Title

Subtitle

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Abstract

lorem ipsum or something

Contents

1 Introduction

General ideas: Full fluid dynamics simulations of the lungs is difficult. Thankfully, we don't need that. We can get pretty accurate results by building on Brody's work, simulating airflow as 1D and incompressible.

Using this model allows us to model how the airflow changes under a number of conditions, e.g. damage to the alveoli in a region, constricting bronchioles, etc.

2 Background

3 Methods

This section is mostly some temporary filler at the moment, just serving as a place to jot down the formulae that we're using inside the simulation *right now*.

3.1 Simultaneous Equations

There are essentially four simultaneous equations that we use to evaluate the state of the system at a given point in time. This section provides a summary and brief description of each, the first of which is the following:

$$P_{\text{parent}(i)} - P_i = R(i)Q_i \tag{1}$$

which specifies that the pressure differential between the distal end of branch i and its parent must equal the pressure from the resistance from the flow through this branch i. For the "root" branch, $P_{\text{parent}(i)}$ is the pressure at the trachea – typically atmospheric pressure.

The resistance term $R(i)$ is defined as following function, as given by Pedley et al. (1970):

$$R(i) = \frac{2\mu L_i c}{\pi r_i^4} \left(\frac{4\rho |Q_i|}{\mu \pi L_i} \right)^{\frac{1}{2}}$$

The parenthesized term corresponds to the Reynold's number of the flow, scaled by the ratio of the diameter of the branch to its length L_i. r_i is the radius of branch i, μ is the viscosity of the air, and $c = 1.85$ is a correction constant.

The second equation ensures incompressibility; the flow through a bifurcation must equal the sum of the flow through its children:

$$Q_i = \sum Q_{\text{child}} \tag{2}$$

where each *child* refers to any branch c with $\text{parent}(c) = i$.

The third equation maintains that the volume of an acinar region changes with the flow into or out of it for the given timestep:

$$V_i^t = V_i^{t-1} + dtQ_i^t \tag{3}$$

where dt is the timestep size, t refers to the current timestep, and V_i is the volume of the acinar region of branch i.

The final equation defines the elastic force of each acinar region, relating the pressure it exerts on its branch to the volume of the region itself and the pressure outside it:

$$P_i = \frac{1}{C_i} V_i + P_{pl}(t) \tag{4}$$

where $P_{pl}(t)$ is the pleural pressure (i.e. the "pressure" from the diaphragm, outside the acinar region) at the current time and C_i is the *compliance* of the acinar region of branch i. The pleural pressure changes over time to mimic human breathing patterns – hence why it is parameterised by t.

At each timestep in the simulation, all of the simultaneous equations are grouped into a single optimization function $f(\boldsymbol{x}) = (\mathrm{EQ}_1..., \mathrm{EQ}_2..., \mathrm{EQ}_3..., \mathrm{EQ}_4)$, where each EQ_i corresponds to the repeated instances of the ith equation above, normalized so the right-hand-side equals zero. Thus the solution exists at $\mathbf{0}$, and we use Euler's method to find an approximate solution, within $\|f(\boldsymbol{x})\|^2 \leq tol$ and $\|dx\|^2 \leq tol$, with a tolerance of 10^{-6}.

The input $\boldsymbol{x}$ is arranged with the values $(P..., Q..., V...)$ for each applicable branch – i.e. using P and Q from all branches and V from each acinar region.

The above descriptions are *nearly* correct – they would work, but there were a few adjustments made to the inputs and equations in practice for better numerical stability. These are discussed in the next section.

3.2 Units & Numerical stability

It is worth making explict the units used for each value. After careful consideration, these were considered to provide the best trade-off of familiar units and those with values of magnitude close to one, where floating-point accuracy is maximized. As we will see momentarily, the spread was still quite wide. The chosen units were:

Type of thing ???	Units
Distance	m
Volume	m^3
Flow velocity	$\frac{\mathrm{m}^3}{\mathrm{s}}$
Density	$\frac{\mathrm{kg}}{\mathrm{m}^3}$
Pressure	Pascals $\left(\frac{\mathrm{kg}}{\mathrm{m}\cdot\mathrm{s}^2}\right)$
Compliance	$\frac{\mathrm{m}^3}{\mathrm{Pascal}} \left(\frac{\mathrm{m}^4\cdot\mathrm{s}^2}{\mathrm{kg}}\right)$
Resistance	$\frac{\mathrm{kg}}{\mathrm{m}^4\cdot\mathrm{s}}$
Viscosity	$\frac{\mathrm{kg}}{\mathrm{m}\cdot\mathrm{s}}$

One of the challenges with using these units is that some values are at a much greater magnitude than the others. For example, the pressure inside each branch is close to atmospheric pressure – or about 10^5 Pascals, but pressure *gradients* are typically much smaller.

In practice, this can mean that if the dx from our Euler step is too small, the pressure won't change; it doesn't have the necessary precision at that magnitude.

To ameliorate this issue, we define two new values: $\hat{P}$ and $\hat{V}$, which are given by:

$$\hat{P} = P - P_{\mathrm{atm}} \tag{5}$$

where P_{atm} is is atmospheric pressure; and:

$$\hat{V} = V - V|_{P=P_{\text{atm}}} \tag{6}$$
$$= C(\hat{P} - P_{pl}) \tag{7}$$

Note that the definition of $\hat{V}$ would be the result of simply substituting $\hat{P}$ for P in 4. Applying these substitutions gives the following equations, equivalent to their counterparts above:

$$\hat{P}_{\text{parent}} - \hat{P}_i = R(i)Q_i \tag{8}$$

$$Q_i = \sum Q_{\text{child}} \tag{9}$$

$$\hat{V}_i^t = \hat{V}_i^{t-1} + dtQ_i^t \tag{10}$$

$$\hat{P}_i = \frac{1}{C_i}\hat{V}_i + P_{pl}(t) \tag{11}$$

Representing the pressure and volume by their *offset* from values at atmospheric pressure causes them to cluster much closer to zero – the magnitude of the mean is significantly decreased, relative to the variance of the values. This of course greatly improves the accuracy of each Euler step.