MA2C03: TUTORIAL 10 PROBLEMS
GRAPH THEORY

1) Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bd, be, ac, cd, and ae.

(a) Draw this graph. Write down its incidence table and its incidence matrix.
(b) Write down this graph's adjacency table and its adjacency matrix.
(c) Is this graph complete? Justify your answer.
(d) Is this graph bipartite? Justify your answer.
(e) Is this graph regular? Justify your answer.
(f) Does this graph have any regular subgraph? Justify your answer.
(g) Give an example of an isomorphism from the graph (V, E) specified at the beginning of this problem to the graph (V', E') with vertices p, q, r, s, and t, and edges pq, ps, rt, st, rs, and rq.

Solution: Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bd, be, ac, cd, and ae.

(a) The graph is drawn at the end of the solutions. If we keep the same order of the vertices and edges given in the statement of the problem, the incidence table is:

	ab	bd	be	ac	cd	ae
a	1	0	0	1	0	1
b	1	1	1	0	0	0
c	0	0	0	1	1	0
d	0	1	0	0	1	0
e	0	0	1	0	0	1

The corresponding incidence matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

(b) If we keep the same order of the vertices given in the statement of the problem, the adjacency table is:

	a	b	c	d	e
a	0	1	1	0	1
b	1	0	0	1	1
c	1	0	0	1	0
d	0	1	1	0	0
e	1	1	0	0	0

The corresponding adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(c) No, as for example edge bc does not belong to the graph, so not every vertex is connected to every other vertex.

(d) No, as the graph contains the complete subgraph $V' = \{a, b, e\}$ and $E' = \{ab, ae, be\}$, which cannot be partitioned.

(e) No, as vertices a and b have degree 3, whereas the other vertices have degree 2.

(f) Any two vertices that have an edge between them taken with that edge form a regular subgraph (1-regular) as do $\{a, b, e\}$ and the edges between them (2-regular) and $\{a, b, c, d\}$ and the edges between them (2-regular).

(g) It does NOT suffice to show the two graphs have the same structure. An isomorphism is a MAP, so you must provide the map on vertices. Two possible isomorphisms are $\varphi(a) = s$, $\varphi(b) = r$, $\varphi(c) = p$, $\varphi(d) = q$, and $\varphi(e) = t$ or the following: $\varphi(a) = r$, $\varphi(b) = s$, $\varphi(c) = q$, $\varphi(d) = p$, and $\varphi(e) = t$.