LMIs in Control Systems
— Analysis, Design and Applications

Solution Manual

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Preface

This is the solutions manual to accompany the book entitled *LMIs in Control Systems—Analysis, Design and Applications*. In total, there are 80 exercise problems, and these involve conceptional, proving and computational types. This manual gives solutions to all these problems, and for some particular problems more than one solutions are provided.

We remark that many exercise problems given in this book may have more than one solutions—it is natural to prove a conclusion through different ways, and a feasibility problem may certainly have many or even infinite number of solutions. From this point of view, this manual is only a reference, solutions different from the ones given in this manual may be certainly also correct.

In this solution manual, we have used the same notations as in the book mentioned above. Please refer to the List of Notations in that book if needed.

It is assumed that students and instructors have access to the LMI Lab in the Matlab Robust Control Toolbox. All of the computer solutions in this solution manual were developed and tested on a Window 2007 platform using Matlab 7.11.0.584 Release 2010b and the Robust Control Toolbox Version 3.5. Please note that with some of the computational problems slightly different solutions may be obtained if a different version of Matlab is used.

We have not given too many exercise problems for each chapter. Therefore, it is very important that students really have tried all the problems. With usage of this manual, we strongly encourage the students to work on the exercise problems on their own first. Please do not refer to this manual until you have worked out a solution to a problem and then you can compare your solution with the one given in the manual, or until you really feel that you can not solve the problem after sufficient effort and trials, and then you can check the solution given in the manual and see what aspects you have neglected. In the latter case, it is also suggested that students check up some similar examples in the book before referring to this manual. "Easy come, easy go!" In this way you can master the contents in the book much better.

Through writing of the book and preparation of this solution manual, we have got the following available:

- Matlab codes for all the computational exercise problems in the form of a set of M files, and
- Matlab codes for all the computational examples in the form of a set of M files.

These may be asked for through our emails, or by contacting the publisher.

Many persons have helped with the preparation of this manual. Mr. Long Zhang, Mr. Feng Zhang, Mrs. Shi Li, Mrs. Chunyan Gao, Mrs. Xiaoling Liang, and Mrs. Fengyu Fu, who are all the PhD students of the first author, all helped with the solutions to the exercises in the book and also with the tests of the computational ones on computer. Their help is very much appreciated indeed.

The first author would also like to take this opportunity to gratefully acknowledge again the financial support kindly provided by the many sponsors, including NSFC, the National Natural Science Foundation of China, the Chinese Ministry of Science and Technology, and the Ministry of Education of China, for projects funded by the Program of the National Science Fund for Distinguished Young Scholars, the Innovative Scientific Research Team Program, the National Key Basic Research and Development Program (973 Program), and also the Program of ChangJiang Scholars. Thanks would also go to the Aerospace Industry Companies of China for funded projects which provide the background and data for the application chapters.

At the last, let us thank in advance all the users of this solution manual. We would be indeed very grateful if users could possibly provide via our emails feedback about any problems found in this manual, including possible incompatibilities with different available computer platforms, any new solutions to certain exercise problems, and, of course, any comments and suggestions for the future editions. Your help will certainly make any future editions of the book much better.

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Chapter 1

Introduction

Exercise 1.1

Let $A \in \mathbb{S}^m$. Show that for arbitrary $M \in \mathbb{R}^{m\times n}$, $A \leq 0$ implies $M^{\mathrm{T}}AM \leq 0$.

Solution I. Since $A \leq 0$, we have

$$y^{\mathrm{T}}Ay \leq 0, \ \forall y \in \mathbb{R}^m.$$

Therefore, for arbitrary $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m\times n}$, there holds

$$x^{\mathrm{T}}M^{\mathrm{T}}AMx = (Mx)^{\mathrm{T}}AMx \leq 0.$$

This completes the proof.

Solution II. Since $A \leq 0$, there exists a matrix $T \in \mathbb{R}^{m\times m}$ such that

$$A = -T^{\mathrm{T}}T.$$

Therefore, for arbitrary $M \in \mathbb{R}^{m\times n}$, there holds

$$M^{\mathrm{T}}AM = -(TM)^{\mathrm{T}}(TM) \leq 0.$$

This completes the proof.

Remark. On the other side, if for arbitrary $M \in \mathbb{R}^{m\times n}$, there holds $M^{\mathrm{T}}AM \leq 0$, we can simply choose $M = I$, the identity, and obtain $A \leq 0$. Therefore, we actually have the conclusion that $A \leq 0$ if and only if $M^{\mathrm{T}}AM \leq 0$ for arbitrary $M \in \mathbb{R}^{m\times n}$.

Exercise 1.2 (Duan and Patton (1998), Zhang and Yang (2003), page 175)

Let $A \in \mathbb{C}^{n\times n}$. Show that A is Hurwitz stable if $A + A^{\mathrm{H}} < 0$.

Solution. First, we remark that, like the case for a real matrix, a complex square matrix is called Hurwitz stable if all its eigenvalues have negative real parts. Let λ be an eigenvalue of A, x be a corresponding eigenvector, then we have

$$Ax = \lambda x,$$

and further

$$x^{\mathrm{H}}(A^{\mathrm{H}} + A)x = (\lambda + \bar{\lambda})x^{\mathrm{H}}x.$$

Thus $A^{\mathrm{H}} + A < 0$ implies

$$\operatorname{Re}\lambda(A) = \frac{\lambda + \bar{\lambda}}{2} < 0.$$

This completes the proof.

Exercise 1.3 (Duan and Patton (1998))

Let $A \in \mathbb{R}^{n\times n}$. Show that A is Hurwitz stable if and only if

$$A = PQ, \tag{s1.1}$$

with $P > 0$ and Q being some matrix satisfying $Q + Q^{\mathrm{T}} < 0$.

Solution. Suppose $A = PQ$ holds with $P > 0$ and Q satisfying

$$Q + Q^{\mathrm{T}} < 0. \tag{s1.2}$$

Let

$$\hat{P} = P^{-1} > 0,$$

then it is easy to see

$$A^{\mathrm{T}}\hat{P} + \hat{P}A = Q + Q^{\mathrm{T}} < 0.$$

Therefore, the matrix A is Hurwitz stable.

Conversely, if A is Hurwitz stable, then there exists a matrix $\hat{P} > 0$, such that

$$A^{\mathrm{T}}\hat{P} + \hat{P}A < 0. \tag{s1.3}$$

Let

$$Q = \hat{P}A,$$

we can easily get (s1.1), with $P = \hat{P}^{-1} > 0$, and the matrix Q obviously satisfies (s1.2) because of (s1.3).

Exercise 1.4

Give an example to show that certain set of nonlinear inequalities can be converted into LMIs.

Solution. Let $Q(x) = Q^{\mathrm{T}}(x),\ R(x) = R^{\mathrm{T}}(x)$ and $S(x)$ depend affinely on x. It is clearly that

$$Q(x) - S(x)R(x)^{-1}S^{\mathrm{T}}(x) > 0,\ R(x) > 0,$$

is quadratic with respect to $S(x)$. Using Schur completion lemma the above two relations can be equivalently converted into

$$\begin{bmatrix} Q(x) & S(x) \\ S^{\mathrm{T}}(x) & R(x) \end{bmatrix} > 0,$$

which is now linear in $S(x)$.

Exercise 1.5

Verify for which integer i the following inequality is true:

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} > \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solution I. Let

$$\Theta = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & i-1 \\ i-1 & 1 \end{bmatrix}, \qquad \text{(s1.4)}$$

then we know

$$sI - \Theta = \begin{bmatrix} s-1 & -i+1 \\ -i+1 & s-1 \end{bmatrix},$$

$$\det(sI - \Theta) = (s-1)^2 - (i-1)^2 = (s-i)(s+i-2).$$

Thus

$$\lambda(\Theta) = \{i, 2-i\},$$

which indicates $\Theta > 0$ if and only if $i = 1$. Therefore the conclusion holds if and only if $i = 1$.

Solution II. It follows from (s1.4) and the Schur complement lemma that $\Theta > 0$ if and only if

$$1 - (i-1)^2 > 0,$$

which is equivalent to

$$(i-1)^2 < 1.$$

Obviously, this holds if and only if $i = 1$.

Exercise 1.6

Consider the combined constraints (in the unknown x) of the form

$$\begin{cases} F(x) < 0 \\ Ax = a \end{cases}, \qquad \text{(s1.5)}$$

where the affine function $F:\mathbb{R}^n \to \mathbb{S}^m$, matrix $A \in \mathbb{R}^{m\times n}$ and vector $a \in \mathbb{R}^m$ are given, and the equation $Ax = a$ has a solution. Show that (s1.5) can be converted into an LMI.

Solution. Suppose $\operatorname{rank} A = r$, then it is well-known that all the solution vectors of the equation $Ax = a$ constitute a manifold, of dimension r, in $\mathbb{R}^n$, and a general form of all the solutions can be written as

$$x = x_0 + z_1 e_1 + z_2 e_2 + \cdots + z_r e_r,$$

where x_0 is a particular solution to the matrix equation $Ax = a$, while $e_1, e_2, \ldots, e_r$ are a set of linearly independent solutions to the homogeneous equation $Ax = 0$, and z_i, $i = 1, 2, \ldots, r$, are a series of arbitrary scalars.

Let

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}, \\ x_0 &= \begin{bmatrix} x_1^0 & x_2^0 & \cdots & x_n^0 \end{bmatrix}^{\mathrm{T}}, \\ e_i &= \begin{bmatrix} e_{1i} & e_{2i} & \cdots & e_{ni} \end{bmatrix}^{\mathrm{T}},\ i = 1, 2, \ldots, r, \end{aligned}$$

then the components of vector x can be written as

$$x_j = x_j^0 + z_1 e_{j1} + z_2 e_{j2} + \cdots + z_r e_{jr},\ j = 1, 2, \ldots, n, \tag{s1.6}$$

and the affine function F can be expressed as

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n. \tag{s1.7}$$

Substituting (s1.6) into (s1.7), yields,

$$\begin{aligned} F(x) &= F_0 + \left(x_1^0 + z_1 e_{11} + z_2 e_{12} + \cdots + z_r e_{1r}\right) F_1 \\ &\quad + \left(x_2^0 + z_1 e_{21} + z_2 e_{22} + \cdots + z_r e_{2r}\right) F_2 \\ &\quad + \cdots + \left(x_n^0 + z_1 e_{n1} + z_2 e_{n2} + \cdots + z_r e_{nr}\right) F_n \\ &= F_0 + x_1^0 F_1 + \cdots + x_n^0 F_n \\ &\quad + z_1 \left(e_{11} F_1 + e_{21} F_2 + \cdots + e_{n1} F_n\right) \\ &\quad + \cdots + z_r \left(e_{1r} F_1 + e_{2r} F_2 + \cdots + e_{nr} F_n\right). \end{aligned}$$

Put

$$\begin{aligned} &\widetilde{F}_0 = F_0 + x_1^0 F_1 + \cdots + x_n^0 F_n, \\ &\widetilde{F}_i = e_{1i} F_1 + e_{2i} F_2 + \cdots + e_{ni} F_n,\ i = 1, 2, \ldots, r, \\ &z = \begin{bmatrix} z_1 & z_2 & \cdots & z_r \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

we finally have

$$F(x) = \widetilde{F}_0 + z_1 \widetilde{F}_1 + \cdots + z_r \widetilde{F}_r \overset{\triangle}{=} \widetilde{F}(z).$$

This implies that $x \in \mathbb{R}^n$ satisfies (1.22) if and only if $\widetilde{F}(z) < 0$, $z \in \mathbb{R}^r$.

Exercise 1.7

Write the Hermite matrix $A \in \mathbb{C}^{n\times n}$ as $X + iY$ with real X and Y. Show that $A < 0$ only if $X < 0$.

Solution. Considering the conjugate symmetry of matrix A, we know that

$$X^{\mathrm{T}} = X, \quad Y^{\mathrm{T}} = -Y.$$

Therefore, for arbitrary $z \in \mathbb{R}^n$, we have

$$\left(z^{\mathrm{T}} Y z\right)^{\mathrm{T}} = z^{\mathrm{T}} Y^{\mathrm{T}} z = -\left(z^{\mathrm{T}} Y z\right),$$

which results in

$$z^{\mathrm{T}} Y z = 0, \ \forall z \in \mathbb{R}^n. \tag{s1.8}$$

Using the above relation, we further have

$$\begin{aligned} z^{\mathrm{T}} (X + iY) z &= z^{\mathrm{T}} X z + i z^{\mathrm{T}} Y z \\ &= z^{\mathrm{T}} X z. \end{aligned} \tag{s1.9}$$

When $A < 0$, we have

$$z^{\mathrm{T}} (X + iY) z < 0, \ \forall z \in \mathbb{R}^n, \ z \neq 0$$

this, together with (s1.9), implies

$$z^{\mathrm{T}} X z < 0, \ \forall z \in \mathbb{R}^n, \ z \neq 0.$$

This gives the negative definiteness of X.

Exercise 1.8

Let A, B be symmetric matrices of the same dimension. Show

1. $A > B$ implies $\lambda_{\max}(A) > \lambda_{\max}(B)$,
2. $\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B)$.

Solution. Let $M \in \mathbb{S}^n$, $\lambda_{\max}(M)$ be the maximum eigenvalue of matrix M. We can easily show that

$$\lambda_{\max}(M) I \geq M.$$

Then, there holds

$$\lambda_{\max}(M) x^{\mathrm{T}} x \geq x^{\mathrm{T}} M x, \quad \forall x \in \mathbb{R}^n. \tag{s1.10}$$

Proof of conclusion 1

Let x be the eigenvector of matrix B corresponding to the eigenvalue $\lambda_{\max}(B)$, then

$$Bx = \lambda_{\max}(B) x.$$

Considering $A > B$, we have

$$x^{\mathrm{T}}(A - B)x > 0,$$

which means

$$x^{\mathrm{T}}Ax > x^{\mathrm{T}}Bx = \lambda_{\max}(B)x^{\mathrm{T}}x. \tag{s1.11}$$

On the other hand, using (s1.10) and (s1.11), gives

$$\lambda_{\max}(A)x^{\mathrm{T}}x > \lambda_{\max}(B)x^{\mathrm{T}}x,$$

which implies $\lambda_{\max}(A) > \lambda_{\max}(B)$, in view of $x^{\mathrm{T}}x > 0,\ x \neq 0$.

Proof of conclusion 2

Let x be the eigenvector of matrix $A+B$ corresponding to the eigenvalue $\lambda_{\max}(A+B)$, then

$$\lambda_{\max}(A+B)x = (A+B)x,$$

from which we have

$$\lambda_{\max}(A+B)x^{\mathrm{T}}x = x^{\mathrm{T}}Ax + x^{\mathrm{T}}Bx. \tag{s1.12}$$

Using (s1.10) again, we obtain

$$x^{\mathrm{T}}Ax \leq \lambda_{\max}(A)x^{\mathrm{T}}x,\ x^{\mathrm{T}}Bx \leq \lambda_{\max}(B)x^{\mathrm{T}}x. \tag{s1.13}$$

Combining (s1.12) with (s1.13), yields

$$\lambda_{\max}(A+B)x^{\mathrm{T}}x \leq (\lambda_{\max}(A) + \lambda_{\max}(B))\,x^{\mathrm{T}}x,$$

which clear implies, in view of $x^{\mathrm{T}}x > 0,\ x \neq 0$, the relation to be proven.

Chapter 2

Technical Lemmas

Exercise 2.1

Let $c(x) \in \mathbb{R}^n$ and $P(x) = P^{\mathrm{T}}(x) \in \mathbb{R}^{n\times n}$ depend affinely on x, and $P(x)$ is nonsingular for all x. Find the equivalent LMIs for the following constraints:

$$c^{\mathrm{T}}(x)P^{-1}(x)c(x) < 1, \quad P(x) > 0.$$

Solution. Rewrite the above relations as

$$c^{\mathrm{T}}(x)P^{-1}(x)c(x) - 1 < 0, \quad P(x) > 0.$$

Then, using the Schur complement lemma, these two relations can be shown to be equivalent with the following LMI condition:

$$\begin{bmatrix} -P(x) & c(x) \\ c^{\mathrm{T}}(x) & -1 \end{bmatrix} < 0,$$

or

$$\begin{bmatrix} P(x) & c(x) \\ c^{\mathrm{T}}(x) & 1 \end{bmatrix} > 0.$$

Exercise 2.2

Let $P(x) \in \mathbb{S}^{n\times n}$ and $Q(x) \in \mathbb{R}^{n\times p}$ depend affinely on x. Convert the following constraints

$$\mathrm{trace}\left(Q^{\mathrm{T}}(x)P^{-1}(x)Q(x)\right) < 1, \; P(x) > 0 \tag{s2.1}$$

into a set of LMIs by introducing a new (slack) matrix variable $X \in \mathbb{S}^{p\times p}$.

Solution. According to Lemma 2.13, the first inequality in (s2.1) is equivalent to

$$Q^{\mathrm{T}}(x)P^{-1}(x)Q(x) < X, \tag{s2.2}$$

and

$$\operatorname{trace}(X) < 1. \tag{s2.3}$$

Applying Schur complement lemma to inequality (s2.2), with the condition $P(x) > 0$, yields

$$\begin{bmatrix} -X & Q^{\mathrm{T}}(x) \\ Q(x) & -P(x) \end{bmatrix} < 0.$$

Therefore, (s2.1) is equivalent to the following set of LMIs:

$$\begin{cases} \begin{bmatrix} -X & Q^{\mathrm{T}}(x) \\ Q(x) & -P(x) \end{bmatrix} < 0 \\ \operatorname{trace}(X) < 1 \end{cases}.$$

Exercise 2.3 (Wang and Zhao (2007))

Let

$$\Lambda = \left\{ \alpha = [\alpha_1 \ \ \alpha_2]^{\mathrm{T}} \in \mathbb{R}^2 \middle| \alpha_1 + \alpha_2 = 1, \alpha_1, \alpha_2 \geq 0 \right\}.$$

Show $P(\alpha) = \alpha_1 P_1 + \alpha_2 P_2 > 0$ for any $\alpha \in \Lambda$ if and only if $P_1 > 0$ and $P_2 > 0$.

Solution. *Necessity*. Since

$$P(\alpha) = \alpha_1 P_1 + \alpha_2 P_2 > 0, \ \forall \alpha \in \Lambda,$$

choosing $\alpha_1 = 1, \ \alpha_2 = 0$ gives $P_1 > 0$, while choosing $\alpha_1 = 0, \ \alpha_2 = 1$ gives $P_2 > 0$.

Sufficiency. Since $P_1 > 0$, $P_2 > 0$, we have

$$\alpha_1 P_1 \geq 0, \ \alpha_2 P_2 \geq 0, \ \forall \alpha \in \Lambda. \tag{s2.4}$$

Therefore,

$$P(\alpha) = \alpha_1 P_1 + \alpha_2 P_2 \geq 0.$$

Further, note that $\alpha_1 + \alpha_2 = 1$, the two equalities in (s2.4) do not simultaneously hold. This implies the strict inequality in the above relation.

Exercise 2.4 (Xu and Yang (2000))

Let

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \in \mathbb{R}^{n \times n}, \ \ M_1 \in \mathbb{R}^{m \times m},$$

and M_4 be invertible. Show that $M + M^{\mathrm{T}} < 0$ implies

$$M_1 + M_1^{\mathrm{T}} - M_2 M_4^{-1} M_3 - M_3^{\mathrm{T}} M_4^{-\mathrm{T}} M_2^{\mathrm{T}} < 0.$$

Solution. Since

$$M+M^{\mathrm{T}}=\left[\begin{array}{cc} M_1+M_1^{\mathrm{T}} & M_2+M_3^{\mathrm{T}} \\ M_2^{\mathrm{T}}+M_3 & M_4+M_4^{\mathrm{T}} \end{array}\right]<0, \tag{s2.5}$$

then according to Corollary 2.2, we have

$$M_1+M_1^{\mathrm{T}}<0, \quad M_4+M_4^{\mathrm{T}}<0. \tag{s2.6}$$

With (s2.5) and (s2.6), further applying Schur complement lemma to $M+M^{\mathrm{T}}<0$, yields

$$S_{ch}(M_4+M_4^{\mathrm{T}})<0. \tag{s2.7}$$

According to the definition of Schur complement, we have

$$\begin{aligned} & S_{ch}(M_4+M_4^{\mathrm{T}}) \\ = \; & M_1+M_1^{\mathrm{T}}-\left(M_2+M_3^{\mathrm{T}}\right)\left(M_4+M_4^{\mathrm{T}}\right)^{-1}\left(M_2^{\mathrm{T}}+M_3\right) \\ = \; & M_1+M_1^{\mathrm{T}}+\Phi_1+\Phi_2, \end{aligned} \tag{s2.8}$$

where

$$\begin{aligned} \Phi_1 &= -M_2\left(M_4+M_4^{\mathrm{T}}\right)^{-1}M_2^{\mathrm{T}}-M_3^{\mathrm{T}}\left(M_4+M_4^{\mathrm{T}}\right)^{-\mathrm{T}}M_3, \\ \Phi_2 &= -M_2\left(M_4+M_4^{\mathrm{T}}\right)^{-1}M_3-M_3^{\mathrm{T}}\left(M_4+M_4^{\mathrm{T}}\right)^{-\mathrm{T}}M_2^{\mathrm{T}}, \end{aligned} \tag{s2.9}$$

Furthermore, using Corollary 2.1, that is, the matrix inversion lemma, yields,

$$\begin{aligned} \left(M_4+M_4^{\mathrm{T}}\right)^{-1} &= M_4^{-1}-M_4^{-1}\left(M_4^{-\mathrm{T}}+M_4^{-1}\right)^{-1}M_4^{-1} \\ &= M_4^{-1}+M_4^{-1}HM_4^{-1}, \end{aligned} \tag{s2.10}$$

where

$$H=-\left(M_4^{-\mathrm{T}}+M_4^{-1}\right)^{-1}=-M_4\left(M_4+M_4^{\mathrm{T}}\right)^{-1}M_4^{\mathrm{T}}.$$

Substituting (s2.10) into (s2.9), gives

$$\begin{aligned} \Phi_2 &= -M_2\left(M_4^{-1}+M_4^{-1}HM_4^{-1}\right)M_3 \\ & \quad -M_3^{\mathrm{T}}\left(M_4^{-\mathrm{T}}+M_4^{-\mathrm{T}}HM_4^{-\mathrm{T}}\right)M_2^{\mathrm{T}} \\ &= \Psi_1-\Psi_2, \end{aligned} \tag{s2.11}$$

where

$$\begin{aligned} \Psi_1 &= -M_2M_4^{-1}M_3-M_3^{\mathrm{T}}M_4^{-\mathrm{T}}M_2^{\mathrm{T}}, \\ \Psi_2 &= M_2M_4^{-1}HM_4^{-1}M_3+M_3^{\mathrm{T}}M_4^{-\mathrm{T}}HM_4^{-\mathrm{T}}M_2^{\mathrm{T}}. \end{aligned}$$

On the other hand, considering that M_4 is nonsingular and $M_4 + M_4^{\mathrm{T}} < 0$, we easily observe that $H > 0$. Using Lemma 2.1, it results in that

$$\begin{aligned}
\Psi_2 &\leq M_2 M_4^{-1} H M_4^{-\mathrm{T}} M_2^{\mathrm{T}} + M_3^{\mathrm{T}} M_4^{-\mathrm{T}} H M_4^{-1} M_3 \\
&= -M_2 \left(M_4^{\mathrm{T}} + M_4\right)^{-1} M_2^{\mathrm{T}} - M_3^{\mathrm{T}} \left(M_4^{\mathrm{T}} + M_4\right)^{-\mathrm{T}} M_3 \\
&= \Phi_1.
\end{aligned}$$

Substituting the above inequality into (s2.8), and using (s2.11), yields

$$\begin{aligned}
S_{ch}(M_4 + M_4^{\mathrm{T}}) &= M_1 + M_1^{\mathrm{T}} + \Phi_1 + \Phi_2 \\
&\geq M_1 + M_1^{\mathrm{T}} + \Psi_2 + \Psi_1 - \Psi_2 \\
&= M_1 + M_1^{\mathrm{T}} + \Psi_1,
\end{aligned}$$

which, together with (s2.7), implies

$$M_1 + M_1^{\mathrm{T}} + \Psi_1 \leq S_{ch}(M_4 + M_4^{\mathrm{T}}) < 0.$$

This is the inequality to be shown. The proof is then finished.

Exercise 2.5 (Yu (2002), Page 128)

Let A be an arbitrary square matrix, and Q be some symmetric matrix. Show that there exists a $P > 0$ satisfying

$$A^{\mathrm{T}} P A - P + Q < 0 \tag{s2.12}$$

if and only if there exists a symmetric matrix X such that

$$\begin{bmatrix} -X & AX \\ XA^{\mathrm{T}} & -X + XQX \end{bmatrix} < 0. \tag{s2.13}$$

Solution. Let

$$\Phi(X) = \begin{bmatrix} I & 0 \\ 0 & X^{-1} \end{bmatrix} \begin{bmatrix} -X & AX \\ XA^{\mathrm{T}} & -X + XQX \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X^{-1} \end{bmatrix},$$

then (s2.13) holds if and only if $\Phi(X) < 0$. Note that

$$\Phi = \begin{bmatrix} -X & A \\ A^{\mathrm{T}} & -X^{-1} + Q \end{bmatrix},$$

applying Schur complement lemma to the above matrix Φ, we know that there exists an $X > 0$ such that $\Phi(X) < 0$ if and only if there exists an $X > 0$ satisfying

$$-X^{-1} + Q + A^{\mathrm{T}} X^{-1} A < 0.$$

Letting $P = X^{-1}$, the above inequality is turned into (s2.12). Therefore, the conclusion holds true.

Exercise 2.6

Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Work out by hand the following using the matrix inversion lemma and the Schur complement lemma:

1. find out $\det(A)$ and A^{-1} (if exists);
2. judge the negative definiteness of A.

Solutions. Let

$$A_{11} = -2, \; A_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; A_{22} = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix},$$

then we have

$$\begin{aligned} S_{ch}(A_{11}) &= A_{22} - A_{21}(A_{11})^{-1} A_{12} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

Therefore, it follows from the matrix inversion lemma that

$$\begin{aligned} \det A &= \det A_{11} \det S_{ch}(A_{11}) \\ &= -2 \det \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix} \\ &= -3. \end{aligned}$$

Thus the matrix A is invertible, and A^{-1} is given by

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) \\ -S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & S_{ch}^{-1}(A_{11}) \end{bmatrix}.$$

Since

$$S_{ch}^{-1}(A_{11}) = \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = -\frac{1}{3} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix},$$
$$A_{11}^{-1} A_{12} = -\frac{1}{2} \begin{bmatrix} 1 & 0 \end{bmatrix},$$

we have

$$A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) = \frac{1}{6} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$A_{11}^{-1}A_{12}S_{ch}^{-1}(A_{11})A_{21}A_{11}^{-1} = -\frac{1}{6}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{6},$$

thus

$$\begin{aligned} A^{-1} &= \begin{bmatrix} -\frac{1}{2}-\frac{1}{6} & -\frac{1}{3}\begin{bmatrix} 1 & 1 \end{bmatrix} \\ -\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} & -\frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \end{bmatrix} \\ &= -\frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}. \end{aligned}$$

Finally, since

$$\begin{aligned} &A_{11} = -2 < 0, \\ &S_{ch}(A_{11}) = \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix} < 0. \end{aligned}$$

It follows from the Schur complement lemma that the matrix A is symmetric negative definite.

Chapter 3

Review of Optimization Theory

Exercise 3.1

Let X be a linear space, $E_i \subset X$, $i = 1, 2, \ldots, n$, are convex sets. Find out whether $\bigcup_{i=1}^{n} E_i$ (or $\bigcap_{i=1}^{n} E_i$) is convex or not.

Solution. It can be easily shown that

$$\Omega = \bigcap_{i=1}^{n} E_i$$

is a convex set. Let $x, y \in \Omega$, then we have

$$x, y \in E_i, \ i = 1, 2, \ldots, n.$$

Since E_i, $i = 1, 2, \ldots, n$, are convex, for arbitrary $1 \leq \theta \leq 1$, there holds

$$z = \theta x + (1 - \theta) y \in E_i, \ i = 1, 2, \ldots, n.$$

Therefore, $z \in \Omega$, and Ω is then convex.

Unlike the intersection, the union $\bigcup_{i=1}^{n} E_i$ may not be convex (see Figure s3.1).

Exercise 3.2

Prove Proposition 3.1, that is, let f_i, $i = 1, 2, \ldots, m$, be a group of convex functions defined over the convex set Ω, then for arbitrary scalars $\alpha_i \geq 0$, $i = 1, 2, \ldots, m$, the combination $g = \sum_{i=1}^{m} \alpha_i f_i$ is also a convex function over Ω.

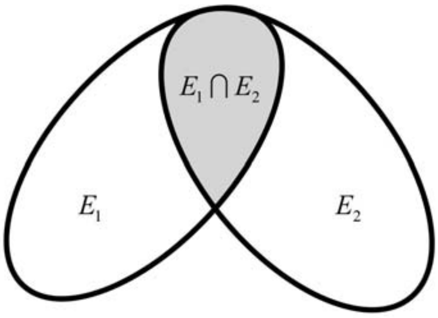

Figure s3.1: $E_1 \cap E_2$ and $E_1 \cup E_2$

Solution. Since $f_i,\ i = 1, 2, \ldots, m$, are a group of convex functions defined over the convex set Ω, for any $x,\ y \in \Omega$ and $0 \le \theta \le 1$, we have

$$\begin{aligned}
& g(\theta x + (1-\theta)y) \\
= \ & \sum_{i=1}^{m} \alpha_i f_i(\theta x + (1-\theta)y) \\
\le \ & \sum_{i=1}^{m} \alpha_i \left(\theta f_i(x) + (1-\theta) f_i(y)\right) \\
= \ & \theta \sum_{i=1}^{m} \alpha_i f_i(x) + (1-\theta) \sum_{i=1}^{m} \alpha_i f_i(y) \\
= \ & \theta g(x) + (1-\theta) g(y).
\end{aligned}$$

It thus follows from the definition of convex functions that g is also a convex function.

Exercise 3.3

Prove Proposition 3.2, that is, let f be a convex function defined over the convex set Ω, then for arbitrary scalar α the set $\Omega_\alpha = \{x \mid f(x) < \alpha\}$ is convex.

Solution. Since f is a convex function over the convex set Ω, we have for any $x, y \in \Omega_\alpha$ and $0 \le \theta \le 1$,

$$\begin{aligned}
f(\theta x + (1-\theta)y) & \leqslant \theta f(x) + (1-\theta) f(y) \\
& < \theta\alpha + (1-\theta)\alpha \\
& = \alpha,
\end{aligned}$$

so

$$\theta x + (1-\theta)y \in \Omega_\alpha,$$

this completes the proof.

Exercise 3.4 (Feng (1995), Page 1)

Given $b_i \in \mathbb{R}^n$, $\beta_i \in \mathbb{R}$, let

$$M_i = \left\{ x \in \mathbb{R}^n \,\middle|\, b_i^{\mathrm{T}} x \le \beta_i \right\}, \ i = 1, 2.$$

Show that $M_1 \cap M_2$ is a convex set.

Solution I. It is obvious that both M_1 and M_2 are halfspaces in $\mathbb{R}^n$, and hence are both convex according to Theorem 3.3. Therefore, $M_1 \cap M_2$ is convex.

Solution II. Note that

$$\begin{aligned} M_1 \cap M_2 &= \left\{ x \in \mathbb{R}^n \,\middle|\, b_i^{\mathrm{T}} x \le \beta_i, \ i = 1, 2 \right\} \\ &= \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{bmatrix} b_1^{\mathrm{T}} \\ b_2^{\mathrm{T}} \end{bmatrix} x \preceq \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right\}, \end{aligned}$$

is still a halfspace, and hence is still convex.

Exercise 3.5 (Feng (1995), Page 100)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive homogeneous function, that is,

$$f(ax) = af(x), \ \text{for } a \in \mathbb{R}, \ x \in \mathbb{R}^n.$$

Show that f is a convex function if and only if

$$f(x+y) \le f(x) + f(y) \tag{s3.1}$$

for any $x, y \in \mathbb{R}^n$.

Solution. *Sufficiency*: For any $x, y \in \mathbb{R}^n$ and $0 \le \theta \le 1$, it follows from (s3.1) and the definition of a positive homogeneous function that

$$\begin{aligned} f(\theta x + (1-\theta)y) &\le f(\theta x) + f((1-\theta)y) \\ &= \theta f(x) + (1-\theta) f(y). \end{aligned}$$

This shows that $f(x)$ is convex.

Necessity: Let $f(x)$ be positive homogeneous and convex, then we have

$$\begin{aligned} f(x+y) &= 2f\left(\frac{x+y}{2}\right) \\ &\le 2\left(f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right)\right) \\ &= 2\left(\frac{1}{2} f(x) + \frac{1}{2} f(y)\right) \\ &= f(x) + f(y). \end{aligned}$$

Exercise 3.6

Prove that the following functions are convex:

1. $f(x) = e^{ax}, \ \ a \in \mathbb{R}, \ a \neq 0,$
2. $f(x) = |x|^p, \ \ p \geq 1.$

Solution. 1. When $a \in \mathbb{R}, \ a \neq 0$, we have

$$\frac{\mathrm{d}^2 (e^{ax})}{\mathrm{d}x^2} = a^2 e^{ax} > 0, \ \forall x \in \mathbb{R}.$$

In view of Theorem 3.7, $f(x) = e^{ax}$ is a convex function.

2. Since

$$f(x) = |x|^p = \begin{cases} x^p, & x > 0 \\ 0, & x = 0 \\ (-x)^p, & x < 0 \end{cases},$$

we can easily obtain

$$\frac{\mathrm{d} f(x)}{\mathrm{d}x} = \begin{cases} px^{p-1}, & x > 0 \\ 0, & x = 0 \\ -p(-x)^{p-1}, & x < 0 \end{cases}.$$

In view of $p > 1$, we have

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} = \begin{cases} p(p-1)x^{p-2} > 0, & x > 0 \\ p(p-1)(-x)^{p-2} > 0, & x < 0 \end{cases},$$

and thus

$$\frac{\mathrm{d}^2 f(0^+)}{\mathrm{d}x^2} = \lim_{x \to 0^+} \frac{px^{p-1}}{x} = \lim_{x \to 0^+} px^{p-2} = \begin{cases} 0, & p > 2 \\ p, & p = 2 \\ +\infty, & 1 < p < 2 \end{cases},$$

$$\frac{\mathrm{d}^2 f(0^-)}{\mathrm{d}x^2} = \lim_{x \to 0^-} \frac{-p(-x)^{p-1}}{x} = \lim_{x \to 0^-} p(-x)^{p-2} = \begin{cases} 0, & p > 2 \\ p, & p = 2 \\ +\infty, & 1 < p < 2 \end{cases}.$$

Combining the above two relations, gives

$$\frac{\mathrm{d}^2 f(0)}{\mathrm{d}x^2} = \begin{cases} 0, & p > 2 \\ p, & p = 2 \\ +\infty, & 1 < p < 2 \end{cases},$$

from which we have

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} \geq 0.$$

Therefore, it follows from Theorem 3.7 again that $f(x) = |x|^p, \ p \geq 1$, is a convex function too.

Exercise 3.7

Show that Problem 3.4 is a convex optimization problem.

Solution. Firstly, we show that $f(x)$ is a convex function. Let

$$\begin{array}{l} P = [p_{ij}]_{n\times n},\ p_{ij} = p_{ji}, \\ q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}^{\mathrm{T}}, \end{array}$$

then

$$\begin{aligned}
f(x) &= \frac{1}{2}x^{\mathrm{T}}Px + q^{\mathrm{T}}x + r \\
&= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} x_1p_{11} + x_2p_{12} + \cdots + x_np_{1n} \\ x_1p_{21} + x_2p_{22} + \cdots + x_np_{2n} \\ \vdots \\ x_1p_{n1} + x_2p_{n2} + \cdots + x_np_{nn} \end{bmatrix} \\
&\quad + x_1q_1 + x_2q_2 + \cdots + x_nq_n + r \\
&= \frac{1}{2}x_1\left(x_1p_{11} + x_2p_{12} + \cdots + x_np_{1n}\right) \\
&\quad + \frac{1}{2}x_2\left(x_1p_{21} + x_2p_{22} + \cdots + x_np_{2n}\right) \\
&\quad + \cdots + \frac{1}{2}x_n\left(x_1p_{n1} + x_2p_{n2} + \cdots + x_np_{nn}\right) \\
&\quad + x_1q_1 + x_2q_2 + \cdots + x_nq_n + r,
\end{aligned}$$

thus

$$\begin{aligned}
\frac{\partial f}{\partial x_j} &= \frac{1}{2}(x_1p_{1j} + x_2p_{2j} + \cdots + x_{j-1}p_{j-1,j}) \\
&\quad + \frac{1}{2}(x_1p_{j1} + x_2p_{j2} + \cdots + x_{j-1}p_{j,j-1} + 2x_jp_{jj} \\
&\qquad + x_{j+1}p_{j,j+1} + \cdots + x_np_{jn}) \\
&\quad + \frac{1}{2}\left(x_{j+1}p_{j+1,j} + \cdots + x_np_{nj}\right) + q_j \\
&= x_1p_{1j} + x_2p_{2j} + \cdots + x_np_{nj} + q_j,\ j = 1, 2, ..., n,
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) \\
&= \frac{\partial}{\partial x_i}\left(x_1p_{1j} + x_2p_{2j} + \cdots + x_np_{nj} + q_j\right) \\
&= p_{ij},\ i, j = 1, 2, ..., n.
\end{aligned}$$

Therefore,

$$\nabla^2 f(x) = P \geq 0.$$

It then follows from Theorem 3.7 that $f(x)$ is a convex function.

Next, noticing that $\{x \mid Ax = b\}$ is a convex set, we need only to show that $\{x \mid Gx \preceq h\}$ is a convex set. Denote

$$\begin{aligned} &G = [g_{ij}]_{n\times n}, \\ &h = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

then

$$Gx - h = \begin{bmatrix} x_1 g_{11} + x_2 g_{12} + \cdots + x_n g_{1n} - h_1 \\ x_1 g_{21} + x_2 g_{22} + \cdots + x_n g_{2n} - h_2 \\ \vdots \\ x_1 g_{n1} + x_2 g_{n2} + \cdots + x_n g_{nn} - h_n \end{bmatrix}.$$

Further denote

$$g_i(x) = x_1 g_{i1} + x_2 g_{i2} + \cdots + x_n g_{in} - h_i,\ i = 1, 2, \ldots, n,$$

then we need to show that $\{x \mid g_i(x) \le 0\}$ is convex. To achieve this, arbitrarily choose

$$x, y \in \{x \mid g_i(x) \le 0\},$$

then we have

$$g_i(x) \le 0, \quad g_i(y) \le 0.$$

Further denote

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}, \\ y &= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

and define

$$z = \theta x + (1-\theta) y = \begin{bmatrix} \theta x_1 + (1-\theta) y_1 \\ \theta x_2 + (1-\theta) y_2 \\ \vdots \\ \theta x_n + (1-\theta) y_n \end{bmatrix},$$

then

$$\begin{aligned} g_i(z) &= (\theta x_1 + (1-\theta) y_1)\, g_{i1} + (\theta x_2 + (1-\theta) y_2)\, g_{i2} \\ &\quad + \cdots + (\theta x_n + (1-\theta) y_n)\, g_{in} - h_i, \\ &= \theta\,(x_1 g_{i1} + x_2 g_{i2} + \cdots + x_n g_{in} - h_i) \\ &\quad + (1-\theta)\,(y_1 g_{i1} + y_2 g_{i2} + \cdots + y_n g_{in} - h_i) \\ &= \theta g_i(x) + (1-\theta) g_i(y) \\ &\le 0. \end{aligned}$$

This shows the convexity of $\{x \mid g_i(x) \le 0\}$. Therefore the optimization problem is indeed a convex one.

Chapter 4

Stability Analysis

Exercise 4.1

What are the relations among the concepts of robust stability, quadratic stability, and affine quadratic stability?

Solution. Quadratic stability and affine quadratic stability are both concerned with stability of a family of systems. When the same family of systems are considered, quadratic stability generally implies affine quadratic stability.

Robust stability is concerned with the stability of a system subject to uncertainties. In the case that the family of systems treated in quadratic stability and/or affine quadratic stability is really the formulation of a system subject to uncertainties, both quadratic stability and affine quadratic stability are in fact some specific types of robust stability.

Exercise 4.2

Let $A \in \mathbb{R}^{n\times n}$ and $B = \dfrac{1}{2}(A + A^{\mathrm{T}})$. Show that

1. if B is Hurwitz stable, then A is also Hurwitz stable;
2. if B is Schur stable, and A is symmetric positive definite, then A is also Schur stable.

Solution. 1. Since B is Hurwitz stable, we have

$$\lambda(B) = \mathrm{Re}\,\lambda(B) < 0.$$

Further note that

$$\lambda_{\min}(B) \le \mathrm{Re}(\lambda(A)) \le \lambda_{\max}(B),$$

we immediately get the conclusion.

2. Since B is Schur stable, we have

$$|\lambda(B)| < 1.$$

Further, since A and B are both symmetric, $\lambda(A)$ and $\lambda(B)$ are all real numbers. Thus combining the above relation with

$$\lambda_{\min}(B) \le \lambda(A) \le \lambda_{\max}(B),$$

produces

$$-1 < \lambda_{\min}(B) \le \lambda(A) \le \lambda_{\max}(B) < 1.$$

This gives the Schur stability of the matrix A.

Exercise 4.3

Verify using the LMI technique the stability of the following continuous-time linear system

$$\dot{x}(t) = \begin{bmatrix} -0.0180 & -0.2077 & -0.7150 \\ -0.5814 & -4.2900 & 0 \\ 1.0670 & 4.2730 & -6.6540 \end{bmatrix} x(t).$$

Solution. Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (4.2), that is,

$$\begin{cases} P > 0 \\ A^{\mathrm{T}}P + PA < 0 \end{cases}, \tag{s4.1}$$

gives the following solution

$$P = \begin{bmatrix} 1.1870 & -0.1557 & -0.1232 \\ -0.1557 & 0.2089 & 0.0584 \\ -0.1232 & 0.0584 & 0.1065 \end{bmatrix} > 0.$$

Therefore, the system is stable. In fact, by the Matlab function `eig` it can be verified that the system poles are -0.0417, -4.5200 and -6.4003, which are all stable ones.

Exercise 4.4

Show that the following regions are LMI ones:

$$\begin{aligned} \mathbb{D}_1 &= \{x + yi \mid -r \le y \le r,\ r > 0\}, \\ \mathbb{D}_2 &= \{x + yi \mid x \ge ay^2 + c,\ a > 0\}, \\ \mathbb{D}_3 &= \{x + yi \mid (x+q)^2 + p^2y^2 < r^2\}. \end{aligned}$$

Solution. Let

$$s = x + iy,$$

then it is obvious that

$$x = \frac{1}{2}(s+\bar{s}),\ y = -\frac{1}{2}i(s-\bar{s}).$$

Using the above relations and the Schur complement lemma, we have

$$\begin{aligned}
\mathbb{D}_1 &= \{x+yi \,|-r \le y \le r,\ r>0\} \\
&= \left\{x+yi \,\middle|\, y^2 \le r^2,\ r>0\right\} \\
&= \left\{x+yi \,\middle|\, -\frac{1}{4}(s-\bar{s})^2 \le r^2,\ r>0\right\} \\
&= \left\{x+yi \,\middle|\, -\frac{1}{2r}(s-\bar{s})^2 - 2r \le 0,\ r>0\right\} \\
&= \left\{x+yi \,\middle|\, \begin{bmatrix} -2r & s-\bar{s} \\ -(s-\bar{s}) & -2r \end{bmatrix} \le 0,\ r>0\right\} \\
&= \left\{x+yi \,\middle|\, \begin{bmatrix} -2r & 0 \\ 0 & -2r \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bar{s} \le 0,\ r>0\right\} \\
&= \left\{s \,\middle|\, L_1 + sM_1 + \bar{s}M_1^{\mathrm{T}} < 0\right\},
\end{aligned}$$

where

$$L_1 = \begin{bmatrix} -2r & 0 \\ 0 & -2r \end{bmatrix},\ M_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$$\begin{aligned}
\mathbb{D}_2 &= \left\{x+yi \,\middle|\, x \ge ay^2 + c,\ a>0\right\} \\
&= \left\{x+yi \,\middle|\, \frac{1}{2}(s+\bar{s}) \ge -\frac{1}{4}a(s-\bar{s})^2 + c,\ a>0\right\} \\
&= \left\{x+yi \,\middle|\, -2(s+\bar{s}) - a(s-\bar{s})^2 + 4c \le 0,\ a>0\right\} \\
&= \left\{x+yi \,\middle|\, \begin{bmatrix} -2(s+\bar{s})+4c & a(s-\bar{s}) \\ -a(s-\bar{s}) & -a \end{bmatrix} \le 0,\ a>0\right\} \\
&= \left\{x+yi \,\middle|\, \begin{bmatrix} 4c & 0 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} -2 & a \\ -a & 0 \end{bmatrix} s + \begin{bmatrix} -2 & -a \\ a & 0 \end{bmatrix} \bar{s} \le 0,\ a>0\right\} \\
&= \left\{s \,\middle|\, L_2 + sM_2 + \bar{s}M_2^{\mathrm{T}} < 0\right\},
\end{aligned}$$

where

$$L_2 = \begin{bmatrix} 4c & 0 \\ 0 & -a \end{bmatrix},\ M_2 = \begin{bmatrix} -2 & a \\ -a & 0 \end{bmatrix}.$$

$$\begin{aligned}
\mathbb{D}_3 &= \{x+iy \mid (x+q)^2+p^2y^2<r^2\} \\
&= \{x+iy \mid (x+q+ipy)(x+q-ipy)<r^2\} \\
&= \left\{s \left| \left(\frac{1+p}{2}s+\frac{1-p}{2}\bar{s}+q\right)\left(\frac{1-p}{2}s+\frac{1+p}{2}\bar{s}+q\right)<r^2 \right.\right\} \\
&= \left\{s \left| \begin{bmatrix} -r & \frac{1+p}{2}s+\frac{1-p}{2}\bar{s}+q \\ \frac{1-p}{2}s+\frac{1+p}{2}\bar{s}+q & -r \end{bmatrix} < 0 \right.\right\} \\
&= \left\{s \left| \begin{bmatrix} -r & q \\ q & -r \end{bmatrix} + s\begin{bmatrix} 0 & \frac{1+p}{2} \\ \frac{1-p}{2} & 0 \end{bmatrix} + \bar{s}\begin{bmatrix} 0 & \frac{1+p}{2} \\ \frac{1-p}{2} & 0 \end{bmatrix}^{\mathrm{T}} < 0 \right.\right\} \\
&= \left\{s \left| L+sM+\bar{s}M^{\mathrm{T}}<0 \right.\right\},
\end{aligned}$$

where

$$L=\begin{bmatrix} -r & q \\ q & -r \end{bmatrix},\ M=\begin{bmatrix} 0 & \frac{1+p}{2} \\ \frac{1-p}{2} & 0 \end{bmatrix}.$$

Thus $\mathbb{D}_1$, $\mathbb{D}_2$ and $\mathbb{D}_3$ are all LMI regions.

Exercise 4.5

Consider the following advanced (CCV-type) fighter aircraft system (Syrmos and Lewis (1993), Duan (2003)):

$$\dot{x}(t)=Ax(t)+Bu(t)$$

with

$$\begin{aligned}
A &= \begin{bmatrix} -1.3410 & 0.9933 & 0 & -0.1689 & -0.2518 \\ 43.2230 & -0.8693 & 0 & -17.2510 & -1.5766 \\ 1.3410 & 0.0067 & 0 & 0.1689 & 0.2518 \\ 0 & 0 & 0 & -20.0000 & 0 \\ 0 & 0 & 0 & 0 & -20.0000 \end{bmatrix}, \\
B &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}.
\end{aligned}$$

Judge the open-loop stability of the above system.

Solution. We suffice to solve the corresponding LMI problem (4.2), that is problem (s4.1). Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox, we find that these LMI constraints are infeasible. Thus the system is unstable by Proposition 4.1.

In fact, by the Matlab function `eig`, it can be verified that the system poles are

$$\lambda(A)=\{0, 5.4515, -7.6618, -20.0000, -20.0000\},$$

where the first two poles are unstable.

Exercise 4.6

Consider a system in the form of (4.22)-(4.24), with $\Delta = \Delta_P$ given by (4.27), where

$$\begin{aligned} A_0 &= \begin{bmatrix} -2.1333 & -1.3333 \\ -0.6667 & -1.4667 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}. \end{aligned}$$

Further, let us assume that the set of perturbation parameters form a polytope in the form of (4.27), with $k = 2$. Judge the quadratic stability of the system.

Solution. According to theory of quadratic stability, this is required to solve the corresponding LMI Problem (4.32), that is,

$$(A_0 + A_i)^{\mathrm{T}} P + P(A_0 + A_i) < 0, \; i = 1, \ldots, k. \tag{s4.2}$$

Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox, gives the solution

$$P = \begin{bmatrix} 0.4740 & -0.3061 \\ -0.3061 & 0.8627 \end{bmatrix} > 0.$$

Thus, the system is quadratically Hurwitz stable.

Exercise 4.7

Verify using the LMI technique the stability of the following time-delay linear system

$$\dot{x}(t) = \begin{bmatrix} -3 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x(t-d).$$

Solution. Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (4.48), that is,

$$\begin{cases} P > 0 \\ \begin{bmatrix} A^{\mathrm{T}} P + PA + S & PA_d \\ A_d^{\mathrm{T}} P & -S \end{bmatrix} < 0 \end{cases}, \tag{s4.3}$$

yields the following solutions

$$
\begin{aligned}
P &= \begin{bmatrix} 0.4651 & -0.0693 & 0.1283 \\ -0.0693 & 0.3271 & -0.0087 \\ 0.1283 & -0.0087 & 0.3693 \end{bmatrix} > 0, \\
S &= \begin{bmatrix} 1.2781 & -0.1356 & 0.0213 \\ -0.1356 & 1.0876 & 0.0029 \\ 0.0213 & 0.0029 & 1.3327 \end{bmatrix} > 0.
\end{aligned}
$$

Thus, we can conclude that this time-delay system is stable.

Exercise 4.8

Judge the stability of the following time-delay linear system

$$\dot{x}(t) = \begin{bmatrix} -5 & 1 \\ 0 & -10 \end{bmatrix} x(t) + \begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix} x(t-d(t)) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t).$$

Solution. For this system, applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (4.48), that is, problem (s4.3), yields the following parameters

$$
\begin{aligned}
P &= \begin{bmatrix} 0.1574 & -0.0033 \\ -0.0033 & 0.0883 \end{bmatrix} > 0, \\
S &= \begin{bmatrix} 0.8949 & -0.0692 \\ -0.0692 & 0.9607 \end{bmatrix} > 0.
\end{aligned}
$$

Thus, this time-delay system is stable.

Chapter 5

H_∞/H_2 Performance

Exercise 5.1

Prove Corollary 5.3, that is, let

$$G(s) = C(sI - A)^{-1}B + D,$$

then the matrix A is Hurwitz stable if $\|G(s)\|_\infty < \gamma$ for some $0 < \gamma < \infty$.

Solution. If $\|G(s)\|_\infty < \gamma$ for some $\gamma > 0$, then it follows from Theorem 5.4 that one can derive that there exists an $X > 0$ such that

$$AX + XA^{\mathrm{T}} < 0,$$

then all the eigenvalues of A have negative real parts, that is, the matrix A is asymptotically stable.

Exercise 5.2 (Scherer and Weiland (2000))

Let

$$A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}, \quad 0 \le \alpha \le \pi\,.$$

Show that the solution X of the Lyapunov equation

$$A^{\mathrm{T}}X + XA + I = 0 \tag{s5.1}$$

diverges in the sense that $\det(X) \to \infty$ whenever $\alpha \to 0$.

Solution. Note that

$$\begin{aligned} \det(sI - A) &= \det \begin{bmatrix} s - \sin\alpha & -\cos\alpha \\ \cos\alpha & s - \sin\alpha \end{bmatrix} \\ &= s^2 - 2s\sin\alpha + 1, \end{aligned}$$

we have

$$\mathrm{Re}\lambda(A) = \frac{1}{2}(\lambda_1 + \lambda_2) = \sin\alpha > 0,\ 0 \le \alpha \le \pi\,.$$

Therefore, there exists a unique solution to the Lyapunov equation (s5.1), which is given by

$$\begin{aligned} X &= -\int_{-\infty}^{0} e^{A^{\mathrm{T}}t} e^{At} \mathrm{d}t \\ &= -\int_{-\infty}^{0} e^{(A^{\mathrm{T}}+A)t} \mathrm{d}t \\ &= -(A^{\mathrm{T}} + A)^{-1} \\ &= -\frac{1}{2\sin\alpha} I_2. \end{aligned} \tag{s5.2}$$

With this expression of X, it is obvious to see that

$$\lim_{\alpha\to 0} \det(X) = \infty.$$

Remark 1. The above proof uses this well-known conclusion: The general Lyapunov matrix equation

$$A^{\mathrm{T}}X + XA + Q = 0$$

has a unique solution if and only if

$$\alpha + \beta \neq 0,\ \forall \alpha, \beta \in \lambda(A),$$

and an analytic solution is given by

$$X = \int_{0}^{\infty} e^{A^{\mathrm{T}}t} Q e^{At} \mathrm{d}t,\ \ \text{when } \lambda(A) \in \mathbb{C}^-,$$

or

$$X = -\int_{-\infty}^{0} e^{A^{\mathrm{T}}t} Q e^{At} \mathrm{d}t,\ \ \text{when } \lambda(A) \in \mathbb{C}^+.$$

Remark 2. The analytic solution (s5.2) to the particular Lyapunov equation (s5.1) can be easily derived. In fact, noting that

$$A^{\mathrm{T}} e^{A^{\mathrm{T}}t} e^{At} + e^{A^{\mathrm{T}}t} e^{At} A = \frac{\mathrm{d}}{\mathrm{d}t}(e^{A^{\mathrm{T}}t} e^{At}),$$

and taking integration, gives

$$A^{\mathrm{T}}\left(-\int_{-\infty}^{0} e^{A^{\mathrm{T}}t} e^{At} \mathrm{d}t\right) + \left(-\int_{-\infty}^{0} e^{A^{\mathrm{T}}t} e^{At} \mathrm{d}t\right) A + I = 0.$$

This shows that (s5.2) is the solution to the Lyapunov equation (s5.1)

Exercise 5.3

Consider the following system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases},$$

where

$$A = \begin{bmatrix} -5 & 1 & 2 \\ 1 & -9 & 1 \\ -1 & -10 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find a minimal γ such that the system transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

satisfies $||G(s)||_\infty < \gamma$.

Solution. This is to solve the corresponding LMI problem (5.30), that is,

$$\begin{cases} \min & \gamma \\ \text{s.t.} & P > 0 \\ & \begin{bmatrix} A^{\mathrm{T}}P + PA & PB & C^{\mathrm{T}} \\ B^{\mathrm{T}}P & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{bmatrix} < 0 \end{cases}. \tag{s5.3}$$

With the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox, we obtain for the above problem $\gamma = 1.4544$, and

$$P = \begin{bmatrix} 15.8703 & -3.1830 & -0.6907 \\ -3.1830 & 1.5792 & 0.0662 \\ -0.6907 & 0.0662 & 1.0878 \end{bmatrix} > 0.$$

By directly using the Matlab function `norm` to the system, we obtain

$$||G(s)||_\infty = 1.4479.$$

Obviously, γ is a very good estimate of $||G(s)||_\infty$.

Exercise 5.4

Consider the system in Exercise 5.3 again. Find a minimal γ such that the system transfer function

$$G(s) = C(sI - A)^{-1}B$$

satisfies $||G(s)||_2 < \gamma$.

Solution. Solving the corresponding LMI problem (5.47), that is,

$$\begin{cases} \min & \rho \\ \text{s.t.} & A^{\mathrm{T}}Y + YA + C^{\mathrm{T}}C < 0 \\ & \operatorname{trace}\left(B^{\mathrm{T}}YB\right) < \rho \\ & Y > 0 \end{cases}, \tag{s5.4}$$

with the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox, we can obtain

$$\gamma = \sqrt{\rho} = 0.3346,\ Y = \begin{bmatrix} 0.1048 & 0.0226 & -0.0001 \\ 0.0226 & 0.1119 & -0.0985 \\ -0.0001 & -0.0985 & 0.1339 \end{bmatrix} > 0.$$

By directly using the Matlab function `norm` to the system, we obtain

$$||G(s)||_2 = 0.3345.$$

Obviously, γ is a very good estimate of $||G(s)||_2$.

Exercise 5.5

Consider a jet transport aircraft system studied by Liu et al. (2000a) and Grace et al. (1992), whose model is in the form of (5.15) with $D = 0$, and

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1.0000 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0729 & 0.0001 \\ -4.7500 & 1.2300 \\ 1.5300 & 10.6300 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find a minimal γ, such that the system transfer function

$$G(s) = C(sI - A)^{-1}B$$

satisfies

$$||G(s)||_\infty < \gamma.$$

Solution. Solving the corresponding LMI problem (5.30), that is, problem (s5.3), with the Matlab function `mincx` in the Matlab LMI toolbox, we obtain $\gamma = 4664.8885$ and

$$X = \begin{bmatrix} 54.3553 & 58.9468 & 17.6259 & -2.6952 \\ 58.9468 & 111.1641 & 26.5342 & -4.6500 \\ 17.6259 & 26.5342 & 8.5877 & -1.1317 \\ -2.6952 & -4.6500 & -1.1317 & 0.2260 \end{bmatrix} > 0.$$

By directly using the Matlab function `norm` to the system, we obtain

$$||G\,(s)\,||_\infty = 4664.4229.$$

Obviously, γ is an approximate estimate of $||G\,(s)\,||_\infty$.

Exercise 5.6

Consider the system in Exercise 5.5. Find a minimal ρ, such that

$$||G\,(s)\,||_2^2 < \rho.$$

Solution. Solving the corresponding LMI problem (5.47), that is, problem (s5.4), with the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox, we obtain $\rho = 80687.0224$ and

$$Y = \begin{bmatrix} 79.3837 & -85.6392 & -18.1825 & -12.0798 \\ -85.6392 & 2634.47782 & 492.7992 & 314.8196 \\ -18.1825 & 492.7992 & 98.7568 & 63.0236 \\ -12.0798 & 314.8196 & 63.0236 & 41.0452 \end{bmatrix} > 0.$$

By directly using the Matlab function `norm` to the system, we obtain

$$||G\,(s)\,||_2 = 283.4034.$$

Obviously, $\gamma = \sqrt{\rho} = 284.0546$ is an approximate estimate of $||G\,(s)\,||_2$.

Exercise 5.7

Consider the transfer function

$$G(s) = C^{\mathrm{T}}(sI - A)^{-1}B,$$

with

$$A = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

First find $\|G(s)\|_2$ using Lemma 5.1, and then compute it via solving (5.46) or (5.47), and compare the values of $\|G(s)\|_2$ obtained through the two different ways.

Solution. Solving the Lyapunov equation (5.4) with the given parameters, yields

$$Y = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

so by Lemma 5.1,

$$\|G(s)\|_2^2 = \mathrm{trace}(B^{\mathrm{T}}YB) = 0.5.$$

On the other hand, solving the corresponding optimization problem (5.47), that is, problem (s5.4), with the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox, we get

$$\begin{aligned} Y &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \\ \rho &= 0.5. \end{aligned}$$

It is obvious that we get the same result for $\|G(s)\|_2$ by the two different methods.

Chapter 6

Property Analysis

Exercise 6.1

Prove Corollary 6.1, that is, the linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

is passive only if the matrix A is stable.

Solution. It follows from Theorem 6.14 that if the above linear system is passive, then there exists a matrix $P \in \mathbb{S}^n$ such that (6.44), that is,

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB - C^{\mathrm{T}} \\ B^{\mathrm{T}}P - C & -D^{\mathrm{T}} - D \end{bmatrix} \leq 0, \tag{s6.1}$$

holds. From this we have

$$A^{\mathrm{T}}P + PA \leq 0.$$

Thus by Lyapunov stability theory, we know that A is stable.

Exercise 6.2

Verify the stabilizability of the following linear system using LMI technique:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u.$$

Solution. For this system, we have

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (6.7), that is,

$$AP + PA^{\mathrm{T}} < BB^{\mathrm{T}}, \tag{s6.2}$$

gives the following solution

$$P = \begin{bmatrix} 0.6897 & 0.0329 & -0.0136 \\ 0.0329 & 0.2357 & -0.1428 \\ -0.0136 & -0.1428 & 0.2671 \end{bmatrix} > 0.$$

Thus the system is stabilizable.

Remark. This conclusion can also be easily verified by checking

$$\operatorname{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = 3, \quad \forall s \in \lambda(A) \cap \mathbb{C}^{+}.$$

It can be easily obtained that

$$\lambda(A) = \{-1,\ 1,\ 2\},$$

and it can be easily checked that

$$\operatorname{rank}\begin{bmatrix} I - A & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 2I - A & B \end{bmatrix} = 3.$$

The system is indeed stabilizable.

Exercise 6.3

Verify using LMI technique the detectability of the following system

$$\begin{cases} \dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} x \\ y = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 8 \end{bmatrix} x \end{cases}.$$

Solution. For this system, we have

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 8 \end{bmatrix}.$$

Applying the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (6.12), that is,

$$A^{\mathrm{T}}P + PA < \gamma C^{\mathrm{T}}C, \quad \gamma > 0, \tag{s6.3}$$

gives the following solution

$$P = 10^3 \times \begin{bmatrix} 1.2854 & 0.1798 & -0.0044 \\ 0.1798 & 1.3342 & -0.0048 \\ -0.0044 & -0.0048 & 1.2933 \end{bmatrix} > 0.$$

Thus the system is detectable.

This conclusion actually can be easily seen from the fact that the system has only one multiple pole $s = -2$, and hence does not have an unstable pole.

Exercise 6.4

Consider the following linear system (Duan (2003))

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

with

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -2 & 10 \\ 0 & 1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Determine whether or not this system is stabilizable and detectable.

Solution. Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (6.7), that is, problem (s6.2), gives the following solution

$$P = \begin{bmatrix} 0.5596 & -0.2917 & -0.2423 \\ -0.2917 & 1.2674 & 0.5964 \\ -0.2423 & 0.5964 & 0.3377 \end{bmatrix} > 0.$$

Thus the system is stabilizable.

Applying the function `feasp` in the Matlab LMI toolbox to the corresponding LMI problem (6.12), that is, problem (s6.3), gives the following solution

$$P = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.0662 & -0.1474 \\ 0 & -0.1474 & 0.4576 \end{bmatrix} > 0.$$

Thus the system is also detectable.

Exercise 6.5

Consider the following linear system

$$G(s) = \frac{1}{s^2 + 3s + 2}. \tag{s6.4}$$

1. Determine whether or not this system is passive.
2. Determine whether or not this system is bounded-real.

Solution. A minimal realization of $G(s)$ can be easily obtained as

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D = 0.$$

Clearly, A is stable.

Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (6.44), that is, problem (s6.1), with the above given parameters, we find that these LMI constraints are infeasible. By Theorem 6.14, this linear system is not passive.

Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (6.50), that is,

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB & C^{\mathrm{T}} \\ B^{\mathrm{T}}P & -I & D^{\mathrm{T}} \\ C & D & -I \end{bmatrix} < 0, \tag{s6.5}$$

with the given parameters, gives the following solution

$$P = \begin{bmatrix} 1.6009 & 0.3017 \\ 0.3017 & 0.3647 \end{bmatrix} > 0.$$

By Theorem 6.16, this linear system is bounded-real.

Remark. The bounded-realness of the system can also be verified by solving the corresponding optimal H_∞ problem (5.30), that is, problem (s5.3), which has a solution $\gamma = 0.5002 < 1$ with

$$P = \begin{bmatrix} 3.0036 & 0.9988 \\ 0.9988 & 1.8569 \end{bmatrix} > 0.$$

Exercise 6.6

Consider the jet transport aircraft system in Example 6.2.

1. Determine whether or not this system is passive.
2. Determine whether or not this system is bounded-real.

Solution. Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (6.44), that is, problem (s6.1), with the parameters given in Example 6.2, we find that the LMI constraints in the problem are infeasible. By Theorem 6.14, this linear system is not passive.

Applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (6.50), that is, problem (s6.5) with the parameters given in Example 6.2, we find that these LMI constraints are also infeasible. By Theorem 6.16, this linear system is not bounded-real.

Exercise 6.7

Prove Theorem 6.16 using Theorem 6.12.

Solution. It follows from Theorem 6.12 that, under the controllability assumption, the system (6.27) is dissipative with the supply function defined by (6.28) if and only if there exists a matrix $P > 0$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB \\ B^{\mathrm{T}}P & 0 \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0. \tag{s6.6}$$

Furthermore, it is known that the dissipativity of system (6.27) reduces to the nonexpansivity of the system or the bounded-realness of its transfer function when the matrix Q is taken as

$$Q = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

Inserting the above Q matrix into (s6.6), gives

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB \\ B^{\mathrm{T}}P & 0 \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -C & -D \\ 0 & I \end{bmatrix} < 0, \tag{s6.7}$$

which can be arranged into the following form:

$$\begin{bmatrix} PA + A^{\mathrm{T}}P + C^{\mathrm{T}}C & PB + C^{\mathrm{T}}D \\ B^{\mathrm{T}}P + D^{\mathrm{T}}C & D^{\mathrm{T}}D - I \end{bmatrix} < 0.$$

This is the condition (6.51) for the bounded-realness of the system.

On the other side, it is easy to verify

$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -C & -D \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}.$$

Therefore, (s6.7) can be written as

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB \\ B^{\mathrm{T}}P & -I \end{bmatrix} - \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0,$$

which can be converted, in view of the Schur complement lemma, into

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB & C^{\mathrm{T}} \\ B^{\mathrm{T}}P & -I & D^{\mathrm{T}} \\ C & D & -I \end{bmatrix} < 0.$$

This is the condition (6.50) for the bounded-realness of the system.

Chapter 7

Feedback Stabilization

Exercise 7.1

Prove Theorem 7.1.

Solution. 1. By Lyapunov stability theory, the problem has a solution if and only if there exists a symmetric positive matrix P such that

$$AP + BKP + (AP + BKP)^{\mathrm{T}} < 0.$$

Let $W = KP$, then the above inequality becomes

$$AP + PA^{\mathrm{T}} + BW + (BW)^{\mathrm{T}} < 0. \tag{s7.1}$$

On the other side, suppose that there exist a symmetric positive definite matrix P and a matrix W satisfying (7.2), that is, (s7.1), then we have

$$\left(A + BWP^{-1}\right) P + P\left(A + BWP^{-1}\right)^{\mathrm{T}} < 0.$$

Thus a solution to the problem is given by $K = WP^{-1}$.

2. Let (A, B) be stabilizable, then under this stabilizability condition, there exists a positive definite matrix X satisfying the following Riccati matrix equation

$$A^{\mathrm{T}}X + XA - XBB^{\mathrm{T}}X + I = 0.$$

Pre- and post-multiplying by $P = X^{-1}$ both sides of the above equation gives

$$PA^{\mathrm{T}} + AP - BB^{\mathrm{T}} = -PP < 0.$$

On the other side, when

$$PA^{\mathrm{T}} + AP - BB^{\mathrm{T}} < 0$$

holds, put

$$K = -\frac{1}{2} B^{\mathrm{T}} P^{-1},$$

then we can varify

$$\begin{aligned}
& (A + BK) P + P (A + BK) \\
= & \left(A - \frac{1}{2} B B^{\mathrm{T}} P^{-1}\right) P + P \left(A - \frac{1}{2} B B^{\mathrm{T}} P^{-1}\right)^{\mathrm{T}} \\
= & \left(AP - \frac{1}{2} B B^{\mathrm{T}}\right) + \left(AP - \frac{1}{2} B B^{\mathrm{T}}\right)^{\mathrm{T}} \\
= & P A^{\mathrm{T}} + AP - B B^{\mathrm{T}} \\
< & \ 0.
\end{aligned}$$

Then (A, B) is stabilizable, and a stabilizing state feedback gain is given by

$$K = -\frac{1}{2} B^{\mathrm{T}} P^{-1}.$$

Remark. The above proof of the second conclusion has used the following well-known conclusion: Let (A, B) be stabilizable, then there exists a positive definite matrix X satisfying the following Riccati matrix equation

$$A^{\mathrm{T}} X + X A - X B R B^{\mathrm{T}} X + Q = 0,$$

where Q and R are symetric positive definite matrices of appropraite dimensions.

Exercise 7.2

Find a stabilizing state feedback controller for the following continuous-time linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 250 & 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t).$$

Solution. By using the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (7.2), that is,

$$AP + P A^{\mathrm{T}} + BW + W^{\mathrm{T}} B^{\mathrm{T}} < 0, \tag{s7.2}$$

a pair of solutions are obtained as

$$\begin{aligned}
P &= \begin{bmatrix} 1.2865 & -0.4288 & -0.3216 \\ -0.4288 & 0.6433 & -0.4288 \\ -0.3216 & -0.4288 & 1.2865 \end{bmatrix} > 0, \\
W &= \begin{bmatrix} -32.2806 & 10.3778 & 8.6196 \end{bmatrix},
\end{aligned}$$

and thus a stabilizing feedback gain is given by

$$K = WP^{-1} = \begin{bmatrix} -25.2121 & -0.5273 & 0.2212 \end{bmatrix}.$$

It can be verified using the Matlab function `eig` that the closed-loop poles are -0.5177 and $-1.1351 \pm 1.6760i$, which are indeed all stable.

Exercise 7.3

Find a stabilizing state feedback controller for the following discrete-time linear system

$$x(k+1) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0.5 & 0 \\ 1 & 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(k).$$

Solution. Applying the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (7.5), that is,

$$\begin{bmatrix} -P & AP + BW \\ PA^{\mathrm{T}} + W^{\mathrm{T}}B^{\mathrm{T}} & -P \end{bmatrix} < 0, \tag{s7.3}$$

yields

$$\begin{aligned} P &= \begin{bmatrix} 227.6393 & 140.8398 & 53.9320 \\ 140.8398 & 276.6322 & -140.8398 \\ 53.9320 & -140.8398 & 227.6393 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} -86.9078 & -417.4720 & 368.4792 \\ -121.8137 & -142.9807 & 121.8137 \end{bmatrix}, \end{aligned}$$

thus a stabilizing feedback gain is obtained as

$$K = WP^{-1} = \begin{bmatrix} 0.0000 & -1.0000 & 1.0000 \\ -1.6179 & 1.1306 & 1.6179 \end{bmatrix}.$$

It can be verified using the Matlab function `eig` that the set of closed-loop eigenvalues is

$$\lambda(A) = \{-0.4285, 0.0591.0\},$$

which contains all stable ones in the discrete-time sense.

Exercise 7.4

Verify that the following linear system is $\mathbb{H}_{(1,3)}$-stable:

$$\dot{x}(t) = \begin{bmatrix} -1 & -2 \\ 1.5 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} u(t).$$

Solution. By definition,

$$\mathbb{H}_{(1,3)} = \{x + yj \mid -3 < x < -1 < 0\},$$

that is

$$\alpha = 1,\ \beta = 3.$$

By using the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (7.8), that is,

$$\begin{cases} AP + PA^{\mathrm{T}} + BW + W^{\mathrm{T}}B^{\mathrm{T}} + 2\alpha P < 0 \\ -AP - PA^{\mathrm{T}} - BW - W^{\mathrm{T}}B^{\mathrm{T}} - 2\beta P < 0 \end{cases}, \tag{s7.4}$$

the parameter matrices are obtained as

$$\begin{aligned} P &= \begin{bmatrix} 42.7965 & 0.0000 \\ 0.0000 & 42.7965 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} -42.7965 & 85.5930 \end{bmatrix}. \end{aligned}$$

Hence a feedback gain is given by

$$K = WP^{-1} = \begin{bmatrix} -1 & 2 \end{bmatrix}.$$

It can be computed using the Matlab function `eig` that the corresponding closed-loop poles are -2 and -2, which indeed locate in the region $\mathbb{H}_{(1,3)}$.

Exercise 7.5

Find for the following linear system

$$\dot{x}(t) = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t),$$

a state feedback controller such that the closed-loop system is $\mathbb{D}_{(2,1)}$-stable.

Solution. By definition, we have

$$\mathbb{D}_{(2,1)} = \left\{ x + jy \,\middle|\, (x+2)^2 + y^2 < 1 \right\},$$

and

$$q = 2,\ r = 1.$$

By using the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (7.9), that is,

$$\begin{bmatrix} -rP & qP + AP + BW \\ qP + PA^{\mathrm{T}} + W^{\mathrm{T}}B^{\mathrm{T}} & -rP \end{bmatrix} < 0, \tag{s7.5}$$

a pair of parameter matrices are obtained as

$$\begin{aligned} P &= \begin{bmatrix} 0.6401 & -0.5954 \\ -0.5954 & 0.9035 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} 1.0333 & -1.2936 \end{bmatrix}. \end{aligned}$$

Hence a feedback gain is given by

$$K = WP^{-1} = \begin{bmatrix} 0.7301 & -0.9506 \end{bmatrix}.$$

It can be computed using the Matlab function `eig` that the corresponding closed-loop poles are -2.2065 and -1.9645, which are indeed located in the region $\mathbb{D}_{(2,1)}$.

Exercise 7.6

Let the matrices associated with the characteristic function of the LMI region $\mathbb{D}$ be

$$L = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \ M = \frac{1}{4}\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}.$$

Find a $\mathbb{D}$-stabilizing state feedback controller for the following linear system

$$\dot{x}(t) = \begin{bmatrix} -2.5 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$$

Solution. By definition, we have

$$\begin{aligned} \mathbb{D} &= \{s \mid s \in \mathbb{C},\ L + sM + \bar{s}M^{\mathrm{T}} < 0\} \\ &= \left\{ s \mid s \in \mathbb{C},\ \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 3s \\ s & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & \bar{s} \\ 3\bar{s} & 0 \end{bmatrix} < 0 \right\} \\ &= \left\{ s \mid s \in \mathbb{C},\ \begin{bmatrix} -1 & 2 + \frac{3}{4}s + \frac{1}{4}\bar{s} \\ 2 + \frac{1}{4}s + \frac{3}{4}\bar{s} & -1 \end{bmatrix} < 0 \right\} \\ &= \left\{ x + yj \mid x, y \in \mathbb{R},\ (x+2)^2 + \frac{1}{4}y^2 < 1 \right\}, \end{aligned}$$

which is obviously an ellipsoid (see Figure 7.2). With this system and the above given region, the corresponding LMI

$$L \otimes P + M \otimes AP + M^{\mathrm{T}} \otimes PA^{\mathrm{T}} + M \otimes BW + M^{\mathrm{T}} \otimes W^{\mathrm{T}}B^{\mathrm{T}} < 0,$$

can be obtained as

$$\begin{bmatrix} -P & 2P + \frac{3}{4}AP + \frac{1}{4}PA^{\mathrm{T}} + \frac{3}{4}BW + \frac{1}{4}W^{\mathrm{T}}B^{\mathrm{T}} \\ * & -P \end{bmatrix} < 0.$$

By using the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox, the above LMI is solved, and the parameter matrices are obtained as

$$\begin{aligned} P &= \begin{bmatrix} 103.5474 & -11.1412 \\ -11.1412 & 24.0791 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} 66.7716 & -64.5560 \end{bmatrix}. \end{aligned}$$

Hence a feedback gain is given by

$$K = WP^{-1} = \begin{bmatrix} 0.3750 & -2.5075 \end{bmatrix}.$$

It can be computed using the Matlab function `eig` that the corresponding closed-loop poles are $-1.8162+0.9193i$ and $-1.8162-0.9193i$, which indeed locate in the LMI region $\mathbb{D}$.

Exercise 7.7

Consider a continuous-time linear system (Liu et al. (2000b) and Wilson et al. (1992)) in the form of (7.1) with

$$\begin{aligned} A &= \begin{bmatrix} -0.5010 & -0.9850 & 0.1740 & 0 \\ 16.8300 & -0.5750 & 0.0123 & 0 \\ -3227 & 0.3210 & -2.1000 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.1090 & 0.0070 \\ -132.8000 & 27.1900 \\ -1620 & -1240 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Design a stabilizing state feedback controller for the system.

Solution. By using the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox, the corresponding LMI problem (7.2), that is, problem (s7.2), is solved and a pair of solutions are obtained as

$$\begin{aligned} P &= \begin{bmatrix} 87.9987 & -2.0649 & 1.8473 & -0.8173 \\ -2.06499 & 90.3057 & 0.0011 & -1.4611 \\ 1.8473 & 0.00119 & 90.3083 & -30.0167 \\ -0.8173 & -1.4611 & -30.0167 & 90.3082 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} -28.7077 & 0.3730 & 10.8947 & 0.2846 \\ -191.4952 & 3.3490 & -19.1572 & 1.8786 \end{bmatrix}. \end{aligned}$$

Thus a stabilizing feedback gain is given by

$$K = WP^{-1} = \begin{bmatrix} -0.3289 & -0.0026 & 0.1432 & 0.0477 \\ -2.1731 & -0.0136 & -0.1882 & -0.0616 \end{bmatrix}.$$

It can be verified using the Matlab function `eig` that the closed-loop poles are -0.3669, -0.4914 and $-0.5519 \pm 22.7811i$, which are indeed all stable.

Exercise 7.8

Consider the following linear system, which was introduced in Duan and Patton (2001):

$$\dot{x}(t) = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 2 & 5 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t).$$

Design a state feedback by LMI method such that the closed-system eigenvalues are located in the region

$$\mathbb{H}_{(1,3)} = \{x + yj \mid -3 < x < -1 < 0\}.$$

Solution. By using the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (7.8), that is, problem (s7.4), with

$$\alpha = 1, \ \beta = 3,$$

a pair of parameter matrices are obtained as

$$\begin{aligned} P &= \begin{bmatrix} 26.7584 & -3.3654 & 2.4122 \\ -3.3654 & 9.3318 & -10.0962 \\ 2.4122 & -10.0962 & 33.1909 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} -52.6458 & -3.9370 & -61.6737 \\ -53.2395 & -17.1617 & -208.3329 \end{bmatrix}. \end{aligned}$$

Hence a feedback gain is given by

$$K = WP^{-1} = \begin{bmatrix} -2.2650 & -4.5774 & -3.0859 \\ -2.8242 & -14.0506 & -10.3456 \end{bmatrix}.$$

In fact, it can be verified that the corresponding closed-loop poles are $-1.8308 \pm 5.1681i$ and -1.9489, which are indeed located in the region $\mathbb{H}_{(1,3)}$.

Chapter 8

H_∞/H_2 Control

Exercise 8.1

Consider the linear system in the form of (8.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ D_1 = 1,\ D_2 = 0.05.$$

Design a state feedback control law $u = Kx$, such that the closed-loop system is stable, and the transfer function matrix

$$G_{zw}(s) = (C + D_1 K)\left(sI - (A + B_1 K)\right)^{-1} B_2 + D_2$$

satisfies

$$\|G_{zw}(s)\|_\infty < \gamma$$

for a minimal γ.

Solution. By using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (8.8), that is,

$$\begin{cases} \min & \gamma \\ \text{s.t.} & X > 0 \\ & \begin{bmatrix} (AX + B_1 W)^{\mathrm{T}} + AX + B_1 W & * & * \\ B_2^{\mathrm{T}} & -\gamma I & * \\ CX + D_1 W & D_2 & -\gamma I \end{bmatrix} < 0 \end{cases}, \tag{s8.1}$$

a set of parameters are obtained as

$$\begin{aligned} X &= 10^8 \times \begin{bmatrix} 2.0132 & -0.4123 & -0.4740 \\ -0.4123 & 0.7971 & -0.2953 \\ -0.4740 & -0.2953 & 0.6627 \end{bmatrix} > 0, \\ W &= 10^8 \times \begin{bmatrix} 2.0132 & -0.4123 & -0.4740 \end{bmatrix}, \end{aligned}$$

and

$$\gamma = 0.05.$$

Thus the corresponding state feedback gain is

$$K = WX^{-1} = \begin{bmatrix} -1.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Exercise 8.2

Consider the linear system in the form of (8.10) with

$$A_1 = \begin{bmatrix} -5 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.05 \\ 0 \\ 0.03 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}.$$

Design a state feedback control law $u = Kx$, such that the closed-loop system is stable, and the transfer function matrix

$$G_{zw}(s) = (C + DK)\,(sI - (A + B_1K))^{-1}\,B_2$$

satisfies

$$\|G_{zw}(s)\|_2 < \gamma$$

for a minimal γ .

Solution. By using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (8.16), that is,

$$\begin{cases} \min & \rho \\ \text{s.t.} & AX + B_1W + (AX + B_1W)^{\mathrm{T}} + B_2B_2^{\mathrm{T}} < 0 \\ & \begin{bmatrix} -Z & CX + DW \\ (CX + DW)^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ & \operatorname{trace}(Z) < \rho \end{cases}, \tag{s8.2}$$

a set of parameters are obtained as

$$\gamma = 0.0784,\ \rho = 0.0061,$$

and

$$\begin{aligned} X &= 10^7 \times \begin{bmatrix} 6.7972 & 0.6001 & -1.2617 \\ 0.6001 & 0.0530 & -0.1114 \\ -1.2617 & -0.1114 & 0.2342 \end{bmatrix} > 0, \\ W &= 10^7 \times \begin{bmatrix} 0.0614 & 0.0054 & -0.0114 \\ -2.2862 & -0.2019 & 0.4244 \end{bmatrix}, \\ Z &= \begin{bmatrix} 0.0018 & 0.0021 \\ 0.0021 & 0.0044 \end{bmatrix}, \end{aligned}$$

and the corresponding state feedback gain is then given by

$$K = WX^{-1} = \begin{bmatrix} -0.3606 & -4.8931 & -4.3183 \\ -0.2516 & 0.8285 & 0.8506 \end{bmatrix}.$$

Exercise 8.3

Consider a linear system in the form of (8.18) with the following parameters

$$A = \begin{bmatrix} 1.95 & 0.78 \\ 0.76 & 1.87 \end{bmatrix},\ B_1 = \begin{bmatrix} 1.50 \\ 0.45 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.01 \\ 0.03 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.95 & 0.78 \end{bmatrix},\ D_1 = 0.50,\ D_2 = 0.02,$$

and the following parameter uncertainties:

$$\begin{bmatrix} \Delta A & \Delta B_1 \end{bmatrix} = HF \begin{bmatrix} E_1 & E_2 \end{bmatrix},\ F^{\mathrm{T}} F \le I,$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ E_1 = \begin{bmatrix} 0.04 & 0.02 \\ -0.01 & 0.01 \end{bmatrix},\ E_2 = \begin{bmatrix} 0.03 \\ 0.01 \end{bmatrix}.$$

Design a state feedback control law $u = Kx$, such that the closed-loop system is asymptotically stable, and the transfer function matrix

$$G_{zw}(s) = (C + D_1 K)\,(sI - [(A + \Delta A) + (B_1 + \Delta B_1)\,K])^{-1} B_2 + D_2$$

satisfies

$$\|G_{zw}(s)\|_{\infty} < \gamma$$

for a minimal γ.

Solution. By using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (8.29), that is,

$$\left\{ \begin{array}{ll} \min & \gamma \\ \text{s.t.} & X > 0 \\ & \begin{bmatrix} \Psi(X, W) & * & * & * \\ B_2^{\mathrm{T}} & -\gamma I & * & * \\ CX + D_1 W & D_2 & -\gamma I & * \\ E_1 X + E_2 W & 0 & 0 & -\alpha I \end{bmatrix} < 0 \end{array} \right., \tag{s8.3}$$

a set of parameters are obtained as

$$\gamma = 0.0221,\ \alpha = 0.0020,$$

and

$$\begin{aligned} X &= \begin{bmatrix} 3.0808 & 0.1375 \\ 0.1375 & 0.0062 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} -6.0895 & -0.3311 \end{bmatrix}, \end{aligned}$$

and the corresponding state feedback gain is then given by

$$K = WX^{-1} = \begin{bmatrix} 161.4996 & -3661.5624 \end{bmatrix}.$$

Exercise 8.4

Consider a linear system in the form of (8.18) with the following parameters

$$A = \begin{bmatrix} 1.7465 & -3.2684 & 6.2368 \\ 3.0268 & 9.4581 & 2.3659 \\ 4.3658 & 3.2478 & -2.3724 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.2369 & 1.4533 \\ 0.2361 & -3.4874 \\ 1.3687 & -5.2365 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.0003 \\ 0.0036 \\ 0.1578 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.7635 & 0.3547 & 5.2654 \\ 1.7566 & -0.5348 & 1.6366 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1.0024 & 0.7853 \\ 0.1365 & 1.3959 \end{bmatrix},\ D_2 = 0,$$

and the following parameter uncertainties:

$$\begin{bmatrix} \Delta A & \Delta B_1 \end{bmatrix} = HF \begin{bmatrix} E_1 & E_2 \end{bmatrix},\ F^{\mathrm{T}}F \le I,$$

$$E_1 = \begin{bmatrix} 0.0473 & -0.1401 & 0.4006 \\ 0.1227 & -0.0006 & 0.1456 \\ 0.1440 & -0.0811 & 0.2366 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.5478 & 0.4537 \\ 0.0237 & 0.1014 \\ 0.0124 & 0.3613 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Design a state feedback control law $u = Kx$, such that the closed-loop system is asymptotically stable, and the transfer function matrix

$$G_{zw}(s) = (C + D_1K)\left(sI - [(A + \Delta A) + (B_1 + \Delta B_1)K]\right)^{-1} B_2$$

satisfies

$$\|G_{zw}(s)\|_2 < \gamma$$

for a minimal γ .

Solution. By using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the corresponding LMI problem (8.33), that is,

$$\begin{cases} \min & \rho \\ \text{s.t.} & X > 0 \\ & \begin{bmatrix} \langle AX + B_1W\rangle_s + B_2B_2^{\mathrm{T}} + \beta HH^{\mathrm{T}} & * \\ E_1X + E_2W & -\beta I \end{bmatrix} < 0 \\ & \begin{bmatrix} -Z & * \\ (CX + D_1W)^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ & \operatorname{trace}(Z) < \rho \end{cases}, \qquad \text{(s8.4)}$$

a group of the parameters are obtained as

$$\rho = 0.1013, \ \beta = 0.0037,$$

and

$$\begin{aligned} X &= \begin{bmatrix} 0.0082 & 0.0024 & -0.0059 \\ 0.0024 & 0.0013 & -0.0001 \\ -0.0059 & -0.0001 & 0.0102 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} 0.0086 & -0.0090 & -0.0436 \\ 0.0111 & 0.0057 & -0.0051 \end{bmatrix}, \\ Z &= \begin{bmatrix} 0.0067 & -0.0215 \\ -0.0215 & 0.0946 \end{bmatrix}. \end{aligned}$$

The corresponding state feedback gain is then given by

$$K = \begin{bmatrix} 1.8432 & -10.9663 & -3.3234 \\ -4.3447 & 12.6118 & -2.8786 \end{bmatrix},$$

while the corresponding attenuation level is given by

$$\gamma = \sqrt{\rho} = 0.3183.$$

Exercise 8.5

Consider a linear system in the form of

$$\begin{cases} \dot{x} = Ax + B_1 u + B_2 w \\ y = Cx + D_1 u + D_2 w \end{cases},$$

where w is a measurable disturbance modelled by

$$\dot{w} = Fw + Dv,$$

with v being a driving disturbance. Give the solution to the design of a feedback controller in the form of

$$u = K_1 x + K_2 w,$$

such that $\left\|G_{yv}(s)\right\|_\infty$ and $\left\|G_{yv}(s)\right\|_2$ are minimized.

Solution. Let

$$z = \begin{bmatrix} x^{\mathrm{T}} & w^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

then the system can be rewritten as

$$\begin{cases} \dot{z} = \bar{A}z + \bar{B}_1 u + \bar{B}_2 v \\ y = \bar{C}z + \bar{D}u \end{cases},$$

where

$$\bar{A} = \begin{bmatrix} A & B_2 \\ 0 & F \end{bmatrix}, \ \bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \ \bar{B}_2 = \begin{bmatrix} 0 \\ D \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} C & D_2 \end{bmatrix}, \quad \bar{D} = D_1,$$

and the feedback controller can be rewritten as

$$u = Kz,$$

with

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}.$$

Then the problem of finding a controller in the form of (8.2) such that $||G(s)||_\infty < \gamma$ is satisfied can be sought via the following optimization:

$$\begin{cases} \min & \gamma \\ \text{s.t.} & X > 0 \\ & \begin{bmatrix} \left(\bar{A}X + \bar{B}_1 W\right)^{\mathrm{T}} + \bar{A}X + \bar{B}_1 W & * & * \\ \bar{B}_2^{\mathrm{T}} & -\gamma I & * \\ \bar{C}X + \bar{D}W & 0 & -\gamma I \end{bmatrix} < 0 \end{cases},$$

and the problem of finding a controller in the form of (8.2) such that $||G(s)||_2 < \gamma$ is met can be solved via the following optimization:

$$\begin{cases} \min & \rho \\ \text{s.t.} & \bar{A}X + \bar{B}_1 W + \left(\bar{A}X + \bar{B}_1 W\right)^{\mathrm{T}} + \bar{B}_2 \bar{B}_2^{\mathrm{T}} < 0 \\ & \begin{bmatrix} -Z & \bar{C}X + \bar{D}W \\ \left(\bar{C}X + \bar{D}W\right)^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ & \mathrm{trace}(Z) < \rho \end{cases},$$

where $\rho = \gamma^2$.

Exercise 8.6

Solve the problem in Example 8.1 using Theorem 8.2, and compare the obtained results with those given in the example.

Solution. Using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the optimization problem in Theorem 8.2, that is,

$$\begin{cases} \min & \gamma \\ \text{s.t.} & \begin{bmatrix} -V - V^{\mathrm{T}} & * & * & * & * \\ AV + B_1 W + X & -X & * & * & * \\ CV + D_1 W & 0 & -\gamma I_m & * & * \\ V & 0 & 0 & -X & * \\ 0 & B_2^{\mathrm{T}} & D_2^{\mathrm{T}} & 0 & -\gamma I_r \end{bmatrix} < 0 \end{cases},$$

we obtain the following set of parameters

$$
\begin{aligned}
X &= 10^8 \times \begin{bmatrix} 2.4757 & -1.3067 & 1.5577 \\ -1.3067 & 3.5849 & -0.1790 \\ 1.5577 & -0.1790 & 2.5159 \end{bmatrix} > 0, \\
W &= 10^8 \times \begin{bmatrix} -1.7153 & 0.5647 & -0.9705 \\ 0.6358 & -1.0519 & 0.7900 \end{bmatrix}, \\
V &= 10^8 \times \begin{bmatrix} 1.0392 & -0.1817 & 0.0765 \\ -0.5776 & 0.7770 & 0.3370 \\ 0.8426 & -0.1474 & 0.6190 \end{bmatrix},
\end{aligned}
$$

and

$$\gamma = 0.0500.$$

The state feedback gain is then given by

$$K = WV^{-1} = \begin{bmatrix} 0.0124 & 0.3916 & -1.7825 \\ -2.0124 & -1.3916 & 2.2825 \end{bmatrix}.$$

It is obvious that the above obtained feedback gain is different from the one given in Example 8.1, but the obtained parameter γ is the same as that given in Example 8.1.

Exercise 8.7

Solve the problem in Example 8.2 using Theorem 8.4, and compare the obtained results with those given in the example.

Solution. Applying the Matlab function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the optimization problem in Theorem 8.4, that is,

$$
\begin{cases}
\min & \rho \\
\text{s.t.} & \operatorname{trace}(Z) < \rho \\
& \begin{bmatrix} -Z & B_2^{\mathrm{T}} \\ B_2 & -P \end{bmatrix} < 0 \\
& \begin{bmatrix} -\left(V + V^{\mathrm{T}}\right) & * & * & * \\ AV + B_1 W + P & -P & * & * \\ CV + DW & 0 & -I & * \\ V & 0 & 0 & -P \end{bmatrix} < 0
\end{cases},
$$

we get the following set of parameters

$$
\begin{aligned}
P &= 10^8 \times \begin{bmatrix} 5.8790 & -0.7166 & -0.0671 \\ -0.7166 & 0.6440 & 1.6888 \\ -0.0671 & 1.6888 & 5.0745 \end{bmatrix} > 0, \\
W &= 10^8 \times \begin{bmatrix} -1.3573 & 0.2473 & 0.2625 \\ -0.4400 & -0.7519 & -2.4268 \end{bmatrix}, \\
V &= 10^8 \times \begin{bmatrix} 1.3472 & -0.0553 & 0.3135 \\ -0.0101 & 0.1920 & 0.5760 \\ 0.4500 & 0.5599 & 1.8508 \end{bmatrix}, \\
Z &= \rho = 0.00033532,
\end{aligned}
$$

and the corresponding state feedback gain is obtained as

$$
K = WV^{-1} = \begin{bmatrix} -0.9112 & 1.7521 & -0.2491 \\ -0.1064 & -1.9006 & -0.7017 \end{bmatrix},
$$

the corresponding minimal attenuation level is

$$
\gamma = \sqrt{\rho} = 0.0183.
$$

As assured by the theory, the above obtained attenuation level γ is the same as that obtained in Example 8.2. Yet it can be noted that even the gain matrix obtained above is also very close to that given in Example 8.2.

Chapter 9

State Observation and Filtering

Exercise 9.1

Prove Lemma 9.1.

Solution. Since

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} sI-A \\ C \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} sI-A_{11} & -A_{12} \\ -A_{21} & sI-A_{22} \\ I & 0 \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} 0 & sI-A_{22} \\ 0 & A_{12} \\ I & 0 \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} sI-A_{22} \\ A_{12} \end{bmatrix} + m,
\end{aligned}$$

it is clearly seen that the matrix pair (A_{22}, A_{12}) is detectable if and only if (A, C) is detectable.

Exercise 9.2

Consider the following linear system, which has been studied by Duan (2003):

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -2 & 10 \\ 0 & 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \end{cases}.$$

Design a full-order observer for the system using LMI technique.

Solution. Applying function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the full-order observer design problem (9.4), that is

$$PA + A^{\mathrm{T}}P + WC + C^{\mathrm{T}}W^{\mathrm{T}} < 0, \tag{s9.1}$$

we get the parameters as

$$P = \begin{bmatrix} 1.8774 & 0 & 0 \\ 0 & 0.6917 & 0.2964 \\ 0 & 0.2964 & 1.8774 \end{bmatrix} > 0,$$

$$W = \begin{bmatrix} 0 & 0 \\ 0 & -7.6083 \\ 0 & -0.1482 \end{bmatrix},$$

and the corresponding observer gain is

$$L = P^{-1}W = \begin{bmatrix} 0 & 0 \\ 0 & -11.7621 \\ 0 & 1.7782 \end{bmatrix}.$$

By using the Matlab function `eig`, the set of observer poles is obtained as

$$\lambda(A+LC) = \{-1.1109 \pm 0.9857i, -0.5000\},$$

and the corresponding observer error $e = x - \hat{x}$ is shown in Figure s9.1.

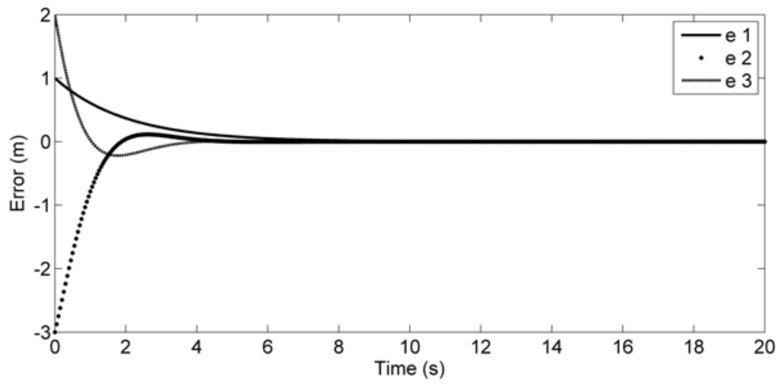

Figure s9.1: The observer error in Exercise 9.2

Exercise 9.3

Design a reduced-order observer for the benchmark system in Example 7.1 using LMI technique.

Solution. Choosing

$$T = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix},$$

we can easily obtain, according to (9.9) and (9.10),

$$A_{11} = 1,\ A_{12} = \begin{bmatrix} 0 & \cdots & 0 & 20 \end{bmatrix},\ B_1 = 0,$$

$$A_{22} = \begin{bmatrix} 20 & & & \\ 20 & 19 & & \\ & \ddots & \ddots & \\ & & 20 & 2 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Applying the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the reduced-order observer design problem (9.13), that is

$$PA_{22} + A_{22}^{\mathrm{T}}P + WA_{12} + A_{12}^{\mathrm{T}}W^{\mathrm{T}} < 0, \tag{s9.2}$$

we get the parameters P and W. Based on the obtained parameter matrices W and P, the observer coefficient matrices are obtained as

$$H = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}},\ M = I_{19},$$

$$F = \begin{bmatrix} \theta_{(20,3)} & \big| & \theta \end{bmatrix},\ \theta_{(20,3)} = \begin{bmatrix} 20 & 0 & & \\ 20 & 19 & & \\ & \ddots & \ddots & \\ & & 20 & 3 \\ & & 0 & 20 \end{bmatrix},$$

$$\theta = \begin{bmatrix} -12699192.6651 \\ -95093503.2849 \\ -345931974.5883 \\ -812993674.5255 \\ -1384469773.5850 \\ -1815943639.8557 \\ -1903377074.9409 \\ -1632061220.0369 \\ -1162150076.4196 \\ -693459455.6043 \\ -348210637.3020 \\ -147113465.5746 \\ -52048176.2083 \\ -15265852.1050 \\ -3648219.1399 \\ -690806.2202 \\ -98755.2805 \\ -9910.3363 \\ -524.3129 \end{bmatrix},\ G = \begin{bmatrix} -322123220.9771 \\ -2404163588.8902 \\ -8714287574.9428 \\ -20398126527.0344 \\ -34581869947.4999 \\ -45132098995.7062 \\ -47035457872.8624 \\ -40066130912.0113 \\ -28311485822.1543 \\ -16739953490.7916 \\ -8313233494.7291 \\ -3464329804.4732 \\ -1204351033.7292 \\ -345102822.4509 \\ -79827335.1893 \\ -14392631.3086 \\ -1893189.4528 \\ -161050.5844 \\ -3913.6121 \end{bmatrix},$$

$$N = \begin{bmatrix} 634959.6333 \\ 4754675.1642 \\ 17296598.7294 \\ 40649683.7263 \\ 69223488.6792 \\ 90797181.9928 \\ 95168853.7470 \\ 81603061.0018 \\ 58107503.8210 \\ 34672972.7802 \\ 17410531.8651 \\ 7355673.2787 \\ 2602408.8104 \\ 763292.6052 \\ 182410.9570 \\ 34540.3110 \\ 4937.7640 \\ 495.5168 \\ 26.3156 \end{bmatrix}, \; L = \begin{bmatrix} -634959.6333 \\ -4754675.1642 \\ -17296598.7294 \\ -40649683.7263 \\ -69223488.6792 \\ -90797181.9928 \\ -95168853.7470 \\ -81603061.0018 \\ -58107503.8210 \\ -34672972.7802 \\ -17410531.8651 \\ -7355673.2787 \\ -2602408.8104 \\ -763292.6052 \\ -182410.9570 \\ -34540.3110 \\ -4937.7640 \\ -495.5168 \\ -26.3156 \end{bmatrix}.$$

By using the Matlab function `eig`, the set of reduced-order observer poles are obtained as

$$\begin{array}{rr} -13.8369 \pm 64.4734i, & -5.3192 \pm 37.5717i, \\ -2.7359 \pm 26.4142i, & -1.7640 \pm 20.3335i, \\ -1.5544 \pm 16.2480i, & -1.7373 \pm 12.2067i, \\ -2.1014 \pm 7.8307i, & -2.8199 \pm 3.2224i, \\ -124.7140 \pm 260.7072i, & -4.1468, \end{array}$$

which are all stable.

Exercise 9.4

Consider a linear system in the form of (9.22) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; D_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find an observer gain matrix L, such that the transfer function of the observation error system (9.25), that is,

$$G_{\tilde{z}w}(s) = C_2 \left(sI - A - LC_1\right)^{-1} \left(B_2 + LD_2\right),$$

satisfies $||G_{\tilde{z}w}(s)||_\infty < 0.5$.

Solution. Applying the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the full-order state H_∞ observer design problem (9.28), that is

$$\begin{bmatrix} A^{\mathrm{T}}P + C_1^{\mathrm{T}}W^{\mathrm{T}} + PA + WC_1 & PB_2 + WD_2 & C_2^{\mathrm{T}} \\ (PB_2 + WD_2)^{\mathrm{T}} & -\gamma I & 0 \\ C_2 & 0 & -\gamma I \end{bmatrix} < 0, \qquad \text{(s9.3)}$$

we get the parameters as

$$\begin{aligned} P &= \begin{bmatrix} 16.1691 & -8.9944 & -2.4561 \\ -8.9944 & 7.0812 & 6.0254 \\ -2.4561 & 6.0254 & 20.5784 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} 2.3472 & 35.1314 \\ -6.1241 & -3.5693 \\ -20.5784 & -6.0573 \end{bmatrix}, \end{aligned}$$

and the corresponding observer gain is

$$L = P^{-1}W = \begin{bmatrix} -0.0876 & 10.5512 \\ -0.1550 & 16.0843 \\ -0.9651 & -3.7445 \end{bmatrix}.$$

Exercise 9.5

Consider the linear system in Exercise 9.4 again. Find a matrix L and a minimal attenuation level γ, such that the transfer function of the observation error system (9.25) satisfies $||G_{\tilde{z}w}(s)||_2 < \gamma$.

Solution. Applying the Matlab function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the full-order H_2 optimal state observer problem (9.40), that is,

$$\begin{cases} \min & \rho \\ \text{s.t.} & \begin{bmatrix} XA + WC_1 + (XA + WC_1)^{\mathrm{T}} & XB_2 + WD_2 \\ (XB_2 + WD_2)^{\mathrm{T}} & -I \end{bmatrix} < 0 \\ & \begin{bmatrix} -Q & C_2 \\ C_2^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ & \operatorname{trace}(Q) < \rho \end{cases}, \qquad \text{(s9.4)}$$

we get the optimal parameters as

$$
\begin{aligned}
X &= 10^8 \times \begin{bmatrix} 4.3152 & -2.6687 & -0.8054 \\ -2.6687 & 2.4869 & 1.6284 \\ -0.8054 & 1.6284 & 4.4242 \end{bmatrix} > 0, \\
Q &= 10^{-7} \times \begin{bmatrix} 0.1860 & -0.0492 \\ -0.0492 & 0.0364 \end{bmatrix} > 0, \\
W &= 10^8 \times \begin{bmatrix} 0.8054 & 3.9893 \\ -1.6284 & 0.5155 \\ -4.4242 & -2.4726 \end{bmatrix},
\end{aligned}
$$

and

$$\gamma = 1.4916 \times 10^{-4}.$$

Thus the corresponding observer gain is

$$L = X^{-1}W = \begin{bmatrix} 0.0000 & 4.4901 \\ -0.0000 & 6.3982 \\ -1.0000 & -2.0965 \end{bmatrix}.$$

Comparing the above obtained results with those in Exercise 9.4, for the same system, we clearly see that the obtained H_2 optimal observer gives a much smaller disturbance attenuation level than that of the H_∞ optimal observer. It is suggested that readers try some specific simulations to see the different disturbance attenuation effect with these two observers.

Exercise 9.6

Consider a linear system in the form of (9.42) with the following parameters:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -5 \end{bmatrix}, \ B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find an H_∞ filter for the system using the LMI technique.

Solution. Applying the Matlab function `mincx` in the LMI Lab in the Matlab

Robust Control Toolbox to the full-order H_∞ filtering problem (9.55), that is,

$$\begin{cases} \min & \gamma \\ \text{s.t.} & X > 0 \\ & R - X > 0 \\ & \begin{bmatrix} RA + A^{\mathrm{T}}R + ZC + C^{\mathrm{T}}Z^{\mathrm{T}} & * & * & * \\ M^{\mathrm{T}} + ZC + XA & M^{\mathrm{T}} + M & * & * \\ B^{\mathrm{T}}R + D^{\mathrm{T}}Z^{\mathrm{T}} & B^{\mathrm{T}}X + D^{\mathrm{T}}Z^{\mathrm{T}} & -\gamma I & * \\ L - D_f C & -N & -D_f D & -\gamma I \end{bmatrix} < 0 \end{cases}, \tag{s9.5}$$

we obtain the set of optimal parameters γ, R, X, M, N, Z and D_f. Based on this set of parameters, the coefficient matrices of the observer can be computed according to

$$A_f = X^{-1}M,\ B_f = X^{-1}Z,\ C_f = N,$$

as

$$\begin{aligned} A_f &= \begin{bmatrix} -4.0000 & 4.8141 & 0.8141 \\ -1.0000 & -2.6079 & -2.6079 \\ -1.0000 & 7.7704 & 0.7704 \end{bmatrix}, \\ B_f &= \begin{bmatrix} -3.0000 & 3.8141 \\ -1.0000 & -1.6079 \\ -2.0000 & 7.7704 \end{bmatrix}, \\ C_f &= \begin{bmatrix} -1.0000 & -0.5873 & 0.4127 \\ 0.0000 & 0.5166 & -0.4834 \end{bmatrix}. \end{aligned}$$

While the parameters D_f and γ are directly given by the solution to the optimization problem, as

$$\begin{aligned} D_f &= \begin{bmatrix} -0.0000 & 0.4127 \\ 0.0000 & 0.5166 \end{bmatrix}, \\ \gamma &= 1.4981 \times 10^{-7}. \end{aligned}$$

Then the optimal H_∞ filter is given by (9.43), that is,

$$\begin{cases} \dot{\varsigma} = A_f \varsigma + B_f y \\ \hat{z} = C_f \varsigma + D_f y \end{cases}. \tag{s9.6}$$

Note that the attenuation level γ is very small, the disturbance in the system is really almost decoupled.

Exercise 9.7

Consider the linear system in Exercise 9.6 again. Find an H_2 filter for the system.

Solution. Applying the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the full-order H_2 filtering problem (9.75), that is,

$$\left\{\begin{array}{ll} \min & \rho \\ \text{s.t.} & X > 0 \\ & R - X > 0 \\ & \operatorname{trace}(Q) < \rho \\ & \begin{bmatrix} -Q & * & * \\ L^{\mathrm{T}} & -R & * \\ -N^{\mathrm{T}} & -X & -X \end{bmatrix} < 0 \\ & \begin{bmatrix} RA + A^{\mathrm{T}}R + ZC + C^{\mathrm{T}}Z^{\mathrm{T}} & * & * \\ M^{\mathrm{T}} + ZC + XA & M^{\mathrm{T}} + M & * \\ B^{\mathrm{T}}R + D^{\mathrm{T}}Z^{\mathrm{T}} & B^{\mathrm{T}}X + D^{\mathrm{T}}Z^{\mathrm{T}} & -I \end{bmatrix} < 0 \end{array}\right. \tag{s9.7}$$

we can get the set of optimal parameters $\rho,\ R,\ X,\ M,\ N,\ Z$ and Q. Then the coefficients of the observer can be computed according to

$$A_f = X^{-1}M,\ B_f = X^{-1}Z,\ C_f = N,$$

as

$$\begin{aligned} A_f &= \begin{bmatrix} -3.9998 & 2.0084 & -1.9889 \\ -1.0000 & -0.7776 & -0.7782 \\ -0.9998 & 2.2665 & -4.7302 \end{bmatrix}, \\ B_f &= \begin{bmatrix} -2.9999 & 1.0099 \\ -1.0000 & 0.2220 \\ -1.9999 & 2.2684 \end{bmatrix}, \\ C_f &= \begin{bmatrix} -1.0000 & -1.0000 & -0.0000 \\ 0.0000 & -0.0000 & -1.0000 \end{bmatrix}. \end{aligned}$$

Then the optimal H_2 filter is given by (9.57), that is,

$$\left\{\begin{array}{l} \dot{\varsigma} = A_f \varsigma + B_f y \\ \hat{z} = C_f \varsigma \end{array}\right. ,$$

and the corresponding attenuation level is given by

$$\gamma = \sqrt{\rho} = 1.4579 \times 10^{-4}.$$

Chapter 10

Multiple Objective Designs

Exercise 10.1

Consider a linear system in the form of (10.3) with the following parameters (Duan (2003)):

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -2 & 10 \\ 0 & 1 & -2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solve the problem of insensitive strip region design with minimum gain for the case of $\gamma_1 = -4$ and $\gamma_2 = -1$.

Solution. According to theory in Section 10.1, this suffices to solve the LMI problem (10.6), that is,

$$\begin{cases} \min & \gamma \\ \text{s.t.} & \begin{bmatrix} -\gamma I & K \\ K^{\mathrm{T}} & -\gamma I \end{bmatrix} < 0 \\ & 2\gamma_1 I < (A+BKC)^{\mathrm{T}} + (A+BKC) < 2\gamma_2 I \end{cases}.$$

Using the Matlab function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox, a solution is found to be

$$\begin{aligned} \gamma &= 3.6440, \\ K &= \begin{bmatrix} -1.7119 & 3.2052 \\ -2.0475 & -1.2838 \end{bmatrix}. \end{aligned}$$

With the help of the Matlab function `eig`, we can find,

$$\begin{aligned} \alpha_1 &= \lambda_{\min}\left(\frac{(A+BKC)^{\mathrm{T}} + (A+BKC)}{2}\right) = -3.9999, \\ \alpha_2 &= \lambda_{\max}\left(\frac{(A+BKC)^{\mathrm{T}} + (A+BKC)}{2}\right) = -1.0508, \end{aligned}$$

and

$$\lambda(A+BKC) = \{-2.6399,\ -1.9279 \pm 2.1485i\}.$$

We thus indeed have

$$\lambda(A+BKC) \subset \mathbb{H}_{(-\alpha_2,-\alpha_1)} \subset \mathbb{H}_{(-\gamma_2,-\gamma_1)}.$$

Exercise 10.2

Consider a discrete-time linear system in the form of (10.7) with the following parameters:

$$A = \begin{bmatrix} 0.9696 & 0.0202 \\ 0.0404 & 0.9898 \end{bmatrix},\ B = \begin{bmatrix} 50500 \\ 50500 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

This is a model describing the population distribution in a certain country (Zheng (2002), page 25-27). The system variables are

- $x_1(k)$, city population in the kth year;
- $x_2(k)$, rural population in the kth year;
- $y(k)$, total population in the kth year; and
- $u(k)$, control policy taken in the kth year.

Note that

$$\lambda(A) = \{0.9494,\ 1.0100\},$$

the system is not stable, and the purpose of control is to make the system robustly stable with minimal control effort. In doing so, we suffice to solve the problem of insensitive disk region design with minimum gain for the case of $q = 0$ and $\gamma_0 = 1$.

Solution. According to theory in Section 10.1, this suffices to solve the LMI problem (10.10), that is,

$$\left\{ \begin{array}{ll} \min & \gamma \\ \text{s.t.} & \begin{bmatrix} -\gamma I & K \\ K^{\mathrm{T}} & -\gamma I \end{bmatrix} < 0 \\ & \begin{bmatrix} -\gamma_0 I & A+BKC+qI \\ (A+BKC+qI)^{\mathrm{T}} & -\gamma_0 I \end{bmatrix} < 0 \end{array} \right. .$$

Using the Matlab function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox, a solution is found to be

$$\begin{array}{rcl} \gamma & = & 0.0121, \\ K & = & -1.1958 \times 10^{-7}. \end{array}$$

With the help of the Matlab function `eig`, we can find,

$$\lambda(A+BKC) = \{0.9494,\ 0.9979\}.$$

Further note that

$$\eta = \|A+BKC\| = 1.0000.$$

We thus indeed have

$$\lambda(A+BKC) \subset \mathbb{D}_{(0,\eta)} = \mathbb{D}_{(0,\gamma_0)}.$$

Exercise 10.3

Consider the linear system in the form of (10.11) with

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & -1 & -3 \\ 4 & 0 & -1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 3 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0.2 \\ -0.5 \end{bmatrix},$$

$$C_\infty = \begin{bmatrix} 2 & 1 & -0.5 \end{bmatrix},\ D_{\infty 1} = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix},\ D_{\infty 2} = 0.05,$$

$$C_2 = \begin{bmatrix} 0.1 & 0 & -0.1 \\ 0 & 0.2 & 0.3 \end{bmatrix},\ D_{21} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}.$$

Design a state feedback control law, such that

1. the closed-loop poles locate in the LMI region

$$\mathbb{D} = \left\{ x+yj \,\middle|\, x, y \in R,\ (x+2)^2 + \frac{1}{4}y^2 < 1 \right\};$$

2. the following objective is minimized:

$$J = \left\| G_{z_\infty w}(s) \right\|_\infty + \left\| G_{z_2 w}(s) \right\|_2^2,$$

where

$$\begin{aligned} G_{z_\infty w}(s) &= (C_\infty + D_{\infty 1}K)(sI - (A+B_1K))^{-1} B_2 + D_{\infty 2}, \\ G_{z_2 w}(s) &= (C_2 + D_{21}K)(sI - (A+B_1K))^{-1} B_2. \end{aligned}$$

Solution. This corresponds to solving the optimization problem (10.27), that is,

$$\begin{cases} \min & c_2\gamma_2^2 + c_\infty \gamma_\infty \\ \text{s.t.} & \operatorname{trace}(Z) < \gamma_2^2 \\ & \begin{bmatrix} -Z & * \\ (C_2X + D_{21}W)^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ & AX + B_1W + (AX+B_1W)^{\mathrm{T}} + B_2B_2^{\mathrm{T}} < 0 \\ & \begin{bmatrix} (AX+B_1W)^{\mathrm{T}} + AX + B_1W & * & * \\ B_2^{\mathrm{T}} & -\gamma_\infty I & * \\ C_\infty X + D_{\infty 1}W & D_{\infty 2} & -\gamma_\infty I \end{bmatrix} < 0 \\ & L \otimes X + M \otimes (AX+B_1W) + M^{\mathrm{T}} \otimes (AX+B_1W)^{\mathrm{T}} < 0 \end{cases}, \tag{s10.1}$$

with the weighting factors being chosen as

$$c_2 = c_\infty = 1.$$

Using the Matlab function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to (s10.1), we obtain

$$\begin{aligned} X &= \begin{bmatrix} 14.6607 & -22.7987 & 8.7135 \\ -22.7987 & 37.3730 & -14.6590 \\ 8.7135 & -14.6590 & 5.8724 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} -26.6020 & 45.8680 & -18.5122 \\ 12.8216 & -26.5203 & 11.5575 \end{bmatrix}, \\ Z &= \begin{bmatrix} 0.0472 & 0.0651 \\ 0.0651 & 0.2386 \end{bmatrix}, \end{aligned}$$

and $J = 0.6792$, with

$$\gamma_\infty = 0.3930, \ \gamma_2^2 = 0.2862.$$

The corresponding state feedback gain is given as

$$K = WX^{-1} = \begin{bmatrix} 1.2470 & 1.2340 & -1.9224 \\ -3.6859 & -1.9614 & 2.5411 \end{bmatrix}.$$

Exercise 10.4

Consider a linear system in the form of (10.28) with the following parameters

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ -1 & 5 \end{bmatrix},$$

$$C_\infty = \begin{bmatrix} 0.2 & 1 & -1 \end{bmatrix}, \ D_{\infty 1} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \ D_{\infty 2} = \begin{bmatrix} 0.01 & -0.02 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.1 & 0 & 0.2 \end{bmatrix}, \ D_{21} = \begin{bmatrix} -0.1 & 0 \end{bmatrix},$$

and the following parameter uncertainties

$$\begin{bmatrix} \Delta A & \Delta B_1 \end{bmatrix} = HF \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \ F^{\mathrm{T}} F \le I,$$

$$H = \begin{bmatrix} 0 & 0 & -0.2 \\ 0 & -0.1 & 0.1 \\ 0.2 & 0 & 0.1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \\ 0.2 & 0.1 & 0.1 \end{bmatrix}, \ E_2 = \begin{bmatrix} -0.1 & 0.5 \\ 0.2 & 0.1 \\ -0.2 & 0 \end{bmatrix}.$$

Design a state feedback control law, such that

1. the closed-loop poles locate in the LMI region

$$\mathbb{D} = \left\{ x + yj \,\middle|\, x, y \in R,\ (x+3)^2 + 4y^2 < 4 \right\};$$

2. the following objective is minimized:

$$J = 5 \left\| \tilde{G}_{z_\infty w}(s) \right\|_\infty + \left\| \tilde{G}_{z_2 w}(s) \right\|_2^2,$$

where

$$\begin{aligned} \tilde{G}_{z_\infty w}(s) &= (C_\infty + D_{\infty 1} K)(sI - A_c)^{-1} B_2 + D_{\infty 2}, \\ \tilde{G}_{z_2 w}(s) &= (C_2 + D_{21} K)(sI - A_c)^{-1} B_2, \end{aligned}$$

with

$$A_c = (A + \Delta A) + (B_1 + \Delta B_1) K.$$

Solution. This corresponds to solving the optimization problem (10.42), that is,

$$\begin{cases} \min & c_2 \gamma_2^2 + c_\infty \gamma_\infty \\ \text{s.t.} & \operatorname{trace}(Z) < \gamma_2^2 \\ & \begin{bmatrix} -Z & * \\ (C_2 X + D_{21} W)^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ & \begin{bmatrix} \langle AX + B_1 W \rangle_s + B_2 B_2^{\mathrm{T}} + \beta H H^{\mathrm{T}} & * \\ E_1 X + E_2 W & -\beta I \end{bmatrix} < 0 \\ & \begin{bmatrix} \Psi(X, W) & * & * & * \\ B_2^{\mathrm{T}} & -\gamma_\infty I & * & * \\ C_\infty X + D_{\infty 1} W & D_{\infty 2} & -\gamma_\infty I & * \\ E_1 X + E_2 W & 0 & 0 & -\alpha I \end{bmatrix} < 0 \\ & L \otimes X + M \otimes (AX + B_1 W) + M^{\mathrm{T}} \otimes (AX + B_1 W)^{\mathrm{T}} < 0 \end{cases}, \tag{s10.2}$$

were

$$\Psi(X, W) = \langle AX + B_1 W \rangle_s + \alpha H H^{\mathrm{T}},$$

and the weighting factors are taken as

$$c_\infty = 5,\ c_2 = 1.$$

Using the Matlab function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to (s10.2), we obtain

$$\begin{aligned} X &= \begin{bmatrix} 188.0126 & 432.4015 & 330.7401 \\ 432.4015 & 1655.9543 & 1305.2245 \\ 330.7401 & 1305.2245 & 1042.5049 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} 775.5700 & 2783.7210 & 2206.4627 \\ -1423.4684 & -4435.0286 & -3406.1026 \end{bmatrix}, \end{aligned}$$

$$Z = 0.4405,\ \alpha = 368.3308,\ \beta = 1176.0241,$$

and $J = 3.7948$, with

$$\gamma_{\infty} = 0.6705,\ \gamma_2^2 = 0.4423.$$

The corresponding state feedback gain is then given by

$$K = WX^{-1} = \begin{bmatrix} 0.8529 & 0.2566 & 1.5247 \\ -2.9483 & -5.3466 & 4.3621 \end{bmatrix}.$$

Chapter 11

Missile Attitude Control

Exercise 11.1

Find out the open-loop stability of the non-rotating missile attitude system (11.9) with matrix A given in (11.16).

Solution. By the Lyapunov stability theory, this requires to find out if there exists a symmetric positive matrix P satisfying the following Lyapunov inequality:

$$A^{\mathrm{T}}P + PA < 0.$$

Using the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the above feasibility problem, we obtain a solution

$$P = \begin{bmatrix} 2.3840 & -0.0134 & 0.0014 & 0.0027 & -0.1635 \\ -0.0134 & 0.0386 & 0.0001 & 0.0002 & 0.0483 \\ 0.0014 & 0.0001 & 0.0124 & 0.0071 & 0.0013 \\ 0.0027 & 0.0002 & 0.0071 & 0.0219 & 0.0015 \\ -0.1635 & 0.0483 & 0.0013 & 0.0015 & 0.1021 \end{bmatrix} > 0.$$

Therefore, the open-loop system is already stable.

Remark. Using the Matlab function `eig`, we can compute the set of eigenvalues of the system as

$$\lambda(A) = \{-50,\ -40,\ -20,\ -0.3300 \pm 7.8722i\}.$$

From this we can also see that the system is stable.

Exercise 11.2

For the following linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, \tag{s11.1}$$

a controller is designed in the form of (11.26), with K enabling the asymptotical stability of the system

$$\dot{x} = (A + BK)\,x$$

and G given by (11.27) and (11.28). Show that the asymptotical tracking relation (11.29) holds.

Solution. Let Z and H be given by (11.28), then we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z \\ H \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

which gives

$$\begin{aligned} AZ + BH &= 0, \quad &\text{(s11.2)} \\ CZ + DH &= I. \quad &\text{(s11.3)} \end{aligned}$$

Define

$$\begin{aligned} \delta x &= x - Zy_r, \\ \delta u &= u - Hy_r, \\ \delta y &= y - y_r, \end{aligned}$$

and use (s11.2) and (s11.3), we can derive

$$\begin{aligned} \delta\dot{x} &= \dot{x} \\ &= A\,(x - Zy_r) + B\,(u - Hy_r) + (AZ + BH)\,y_r \\ &= A\delta x + B\delta u, \end{aligned}$$

and

$$\begin{aligned} \delta y &= y - y_r \\ &= C\,(x - Zy_r) + D\,(u - Hy_r) + (CZ + DH - I)\,y_r \\ &= C\delta x + D\delta u, \end{aligned}$$

which give the following linear system with the same coefficient matrices as system (s11.1):

$$\begin{cases} \delta\dot{x} = A\delta x + B\delta u \\ \delta y = C\delta x + D\delta u \end{cases}.$$

Therefore, when K is chosen to stabilize system (s11.1), it also stabilizes the above system. Thus the following closed-loop system

$$\begin{cases} \delta\dot{x} = (A + BK)\,\delta x \\ \delta y = (C + DK)\,\delta x \end{cases},$$

is asymptotically stable, and as a consequence, we have

$$\lim_{t \longrightarrow \infty} \delta y(t) = \lim_{t \longrightarrow \infty} (y(t) - y_r) = 0.$$

This is the asymptotical tracking relation to be shown.

Exercise 11.3

Consider the pitch channel attitude model (11.3)-(11.8) of the type of non-rotating missiles with

$$A = \begin{bmatrix} -1.2950 & 1 & -0.2430 \\ 0.1920 & -1.6770 & -0.0467 \\ 0 & 0 & 20 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.1745 & 0 \\ 0.0335 & -0.1309 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ -3.2253 & 0 & -0.6052 \end{bmatrix},$$

$$D_1 = 0,\ D_2 = \begin{bmatrix} 0 & 0 \\ -0.0025 & 0 \end{bmatrix}.$$

Find a state feedback control law $u = Kx$ such that the closed-loop system is stable, and the transfer function matrix $G_{yd}(s)$ satisfies

$$||G_{yd}(s)||_\infty < \gamma$$

for a minimum positive scalar γ.

Solution. By using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the optimization problem (11.14), that is

$$\left\{ \begin{array}{ll} \min & \gamma \\ \text{s.t.} & X > 0 \\ & \begin{bmatrix} (AX + B_1W)^{\mathrm{T}} + AX + B_1W & * & * \\ B_2^{\mathrm{T}} & -\gamma I & * \\ CX + D_1W & D_2 & -\gamma I \end{bmatrix} < 0 \end{array} \right. , \tag{s11.4}$$

a set of optimal parameters are obtained as

$$\gamma = 0.3853,$$

and

$$\begin{aligned} X &= \begin{bmatrix} 0.9695 & 1.8506 & -4.9204 \\ -1.8506 & 3.5325 & 9.9322 \\ -4.9204 & 9.9322 & 12284.8217 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} 153.8089 & -304.3780 & -148103104.3840 \end{bmatrix}. \end{aligned}$$

Thus the state feedback gain is given by

$$K = WX^{-1} = \begin{bmatrix} 320635991.5092 & 168030644.9787 & -19484.5720 \end{bmatrix}.$$

Note that the attenuation level γ is very small, the disturbance d is effectively attenuated. However, with this optimal solution the magnitude of the feedback gain matrix K is obviously large ($\|K\|_2 = 361996874.4101$), which is not desirable in applications. Therefore, in the following we will seek a feasible solution instead.

Limiting the attenuation level by $\gamma < 0.5$, and applying the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the feasibility problem (11.15), we obtain the following optimal parameters

$$\gamma = 0.4623,$$

and

$$\begin{aligned} X &= \begin{bmatrix} 0.8697 & -1.7153 & -3.7054 \\ -1.7153 & 3.5386 & 8.0159 \\ -3.7054 & 8.0159 & 25.9487 \end{bmatrix} > 0, \\ W &= \begin{bmatrix} 70.8908 & -155.7606 & -593.9321 \end{bmatrix}. \end{aligned}$$

Thus the state feedback gain is given by

$$K = WX^{-1} = \begin{bmatrix} 28.3779 & 41.3309 & -31.6041 \end{bmatrix}.$$

Obviously, this gain matrix possesses a reasonable magnitude and meanwhile also gives an acceptable attenuation level γ.

Exercise 11.4

Let $A, A_0 \in \mathbb{R}^{n\times n}$. Show that

$$A(\omega) = A + A_0\omega, \ \omega \in [\alpha, \beta],$$

is quadratically stable if $\{A + A_0\alpha, \ A + A_0\beta\}$ is quadratically stable. (Note: This is related with Subsection 11.3.2, in which the quadratic stabilization of system (11.18) with $\omega_x \in [-400, 400]$ is treated.)

Solution I. Let $\{A + A_0\alpha, \ A + A_0\beta\}$ be quadratically stable, then there exists a positive definite matrix P satisfying

$$(A + A_0\alpha)^{\mathrm{T}} P + P(A + A_0\alpha) < 0,$$

$$(A + A_0\beta)^{\mathrm{T}} P + P(A + A_0\beta) < 0.$$

Therefore, for $0 \le \theta \le 1$, we have

$$(A\theta + A_0\theta\alpha)^{\mathrm{T}} P + P(A\theta + A_0\theta\alpha) < 0,$$

$$(A(1-\theta) + A_0(1-\theta)\beta)^{\mathrm{T}} P + P(A(1-\theta) + A_0(1-\theta)\beta) < 0.$$

Adding the above two inequalities side by side, and denoting

$$\omega = \theta\alpha + (1-\theta)\beta,$$

Table s11.1: Parameters at characteristic points

Times (s)	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$c_1(t)$	1.264	1.600	1.636	1.635	1.607	0.936	0.644
$c_3(t)$	1787.048	1832.067	2128.877	2231.985	3045.292	1329.481	818.706

gives

$$(A + A_0\omega)^{\mathrm{T}} P + P(A + A_0\omega) < 0, \quad \omega \in [\alpha, \beta].$$

This states that $A + A_0\omega$, $\omega \in [\alpha, \beta]$, is quadratically stable.

Solution II. It is obvious that $A + A_0\omega$, $\omega \in [\alpha, \beta]$, is a family of systems with one varying parameter. As a matter of fact, this is an interval system. Therefore, it is quadratically stable if and only if it is stable at the two extreme points α and β. Thus the conclusion is true.

Exercise 11.5

The closed-loop poles at each operating point with different values of ω_x are listed in Table 11.3, and they can be observed to be all stable. Is this fact sufficient for the stability of the designed closed-loop missile attitude system? Why?

Solution. This fact is not sufficient for the stability of the designed closed-loop missile attitude system.

The stability of a time-varying system, even if linear, is very complicated. The stability of a linear time varying system is no longer completely determined by the eigenvalues of the system, it also has relation with the eigenvectors. There are such linear time varying systems whose eigenvalues are constants and all have negative real parts, yet the system is not stable. Therefore, the stability of the designed closed-loop missile attitude system can not be assured by the fact that the closed-loop eigenvalues along ω_x are all stable.

It should be noted that the stability of the designed attitude system is really ensured by the quadratic stability theory.

Exercise 11.6

Consider the rolling channel model (11.11) of the type of BTT missiles. The seven operating points along the whole trajectory are the same as in Table 11.1, while the parameters of the system at these operating points are given in Table s11.1. Following the lines in Section 11.3, complete the attitude control system design for this rolling channel using the quadratic stabilization approach.

Solution. As shown in Section 11.1 that, when the state vector and the input variable are taken, respectively, as

$$x = \begin{bmatrix} \omega_x \\ \phi \end{bmatrix}, \quad u = \delta_x,$$

the state-space model for the rolling channel attitude system is

$$\dot{x} = A(t)x + B(t)u,$$

where the system matrix and the control distribution matrix are

$$A(t) = \begin{bmatrix} -c_1(t) & 0 \\ 1 & 0 \end{bmatrix}, \; B(t) = \begin{bmatrix} -c_3(t) \\ 0 \end{bmatrix}.$$

With the values given in Table s11.1, we can obtain 7 constant linear systems:

$$\dot{x} = A_i x + B_i u, \quad i = 1, 2, ..., 7,$$

with the following coefficients:

$$A_1 = A(t_1) = \begin{bmatrix} -1.264 & 0 \\ 1 & 0 \end{bmatrix}, \; B_1 = B(t_1) = \begin{bmatrix} -1787.048 \\ 0 \end{bmatrix},$$

$$A_2 = A(t_2) = \begin{bmatrix} -1.600 & 0 \\ 1 & 0 \end{bmatrix}, \; B_2 = B(t_2) = \begin{bmatrix} -1832.067 \\ 0 \end{bmatrix},$$

$$A_3 = A(t_3) = \begin{bmatrix} -1.636 & 0 \\ 1 & 0 \end{bmatrix}, \; B_3 = B(t_3) = \begin{bmatrix} -2128.877 \\ 0 \end{bmatrix},$$

$$A_4 = A(t_4) = \begin{bmatrix} -1.635 & 0 \\ 1 & 0 \end{bmatrix}, \; B_4 = B(t_4) = \begin{bmatrix} -2231.985 \\ 0 \end{bmatrix},$$

$$A_5 = A(t_5) = \begin{bmatrix} -1.607 & 0 \\ 1 & 0 \end{bmatrix}, \; B_5 = B(t_5) = \begin{bmatrix} -3045.292 \\ 0 \end{bmatrix},$$

$$A_6 = A(t_6) = \begin{bmatrix} -0.936 & 0 \\ 1 & 0 \end{bmatrix}, \; B_6 = B(t_6) = \begin{bmatrix} -1329.481 \\ 0 \end{bmatrix},$$

$$A_7 = A(t_7) = \begin{bmatrix} -0.644 & 0 \\ 1 & 0 \end{bmatrix}, \; B_7 = B(t_7) = \begin{bmatrix} -818.706 \\ 0 \end{bmatrix}.$$

The design purpose is to solve a quadratic stabilization problem, that is, find a constant state feedback controller

$$u = Kx$$

stabilizing simultaneously all the above 7 systems. To do this, according to theory of quadratic stabilization, we suffice to solve the following set of LMIs:

$$\begin{cases} A_i P + P A_i^{\mathrm{T}} + B_i W + W^{\mathrm{T}} B_i^{\mathrm{T}} < 0, \; i = 1, 2, \cdots, 7 \\ P > 0 \end{cases}.$$

By applying the Matlab function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the above set of LMIs, the two parameter matrices are obtain as

$$P = \begin{bmatrix} 0.4065 & -0.5538 \\ -0.5538 & 1.2368 \end{bmatrix} > 0$$

and

$$W = \left[\begin{array}{cc} 1.5406 \times 10^{-5} & 5.7139 \times 10^{-4} \end{array} \right].$$

Therefore, the feedback control gain matrix is given by

$$K = WP^{-1} = \left[\begin{array}{cc} 0.0017 & 0.0012 \end{array} \right].$$

Remark. With the above designed controller, the closed-loop system is stable for all parameters (c_1, c_3) lying in the convex hull of the seven points $(c_1(t_i), c_3(t_i))$, $i = 1, 2, ..., 7$, that is,

$$\Omega = \text{conv}\left\{(c_1(t_i), c_3(t_i)),\ i = 1, 2, ..., 7\right\}.$$

This set of parameters is shown in Figure s11.1.

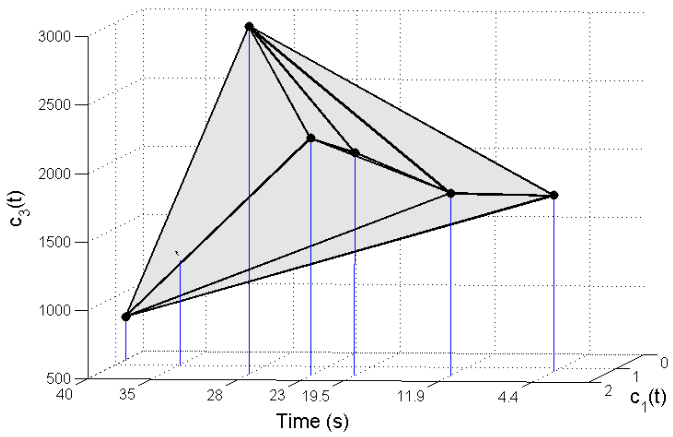

Figure s11.1: Admissible set of parameters

Chapter 12

Satellite Control

Exercise 12.1

For the satellite system (12.6), show that the gravity torques are given by the formulas in (12.7).

Solution. As is seen in Figure s12.1, O is the mass center of the satellite, $Oxyz$ is the spacecraft coordinate system, R is the radius of the orbit. Taking an element of mass dm, and denoting the distances from dm to the mass center O and the center of the earth by ρ and r, respectively, then we have the gravity of this element mass as

$$\mathrm{d}W = -G\frac{M_e \mathrm{d}m}{r^3}\vec{r} = -\frac{\mu \mathrm{d}m}{r^3}\vec{r},$$

where G is the coefficient of the universal gravitation, M_e is the mass of the earth, $\mu = GM_e$ is the gravitational constant. Let B be the volume of the satellite, then the gravity torque of the satellite is obviously given by

$$\begin{aligned} M_g &= \int_B \vec{\rho} \times \mathrm{d}W \\ &= -\mu \int_B \vec{\rho} \times \frac{\vec{r}}{r^3}\mathrm{d}m \\ &= -\mu \int_B \vec{\rho} \times \frac{\vec{R}+\vec{\rho}}{r^3}\mathrm{d}m \\ &\approx -\mu \int_B \frac{\vec{\rho} \times \vec{R}}{r^3}\mathrm{d}m. \end{aligned} \tag{s12.1}$$

Further, note that

$$r^2 = R^2 + \rho^2 + 2\vec{R}\cdot\vec{\rho},$$

we can get

$$r^{-3} \approx R^{-3}\left(1 - \frac{3\vec{R}\cdot\vec{\rho}}{R^2}\right) = R^{-3} - \frac{3\vec{R}\cdot\vec{\rho}}{R^5}.$$

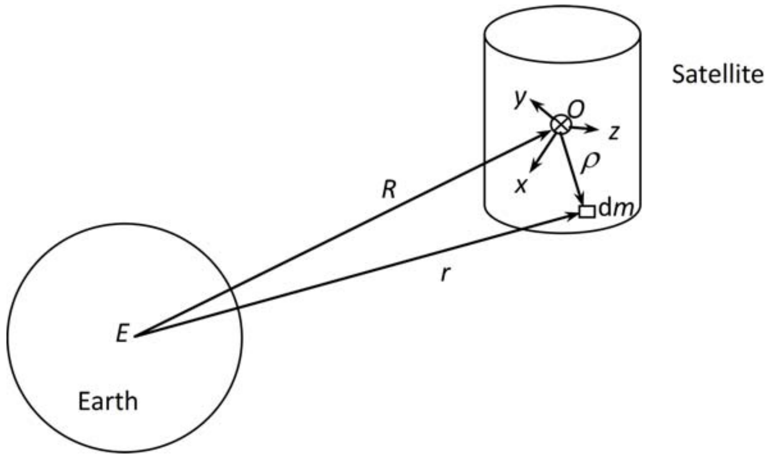

Figure s12.1: Gravity torques on a satellite

Substituting the above into (s12.1), gives

$$M_g = -\frac{\mu}{R^3}\int_B \vec{\rho}\mathrm{d}m \times \vec{R} + \frac{3\mu}{R^5}\int_B \vec{R}\cdot\vec{\rho}\vec{\rho}\times\vec{R}\mathrm{d}m.$$

Since O is the mass center of B, the first term in the right hand side of the above formula is zero. Further, in view of

$$\vec{\rho}\cdot\left(\vec{\rho}\times\vec{R}\right) = \vec{R}\cdot(\vec{\rho}\times\vec{\rho}) = 0,$$

and the general relation

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\left(\vec{a}\cdot\vec{b}\right),$$

we have

$$\begin{aligned}
M_g &= \frac{3\mu}{R^5}\int_B \vec{R}\cdot\vec{\rho}\vec{\rho}\times\vec{R}\mathrm{d}m \\
&= \frac{3\mu}{R^5}\int_B \left[\left(\vec{\rho}\times\vec{R}\right)\left(\vec{\rho}\cdot\vec{R}\right) - \vec{R}\left(\vec{\rho}\cdot\left(\vec{\rho}\times\vec{R}\right)\right)\right]\mathrm{d}m \\
&= \frac{3\mu}{R^5}\int_B \vec{\rho}\times\left[\left(\vec{\rho}\times\vec{R}\right)\times\vec{R}\right]\mathrm{d}m \\
&= \frac{3\mu}{R^5}\int_B \vec{\rho}\times\left[\vec{R}\times\left(\vec{R}\times\vec{\rho}\right)\right]\mathrm{d}m \\
&= \frac{3\mu}{R^5}\vec{R}\times\int_B \vec{\rho}\times\left(\vec{R}\times\vec{\rho}\right)\mathrm{d}m.
\end{aligned}$$

Further, it can be shown that

$$\int_B \vec{\rho}\times\left(\vec{R}\times\vec{\rho}\right)\mathrm{d}m = \mathbb{J}\cdot\vec{R},$$

where

$$\mathbb{J} = \int_B (\vec{\rho} \cdot \vec{\rho} I_3 - \vec{\rho}\vec{\rho})\, \mathrm{d}m.$$

We thus have

$$M_g = \frac{3\mu}{R^5} \vec{R} \times \mathbb{J} \cdot \vec{R}, \tag{s12.2}$$

Denote

$$\vec{R} = R \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^{\mathrm{T}},$$

with $\alpha_i,\ i = 1, 2, 3$, being the directional cosines of $\vec{R}$, then the above (s12.2) can be written in the following matrix form:

$$M_g = \frac{3\mu}{R^3} \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} J \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. \tag{s12.3}$$

Assuming that the coordinate system coincide with the axes of the satellite, we have

$$J = \mathrm{diag}\left(J_x, J_y, J_z\right),$$

substituting this into (s12.3), yields the following general expression for the gravity torque:

$$M_g = \frac{3\mu}{R^3} \begin{bmatrix} \left(J_z - J_y\right) \alpha_2 \alpha_3 \\ \left(J_x - J_z\right) \alpha_1 \alpha_3 \\ \left(J_y - J_x\right) \alpha_1 \alpha_2 \end{bmatrix}.$$

When the turning of the satellite obey the following order

$$z\,(\psi) \longrightarrow y\,(\vartheta) \longrightarrow x\,(\varphi),$$

we have

$$\vec{R} = R \begin{bmatrix} \sin\vartheta & -\sin\varphi\cos\vartheta & -\cos\varphi\cos\vartheta \end{bmatrix}^{\mathrm{T}}.$$

Therefore, there hold

$$\alpha_1 = \sin\vartheta,\ \alpha_2 = -\sin\varphi\cos\vartheta,\ \alpha_3 = -\cos\varphi\cos\vartheta,$$

in this case the gravity torque is further given by

$$\begin{aligned} M_g &= \frac{3\mu}{R^3} \begin{bmatrix} \left(J_z - J_y\right) \sin 2\varphi \cos^2\vartheta \\ \left(J_z - J_x\right) \cos\varphi \sin 2\vartheta \\ \left(J_x - J_y\right) \sin\varphi \sin 2\vartheta \end{bmatrix} \\ &\approx \frac{3\mu}{R^3} \begin{bmatrix} \left(J_z - J_y\right) \varphi \\ \left(J_z - J_x\right) \vartheta \\ 0 \end{bmatrix}. \end{aligned}$$

This is the relation to be shown.

Exercise 12.2

With the parameters given in (12.17),

1. check the stability of the open-loop system (12.10)-(12.15);
2. find out the value of $\left\| G^o_{z_\infty d} \right\|_\infty$ and $\left\| G^o_{z_2 d} \right\|_2$, with

$$G^o_{z_\infty d} = C_1 (sI - A)^{-1} B_2 + D_2,$$

and

$$G^o_{z_2 d} = C_2 (sI - A)^{-1} B_2,$$

and compare the obtained values with those corresponding to the closed-loop system designed in Section 12.2.

Solution. By the Lyapunov stability theory, this requires to find out if there exists a symmetric positive matrix P satisfying the following Lyapunov inequality:

$$A^{\mathrm{T}} P + PA < 0.$$

Unfortunately, using the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox, we find the problem infeasible. Therefore, this open-loop system is not stable. As a matter of fact, using the Matlab function `eig`, we obtain

$$\lambda(A) = \{\pm 0.0000142, \pm 0.0000724i, \pm 0.0001029\},$$

which clearly contains unstable open-loop poles.

With the system (A, B_2, C_1, D_2), we find that the H_∞ optimization problem (5.30), that is, problem (s5.3), is infeasible. While with the system (A, B_2, C_2), we find that the H_2 optimization problem (5.46) is also infeasible. This means that these values do not exist, or in other words, are infinities.

With the open-loop system (12.10)-(12.15), both the H_∞ and H_2 indices are infinities. However, as is seen in Section 12.2 that, after proper designs, the H_∞ index can be reduced to 0.001, while the H_2 index can be reduced to 0.003.

Remark. Using the Matlab function `norm`, we can directly obtain

$$\left\| G^o_{z_\infty d} \right\|_\infty = 2648604001826.497, \quad \left\| G^o_{z_2 d} \right\|_2 = \infty.$$

These are consistent with the above conclusions obtained. In fact, these results should have been anticipated because the open-loop system is not stable!

Exercise 12.3

It is seen from Subsection 12.2.1 that the obtained optimal solution does not give satisfactory response performance, while a feasible solution to the altered LMI in (12.18) does. Find another solution for the problem by reasonably altering the original LMI condition.

Solution. Let us replace the first inequality in the optimization problem (12.16) with the following one:

$$0 < X < 10I.$$

Using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the altered optimization problem, that is,

$$\begin{cases} \min & \gamma_\infty \\ \text{s.t.} & X > 0 \\ & X < 10I \\ & \begin{bmatrix} (AX+B_1W)^{\mathrm{T}}+AX+B_1W & * & * \\ B_2^{\mathrm{T}} & -\gamma_\infty I & * \\ C_1X+D_1W & D_2 & -\gamma_\infty I \end{bmatrix} < 0 \end{cases}, \tag{s12.4}$$

we obtain $\gamma_\infty = 0.0010$ and

$$X = \begin{bmatrix} 6.6603 & 0 & 0.0000 & -0.2184 & 0 & -0.0000 \\ 0 & 6.6671 & 0 & 0 & 0.1070 & 0 \\ 0.0000 & 0 & 6.6672 & -0.0000 & 0 & -0.1067 \\ -0.2184 & 0 & -0.0000 & 0.0143 & 0 & 0.0000 \\ 0 & -0.1070 & 0 & 0 & 0.0034 & 0 \\ -0.0000 & 0 & -0.1067 & 0.0000 & 0 & 0.0034 \end{bmatrix},$$

$$W = \begin{bmatrix} 0.0000 & 0 & 0.0081 & -0.9711 & 0 & -0.0003 \\ 0 & -0.0002 & 0 & 0 & -0.3317 & 0 \\ -0.0166 & 0 & 0.0001 & 0.0011 & 0 & -0.3301 \end{bmatrix}.$$

The corresponding state feedback gain is given by

$$\begin{aligned} K &= WX^{-1} \\ &= \begin{bmatrix} -4.4357 & 0 & -0.0000 & -135.2838 & 0 & -0.0762 \\ 0 & -3.1024 & 0 & 0 & -193.2187 & 0 \\ -0.0000 & 0 & -3.0973 & 0.0762 & 0 & -193.5263 \end{bmatrix}. \end{aligned}$$

Remark. By now we have given three different solutions to the problem treated in Subsection 12.2.1. It is interesting to note that with all these three solutions the attenuation level is the same when it is displayed with 4 digits after the decimal place.

Exercise 12.4

Let $\omega_0 = \pi/12$(rad/h). Solve the H_∞ spacecraft rendezvous control problem described in Section 12.4.

Solution. As described in Section 12.4, the problem is to solve the H_∞ problem of the following system

$$\begin{cases} \dot{x} = Ax + B_1 u + B_2 d \\ y = Cx \end{cases}, \tag{s12.5}$$

with

$$\begin{cases} A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega_0^2 & 0 & 0 & 0 & 2\omega_0 & 0 \\ 0 & 0 & 0 & -2\omega_0 & 0 & 0 \\ 0 & 0 & -\omega_0^2 & 0 & 0 & 0 \end{bmatrix} \\ B_1 = B_2 = \begin{bmatrix} 0_{3\times 3} \\ I_3 \end{bmatrix} \\ C = \begin{bmatrix} I_3 & 0_{3\times 3} \end{bmatrix} \end{cases}. \tag{s12.6}$$

That is, to find a controller for the system in the form of

$$u = Kx,$$

such that $\left\| G_{yd}(s) \right\|_\infty$ is minimized, where

$$G_{yd}(s) = C(sI - A - B_1 K)^{-1} B_2.$$

The optimal solution. Using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the optimization problem (12.16), that is,

$$\begin{cases} \min & \gamma_\infty \\ \text{s.t.} & X > 0 \\ & \begin{bmatrix} (AX + B_1 W)^{\mathrm{T}} + AX + B_1 W & * & * \\ B_2^{\mathrm{T}} & -\gamma_\infty I & * \\ C_1 X + D_1 W & D_2 & -\gamma_\infty I \end{bmatrix} < 0 \end{cases}, \tag{s12.7}$$

we obtain $\gamma_\infty = 8.9360 \times 10^{-10}$ and

$$X = \begin{bmatrix} 0.0000 & 0 & 0 & -0.1078 & 0 & 0 \\ 0 & 0.0000 & 0 & 0 & -0.1077 & 0 \\ 0 & 0 & 0.0000 & 0 & 0 & -0.1076 \\ -0.1078 & 0 & 0 & 62966168.6657 & -9585.1499 & 0 \\ 0 & -0.1077 & 0 & -9585.1499 & 62965966.2773 & 0 \\ 0 & 0 & -0.1076 & 0 & 0 & 62966066.2948 \end{bmatrix} > 0,$$

$$W = \left[\begin{array}{rrrrrr} -62965462.5109 & 9585.1846 & 0 & -566922248.2043 & -356.8600 & 0 \\ 9585.0938 & -62965261.4015 & 0 & -356.8600 & -566825320.1972 & 0 \\ 0 & 0 & -62965361.8263 & 0 & 0 & -566789501.6198 \end{array}\right],$$

and the corresponding state feedback gain is given by

$$K_0 = WX^{-1} = \left[\begin{array}{rrrrrr} -8092406813415.824 & 1154703976.714 & 0 & -13864.279 & 0.019 & 0 \\ 1154025596.286 & -8096762227239.340 & 0 & 0.261 & -13857.523 & 0 \\ 0 & 0 & -8098559379702.618 & 0 & 0 & -13854.624 \end{array}\right].$$

Note that the attenuation level γ_∞ is very small, eventually the effect of the disturbance d must be effectively attenuated. However, with this optimal solution the feedback gain matrix K_0 is very large ($\|K_0\|_2 = 8098559379702.618$), which is not desirable in applications.

A feasible solution. Instead of solving the optimization problem (s12.7), let us turn to solve the following feasibility problem with the added attenuation level restriction $\gamma_\infty < 0.5$:

$$\begin{cases} X > 0 \\ \gamma_\infty < 0.5 \\ \begin{bmatrix} (AX + B_1W)^{\mathrm{T}} + AX + B_1W & * & * \\ B_2^{\mathrm{T}} & -\gamma_\infty I & * \\ C_1X + D_1W & D_2 & -\gamma_\infty I \end{bmatrix} < 0 \end{cases}. \qquad \text{(s12.8)}$$

Using the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the above altered feasible problem, we obtain $\gamma_\infty = 0.0429$ and

$$X = \begin{bmatrix} 0.2528 & 0.0000 & 0 & -1.2820 & -0.0000 & 0 \\ 0.0000 & 0.2528 & 0 & -0.0000 & -1.2820 & 0 \\ 0 & 0 & 0.2528 & 0 & 0 & -1.2820 \\ -1.2820 & -0.0000 & 0 & 37.4963 & 0.0000 & 0 \\ -0.0000 & -1.2820 & 0 & 0.0000 & 37.4963 & 0 \\ 0 & 0 & -1.2820 & 0 & 0 & 37.4963 \end{bmatrix},$$

$$W = \left[\begin{array}{rrrrrr} -31.6233 & 0.6712 & 0 & -36.9180 & 11.1032 & 0 \\ -0.6712 & -31.5713 & 0 & -11.1032 & -37.1816 & 0 \\ 0 & 0 & -31.5540 & 0 & 0 & -37.2695 \end{array}\right],$$

and the corresponding state feedback gain is given by

$$K = WX^{-1} = \begin{bmatrix} -157.3358 & 5.0276 & 0 & -6.3637 & 0.4680 & 0 \\ -5.0276 & -157.1302 & 0 & -0.4680 & -6.3637 & 0 \\ 0 & 0 & -157.0617 & 0 & 0 & -6.3637 \end{bmatrix}.$$

With this solution, we have indeed obtained a gain matrix with a reasonable magnitude, but with the cost of loosing the disturbance attenuation degree.

Exercise 12.5

Let $\omega_0 = \pi/12(\text{rad/h})$. Solve the H_2 spacecraft rendezvous problem described in Section 12.4.

Solution. As described in Section 12.4, the problem is to solve the H_2 problem of system (s12.5)-(s12.6). That is, to find a controller for the system in the form of

$$u = Kx,$$

such that $\left\| G_{yd}(s) \right\|_2$ is minimized, where

$$G_{yd}(s) = C(sI - A - B_1K)^{-1}B_2.$$

The optimal solution. Using the function `mincx` in the LMI Lab in the Matlab Robust Control Toolbox to the optimization problem (12.19), that is,

$$\begin{cases} \min & \rho \\ \text{s.t.} & AX + B_1W + (AX + B_1W)^{\mathrm{T}} + B_2B_2^{\mathrm{T}} < 0 \\ & \begin{bmatrix} -Z & C_2X \\ (C_2X)^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ & \operatorname{trace}(Z) < \rho \end{cases}, \tag{s12.9}$$

we obtain $\rho = 4.8872 \times 10^{-11}$ and

$$X = \begin{bmatrix} 0.0000 & 0 & 0 & -0.0301 & 0 & 0 \\ 0 & 0.0000 & 0 & 0 & -0.0301 & 0 \\ 0 & 0 & 0.0000 & 0 & 0 & -0.0302 \\ -0.0301 & 0 & 0 & 289840798.7778 & 47.5829 & 0 \\ 0 & -0.0301 & 0 & 47.5829 & 290047706.4293 & 0 \\ 0 & 0 & -0.0302 & 0 & 0 & 2.89604530.4523 \end{bmatrix} > 0,$$

$$W = \left[\begin{array}{rrr} -289840798.7228 & -47.5829 & 0 \\ -47.5829 & -290047706.3744 & 0 \\ 0 & 0 & -289604530.3973 \end{array} \right.$$
$$\left. \begin{array}{rrr} -316263464.6513 & -7.4282 & 0 \\ -7.4282 & -316263464.6445 & 0 \\ 0 & 0 & -316263464.6479 \end{array} \right],$$

$$\gamma_2 = \sqrt{\rho} = 6.9908 \times 10^{-6},$$

and the corresponding state feedback gain is given by

$$K_0 = W X^{-1} = 10^6 \times \left[\begin{array}{rr} -300011554673516 & -6119642342 \\ -6118914458 & -331055773687139 \\ 0 & 0 \end{array} \right.$$
$$\left. \begin{array}{rrrr} 0 & -31197 & -0.62 & 0 \\ 0 & -0.62 & -34333 & 0 \\ -270816895288647 & 0 & 0 & -28249 \end{array} \right].$$

Note that the attenuation level γ_2 is very small, eventually the effect of the disturbance d must be effectively attenuated. However, with this optimal solution the feedback gain matrix K_0 is very large ($\|K_0\|_2 = 331055774893340000000$), which is not desirable in applications.

A feasible solution. Instead of solving the optimization problem (s12.9), let us solve the feasibility problem with the attenuation level restriction $\rho < 0.1$. Using the function `feasp` in the LMI Lab in the Matlab Robust Control Toolbox to the altered feasible problem, that is,

$$\left\{ \begin{array}{l} AX + B_1 W + (AX + B_1 W)^{\mathrm{T}} + B_2 B_2^{\mathrm{T}} < 0 \\ \begin{bmatrix} -Z & C_2 X \\ (C_2 X)^{\mathrm{T}} & -X \end{bmatrix} < 0 \\ \operatorname{trace}(Z) < \rho \\ \rho < 0.1 \end{array} \right. , \tag{s12.10}$$

we obtain $\rho = 0.0808$ and

$$X = \begin{bmatrix} 0.0078 & -0.0000 & 0 & -0.0659 & 0.0000 & 0 \\ -0.0000 & 0.0078 & 0 & 0.0000 & -0.0659 & 0 \\ 0 & 0 & 0.0078 & 0 & 0 & -0.0659 \\ -0.0659 & 0.0000 & 0 & 1.0041 & 0.0000 & 0 \\ 0.0000 & -0.0659 & 0 & 0.0000 & 1.0041 & 0 \\ 0 & 0 & -0.0659 & 0 & 0 & 1.0041 \end{bmatrix},$$

$$W = \begin{bmatrix} -1.0057 & 0.0345 & 0 & -0.8177 & -170.9200 & 0 \\ -0.0345 & -1.0041 & 0 & 170.9200 & -0.8313 & 0 \\ 0 & 0 & -1.0036 & 0 & 0 & -0.8358 \end{bmatrix}.$$

Thus,

$$\gamma_2 = \sqrt{\rho} = 0.2843,$$

and the corresponding state feedback gain is given by

$$K = WX^{-1} = \left[\begin{array}{cccccc} -301.1067 & -3178.9741 & 0 & -20.5794 & -378.8900 & 0 \\ 3178.9741 & -300.9011 & 0 & 378.8900 & -20.5794 & 0 \\ 0 & 0 & -300.8326 & 0 & 0 & -20.5794 \end{array}\right].$$

With this solution, we have indeed obtained a gain matrix with a reasonable magnitude, but with the cost of loosing some of the disturbance attenuation degree.

Concluding Remarks

In this manual, we have provided solutions to all the 80 exercise problems appeared at the end of each chapter of the accompany book which is entitled *LMIs in Control Systems—Analysis, Design, and Applications*. To finish this manual, let us further address on some important issues and make them clearer.

Numerical reliability

There are some problems (see Table s.1), which really can be easily solved via other, sometimes, maybe more direct methods. While these problems are solved in this solution manual using LMI techniques. On one hand we are introducing and practicing the LMI techniques, and on the other hand, it is the problem of numerical reliability which really matters. LMI techniques are numerically efficient and reliable!

We have shown in Section 3.6.3 a counter example for numerical reliability with a pole assignment benchmark problem. The solution obtained with the best pole assignment algorithm to a very simple problem could have gone so far away from the true one, and what is worse, the program may even refuse to produce a result. Nevertheless, with LMI techniques, we can always have a reliable solution even when the problem size is very large. The same system in the pole assignment benchmark problem is considered in the book for certain other design problems, such as state feedback stabilization (Example 7.1), D-stabilization (Example 7.3) and observer design (Example 9.1), and also used in this solution manual for reduced-order observer design (Exercise 9.3), all using LMI techniques, and in all cases the LMI techniques just work perfectly.

We do admit that incorrect results due to numerical unreliability do not often occur, yet this does not mean that numerical reliability is not an important issue. We want to be sure about what we have got from a computer, and a mistake due to numerical unreliability in a practical application may sometimes cause a huge damage.

Regarding this issue, please also refer to Remark 4.2, which actually gives a partial answer.

Open-loop problems

In this manual, there are some problems which are involved with checking certain properties of an open-loop system (see Table s.2). For such problems, some of the

Table s.1: Problems can be solved by methods other than LMIs

Exercise	Description
4.3	Hurwitz stability of a linear system
4.5	Open-loop stability of a linear system
5.3	H_∞ index of a given transfer function
5.4	H_2 index of a given transfer function
5.5	H_∞ index of a given transfer function
5.6	H_2 index of a given transfer function
5.7	H_2 index of a given transfer function
6.2	Stabilizability of a linear system
6.3	Detectability of a linear system
6.4	Stabilizability and detectability of a linear system
11.1	Stability of a linear system
12.2	Stability, H_∞ and H_2 indices of a linear system

Table s.2: Open-loop problems

Exercise	Description	Feasibility
4.3	Hurwitz stability	yes
4.5	Hurwitz stability	no
4.6	Quadratic stability	yes
4.7	Stability of a time-delay system	yes
4.8	Stability of a time-delay system	yes
6.5	Passivity	no
6.5	Bounded-realness	yes
6.6	Passivity and bounded-realness	no
11.1	Stability of a linear system	yes
12.2	Stability, H_∞ and H_2 indices	no

results are negative, that is, some of the problems do not admit a feasible solution. We point out that such a phenomenon is just normal.

Stability, passivity, bounded-realness, etc. are all good properties of a linear system. These properties are usually achieved through feedback designs, while it is asking too much to require an open-loop system to possess such properties. This is actually what the importance of the control systems design techniques lies in. Note that most of the systems involved in the exercises listed in Table s.2 are numerical ones. If, by any chance, like the case of Exercise 11.1, we indeed encounter a practical open-loop system which possesses such properties, then we are lucky. The system without feedback control design has already been a good one in certain sense.

LMI approaches

Users of this manual might have noticed this already. There are several exercise problems in this manual, and two problems in the application part of the accompany

Table s.3: Some H_∞/H_2 control problems

Problem	**Description**	**Index γ**	$\|K\|_2$
Exercise 11.3	H_∞ control	0.3853	$3.619968744101 \times 10^8$
Exercise 12.4	H_∞ control	8.9360×10^{-10}	$8.098559379702618 \times 10^{12}$
Exercise 12.5	H_2 control	6.9908×10^{-6}	$3.3105577489334 \times 10^{20}$
Section 11.2.2	H_∞ control	0.01	$8.986351649196064 \times 10^6$
Section 12.2.2	H_2 control	5.80569×10^{-6}	$8.759253920405163 \times 10^{14}$

book, involving minimization of an attenuation level γ, and the LMI approach has indeed produced a minimal value for it. This is what we have expected. However, meanwhile the LMI approach has also produced a feedback gain with a very large magnitude (see Table s.3). This is something we do not expect. Because of this, some one may say: “LMI approaches are not good since they produce with many of the control systems design problems feedback gains with too large magnitudes.”

Are LMI approaches really good or not? When you want a minimal γ, an LMI approach produces it. What is better than an approach which can always produce you with the best (globally optimal) result? It produces at the same time a gain matrix with a large magnitude, because you did not tell it not to. What an approach would it be if it could provide you with things you did not ask? no matter good or bad.

In many cases, controller design via minimizing a single objective does not give a good result in the sense that, although this particular objective is minimized, some other objectives may become extremely worse, just like the cases listed in Table s.3. There are always some compromises between the different objectives, like the indices of the attenuation level and the gain magnitude.

If we want in the meanwhile a gain matrix with a reasonable magnitude, we should add certain extra constraints as is treated in Sections 11.2.2, 12.2.2 and in the solutions to Exercises 11.3, 12.4 and 12.5, and then let the LMI approach to seek the solution for you.

Regarding this issue, we have given in Section 12.4 some more comments.

As you have found, in the book and this manual, and maybe some elsewhere, LMI approaches are numerally efficient and reliable, and provide globally optimal solutions to the formulated convex optimization problems. LMI today has been a real technology, and has become a very powerful tool for control systems analysis and designs. To conclude, let us cite again the words in Doyle et al. (1991):

LMIs play the same central role in the postmodern theory as Lyapunov and Riccati equations played in the modern, and in turn various graphical techniques such as Bode, Nyquist and Nichols plots played in the classical.

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