^2 + \rho\varphi^2(\sigma)] \le 0, \quad 0 < k \ll 1,
\end{aligned}$$

and $\left.\dfrac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.41)} = 0$ if and only if $x = 0$ and $\mu = 0$. As a result, we write

$$S := \left\{(x,\mu) \in H : \left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.41)} = 0\right\} = \{0,0\},$$

$$W(x(t),\mu(t)) \leq W(x(0),\mu(0)), \quad t \geq 0.$$

Therefore, every solution $(x,\mu) \in H$ is bounded. The orbits of (11.41) form a precompact subset of H. The invariance principle of Hale [40] (p. 50) yields $(x(t), \mu(t)) \to 0$ as $t \to +\infty$. This concludes the proof. □

In the following, we will apply the idea of Ladas and Lakshmikantham [54] to generalize the results to the system with nonlinear phase equation:

$$\begin{aligned} \dot{x} &= f(x) + b\mu, \\ \dot{\mu} &= \varphi(\sigma), \\ \sigma &= (c,x) - \rho\mu, \end{aligned} \tag{11.48}$$

where b, c, μ are defined as in (11.41) and $f : X \to X$ is a continuous Frechet differentiable function whose Frechet derivative at x is $A(x)$. To ensure that the solution of (11.48) exists, we shall always assume, for example, that $\varphi : \mathrm{R} \to \mathrm{R}$ is continuous, uniformly Lipschitzian and that $-f$ is a monotone function. In other words, there exists a constant M such that

$$(f(u) - f(v), u - v) \leq M|u - v|^2, \quad u, v \in X.$$

We now state a basic stability comparison theorem for the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = l(t,x), \tag{11.49}$$

where $l : \mathrm{R}^+ \times X \to X$ is continuous.

Lemma 11.16. *Assume that the following conditions hold:*

1. $V \in (R^+ \times X, R^+)$ and for (t,x_1) and $(t,x_2) \in R^+ \times X$, there exists

$$|V(t,x_1) - V(t,x_2)| \leq L(t)|x_1 - x_2|,$$

where $L(t) \geq 0$ and is continuous on R^+

2. There exists a function $g \in C[R^+ \times R^+, R]$ such that for each $(t,x) \in R^+ \times R$,

$$\begin{aligned} D^+V(t,x) &:= \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h, x+hl(t,x)) - V(t,x)] \\ &\leq g(t, V(t,x)) \end{aligned}$$

3. For each $(t_0, r_0) \in R^+ \times R^+$, the maximal solution $r(t,t_0,r_0)$ of the scalar initial value problem

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= g(t,r), \\ r(t_0) &= r_0 \end{aligned} \tag{11.50}$$

exists for $t > t_0$

4. $f(t,0) \equiv 0, g(t,0) \equiv 0$ *and* $V(t,0) \equiv 0$, $t \in R^+$
5. There exist functions $a(r)$ *and* $b(r) \in K$ *such that*

$$b(\|x\|) \le V(t,x) \le a(\|x\|) \quad \textit{for} \quad (t,x) \in R^+ \times X$$

Then if the zero solution of (11.50) is uniformly asymptotically stable in the large then the zero solution of (11.49) has the same property.

Theorem 11.17. *Let in (11.48)* $f(0) = 0$ *and* $\varphi(0) = 0$. *Again let* $A(x)$ *be the Frechet derivative of* $f(x)$ *at x. Suppose that*

1. There exists a symmetric positive definite operator P such that

$$((PA(x) + A^*(x)P)y, y) \le -\lambda |y|^2$$

for all x and y in X and some $\lambda > 0$, *where* A^* *is the adjoint of A*
2. $\varphi(s)\, sgn(s) > 0$, $\varphi(s)\, sgn(s) \to +\infty$ *as* $|s| \to +\infty$,

$$\varphi'(s) \ge \frac{1}{2}\lambda_1 \quad \textit{for some} \quad \lambda_1 \ge |P|$$

3. $|f(x) + b\mu| \to +\infty$ *as* $|x| + |\mu| \to +\infty$
4. $\lambda\rho > \left|Pb + \frac{c}{2}\right|^2$

Then the zero solution of system (11.48) is absolutely stable.

Proof. Let $H = X \times$R be equipped with the inner product $(\cdot,\cdot)$ defined by

$$((x_1,r_1),(x_2,r_2))_H = (x_1,x_2) + r_1 r_2.$$

Let $V : H \to R$ be defined by

$$V = W + U,$$

where

$$W = (f(x) + b\mu, p(f(x) + b\mu)),$$
$$U = \int_0^\sigma \varphi(s) \mathrm{d}s.$$

Since P is positive definite and symmetric, there exist positive constants λ_1 and λ_2 such that

$$\lambda_2 |f(x) + b\mu| \le \lambda_1 |f(x) + b\mu|^2,$$

with $\lambda_1 \ge |P|$. Hence

$$\lambda_2 |f(x) + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s$$
$$\le V(x,\mu) \le \lambda_1 |f(x) + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s. \tag{11.51}$$

It follows from the condition (2) that $\int_0^\sigma \varphi(s)ds \to +\infty$ as $\sigma \to +\infty$. Thus, by (3) and (11.51), we find that the condition (5) of Lemma 11.16 is satisfied. Noting

$$\begin{aligned}
\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.48)} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(f(x)+b\mu), P(f(x)+b\mu)\right) \\
&\quad + \left(f(x)+b\mu, P\frac{\mathrm{d}}{\mathrm{d}t}(f(x)+b\mu)\right) \\
&= (A(x)[f(x)+b\mu]+b\varphi(\sigma), P[f(x)+b\mu]) \\
&\quad + (f(x)+b\mu, P\{A(x)[f(x)+b\mu]+b\varphi(\sigma)\}) \\
&= (PA(x))[f(x)+b\mu], f(x)+b\mu) \\
&\quad + (A^*(x)P[f(x)+b\mu], f(x)+b\mu) \\
&\quad + 2(Pb\varphi(\sigma), f(x)+b\mu) \\
&\leq -\lambda|f(x)+b\mu|^2 + 2(Pb\varphi(\sigma), f(x)+b\mu),
\end{aligned}$$

and

$$\left.\frac{\mathrm{d}U}{\mathrm{d}t}\right|_{(11.48)} = \varphi(\sigma)\dot{\sigma} = \varphi(\sigma)(c, f(x)+b\mu) - \rho\varphi^2(\sigma),$$

we obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.48)} &\leq \left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(11.48)} + \left.\frac{\mathrm{d}U}{\mathrm{d}t}\right|_{(11.48)} \\
&\leq -\lambda|f(x)+b\mu|^2 + 2\varphi(\sigma)\left(Pb+\frac{c}{2}, b(x)+b\mu\right) - \rho\varphi^2(\sigma) \\
&\leq -\lambda|f(x)+b\mu|^2 + 2|\varphi(\sigma)|\left|Pb+\frac{c}{2}\right||f(x)+b\mu| - \rho\varphi^2(\sigma).
\end{aligned}$$

The condition $\lambda\rho > \left|Pb+\frac{c}{2}\right|^2$ implies that there exists $0 < \lambda_2 \ll 1$ such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.48)} \leq -\lambda_3[|f(x)+b\mu|^2 + \varphi^2(\sigma)].$$

By virtue of (11.51), we have the inequality

$$\left\{V(x,u) - \int_0^\sigma \varphi(s)\mathrm{d}s\right\}\frac{1}{\lambda_1} \leq |f(x)+b\mu|^2,$$

and thus

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.48)} \leq -\lambda_3\left\{\frac{V}{\lambda_1} - \frac{1}{\lambda_1}\int_0^\sigma \varphi(s)\mathrm{d}s + \varphi^2(\sigma)\right\}.$$

Since $\varphi^2(0) = 0$, it follows that

$$\varphi^2(\sigma) = 2\int_0^\sigma \varphi'(s)\varphi(s)\mathrm{d}s.$$

Consequently,

$$\frac{1}{\lambda_1}\int_0^{\sigma}\varphi(s)\mathrm{d}s-\varphi^2(s)=\int_0^{\sigma}\left[\frac{1}{\lambda_1}-2\varphi'(s)\right]\varphi(s)\mathrm{d}s\le 0.$$

Using $\varphi(s)\,\mathrm{sgn}(s)>0$ and $\frac{1}{2}\lambda_1-\varphi'(s)\le 0$, the condition (2), and the above remarks, we obtain the final inequality

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(11.48)}\le -\frac{\lambda_3}{\lambda_1}V(x,u).$$

Thus, the comparison equation takes the form

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t}&=-\frac{\lambda_3}{\lambda_1}r(t),\\ r(t_0)&=r_0.\end{aligned}\tag{11.52}$$

It is easy to prove that the solution

$$r(t)=r_0\exp\left[-\frac{\lambda_3}{\lambda_1}(t-t_0)\right],\quad t\ge t_0$$

of (11.52) is uniformly asymptotically stable. Theorem 3.1 in [54] yields the absolute stability of the zero solution of (11.48). □

11.6 Lurie Systems Described by PFDE

In this section, the necessary and sufficient condition for absolute stability of Lurie control system is extended to a more general Lurie system described by abstract functional differential equation, which is also called partial functional differential equation [42].

Let X be a real Banach space, $X^n=\overbrace{X\times\cdots\times X}^{n}$, $X^m=\overbrace{X\times\cdots\times X}^{m}$, $X^p=\overbrace{X\times\cdots\times X}^{p}$, $m+p=n$, $C=C([-r,0],X)$, $C^k=C([-r,0],X^k)\,k=m,p,n$, $\varphi\in C^n$, $\psi\in C^m$, $\eta\in C^p$. Denote $\|\varphi\|_{C^n}=\sup\limits_{-r\le\theta\le 0}|\varphi|_{X^n}$, $\|\psi\|_{C^m}=\sup\limits_{-r\le\theta\le 0}|\psi|_{X^m}$, $\|\eta\|_{C^m}=\sup\limits_{-r\le\theta\le 0}|\eta|_{X^p}$.

Consider the initial value problem of the following abstract functional differential equation:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t}&=Au+L(t,u_t)+f(t,u_t),\quad t\ge t_0>0,\\ u&=\varphi(t),\quad t\in[t_0-r,t_0],\end{aligned}\tag{11.53}$$

where $u=(u_1,\cdots,u_n)^{\mathrm{T}}\in X^n$, $x=(u_1,\cdots,u_m)^{\mathrm{T}}\in X^m$, $y=(u_{m+1},\cdots,u_n)\in X^p$, $L=(L_1,\cdots,L_m)^{\mathrm{T}}$, $\varphi=(\varphi_1,\cdots,\varphi_n)^{\mathrm{T}}=(\psi_1,\cdots,\psi_m,\eta_{m+1},\cdots,\eta_n)^{\mathrm{T}}=(\psi,\eta)^{\mathrm{T}}$, $u_t=u(t+\theta)\ \theta\in[t_0-r,t_0]$, A is a linear operator on X^n.

Assume equation (11.53) satisfies the global existence condition of the solution of Cauchy problem, we have the following two lemmas.

Lemma 11.18. *If the partial variable solution of following linear partial function differential equation*

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= Av + L(t, v_t), \quad t \geq t_0 > 0, \\ v &= \varphi(t), \quad t \in [t_0 - r, t_0], \end{aligned} \tag{11.54}$$

satisfies the following inequality:

$$\|x_t(t_0, \varphi)\|_{C^m} \leq K(t_0)\|\varphi\|_{C^m} e^{-\int_{t_0}^{t} \varepsilon(s)\mathrm{d}s}, \tag{11.55}$$

where $\varepsilon(t) \in C[R^+, R]$, $K(t) \in C[R^+, R^+]$, then there exists a Lyapunov function

$$V(t, \varphi) \in C[R^+ \times C^n, R^+],$$

satisfying the following conditions:

1. $\|\psi\|_{C^m} \leq V(t, \varphi) \leq K(t)(\|\psi\|_{C^m} + \|\eta\|_{C^p})$
2. $\|V(t, \varphi) - V(t, \tilde{\varphi})\| \leq K(t)(\|\psi - \tilde{\psi}\|_{C^m} + \|\eta - \tilde{\eta}\|_{C^p})$
3. $D^+V|_{(11.53)} \leq -\varepsilon(t)V(t, \varphi)$

Proof. Construct the following Lyapunov function

$$V(t, \varphi) = \sup_{\tau \geq 0} \|x_{t+\tau}(t, \varphi)\|_{C^m} e^{\int_t^{t+\tau} \varepsilon(s)\mathrm{d}s}.$$

It is obvious that the condition (1) is satisfied. We then have

$$\begin{aligned} \|V(t, \varphi) - V(t, \tilde{\varphi})\| &= \sup_{\tau \geq 0} \|x_{t+\tau}(t, \varphi)\|_{C^m} e^{\int_t^{t+\tau} \varepsilon(s)\mathrm{d}s} \\ &\quad - \sup_{\tau \geq 0} \|x_{t+\tau}(t, \tilde{\varphi})\|_{C^m} e^{\int_t^{t+\tau} \varepsilon(s)\mathrm{d}s} \\ &\leq \sup_{\tau \geq 0} \|x_{t+\tau}(t, \varphi - \tilde{\varphi}))\|_{C^m} e^{\int_t^{t+\tau} \varepsilon(s)\mathrm{d}s} \\ &\leq K(t)\|\varphi - \tilde{\varphi}\|_{C^n} \leq K(\|\psi - \tilde{\psi}\|_{C^m} + \|\eta - \tilde{\eta}\|_{C^p}). \end{aligned}$$

Thus, the condition (2) is satisfied. Further, we obtain

$$\begin{aligned} D^+V|_{(11.53)} &= \overline{\lim_{h \to 0^+}} \frac{1}{h}(V(t+h, V_{t+h}(t, \varphi) - V(t, \varphi)) \\ &= \overline{\lim_{h \to 0^+}} \frac{1}{h} \Bigg[\sup_{\tau \geq 0} \|x_{t+h+\tau}(t+h, V_{t+h}(t, \varphi))\|_{C^m} e^{\int_{t+h}^{t+h+\tau} \varepsilon(s)\mathrm{d}s} \\ &\quad - \sup_{\tau \geq 0} \|x_{t+h}(t, \varphi)\|_{C^m} e^{\int_t^{t+\tau} \varepsilon(s)\mathrm{d}s} \Bigg] \\ &= \overline{\lim_{h \to 0^+}} \frac{1}{h} \Bigg[\sup_{\tau \geq h} \|x_{t+\tau}(t, \varphi)\|_{C^m} e^{\int_{t+h}^{t+h+\varepsilon} \varepsilon(s)\mathrm{d}s} \\ &\quad - \sup_{\tau \geq 0} \|x_{t+h}(t, \varphi)\|_{C^m} e^{\int_t^{t+\tau} \varepsilon(s)\mathrm{d}s} \Bigg] \end{aligned}$$

$$\leq \overline{\lim_{h\to 0^+}} \frac{1}{h} (\sup_{\tau\geq 0} \|x_{t+\tau}(t,\varphi)\|)_{C^m} e^{\int_t^{t+\tau} \varepsilon(s)\mathrm{d}s} \left(e^{-\int_t^{t+h} \varepsilon(s)\mathrm{d}s} - 1 \right)$$
$$\leq -\varepsilon(t)V(t,\varphi).$$

Thus, the condition (3) is satisfied. □

Lemma 11.19. *Suppose that the following conditions are satisfied:*

1. The partial variable solution $x_t(t_0,\varphi)$ satisfies equation (11.55) and there exists $W(t,r) \in C[R^+ \times R^+, R^+]$, which is monotonically nondecreasing w.r.t. r, satisfying

$$|f(t,\varphi)|_{X^n} \leq W(t, \|\psi\|_{C^m})$$

2. The initial value problem of the following system

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= -\varepsilon(t)z + K(t)W(t,z), \\ z(t_0) &= z_0 \geq V(t_0,\varphi), \end{aligned} \tag{11.56}$$

has a maximum right solution $Z_m(t,t_0,z_0)$ in R^+.

Then there exists the following estimation for the partial variable solution $x_t(t_0,\varphi)$ of system (11.53):

$$\|x_t(t_0,\varphi)\|_{C^m} \leq Z_m(t,t_0,z_0), \quad t \geq t_0. \tag{11.57}$$

Proof. From Lemma 11.18, we know that there exists a Lyapunov function $V(t,\varphi)$ satisfying the conditions (1), (2), and (3). Then,

$$\begin{aligned} D^+V_{(11.53)} &\leq \overline{\lim_{h\to 0^+}} \frac{1}{h}(V(t+h, V_{t+h}(t,\varphi)) - V(t,\varphi)) \\ &\quad + \overline{\lim_{h\to 0^+}} \frac{1}{h}(V(t+h, u_{t+h}(t,\varphi) - V(t+h, V_{t+h}(t,\varphi)) \\ &\leq -\varepsilon(t)V(t,\varphi) - V \overline{\lim_{h\to 0^+}} \frac{\|u_{t+h} - V_{t+h}\|_{C^m}}{h}. \end{aligned} \tag{11.58}$$

From the formula of variation of constants, we have

$$u_{t+h} - V_{t+h} = \int_t^{t+h} U(t+h,x) f(s,u_s)\mathrm{d}s.$$

Thus,

$$\overline{\lim_{h\to 0^+}} \frac{\|u_{t+h} - V_{t+h}\|_{C^m}}{h} \leq |f(t,\varphi)|_{X^n}. \tag{11.59}$$

Substituting equation (11.59) into (11.58) yields

$$\begin{aligned} D^+V_{(11.53)} &\leq -\varepsilon(t)V(t,\varphi) + K(t)|f(t,\varphi)|_{X^n} \\ &\leq -\varepsilon(t)V(t,\varphi) + K(t)W(t, \|\psi\|_{C^n}) \\ &\leq -\varepsilon(t)V(t,\varphi) + K(t)W(t, V(t,\varphi)). \end{aligned}$$

Noting that $\varphi \in C^n$ is arbitrary, we have

$$D^+V(t,u_t(t_0,\varphi)) \le -\varepsilon(t)V(t,u_t(t_0,\varphi)) + K(t)W(t,V(t,u_t(t_0,\varphi)).$$

From comparison principle and the condition (1) of Lemma 11.18 it is easy to obtain that

$$\|x_t(t_0,\varphi)\|_{C^m} \le V(t,u_t(t_0,\varphi)) \le Z_m(t,t_0,z_0). \qquad \square$$

Defining the inner product (x,x) in Banach space X results in a Hilbert space.

Now consider the following Lurie control system described by the abstract functional differential equation:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= Au + L(t,u_t) + bf(t,\sigma_t), \quad t \ge t_0 \ge 0, \\ u &= \varphi, \quad t \in [t_0 - r, t_0], \quad \sigma_{t_0} = c^{\mathrm{T}}u, \end{aligned} \tag{11.60}$$

where $L = \int_{-r}^{0} d\eta(t,\theta)\varphi(\theta)$ is a Stieltjes integral, $f \in C[I \times C^n, X]$, $0 < (\sigma_t, f(t,\sigma_t) \le k(\sigma_t,\sigma_t)$ when $\sigma_t \ne 0$, $u = (u_1,\cdots,u_n)^{\mathrm{T}}$, $b = (b_1,\cdots,b_n)^{\mathrm{T}}$, $c = (c_1,\cdots,c_n)^{\mathrm{T}}$.

Definition 11.20. *If for any $f(\cdot)$ satisfying $(f(t,\sigma_t),\sigma_t) > 0$, $\sigma_t \ne 0$, then the zero solution of system (11.60) is globally asymptotically stable, the zero solution of equation (11.60) is absolutely stable.*

Suppose $C_n \ne 0$. Under the transformation

$$W = G(g_{ij})u,$$

where

$$g_{ij} = \begin{cases} 1 & i = j = 1,\cdots,n-1, \\ 0 & i \ne j,\, i = 1,\cdots,n-1,\, j = 1,\cdots,n, \\ c_j & j = 1,\cdots,n, \quad i = n, \end{cases}$$

then system (11.60) is transformed into

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t} &= \tilde{A}w + \tilde{L}(t,w_t) + \tilde{b}f(t,w_{n,t}), \quad t \ge t_0, \\ w &= \tilde{\varphi}, \quad t \in [t_0 - r, t_0], \end{aligned} \tag{11.61}$$

where $w = (w_1,\cdots,w_n)^{\mathrm{T}} = (u_1,\cdots,u_{n-1},\sigma)^{\mathrm{T}}$, $\tilde{A}$ and $\tilde{L}$ have the same properties as A and L, respectively. Then the zero solution of system (11.60) is absolutely stable if and only if the zero solution of system (11.61) is absolutely stable.

Theorem 11.21. *If the solution $v_t(t_0,\varphi)$ of the following linear system*

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= Av + L(t,v_t), \quad t \ge t_0, \\ v(t) &= \varphi, \quad t \in [t_0 - \tau, t_0], \end{aligned} \tag{11.62}$$

satisfies the following inequality

$$\|v_t(t_0,\varphi)\|_{C^n} \le K\|\varphi\|_{C^n} e^{-\alpha(t-t_0)}, \quad \alpha > 0,$$

then the zero solution of (11.60) is absolutely stable if and only if the zero solution of (11.61) is absolutely stable w.r.t. the single variable $W_{nt}(t_0,\varphi)$.

Proof. *Necessity*. Assume that the zero solution of (11.60) is absolutely stable and $\|c\| = M$, $\forall \varepsilon > 0$, then there exists $\delta(\varepsilon) > 0$ satisfying

$$\|u_t(t,\varphi)\|_{C^n} < \varepsilon/M, \quad \text{if } t \ge t_0,\ \|\varphi\|_{C^n} < \delta.$$

Since $W = Gu$, $\tilde{\varphi} = G\varphi$, or $\varphi = c_T^{-1}\tilde{\varphi}$, we have

$$\|\hat{W}_{nt}(t_0,\tilde{\varphi})\|_C \le \|C\|\,\|u_t(t_0,\varphi)\|_{C^n} < \varepsilon, \quad t \ge t_0, \quad \text{if} \quad \|\tilde{\varphi}\|_{C^n} < \delta/\|G^{-1}\|.$$

For any $\varphi \in C^n$, from the condition $\lim\limits_{t\to+\infty} \|u_t(t_0,\varphi)\|_{C^n} = 0$, we have

$$\lim_{t\to+\infty} \|W_{nt}(t_0,\tilde{\varphi})\|_C \le M \lim_{t\to+\infty} \|u_t(t_0,\varphi)\|_{C^n} = 0.$$

The necessity is proved.

Sufficiency. Suppose that the zero solution of (11.61) is absolutely stable w.r.t. $W_{nt}(t,\tilde{\varphi})$. According to the formula of variation of constants [42], any non-zero solution of (11.60) can be expressed as

$$\begin{aligned} u_t(t_0,\varphi) &= u(t,t_0)\varphi + \int_{t_0}^{t} u(t,s)X_0\, b\, f(s,\sigma_s)\mathrm{d}s \\ &= u(t,t_0)\varphi + \int_{t_0}^{t} u(t,t_0)X_0\, b\, f(s,W_{ns}(t_0,\tilde{\varphi})\mathrm{d}s, \end{aligned} \tag{11.63}$$

where $X_0 :\ [-r,0] \to B(x,x)$ is given by $X_0(0) = 0$, and $X_0(\theta) = I_d$ for $-r \le \theta \le 0$, where I_d is a unit matrix. Thus,

$$\|u_t(t_0,\varphi)\|_{C^n} \le Ke^{-\alpha(t-t_0)}\|\varphi\|_{C^n} + \int_{t_0}^{t} Ke^{-\alpha(t-s)}\|b\|\,|f(s,W_{ns}(t_0,\tilde{\varphi})|_X \mathrm{d}s.$$

$\forall \varepsilon > 0$, due to $W_{nt}(t_0,\hat{\varphi}) \to 0$ as $t \to +\infty$, there exists $t^* > t_0$ satisfying

$$\int_{t^*}^{t} Ke^{-\alpha(t-s)}\|b\|\,|f(s,W_{ns}(t_0,\tilde{\varphi})|_X \mathrm{d}s < \varepsilon/3.$$

Furthermore, we know that f is continuous, the solution of (11.60) is continuous w.r.t. the initial conditions, and its zero solution is absolutely stable w.r.t. $W_{ns}(t_0,\tilde{\varphi})$, then there exists $\delta_1(\varepsilon) > 0$ satisfying

$$\int_{t}^{t^*} Ke^{-\alpha(t-s)}\|b\|\,|f(s,W_{ns}(t_0,\hat{\varphi})|_X \mathrm{d}s < \varepsilon/3,$$

if $\|\varphi\|_{C^n} < \delta_1(\varepsilon)$ and $\|\tilde{\varphi}\|_{C^n}$ is sufficiently small. Let $\delta_2(\varepsilon) < \varepsilon/3k$, and $\delta(\varepsilon) = \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$, we have

$$\|u_t(t_0,\varphi)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \text{if} \quad \|\varphi\|_{C^n} < \delta(\varepsilon).$$

Thus, the zero solution of system (11.60) is stable.

To prove the zero solution of equation (11.60) being absolutely stable, we need to show

$$\|u_t(t_0,\varphi)\|_{C^n} \to 0, \quad \text{as} \quad t \to +\infty.$$

From the condition

$$\begin{aligned} u_t(t_0,\varphi) &= u(t,t_0)\varphi + \int_{t_0}^{t} u(t,s)X_0\, b\, f(s,\sigma_s)\mathrm{d}s \\ &= u(t,t_0)\varphi + \int_{t_0}^{t} u(t,s)X_0\, b\, f(s, W_{ns}(t_0,\tilde{\varphi}))\mathrm{d}s, \end{aligned}$$

we have

$$\begin{aligned} \|u_t(t_0,\varphi)\|_{\mathrm{C^n}} &\le Ke^{-\alpha(t-t_0)}\|\varphi\| + \int_{t_0}^{t} Ke^{-\alpha(t-s)}\|b\|\,|f(s,W_{ns}(t_0,\tilde{\varphi}))|_X \mathrm{d}s \\ &= Ke^{-\alpha(t-t_0)}\|\varphi\|_{\mathrm{C^n}} + \frac{\int_{t_0}^{t} Ke^{\alpha s}\|b\|\,|f(s,W_{ns}(t_0,\tilde{\varphi}))|_X \mathrm{d}s}{e^{\alpha t}}. \end{aligned}$$

Applying the L'Hospital's rule yields

$$\begin{aligned} 0 &\le \lim_{t\to+\infty} \|u_t(t_0,\varphi)\| \\ &\le \lim_{t\to+\infty} Ke^{-\alpha(t-t_0)}\|\varphi\|_{\mathrm{C^n}} + \lim_{t\to+\infty} \|b\| k \frac{\|f(t,W_{nt}(t_0,\hat{\varphi}))|}{\alpha} \\ &= 0. \end{aligned}$$

Thus, the zero solution of (11.60) is absolutely stable. □

12

Absolute Stability of Hopfield Neural Network

In this chapter, we first discuss the relationship between the stability of Hopfield neural network, Lyapunov stability, and the invariant principle in the sense of LaSalle. Next, we describe the connection and difference between the Hopfield neural network and the Lurie control systems with multiple nonlinear controllers. Then, we introduce the concept of absolute stability for neural networks, and present the sufficient and necessary conditions for two types of neural networks. Finally, we discuss various sufficient conditions for the absolute stability of Hopfield neural network. Partial materials are chosen Forti et al. [27, 28] and Kaskurewicz et al. [50] (Sects. 12.3 and 12.4), Liao et al. [87, 88, 90] (Sect. 12.5), and Liu [98] (Sect. 12.6).

12.1 Hopfield Neural Network

Recently, the study of neural networks and applications has attracted many researchers from different disciplines. Neural networks have special nonlinear structure and method of information processing, in analogy of human being's brain. Great success has been achieved in many different areas and may solve some difficult problems, which are difficult to be solved by conventional methods. In particular, the development of Hopfield neural network has motivated a new high tide in the study of neural network.

In the 1980s of last century, Hopfield and Tank [43] developed a new continuous neural network model – the Hopfield model, described by the differential equations:

$$
\begin{aligned}
C_i \dot{u}_i &= -\frac{u_i}{R_i} + \sum_{j=1}^{n} T_{ij} V_j + I_i\,, \\
V_i &= g_i(u_i),
\end{aligned}
\tag{12.1}
$$

where $T_{ij} = T_{ji}$ $(i, j = 1, 2, \ldots, n)$, the resister R_i and the capacitor C_i are parallel connected, simulating the output time constant of the ith biological neuron, the conductor T $(T_{ij})_{n \times n}$ is an $n \times n$ matrix, called weight matrix or connection matrix, describing the strength of connections between neurons; u_i is the input voltage to the ith neuron, and V_i is the output; and

$$V_i = g_i(u_i), \quad i = 1, 2, \ldots, n,$$

is the ith nonlinear, continuously differentiable and monotone increasing function, i.e., $g_i'(u_i) > 0$.

Under the symmetry $T_{ij} = T_{ji}$ $(i, j = 1, 2, \ldots, n)$, Hopfield constructed the following so-called computational energy function:

$$E(V) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij}V_iV_j - \sum_{i=1}^{n} V_iI_i + \sum_{i=1}^{n}\frac{1}{R_i}\int_0^{V_i} g_i^{-1}(s)\,\mathrm{d}s. \tag{12.2}$$

Differentiating E with respect to time t along the trajectory of (12.1) yields

$$\begin{aligned}
\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{(12.1)} &= \sum_{i=1}^{n}\frac{\partial E}{\partial V_i}\frac{\mathrm{d}V_i}{\mathrm{d}t} \\
&= \sum_{i=1}^{n}\Big[-\frac{1}{2}\sum_{j=1}^{n}T_{ij}V_j - \frac{1}{2}\sum_{j=1}^{n}T_{ji}V_j + \frac{u_i}{R_i} - I_i\Big]\frac{\mathrm{d}V_i}{\mathrm{d}t} \\
&= \sum_{i=1}^{n}\Big[-\frac{1}{2}\sum_{j=1}^{n}(T_{ij}-T_{ji})V_j - \Big(\sum_{j=1}^{n}T_{ij}V_j - \frac{u_i}{R_i} + I_i\Big)\Big]\frac{\mathrm{d}V_i}{\mathrm{d}t} \\
&= \sum_{i=1}^{n} -C_i\frac{\mathrm{d}u_i}{\mathrm{d}t}\frac{\mathrm{d}V_i}{\mathrm{d}t} \\
&= -\sum_{i=1}^{n}C_i\,(g_i(u_i))'\left(\frac{\mathrm{d}u_i}{\mathrm{d}t}\right)^2 \le 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{(12.1)} = 0 &\iff \frac{\mathrm{d}u_i}{\mathrm{d}t} = 0\ (i = 1, 2, \ldots, n) \\
&\iff -\frac{u_i}{R_i} + \sum_{j=1}^{n}T_{ij}V_j + I_i = 0\ (i = 1, 2, \ldots, n).
\end{aligned} \tag{12.3}$$

According to (12.3), we still cannot assure that the equilibirum point $u = u^*$ is stable in the sense of Lyapunov, as different solutions starting from different initial points may converge to different equilibrium point u^*. Also, we cannot conclude that u^* must be the minimal point of $E(v)$. It might be a reflection point since first-order derivative being zero is only the necessary (not sufficient) condition to have minimal value.

It should be noted that Hopfield's stability means "movements go to equilibrium," or "the attraction of the set for equilibrium points." It is different from Lyapunov stability. In Lyapunov stability theory, the equilibrium point is known, and the Lyapunov function and its derivative take opposite signs in the neighborhood of the equilibrium point, while the equilibrium point of Hopfield network is unknown, and it is not required that E and $\frac{\mathrm{d}E}{\mathrm{d}t}$ should have opposite signs.

LaSalle first proposed the LaSalle's invariant principle [58] that a solution $x(t, t_0, x_0)$ starting from some bounded region in R^n is kept bounded in this region, and one then constructs a Lyapunov function V such $\dot{V} \le 0$ to show that this solution is asymptotically approaches a large object (the largest invariant set). However, the LaSalle's principle does not mention stability. Therefore, it is necessary to distinguish the difference between the Hopfield stability, Lyapunov stability, and LaSalle invariant principle.

The advantage of Hopfield's method is based on the attraction of electronic network and dynamical systems and simulation of differential equations to quickly, automatically find the fixed points (or equilibrium points) of system (12.1). This provides new ideas in developing NN computers, which is the most difficult part encountered using other methodologies, and this is perhaps most attractive part of neural networks. More precisely, the stability of Hopfiled neural network is to use the interaction of equilibrium points to find the equilibrium points, rather than studying the stability of equilibrium points.

The stability of neural networks is one of the fundamental problems in both theoretical development and practical applications. It not only provides the theoretical basis for optimal computations, but is also fundamental in studying convergence of most learning and training methods, as these problems are finally transformed to ones of considering stability and attractive property.

12.2 Relation and Difference of Hopfield Neural Network and Lurie System

For convenience in comparison of Hopfield neural network and Lurie systems with multiple nonlinear controls, we introduce some transformations to system (12.1). Let

$$d_i = \frac{1}{C_i R_i}, \quad b_{ij} = \frac{T_{ij}}{C_i},$$

then the Hopfield neural network (12.1) becomes

$$\dot{u}_i = -d_i u_i + \sum_{i=1}^{n} b_{ij} g_j(u_j) + \frac{I_i}{C_i}. \tag{12.4}$$

The original network defines the function $g_j(u_j)$ as Sigmoidal type of function, i.e., $D^+ g_i(u_i) > 0$ and $|g_i(u_i)| \le K$. Now we consider a more general class function.

$$g_i \in F_{[0,k]} = \left\{ g_i | 0 < u_i g_i(u_i) < k_i u_i^2, \ g_i \text{ is continuous and } g_i(0) = 0 \right\}.$$

Therefore, the nonlinear functions in neural networks and Lurie control systems are basically same. More precisely, a nonlinear function in neural network requires that the function is monotone increasing and has its maximum slope at the origin. So strictly speaking, the nonlinear functions in neural networks belong to a subset of the nonlinear functions of Lurie control systems.

Let $u_i = u_i^*$ be an equilibrium point of (12.4), and

$$f(x_i) = g_i(u_i) - g_i(u_i^*) = g_i(x_i + u_i^*) - g_i(u_i^*),$$

then (12.4) can be rewritten as

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j). \tag{12.5}$$

Since $f_i(0) = g_i(u_i^*) - g_i(u_i^*) = 0$ and $f_i' = g_i' > 0$ $(i = 1,2,\ldots,n)$, we have

$$f_i \in F_{[0,k]} = \Big\{ f_i | 0 < x_i f_i \leq k_i x_i^2,\ \ f_i(0) = 0,\ \ f \text{ is continuous} \Big\}.$$

The above discussions give the following results.

1. Hopfield neural network is actually a special case of the more general Lurie systems with multiple nonlinear controls, described by

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} b_{ij} f_j(\sigma_j), \\ \sigma_j &= \sum_{i=1}^{n} c_i x_i, \end{aligned} \tag{12.6}$$

   in which letting $a_{ij} = 0,\ i \neq j,\ a_{ii} = -d_i < 0$, and $\sigma_j = x_j,\ c_j = 1, c_i = 0,\ i \neq j$ yields the Hopfield neural network (12.5).
2. The Lurie control system (12.6) does not require $a_{ii} < 0$ for all i, nor does the nonlinear function $f_j(\sigma_j)$ have to be monotone. It only requires $\sigma_i f(\sigma_i) > 0$, $\sigma \neq 0$ (i.e., the function is located in the first and third quadrants of the σ-$f(\sigma)$ plane), and is thus more general than the neural network (12.4).
3. However, the study of the Lurie systems with multiple nonlinear controls is far behind that with single control. As pointed out in a SIAM research report (1988) *The Future of Control Theory – Mathematical Prospect* that although the study on the stability of nonlinear control systems has received considerable attentions and many results have been obtained, most of achievements are limited to the systems with single control, such as Popov criterion, Lyapunov method, etc. The problem for the Lurie systems with multiple controls has not been solved with satisfactory. This indicates the difficulty in generalizing the study from the case of single control to multiple controls.
4. Note that the Hopfield neural network (12.4) contains an input I_i, while the input in the Lurie system (12.5) is zero. However, this input can be removed by a simple translation, and thus (12.4) becomes (12.5). Since the activation functions $g_i(u_i)$ and $f_i(x_i)$ are the same type of functions, the absolute stabilities of the equilibrium point $u = u^*$ of (12.4) and $x = 0$ of (12.5) are not different.
5. The development of neural network (12.5) provides a useful practical model for the study of Lurie systems with multiple controls, and will greatly promote the study of Lurie systems with multiple controls.

The study of the global stability of the neural network (12.5) has opened a new area for optimal computations. Therefore, it is natural to combine studies on the stability of neural networks and the absolute stability of Lurie control systems.

12.3 Sufficient and Necessary Conditions for Hopfield Neural Network

In this section, we introduce the definition of absolute stability of neural network and the sufficient and necessary conditions for the absolute stability of one type of most original Hopfield neural network [27, 28, 50].

The neural approach for solving optimization problems has attracted considerable attention in recent years. Some crucial drawbacks have seriously limited its applicability. one main drawback is that spurious suboptimal responses, because of the existence of many stable equilibrium points.

The main features that a neural optimizer of the Hopfield type should posses are as follows.

1. The interconnection matrix should be symmetric. The property of symmetry is indeed inherently related to the optimization capabilities.
2. There should be an unique equilibrium point, which is globally asymptotically stable (GAS), i.e., locally stable and attracting all trajectories of motion. GAS is a necessary property to avoid the presence of spurious responses and to guarantee convergence toward the global optimal solution.
3. The neural network should be absolutely stable (ABST).

By ABST, it is meant that there is a GAS equilibrium point for *every* neuron activation function belonging to the class S of Sigmoidal (i.e., bounded increasing) functions and for every constant input vector to the neural network. ABST is important because in practical problems the neuron activation is known to belong to the class S, but its shape is not specified exactly. ABST neural networks are best suited for optimization problems, being devoid of spurious responses for every choice of the activation function and of the input vector.

Consider the neural networks, described by the system of differential equations:

$$\dot{x} = -Dx + Tg(x) + I, \tag{12.7}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$ is the state variable, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is an $n \times n$ matrix with diagonal entries $d_i > 0,\ i = 1, 2, \ldots, n,\ T = [T_{ij}]$ is an $n \times n$ constant matrix, $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))^{\mathrm{T}} : R^n \to R^n$ is a nonlinear diagonal mapping and $I = (I_1, I_2, \ldots, I_n)^{\mathrm{T}} \in R^n$ is a constant vector.

According to the usual assumption on the neuron activations, we require that g belongs to the class S of Sigmoidal functions, defined by the property that $g \in S$ if for $i = 1, 2, \ldots, n,\ g_i(R) \equiv (a_i, b_i),\ a_i \in R,\ b_i \in R,\ a_i < b_i$, or expressed as $g_i \in C[R, (a_i, b_i)]$.

We will analyze the properties of global asymptotic stability and absolute stability, as defined below. An equilibrium point is a constant solution of (12.7) and hence satisfies the algebraic equation:

$$H(x) = -Dx + Tg(x) + I = 0. \tag{12.8}$$

For a given $g \in S$, let $x(t; t_0, x_0)$ denote the solution, which is uniquely determined by the initial condition $x_0 \in R^n$ at $t = t_0$. Under the hypothesis $g \in S$, any solution of (12.7) is ultimately bounded and hence defined for all $t \geq t_0$ (Lemma 12.3). Consider an equilibrium point x^* for (12.7).

Definition 12.1. *The equilibrium x^* is said to be GAS if it is locally stable in the sense of Lyapunov and globally attractive, where global attractive means that* $\lim_{t\to\infty} x(t;t_0,x_0) = x^*$, *for every initial condition $x_0 \in R^n$.*

Definition 12.2. *System (12.7) is said to be ABST if it possesses a GAS equilibrium point for every function $g \in S$ and for every input vector $I \in R^n$. When system (12.7) is ABST, then the vector field defined by the right-hand side of system (12.7) is said to be structurally stable.*

The concept of GAS (and that of ABST) involves a static aspect (i.e., the uniqueness of the equilibrium point, which is a necessary condition for GAS) and a dynamic aspect (i.e., local stability and global interactivity of the unique equilibrium).

We find it convenient to analyze the problem of uniqueness of the equilibrium point in the general case of both symmetric and nonsymmetric connection matrices.

We recall that P_0 denotes the class of square matrices A defined by one of the following equivalent properties:

(P_0-(i)) All principal minors of A are nonnegative
(P_0-(ii)) Every real eigenvalues of A as well as of each principal submatrix of A is nonnegative
(P_0-(iii)) $\det(K+A) \neq 0$ for every diagonal matrix $K = \mathrm{diag}(K_1,\dots,K_n)$ with $K_i > 0$, $i = 1,2,\dots,n$

Lemma 12.3. *If $g \in S$ and $-T \in P_0$, then the function H defined in (12.8) is diffeomorphism of R^n onto R^n, i.e., H is globally one to one onto and the inverse function H^{-1} is $C^1(R^n)$.*

Proof. We state that a C^1 function $H = (H_1,\dots,H_n)^{\mathrm{T}}$ is a diffeomorphism of R^n onto R^n if and only if: (1) $\det J_H(x) \neq 0$ for all $x \in R^n$ (J_H is the Jacobian of H) and: (2) $\lim_{\|x\|\to\infty} \|H(x)\| = +\infty$. The Jacobian of H is given by $J_H(x) = -D + T\,\mathrm{diag}(g_1'(x_1),\dots,g_n'(x_n))$. Since $-T \in P_0$, from P_0-(i), it easily follows that $-T\,\mathrm{diag}(g_1'(x_1),\dots,g_n'(x_n)) \in P_0$ for all $x \in R^n$. In fact $g_i'(x_i) > 0$ and the multiplication of $-T$ by a diagonal matrix with positive diagonal entries does not alter the sign of the principal minors of $-T$. Denote with P the class of matrices defined by the property that all of their principal minors are positive. Then we can obtain $D - T\,\mathrm{diag}(g_1'(x_1),\dots,g_n'(x_n)) \in P$. This implies that $\det(D - T\,\mathrm{diag}(g_1'(x_1),\dots,g_n'(x_n))) = -\det J_H(x) > 0$ for all $x \in R^n$ and conclusion (1) holds.

Note that since $g \in S$, then g is a bounded function and so is the function $T\,g + I$. Therefore, there exists $m > 0$ such that $\|T\,g(x) + I\| < m$, for all $x \in R^n$. Being D a diagonal matrix with positive diagonal entries, we have $\lim_{\|x\|\to\infty} \|-Dx\| = +\infty$. Then, $\|H(x)\| = \|-Dx + T\,g(x) + I\| \geq \|-Dx\| - m$, from which $\lim_{\|x\|\to\infty} \|H(x)\| = +\infty$ and hence also conclusion 2) is true. $\square$

Theorem 12.4. *System (12.7) has an unique equilibrium point for each $g \in S$ and for each $I \in R^n$ if and only if the connection matrix $-T \in P_0$.*

Proof. *Sufficiency*. From Lemma 12.3 the function H in (12.8) is globally one-to-one and onto. Then (12.8), $H(x) = 0$, has an unique solution and hence (12.7) has unique equilibrium point.

Necessity. Suppose now that $-T \notin P_0$. We show that we can find a $g \in S$ and a vector I for which (12.8) has more than one solution. If $-T \notin P_0$, from P_0-(i) also matrix $-D^{-1}T \notin P_0$, since D^{-1} is a diagonal matrix with positive diagonal entries. Hence, from P_0-(iii), there exists a diagonal matrix $\tilde{K} = \mathrm{diag}(\tilde{K}_1, \ldots, \tilde{K}_n)$ with $\tilde{K}_i > 0, i = 1, 2, \ldots, n$, for which $\det(\tilde{K} - D^{-1}T) = 0$. Since $\tilde{K} - D^{-1}T = D^{-1}(D\tilde{K} - T) = D^{-1}(D - T\tilde{K}^{-1})\tilde{K}$, it results in $\det(\tilde{K} - D^{-1}T) = \det(D^{-1})\det(D - T\tilde{K}^{-1})\det(\tilde{K})$. Being $\det(D^{-1}) \neq 0$ and also $\det(\tilde{K}) \neq 0$, we therefore obtain $\det(D - T\tilde{K}^{-1}) = 0$.

Now consider a function $\tilde{g}(x) = (\tilde{g}_1(x), \ldots, \tilde{g}_n(x))^{\mathrm{T}}$, which belongs to S and is such that $\tilde{g}(x) = \tilde{K}^{-1}x$, for $\|x\| \leq 1$. If we choose $I = 0$, the solutions of (12.8) in the set $\|x\| \leq 1$ are solutions of the linear system $-(D - T\tilde{K}^{-1})x = 0$. However, as $\det(D - T\tilde{K}^{-1}) = 0$, we get that for these choices of g and I, (12.8) has infinitely many solutions for $\|x\| \leq 1$. □

To prove the following Theorem 12.6, we need the following lemma.

Lemma 12.5. *Let $g \in S$; then any solution $x(t; t_0, x_0)$ of (12.7) is bounded and hence defined for $t \geq t_0$. It results in*

$$\|x(t; t_0, x_0)\| \leq \sqrt{n}\left(\|x_0\| + \frac{k\|T\|}{d_{\min}}\right) \qquad for \quad t \geq t_0,$$

where k is such that $\|g(x)\| < k$ for all $x \in R^n$ and $d_{\min} > 0$, $d_{\min} = \min\{d_1, \ldots, d_n\}$.

Proof. Let $x(t) = x(t; t_0, x_0)$ be solution of (12.7). By the variation of constants formula, $x(t)$ satisfies

$$x(t) = Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^{t} Y(t)Y^{-1}(s)\,T\,g(x(s))\,\mathrm{d}s, \tag{12.9}$$

where $Y(t)$ is any fundamental matrix of $\dot{y} = -Dy$. We can choose $Y(t) = \mathrm{diag}(e^{-d_1 t}, \ldots, e^{-d_n t})$. For $g \in S$, there exists $k > 0$ such that $\|g(x)\| \leq k$ for all $x \in R^n$. Therefore, from (12.9), and for $t > t_0$:

$$\|x(t)\| \leq \|Y(t)Y^{-1}(t_0)\|\,\|x_0\| + k\,\|T\| \int_{t_0}^{t} \|Y(t)Y^{-1}(s)\|\,\mathrm{d}s.$$

We have

$$\begin{aligned}\|Y(t)Y^{-1}(t_0)\| &= \left(\sum_{i=1}^{n} e^{-2d_i(t-t_0)}\right)^{1/2} \leq \left(n\sum_{i=1}^{n} e^{-2d_{\min}(t-t_0)}\right)^{1/2} \\ &= \sqrt{n}e^{-d_{\min}(t-t_0)}\end{aligned}$$

for all $t \geq t_0$, where $d_{\min} = \min\{d_1, \ldots, d_n\} > 0$. Similarly, we obtain

$$\|Y(t)Y^{-1}(s)\| \leq \sqrt{n}e^{-d_{\min}(t-s)}$$

for $t \geq s$. Therefore, for $t \geq t_0$,

$$\|x(t)\| \leq \sqrt{n}e^{-d_{\min}(t-t_0)}\|x_0\| + \sqrt{n}k\|T\| \int_{t_0}^{t} e^{-d_{\min}(t-s)}\,\mathrm{d}s,$$

which by integration yields the desired result, $\|x(t)\| \leq \sqrt{n}\left(\|x_0\| + \frac{k\|T\|}{d_{\min}}\right)$, for $t \geq t_0$. □

Theorem 12.6. *For a symmetric connection matrix T, system (12.7) is ABST if and only if $-T \in P_0$, i.e., if and only if T is negative semidefinite. When these conditions are satisfied, the Hopfield-Tank vector field defined by the right-hand side of (12.7) is structurally stable.*

Proof. *Sufficiency*. Suppose T is negative semidefinite. Fix $g \in S$ and $I \in R^n$ and let x_e be the unique equilibrium point of (12.7) on the basis of Theorem 12.4. With the coordinate change $z = x - x_e$, (12.7) can be transformed into the following system, having an unique equilibrium point at $z = 0$,

$$\dot{z} = -Dz + TG(x), \tag{12.10}$$

where $G(z) = (G_1(z_1), \ldots, G_n(z_n))^{\mathrm{T}} = g(z+x) - g(x) := g^*(z)$ and $g^*(0) = g(x) - g(x) = 0$. The function $G \in S$ and also $G(0) = 0$. Thus for (12.10) we can construct radially unbounded, positive definite Lyapunov function:

$$V = \sum_{i=1}^{n} \int_0^{z_i} G_i(\xi)\,\mathrm{d}\xi.$$

Then we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(12.10)} = -\sum_{i=1}^{n} d_i z_i G_i(\xi) + G^{\mathrm{T}}(z)TG(z) \leq -\sum_{i=1}^{n} d_i z_i G_i(z_i) < 0$$

when $\|z\| \neq 0$. Thus, the zero solution of (12.10) is absolutely stable, and so the equilibrium point $x = x^*$ of (12.7) is absolutely stable.

Necessity. Since a necessary condition for ABST is that there is an unique equilibrium point for every $g \in S$ and for every $I \in R^n$, the necessity part is a direct consequence of Theorem 12.4. The proof is completed. □

Since a necessary condition for GAS is that there is an unique equilibrium point, it readily follows from Theorem 12.4 that a necessary condition for ABST (valid both for symmetric and nonsymmetric connection matrices) is that $-T \in P_0$. The question arises whether $-T \in P_0$ is also sufficient for ABST.

For nonsymmetric matrices T, the condition $-T \in P_0$ is not in general sufficient for ABST, as the following example of a 3-neuron network shows. The example is relative to a case where for a given network with $-T \in P_0$, the unique equilibrium of (12.7) is unstable, so that ABST does not hold.

Example 12.7. Consider the third-order neural system of the type (12.7):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = -d \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 & -\alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{bmatrix} \begin{pmatrix} g_1(x_1) \\ g_2(x_2) \\ g_3(x_3) \end{pmatrix}, \tag{12.11}$$

where it is assumed that $D = \mathrm{diag}(d,d,d)$ with $d > 0$ and $I = 0$. Furthermore, $g(x) \in S$ and is such that $g_i(0) = 0$, $g_i'(0) = 1$, $i = 1,2,3$. Also assume that α, β, γ are positive constants. It can be verified that $-T \in P_0$, so that from Theorem 12.4, (12.11) has the origin $x = 0$ as its unique equilibrium.

Let us prove that for sufficiently small $d > 0$, $x = 0$ is unstable. In our hypothesis, the Jacobian at $x = 0$ is $J_H(0) = -D + T$. The matrix T has two complex conjugate eigenvalues with a positive real part $(\alpha\beta\gamma)^{1/3}/2$. Since the diagonal matrix $-D = -\mathrm{diag}(d,d,d)$ in $J_H(0)$ translates the eigenvalues of T by the negative quantity $-d$, we obtain that for $0 < d < (\alpha\beta\gamma)^{1/3}/2$, $J_H(0)$ has two complex eigenvalues with positive real part. Hence $x = 0$ is unstable and (12.11) is not ABST.

Different from the nonsymmetric case, for symmetric matrices T, the fact that $-T \in P_0$ (which is equivalent on the basis of P_0-(iii) to the fact that T is negative semidefinite) characterizes both the static and dynamical property of ABST, as the following result shows.

Since the above argument is true for any choices of $g \in S$ and for any vector $I \in R^n$, we have proved that (12.10) is ABST.

In the following, we introduce a sufficient condition, obtained by Kaskurewicz and Bhaya [50], for the absolute stability, which is an improved proof for Theorem 12.8.

Theorem 12.8. [50] *If $T \in D_0$, i.e., there exists a positive diagonal matrix $Q =$ diga$(q_1, \cdots, q_n)$ satisfying $QT + T^{\mathrm{T}}Q \le 0$, then system (12.7) is ABST w.r.t. $g \in S$. Furthermore, the Hopfield-Tank vector filed, defined by the right-hand side of (12.7) is structurally stable.*

Proof. Since $T \in D_0$ implies $-T \in P_0$, it follows that x^* is the unique equilibrium point of (12.7). We now use the radially unbounded, positive definite Lyapunov function:

$$V(z) = 2\sum_{i=1}^{n} q_i \int_0^{z_i} G_i(\xi)\,\mathrm{d}\xi.$$

Along the trajectories of (12.7), the time derivative of $V(z)$ is given by

$$\frac{\mathrm{d}V(z)}{\mathrm{d}t} = G(z)\,(QT + T^{\mathrm{T}}Q)\,G(z) - G(z)\,QDz. \tag{12.12}$$

Since $T \in D_0$, there exists a positive diagonal matrix $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n)$ such that the first term on the right-hand side of (12.12) is negative semidefinite and $G(z)QDz$ is negative definite for all $G \in S$. This ensures that system (12.7) is ABST w.r.t. $g \in S$, and the Hopfield-Tank Vector field defined by the right-hand side of (12.7) is structurally stable. □

12.4 Absolute Stability of Cooperative Hopfield Neural Network

In Sect. 12.3, we have introduced the sufficient and necessary conditions for the absolute stability of a special type of Hopfield neural network with weighted matrices [28]. More precisely, these conditions are sufficient conditions as additional constraints are posed. Therefore, we should look for other sufficient and necessary conditions without these constraints, but under some other different constraints.

According to the definition of absolute stability, we first need to prove the existence and uniqueness of equilibrium point for any $g_i \in S$ and any input I_i, and then prove that it is globally asymptotically stable. Proving the existence and uniqueness of the equilibrium point belongs to functional analysis and computational mathematics, and is usually difficult. However, because of the special property of the Sigmoidal function, the existence of the equilibrium point can be shown using the Brown fixed point theory, while the uniqueness of the equilibrium point is the necessary condition for the global asymptotic stability. Thus, once the global asymptotic stability is asserted, the uniqueness of the equilibrium point is also proved. When non-Sigmoidal functions are used as activation, the situation is different. Therefore, next we will consider new sufficient and necessary conditions for Hopfield neural network.

In the previous section, the weight matrix T is assumed symmetric. In this section, we introduce the sufficient and necessary conditions for the absolute stability of cooperative Hopfield neural network. Again, consider system (12.7):

$$\dot{x} = -Dx + T\,g(x) + I, \tag{12.13}$$

where the definitions for D, g, and I are the same as that defined in the previous section. The only different is for T, which now has different constraints, defined below.

Definition 12.9. *If in the Hopfield neural network (12.13), the off-diagonal elements of the weighted matrix T are nonnegative, i.e., $T_{ij} \geq 0,\ i \neq j$, then system (12.13) is called cooperative neural network.*

Note that T defined above can be either symmetric or nonsymmetric. Thus what to be discussed in this section is not covered in the previous section. Cooperative neural networks are widely employed for cooperative learning and other computational tasks.

Let $g \in S$ and $|g(x)| \leq k|x|,\ k > 0$.

Theorem 12.10. *For a cooperative Hopfield neural network (12.13) to be ABST if and only if $-T \in K_0$, where K_0 denotes the class of (symmetric or nonsymmetric) matrices with nonpositive off-diagonal entries and with nonnegative principal minors.*

Proof. *Sufficiency*. Assume $-T \in K_0$. Fix any function $g \in S$ and any vector $I \in R^n$. Since $-T \in K_0$ implies $-T \in P_0$, system (12.13) has a unique equilibrium point x^*.

By means of the coordinate change $z = x - x^*$, system (12.13) can be transformed into the following system, having an unique equilibrium at $z = 0$,

$$\dot{z} = -Dz + TG(z), \tag{12.14}$$

where $(G(z) = (G_1(z_1),\ldots,G_n(z_n))^{\mathrm{T}} = g(z+x^*) - g(x^*) := g^*(z)$ and $g^*(0) = g(x^*) - g(x^*) = 0$. The function $G \in S$ and also $G(0) = 0$.

For $G \in S$, there exist k_i, $0 < k_i < \infty$, $i = 1,\ldots,n$, such that $|G_i(z_i)| \le k_i z_i$. Choose a positive ε satisfying $\varepsilon < d_i/k_i$ and consider the functions

$$\mathscr{D}_i(z_i) = d_i z_i - \varepsilon G_i(z_i) \qquad i = 1,\ldots,n.$$

Now rewrite system (12.13) by adding and subtracting the same term $\varepsilon G(z)$ to obtain (I_n is the identity matrix)

$$\dot{z} = -(Dz - \varepsilon G(z)) + (T - \varepsilon I_n)G(z) = -\mathscr{D}(z) + (T - \varepsilon I_n)G(z). \tag{12.15}$$

Since $-T \in K_0$, then $-T + \varepsilon I_n \in K$. From these we get that $z = 0$ is GAS for (12.15) and hence that x^* is GAS for (12.13).

Necessity. Since a necessary condition for ABST is that there exists a unique equilibrium point for all $g \in S$ and for all $I \in R^n$, the necessity part is an immediate consequence of Theorem 12.6 and the fact that for a cooperative neural network, $-T \in K_0$ is equivalent to $-T \in P_0$. □

12.5 Sufficient Conditions for Absolutely Exponential Stability

In this section, we turn to investigate the sufficient conditions for the absolutely exponential stability of Hopfield neural networks [87]. First, we consider the Hopfield neural networks with finite gains, and then that with finite gains.

12.5.1 Hopfield Neural Networks with Finite Gains

Consider a model of Hopfield neural networks, given by

$$C_i \dot{u}_i = \sum_{j=1}^{n} T_{ij} g_j(u_j) - \frac{u_i}{R_i} + I_i, \tag{12.16}$$

where C_i, T_{ij}, R_i, I_i are the physical parameters depicted in [43]. The circuit scheme of the neural networks is omitted here. For the system (12.16), we merely assume that the activation function $g_i(u_i)$ belong to class S of sigmoidal function, defined by the property that $g_i(R) \equiv (a_i, b_i)$, $a_i \in R$, $b_i \in R$, $a_i < b_i$, but suppose that the gains are finite, i.e., $\sup D^+ g_i(u_i) := M_i < +\infty$. The matrix $T = (T_{ij})_{n\times n}$ needs not to be symmetric. Here, we assume that system (12.16) has a finite equilibrium point $u = u^*$.

Let $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ and the equilibrium be $u^* = (u_1^*, \cdots, u_n^*)^{\mathrm{T}}$, and $S_B = \{g \in S, 0 \le D^+ g_i(u_i) \le M\}$. It is obvious that $S_B \subset S$.

Definition 12.11. *If* $\forall I_i, \forall g_i \in S_B,\ i = 1,\cdots,n$, *the equilibrium solution of (12.16) is globally exponentially stable, then (12.16) is said to be absolutely exponentially stable w.r.t.* S_B.

In the following, we assume

$$\Omega_1 = \operatorname{diag}\left(\frac{1}{R_1} - T_{11}M_1, \cdots, \frac{1}{R_n} - T_{nn}M_n\right)_{n\times n} - (\sigma_{ij}|T_{ij}|M_j)_{n\times n}$$
$$\Omega_2 = \operatorname{diag}\left(\frac{-1}{R_1M_1}, \cdots, \frac{-1}{R_nM_n}\right) + (T_{ij})_{n\times n},$$

where $\sigma_{ij} = 1 - \delta_{ij}$, δ_{ij} denotes the Kronecker operator.

Theorem 12.12. *If the matrix* Ω_1 *is an M-matrix, then (12.16) is absolutely exponentially stable.*

Proof. Since the Sigmoidal function is bounded, thus we can apply the Brouwer fixed point theorem to prove that for any I and $g \in S_B$, the equilibrium solution of (12.16) exists.

Assume $u^* = (u_1^*, \cdots, u_n^*)^{\mathrm{T}}$ is the equilibrium point of (12.16) and $x = (x_1, \cdots, x_n)^{\mathrm{T}} = (u_1 - u_1^*, \cdots, u_n - u_n^*)^{\mathrm{T}},\ f(x_i) = g_i(u_i) - g_i(u_i^*) = g_i(x_i + u_i^*) - g_i(u_i^*),\ i = 1,2,\ldots,n$, then (12.16) changes into

$$C_i\dot{x}_i = \sum_{j=1}^{n} T_{ij}f_j(x_j) - \frac{x_i}{R_i},\ (i = 1,\cdots,n). \tag{12.17}$$

Since Ω_1 is an M-matrix, there exists a group of constants $\xi = (\xi_1, \cdots, \xi_n)^{\mathrm{T}} > 0$, such that

$$\xi_j\left(\frac{-1}{R_j} + T_{jj}M_j\right) + \sum_{i=1}^{n} \xi_i\sigma_{ij}|T_{ij}|M_j < 0,\ j = 1,\cdots,n.$$

Without loss of generality, we assume

$$\xi_jT_{jj} + \sum_{i=1}^{n} \xi_i\sigma_{ij}|T_{ij}| \le 0, \quad j = 1,\cdots,n_0,\ 1 <\le n_0 < n,$$
$$\xi_jT_{jj} + \sum_{i=1}^{n} \xi_i\sigma_{ij}|T_{ij}| > 0, \quad j = n_0 + 1,\cdots,n.$$

Let $\lambda = \min\limits_{1\le k\le n_0, n_0+1\le j\le n}\left[\frac{1}{R_kC_k}, \left(\frac{\xi_j}{R_j} - \xi_jT_{jj}M_j - \sum_{i=1}^{n}\xi_i\sigma_{ij}|T_{ij}|M_j\right)\frac{1}{\xi_jC_j}\right]$.

Constructing the radically unbounded, positive definite Lyapunov function $V(x) = \sum_{i=1}^{n} \xi_iC_i|x_i|$, we find that the right upper Dini derivative of V along the solution to system (12.17) satisfies

$$
\begin{aligned}
D^+V(x)|_{(12.17)} &= \sum_{i=1}^{n} \xi_i C_i \dot{x}_i \operatorname{sgn} x_i \\
&\le \sum_{i=1}^{n} \xi_i \Big[-\frac{1}{R_i}|x_i| + T_{ii}|f_i(x_i)| + \sum_{j=1}^{n} \sigma_{ij}|T_{ij}||f_j(x_j)|\Big] \\
&\le \sum_{j=1}^{n_0} \Big(-\frac{\xi_j}{R_j}\Big)|x_j| + \sum_{j=1}^{n_0} \Big(\xi_j T_{jj} + \sum_{i=1}^{n} \sigma_{ij}|T_{ij}|\Big)|f_j(x_j)| \\
&\quad + \sum_{j=n_0+1}^{n} \Big(-\frac{\xi_j}{R_j}\Big)|x_j| + \sum_{j=n_0+1}^{n} \Big(\xi_j T_{jj} + \sum_{i=1}^{n} \sigma_{ij}|T_{ij}|\Big) M_j |x_j| \\
&\le \sum_{j=1}^{n_0} \Big(-\frac{\xi_j}{R_j}\Big)|x_j| + \sum_{j=n_0+1}^{n} \Big[-\frac{\xi_j}{R_j} + \xi_j T_{jj} M_j + \sum_{i=1}^{n} \sigma_{ij}\xi_i M_j\Big]|x_j| \\
&\le -\lambda V(x).
\end{aligned} \tag{12.18}
$$

Then, $0 \le V(x(t)) \le V_1(x(0))e^{-\varepsilon t}$, and furthermore, we have

$$
|x_i| \le \frac{1}{\min\limits_{1\le i\le n} \xi_i C_i} V(x(0))e^{-\lambda t}.
$$

It implies that the equilibrium of system (12.17) is globally exponentially stable with Lyapunov exponent $-\lambda$, i.e., system (12.17) is absolutely exponentially stable w.r.t. S_B with Lyapunov exponent $-\lambda$. □

Theorem 12.13. *If the matrix Ω_2 is Lyapunov-Volterra stable or called diagonal stable, then system (12.16) is absolutely exponentially stable.*

Proof. The Lyapunov-Volterra stability of matrix Ω_2 implies that there exists a positive definite diagonal matrix $H = \mathrm{diag}(h_1, \cdots, h_n)$ such that $Q = \frac{1}{2}(H\Omega_2 + \Omega_2^{\mathrm{T}} H)$ is negative definite, (Ω_2^{T} denotes the transpose of Ω_2). Let $\lambda \in \Big(0, \min\limits_{1\le i\le n} \frac{1}{R_i C_i}\Big)$ be the maximum positive solution to the following problem:

$$
Q + \lambda\, \mathrm{diag}\Big(\frac{C_1 h_1}{M_1}, \cdots, \frac{C_n h_n}{M_n}\Big) \quad \text{is negative semidefinite.}
$$

Construct Lyapunov function

$$
V(x) = \sum_{i=1}^{n} C_i h_i \int_0^{x_i} f_i(x_i)\mathrm{d}x_i. \tag{12.19}
$$

Obviously, $V(x)$ is positive definite. Now we verify its radical unboundedness. Let $\underline{m}_i = \min\limits_{|x|\le 1} |f_i'(x)|$, $\overline{m}_i = \min[f_i(1), f_i(-1)]$. For arbitrary $x \in \mathrm{R}^n$, without loss of generality, we assume that

$$
|x_i| \le 1, i = 1, \cdots, l_0, \quad |x_i| > 1, i = l_0 + 1, \cdots, n
$$

It follows from the monotone property of $f_i(x_i)$ that

$$V(x)=\sum_{i=1}^{n}C_ih_i\int_0^{x_i}f_i(x_i)\mathrm{d}x_i \geq \sum_{i=1}^{l_0}\frac{1}{2}C_ih_i\underline{m}_ix_i^2+\sum_{l_0=1}^{n}C_ih_i\overline{m}_i|x_i| \to +\infty$$

as $|x|\to+\infty$, indicating that V is radially unbounded.

Also from the monotone property of the function $f_i(x_i)$, we obtain an estimation of the derivative of the Lyapunov function $e^{\lambda t}V$ along the solution to system (12.16), given below:

$$\begin{aligned}
\frac{\mathrm{d}e^{\lambda t}V}{\mathrm{d}t} &= \lambda e^{\lambda t}\sum_{i=1}^{n}C_ih_i\int_0^{x_i}f_i(x_i)\mathrm{d}x_i+e^{\lambda t}\sum_{i=1}^{n}C_ih_if_i(x_i)\frac{\mathrm{d}x_i}{\mathrm{d}t}\\
&\leq e^{\lambda t}\Big[\lambda\sum_{i=1}^{n}C_ih_ix_if_i(x_i)-\sum_{i=1}^{n}\frac{h_i}{R_i}x_if_i(x_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}h_if_i(x_i)T_{ij}f_j(x_j)\Big]\\
&= e^{\lambda t}\Big[-\sum_{i=1}^{n}(\frac{h_i}{R_i}-C_ih_i\lambda)x_if_i(x_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}h_if_i(x_i)T_{ij}f_j(x_j)\Big]\\
&= e^{\lambda t}\Big[-\sum_{i=1}^{n}(\frac{h_i}{R_iM_i}-\frac{C_ih_i\lambda}{M_i})f_i^2(x_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}h_if_i(x_i)T_{ij}f_j(x_j)\Big]\\
&= e^{\lambda t}\left\{\begin{pmatrix}f_1(x_1)\\ \vdots\\ f_n(x_n)\end{pmatrix}^{\mathrm{T}}\left[\frac{1}{2}(H\Omega_2+\Omega_2^{\mathrm{T}}H)+\lambda\,\mathrm{diag}\left(\frac{C_1h_1}{M_1},\cdots,\frac{C_nh_n}{M_n}\right)\right]\right.\\
&\qquad\left.\times\begin{pmatrix}f_1(x_1)\\ \vdots\\ f_n(x_n)\end{pmatrix}\right\}\\
&= e^{\lambda t}\left\{\begin{pmatrix}f_1(x_1)\\ \vdots\\ f_n(x_n)\end{pmatrix}^{\mathrm{T}}\left[Q+\lambda\,\mathrm{diag}\left(\frac{C_1h_1}{M_1},\cdots,\frac{C_nh_n}{M_n}\right)\right]\begin{pmatrix}f_1(x_1)\\ \vdots\\ f_n(x_n)\end{pmatrix}\right\}\\
&\leq 0.
\end{aligned}\tag{12.20}$$

Integrating the inequality (12.20) from 0 to t, we have

$$V(x(t))\leq e^{-\lambda t}V(x(0)):=e^{-\lambda t}V_0.$$

That is,

$$\sum_{i=1}^{l_0}C_ih_i\underline{m}_ix_i^2+\sum_{i=l_0+1}^{n}\frac{1}{2}C_i\,h_i\,\overline{m}_i|x_i|\leq e^{-\lambda t}V_0. \tag{12.21}$$

It follows from (12.21) that

$$|x_i(t)| \le \sqrt{\frac{2}{C_i\xi_i\underline{m}_i}}\sqrt{V_0}e^{-\frac{\lambda}{2}t}, \quad 1 \le i \le l_0,$$
$$|x_i(t)| \le \frac{1}{C_i\xi_i\overline{m}_i}V_0e^{-\lambda t} \le \frac{1}{C_i\xi_i\overline{m}_i}V_0e^{-\dfrac{\lambda}{2}t}, \quad l_0+1 \le i \le n.$$

Setting

$$K = \max_{\substack{1\le i\le l_0 \\ l_0+1\le j\le n}} \left[\sqrt{\tfrac{2}{C_i\xi_i\underline{m}_i}}V_0,\ \frac{1}{C_j\xi_j\overline{m}_j}V_0\right],$$

we obtain $|x_i(t)| \le ke^{-\frac{\lambda}{2}t}$, $i = 1,\cdots,n$, implying that the equilibrium $u = u^*$ is absolutely exponentially stable. The proof is complete. □

12.5.2 Hopfield Neural Networks with Infinite Gains

Now, we consider the sufficient conditions for the absolutely exponential stability of Hopfield neural networks with a class of sigmoidal nonlinear activation function (i.e., the activation function with unbounded gains).

Let $S_{UB} := \{g(x)|g(x) \in C[R,R], D^+g_i(u_i) \ge 0, i = 1,\cdots,n\}$, i.e., $\sup D^+g_i(u_i) \le +\infty$. Consider (12.16) with $g_i(u_i) \in S_{UB}$, it is obvious that $S_B \subset S_{UB}$. If we still use the definition 12.11, $\forall g_i \in S_{UB}$, $\forall I_i \in R$, if the equilibrium of (12.16) is globally exponentially stable, then (12.16) is absolutely exponentially stable w.r.t. S_{UB}.

Theorem 12.14. *If there exist two groups of constants $\xi_i > 0, \eta_i > 0, i = 1,\cdots,n$ such that the matrix A is negative definite, then the following conclusions hold:*

1. System (12.16) is absolutely exponentially stable
2. The Lyapunov exponent is $-\dfrac{\lambda}{\mu}$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{\mathrm{T}} & A_{22} \end{bmatrix}_{2n\times 2n},$$
$$A_{11} = \mathrm{diag}(-\frac{\xi_1}{R_1},\cdots,-\frac{\xi_n}{R_n}),$$
$$A_{12} = \mathrm{diag}(-\frac{\eta_1}{2R_1},\cdots,-\frac{\eta_n}{2R_n})_{n\times n} + (\frac{\xi_iT_{ij}+\xi_jT_{ji}}{2})_{n\times n}, \quad i,j = 1,2,\ldots,n,$$
$$A_{22} = (\frac{\eta_iT_{ij}+\eta_jT_{ji}}{2})_{n\times n}, \quad i,j = 1,2,\ldots,n,$$
$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^{\mathrm{T}} & B_{22} \end{bmatrix}_{2n\times 2n},$$

$$B_{11} = \mathrm{diag}(\frac{C_1\xi_1}{2}, \cdots, \frac{C_n\xi_n}{2})_{n\times n},$$
$$B_{12} = B_{12}^{\mathrm{T}} = \mathrm{diag}(\frac{C_1\eta_1}{2}, \cdots, \frac{C_n\eta_n}{2})_{n\times n},$$
$$B_{22} = O_{n\times n}.$$

Here, $-\lambda$ denotes the maximum eigenvalue of matrix A and μ is the maximum eigenvalue of matrix B, the superscript T represents transpose.

Proof.

1. Let $x = (x_1, \cdots, x_n)^{\mathrm{T}} = (u_1 - u_1^*, \cdots, u_n - u_n^*)^{\mathrm{T}}$. Then,

$$f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*).$$

and system (12.16) can be rewritten as

$$C_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} T_{ij} f_j(x_j) - \frac{x_i}{R_i} \quad (i = 1, \cdots, n). \tag{12.22}$$

The global stability of the equilibrium $u = u^*$ of system (12.16) is equivalent to that of the equilibrium $x = 0$ of system (12.22).

Construct the radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \frac{C_i\xi_i}{2} x_i^2 + \sum_{i=1}^{n} \eta_i C_i \int_0^{x_i} f_i(x_i)\mathrm{d}x_i. \tag{12.23}$$

It is obvious to see that $V(0) = 0$, $V(x) > 0$, for $x \neq 0$, and

$$V(x) \geq \sum_{i=1}^{n} \frac{C_i\xi_i}{2} x_i^2 \to +\infty \quad \text{as} \quad |x| \to +\infty.$$

Thus, $V(x)$ is radially unbounded and positive definite.

The derivative of $V(x)$ along the solution of system (12.22) is

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(12.22)} &= -\sum_{i=1}^{n} \frac{\xi_i}{R_i} x_i^2 + \sum_{i=1}^{n} \xi_i x_i \sum_{j=1}^{n} T_{ij} f_j(x_j) \\
&\quad - \sum_{i=1}^{n} \frac{\eta_i x_i}{R_i} f_i(x_i) + \sum_{i=1}^{n} \eta_i \sum_{j=1}^{n} T_{ij} f_j(x_j) f_i(x_i) \\
&= \begin{pmatrix} x \\ f(x) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{\mathrm{T}} & A_{22} \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix}.
\end{aligned} \tag{12.24}$$

2. Set $V^* = e^{\varepsilon t} V$, $\varepsilon > 0$ is a sufficiently small constant. Then we obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}V^*}{\mathrm{d}t}\right|_{(12.22)} &= \varepsilon e^{\varepsilon t} V + e^{\varepsilon t} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(12.22)} \\
&= e^{\varepsilon t} \left\{ \varepsilon V + \begin{pmatrix} x \\ f(x) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{\mathrm{T}} & A_{22} \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= e^{\varepsilon t}\left\{\varepsilon\left(\sum_{i=1}^{n}\frac{C_i\xi_i}{2}x_i^2+\sum_{i=1}^{n}\eta_i C_i\int_0^{x_i}f_i(x_i)\mathrm{d}x_i\right)\right.\\
&\qquad\left.+\begin{pmatrix} x\\ f(x)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix} A_{11} & A_{12}\\ A_{12}^{\mathrm{T}} & A_{22}\end{bmatrix}\begin{pmatrix} x\\ f(x)\end{pmatrix}\right\}\\
&\le e^{\varepsilon t}\left\{\varepsilon\left(\sum_{i=1}^{n}\frac{C_i\xi_i}{2}x_i^2+\sum_{i=1}^{n}\eta_i C_i x_i f_i(x_i)\right)\right.\\
&\qquad\left.+\begin{pmatrix} x\\ f(x)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} A_{11} & A_{12}\\ A_{12}^{\mathrm{T}} & A_{22}\end{pmatrix}\begin{pmatrix} x\\ f(x)\end{pmatrix}\right\}\\
&= e^{\varepsilon t}\left\{\begin{pmatrix} x\\ f(x)\end{pmatrix}^{T}\begin{bmatrix} A_{11}+B_{11}\varepsilon & A_{12}+B_{12}\varepsilon\\ A_{12}^{\mathrm{T}}+B_{12}^{\mathrm{T}}\varepsilon & A_{22}+B_{22}\varepsilon\end{bmatrix}\begin{pmatrix} x\\ f(x)\end{pmatrix}\right\}. \qquad (12.25)
\end{aligned}
$$

Since the eigenvalues continuously depend on the elements of a matrix, the negative definite property of matrix $\begin{bmatrix} A_{11} & A_{12}\\ A_{12}^{\mathrm{T}} & A_{22}\end{bmatrix}$ implies the same property of $\begin{bmatrix} A_{11}+B_{11}\varepsilon & A_{12}+B_{12}\varepsilon\\ A_{12}^{\mathrm{T}}+B_{12}^{\mathrm{T}}\varepsilon & A_{22}+B_{22}\varepsilon\end{bmatrix}$ when $0<\varepsilon\ll 1$. It follows that

$$
\left.\frac{\mathrm{d}V^*}{\mathrm{d}t}\right|_{(12.22)}\le 0 \quad \text{for} \quad 0<\varepsilon\ll 1. \qquad (12.26)
$$

Integrating (12.26) from 0 to arbitrary t yields

$$
e^{\varepsilon t}V(x(t))=V^*(x(t))\le V^*(x(0)):=V_0^*.
$$

Hence, we have

$$
\sum_{i=1}^{n}\frac{C_i\xi_i}{2}x_i^2\le V(x(t))\le e^{-\varepsilon t}V_0^*, \qquad (12.27)
$$

and then

$$
\sum_{i=1}^{n}x_i^2\le\frac{V_0^*e^{-\varepsilon t}}{\min\limits_{1\le i\le n}\frac{C_i\xi_i}{2}} \quad (\varepsilon>0). \qquad (12.28)
$$

Inequality (12.28) implies that the equilibrium $x=0$ of system (12.22) (or the equilibrium point $u=u^*$ of system (12.16)) is globally exponentially stable, i.e., system (12.16)) is absolutely exponentially stable. Also, the positive constant ε can be considered as a Lyapunov exponent.

3. Let $-\lambda$ and μ be the maximum eigenvalues of matrices A and B, respectively. From the argument of inequality (12.25), we have

$$
\begin{aligned}
\left.\frac{\mathrm{d}V^*}{\mathrm{d}t}\right|_{(12.22)}&\le e^{\varepsilon t}\left\{\begin{pmatrix} x\\ f(x)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix} A_{11} & A_{12}\\ A_{12}^{\mathrm{T}} & A_{22}\end{bmatrix}\begin{pmatrix} x\\ f(x)\end{pmatrix}\right\}\\
&\quad+e^{\varepsilon t}\left\{\varepsilon\left(\sum_{i=1}^{n}\frac{C_i\xi_i}{2}x_i^2+\sum_{i=1}^{n}\eta_i C_i x_i f_i(x_i)\right)\right\}
\end{aligned}
$$

$$\leq e^{\varepsilon t}\left\{-\lambda\left(\sum_{i=1}^{n}x_i^2+\sum_{i=1}^{n}f_i^2(x_i)\right)\right\}$$
$$+ e^{\varepsilon t}\left\{\varepsilon\begin{pmatrix} x \\ f(x)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^{\mathrm{T}} & 0\end{bmatrix}\begin{pmatrix} x \\ f(x)\end{pmatrix}\right\}$$
$$\leq e^{\varepsilon t}[-\lambda+\varepsilon\mu]\left(\sum_{i=1}^{n}x_i^2+\sum_{i=1}^{n}f_i^2(x_i)\right)\leq 0. \tag{12.29}$$

Thus, the inequality (12.27) holds and $\varepsilon = \dfrac{\lambda}{\mu}$ is an estimation of the Lyapunov exponent.

The proof is complete. □

Set $\overset{\circ}{A} = \begin{bmatrix} \overset{\circ}{A}_{11} & \overset{\circ}{A}_{12} \\ \overset{\circ}{A}_{21} & \overset{\circ}{A}_{22}\end{bmatrix}$, where

$$\begin{aligned}
\overset{\circ}{A}_{11} &= -\operatorname{diag}\Big(\frac{1}{R_1},\cdots,\frac{1}{R_n}\Big),\\
\overset{\circ}{A}_{12} &= -\operatorname{diag}\Big(\frac{1}{2R_1},\cdots,\frac{1}{2R_n}\Big)_{n\times n}+\Big(\frac{T_{ij}+T_{ji}}{2}\Big)_{n\times n},\\
\overset{\circ}{A}_{22} &= \Big(\frac{T_{ij}+T_{ji}}{2}\Big)_{n\times n},
\end{aligned}$$

and set $\overset{\circ}{B} = \begin{bmatrix} \overset{\circ}{B}_{11} & \overset{\circ}{B}_{12} \\ \overset{\circ}{B}_{21} & \overset{\circ}{B}_{22}\end{bmatrix}_{2n\times 2n}$,

$$\begin{aligned}
\overset{\circ}{B}_{11} &= = \operatorname{diag}\Big(\frac{C_1}{2},\cdots,\frac{C_n}{2}\Big)_{n\times n},\\
\overset{\circ}{B}_{12} &= \overset{\circ}{B}{}_{12}^{\mathrm{T}} = \operatorname{diag}\Big(\frac{C_1}{2},\cdots,\frac{C_n}{2}\Big)_{n\times n}, \overset{\circ}{B}_{22} = O_{n\times n}.
\end{aligned}$$

Corollary 12.15. *If the matrix $\overset{\circ}{A}$ is negative definite, then the equilibrium $u = u^*$ of system (12.16) is unique and globally exponentially stable, and the Lyapunov exponent can be chosen as $\varepsilon = \frac{\tilde{\lambda}}{\tilde{\mu}}$, where $-\tilde{\lambda}$ and $\tilde{\mu}$ are the maximum eigenvalues of the matrices $\overset{\circ}{A}$ and $\overset{\circ}{B}$, respectively.*

Proof. Taking $\xi_i = \eta_i = 1,\ i = 1,\cdots,n$ in Theorem 12.14 proves this corollary. □

Theorem 12.16. *If there exist two groups of constants $\xi_i > 0, \eta_i > 0,\ i = 1,\cdots,n$ such that the matrix G is negative definite, then the conclusion of theorem 12.14 holds, where*

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^{\mathrm{T}} & G_{22} \end{bmatrix}_{2n\times 2n}, \qquad G_{11} = -\operatorname{diag}\left(\frac{\xi_1}{R_1}, \cdots, \frac{\xi_n}{R_n}\right),$$

$$G_{12} = G_{12}^{\mathrm{T}} = \left((1-\delta_{ij})\frac{\xi_i T_{ij} + \xi_j T_{ji}}{2}\right)_{n\times n}, \quad G_{22} = \left(\frac{\eta_i T_{ij} + \eta_j T_{ji}}{2}\right)_{n\times n},$$

$$\delta_{ij} = \begin{cases} 1 \text{ when } i = j, \\ 0 \text{ when } i \neq j. \end{cases}$$

The estimation (12.28) is also satisfied, where $\varepsilon = \min\left[\min_{1\le i\le n} \frac{\frac{\eta_i}{R_i} - \xi_i T_{ii}}{\eta_i C_i}, \frac{\lambda^*}{\mu^*}\right]$, $-\lambda^*$ *is the maximum eigenvalue of the matrix* G, *and* $\mu^* = \max_{1\le i\le n} \frac{C_i \xi_i}{2}$.

Proof. From the negative definite property of the matrix G, we know that $T_{ii} < 0,\ i = 1,\cdots,n$. Using the Lyapunov function (12.23), analogous to the argument for equation (12.25), we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(12.22)} = \begin{pmatrix} x \\ f(x) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^{\mathrm{T}} & G_{22} \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} + \sum_{i=1}^{n} \xi_i T_{ii} x_i f_i(x_i) - \sum_{i=1}^{n} \frac{\eta_i}{R_i} x_i f_i(x_i). \tag{12.30}$$

Let $V^* = e^{\varepsilon t} V$, analogous to the argument for the inequality (12.25), we obtain

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(12.22)} &= e^{\varepsilon t}\left[\begin{pmatrix} x \\ f(x) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^{\mathrm{T}} & G_{22} \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} + \varepsilon \sum_{i=1}^{n} \frac{\xi_i C_i}{2} x_i^2 \right. \\ &\quad \left. + \sum_{i=1}^{n} \left(\varepsilon \eta_i C_i + \xi_i T_{ii} - \frac{\eta_i}{R_i}\right) x_i f_i(x_i)\right] \\ &\le e^{\varepsilon t}\left[-\lambda^* \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} f_i^2(x_i)\right) + \mu^* \varepsilon \sum_{i=1}^{n} x_i^2\right] \\ &\quad + e^{\varepsilon t} \sum_{i=1}^{n} \left(\varepsilon \eta_i C_i + \xi_i T_{ii} - \frac{\eta_i}{R_i}\right) x_i f_i(x_i). \end{aligned} \tag{12.31}$$

Similar to the proof of Theorem 12.14, the proof of this theorem can be completed. □

12.6 Absolute Stability of Lurie Discrete Delay Neural Networks

In this section, we consider the following Lurie-type neural network model with discrete time delays:

$$\begin{aligned} x(k) &= Ax(k) + B_1 f(\sigma(k)) + B_2 f(\sigma(k - \tau(k))), \\ \sigma(k) &= Cx(k) + D_1 f(\sigma(k)) + D_2 f(\sigma(k - \tau(k))), \end{aligned} \tag{12.32}$$

where $A \in \mathrm{R}^{n\times n}$, $B_1, B_2 \in \mathrm{R}^{n\times \ell}$, $C \in \mathrm{R}^{l\times n}$, $D_1,\ D_2 \in \mathrm{R}^{\ell\times\ell}$, $x \in \mathrm{R}^n$, $f \in \mathrm{R}^l$, $\ell \le n$. L is the number of nonlinear activation functions (i.e., the total number of neurons in the implication layers and output layers of the neural network), the delay $\tau(k) \le h$, is a positive integer.

Assume that the only equilibrium of (12.32) is $x=0$, and the nonlinear activation function satisfies

$$f \in F_{[q_i,k_i]} = \{f | f_i(0) = 0, q_i \le \frac{f_i(\sigma_i)}{\sigma_i} \le u_i\}, \quad f_i \text{ is continuous.}$$

Thus $[f_i(\sigma_i) - q_i\sigma_i][f_i(\sigma_i) - u_i\sigma_i] \le 0$.

In the following, we will discuss the absolute stability of (12.32) in the Lurie sense.

Theorem 12.17. *If the following any one condition is satisfied:*

1. *There exist positive definite symmetric matrices P, Γ, and positive semidefinite diagonal matrices Λ, S such that the following linear matrix inequality (LMI) holds*
$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} < 0, \quad i.e., \quad H is negative definite,$$
or
2. *There exist positive definite symmetric matrices P, Γ, and positive semidefinite diagonal matrices Λ, S such that the following LMI holds*
$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} < 0,$$

where

$$\begin{aligned} &H_{11} = A^{\mathrm{T}}PA - P, && H_{12} = A^{\mathrm{T}}PB_1 + C^{\mathrm{T}}\Lambda, && H_{13} = A^{\mathrm{T}}PB_2, \\ &H_{21} = B_1^{\mathrm{T}}PA + \Lambda C, && H_{22} = B_1^{\mathrm{T}}PB_1 + 2\Lambda D_1 + \Gamma, && H_{23} = B_1^{\mathrm{T}}PB_2 + \Lambda D_2, \\ &H_{31} = B_2^{\mathrm{T}}PA, && H_{32} = B_2^{\mathrm{T}}PB_1 + D_2^{\mathrm{T}}\Lambda, && H_{33} = B_2^{\mathrm{T}}PB_2 - \Gamma, \end{aligned}$$

$$\begin{aligned} G_{11} &= A^{\mathrm{T}}PA - P - 2C^{\mathrm{T}}SQUC, \\ G_{12} &= A^{\mathrm{T}}PB_1 + C^{\mathrm{T}}\Lambda - 2C^{\mathrm{T}}SQUD_1 + C^{\mathrm{T}}(Q+U)S, \\ G_{13} &= A^{\mathrm{T}}PB_2 - 2C^{\mathrm{T}}SQUD_2, \\ G_{22} &= B_1^{\mathrm{T}}PB_1 + 2\Lambda D_1 + \Gamma - 2D_1^{\mathrm{T}}SQUD_1 - 2S + D_1^{\mathrm{T}}(Q+U)S + S(Q+U)D_1, \\ G_{23} &= B_1^{\mathrm{T}}PB_2 + \Lambda D_2 - 2D_1^{\mathrm{T}}SQUD_2 + S(Q+U)D_2, \\ G_{33} &= B_2^{\mathrm{T}}PB_2 - \Gamma - 2D_2^{\mathrm{T}}SQUD_2, \quad G_{21} = G_{12}^{\mathrm{T}}, \quad G_{31} = G_{13}^{\mathrm{T}}, \quad G_{32} = G_{23}^{\mathrm{T}}, \end{aligned}$$

$$Q = \mathrm{diag}(q_1, \cdots, q_n),\ U = \mathrm{diag}(u_1, \cdots, u_\ell),$$
$$\lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n),\ S = \mathrm{diag}(s_1, \cdots, s_\ell).$$

Then the zero solution of (12.32) is absolutely stable in the sense of Lurie.

Proof. Construct the following radially unbounded, positive definite Lyapunov function

$$V(x_k) = x_k^{\mathrm{T}} P x_k + \sum_{i=k-h}^{k-1} f^{\mathrm{T}}(\sigma(i)) F f(\sigma(i)) + 2\sum_{i=1}^{\ell} \lambda_i \sum_{j=0}^{k-1} f_i(\sigma_i(j))\sigma_i(j).$$

Then,

$$\begin{aligned}
\Delta V(x_k) &= x_{k+1}^{\mathrm{T}} P x_{k+1} - x_k^{\mathrm{T}} P x_k + 2\sum_{i=1}^{n} \lambda_i f_i(\sigma_i(k))\sigma_i(k) + f^{\mathrm{T}}(\sigma(k)) \Gamma f(\sigma(h)) \\
&\quad - f^{\mathrm{T}}(\sigma(k-h)) \Gamma f(\sigma(k-h)) \\
&= \Big[A x_k + B_1 f(\sigma(k)) + B_2 f^{\mathrm{T}}(\sigma(k-h))\Big] P \\
&\quad \times \Big[A x_k + B_1 f(\sigma(k)) + B_2 f(\sigma(k-h))\Big] \\
&\quad - x_k^{\mathrm{T}} P x_k + 2\sum_{i=1}^{\ell} \lambda_i f_i(\sigma_i(k)) \left[C_i x_k + D_{1i} f(\sigma(k)) + D_{2i} f(\sigma(k-h))\right] \\
&\quad + f^{\mathrm{T}}(\sigma(k)) \Gamma f(\sigma(k)) - f^{\mathrm{T}}(\sigma(k-h)) \Gamma f(\sigma(k-h)) \\
&= \begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}} P A - P & A^{\mathrm{T}} P B_1 + C^{\mathrm{T}} \Lambda & A^{\mathrm{T}} P B_2 \\ B_1^{\mathrm{T}} P A + \Lambda C & B_1^{\mathrm{T}} P B_1 + 2\Lambda D_1 + \Gamma & B_1^{\mathrm{T}} P B_2 + \Lambda D_2 \\ B_2^{\mathrm{T}} P A & B_2^{\mathrm{T}} P B_1 + D_2^{\mathrm{T}} \Lambda & B_2^{\mathrm{T}} P B_2 - \Gamma \end{bmatrix} \\
&\quad \times \begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix}.
\end{aligned} \tag{12.33}$$

Thus, when $H < 0$, the conditions of Theorem 12.17 imply that the conclusion is true. Otherwise, from

$$[f_i(\sigma(k)) - q_i \sigma_i(k)][f_i(\sigma(k)) - u_i \sigma_i(k)] \le 0,$$

we have

$$\begin{aligned}
&\Big[f_i(\sigma(k)) - q_i C_i x_k - q_i D_{1i} f(\sigma(k)) - q_i D_{2i} f(\sigma(k-h))\Big] \\
&\quad \times \Big[f_i(\sigma(k)) - u_i C_i x_k - u_i D_{1i} f(\sigma(k)) - u_i D_{2i} f(\sigma(k-h))\Big] \le 0.
\end{aligned}$$

Applying the S-method, we can rewrite (12.33) as

$$\begin{aligned}
\Delta V(x_k) &= \begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}} P A - P & A^{\mathrm{T}} P B_1 + C^{\mathrm{T}} \Lambda & A^{\mathrm{T}} P B_2 \\ B_1^{\mathrm{T}} P A + \Lambda C & B_1^{\mathrm{T}} P B_1 + 2\Lambda D_1 + \Gamma & B_1^{\mathrm{T}} P B_2 + \Lambda D_2 \\ B_2^{\mathrm{T}} P A & B_2^{\mathrm{T}} P B_1 + D_2^{\mathrm{T}} \Lambda & B_2^{\mathrm{T}} P B_2 - \Gamma \end{bmatrix} \\
&\quad \times \begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&-\begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{23} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} \begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix} \\
&\leq \begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix}^{\mathrm{T}} G \begin{pmatrix} x_k \\ f(\sigma(k)) \\ f(\sigma(k-h)) \end{pmatrix} < 0 \quad \text{when} \quad G < 0,
\end{aligned}
\tag{12.34}
$$

where

$$
\begin{aligned}
W_{11} &= 2C^{\mathrm{T}}SQUC, \\
W_{22} &= 2D_1^{\mathrm{T}}SQUD_1 + 2S - D_1^{\mathrm{T}}(Q+U)S - S(Q+U)D_1, \\
W_{33} &= 2D_2^{\mathrm{T}}SQUD_2, \\
W_{12} &= W_{21}^{\mathrm{T}} = 2C^{\mathrm{T}}SQUD_1 - C^{\mathrm{T}}(Q+U)T, \\
W_{13} &= W_{31}^{\mathrm{T}} = 2C^{\mathrm{T}}TQUD_2, \\
W_{23} &= W_{32}^{\mathrm{T}} = 2D_1^{\mathrm{T}}SQUD_2 - S(Q+U)D_2.
\end{aligned}
$$

Thus, the conditions implied by the theorem are true. The theorem is proved. □

When $B_1 = 0, D_1 = 0$, (12.32) is reduced to the common neural network:

$$
\begin{cases} x(k+1) = Ax(k) + B_2 f(\sigma(k-\tau(k))), \\ \sigma(k) \quad = Cx(k) + D_{21} f(\sigma(k-\tau(k))), \end{cases}
\tag{12.35}
$$

where $q_i \leq \frac{f_i(\sigma_i(k))}{\sigma_i(k)} \leq u_i$.

Corollary 12.18. *If there exist symmetric positive definite matrices P, Γ, and positive semidefinite diagonal matrices Λ, S such that the following LMI holds*

$$
\begin{bmatrix} A^{\mathrm{T}}PA - P - 2C^{\mathrm{T}}SQUC & C^{\mathrm{T}}A + C^{\mathrm{T}}(Q+U)S & A^{\mathrm{T}}PB_2 - 2C^{\mathrm{T}}SQUD_2 \\ (CA^{\mathrm{T}} + C^{\mathrm{T}}(Q+U)S)^{\mathrm{T}} & \Gamma - 2S & \Lambda D_2 + S(Q+U)D_2 \\ (A^{\mathrm{T}}PB_2 - 2C^{\mathrm{T}}SQUD_2)^{\mathrm{T}} & (\Lambda D_2 + S(Q+U)D_2)^{\mathrm{T}} & B_2PB_2 - \Gamma - 2D_2SQUD_2 \end{bmatrix} < 0,
$$

then the zero solution of (12.35) is absolutely stable.

Example 12.19. Consider the absolute stability of the zero solution of the following discrete time delay neural network:

$$
x(k+1) = Ax(k) + B_1 f(x_k)) + B_2 f(x(k-2)),
\tag{12.36}
$$

where

$$
\begin{gathered}
f_i(x_i) = \tfrac{1}{2}(|x_i+1| - |x_i-1|), \quad i = 1,2, \\
A = \begin{bmatrix} -0.9 & 0.0 \\ 0.0 & -0.75 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.1 \\ 1.0 & -0.1 \end{bmatrix}, \quad U = \mathrm{diag}(1,1), \\
\varphi = 0,\ C = I_2,\ D_1 = D_2 = 0,\ L = 2.
\end{gathered}
$$

With the helped of Matlab, we obtain the solution:

$$P = \begin{bmatrix} 489.6145 & 39.7324 \\ 39.7324 & 57.6801 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 236.9766 & 25.6638 \\ 25.6638 & 34.3699 \end{bmatrix},$$

$$\Lambda = \mathrm{diag}(9.9645, 1.4662), \quad T = \mathrm{diag}(233.1834, 66.4001).$$

P and Γ are positive definite matrices, and Λ and T are positive diagonal matrices. The conditions of Theorem 12.17 are satisfied. Thus, the zero solution of (12.36) is absolutely stable in the sense of Lurie.

13

Application to Chaos Control and Chaos Synchronization

Since Pecora and Carroll [117, 118] first designed an analog electrical circuit to realize chaos synchronization, many researchers have extensively studied the property of chaos synchronization and possible applications in practice. This has changed a long time viewpoint: chaos cannot be controlled, nor synchronized.

Although many results about chaos synchronization have been obtained on the basis of stability theory, general mathematical theory and methodology are still under development. Recently, Curran and Chua [20] suggested that different chaos synchronization methods should be unified to establish a fundamental mathematical theory on the basis of the absolute stability theory of Lurie control systems. The authors and their coworkers have also studied chaos synchronization following Curran and Chua's idea [81, 82, 154]. For the Chua's chaotic circuit, we have recently found that it could be transformed into a type of Lurie system, and thus the theory and methodology developed by Liao [72–77] can be used to study the synchronization of two Chua's circuits. Chua's circuit is the first electrical circuit to realize chaos in experiment, which exhibits very rich complex dynamical behavior, and yet has very high potential in real applications.

In this chapter, as an application, we will apply the absolute stability of Lurie control systems developed in previous chapters to study the globally exponential synchronization of two Chua's chaotic circuits [95]. Also we propose and develop the theory and methodology of absolutely exponential stability, and investigate the global synchronization of two chaotic systems with feedback controls [81, 82, 89]. The materials presented in this chapter are mainly chosen from Liao and Yu [95] (Sects. 13.1–13.5), Liao et al. [89] (Sect. 13.6.1), and Liao et al. [84] (Sect. 13.6.2).

13.1 The Relation of Chua's Circuit and Lurie System

Consider the following general Lurie control system in which the feedback state σ has been changed to the state viable x_1:

$$\dot{x} = Ax + bf(x_1), \quad x \in R^n, \quad A \in R^{n \times n}, \quad b \in R^n, \quad x_1 \in R^1, \tag{13.1}$$

where the dot denotes differentiation w.r.t. time t,

$$f(x_1) \in F_{[0,L]} \overset{\triangle}{=} \left\{ x_1 | \, 0 < x_1 f(x_1) \le L x_1^2, \; f(0) = 0, \; L < +\infty \right\},$$

$f(x_1)$ is a scalar, continuous function of x_1.

X. Liao and P. Yu, *Absolute Stability of Nonlinear Control Systems, Second Edition*.

Definition 13.1. $\forall\, f(x_1) \in F_{[0,L]}$, *if the zero solution of (13.1) (i.e., $x = 0,\ x_1 = 0$) is globally exponentially stable (globally asymptotically stable), then the zero solution is called absolutely exponentially stable (absolutely stable) in Hurwitz angle* $[0,L]$.

Definition 13.2. $\forall\, f(x_1) \in F_{[0,L]}$, *if the zero solution of (13.1) is globally exponentially stable (globally asymptotically stable) with respect to (w.r.t.) the variable x_1, then the zero solution is called absolutely exponentially stable (absolutely stable) w.r.t. x_1 in Hurwitz angle* $[0,L]$.

Lemma 13.3. *If A is a Hurwitz matrix, then the zero solution of (13.1) is absolutely exponentially stable w.r.t. all state variables if and only if the zero solution of (13.1) is absolutely exponentially stable w.r.t. x_1 in Hurwitz angle* $[0,L]$.

Proof. *Necessity*. Necessity is obvious, since $\sum_{i=1}^{n} x_i^2(t,t_0,x_0) \le M(x_0)\, e^{-\alpha\,(t-t_0)}$, and thus particularly, $x_1^2(t,t_0,x_0) \le M(x_0)\, e^{-\alpha\,(t-t_0)}$.

For sufficiency, from the given condition, we know that there exist constants $M \ge 1$ and $\alpha > 0$ such that

$$\|\, e^{A(t-t_0)}\,\| \le M\, e^{-\alpha\,(t-t_0)} \quad \text{and} \quad |x_1(t)| \le h\, e^{-\tilde{\alpha}\,(t-t_0)},$$

where $x_1(t) = x_1(t,t_0,x_0)$ and $\tilde{\alpha} > 0$ is a constant. Without loss of generality, we may assume $\tilde{\alpha} \ne \alpha$. (Otherwise, if $\tilde{\alpha} = \alpha$, one can always change it to, say, $\tilde{\alpha} = \frac{1}{2}\,\alpha$, such that the second inequality still holds.)

Since the general solution of (13.1) can be written as

$$x(t) = x(t,t_0,x_0) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} b\, f(x_1(\tau))\, \mathrm{d}\tau,$$

we obtain

$$\begin{aligned} \|x(t)\| = \|x(t,t_0,x_0)\| \;&\le M\, e^{-\alpha\,(t-t_0)} \|x_0\| \\ &\quad + \int_{t_0}^{t} M\,\|b\|\, L h\, e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(\tau-t_0)}\, \mathrm{d}\tau. \end{aligned} \tag{13.2}$$

It is seen that the first term in (13.2) has a negative exponential estimation, thus we only need to prove that the second term in (13.2) also has a negative exponential estimation. To achieve this, let $M \;\|\, b \,\|\; Lh := c$, then

$$\begin{aligned} \int_{t_0}^{t} c\, e^{-\alpha(t-\tau)} e^{-\tilde{\alpha}(\tau-t_0)}\, \mathrm{d}\tau &= c\, e^{-\alpha t} \int_{t_0}^{t} e^{(\alpha-\tilde{\alpha})\,\tau} e^{\tilde{\alpha} t_0}\, \mathrm{d}\tau \\ &= c\, e^{-\alpha t} \left[\frac{e^{(\alpha-\tilde{\alpha})\,t} - e^{(\alpha-\tilde{\alpha})\,t_0}}{\alpha - \tilde{\alpha}} \right] e^{\tilde{\alpha} t_0} \\ &= \frac{c}{\alpha-\tilde{\alpha}} \left[e^{-\tilde{\alpha}(t-t_0)} - e^{-\alpha(t-t_0)} \right] \\ &\le \begin{cases} \dfrac{c}{\alpha-\tilde{\alpha}}\, e^{-\tilde{\alpha}(t-t_0)} & \text{for } \ \alpha > \tilde{\alpha}, \\[2ex] \dfrac{-c}{\alpha-\tilde{\alpha}}\, e^{-\alpha(t-t_0)} & \text{for } \ \alpha < \tilde{\alpha}. \end{cases} \end{aligned} \tag{13.3}$$

Equation (13.3) indicates that the second term in (13.2) indeed has a negative exponential estimation, and therefore, the zero solution of (13.1) is absolutely exponentially stable in Hurwitz angle $[0,L]$. This finishes the proof of Lemma 13.3. □

Next, we establish the relation between Chua's circuit and Lurie system. Chua's circuit is described by the following differential equations [11, 20]:

$$\begin{aligned}\dot{x} &= p\left[-x+y-f(x)\right],\\ \dot{y} &= x-y+z,\\ \dot{z} &= -qy,\end{aligned}\tag{13.4}$$

where

$$f(x) = bx + \frac{1}{2}(a-b)\left(|x+E| - |x-E|\right),$$

and $p = 10.0$, $q = 14.87$, $a = -1.27$, $b = -0.68$, E is a positive constant.

On the basis of (13.4), consider the synchronization of Chua's circuits, which consist of two chaotic systems: One is the drive system, given by

$$\begin{aligned}\dot{x}_{\mathrm{d}} &= p\left[-x_{\mathrm{d}}+y_{\mathrm{d}}-f(x_{\mathrm{d}})\right],\\ \dot{y}_{\mathrm{d}} &= x_{\mathrm{d}}-y_{\mathrm{d}}+z_{\mathrm{d}},\\ \dot{z}_{\mathrm{d}} &= -qy_{\mathrm{d}},\end{aligned}\tag{13.5}$$

where the subscript d denotes the drive system; and the other one is the response system, described by

$$\begin{aligned}\dot{x}_{\mathrm{r}} &= p\left[-x_{\mathrm{r}}+y_{\mathrm{r}}-f(x_{\mathrm{r}})\right]+u_1(x_{\mathrm{d}}-x_{\mathrm{r}},y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}),\\ \dot{y}_{\mathrm{r}} &= x_{\mathrm{r}}-y_{\mathrm{r}}+z_{\mathrm{r}}+u_2(x_{\mathrm{d}}-x_{\mathrm{r}},y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}),\\ \dot{z}_{\mathrm{r}} &= -qy_{\mathrm{r}}+u_3(x_{\mathrm{d}}-x_{\mathrm{r}},y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}),\end{aligned}\tag{13.6}$$

where the subscript r indicates the response system, u_1, u_2, and u_3 represent feedback controls to be determined.

Letting

$$e_x = x_{\mathrm{d}} - x_{\mathrm{r}}, \quad e_y = y_{\mathrm{d}} - y_{\mathrm{r}}, \quad e_z = z_{\mathrm{d}} - z_{\mathrm{r}}$$

yields the following error system:

$$\begin{aligned}\dot{e}_x &= -pe_x+pe_y-p\left[f(x_{\mathrm{d}})-f(x_{\mathrm{r}})\right]-u_1(x_{\mathrm{d}}-x_{\mathrm{r}},y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}),\\ \dot{e}_y &= e_x-e_y+e_z-u_2(x_{\mathrm{d}}-x_{\mathrm{r}},y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}),\\ \dot{e}_z &= -qe_y-u_3(x_{\mathrm{d}}-x_{\mathrm{r}},y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}).\end{aligned}\tag{13.7}$$

The basic idea here is to choose simple feedback controls u_1, u_2, and u_3 such that the zero solution of the error system (13.7) is globally exponentially stable in Hurwitz angle $[0,L]$, and therefore the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$.

Definition 13.4. *If* $\forall\ x_{\mathrm{d}}(0),\ y_{\mathrm{d}}(0),\ z_{\mathrm{d}}(0) \in R^3$*, and the corresponding* $x_{\mathrm{r}}(0),\ y_{\mathrm{r}}(0),\ z_{\mathrm{r}}(0) \in R^3$*, the zero solution of the error system (13.7) is absolutely exponentially stable, then the two systems (13.5) and (13.6) are globally exponentially synchronized.*

Next, we shall show that the error system (13.7) can be considered as a standard Lurie system. In fact, on the one hand, if all controls u_1, u_2, and u_3 are merely linear functions of e_x, e_y, and e_z, then these controls can be included in the linear part A of the Lurie system. On the other hand, if u_1 contains $f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})$, then

$$f(x_{\mathrm{d}}) - f(x_{\mathrm{r}}) = k(x_{\mathrm{d}}, x_{\mathrm{r}})\,(x_{\mathrm{d}} - x_{\mathrm{r}}) = k(x_{\mathrm{d}}, x_{\mathrm{r}})\,e_x.$$

Since $a \le k(x_{\mathrm{d}}, x_{\mathrm{r}}) \le b < 0$, we have

$$a \le \frac{f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})}{x_{\mathrm{d}} - x_{\mathrm{r}}} = \frac{k(x_{\mathrm{d}}, x_{\mathrm{r}})\,e_x}{x_{\mathrm{d}} - x_{\mathrm{r}}} \le b < 0,$$

which does not satisfy the condition of Lurie function. However, because

$$0 \le -b \le \frac{-\left[f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})\right]}{x_{\mathrm{d}} - x_{\mathrm{r}}} = \frac{-k(x_{\mathrm{d}}, x_{\mathrm{r}})\,e_x}{x_{\mathrm{d}} - x_{\mathrm{r}}} \le -a := L < +\infty,$$

we can define

$$g(e_x) := -\left[f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})\right] \in F_{[0,L]},$$

which satisfies the condition of Lurie function. Therefore, the error system (13.7) can be always transformed into a standard Lurie system, given by

$$\begin{aligned}
\dot{e}_x &= -p\,e_x + p\,e_y + p\,g(e_x) - u_1(x_{\mathrm{d}} - x_{\mathrm{r}}, y_{\mathrm{d}} - y_{\mathrm{r}}, z_{\mathrm{d}} - z_{\mathrm{r}}),\\
\dot{e}_y &= e_x - e_y + e_z - u_2(x_{\mathrm{d}} - x_{\mathrm{r}}, y_{\mathrm{d}} - y_{\mathrm{r}}, z_{\mathrm{d}} - z_{\mathrm{r}}),\\
\dot{e}_z &= -q\,e_y - u_3(x_{\mathrm{d}} - x_{\mathrm{r}}, y_{\mathrm{d}} - y_{\mathrm{r}}, z_{\mathrm{d}} - z_{\mathrm{r}}).
\end{aligned} \tag{13.8}$$

13.2 Globally Exponent Synchronization of Two Chua's Chaotic Circuits

In this section, we shall show how to choose the feedback controls u_1, u_2, and u_3 such that the drive-response systems (13.5) and (13.6) are globally exponentially synchronized. We first consider linear feedback control laws, and then nonlinear feedback control laws.

13.2.1 Linear Feedback Control

Theorem 13.5. *In the response system (13.6), take the following feedback controls:*

$$u_1 = p\,\delta_x\,(x_{\mathrm{d}} - x_{\mathrm{r}}) + p\,(y_{\mathrm{d}} - y_{\mathrm{r}}) \qquad (\delta_x > -(1+a)), \qquad u_2 = u_3 = 0. \tag{13.9}$$

Then, the zero solution of (13.7) is globally exponentially stable in Hurwitz angle $[0, L]$*, and thus the two systems (13.5) and (13.6) are absolutely exponentially synchronized in Hurwitz angle* $[0, L]$.

Proof. Let $\tilde{\delta}_x = \delta_x + (1+a)$, i.e., $\delta_x = \tilde{\delta}_x - (1+a)$. Consider the first equation of (13.7) and construct the radially unbounded, positive definite Lyapunov function for this equation: $V = \frac{1}{2} e_x^2$, then we have

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.7)} = e_x \dot{e}_x &= -p e_x^2 + p e_x e_y - p e_x e_y - e_x p\,[f(x_{\rm d}) - f(x_{\rm r})] - p\,\delta_x e_x^2 \\ &= -p e_x^2 - p k(x_{\rm d}, x_{\rm r})\, e_x^2 - p\,\delta_x e_x^2 \\ &\le (-p - pa)\, e_x^2 - p\left[\tilde{\delta}_x - (1+a)\right] e_x^2 \\ &= -p\,\tilde{\delta}_x e_x^2 \\ &= -2p\,\tilde{\delta}_x V,\end{aligned}$$

which implies that $\frac{1}{2} e_x^2(t) = V(t) \le V(0)\, e^{-2p\tilde{\delta}_x t}$, i.e.,

$$e_x^2(t) = 2V(t) \le 2V(0)\, e^{-2p\tilde{\delta}_x t}. \qquad (13.10)$$

Equation (13.10) indicates that the zero solution of the error system (13.7) is absolutely exponentially stable in Hurwitz angle $[0, L]$ w.r.t. partial variable e_x (i.e., for any $k(x_{\rm d}, x_{\rm r})\,\delta_x$).

Next, consider the following matrix:

$$A = \begin{bmatrix} -p - p\,\delta_x & 0 & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix}$$

which gives the characteristic polynomial:

$$\begin{aligned}\det(\lambda I - A) &= \begin{bmatrix} \lambda + p + p\,\delta_x & 0 & 0 \\ -1 & \lambda + 1 & -1 \\ 0 & q & \lambda \end{bmatrix} \\ &= \lambda^3 + (p + p\,\delta_x + 1)\,\lambda^2 + (p + p\,\delta_x + q)\,\lambda + (p + p\,\delta_x)\,q. \qquad (13.11)\end{aligned}$$

According to the well-known Hurwitz criterion [68], the sufficient and necessary conditions for matrix A being a Hurwitz matrix (i.e., all eigenvalues of A have negative real parts) are

$$0 < (p + p\,\delta_x)\,q < (p + p\,\delta_x + 1)\,(p + p\,\delta_x + q).$$

Obviously, the above inequalities hold under the given conditions. Hence, by Lemma 13.3, the zero solution of system (13.7) is absolutely, exponentially stable in Hurwitz angle $[0, L]$ (i.e., for any $k(x_{\rm d}, x_{\rm r})\,\delta_x$), and therefore, the drive-response systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0, L]$. The proof of Theorem 13.5 is complete. □

Theorem 13.6. *In the response system (13.6), choose the following feedback controls:*

$$
\begin{aligned}
u_1 &= p\,\delta_x(x_\mathrm{d}-x_\mathrm{r})+\tilde{p}\,(y_\mathrm{d}-y_\mathrm{r}) \ \ (\delta_x > -(1+a),\ \tilde{p}>p),\\
u_2 &= z_\mathrm{d}-z_\mathrm{r},\\
u_3 &= \delta_z(z_\mathrm{d}-z_\mathrm{r}),
\end{aligned} \tag{13.12}
$$

where $\delta_z > 0$ is an arbitrary real number. Then, the zero solution of (13.7) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and thus the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$.

Proof. Under the controls given in (13.12), the error system (13.7) becomes

$$
\begin{aligned}
\dot{e}_x &= -p\,(e_x-e_y)-\tilde{p}\,e_y-p\,\delta_x\,e_x-p\,[\,f(x_\mathrm{d})-f(x_\mathrm{r})\,],\\
\dot{e}_y &= e_x-e_y\\
\dot{e}_z &= -q\,e_y-\delta_z\,e_z\,.
\end{aligned} \tag{13.13}
$$

Let $\tilde{p}-p=r>0,\ \tilde{\delta}_x=\delta_x+(1+a)$. Then, consider the first two equations of (13.13) and construct the radially unbounded, positive definite Lyapunov function for (e_x,e_θ):

$$
V=\frac{1}{2}\,(e_x^2+r\,e_y^2),
$$

from which we obtain

$$
\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.13)} &\le -(p+p\,\delta_x+p\,a)\,e_x^2-r\,e_x\,e_y+r\,e_x\,e_y-r\,e_y^2\\
&\le -p\left[1+a+(\tilde{\delta}_x-(1+a)\right]e_x^2-r\,e_y^2\\
&= -p\,\tilde{\delta}_x\,e_x^2-r\,e_y^2\\
&\le \begin{cases} -e_x^2-r\,e_y^2=-2V & \text{for } \ p\,\tilde{\delta}_x\ge 1,\\ -p\,\tilde{\delta}_x\,(e_x^2+r\,e_y^2)=-2\,p\,\tilde{\delta}_x\,V & \text{for } \ p\,\tilde{\delta}_x<1. \end{cases}
\end{aligned}
$$

Thus,

$$
V(t)\le \begin{cases} V(t_0)\,e^{-2\,(t-t_0)} & \text{for } \ p\,\tilde{\delta}_x\ge 1,\\ V(t_0)\,e^{-2\,p\,\tilde{\delta}_x(t-t_0)} & \text{for } \ p\,\tilde{\delta}_x<1, \end{cases}
$$

which, in turn, results in

$$
e_x^2(t)\le 2\,V(t)\le \begin{cases} 2\,V(t_0)\,e^{-2\,(t-t_0)} & \text{for } \ p\,\tilde{\delta}_x\ge 1,\\ 2\,V(t_0)\,e^{-2\,p\,\tilde{\delta}_x(t-t_0)} & \text{for } \ p\,\tilde{\delta}_x<1. \end{cases}
$$

Therefore, the zero solution of system (13.13) is absolutely exponentially stable w.r.t. partial variables e_x and e_y in Hurwitz angle $[0,L]$.

Next, from the linear part of system (13.13), we have the matrix:

$$A = \begin{bmatrix} -p - p\,\delta_x & -r & 0 \\ 1 & -1 & 0 \\ 0 & -q & -\delta_z \end{bmatrix},$$

which yields the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \begin{bmatrix} \lambda + p + p\,\delta_x & r & 0 \\ -1 & \lambda + 1 & 0 \\ 0 & q & \lambda + \delta_z \end{bmatrix} \\ &= (\lambda + \delta_z)\left[\lambda^2 + (p + p\,\delta_x + 1)\,\lambda + p + p\,\delta_x + r\right] \end{aligned}$$

It follows from the given conditions: $\delta_z > 0,\ p + p\,\delta_x + 1 > 0$ and $p + p\,\delta_x + r > 0$ that A is a Hurwitz matrix. Thus, by Lemma 13.3, the conclusion of Theorem 13.6 is true. □

Theorem 13.7. *In the response system (13.6), take the following feedback controls:*

$$\begin{aligned} u_1 &= p\,\delta_x(x_{\mathrm{d}} - x_{\mathrm{r}}) \quad (\delta_x > -(1+a)), \\ u_2 &= l\,(x_{\mathrm{d}} - x_{\mathrm{r}}) + (z_{\mathrm{d}} - z_{\mathrm{r}}), \quad (l > 1), \\ u_3 &= \delta_z\,(z_{\mathrm{d}} - z_{\mathrm{r}}) \quad (\delta_z > 0). \end{aligned} \tag{13.14}$$

Then, the zero solution of (13.7) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and thus the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$.

Proof. Under the controls given in (13.14), the error system (13.7) can be rewritten as

$$\begin{aligned} \dot{e}_x &= -p\,(e_x - e_y) - p\,\delta_x\, e_x - p\,[f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})]\,, \\ \dot{e}_y &= (1 - l)\,e_x - e_y\,, \\ \dot{e}_z &= -q\,e_y - \delta_z\, e_z\,. \end{aligned} \tag{13.15}$$

Let $\tilde{\delta}_x = \delta_x + (1+a)$. Then, consider the first two equations of (13.15) and construct the Lyapunov function for these two equations:

$$V = \frac{1}{2p}\,e_x^2 + \frac{1}{2\,(l-1)}\,e_y^2,$$

which is radially unbounded, positive definite for (e_x, e_y). It is easy to show that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.15)} \leq -(1 + a + \delta_x)\,e_x^2 + e_x\,e_y - e_x\,e_y - e_y^2 = -\,\tilde{\delta}_x\,e_x^2 - e_y^2.$$

Similar to the proof of Theorem 13.6, a simple algebraic manipulation leads to the conclusion: the zero solution of system (13.15) is absolutely exponentially stable w.r.t. partial variables e_x and e_y in Hurwitz angle $[0, L]$.

Next, similarly, the linear matrix of (13.15):

$$A = \begin{bmatrix} -p - p\,\delta_x & p & 0 \\ 1 - l & -1 & 0 \\ 0 & -q & -\delta_z \end{bmatrix}$$

yields the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \begin{bmatrix} \lambda + p + p\,\delta_x & -p & 0 \\ l - 1 & \lambda + 1 & 0 \\ 0 & q & \lambda + \delta_z \end{bmatrix} \\ &= (\lambda + \delta_z)\left[\lambda^2 + (p + p\,\delta_x + 1)\,\lambda + p + p\,\delta_x + p\,(l - 1)\right]. \end{aligned}$$

It is obvious that A is a Hurwitz matrix. Thus, by Lemma 13.3, the zero solution of system (13.15) is absolutely exponentially stable in Hurwitz angle $[0, L]$, and therefore, the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0, L]$. □

Theorem 13.8. *In the response system (13.6), take the following feedback controls:*

$$\begin{aligned} u_1 &= p\,\delta_x(x_{\mathrm{d}} - x_{\mathrm{r}}) \quad (\delta_x > -(1 + a)), \\ u_2 &= \delta_y(x_{\mathrm{d}} - x_{\mathrm{r}}) + (z_{\mathrm{d}} - z_{\mathrm{r}}) \qquad (\delta_y > \frac{p}{p + pa + p\,\delta_x} - 1), \\ u_3 &= \delta_z\,(z_{\mathrm{d}} - z_{\mathrm{r}}) \quad (\delta_z > 0). \end{aligned} \tag{13.16}$$

Then, the zero solution of (13.7) is absolutely exponentially stable in Hurwitz angle $[0, L]$, and thus the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0, L]$.

Proof. Under the controls given in (13.16), the error system (13.7) becomes

$$\begin{aligned} \dot{e}_x &= -p\,(e_x - e_y) - p\,\delta_x e_x - p\,[f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})], \\ \dot{e}_y &= e_x - e_y - \delta_y e_y, \\ \dot{e}_z &= -q\,e_y - \delta_z e_z. \end{aligned} \tag{13.17}$$

From the given conditions: $-p - pa - p\,\delta_x < 0$, $-1 - \delta_y < 0$, $\delta_y > \frac{p}{p+pa+p\,\delta_x} - 1$, we have $(p + pa + p\,\delta_x)(1 + \delta_y) > p$. Further, using the sufficient and necessary conditions for a second-order real matrix to be Lyapunov-Voltlor stable, one can conclude that there exists $\xi > 0$ such that

$$\Omega = \begin{bmatrix} -2\,(p + pa + p\,\delta_x) & p + \xi \\ p + \xi & -2\,\xi\,(1 + \delta_y) \end{bmatrix}$$

is negative definite. Let $\lambda_{\max}(\Omega)$ denote the largest eigenvalue of Ω. Then, similarly, consider the first two equations of (13.17) and construct the Lyapunov function for these two equations: $V = (e_x^2 + \xi\, e_y^2)$. Thus, we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\,e_x\,\dot{e}_x + 2\,e_y\,\dot{e}_y \\ &\leq \begin{pmatrix} e_x \\ e_y \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2\,(p + pa + p\,\delta_x) & p+\xi \\ p+\xi & -2\,\xi\,(1+\delta_y) \end{bmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix} \\ &\leq \lambda_{\max}(\Omega)\,(e_x^2 + e_y^2),\end{aligned}$$

which implies that the zero solution of system (13.17) is absolutely exponentially stable in Hurwitz angle $[0,L]$ w.r.t. partial variables e_x and e_y. In particular, it is absolutely exponentially stable w.r.t. e_x in Hurwitz angle $[0,L]$.

By considering the linear matrix of system (13.17):

$$A = \begin{bmatrix} -p - p\,\delta_x & p & 0 \\ 1 & -1-\delta_y & 0 \\ 0 & -q & -\delta_z \end{bmatrix},$$

we can similarly prove that A is a Hurwitz matrix. Therefore, by Lemma 13.3, the conclusion of Theorem 13.8 is true. □

In the following, we give another theorem, which is completely different from Theorems 13.5–13.8.

Theorem 13.9. *In the response system (13.6), choose the following feedback controls:*

$$u_1 = p\,\delta_x(x_{\mathrm{d}} - x_{\mathrm{r}}) \quad (\delta_x > -(1+a)), \quad u_2 = x_{\mathrm{d}} - x_{\mathrm{r}}, \quad u_3 = 0. \tag{13.18}$$

Then, the zero solution of (13.7) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and thus the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$.

Proof. Under the controls given in (13.18), the error system (13.7) becomes

$$\begin{aligned}\dot{e}_x &= -p\,(e_x - e_y) - p\,\delta_x\,e_x^2 - p\,[f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})]\,, \\ \dot{e}_y &= -e_y + e_z\,, \\ \dot{e}_z &= -q\,e_y\,.\end{aligned} \tag{13.19}$$

Consider the second and third equations of (13.19) and construct the Lyapunov function for these two equations:

$$V = e_y^2 + \tfrac{1}{q}\,e_z^2 - \varepsilon\,e_y\,e_z = \begin{pmatrix} e_y \\ e_z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & -\dfrac{1}{2}\varepsilon \\ -\dfrac{1}{2}\varepsilon & \dfrac{1}{q} \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} := \begin{pmatrix} e_y \\ e_z \end{pmatrix}^{\mathrm{T}} W \begin{pmatrix} e_y \\ e_z \end{pmatrix}.$$

It is easy to see that V is positive definite when $0 < \varepsilon \ll 1$. Further, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.19)} &= -2\,e_y^2 + 2\,e_y\,e_z - 2\,e_y\,e_z - \varepsilon\,\dot{e}_y\,e_z - \varepsilon\,e_y\,\dot{e}_z \\
&= -2\,e_y^2 - \varepsilon\,(-e_y + e_z)\,e_z + \varepsilon\,q\,e_y^2 \\
&= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2+\varepsilon q & \frac{1}{2}\varepsilon \\ \frac{1}{2}\varepsilon & -\varepsilon \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \\
&:= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^{\mathrm{T}} Q \begin{pmatrix} e_y \\ e_z \end{pmatrix} < 0 \qquad (\text{when } \ e_y^2 + e_z^2 \neq 0 \ \text{ and } \ 0 < \varepsilon \ll 1).
\end{aligned}$$

Thus,

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq \lambda_{\max}(Q)\,(e_y^2 + e_z^2) \leq \frac{\lambda_{\max}(Q)}{\lambda_{\max}(W)}\,V,$$

and so $V(t) \leq V(t_0)\,e^{\frac{\lambda_{\max}(Q)}{\lambda_{\max}(W)}\,(t-t_0)}$. Also note that $V \geq \lambda_{\min}(W)\,(e_y^2 + e_z^2)$. Combining these results yields

$$\lambda_{\min}(W)\,(e_y^2 + e_z^2) \leq V(t) \leq V(t_0)\,e^{\frac{\lambda_{\max}(Q)}{\lambda_{\max}(W)}\,(t-t_0)},$$

where $\lambda_{\min}(W)$ and $\lambda_{\max}(Q)$ denote, respectively, the minimum eigenvalue of W and the maximum eigenvalue of Q. This shows that the zero solution of system (13.19) is absolutely exponentially stable w.r.t. e_y and e_z in Hurwitz angle $[0, L]$.

Next, consider the first equation of (13.19) and construct the Lyapunov function $V = \frac{1}{2}\,e_x^2$, and choose $0 < \varepsilon \ll 1$ such that $(\frac{\varepsilon}{2})^2 < 1 + a + \delta_x$, then

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.19)} &\leq -(p + p\,a + p\,\delta_x)\,e_x^2 + p\,e_x\,e_y \\
&\leq -(p + p\,a + p\,\delta_x)\,e_x^2 + p\left(\frac{\varepsilon}{2}\right)^2 e_x^2 + p\left(\frac{1}{\varepsilon}\right)^2 e_y^2 \\
&= -\left[p + p\,a + p\,\delta_x - p\left(\frac{\varepsilon}{2}\right)^2\right] e_x^2 + p\,\frac{1}{\varepsilon^2}\,e_y^2 \\
&= -2\left[p + p\,a + p\,\delta_x - p\left(\frac{\varepsilon}{2}\right)^2\right] V + p\left(\frac{1}{\varepsilon}\right)^2 e_y^2.
\end{aligned} \tag{13.20}$$

Let $V(t_0) = V_0$, Consider the following initial value problem:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -2\left[p + p\,a + p\,\delta_x - p\left(\frac{\varepsilon}{2}\right)^2\right] U + p\left(\frac{1}{\varepsilon}\right)^2 e_y^2 \qquad \text{with } \ U(t_0) = V_0, \tag{13.21}$$

which is used for comparison with (13.20). So $V(t) \leq U(t)$. Following the proof of Lemma 13.3, one can show that $U(t)$ has a negative exponential estimation since e_y^2 has a negative exponential estimation. Therefore, $V(t)$ and $e_x^2(t)$ have negative exponential estimations. This implies that the zero solution of system (13.19) is also absolutely exponentially stable w.r.t. e_x in Hurwitz angle $[0,L]$, and thus the zero solution of system (13.19) is absolutely exponentially stable in Hurwitz angle $[0,L]$. Therefore, the drive-response systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$. □

To end this section, we present a general result for existence, and then apply it to find some useful conditions for practical applications.

In the response system (13.6), take the following feedback controls:

$$
\begin{aligned}
u_1 &= p\,\delta_x\,(x_{\mathrm{d}} - x_{\mathrm{r}}) \quad (\delta_x > 0),\\
u_2 &= \delta_y\,(y_{\mathrm{d}} - y_{\mathrm{r}}) \quad (\delta_y > 0),\\
u_3 &= q\,\delta_z\,(z_{\mathrm{d}} - z_{\mathrm{r}}) \quad (\delta_z > 0).
\end{aligned}
\tag{13.22}
$$

Then, the error system (13.7) becomes

$$
\begin{aligned}
\dot{e}_x &= p\,[e_y - e_x - (f(x_{\mathrm{d}}) - f(x_{\mathrm{r}}))] - p\,\delta_x\,e_x\,,\\
\dot{e}_y &= e_x - e_y + e_z - \delta_y\,e_y\,,\\
\dot{e}_z &= -q\,e_y - q\,\delta_z\,e_z\,.
\end{aligned}
\tag{13.23}
$$

Theorem 13.10. *There always exist $\delta_x \geq 0,\ \delta_y \geq 0,\ \delta_z \geq 0$, such that the zero solution of (13.23) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and thus the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$.*

Proof. Let the radially unbounded, positive definite Lyapunov function be: $V = \frac{1}{p}\,e_x^2 + e_y^2 + \frac{1}{q}\,e_z^2$, then

$$
\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.23)} &= 2\,e_x\,[e_y - e_x - k(x_{\mathrm{d}},x_{\mathrm{r}})\,e_x - \delta_x\,e_x]\\
&\quad + 2\,e_y\,(e_x - e_y + e_z - \delta_y\,e_y) - 2\,e_z\,(-e_y - \delta_z\,e_z)\\
&= -\,2\,e_x^2 - 2\,k(x_{\mathrm{d}},x_{\mathrm{r}})\,e_x^2 - 2\,\delta_x\,e_x^2 + 4\,e_x\,e_y - 2\,e_y^2\\
&\quad - 2\,\delta_y\,e_y^2 + 2\,e_y\,e_z - 2\,e_y\,e_z - 2\,\delta_z\,e_z^2\\
&= \begin{pmatrix} e_x\\ e_y\\ e_z \end{pmatrix}^{\mathrm{T}}
\begin{bmatrix} -2\,(1 + k(x_{\mathrm{d}},x_{\mathrm{r}}) + \delta_x) & 2 & 0\\ 2 & -2\,(1+\delta_y) & 0\\ 0 & 0 & -2\,\delta_z \end{bmatrix}
\begin{pmatrix} e_x\\ e_y\\ e_z \end{pmatrix}\\
&\leq \begin{pmatrix} e_x\\ e_y\\ e_z \end{pmatrix}^{\mathrm{T}}
\begin{bmatrix} -2\,(1 + a + \delta_x) & 2 & 0\\ 2 & -2\,(1+\delta_y) & 0\\ 0 & 0 & -2\,\delta_z \end{bmatrix}
\begin{pmatrix} e_x\\ e_y\\ e_z \end{pmatrix}.
\end{aligned}
\tag{13.24}
$$

Thus, when $e_x^2 + e_y^2 + e_z^2 \neq 0$, as long as the parameters are chosen to satisfy $(1+\delta_y) > 0$, $(1+a+\delta_x) > 0$ and $(1+\delta_y)\,(1+a+\delta_x) > 1$, i.e., $a + \delta_x + \delta_y + a\,\delta_y + \delta_x\,\delta_y > 0$, then

$\frac{\mathrm{d}V}{\mathrm{d}t} < 0$. For example, when $a+\delta_x > 0,\ \delta_y = 0,\ \delta_z > 0$, or $-1 < a+\delta_x < 0,\ \delta_y > \frac{1}{1+a+\delta_x}+1,\ \delta_z > 0$, we have $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$. Thus, the zero solution of system (13.23) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and so the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$. □

13.2.2 Nonlinear Feedback Control

Now, we turn to apply nonlinear feedback controls to obtain globally exponential synchronization. To achieve this, in the response system (13.6), let

$$u_1 = -l\,[\,f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})\,] := l\,g(e_x) \quad (l > p), \qquad u_2 = u_3 = 0, \tag{13.25}$$

where $g(e_x) = -\,[\,f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})\,] = -k(x_{\mathrm{d}},x_{\mathrm{r}})\,e_x$, satisfying $0 < -h \le \frac{g(e_x)}{e_x} \le -a := L < +\infty$. Then, system (13.7) becomes

$$\begin{aligned} \dot{e}_x &= -p\,(e_x - e_y) - (l-p)\,g(e_x), \\ \dot{e}_y &= e_x - e_y - e_z\,, \\ \dot{e}_z &= -q\,e_y\,. \end{aligned} \tag{13.26}$$

Further, we can rewrite (13.26) as a standard Lurie direct control system:

$$\begin{aligned} &\begin{pmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{pmatrix} = \begin{bmatrix} -p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} + \begin{pmatrix} -\delta \\ 0 \\ 0 \end{pmatrix} g(e_x) := A\,e + h\,g(e_x), \\ &\sigma := c^{\mathrm{T}} e, \end{aligned} \tag{13.27}$$

where $\delta = l - p$, and

$$A = \begin{bmatrix} -p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix}, \quad h = \begin{pmatrix} -\delta \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}.$$

The characteristic polynomial is

$$\det(\lambda I - A) = \lambda^3 + (p+1)\,\lambda^2 + q\lambda + pq = 0. \tag{13.28}$$

According to Hurwitz criterion, the sufficient and necessary conditions for (13.28) being a Hurwitz polynomial are

$$q > 0, \quad 0 < pq < (p+1)\,q = pq + q.$$

Since $p > 0,\ q > 0$, the above conditions obviously hold, indicating that A is a Hurwitz matrix. Thus, by Lemma 13.3, the sufficient and necessary conditions for the zero solution of system (13.27) being absolutely exponentially stable are that the zero solution of system (13.27) is absolutely exponentially stable w.r.t. partial variable e_x.

Theorem 13.11. *If there exist $\tau > 0$ and $\varepsilon > 0$ such that*

$$H(\tau) = \begin{bmatrix} & & & -R_{11}\,\delta - \dfrac{p}{2} + \tau \\ & RA + A^{\mathrm{T}}R & & -R_{21}\,\delta + \dfrac{p}{2} \\ & & & -R_{31}\,\delta \\ -R_{11}\,\delta - \dfrac{p}{2} + \tau & -R_{21}\,\delta + \dfrac{p}{2} & -R_{31}\,\delta & -\delta + \dfrac{1}{|a| + \varepsilon} \end{bmatrix}$$

is negative definite, then the zero solution of system (13.27) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and thus the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$. Here, R is the symmetric, positive definite solution of the Lyapunov matrix equation:

$$RA + A^{\mathrm{T}}R = B, \tag{13.29}$$

where B is a symmetric, negative definite matrix. R_{11}, R_{21}, and R_{31} are the first row elements of R.

Proof. Construct the Lurie-Lyapunov function: $V = e^{\mathrm{T}}Re + \int_0^{e_x} g(e_x)\,\mathrm{d}e_x$, and then differentiate V along the solution of system (13.27) with the aid of S programming to obtain

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= e^{\mathrm{T}}(RA + A^{\mathrm{T}}R)e + 2\left(Rh + \frac{1}{2}A^{\mathrm{T}}C\right)^{\mathrm{T}} e\,g(e_x) - \delta\,g^2(e_x) \\
&= e^{\mathrm{T}}(RA + A^{\mathrm{T}}R)e + \left[2\begin{pmatrix} -R_{11}\,\delta \\ -R_{21}\,\delta \\ -R_{31}\,\delta \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} -p \\ 0 \\ 0 \end{pmatrix}^{\mathrm{T}}\right] e\,g(e_x) - \delta\,g^2(e_x) \\
&= \begin{pmatrix} e_x \\ e_y \\ e_z \\ g(e_x) \end{pmatrix}^{T} \begin{bmatrix} & & & -R_{11}\delta - \dfrac{p}{2} + \tau \\ & RA + A^{\mathrm{T}}R & & -R_{21}\,\delta + \dfrac{p}{2} \\ & & & -R_{31}\,\delta \\ -R_{11}\delta - \dfrac{p}{2} + \tau & -R_{21}\delta + \dfrac{p}{2} & -R_{31}\,\delta & -\delta + \dfrac{1}{|a| + \varepsilon} \end{bmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \\ g(e_x) \end{pmatrix} \\
&\quad - \tau\,g(e_x)\left[e_x - \frac{1}{|a| + \varepsilon}\,g(e_x)\right] \\
&\leq \lambda_{\max}(H)\,(e_x,\, e_y,\, e_z,\, g(e_x))\,(e_x,\, e_y,\, e_z,\, g(e_x))^{\mathrm{T}} \\
&\quad - \tau\,g(e_x)\left(e_x - \frac{|a|\,e_x}{|a| + \varepsilon}\right) \\
&\leq \lambda_{\max}(H)\,e^{\mathrm{T}}e - \tau\,g(e_x)\left(\frac{\varepsilon}{|a| + \varepsilon}e_x\right) \\
&\leq \frac{\lambda_{\max}(H)}{\lambda_{\max}(R)}\,e^{\mathrm{T}}Re - \frac{\tau\,\varepsilon}{|a| + \varepsilon}\,e_x\,g(e_x).
\end{aligned} \tag{13.30}$$

One may choose $\xi > 0$ such that $\frac{1}{2}\xi\,|a| \le |b|$. Then $e_x\,g(e_x) \ge \xi \int_0^{e_x} g(e_x)\,\mathrm{d}e_x$. In fact, since $|b|\,e_x^2 \le e_x\,g(e_x) \le |a|\,e_x^2$, we have

$$\xi\,\frac{|b|}{2}\,e_x^2 \le \int_0^{e_x} \xi\,g(e_x)\,\mathrm{d}e_x \le \xi\,\frac{|a|}{2}\,e_x^2 \le |b|\,e_x^2 \le e_x\,g(e_x).$$

Thus, inequality (13.30) can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\lambda_{\max}(H)}{\lambda_{\max}(R)}\,e^{\mathrm{T}}Re - \frac{\tau\,\varepsilon\,\xi}{|a|+\varepsilon}\int_0^{e_x} g(e_x)\,\mathrm{d}e_x \\
&\le \max\left[\frac{\lambda_{\max}(H)}{\lambda_{\max}(R)},\ -\frac{\tau\,\varepsilon\,\xi}{|a|+\varepsilon}\right]\left[e^{\mathrm{T}}Re + \int_0^{e_x} g(e_x)\,\mathrm{d}e_x\right] \\
&\le \max\left[\frac{\lambda_{\max}(H)}{\lambda_{\max}(R)},\ -\frac{\tau\,\varepsilon\,\xi}{|a|+\varepsilon}\right]V.
\end{aligned}$$

Hence,

$$V(t) \le V(t_0)\,e^{\max\left[\frac{\lambda_{\max}(H)}{\lambda_{\max}(R)},\ -\frac{\tau\,\varepsilon\,\xi}{|a|+\varepsilon}\right](t-t_0)},$$

which, in turn, results in

$$e^{\mathrm{T}}(t)\,e(t) \le \frac{1}{\lambda_{\mathrm{mim}}(R)}\,V(t) \le \frac{1}{\lambda_{\mathrm{mim}}(R)}\,V(t_0)\,e^{\max\left[\frac{\lambda_{\max}(H)}{\lambda_{\max}(R)},\ -\frac{\tau\,\varepsilon\,\xi}{|a|+\varepsilon}\right](t-t_0)}. \tag{13.31}$$

Equation (13.31) implies that the zero solution of (13.27) is absolutely exponentially stable. □

Because A is a Hurwitz matrix, the Lyapunov matrix equation (13.29) always has symmetric, positive definite matrix solution. Thus, to apply Theorem 13.11, one only needs to verify if there exist $\tau > 0$ and $\varepsilon > 0$ such that H is negative definite. In the following, we derive an explicit condition for proving the existence of such τ and ε.

Theorem 13.12. *Let*

$$\alpha = -\left[R_{11}\,\delta + \frac{p+\tau}{2},\ R_{21}\,\delta - \frac{p}{2},\ R_{31}\,\delta\right]^{\mathrm{T}} = \left(\frac{A^{\mathrm{T}}c}{2} + Rh + \frac{\tau\,c}{2}\right)^{\mathrm{T}}.$$

If there exist $\tau > 0$ *and* $\varepsilon > 0$ *such that*

$$d^{\mathrm{T}}B^{-1}d + \delta + \frac{1}{|a|+\varepsilon} < 0, \tag{13.32}$$

then the zero solution of system (13.27) is absolutely exponentially stable, and thus the two systems (13.5) and (13.6) are globally exponentially synchronized.

Proof. Because the $H(\tau)$ given in Theorem 13.11 is negative definite, this implies that the condition (13.32) is satisfied. □

Now we expand the inequality (13.32) as a quadratic polynomial of τ:

$$\Big(\frac{A^{\mathrm{T}}c}{2}+Rh+\frac{\tau c}{2}\Big)^{\mathrm{T}}B^{-1}\Big(\frac{A^{\mathrm{T}}c}{2}+Rh+\frac{\tau c}{2}\Big)+\delta+\frac{\tau}{|a|+\varepsilon}<0,$$

which can be rewritten as

$$\begin{aligned}\Big(\frac{c}{2}\Big)^{\mathrm{T}}B^{-1}\Big(\frac{c}{2}\Big)\tau^2+\Big[\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)^{\mathrm{T}}B^{-1}\frac{c}{2}+\frac{c^{\mathrm{T}}}{2}B^{-1}\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)+\frac{1}{|a|+\varepsilon}\Big]\tau\\+\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)^{\mathrm{T}}B^{-1}\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)+\delta<0,\end{aligned}\tag{13.33}$$

from which we can obtain simpler conditions for the existence of τ as follows.

Corollary 13.13. *If one of the following conditions is satisfied, then the quadratic inequality (13.32) has positive solutions for τ:*

(1) $\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)^{\mathrm{T}}B^{-1}\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)+\delta<0$

(2) $2\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)^{\mathrm{T}}B^{-1}\frac{c}{2}+\frac{1}{|a|+\varepsilon}<0$

$$\begin{aligned}&\Big[2\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)^{\mathrm{T}}B^{-1}\frac{c}{2}+\frac{1}{|a|+\varepsilon}\Big]^2\\&\quad-c^{\mathrm{T}}B^{-1}c\Big[2\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)^{\mathrm{T}}B^{-1}\Big(\frac{A^{\mathrm{T}}c}{2}+Rh\Big)+\delta\Big]\geq 0.\end{aligned}$$

The above conditions indicate that the inequality (13.33) has positive solutions for τ. Thus, the zero solution of (13.27) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and so the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$.

13.3 Globally Exponential Synchronization w.r.t. Partial State Variables

Chua's circuits are perhaps the earliest developed system from which chaos synchronization was observed [117], via a state signal taken from the transmitter system to drive the response system. In other words, some of the states of the response system are exactly same as that of the drive system. For example, let $x_{\mathrm{d}}=x_{\mathrm{r}}$. Then, consider the synchronizations between the two pairs of $(y_{\mathrm{d}},y_{\mathrm{r}})$ and $(z_{\mathrm{d}},z_{\mathrm{r}})$. In this section, we use partial states stability theory and methodology [68, 95] to study the globally exponential synchronization w.r.t. partial state variables.

(1) For $x_{\mathrm{r}}=x_{\mathrm{d}}$, system (13.7) becomes

$$\begin{aligned}\dot e_y&=-e_y+e_z-u_2(y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}),\\ \dot e_z&=-qe_y-u_3(y_{\mathrm{d}}-y_{\mathrm{r}},z_{\mathrm{d}}-z_{\mathrm{r}}).\end{aligned}\tag{13.34}$$

Theorem 13.14. *In (13.34), let $u_2 = u_3 = 0$. Then, the zero solution of system (13.34) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and thus when $x_{\mathrm{r}} = x_{\mathrm{d}}$ (i.e., without feedback control), the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$ between the two pairs of $(y_{\mathrm{d}}, y_{\mathrm{r}})$ and $(z_{\mathrm{d}}, z_{\mathrm{r}})$.*

Proof. Construct the radially unbounded, positive definite Lyapunov function for system (13.34) as follows:

$$V = e_y^2 + \frac{1}{q} e_z^2 - \varepsilon\, e_y e_z = \begin{pmatrix} e_y \\ e_z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \frac{1}{q} \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad (\text{for } 0 < \varepsilon \ll 1).$$

Then,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= -2e_y^2 + 2e_y e_z - 2e_y e_z - \varepsilon\, e_z(-e_y + e_z) - \varepsilon(-q e_y) e_y \\ &= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2+\varepsilon q & -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & -\varepsilon \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} < 0 \quad (\text{when } e_y^2 + e_z^2 \neq 0 \text{ for } 0 < \varepsilon \ll 1). \end{aligned}$$

It is easy to see that one can choose an $\varepsilon = \varepsilon_0$ such that V is positive definite while $\frac{\mathrm{d}V}{\mathrm{d}t}$ is negative definite. Thus, the zero solution of system (13.34) is absolutely exponentially stable, implying that when $x_{\mathrm{r}} = x_{\mathrm{d}}$ (i.e., without feedback control), the two systems (13.5) and (13.6) are globally exponentially synchronized between the two pairs of $(y_{\mathrm{d}}, y_{\mathrm{r}})$ and $(z_{\mathrm{d}}, z_{\mathrm{r}})$. □

(2) For $y_{\mathrm{r}} = y_{\mathrm{d}}$, system (13.7) becomes

$$\begin{aligned} \dot{e}_x &= -p\,e_x - p\,[f(x_{\mathrm{d}}) - f(x_{\mathrm{r}})] - u_1(x_{\mathrm{d}} - x_{\mathrm{r}},\, z_{\mathrm{d}} - z_{\mathrm{r}}), \\ \dot{e}_z &= -u_3(x_{\mathrm{d}} - x_{\mathrm{r}},\, z_{\mathrm{d}} - z_{\mathrm{r}}). \end{aligned} \tag{13.35}$$

Theorem 13.15. *In (13.35), choose*

$$u_1 = p\,\delta_x(x_{\mathrm{d}} - x_{\mathrm{r}})\ (\delta_x > -(1+a)), \quad u_3 = -\,\delta_z(z_{\mathrm{d}} - z_{\mathrm{r}})\ (\delta_z > 0). \tag{13.36}$$

Then, the zero solution of system (13.35) is absolutely exponentially stable, and thus when $y_{\mathrm{r}} = y_{\mathrm{d}}$, the two systems (13.5) and (13.6) are globally exponentially synchronized between the two pairs of $(x_{\mathrm{d}}, x_{\mathrm{r}})$ and $(z_{\mathrm{d}}, z_{\mathrm{r}})$.

Proof. Construct the radially unbounded, positive definite Lyapunov function:

$$V = \frac{1}{2} e_x^2 + \frac{1}{2} e_z^2,$$

which yields

$$\frac{\mathrm{d}V}{\mathrm{d}t} = e_x \dot{e}_x + e_z \dot{e}_z \leq -p\,e_x^2 - p\,a\,e_x^2 - p\,\delta_x e_x^2 - \delta_z e_z^2 := -\,p\,\tilde{\delta}_x e_x^2 - \delta_z e_z^2,$$

where $\tilde{\delta}_x = (\delta_x + 1 + a)$. Obviously, the conclusion is true. □

(3) For $z_\mathrm{r} = z_\mathrm{d}$, system (13.7) becomes

$$\begin{aligned}\dot{e}_x &= -p\,e_x + p\,e_y - p\,[f(x_\mathrm{d}) - f(x_\mathrm{r})] - u_1(x_\mathrm{d} - x_\mathrm{r}, y_\mathrm{d} - y_\mathrm{r}),\\ \dot{e}_y &= e_x - e_y - u_2(x_\mathrm{d} - x_\mathrm{r}, y_\mathrm{d} - y_\mathrm{r}).\end{aligned} \tag{13.37}$$

Theorem 13.16. *In (13.37), take either*

(i) $u_1 = p\,\delta_x\,(x_\mathrm{d} - x_\mathrm{r}) \quad (\delta_x > |a|), \qquad u_2 = 0;$ *or*

(ii) $u_1 = p\,\delta_x\,(x_\mathrm{d} - x_\mathrm{r}) \;\; (\delta_x > -(1+a)),$

$$u_2 = \delta_y\,(y_\mathrm{d} - y_\mathrm{r}) \;\; \left(\delta_y > \frac{1}{1+a+\delta_x} - 1\right). \tag{13.38}$$

Then, the zero solution of system (13.37) is absolutely exponentially stable in Hurwitz angle $[0,L]$, and thus when $z_\mathrm{r} = z_\mathrm{d}$, the two systems (13.5) and (13.6) are globally exponentially synchronized in Hurwitz angle $[0,L]$ between the two pairs of $(x_\mathrm{d}, x_\mathrm{r})$ and $(y_\mathrm{d}, y_\mathrm{r})$.

Proof. Construct the radially unbounded, positive definite Lyapunov function:

$$V = \frac{1}{p}\,e_x^2 + e_y^2.$$

Then, for Case (i):

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le -2\,e_x^2 + 2\,e_x e_y + 2\,|a|\,e_x^2 - 2\,\delta_x e_x^2 + 2\,e_x e_y - 2\,e_y^2\\ &\le \begin{pmatrix} e_x\\ e_y\end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2\,(1+\tilde{\delta}_x) & 2\\ 2 & -2\end{bmatrix} \begin{pmatrix} e_x\\ e_y\end{pmatrix} < 0 \qquad (\text{for } \; e_x^2 + e_y^2 \ne 0),\end{aligned}$$

where $\tilde{\delta}_x = \delta_x - |a| > 0$. For Case (ii):

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le \begin{pmatrix} e_x\\ e_y\end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2\,(1+a+\delta_x) & 2\\ 2 & -2\,(1+\delta_y)\end{bmatrix} \begin{pmatrix} e_x\\ e_y\end{pmatrix} < 0 \qquad (\text{for } \; e_x^2 + e_y^2 \ne 0).$$

Hence, the conclusion of Theorem 13.16 is true. □

13.4 Remarks on Nonsynchronization

If no feedback control is applied to the drive-response systems (13.5) and (13.6), or even with a driving signal but without feedback control, then it may fail to synchronize the two systems.

(1) Part of variables are not synchronized. Suppose $y_\mathrm{r} = y_\mathrm{d}$, and take $u_1 = u_3 = 0$ (i.e., without feedback). Then, obviously the zero solution of system (13.35) cannot be asymptotically stable. This is because $e_z(t) \equiv e_z(t_0) \ne 0$, implying that

$\lim_{t\to+\infty} e_z(t) \neq 0$, and therefore when $x_{\mathrm{d}}(0) > E$ and $x_{\mathrm{r}}(0) > E$ $(E > 0)$, $\lim_{t\to+\infty} e_x(t) = +\infty$; when $x_{\mathrm{d}}(0) < -E$ and $x_{\mathrm{r}}(0) < -E$, $\lim_{t\to+\infty} e_x(t) = -\infty$. This indicates that although y_{d} is used as the driving signal, x_{d} and x_{r} (z_{d} and z_{r}) cannot be synchronized.

Let $z_{\mathrm{r}} = z_{\mathrm{d}}$ and take $u_1 = u_2 = 0$ in (13.37). Then, when $|x_{\mathrm{r}}| < E$ and $|x_{\mathrm{d}}| < E$, $(x_{\mathrm{d}}, y_{\mathrm{d}}) = (0,0)$ is an equilibrium point of the system:

$$\dot{x}_{\mathrm{d}} = -px_{\mathrm{d}} - pax_{\mathrm{d}} = -(1+a)px_{\mathrm{d}},$$

$$\dot{y}_{\mathrm{d}} = x_{\mathrm{d}} - y_{\mathrm{d}}.$$

However, obviously, $A = \begin{bmatrix} -(1+a)p & 0 \\ 1 & -1 \end{bmatrix}$ is not a Hurwitz matrix. Thus, $\lim_{t\to+\infty} (x_{\mathrm{r}}^2(t) + y_{\mathrm{r}}^2(t)) \neq 0$, implying that x_{d} and x_{r} (y_{d} and y_{r}) cannot be synchronized, though z_{d} is used as the driving signal.

(2) All variables are not synchronized. In (13.6) take $u_1 = u_2 = u_3 = 0$. Then, $(0,0,0)$ is an equilibrium point of system (13.5). However, when $|x_{\mathrm{r}}| \leq E$, the trace of the matrix

$$A = \begin{bmatrix} -(1+a)p & p & 0 \\ 1 & -1 & 0 \\ 0 & -q & 0 \end{bmatrix}$$

is $-p - pa - 1 > 0$, so A is not a Hurwitz matrix. Thus, $\lim_{t\to+\infty} (x_{\mathrm{r}}^2(t) + y_{\mathrm{r}}^2(t) + z_{\mathrm{r}}^2(t)) \neq 0$.

The above discussions may explain why for a long time people believe that chaotic systems cannot be synchronized. It actually means that chaotic systems cannot be synchronized without feedback control or with an improper feedback control. The first observed chaos synchronization [11] was actually obtained using one variable as a driving signal, while the other two variables can be synchronized. Therefore, chaos synchronization can occur only if certain conditions are satisfied.

13.5 Numerical Simulation Results

In this section, we present several examples using numerical simulations to illustrate the theoretical predictions. The fourth-order Runge-Kutta method is used to obtain the results. As the problem of nonsynchronization, discussed in the previous section, is obvious, we thus only consider synchronization with feedback controls. Further, for definite, choose $E = 1$ for the function $f(x)$ in (13.4).

The second example, depicted in Fig. 13.2, uses the control (13.14) with $\delta_x = 2.5$, $l = 1.5$, and $\delta_z = 0.5$. The initial conditions are given in (13.39). It is seen that the error, e_x, quickly, exponentially converges to zero.

The first example takes the control given in (13.9) and choose $\delta_x = 2.5 > -(1+a) = 2.27$. The simulation results are shown in Fig. 13.1. Note that Chua's circuit exhibits chaos only for the initial values bounded in certain region. For example, the

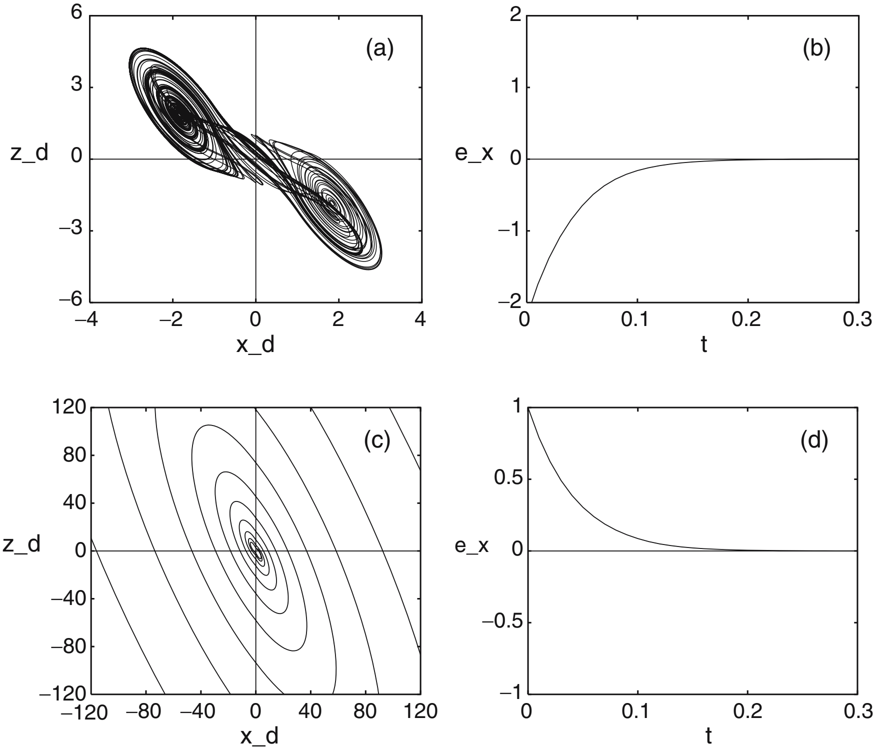

Fig. 13.1 Phase portrait $(x_{\mathrm{d}}, z_{\mathrm{d}})$ (shown in (**a**) and (**c**)) and error time history e_x (shown in (**b**) and (**d**)) of systems (13.5) and (13.6) using the control (13.9) for $\delta_x = 2.5$, with the initial conditions: $x_{\mathrm{d}}(0) = -1.2,\ y_{\mathrm{d}}(0) = -0.5,\ x_{\mathrm{d}}(0) = 1.0;\ x_{\mathrm{r}}(0) = 1.0,\ y_{\mathrm{r}}(0) = 0.2,\ z_{\mathrm{r}}(0) = -1.1$ for (a) and (b); and the initial conditions: $x_{\mathrm{d}}(0) = 1.2,\ y_{\mathrm{d}}(0) = -0.5,\ x_{\mathrm{d}}(0) = 1.0;\ x_{\mathrm{r}}(0) = 0.2,\ y_{\mathrm{r}}(0) = 0.2,\ x_{\mathrm{d}}(0) = -1.1$ for (**c**) and (**d**)

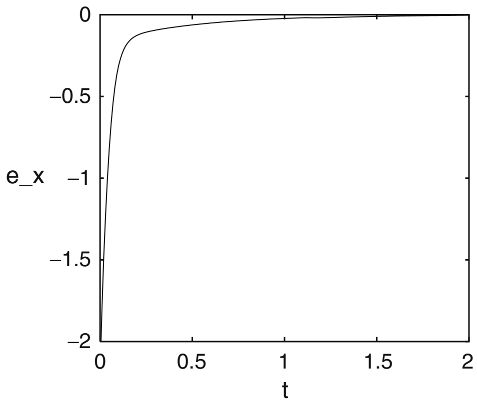

Fig. 13.2 Convergence of e_x of system (13.15) using the control (13.14) for $\delta_x = 2.5,\ l = 1.5,\ \delta_z = 0.5$ with the initial conditions: $x_{\mathrm{d}}(0) = -1.2,\ y_{\mathrm{d}}(0) = -0.5,\ x_{\mathrm{d}}(0) = 1.0;\ x_{\mathrm{r}}(0) = 1.0,\ y_{\mathrm{r}}(0) = 0.2,\ z_{\mathrm{r}}(0) = -1.1$

results given in Figs. 13.1a, b use the following initial values:

$$\begin{aligned} &x_\mathrm{d}(0) = -1.2,\ y_\mathrm{d}(0) = -0.5,\ z_\mathrm{d}(0) = 1.0; \\ &x_\mathrm{r}(0) = 1.0, \quad y_\mathrm{r}(0) = 0.2, \quad z_\mathrm{r}(0) = -1.1, \end{aligned} \tag{13.39}$$

showing a chaotic attractor. When the initial conditions are chosen as:

$$\begin{aligned} &x_\mathrm{d}(0) = 1.2,\ y_\mathrm{d}(0) = -0.5,\ x_\mathrm{d}(0) = 1.0; \\ &x_\mathrm{r}(0) = 0.2,\ y_\mathrm{r}(0) = 0.2, \quad z_\mathrm{r}(0) = -1.1, \end{aligned} \tag{13.40}$$

the trajectory diverges to infinity. However, both two cases show that the error e_x is exponentially converges to zero within a short transient period. (e_y and e_z, not shown in the paper, are also convergent to zero.)

The third example applies the control (13.18) with $\delta_x = 2.5$. The initial conditions are given by

$$\begin{aligned} &x_\mathrm{d}(0) = 1.2, \quad y_\mathrm{d}(0) = -0.5,\ z_\mathrm{d}(0) = -1.0; \\ &x_\mathrm{r}(0) = -1.0,\ y_\mathrm{r}(0) = 0.2, \quad z_\mathrm{r}(0) = 1.1. \end{aligned} \tag{13.41}$$

The results are shown in Fig. 13.3. Again, it can be seen that the error exponentially converges to zero. But unlike the previous two examples, it dies out with normal oscillations.

The next example applies the control (13.22) with the same initial values, given in (13.41), where the control coefficients are chosen as $\delta_x = 0.1$, $\delta_y = 0.2$, and $\delta_z = 0.3$. The error, shown in Fig. 13.4, exponentially converges to zero with some irregular oscillations. It is seen from the above examples that the convergent rate depends upon the initial values. The first two examples converge fast than the third and fourth examples.

The above four examples use linear controls. The next example employs the nonlinear control (13.25) in which $l = 11$ $(>p = 10)$. The error e_x is depicted in Fig. 13.5, again confirming that the error exponentially converges to zero, but for this example, the convergent rate is very slow, compared with the previous examples.

The sixth example is to demonstrate synchronization with respect to partial system variables. It assumes $z_\mathrm{r} = z_\mathrm{d}$, i.e., z_d is used as the driving signal. The control

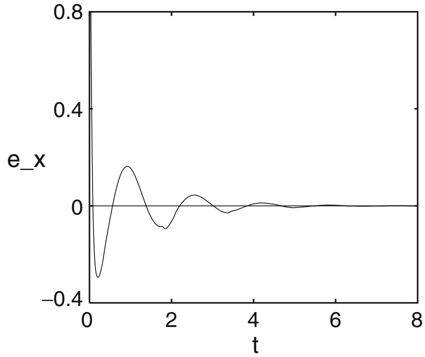

Fig. 13.3 Convergence of e_x of system (13.19) using the control (13.18) for $\delta_x = 2.5$ with the initial conditions: $x_\mathrm{d}(0) = 1.2$, $y_\mathrm{d}(0) = -0.5$, $x_\mathrm{d}(0) = -1.0$; $x_\mathrm{r}(0) = -1.0$, $y_\mathrm{r}(0) = 0.2$, $z_\mathrm{r}(0) = 1.1$

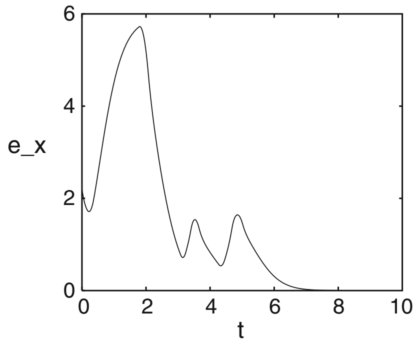

Fig. 13.4 Convergence of e_x of system (13.23) using the control (13.22) for $\delta_x = 0.1$ $\delta_y = 0.2$, $\delta_z = 0.3$ with the initial conditions: $x_{\mathrm{d}}(0) = 1.2$, $y_{\mathrm{d}}(0) = -0.5$, $x_{\mathrm{d}}(0) = -1.0$; $x_{\mathrm{r}}(0) = -1.0$, $y_{\mathrm{r}}(0) = 0.2$, $z_{\mathrm{r}}(0) = 1.1$

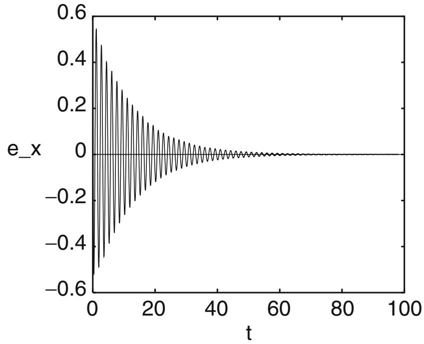

Fig. 13.5 Convergence of e_x of system (13.26) using the control (13.25) for $l = 11.9$ with the initial conditions: $x_{\mathrm{d}}(0) = 1.2$, $y_{\mathrm{d}}(0) = -0.5$, $x_{\mathrm{d}}(0) = -1.0$; $x_{\mathrm{r}}(0) = -1.0$, $y_{\mathrm{r}}(0) = 0.2$, $z_{\mathrm{r}}(0) = 1.1$

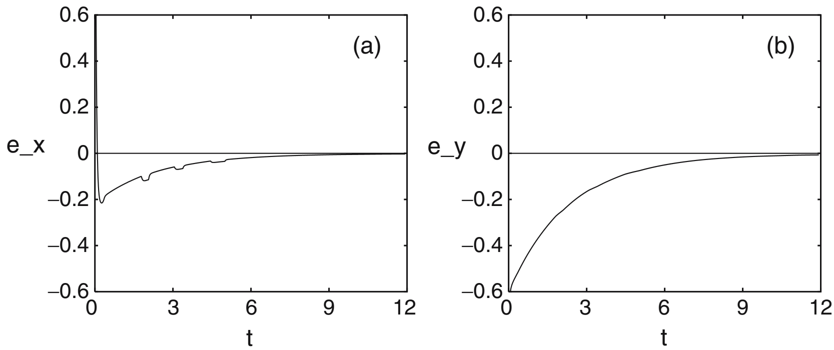

Fig. 13.6 Convergence of the error system (13.37) using control (ii) of (13.16) with $\delta_x = 2.5$, $\delta_y = -0.2$: (**a**) for e_x; and (**b**) for e_y, with the initial conditions: $x_{\mathrm{d}}(0) = 1.2$, $y_{\mathrm{d}}(0) = -0.5$, $x_{\mathrm{d}}(0) = -1.0$; $x_{\mathrm{r}}(0) = -1.0$, $y_{\mathrm{r}}(0) = 0.2$, $z_{\mathrm{r}}(0) = 1.1$

given in case (ii) of (13.16) is used, where $\delta_x = 2.5$, $\delta_y = -0.2$. There is no feedback control for the equation $\dot{z}$ (or $\dot{e}_z$). The initial condition is given in (13.41). The time histories of the error variables e_x and e_y are shown in Fig. 13.6, indicating that the error exponentially converges to zero.

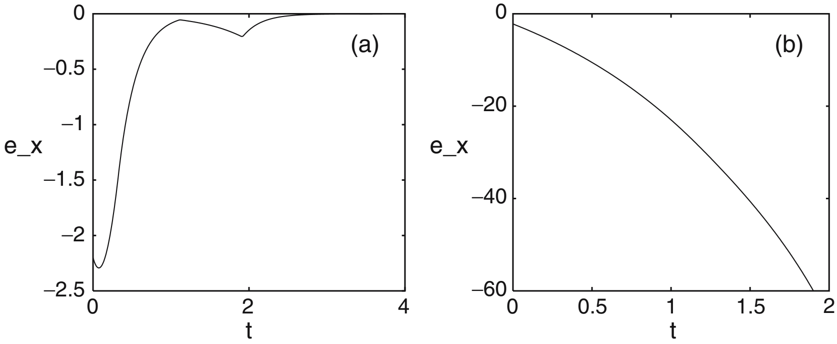

Fig. 13.7 Error time history e_x of systems (13.5) and (13.6) using the control (13.9): (**a**) $\delta_x = 0.1$, convergent; and (**b**) $\delta_x = -0.4$, divergent, for the initial conditions: $x_\mathrm{d}(0) = -1.2$, $y_\mathrm{d}(0) = -0.5$, $x_\mathrm{d}(0) = 1.0$; $x_\mathrm{r}(0) = 1.0$, $y_\mathrm{r}(0) = 0.2$, $z_\mathrm{r}(0) = -1.1$

Finally, we give one more example to show that the conditions given in this paper are sufficient, but not necessary. We use the control (13.9) (which has been used in Example 1, see Fig. 13.1) with different values of δ_x. By Theorem 13.5, it is sufficient to obtain global exponential synchronization if $\delta_x > -(1+a) = 0.27$. In Fig. 13.7, we present two cases: (a) $\delta_x = 0.1$ and (b) $\delta_x = -0.4$. It is seen from Fig. 13.7 that the error still quickly converges to zero when $\delta_x = 0.1 < 0.27$, while it diverges to infinity when $\delta_x = -0.4$. This suggests that the sufficient conditions obtained in this paper might be further improved.

13.6 Master-Slave Synchronization of Two General Lurie Systems

In this section, we study global synchronization of two general Lurie systems. First, we consider the Lurie systems with indirect feedback control [89], and the the Lurie systems with time-delayed feedback control [84].

13.6.1 Indirect Feedback Control

Consider two general Lurie systems with indirect control [89]:

$$\text{Master}: \begin{cases} \dot{x} = Ax + b\xi_1 + d f(\sigma_1), \\ \dot{\xi}_1 = f(\sigma_1), \\ \sigma_1 = c^\mathrm{T} x - \gamma \xi_1; \end{cases} \tag{13.42}$$

$$\text{Slave}: \begin{cases} \dot{y} = Ay + b\xi_2 + d f(\sigma_2), \\ \dot{\xi}_2 = f(\sigma_2), \\ \sigma_2 = c^\mathrm{T} y - \gamma \xi_2. \end{cases} \tag{13.43}$$

Here $A \in R^n$, $x, y \in R^n$, $b, c, d \in R^n$, γ, $\alpha \in R$, and $f(\sigma) \in F_{[0,k]}$.

Let $e(t) = x(t) - y(t)$, $\eta(t) = \xi_1(t) - \xi_2(t)$, $\sigma(t) = \sigma_1(t) - \sigma_2(t)$. Then from (13.42) and (13.43) we have

$$\begin{aligned} \dot{e} &= A\,e(t) + b\,\eta(t) + d\,F(\sigma), \\ \dot{\eta} &= F(\sigma), \\ \sigma &= c^{\mathrm{T}} e(t) - \gamma\eta(t), \end{aligned} \tag{13.44}$$

where

$$0 \le \frac{F(\sigma)}{\sigma} = \frac{f(\sigma_1) - f(\sigma_2)}{\sigma_1 - \sigma_2} = \frac{f(\sigma + \sigma_2) - f(\sigma_2)}{\sigma_1 - \sigma_2} \le k.$$

From the third equation of (13.44) we have

$$\eta = \frac{c^{\mathrm{T}} e - \sigma}{\gamma} \quad \text{and} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = c^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} - \gamma \frac{\mathrm{d}\eta}{\mathrm{d}t} = c^{\mathrm{T}} \frac{\mathrm{d}e}{\mathrm{d}t} - \gamma F(\sigma).$$

Then system (13.44) can be equivalently written as

$$\begin{aligned} \dot{e} &= \left(A + \frac{1}{\gamma} b\, c^{\mathrm{T}}\right) e(t) - \frac{b}{\gamma}\sigma + d\,F(\sigma), \\ \dot{\sigma} &= c^{\mathrm{T}}\left(A + \frac{1}{\gamma} b\, c^{\mathrm{T}}\right) e(t) - \frac{c^{\mathrm{T}} b}{\gamma}\sigma + (c^{\mathrm{T}} d - \gamma) F(\sigma). \end{aligned} \tag{13.45}$$

Thus, if the zero solution of the system (13.44) or (13.45) is absolutely stable, then the systems (13.42) and (13.43) are globally synchronized.

In general, two chaotic systems are not possible to be synchronized without feedback controls. Thus, how to design a feedback control for system (13.45) such that the zero solution of the system (13.44) or (13.45) becomes absolutely stable is a new concept in absolutely stabilizing the Lurie systems, which are not absolutely stable. A simple and easy-applicable feedback control law should be based on some simple algebraic sufficient conditions of absolute stability. Otherwise, even for a general non-Hurwitz matrix, how to choose possible nonconservative feedback control matrix K such that $A + K$ becomes Hurwitz is not easy to verify in practice. Although many computer software like those for solving linear matrix inequality (Matlab) can be used to help solve such kind of problems, only that with all definite parameter values can be considered. For other design purpose such as parameter study (in which may parameters are represented by symbolic notations), computer software cannot provide useful criteria.

In the following, we consider adding as simple feedback controls as possible to system (13.45) to obtain absolute stability. To achieve this, adding $-\alpha I_n e$ and $-\beta_1\sigma$, $-\beta_2 F(\sigma)$, respectively, to the first and second equations of (13.45) yields the controlled system as follows:

$$\begin{aligned} \dot{e} &= \left(A + \frac{1}{\gamma} b\, c^{\mathrm{T}} - \alpha I_n\right) e(t) - \frac{b}{\gamma}\sigma + d\,F(\sigma), \\ \dot{\sigma} &= c^{\mathrm{T}}\left(A + \frac{1}{\gamma} b\, c^{\mathrm{T}}\right) e(t) - \frac{c^{\mathrm{T}} b}{\gamma}\sigma - \beta_1\sigma + (c^{\mathrm{T}} d - \gamma - \beta_2) F(\sigma), \end{aligned} \tag{13.46}$$

where e and σ are state variables.

Theorem 13.17. *The sufficient and necessary conditions for the zero solution of system (13.45) being absolutely stable, i.e., the two systems (13.42) and (13.43) are globally synchronized, are given by*

1. *The matrix* $H = \begin{bmatrix} A + \frac{1}{\gamma} b c^{\mathrm{T}} - \alpha I_n & -\frac{b}{\gamma}\sigma + d \\ c^{\mathrm{T}}\left(A + \frac{1}{\gamma} b c^{\mathrm{T}}\right) & -\frac{c^{\mathrm{T}} b}{\gamma} - \beta_1 - \beta_2 + c^{\mathrm{T}} d - \gamma \end{bmatrix}$ *is a Hurwitz matrix*
2. *The zero solution of the system (13.45) is absolutely stable w.r.t.* σ

Proof. *Necessity*. For (1) we let $f(\sigma) = \sigma$. Then system (13.45) becomes linear and thus H must be a Hurwitz matrix. For (2) it is obvious.

Sufficiency. let the solution of (13.45) be expressed as

$$\begin{pmatrix} \dot{e}(t) \\ \dot{\sigma}(t) \end{pmatrix} = \begin{bmatrix} A + \frac{1}{\gamma} b c^{\mathrm{T}} - \alpha I_n & -\frac{b}{\gamma}\sigma + d \\ c^{\mathrm{T}}\left(A + \frac{1}{\gamma} b c^{\mathrm{T}}\right) & -\frac{c^{\mathrm{T}} b}{\gamma} - \beta_1 - \beta_2 + c^{\mathrm{T}} d - \gamma \end{bmatrix} \begin{pmatrix} e(t) \\ \sigma(t) \end{pmatrix}$$
$$+ \begin{pmatrix} d \\ c^{\mathrm{T}} d - \gamma - \beta_2 \end{pmatrix} F(\sigma(t)) - \begin{pmatrix} d \\ c^{\mathrm{T}} d - \gamma - \beta_2 \end{pmatrix} \sigma(t). \tag{13.47}$$

Then following Theorem 4.3 we know that the conclusion of the theorem is true. The proof is complete. □

Lemma 13.18. *If appropriate values of* α, β_1, β_2 *can be chosen such that*

1. *The matrix* $H_1 = \left(A + \frac{1}{\gamma} b c^{\mathrm{T}} - \alpha I_n\right)$ *is a Hurwitz matrix*
2. *The zero solution of the system (13.45) is absolutely stable w.r.t.* σ

Then the conclusion of Theorem 13.17 *holds.*

Rewrite the H in Theorem 13.17 as

$$H = \begin{bmatrix} H_{m\times m} & H_{m\times(n+1-m)} \\ H_{(n+1-m)\times m} & H_{(n+1-m)\times(n+1-m)} \end{bmatrix}.$$

Then we have

Theorem 13.19. *If appropriate values of* α, β_1, β_2 *can be chosen such that*

1. *The matrix* $H_{m\times m}$ *is a Hurwitz matrix*
2. *The zero solution of the system (13.46) is absolutely stable w.r.t. the partial variables* $e_{m+1}, e_{m+2}, \cdots, e_n, \sigma$

Then the zero solution of system (13.46) is absolutely stable, i.e., the two systems (13.42) and (13.43) are globally synchronized.

Proof. Let

$$e_I = (e_1, e_2, \dots, e_m)^{\mathrm{T}},\ e_{II} = (e_{m+1}, e_{m+2}, \dots, e_n, \sigma)^{\mathrm{T}},$$

$$d_I = (d_1, d_2, \dots, d_m)^{\mathrm{T}},\ d_{II} = (d_{m+1}, d_{m+2}, \dots, d_n, c^{\mathrm{T}}d - \gamma - \beta_2)^{\mathrm{T}},$$

Rewrite system (13.45) as

$$\begin{aligned}\dot{e}_I &= H_{m\times m}\, e_I + H_{m\times(n+1-m)}\, e_{II} + d_I F(\sigma),\\ \dot{e}_{II} &= H_{(n+1-m)\times m}\, e_I + H_{(n+1-m)\times(n+1-m)}\, e_{II} + d_{II} F(\sigma).\end{aligned} \tag{13.48}$$

We can use the method of constant variation to express the solution of the first equation of (13.48) as

$$e_I(t) = e^{H_{m\times m}(t-t_0)} e_I(t_0) + \int_{t_0}^{t} e^{H_{m\times m}(t-\tau)} \Big[H_{m\times(n+1-m)} e_{II} + d_I F(\sigma(\tau))\Big]\, d\tau.$$

One can use the absolute stability of the zero solution of (13.48) w.r.t. e_{II} to complete the proof. □

Corollary 13.20. *If the following conditions are satisfied:*

1. *The condition (1) of Theorem 13.17 holds*
2. *There exist positive definite, radially unbounded Lyapunov function $V(e_{II})$ w.r.t. the partial variables $\sigma_{m+1}, \sigma_{m+1}, \dots, \sigma_{m+1}, \sigma$ such that $\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(13.46)}$ is negative definite w.r.t. e_{II}.*

Then the zero solution of system (13.46) is absolutely stable, i.e., the two systems (13.42) and (13.43) are globally synchronized.

The conclusion of Corollary 13.20 is true simply because the condition (2) of the Lemma 13.18 implies the condition (2) of Theorem 13.19.

Corollary 13.21. *If the following conditions are satisfied:*

1. *The condition (1) of Theorem* 13.19 *hold*
2. *There exist constants $\eta_i \geq 0,\ i = 1, 2, \dots, m$ and $\eta_j > 0,\ j = m+1, m+2, \dots, n$ such that*

$$\eta_j h_{jj} + \sum_{i=1, i\neq j}^{n+1} \eta_i \|h_{ij}\| \leq 0, \quad j = 1, 2, \dots, m,$$

$$\eta_j h_{jj} + \sum_{i=1, i\neq j}^{n+1} \eta_i \|h_{ij}\| < 0, \quad j = m+1, m+2, \dots, n,$$

and

$$\eta_{n+1} h_{(n+1)(n+1)} + \sum_{i=1}^{n} \eta_i \|h_{i(n+1)}\| \leq 0,$$

$$\eta_{n+1} h_{(n+1)(n+1)} + \sum_{i=1}^{n} \eta_i \|d_i\| \leq 0,$$

where at least one of the two inequalities is a strict inequality.

Then the conclusion of Corollary 13.20 holds.

Proof. Construct radially unbounded, positive definite Lyapunov function:

$$V = \sum_{i=1}^{n} \eta_i |e_i| + \eta_{n+1} |\sigma|$$

w.r.t. $e_{m+1}, e_{m+2}, \ldots, e_n, \sigma$. Then we have

$$\begin{aligned} D^+V\big|_{(13.45)} \leq & \sum_{j=1}^{m} \Big(\eta_j h_{jj} + \sum_{i=1, i\neq j}^{n+1} \eta_i |h_{ii}|\Big)|e_j| + \sum_{j=m+1}^{n} \Big(\eta_j h_{jj} + \sum_{i=1, i\neq j}^{n+1} \eta_i |h_{ii}|\Big)|e_j| \\ & + \eta_{n+1} h_{(n+1)(n+1)} + \sum_{i=1}^{n} \eta_i \ h_{i(n+1)} + \eta_{n+1}(c^{\mathrm{T}} d - \gamma) + \sum_{i=1}^{n} \eta_i |d_i| \\ < 0 & \qquad \text{when} \ \sum_{j=m+1}^{n+1} |e_i| + |\sigma| \neq 0. \end{aligned}$$

Therefore, the conclusion of Corollary 13.20 is true. □

13.6.2 Time-Delayed Feedback Control

Finally, we consider more general two master-slave Lurie systems, given by

$$\text{Master}: \begin{cases} \dot{x} = Ax(t) + Bf(Cx(t)), \\ p(t) = Hx(t); \end{cases} \tag{13.49}$$

$$\text{Slave}: \begin{cases} \dot{y} = Ay + Bf(Cy(t)) + u(t), \\ q(t) = Hy(t). \end{cases} \tag{13.50}$$

Here the control is given by

$$u(t) = -K\left(x(t) - y(t)\right) + M\left(p(t-\tau) - q(t-\tau)\right),$$

τ is a constant time delay, $x,\ y \in R^n$, $A,\ B,\ C,\ H,\ K,\ M \in R^{n\times n}$, and $f(\cdot) = (f_1, f_2, \ldots, f_n)^{\mathrm{T}}$, $f_i \in F_{[0,k]}$.

Let $e(t) = x(t) - y(t)$. We then obtain the following error system:

$$\dot{e}(t) = (A+K)\,e(t) + B\tilde{f}(Ce(t), y(t)) + F\left(e(t-\tau)\right), \tag{13.51}$$

where

$$\begin{aligned} F &= -MH, \\ \tilde{f}(Ce,y) &\overset{\triangle}{=} f(Ce + Cy) - f(Cy), \\ C &= (c_1, c_2, \ldots, c_n) \quad c_i \in R^n, \end{aligned}$$

and

$$0 \leq \frac{\tilde{f}(c_i^{\mathrm{T}} e, y)}{c_i^{\mathrm{T}} e} = \frac{f_i(c_i^{\mathrm{T}} e + c_i^{\mathrm{T}} y) - f_i(c_i^{\mathrm{T}} y)}{c_i^{\mathrm{T}} e} \leq k < +\infty, \tag{13.52}$$

for $i = 1, 2, \ldots, n$, $\forall e, y \in R^n$.

From (13.52) we have

$$\tilde{f}(c_i^{\mathrm{T}} e, y)\,(\tilde{f}(c_i^{\mathrm{T}} e, y) - k\,c_i^{\mathrm{T}} e) \leq 0, \quad i = 1, 2, \ldots, n. \tag{13.53}$$

Here, we allow $K \neq 0$, which is more useful in using feedback control to reach synchronization.

Theorem 13.22. *If there exist $n \times n$ constant matrices: $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) > 0$ and positive number ε, $0 < \varepsilon \ll 1$, such that the following matrix is negative definite:*

$$\begin{bmatrix} P(A+K) + (A+K)^{\mathrm{T}} P + Q + \varepsilon I_n & PB + kC^{\mathrm{T}}\Lambda & PF \\ B^{\mathrm{T}} P + k\Lambda C & -2\Lambda & 0 \\ F^{\mathrm{T}} P & 0 & -Q \end{bmatrix} \leq 0,$$

then the zero solution of system (13.51) is absolutely stable, i.e., the two systems (13.49) and (13.50) are globally synchronized.

Proof. Construct the radially unbounded, positive definite Lyapunov function:

$$V(e) = e^{\mathrm{T}} P e + \int_{t-\tau}^{t} e^{\mathrm{T}}(\tau)\,Q\,e(\tau)\,\mathrm{d}\tau. \tag{13.54}$$

Then, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.51)} &= \dot{e}^{\mathrm{T}}(t) P e(t) + e^{\mathrm{T}}(t) P \dot{e}(t) + e^{\mathrm{T}}(t) Q e(t) - e^{\mathrm{T}}(t-\tau) Q e(t-\tau) \\
&\leq e^{\mathrm{T}}(t) \Big[P(A+K) + (A+K^{\mathrm{T}})P)\Big] e(t) + \tilde{f}(C^{\mathrm{T}} e, y)\, B^{\mathrm{T}} P e(t) \\
&\quad + e^{\mathrm{T}}(t) PB \tilde{f}(C^{\mathrm{T}} e, y(t)) + e^{\mathrm{T}}(t-\tau) F^{\mathrm{T}} P e(t) + e^{\mathrm{T}}(t)\, PF\, e(t-\tau) \\
&\quad + e^{\mathrm{T}}(t) Q e(t) - e^{\mathrm{T}}(t-\tau) Q e(t-\tau) \\
&\quad - \sum_{i=1}^{n} \lambda_i \tilde{f}(C^{\mathrm{T}} e, y(t)) \Big[\tilde{f}(C^{\mathrm{T}} e, y(t)) - kC_i^{\mathrm{T}} e\Big] \\
&= \begin{pmatrix} e(t) \\ \tilde{f}(C^{\mathrm{T}} e, y) \\ e(t-\tau) \end{pmatrix}^{T} \begin{bmatrix} P(A+K) + (A+K)^{\mathrm{T}} P + Q + \varepsilon I_n & PB + kC^{\mathrm{T}}\Lambda & PF \\ B^{\mathrm{T}} P + k\Lambda C & -2\Lambda & 0 \\ F^{\mathrm{T}} P & 0 & -Q \end{bmatrix} \\
&\quad \times \begin{pmatrix} e(t) \\ \tilde{f}(C^{\mathrm{T}} e, y) \\ e(t-\tau) \end{pmatrix} \\
&\leq -\varepsilon\, e^{\mathrm{T}}(t)\, e(t) < 0 \qquad \text{when } e(t) \neq 0.
\end{aligned}$$

This completes the proof. □

In Theorem 13.22 taking $Q = I_n$ yields

Corollary 13.23. *If there exist $n \times n$ constant matrices: $P = P^{\mathrm{T}} > 0$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) > 0$ and $0 < \varepsilon \ll 1$ such that*

$$\begin{bmatrix} P(A+K)+(A+K)^{\mathrm{T}}P+(1+\varepsilon)I_n & PB+kC^{\mathrm{T}}\Lambda & PF \\ B^{\mathrm{T}}P+k\Lambda C & -2\Lambda & 0 \\ F^{\mathrm{T}}P & 0 & -I_n \end{bmatrix} \leq 0,$$

then the zero solution of system (13.51) is absolutely stable, i.e., the two systems (13.49) and (13.50) are globally synchronized.

Assume that $C \in R^{n\times n}$ is a nonsingular matrix. Introduce the transform $\xi(t) = C^{\mathrm{T}}e(t)$ into (13.51) to obtain

$$\begin{aligned} \dot{\xi}(t) &= C(A+K)C^{-1}\xi(t) + CB\tilde{f}(\xi(t),y) + CFC^{-1}\xi(t-\tau), \\ &:= \tilde{A}\xi(t) + \tilde{B}\tilde{f}\xi(t),y) + \tilde{F}\xi(t-\tau). \end{aligned} \tag{13.55}$$

Theorem 13.24. *If there exist $n \times n$ constant matrices: $P = P^{\mathrm{T}} > 0$, and a constant ε, $0 < \varepsilon \ll 1$, such that*

$$\begin{bmatrix} P\tilde{A}+\tilde{A}P+G+(1+\varepsilon)I_n & P\tilde{F} \\ \tilde{F}^{\mathrm{T}}P & -I_n \end{bmatrix} \leq 0, \tag{13.56}$$

where $H := 2P\tilde{B} = (h_{ij})_{n\times n}$, $G := (g_{ij})_{n\times n}$, in which

$$g_{ij} = \begin{cases} kh_{ij} & \text{when } h_{ij}\xi_i\tilde{f}_j \geq 0, \quad i \neq j, \\ -kh_{ij} & \text{when } h_{ij}\xi_i\tilde{f}_j < 0, \quad i \neq j, \\ kh_{ii} & \text{when } h_{ii}\xi_i\tilde{f}_i > 0, \\ 0 & \text{when } h_{ii}\xi_i\tilde{f}_i \leq 0, \end{cases}$$

$i, j = 1, 2, \ldots, n$. Then the zero solution of system (13.55) is absolutely stable, i.e., the two systems (13.49) and (13.50) are globally synchronized.

Proof. Construct the radially unbounded, positive definite Lyapunov function:

$$V(e,\xi) = \xi^{\mathrm{T}}P\xi + \int_{t-\tau}^{t} \xi^{\mathrm{T}}(s)\xi(s)\,\mathrm{d}s. \tag{13.57}$$

Differentiating V w.r.t. time t along the trajectory of system (13.55) yields

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(13.55)} &= \xi^{\mathrm{T}}(t)(P\tilde{A}+\tilde{A}^{\mathrm{T}}P)\xi(t) + 2\xi^{\mathrm{T}}(t)P\tilde{B}\tilde{f}(\xi(t),y) \\ &\quad + \xi^{\mathrm{T}}(t-\tau)\tilde{F}^{\mathrm{T}}P\xi(t) + \xi^{\mathrm{T}}(t)P\tilde{F}\xi(t-\tau)\xi^{\mathrm{T}}(t)\xi(t) - \xi^{\mathrm{T}}(t-\tau)\xi(t-\tau) \\ &\leq \xi^{\mathrm{T}}(t)(P\tilde{A}+\tilde{A}^{\mathrm{T}}P)\xi(t) + \xi^{\mathrm{T}}(t)G\xi(t) + \xi^{\mathrm{T}}(t-\tau)\tilde{F}^{\mathrm{T}}P\xi(t) \end{aligned}$$

$$
\begin{aligned}
&+\xi^{\mathrm{T}}(t)P\tilde{F}\xi(t-\tau)+\xi^{\mathrm{T}}(t)\xi(t)-\xi^{\mathrm{T}}(t-\tau)\xi(t-\tau)\\
\le\ &\xi^{\mathrm{T}}(t)(P\tilde{A}+\tilde{A}^{\mathrm{T}}P)\xi(t)+\xi^{\mathrm{T}}(t)G\xi(t)+\xi^{\mathrm{T}}(t-\tau)\tilde{F}^{\mathrm{T}}P\xi(t)\\
&+\xi^{\mathrm{T}}(t)P\tilde{F}^{\mathrm{T}}\xi(t-\tau)+\varepsilon\xi^{\mathrm{T}}(t)\xi(t)-\varepsilon\xi^{\mathrm{T}}(t)\xi(t)\\
\le\ &\begin{pmatrix}\xi(t)\\ e(t-\tau)\end{pmatrix}^{T}\begin{bmatrix}P\tilde{A}+\tilde{A}P+G+(1+\varepsilon)I_n & P\tilde{F}\\ \tilde{F}^{\mathrm{T}}P & -I_n\end{bmatrix}\begin{pmatrix}\xi(t)\\ \xi(t-\tau)\end{pmatrix}-\varepsilon\xi^{\mathrm{T}}(t)\xi(t)\\
\le\ &-\varepsilon\xi^{\mathrm{T}}(t)\xi(t)<0 \qquad \text{when } \ \xi(t)\ne 0,\ t\ge t_0.
\end{aligned}
$$

Thus the conclusion of Theorem 13.24 is true. □

Define

$$
\tilde{G}=(\tilde{g}_{ij})_{n\times n},\quad \text{where}\quad \tilde{g}_{ij}=\begin{cases}2k\tilde{b}_{ij} & \text{when } \ \tilde{b}_{ij}\xi_i\tilde{f}_j\ge 0,\quad i\ne j,\\ -2k\tilde{b}_{ij} & \text{when } \ \tilde{b}_{ij}\xi_i\tilde{f}_j<0,\quad i\ne j,\\ 2k\tilde{b}_{ii} & \text{when } \ \tilde{b}_{ii}\xi_i\tilde{f}_i>0,\\ 0 & \text{when } \ \tilde{b}_{ii}\xi_i\tilde{f}_i\le 0,\end{cases}
$$

$i,j=1,2,\ldots,n$. Then in Theorem 13.24 taking $P=I_n$ gives

Corollary 13.25. *If there exists* $0<\varepsilon\ll 1$ *such that*

$$
\begin{bmatrix}\tilde{A}+\tilde{A}^{\mathrm{T}}+(1+\varepsilon)I_n+\tilde{G} & \tilde{F}\\ \tilde{F}^{\mathrm{T}} & -I_n\end{bmatrix}\le 0,
$$

then the zero solution of system (13.55) is absolutely stable, i.e., the two systems (13.49) and (13.50) are globally synchronized.

In the following, define $W=(w_{ij})_{n\times n}$, where

$$
w_{ij}=\begin{cases}\tilde{a}_{ii}+|\tilde{F}_{ii}| & \text{when } \ \tilde{b}_{ii}\le 0,\\ \tilde{a}_{ii}+k\tilde{b}_{ii}+|\tilde{F}_{ii}| & \text{when } \ \tilde{b}_{ii}>0,\\ |\tilde{a}_{ii}|+k|\tilde{b}_{ij}|+|\tilde{F}_{ii}| & i\ne j,\end{cases}\qquad i,j=1,2,\ldots,n.
$$

Theorem 13.26. *If* $-W$ *is an* M *matrix, then the zero solution of system (13.55) is absolutely stable, i.e., the two systems (13.49) and (13.50) are globally synchronized.*

Proof. Since $-W$ is an M matrix, there exists constant $\xi_1>0$ such that $\sum_{i=1}^{n}\xi_i w_{ij}<0,\ i,j=1,2,\ldots,n$. Construct the radially unbounded, positive definite Lyapunov function:

$$
V=\sum_{i=1}^{n}\xi_i|\xi(t)|+\sum_{i,j=1}^{n}\int_{t-\tau}^{t}|\tilde{F}_{ij}(\xi_j(s))|\,\mathrm{d}s.
$$

Further let

$$|v_{ij}| = \begin{cases} 0 & \text{if } \tilde{b}_{ii} \le 0, \\ k\tilde{b}_{ii} & \text{if } \tilde{b}_{ii} > 0, \\ k\tilde{b}_{ij} & i \ne j, \end{cases} \qquad i, j = 1, 2, \ldots, n.$$

Then we have

$$\begin{aligned} D^+V\big|_{(13.55)} &= \sum_{i=1}^{n} \xi_i D^+|\xi_i(t)| + \sum_{i,j=1}^{n} |\tilde{F}_{ij}(\xi_j(t))| \\ &\quad + \sum_{i,j=1}^{n} |\tilde{F}_{ij}(\xi_j(t))| - \sum_{i,j=1}^{n} |\tilde{F}_{ij}(\xi_j(t-\tau))| \\ &\le \sum_{i=1}^{n} \xi_i \Big[\tilde{a}_{ii}|\xi_i(t)| + \sum_{j=1, j\ne i}^{n} \tilde{q}_{ij}|\xi_j(t)| \\ &\quad + \sum_{j=1}^{n} |v_{ij}||\xi_j(t)| + \sum_{j=1}^{n} |\tilde{F}_{ij}(\xi_j(t-\tau))|\Big] \\ &\quad + \sum_{i,j=1}^{n} |\tilde{F}_{ij}(\xi_j(t))| - \sum_{j=1}^{n} |\tilde{F}_{ij}(\xi_j(t-\tau))| \\ &\le \sum_{j=1}^{n} \Big[\sum_{j=1}^{n} \xi_i w_{ij}|\xi_j(t)|\Big] < 0 \quad \text{when } \xi(t) \ne 0, \ t \ge t_0. \end{aligned}$$

The proof is complete. □

Theorem 13.27. *If there exists a constant matrix $P = P^{\mathrm{T}} > 0$ such that*

$$P\tilde{A} + \tilde{A}P + P\tilde{B}\tilde{B}^{\mathrm{T}}P + (k^2+1)\,I_n < 0,$$

then the zero solution of system (13.55) is absolutely stable, i.e., the two systems (13.49) and (13.50) are globally synchronized.

Proof. Choose the radially unbounded, positive definite Lyapunov function as

$$V = \xi^{\mathrm{T}}(t)\,P\,\xi(t) + \sum_{i=1}^{n} \int_{t-\tau}^{t} \xi_j^2(s)\,\mathrm{d}s.$$

Then we obtain

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(13.51)} &= \dot{\xi}^{\mathrm{T}}(t)\,P\,\xi(t) + \xi^{\mathrm{T}}(t)P\dot{\xi}(t) + \xi^{\mathrm{T}}(t)Q\xi(t) - \xi^{\mathrm{T}}(t-\tau)Q\xi(t-\tau) \\ &= \xi^{\mathrm{T}}(t)(P\tilde{A} + \tilde{A}P)\,\xi(t) + 2\,\xi^{\mathrm{T}}(t)P\tilde{B}\tilde{f}(\xi(t), y(t)) \\ &\quad + 2\,\xi^{\mathrm{T}}(t)P\tilde{F}(t-\tau) + \xi^{\mathrm{T}}(t)\,\xi(t) - \xi^{\mathrm{T}}(t-\tau)\,\xi(t-\tau) \\ &= \xi^{\mathrm{T}}(t)(P\tilde{A} + \tilde{A}^{\mathrm{T}}P + I_n)\,\xi(t) + \xi^{\mathrm{T}}(t)P\tilde{B}\tilde{B}^{\mathrm{T}}P\,\xi(t) \\ &\quad - \Big[\tilde{B}^{\mathrm{T}}P\,\xi(t) - \tilde{f}(\xi(t), y(t)\Big]^{\mathrm{T}}\Big[\tilde{B}^{\mathrm{T}}P\,\xi(t) - \tilde{f}(\xi(t), y(t)\Big] \end{aligned}$$

$$
\begin{aligned}
&+ \tilde{f}^{\mathrm{T}}(\xi(t),y(t))f(\xi(t),y(t)) + \xi^{\mathrm{T}}(t)P\tilde{F}\tilde{F}^{\mathrm{T}}P\xi(t) \\
&- \left[\tilde{F}^{\mathrm{T}}P\xi(t) - \xi(t-\tau)\right]^{\mathrm{T}}\left[\tilde{F}^{\mathrm{T}}P\xi(t) - \xi(t-\tau)\right] \\
\leq\ & \xi^{\mathrm{T}}(t)\left[P\tilde{A} + \tilde{A}^{\mathrm{T}}P + (1+k^2)I_n + P\tilde{B}\tilde{B}^{\mathrm{T}}P + P\tilde{F}\tilde{F}^{\mathrm{T}}P\right]\xi(t) \\
<\ & 0 \qquad \text{when } \ \xi(t) \neq 0, \ \ t \geq t_0,
\end{aligned}
$$

which shows that the conclusion of Theorem 13.27 is true. □

In Theorem 13.27, taking $P = I_n$ yields more practically useful result.

Corollary 13.28. *For any given $\tilde{F}$, let $u_{\max}$ denote the maximum eigenvalue of the following matrix:*

$$
CAC^{-1} + (C^{-1})^{\mathrm{T}}A^{\mathrm{T}}C^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} + \tilde{F}\tilde{F}^{\mathrm{T}} + (1+k^2)I_n,
$$

and let $\lambda > u_{\max}$. Then if choosing $K = -\mathrm{diag}(\frac{\lambda}{2}\cdots\frac{\lambda}{2})$, we obtain $\tilde{A} + \tilde{A}^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} + \tilde{F}\tilde{F}^{\mathrm{T}} + (1+k^2)I_n < 0$, and the conclusion of Theorem 13.27 is true.

Proof. For any given $\xi \in R^n$, $\xi \neq 0$, we have

$$
\begin{aligned}
&\xi^{\mathrm{T}}\left[\tilde{A} + \tilde{A}^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} + \tilde{F}\tilde{F}^{\mathrm{T}} + (1+k^2)I_n\right]\xi \\
&= \xi^{\mathrm{T}}\Big[CAC^{-1} + (C^{-1})^{\mathrm{T}}A^{\mathrm{T}}C^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} + \tilde{F}\tilde{F}^{\mathrm{T}} \\
&\qquad + (1+k^2)I_n + CKC^{-1} + (C^{-1})^{\mathrm{T}}KC^{\mathrm{T}}\Big]\xi \\
&\leq -\lambda\,\xi^{\mathrm{T}}\xi + u_{\max}\xi^{\mathrm{T}}\xi < 0 \qquad \text{when } \ \xi \neq 0.
\end{aligned}
$$

This completes the proof. □

Example 13.29. Consider the following 2D error system in the form of (13.55):

$$
\begin{aligned}
\begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{pmatrix} &= \begin{bmatrix} -4 & -1 \\ \frac{3}{2} & -6 \end{bmatrix}\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} + \begin{bmatrix} -3 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}\begin{pmatrix} f_1(\xi(t),y(t)) \\ f_2(\xi(t),y(t)) \end{pmatrix} \\
&\quad + \begin{bmatrix} 1 & \frac{3}{2} \\ -\frac{1}{2} & -\frac{6}{5} \end{bmatrix}\begin{pmatrix} \xi_1(t-\tau) \\ \xi_2(t-\tau) \end{pmatrix},
\end{aligned} \tag{13.58}
$$

where the three matrices correspond to $\tilde{A}$, $\tilde{B}$, and $\tilde{F}$, respectively. Then $-W = \begin{bmatrix} 3 & -3 \\ -7/2 & 19/5 \end{bmatrix}$ is an M matrix. The conditions of Theorem 13.24 are satisfied. Hence, the zero solution of system (13.58) is absolutely stable.

Example 13.30. Again consider a 2D error system in the form of (13.55), given by

$$\begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{pmatrix} = \begin{bmatrix} -4 & 8 \\ -16 & -4 \end{bmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} f_1(\xi(t), y(t)) \\ f_2(\xi(t), y(t)) \end{pmatrix}$$
$$+ \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} \xi_1(t-\tau) \\ \xi_2(t-\tau) \end{pmatrix}, \tag{13.59}$$

where the three matrices correspond to $\tilde{A}$, $\tilde{B}$, and $\tilde{F}$, respectively. Taking $k = \sqrt{3/2}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$P\tilde{A} + \tilde{A}P + P\tilde{B}\tilde{B}^{\mathrm{T}}P + P\tilde{F}\tilde{F}^{\mathrm{T}}P + (1+k^2)I_2 = \begin{bmatrix} -\frac{1}{2} & -1 \\ -1 & -\frac{13}{4} \end{bmatrix} < 0,$$

which indicates that the conditions of Theorem 13.24 are satisfied. Thus, the zero solution of system (13.59) is absolutely stable.

To end this section, we use an example to demonstrate that linear feedback control without time delay plays more important role in the study of stability than the linear feedback with time delay.

Example 13.31. Consider a system in the form of (13.55):

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} f_1(e_1(t) - e_2(t), y(t)) \\ f_2(e_1(t) - e_2(t), y(t)) \end{pmatrix}$$
$$+ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} e_1(t-\tau) \\ e_2(t-\tau) \end{pmatrix} - \begin{bmatrix} \frac{\lambda}{2} & 0 \\ 0 & \frac{\lambda}{2} \end{bmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix}, \tag{13.60}$$

where $k = 2$ while λ is to be determined. If choose $\lambda = 0$, namely without linear state feedback, then this system cannot satisfy the conditions given in [162]. That is, no matter what time-delayed linear feedback control is applied, the zero solution of system (13.60) cannot be absolutely stabilized.

In fact, if there exist $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ such that $PA + A^{\mathrm{T}}P + Q < 0$, then we must have $PA + A^{\mathrm{T}}P < 0$. Thus, A is a Hurwitz matrix. However, here $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not a Hurwitz matrix. Since

$$B = \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} \frac{\lambda}{2} & 0 \\ 0 & \frac{\lambda}{2} \end{bmatrix},$$

C is nonsingular, we have

$$CAC^{-1} + (C^{-1})^{\mathrm{T}}A^{\mathrm{T}}C^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} + \tilde{F}\tilde{F}^{\mathrm{T}} + (1+k^2)I_2 = \begin{bmatrix} 31 & 8 \\ 8 & 35 \end{bmatrix}$$

which has the maximum eigenvalue $u_{\max} = 35 + 3\sqrt{2}$. Therefore, as long as one chooses $\lambda > u_{\max} = 35 + 3\sqrt{2}$ and $k = \lambda/2$, the conditions in Corollary 13.28 are satisfied, i.e.,

$$\tilde{A} + \tilde{A}^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} + \tilde{F}\tilde{F}^{\mathrm{T}} + (1 + k^2) I_2 < 0.$$

Therefore, the zero solution of system (13.60) is absolutely stable.

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