MA2C03: ASSIGNMENT 3
DUE BY FRIDAY, MARCH 2
IN THE MATHS OFFICE ROOM 0.6

Please write down clearly both your name and your student ID number on everything you hand in.

1) (10 points) Let (V, E) be the graph with vertices a, b, c, d, e, f, g, and h, and edges ab, bc, cd, de, ef, af, bg, and eh.
(a) Is this graph connected? Justify your answer.
(b) Does this graph have an Eulerian trail? Justify your answer.
(c) Does this graph have an Eulerian circuit? Justify your answer.
(d) Does this graph have a Hamiltonian circuit? Justify your answer.
(e) Is this graph a tree? Justify your answer.

1(a) Yes, the graph is connected. There is a walk from any vertex to any other vertex.

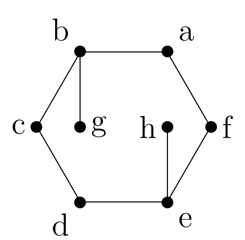
1(b) No, as four vertices have odd degree instead of two: $\deg b = \deg e = 3$ and $\deg g = \deg h = 1$.

1(c) No, as all vertices should have even degree for an Eulerian circuit to exist. Here four vertices have odd degree: b, e, g, and h.

1(d) No, as we have two pendant vertices, g and h. If we had a Hamiltonian circuit, it would have to pass through the two vertices adjacent to the pendant vertices, b and e, twice, which is not allowed.

1(e) No, as there is a circuit $abcdef$.

The graph can be drawn as



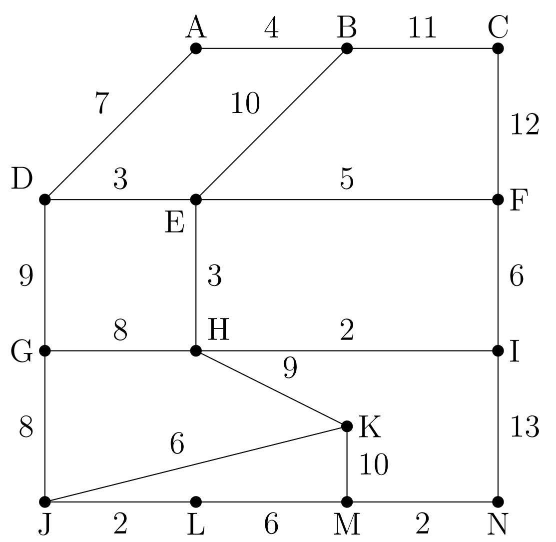
Grading rubric: 2 points for each of the five parts: 1 point for the answer and 1 points for the justification.

2) Consider the connected graph with vertices A, B, C, D, E, F, G, H, I, J, K, L, M and N and with edges, listed with associated costs, in the following table:

HI	JL	MN	EH	AB	EF	FI	LM	JK	AD
2	2	2	3	4	5	6	6	6	7
GH	GJ	HK	DG	KM	BE	DE	BC	CF	IN
8	8	9	9	10	10	11	11	12	13

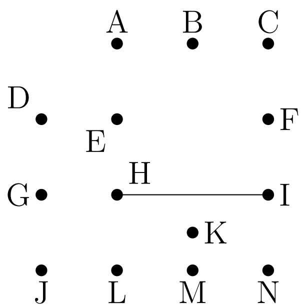
(a) (2 points) Draw the graph and label each edge with its cost.
(b) (9 points) Determine the minimum spanning tree generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, list the edge that is added and draw the graph.
(c) (9 points) Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex F, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, list the edge that is added and draw the graph.

2(a) (2 points) The graph can be drawn as



Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

2(b) The edges are added in the following order: HI, JL, MN, EH, AB, EF, LM, JK, AD, GH, GJ, DG, and BC. This looks like:



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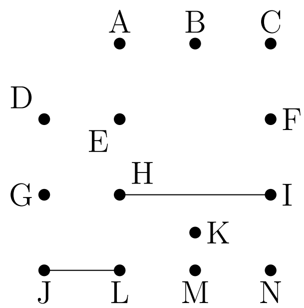

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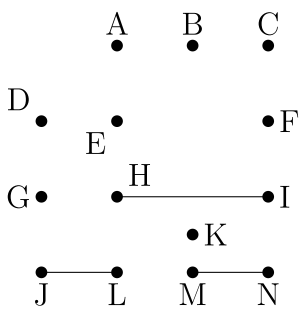

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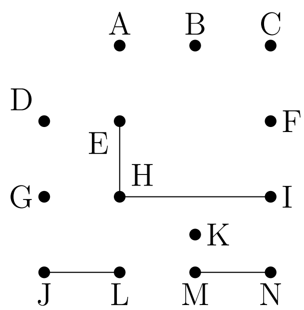

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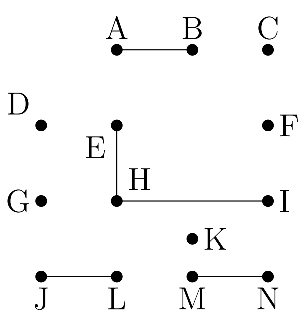

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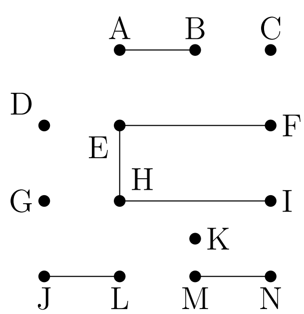

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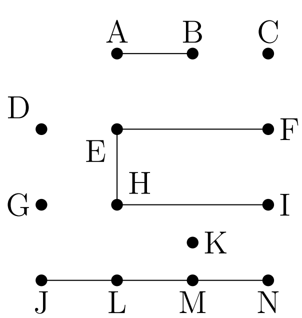

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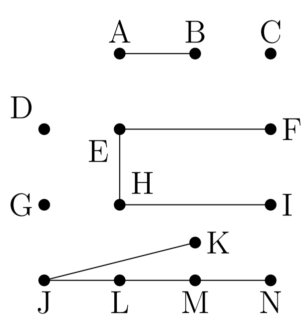

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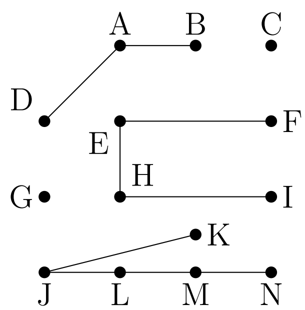

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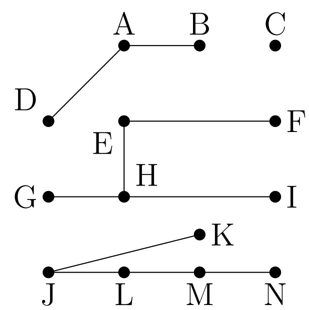

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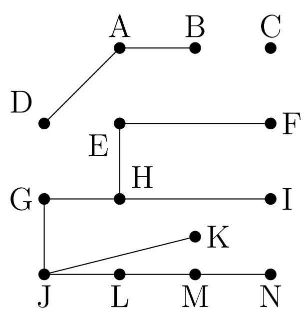

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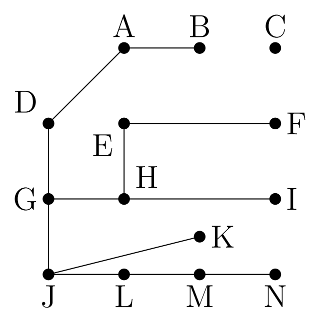

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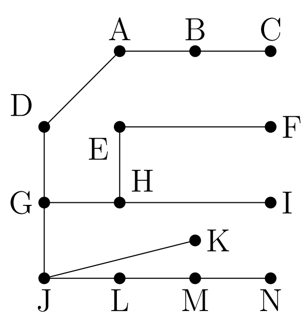
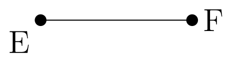


13.

Grading rubric: 9 points: roughly 1 point per correct edge in the correct order.

2(c) The edges are added in the following order: EF, EH, HI, GH, GJ, JL, LM, MN, JK, DG, AD, AB, and BC. This looks like:



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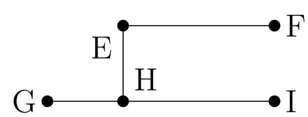

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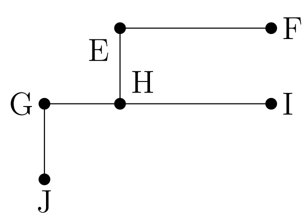

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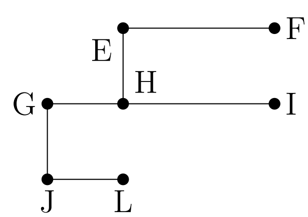

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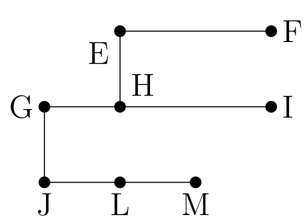

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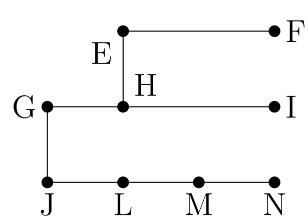

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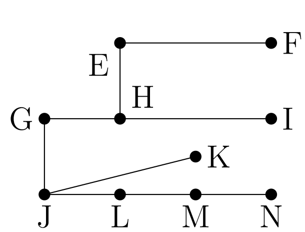

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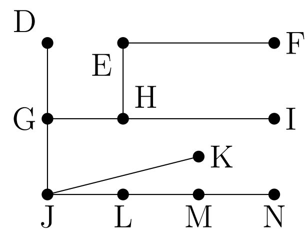

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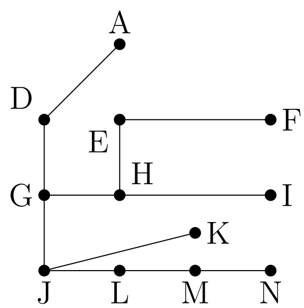

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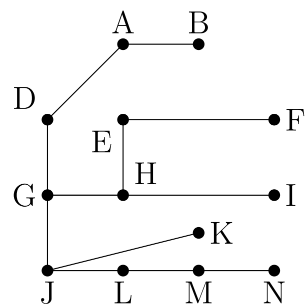

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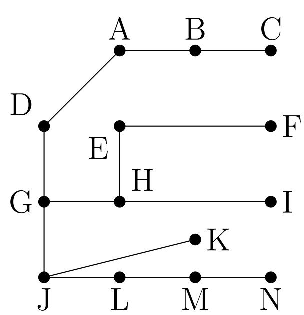


13.

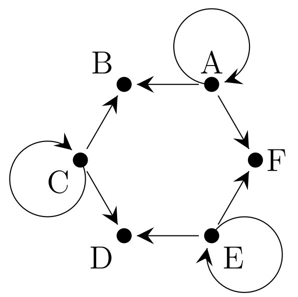
Grading rubric: 9 points: roughly 1 point per correct edge in the correct order.

3) (10 points) Let $(\mathcal{V}, \mathcal{E})$ be the directed graph with vertices A, B, C, D, E, and F and edges (A, A), (A, B) (A, F), (C, B), (C, C), (C, D), (E, D), (E, E), and (E, F).

(a) Draw this graph.
(b) Write down this graph's adjacency matrix.

(c) Give an example of an isomorphism φ from the graph $(\mathcal{V}, \mathcal{E})$ to itself such that $\varphi(C) = C$. Note that an isomorphism of directed graphs should also respect the direction of the edges.

3(a) (2 points) This graph can be drawn as:



Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

3(b) The adjacency matrix is as follows (note that we must stick to the given ordering of vertices; A, B, C, D, E, F):

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

3(c) (6 points) $\varphi(C) = C$, $\varphi(A) = E$, $\varphi(E) = A$, $\varphi(F) = F$, $\varphi(B) = D$, and $\varphi(D) = B$ respects the direction of the edges. Note it takes advantage of the natural symmetry in this directed graph. The identity map $\varphi(C) = C$, $\varphi(A) = A$, $\varphi(E) = E$, $\varphi(F) = F$, $\varphi(B) = B$, and $\varphi(D) = D$ also gives an isomorphism and is another correct solution.

Grading rubric: 1 point for each correct vertex assignment in the isomorphism map.

4) (20 points) Let R be a relation on a set $V = \{a, b, c, d\}$ given by

$$R = \{(a, a), (a, b), (b, d), (b, b), (a, c), (a, d), (b, c), (c, c), (d, c), (d, d)\}.$$

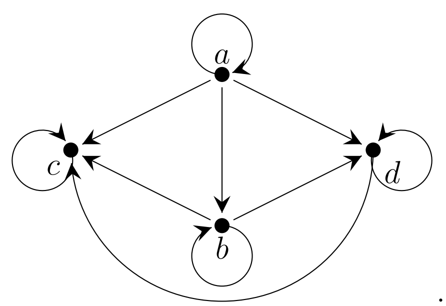
(a) Using the one-to-one correspondence between relations on finite sets and directed graphs, draw the directed graph corresponding to the relation R.

(b) Is R an equivalence relation? Justify your answer.
(c) If R is not an equivalence relation, which ordered pairs would have to be added to R to make it into an equivalence relation?

Let R' be a relation on a set $V = \{a, b, c\}$ with three elements.

(d) What is the minimum number of ordered pairs in R' for the relation R' to be reflexive? What should those pairs be? Justify your answer.
(e) What is the minimum number of ordered pairs in R' for the relation R' to be symmetric? Justify your answer.
(f) What is the minimum number of ordered pairs in R' for the relation R' to be transitive? Justify your answer.

4(a) The graph can be drawn as:



Grading rubric: 2 points: 1 point if the 4 vertices are there, and the extra point for the edges. If only 1, 2 edges are incorrect, full marks given.

4(b) Not an equivalence relation as it fails to be symmetric. For example, the edge (b,a) is not there.

Grading rubric: 2 points: 1 point for the answer and 1 points for the justification.

4(c) To achieve symmetry we must add (b,a), (c,b), (c, d), (d,a), (c,a), and (d,b). Those are all possible edges that can be added, so we know transitivity must hold as well.

Grading rubric: 6 points. Award 1 point for each of the six pairs. No justification is required for this part in fact, so points are awarded strictly for having written down the correct pairs.

4(d) The definition of reflexivity says that $\forall\, x \in V$, we must have $xR'x$. Since we have three elements in V, we must add the pairs (a,a), (b,b), and (c,c), i.e., 3 pairs.

Grading rubric: 3 points: 1 for the answer (3) and 2 for the justification (1 point for the definition of reflexivity and 1 point for applying it here).

4(e) The definition of symmetry says that $\forall\, x, y \in V$ if $xR'y$ then $yR'x$. Note that this statement is an implication, which is vacuously true when the antecedent is false, so the minimum number of pairs is zero.

Grading rubric: 4 points: 1 for the answer (0) and 3 for the justification (1 point for the definition of symmetry and 2 points for applying it here).

4(f) The definition of transitivity says that $\forall\, x, y, z \in V$ if $xR'y$ and $yR'z$, then $xR'z$. Note that this statement is an implication, which is vacuously true when the antecedent is false, so the minimum number of pairs is zero.

Grading rubric: 3 points: 1 for the answer (0) and 2 for the justification (1 point for the definition of transitivity and 1 point for applying it here).