Absolute Stability of Nonlinear Control Systems with Isolated Nonlinearities

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***Abstract*—Absolute stability criteria are used to establish stability for a specific class of nonlinear control systems which arise in various applications in control engineering. Despite many of these criteria being well-established, due to the differences in their derivations, it is not always clear when certain criteria are applicable and the type of stability they guarantee. This paper will review the literature on absolute stability, introduce the Lur'e problem, and will discuss fundamental techniques, theorems, definitions, and provide a comprehensive review of the differences in criteria. The main contribution of this paper is to delineate when certain criteria are applicable and the strength of stability available in each case.**

I. Introduction

Nonlinear control is a field of study that focuses on designing controllers for systems that exhibit nonlinear behaviour. The applications of this field are vast and include the control of distributed networks of agents, flocking in autonomous vehicle swarms, coordination of satellites, and many others. Classical controller design techniques, like linear control techniques, are *inherently unsuitable* for ensuring stability of nonlinear control systems. As a result, more advanced approaches are necessary to address the unique challenges posed by these systems. In the past decade, theories have been developed to tackle these challenges.

Stability theory plays a central role in system engineering, especially in the field of control systems and automation, with regard to both dynamics and control. Stability of a dynamical system, with or without control and disturbance inputs, is a fundamental requirement for its practical value, particularly in most real-world applications. Stability of nonlinear systems can be more challenging to analyse compared to their linear counterparts. Essentially, the stability of a nonlinear system can be defined in one of two ways: either the output remains bounded when the input is bounded, which is known as bounded-input-bounded-output stability, or all initial states within a specific region of the system converge to an equilibrium point as time approaches infinity, also referred to as asymptotic stability.

An important class of nonlinear system are so called *Lur'e systems* which consist of a linear element and a static nonlinear element. Therefore, over the past decade or so, theories have been developed to cope with the new challenges presented by such systems. *Absolute stability* theory is the branch of control theory concerned with establishing stability of a Lur'e system. The stability criteria for nonlinear systems are generally stated in terms of the linear system and apply to every element of a specified class of nonlinearities. Hence, absolute stability theory provides sufficient conditions for robust stability with a given class of uncertain elements. The literature on absolute stability is extensive. A convenient way to distinguish these results is to focus on the allowable class of feedback nonlinearities.

This paper will delve into the different absolute stability criteria that can accommodate various nonlinearities. It will also tackle the type of stability guaranteed by these criteria. However, it is worth noting that this task is not an easy feat, as each stability criterion is derived using different techniques. The article seeks to introduce the main technical ideas and provide a straightforward construction of the main results in an accessible way.

II. Control theory

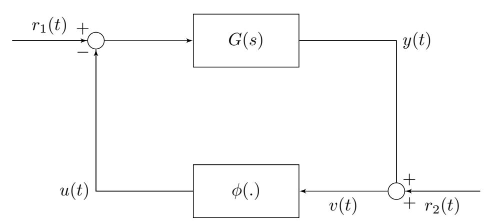

Fig. 1: Lur'e system: a feedback interconnection of a linear system and a static nonlinearity. Positive feedback is assumed.

The system under consideration is shown in Figure [1], which depicts a feedback interconnection containing a linear time invariant (LTI) system $G(s)$ and a static nonlinear feedback $\phi(.)$. LTI systems are a major subclass of control systems, where the parameters do not change with time. LTI systems are particularly amenable to mathematical techniques in the frequency domain, such as the Laplace transform, Fourier transform, Z-transform, Bode plot, Root locus, and Nyquist stability criterion. Due to the multi-variable nature of Lur'e systems, these techniques cannot provide a comprehensive description and analyse stability of these networks.

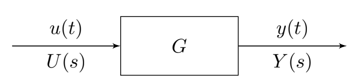

Fig. 2: Linear system G

A linear system G is depicted in Figure [2]. The linear part of the Lure feedback system is represented in various ways, and understanding these representations is important for interpreting the different absolute stability criteria.

1) Convolution integral:

$$y(t) = \int_{-\infty}^{t} g(t-\tau)u(\tau)\,d\tau \tag{1}$$

where $g(t)$ is the impulse response of the system. The convolution integral is instrumental in the proof of the Zames-Falb stability criterion.

2) Transfer functions:

$$Y(s) = G(s)U(s) \tag{2}$$

where $G(s)$ is the systems transfer function. $G(s)$ can also be expressed as a transfer matrix, defined as

$$G(s) \triangleq C(sI-A)^{-1}B + D \tag{3}$$

Transfer functions are crucial to interpret Circle, Popov and Zames-Falb criteria.

3) State-space:

$$G \sim \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \tag{4}$$

where $x(t)$ is an intermediate variable, the state vector linking the state and output equations. State-space representations are relied on by all Lyapunov-based criteria.

Note: although different representations of a system are equivalent and contain the same information, they may be used interchangeably for convenience.

A. Stability

The *stability* of a linear system is completely determined by the eigenvalues of its A matrix. A system is *asymptotically stable* if all the eigenvalues of the A matrix are in the open left half complex plane. A system is *unstable* if any of the eigenvalues of the A matrix are in the open right half complex plane. Stability analysis approaches include the Nyquist stability criterion, the Routh-Hurwitz stability criterion, and others. Nyquist is one of the tests for stability that is most broadly applicable, but it is still only applicable to LTI systems. However, for non-linear systems, there exist generalisations of the Nyquist criterion, such as the Circle criterion. Other stability criteria, such as the Lyapunov approach, can also be used for nonlinear systems.

1) Lyapunov stability: of an equilibrium implies that solutions starting in its vicinity remain sufficiently close to the equilibrium point for all time.

2) Asymptotic stability: is a stronger condition, requiring solutions that start sufficiently close to the equilibrium point to converge to it eventually.

3) Exponential stability: is an even stronger property that guarantees solutions not only converge to the equilibrium point, but do so at a known, or faster, rate.

B. Nonlinear systems

In practice, the majority of systems are somewhat *nonlinear*. This leads to the introduction of the limitation of linear system stability analysis methods, with Laplace and frequency-domain approaches being inapplicable as they are unable to represent the diversity and complexity of some (nonlinear) systems' behaviour.

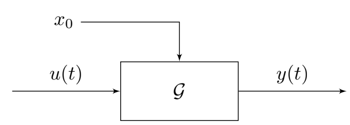

Fig. 3: Nonlinear system $\mathcal{G}$

A nonlinear system $\mathcal{G}$ as seen in Figure [3] is equivalent to the generic nonlinear state-space system:

$$\mathcal{G} \sim \begin{cases} \dot{x} &= f(x,u,t) \\ y &= h(x,u,t) \end{cases} \tag{5}$$

A summary of the distinctions between linear and nonlinear systems and stability is given below in Table I due to the difference in nature of these systems.

Linear systems	Nonlinear systems
$\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t) + Du(t)$	$\dot{x}(t) = f(x(t), u(t))$ $y(t) = h(x(t), u(t))$
– Strictly the Superposition and Homogeneous principle	– Does not follow superposition and homogeneous theorem, at a time
– Only one equilibrium	– Possibly multiple equilibria
– Input/time-dependency	– No input/time-dependency
– Stability is entirely dependent on eigenvalues of A, purely determined by system parameters	– Stability is study of $\dot{x}(t) = f(x(t), u(t))$, function of system parameters, initial conditions and external inputs
– Internal stability $\equiv$ Bounded-input-bounded-output	– Internal stability $\not\equiv$ Bounded-input-bounded-output

TABLE I: Comparison between linear and nonlinear systems

III. Lyapunov Stability

The study of stability in nonlinear systems was first proposed by mathematician Alexandr M. Lyapunov and has since been the subject of extensive research. The central idea of Lyapunov's second method is to find a so-called Lyapunov function satisfying certain properties, which then can be used to prove stability of the nonlinear system. The approach is popular since it enables one to draw conclusions about the system's stability *without* needing to solve the underlying differential equations [1].

Lyapunov's Theorem gives *sufficient* conditions for (asymptotic) stability of nonlinear systems. Consider a nonlinear autonomous system

$$\dot{x} = f(x) \quad f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$$

The origin $x = 0$ is *globally (asymptotically) stable* if there exists a Lyapunov function $V(x) > 0$ such that $\dot{V}(x) < 0$, and no differential equations need to be solved.

The Lyapunov function is positive definite, has a minimum value at the equilibrium point of the system, and is used to measure the stability of a system by assessing how quickly it returns to equilibrium after small perturbations. In the case of nonlinear systems, the Lyapunov function often takes a non-quadratic form, with the requirement that it is positive definite, differentiable, and has a negative definite derivative. Recent research has shown that a certain class of Lyapunov functions, namely the quadratic Lyapunov function, can be used to automate the search for a Lyapunov function and thereby certify the stability of a nonlinear system. However, it is important to note that Lyapunov's second method is only a sufficient condition for stability. Therefore, if the search is unsuccessful, it does not necessarily mean that the system is unstable, but rather that stability has not been established.

If a Lyapunov function can be found for a nonlinear system, the system is considered stable if the function decreases along the trajectories of the system. In other words, the system will tend towards its equilibrium point over time. Lyapunov stability theory can be applied to both autonomous and non-autonomous systems and has been successfully used to analyse the stability of a wide range of nonlinear control problems.

1) Quadratic Lyapunov Functions: Lyapunov's method requires one to choose a positive definite Lyapunov function and then prove that its derivative is negative semi/definite. This is a difficult task in general, but in the case of linear systems, it is both necessary and sufficient to confine one's attention to linear systems. Considering the linear system

$$\dot{x} = f(x) = \underbrace{Ax}_{linear} \quad \Re(\lambda_i(A)) < 0 \quad \forall i \tag{6}$$

If A has all negative eigenvalues, there exists a $P > 0$ satisfying the Lyapunov inequality:

$$A^T P + PA = -Q < 0 \tag{7}$$

It is easy to see that with $V(x) = x' P(x)$, $\dot{V}(x) = x^T(A^T P + PA)x$. Thus for $\dot{V}(x) < 0$ it follows that equation [7] must hold. Although quadratic Lyapunov functions are a small subset of the available choice for Lyapunov functions, they have found favour amongst control engineers since i) many systems can be modelled as a linear system plus a perturbation; and ii) searches over quadratic Lyapunov functions can often be cast as LMI problems which are computationally tractable.

IV. Linear Matrix Inequalities

Linear matrix inequalities (LMIs) are a class of matrix inequalities that are linear or affine in a set of matrix variables. They are a sub-class of semi-definite programming problems with the significant advantage that they are computationally tractable, with various LMI solvers available. LMIs have proven to be of great significance to control theorists, as many linear and some nonlinear stability and performance problems can be cast as LMIs. The first LMI used to examine the stability of a dynamical system dates back to the 1890s, with the analytically solvable Lyapunov inequality on P (as seen in equation [7]).

In the 1940s, Lyapunov's techniques were used by Lure, Postnikov, and others to address the stability of non-linear control systems. Their criteria, taking the form of LMIs, were unintentionally established. Positive-real (PR) lemma was used in the 1960s to reduce the solution of LMIs that appeared in the Lur'e problem to straightforward graphical criteria. As a result, the Popov criterion, Circle criterion, Tsypkin criterion, and their variants were developed. During the 1980s, LMIs that arose in system and control theory were discovered to be expressed as convex optimization problems that computers can solve. Nesterov and Nemirovskii's interior-point approaches were a significant contribution to control theory in 1988 and have been highly efficient in solving problems that could not previously be solved. [9] The basic structure of an LMI is given by

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{8}$$

where $x \in \mathbb{R}^m$ is a variable, and F_0 and F_1 are constant symmetric real matrices. The basic LMIP, the Feasibility Problem, is to find a feasible x, x_{feas}, such that $F(x_{feas}) > 0$ given an LMI $F(x) > 0$.

Often, problems arising in control engineering do not initially take the form of an LMIP, and various tools are used to re-cast them as LMIP. An exhaustive discussion of these techniques is beyond the scope of this report, but several are discussed below. [9].

1) The S-procedure: Enables one to combine several quadratic inequalities into one single inequality (generally with some conservatism). In control engineering there are cases where we would like to ensure a single quadratic function of $x \in \mathbb{R}^m$ is such that $F_0(x) \leq 0$ whenever certain other quadratic functions are positive semi-definite, i.e. when $F_i(x) \geq 0$. In general the S-procedure is conservative, the inequality above implies inequality prior to that, but not vice versa. The usefulness of the S-procedure is in the possibility of including τ_i as variables in an LMIP.

2) Congruence Transformation: This method uses a change of variable and some matrix properties to transform Bilinear Matrix Inequalities (BMIs) to LMIs. This method preserves the definiteness of the matrices that undergo congruence transformation. For a given positive definite matrix $Q \in \mathbb{R}^{n \times n}$, for any real matrix $W \in \mathbb{R}^{n \times n}$ such that $\text{rank}(W) = n$, the matrix inequality $Q < 0$ is satisfied if and only if $WQW^T < 0$.

V. Lur'e Systems

For the majority of the paper, our attention will be directed towards the Lur'e system illustrated in Figure [1], which can be represented in state-space form as follows:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= \phi(y) \end{cases} \tag{9}$$

$\phi(.) : \mathbb{R} \mapsto \mathbb{R}$ is a scalar slope-restricted or sector-bounded nonlinearity with slope conditions $\phi \in \partial[0, \alpha]$ or sector conditions $\phi \in \text{Sector}[0, \beta]$ respectively.

The plant $G(s) := (A, B, C, 0)$ is a minimal realisation, and the nonlinearity $\phi(.)$ is memoryless, locally Lipschitz in y and satisfies a slope-restricted or sector-bound condition. Static nonlinearities have no states and are described by their input-output maps. The idea behind *absolute stability analysis* is to guarantee that the Lur'e system is stable for a class of nonlinearities (to be described shortly) thereby meaning that the system will *absolutely* be stable for any nonlinearity in that class.

1) Slope restrictions and sector bounds: Nonlinearity $\phi(\cdot)$ is said to have the slope restriction $[0, \alpha]$ if

$$0 \leq \frac{\phi(x) - \phi(y)}{x - y} \leq \alpha \qquad \forall x, y \in \mathbb{R}, \qquad \alpha > 0 \tag{10}$$

The shorthand $\phi \in \partial[0, \alpha]$ is used to indication that a function is slope restricted. Similarly, a function $\phi(.)$ is said to have the sector bound $[0, \beta]$ if

$$0 \leq \phi(x)x \leq \beta x^2 \qquad \forall x \in \mathbb{R}, \qquad \beta > 0 \tag{11}$$

with the shorthand $\phi \in$ Sector$[0, \beta]$ [12].

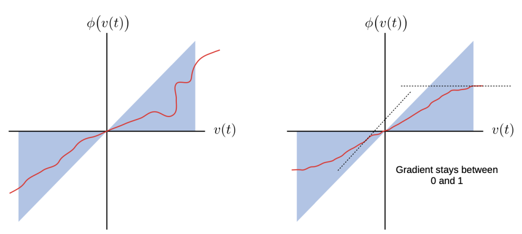

Fig. 4: Sector $[0, 1]$ bound nonlinearity (L), slope restricted nonlinearity (R)

2) Loop Transformation: The standard form of the Lur'e system is achieved by applying a loop transformation to the general Lur'e system. The general Lur'e system has the static nonlinearity bounded by sector $[\alpha, \beta]$, the general sector condition is $\alpha\sigma^2 \leq \sigma\phi(\sigma) \leq \beta\sigma^2 \quad \forall \sigma \in \mathbb{R}$. In summary, the loop transformation maps nonlinearities in sector $[\alpha, \beta]$ into a standard sector, for example, Sector $[0, 1]$.

Definition (Absolute Stability): Suppose ϕ satisfies a sector or slope condition. The system is absolutely stable if the equilibrium point at the origin is globally uniformly asymptotically stable for any nonlinearity in the given sector [1].

VI. Input-output stability

Lyapunov stability techniques have very strong connections with the *input-output* approach to study the stability of nonlinear, uncertain systems. This approach was sparked by Popov who used it to study Lur'e systems. [1] [9] The advantage of the input-output technique is that the systems are not required to have finite-dimensional states; instead, infinite-dimensional convex optimisation problems are produced since the relevant stability criteria are typically most easily represented in the frequency domain. Several methods have been developed to analyse the input-output stability of nonlinear systems, with the popular two methods being passivity and small gain.

A. Small-gain theorem

The small gain theorem states that a system is input-output stable if its input-output map satisfies a certain gain condition. The theorem states that if two nonlinear systems are connected in a feedback loop such that the gain from the output of one system to the input of the other is small, then the closed-loop system is stable. The theorem provides a sufficient condition for stability, which is based on the linearised system. The gain condition is usually expressed as an LMI or a set of LMIs. An advantage of this theorem is that it is a convenient criterion for verifying the BIBO stability and can be applied to almost all kinds of systems (linear and nonlinear, continuous-time and discrete-time, time-delayed, of any dimensions), as long as the mathematical setup is appropriately formulated to meet the theorem conditions. Yet, this criterion's primary drawback is its excessive conservatism. [1]

Despite the generality of the Small Gain Theorem, it has been discovered that this theorem is equivalent to the Circle Criterion when the linear system S_1 is represented by G and the nonlinear system S_2 is represented by a sector-bounded nonlinearity, via loopshifting. Thus, the Small Gain Theorem can be considered an additional way to verify the Circle Criterion.

B. Passivity theorem

Passivity is a concept that originated in circuit theory and has been extended to the analysis of nonlinear systems. A system is said to be *passive* if the abstract energy stored in the system is less than that supplied to it. Passivity is the foundation for the Circle criterion, Popov criterion, and Zames-Falb multipliers. As previously stated, the fundamental disadvantage of the passivity stability theorem is its overconservativity in providing adequate conditions for BIBO stability. One solution is to transform the system to the Lur'e structure and then apply the Circle or Popov criterion under the sector condition (if it can be satisfied), which usually results in less conservative stability conditions.

The connection between passivity and LMIs can be summarised. Considering the state-space system $G(s) := (A, B, C, D)$, such that all $eig(A)$ have negative-real part, (A, B) is controllable, $D + D^T \geq 0$, the following statements are equivalent.

Lemma 1 (Positive-real (PR) Lemma): The system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \tag{12}$$

is *passive*. i.e.

$$\int_0^T u(t)^T y(t) dt \geq 0 \tag{13}$$

for all $u, T \geq 0$

The transfer function $H(s)$ is positive-real, i.e.

$$H(s) + H(s)^* \geq 0 \qquad \forall \mathbb{R}(s) > 0 \tag{14}$$

The LMI [15]:

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leq 0, \qquad P > 0 \qquad (15)$$

in the variable $P = P^T$ is feasible [1].

C. Circle Criterion

A frequency-domain passivity-based method used to analyse the input-output stability of nonlinear feedback systems. The criterion provides a sufficient condition for stability in terms of the gain and phase margins of the open-loop transfer function, and is particularly useful for systems with nonlinearities that are sector-bounded and can be applied to nonlinear time-varying systems.

Theorem 2 (Circle Criterion): Consider the system as described in Figure [1], with nonlinearities independent of time and sector bounded by $[0, \beta]$ with $\beta > 0$ is globally asymptotically stable if the Nyquist locus $G(j\omega) \quad (-\infty < \omega < \infty)$ does not penetrate the disc with diameter $[\frac{-1}{\alpha}, \frac{-1}{\beta}]$ and encircles it as many times anticlockwise as $G(s)$ has unstable poles [13]. In this case, the frequency domain condition for the criterion becomes

$$Re\left\{\frac{1}{\beta} + G(j\omega)\right\} \geq \delta > 0, \qquad \forall \omega \geq 0 \qquad (16)$$

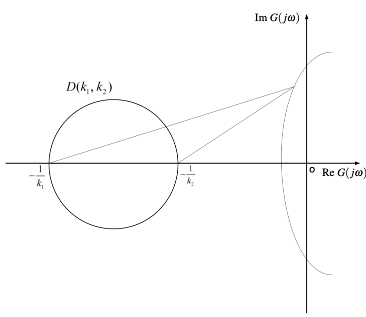

Fig. 5: Graphical representation of the Circle criterion for a SISO case, i.e. when m=1

To summarise, the Circle criterion states that a nonlinear system is input-output stable if its input-output map satisfies a certain passivity condition. [14]

D. Popov Criterion

A frequency-domain method used to analyse the absolute stability of nonlinear systems whose nonlinearity must satisfy an open-sector condition. The Popov criterion is less conservative than the circle criterion and is applicable only to autonomous (that is, time-invariant) systems.

Theorem 3 (Popov Criterion): Consider the system as described in Figure [1], with nonlinearities independent of time and sector bounded by $[0, \beta]$ with $\beta > 0$. The feedback system is absolutely stable if there exists a $\delta > 0$ such that

$$Re\left\{\frac{1}{\beta} + (1 + j\omega q)G(j\omega)\right\} \geq \delta > 0, \qquad \forall \omega \geq 0 \qquad (17)$$

is satisfied for some $q \geq 0$.

A Popov plot is shown in Figure [6], and the condition [17] is satisfied if the plot lies to the right of the line, which goes through point $(-\frac{1}{k}, j0)$ and with a slope of $\frac{1}{a}$. [14]

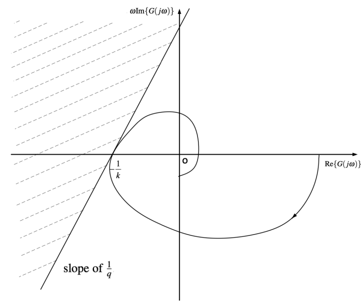

Fig. 6: Graphical representation of the Popov criterion

The Popov criterion, which encompasses a wider range of nonlinear systems, is a generalisation of the circle criterion.

E. Zames-Falb Multipliers

Zames-Falb multipliers are an advanced approach to the analysis of Lur'e systems. Rather than relying on the assumption that the nonlinearity function $\phi(\cdot)$ is sector bounded, a more stringent requirement that it is slope-restricted (i.e., satisfying Equation [10]) is imposed. This approach yields a tighter bound on the nonlinearity, allowing for a more accurate analysis of the system's stability properties. The application of Zames-Falb multipliers involves rearranging the system so that a modified nonlinearity function is created, which is simply monotone increasing [12]. It is worth noting Zames-Falb multipliers offer less conservative results compared to the Circle and Popov Criteria. The fundamental concept behind the Zames-Falb multiplier analysis is the substitution of the frequency domain conditions [16] and [17] with a more general condition involving a multiplier $\mathbf{M}$[1]. However, using Zames-Falb multipliers for stability analysis is more difficult than using the Circle or Popov Criteria because finding or choosing the dynamics of the Zames-Falb multiplier, $\mathbf{M}$, is a challenging task due to the wide range of linear operators that may be used. Zames and Falb have noted that these operators do not necessarily have to be rational or causal [11].

Consider the system in Figure [1] with time-invariant nonlinearity $\phi(\cdot)$ satisfying the slope condition [17]. The feedback system is absolutely stable if there exists a δ such that

$$M(j\omega)(1/\beta + G(j\omega)) \geq \delta > 0 \qquad (18)$$

where the $\mathbf{M}$ = I- $\mathbf{H}$ satisfies the $\mathcal{L}_1$ norm condition $\|\mathbf{H}\|_1 \leq 1$. *Note:* Compared to Theorems [2] and [3] of the Circle and Popov criteria, Zames-Falb multiplier analysis offers significantly greater freedom. However, as discussed earlier, this increased flexibility comes at the cost of increased complexity in analyzing the wide range of available $\mathbf{M}$ functions.

[1]The bold notation $\mathbf{M}$ is used to capture a not necessarily rational and not necessarily causal linear operator i.e. one which does not have a standard state-space or transfer function description.

Zames and Falb established two versions of their main results: one tailored for odd nonlinearities and the other for non-odd nonlinearities. The result for odd nonlinearities is less conservative, as it is applicable to a smaller sub-class of nonlinear elements, whereas the one for non-odd nonlinearities is more general but requires a positive impulse response for the dynamic part of **M**. Turner (2021) introduces these theorems, see Theorems [2-5] [3]. With some effort, the IQC of Megretski and Rantzer [8] can be harnessed to convert these conditions into either frequency domain conditions, (Theorems [2-3] [3]), or matrix inequality conditions as discussed in Section [IV]. In particular, the latter enables efficient use of Zames-Falb multipliers in stability analysis.

F. Park's Criterion

There is another effective approach that has not yet been addressed, namely *Park's Criteria* as described in Park (2002) [2]. While it is beyond the scope of this paper to delve into this topic in depth, we can broadly understand Park's Criteria as a refinement of Popov's Criterion for systems subject to slope restrictions. The added constraint on the nonlinearity often makes Park's approach superior to more conventional methods. The Park criterion utilises both self and cross properties of sector and slope restrictions in the time domain and supplied a Lyapunov function associated with matrix inequalities, which was subsequently relaxed into the frequency domain via KYP lemma. This relaxation process allowed for criterion simplification.

VII. Comparison

Park [2] and Turner and Sofrony [4] make use of two identical examples. These examples serve to illustrate the maximum value of β for a given Lur'e system, for which stability is predicted by various criteria. It is worth noting the lower bound of the sector α is zero. A comprehensive listing of the plants $G(s)$ are presented in Table III, which also includes the original literature reference for each entry.

Example	$G(s)$	Reference
Ex I	$G_1(s) = \frac{s^2-0.2s-0.1}{s^3+2s^2+s+1}$	Haddad and Kapila (1995) [5]
Ex II	$G_2(s) = -G_1(s)$	Park (2002) [2]

TABLE III: Table of transfer functions $P(s)$

Criteria	Ex I	Ex II	Reference
Circle	1.2431	0.7640	[2]
(Scaled) Popov	1.7636	1.0827	[2]
Haddad and Kapila	failed	1.0894	[5]
Suykens et al	1.2431	1.0837	[10]
Park Theorem 1	4.5894	1.0894	[2]
Park Theorem 2	4.5894	1.0894	[2]
Improved Circle	1.5301	0.8629	[4]
Improved Popov	1.9961	1.0891	[4]
Nyquist Value	4.5911	1.0902	[4]

TABLE IV: Sector bounds using various stability criteria

In this section, the two examples can be used to compare the results of the criteria in literature. Table IV presents a comparison of different stability results for the absolute stability problem, most of which have been previously discussed in this paper. Given the intricate search routines required to employ Zames-Falb multipliers, they present a notable challenge to use [11]. Consequently, the following comparisons deliberately disregards these.

A. Results comparison

Park Theorems 1 and 2 yielded the largest bounds and the larger bounds for Ex I and Ex II respectively. It was initially thought that the bounds for Ex I were much larger due to the simplification of the criteria. However, this conjecture was proven to be false as the original and simplified criteria were shown to perform equally well, with the simplified version yielding less conservative results. The results from Park (2002) [2] for Ex I and Ex II virtually exactly produce the Nyquist value results depicted in Table 2 of Turner and Sofrony's [4].

The Improved Circle and Improved Popov criteria were introduced to obtain higher maximum values of β compared to their original counterparts. Turner and Sofrony concluded that the original Circle and Popov criteria may be unreliable and proposed improvements to the existing criteria. Specifically, the improved Circle criterion has been found to provide stability guarantees for a larger value of β compared to the standard Popov criterion. In order to establish asymptotic stability of the origin, Turner and Sofrony further suggested modified versions of the Popov and Circle criteria, which incorporated information about the magnitude bound of the nonlinearity alongside the boundedness condition; the modified criteria do not, however, require slope restrictions to be imposed.

Figure [7] depicts a number line plot of the maximum value of β obtained from various stability criteria for Ex I. The graphical representation of the maximum bounds for the plant is useful for comparison purposes and to highlight the modifications made to the existing criteria. It is evident from the figure that all of the stability criteria produce similar results, except for the Park theorems. It is worth noting that although the Park theorems appear to be outliers, they are reliable, as previously mentioned, and provide an accurate prediction of the maximum value of β.

Similarly, the maximum value of β resulting from the application of different stability criteria for Ex II is depicted in Figure [8]. The figure reveals the close agreement among most of the criteria, with the exception of the Circle and Improved Circle criteria, which appear to be outliers.

It is worth noting that the Nyquist criterion may be deemed unreliable due to its assumption that the nonlinearity is linear. Consequently, it only serves as an upper bound. An interesting point to consider regarding Park's approach is that it can, in certain cases, guarantee stability up to the Nyquist limit for some systems, but not all.

Theorem	Conditions on nonlinearity ϕ	Computation	Domain	System classification	Stability
Circle Criterion	Sector-bounded	Graphical (SISO) Non-graphical (MIMO)	Frequency	Time-variant	Global exponential
Popov Criterion	Sector-bounded	Graphical (SISO) Non-graphical (MIMO)	Frequency	Time-invariant	Global asymptotic
Zames-Falb Multipliers	Slope-restricted ($\Rightarrow$ Sector-bounded)	Non-graphical	Time $\xrightarrow{*}$ Frequency	Time-invariant	Global exponential
Park Criterion [2]	Slope-restricted ($\Rightarrow$ Sector-bounded)	Non-graphical	Time and Frequency	Time-invariant	Global exponential

TABLE II: Theorem Comparison

**Using Parseval's Identity, the time domain form can be written in the frequency domain form [3].*

Fig. 7: Example I Sector Bounds using various stability criteria

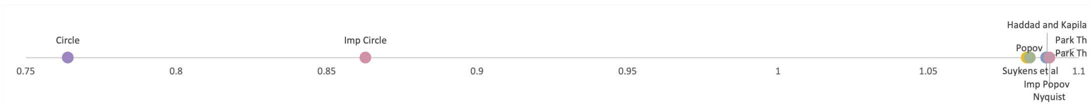

Fig. 8: Example II Sector Bounds using various stability criteria

B. Comparison of Methods, System and Knowledge of Dynamics

The aforementioned criteria can be applied to a diverse range of system types, including time-variant and time-invariant systems, as well as continuous and discrete ones. Certain theorems operate exclusively in the frequency domain, while others function in the time-domain with an associated frequency domain counterpart. Moreover, there are notable distinctions between the techniques concerning their implementation methods, the information on $\phi(\cdot)$ necessary for their operation, and their approaches to stability analysis. Table II displays these findings in a clear and concise manner.

When the nonlinearity of the Lur'e system is only sector-bounded, the Circle and Popov criterion are suitable to assess the absolute stability of the system. Likewise, when the nonlinearity is additionally slope-restricted (implying sector boundedness), Zames-Falb results, as well as Park's are suitable. This is important as a slope-restriction is a tighter condition than sector boundedness, suitable for when one is looking for less conservative results (than the Circle and Popov criteria). To summarise, Zames-Falb multipliers and the Park criterion are superior in terms of lower levels of conservatism [3].

The Popov and Circle criteria rely on the frequency response of the linear system and are built upon classical control tools such as the Nyquist plot and Nyquist criterion. As these criteria were originally developed in the frequency domain, strictly speaking, they guarantee $\mathcal{L}_2$ input-output stability. The Popov criterion provides sufficient conditions for the stability of nonlinear systems in the frequency domain and has a direct graphical interpretation, making it a convenient tool for both analysis and design. However, when the nonlinearity becomes time-varying, the Popov criterion is no longer applicable. On the other hand, the Circle criterion allows for the analysis of absolute stability for time-varying nonlinearities. Both criteria have an attractive graphical form, making them suitable for graphical checks in single-input-single-output (SISO) systems, or in multi-input-multi-output (MIMO) systems using LMIs. However, Zames-Falb and Park criteria do not have graphical representations, and thus, in the absence of readily available computer optimisation, the results of the Popov and Circle criteria were preferred [11].

As previously mentioned, the Popov and Circle criteria operate in the frequency domain. In contrast, the Zames-Falb multiplier analysis requires a combination of time and frequency domain conditions to be satisfied. It is shown in [3], that using Parseval's Identity, the main stability condition can be transformed into the frequency domain, but the accompanying time-domain condition involving the $\mathcal{L}_1$-norm (see Theorems [2-5] [3]) cannot. This tends to make the Zames-Falb Criteria more difficult to use.

Moreover, several variants of the Zames-Falb criterion have been proposed, some of which accommodate time-varying nonlinearities, albeit at the cost of increased complexity, conservatism, and some violation of the slope-restriction. Similarly, the Park criterion operates in the time-domain, and its interpretation in the frequency domain is elaborated upon in Section VII-A. It should be emphasised that the majority of the criteria offer a high degree of stability, with global exponential stability implying global asymptotic stability. Local versions of the Circle, Popov, and Park criteria have also been developed. Theorem 2.2 [15] demonstrates that $\mathcal{L}_2$ input-output stability and global exponential stability are equivalent to the stronger property of exponential input-to-state stability.

Extensions upon the standard criteria (Circle, Popov, Zames-Falb and Park) have been introduced in the recent years [14]. Furthermore, variations of the criteria exist that can enhance stability, such as strengthening the Popov criteria to global exponential stability. These findings can be summarised and

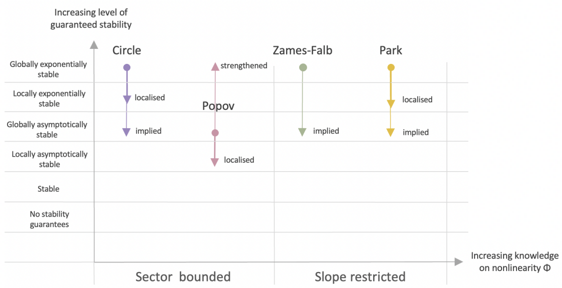

Fig. 9: Absolute stability theorems compared in terms of knowledge on nonlinearity ϕ and guaranteed level of stability.

visualised in Figure [9].

Criteria	Advantages	Limitations
Circle and Popov	– Graphical interpretation (for SISO) – Highly extensible, variations include improved Circle and Popov –Applicable to SISO and MIMO	– Only in frequency domain – Quite conservative – Sometimes doesn't provide stability guarantees when system is stable [6]
Zames-Falb	– General multipliers used to improve predictions on stability and performance – Computer-aided search and optimisation [11] – IQC framework allows additional properties of $\phi(\cdot)$ to be included [11]	– Zames–Falb multipliers must be factorisable [11] – Multipliers must satisfy passivity (positivity) condition and time domain $\mathcal{L}_1$ bound; simultaneously, non-conservatively and efficiently [12]

TABLE V: Advantages and limitations of each criterion

Table V presents a summary of the key advantages and limitations of each theorem discussed in this review. A notable observation is that while the Circle and Popov criteria have a visually appealing form (in the SISO case), they can be overly conservative, sometimes failing to provide stability guarantees even when the feedback interconnection is stable.

The main benefit of Zames-Falb multipliers lies in their ability to enhance stability and performance predictions of the interconnection. Nevertheless, the analysis of systems using Zames-Falb multipliers can be challenging, as it requires meeting a time-domain $\mathcal{L}_1$ bound on the impulse response of the multipliers, while also satisfying a passivity condition. Achieving these conditions simultaneously and non-conservatively can be computationally demanding. Carrasco et al [11] raise a number of important questions and possible developments pertaining to this method, such as complete search, instability criteria, the dual problem, and stability conjecture.

VIII. Conclusion

This survey has provided a summary and comprehensive overview of four main absolute stability criteria: the Circle, Popov, Zames-Falb, and Park criteria. Each method has its own strengths and limitations, with the main trade-offs between the different stability criteria, such as the Circle/Popov Criteria and the Park/Zames-Falb Criteria, involving considerations of conservatism, generality, and complexity.

The former criteria are more conservative but also more general, applying to a wider range of nonlinear systems. On the other hand, the latter criteria are less conservative but also less general, requiring tighter restrictions on slope or sector boundedness. Additionally, the Circle/Popov Criteria are computationally simpler than the Zames-Falb Criteria, making them more practical for systems with large state-dimension. However, the Zames-Falb Criteria may be necessary in certain cases, despite their increased complexity, to ensure system stability. Ultimately, the choice of stability criteria should depend on the specific characteristics of the system and the desired level of performance and safety.

The findings of this survey further supports the potential of these criteria in the future of absolute stability methods. The author notes that the Popov and Park's multiplier, as well as their extensions, remain useful even when the nonlinearity is slope-restricted, which emphasises their advantages and makes it challenging to identify a *superior* criterion [17]. A key research direction for the future is the development of methods that require less knowledge of the system but can still provide strict stability guarantees.

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