CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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References: Sedgewick 12.4, 15.2-15.4
Goodrich & Tamassia 9.2.1-9.2.2

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Outline

1. Lower bound
2. Interpolation Search
3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries

Lower bound for search

The fastest implementations of the dictionary ADT require $\Theta(\log n)$ time to search a dictionary containing n items. Is this the best possible?

Theorem: In the comparison model (on the keys),
$\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof:

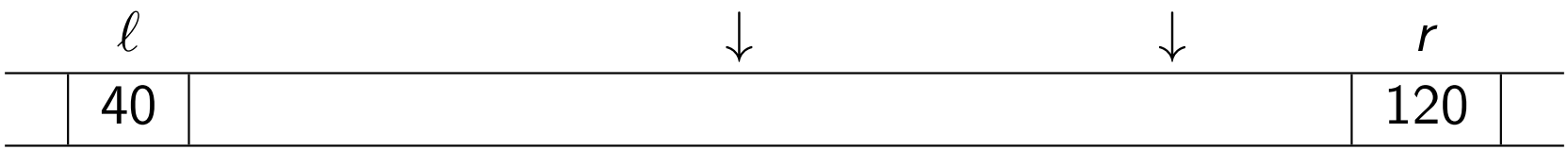
But can we beat the lower bound for special keys?

Interpolation Search: Motivation

Ordered array

- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

binary search($A[\ell, r], k$): Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r-\ell) \rfloor$



Question: If keys are numbers, where would you expect key $k = 100$?

Interpolation Search($A[\ell, r], k$): Compare at index $\ell + \left\lfloor \frac{k-A[\ell]}{A[r]-A[\ell]}(r-\ell) \right\rfloor$

Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash due to $A[\ell] = A[r]$

Interpolation-search(A, n, k)
A: Array of size n, k: key

1. $\ell \leftarrow 0$
2. $r \leftarrow n - 1$
3. **while** $((A[r]! = A[\ell])\&\&(k \geq A[\ell])\&\&(k \leq A[r]))$
4. $\quad m \leftarrow \ell + \frac{k - A[\ell])}{A[r] - A[\ell]} \cdot (r - \ell)$
5. $\quad$ **if** $(A[m] < k) \quad \ell = m + 1$
6. $\quad$ **elsif** $(k < A[m]) \quad r = m - 1$
7. $\quad$ **else** return m
8. **if** $(k = A[\ell])$ return ℓ
9. **else** return "not found"

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500

Search(449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $r = 4$, $m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$, found at $A[4]$

Works well if keys are uniformly distributed:

- Can show: the array in which we recurse into has expected size $\sqrt{n}$.
- Recurrence relation is $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\text{avg})}(n) \in \Theta(\log \log n)$.

But: Worst case performance $\Theta(n)$

Outline

Tries: Introduction

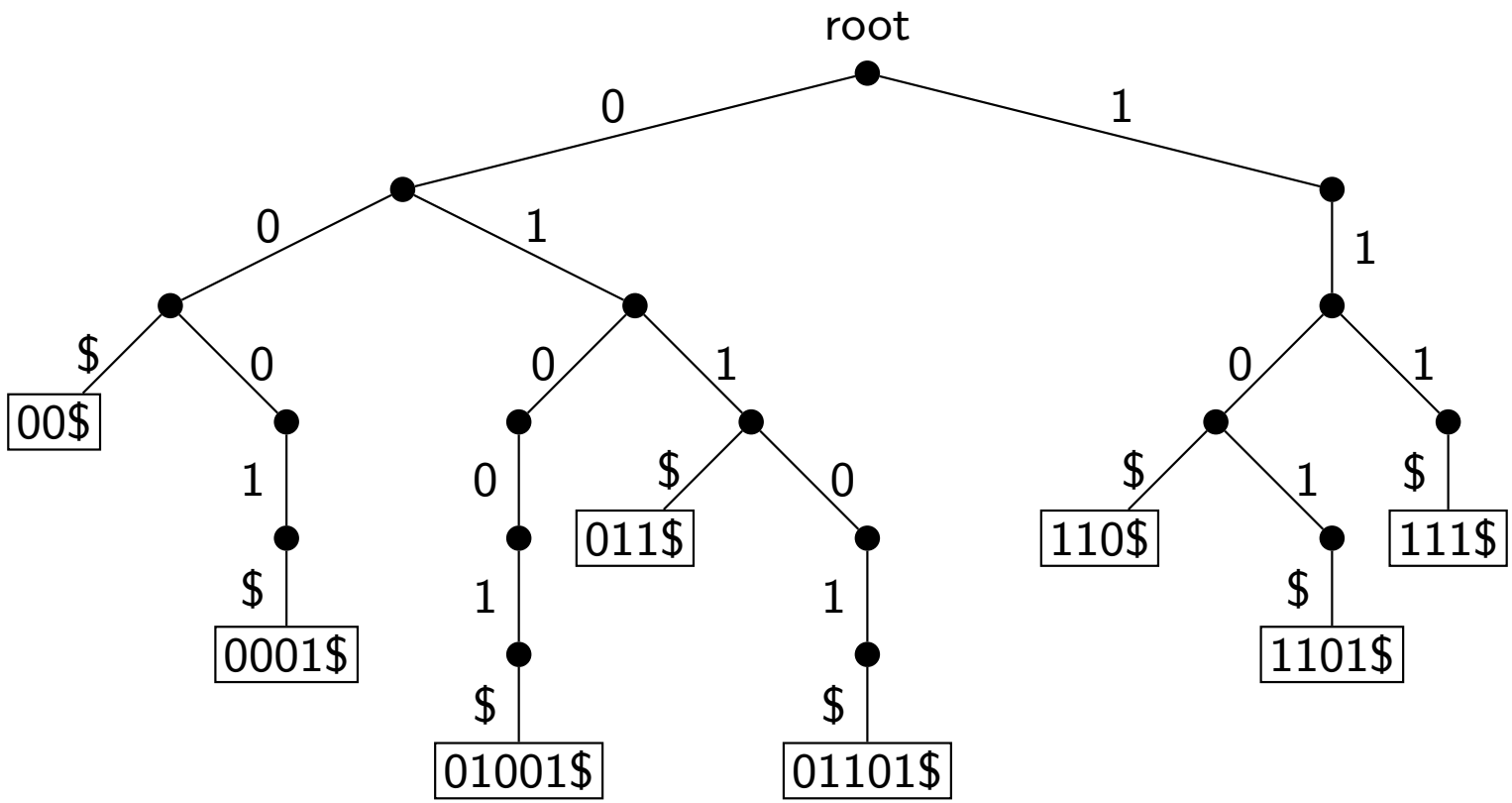
- **Trie (Radix Tree)**: A dictionary for binary strings
  - Comes from retrieval, but pronounced "try"
  - A tree based on **bitwise comparisons**
  - Similar to **radix sort**: use individual bits, not the whole key
- Keys can have different number of bits
- **Assumption:** Dictionary is *prefix-free* (there is no pair of binary strings in the dictionary where one is the prefix of the other):
  - **Prefix** of a string $S[0..n-1]$: a substring $S[0..i]$ of S for some $0 \leq i \leq n-1$
  - This is always satisfied if all strings have the same length.
  - This is always satisfied if all strings end with a special 'end-of-word' character \$.

Tries: structure

Structure of trie:

- Items (keys) are stored **only** in the leaf nodes
- Edge to child is labelled with corresponding bit or \$

Example: A trie for
$S = \{00\$, 0001\$, 01001\$, 011\$, 01101\$, 110\$, 1101\$, 111\$\}$



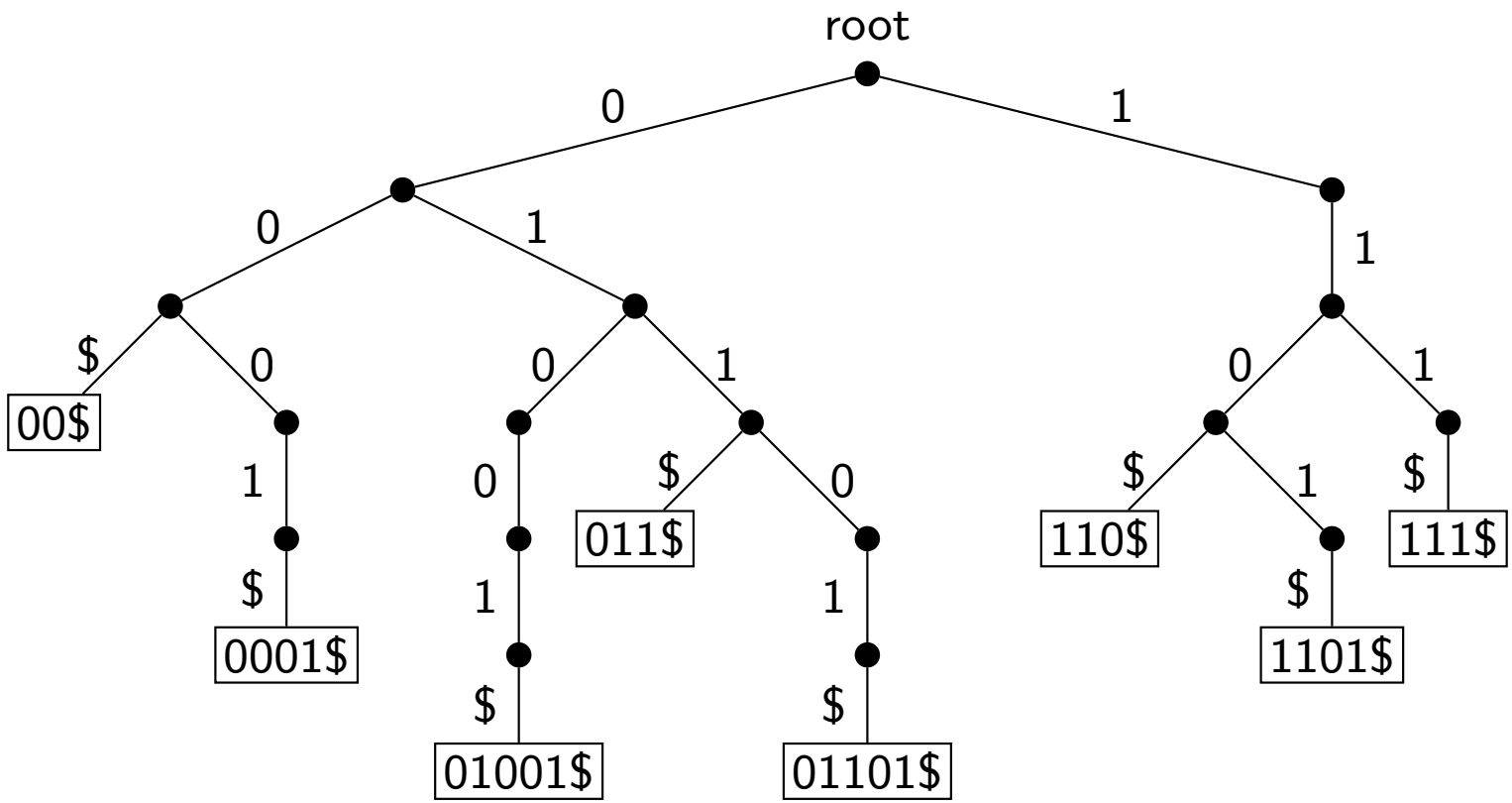
Tries: Search

- start from the root and the most significant bit of x
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- return success if we reach a leaf (it must store x)
- else recurse on the new node and the next bit of x

```
Trie-search(v ← root, d ← 0, x)
v: node of trie; d: level of v, x: word
1.    if v is a leaf
2.        return v
3.    else
4.        let c be child of v labelled with x[d]
5.        if there is no such child
6.            return "not found"
7.        else Trie-search(c, d + 1, x)
```

Tries: Search Example

Example: Search(011$)

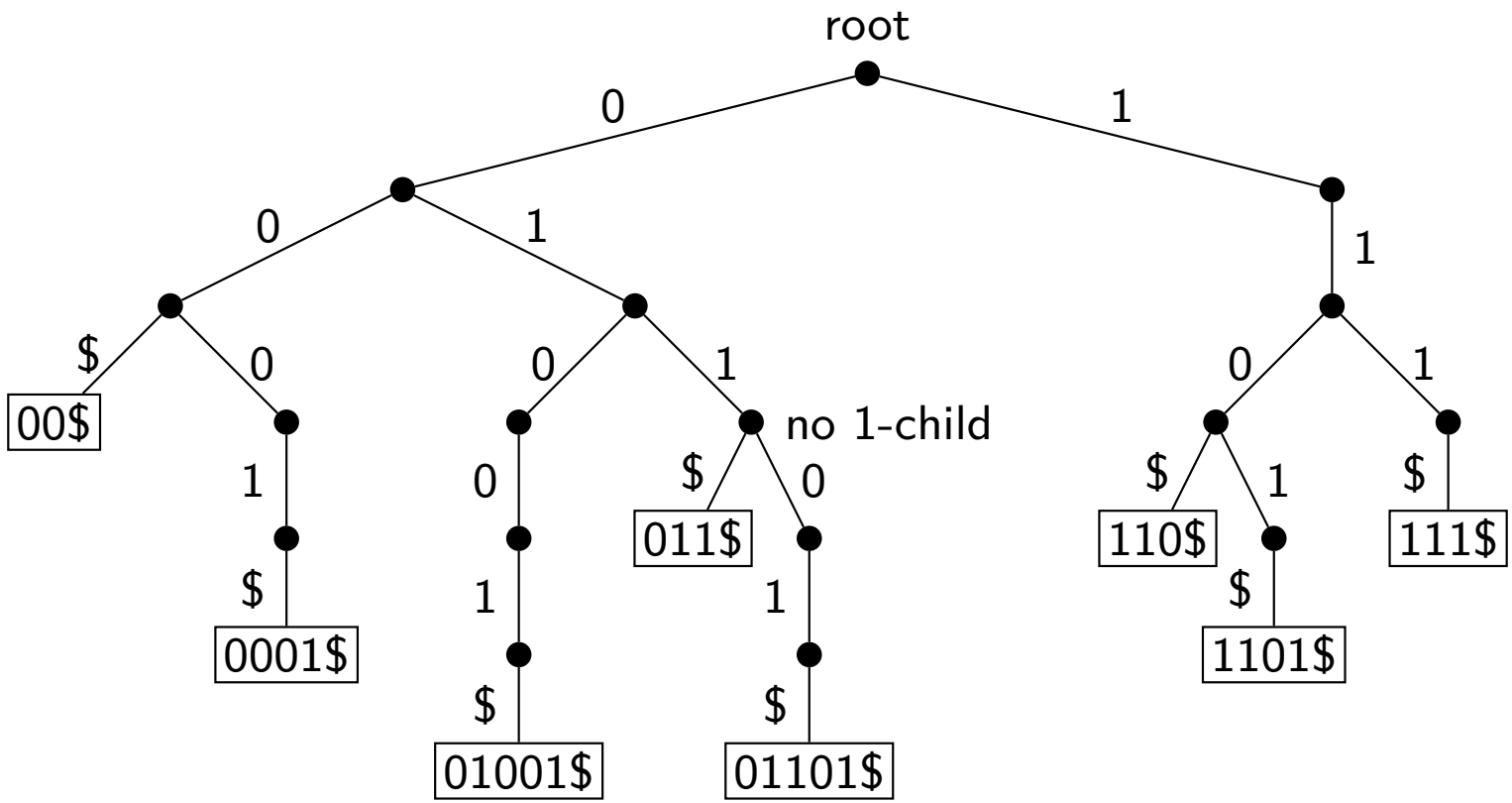


Tries: Insert & Delete

- **Insert(x)**
  - Search for x, this should be unsuccessful
  - Suppose we finish at a node v that is missing a suitable child. Note: x has extra bits left.
  - Expand the trie from the node v by adding necessary nodes that correspond to extra bits of x.
- **Delete(x)**
  - Search for x
  - let v be the leaf where x is found
  - delete v and all ancestors of v until we reach an ancestor that has two children.
- Time Complexity of all operations: $\Theta(|x|)$
  $|x|$: length of binary string x, i.e., the number of bits in x

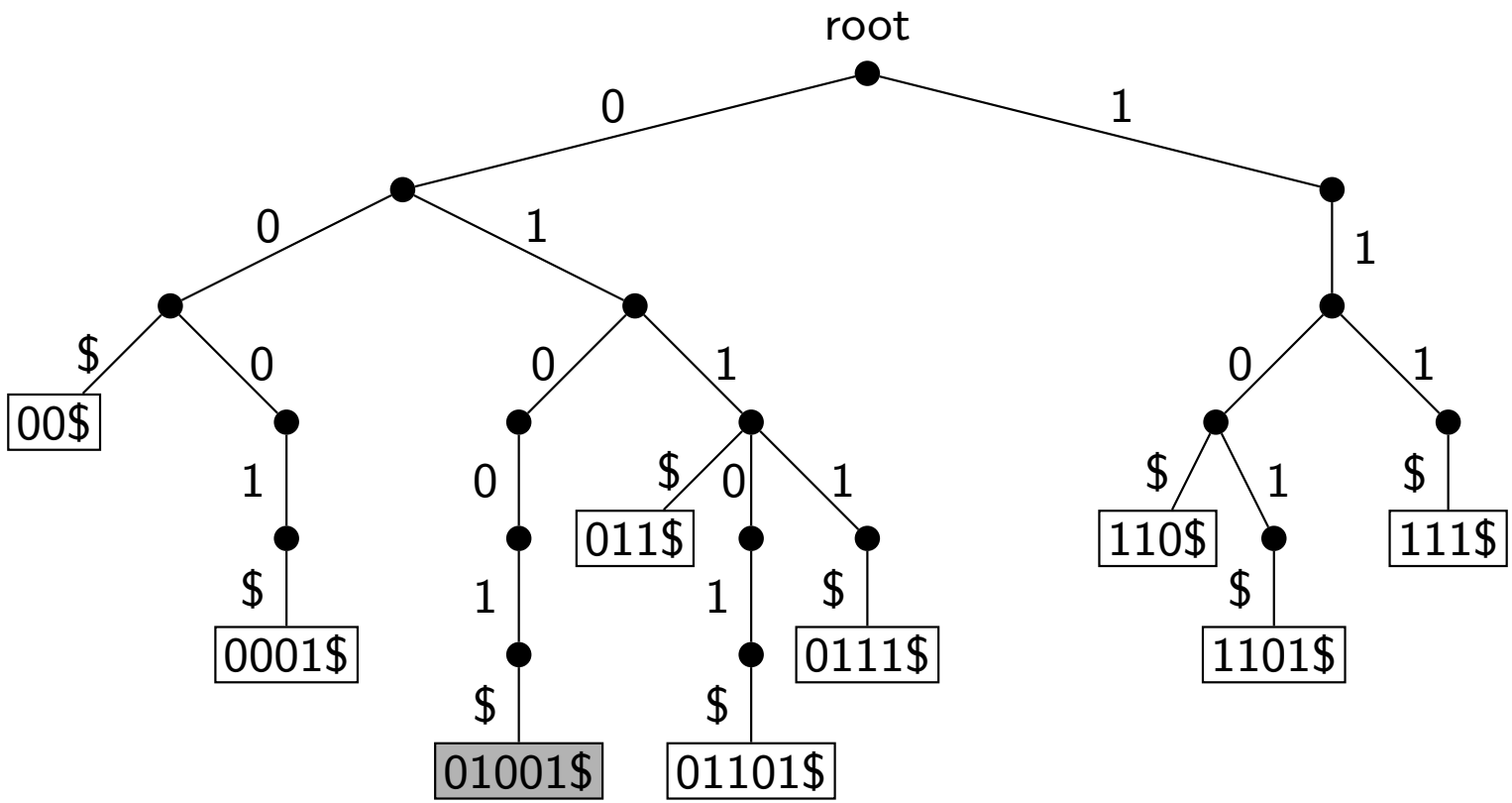
Tries: Insert Example

Example: Insert(0111$)



Tries: Delete Example

Example: Delete(01001$)

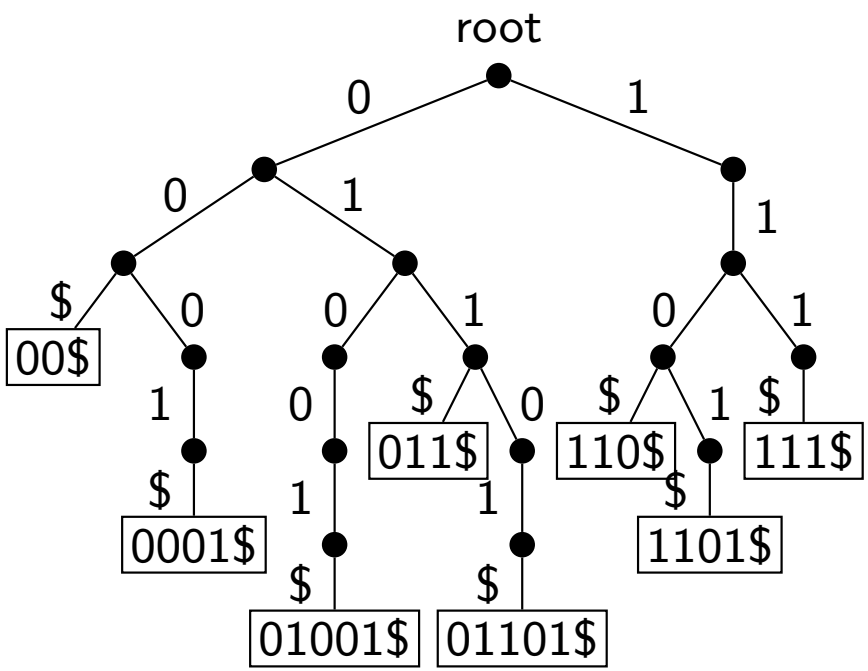


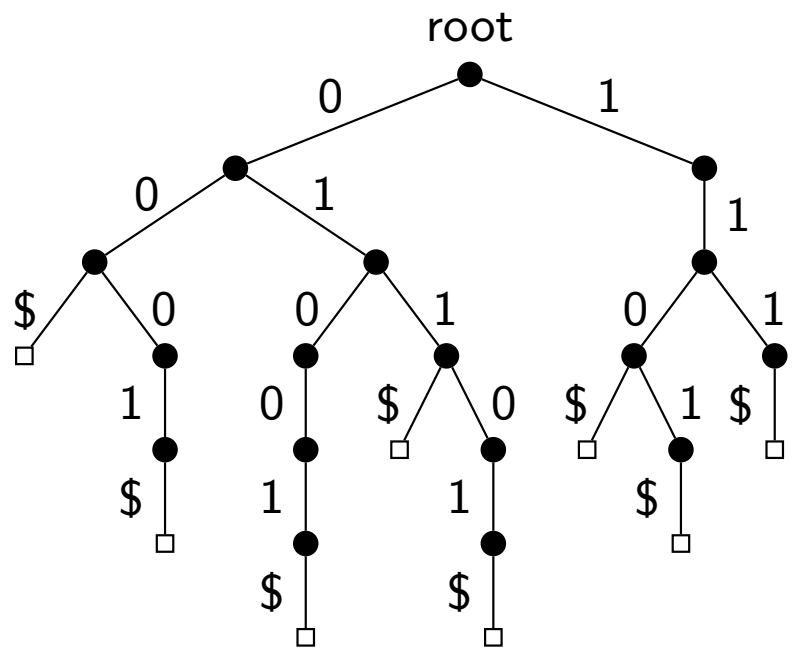
Outline

Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.



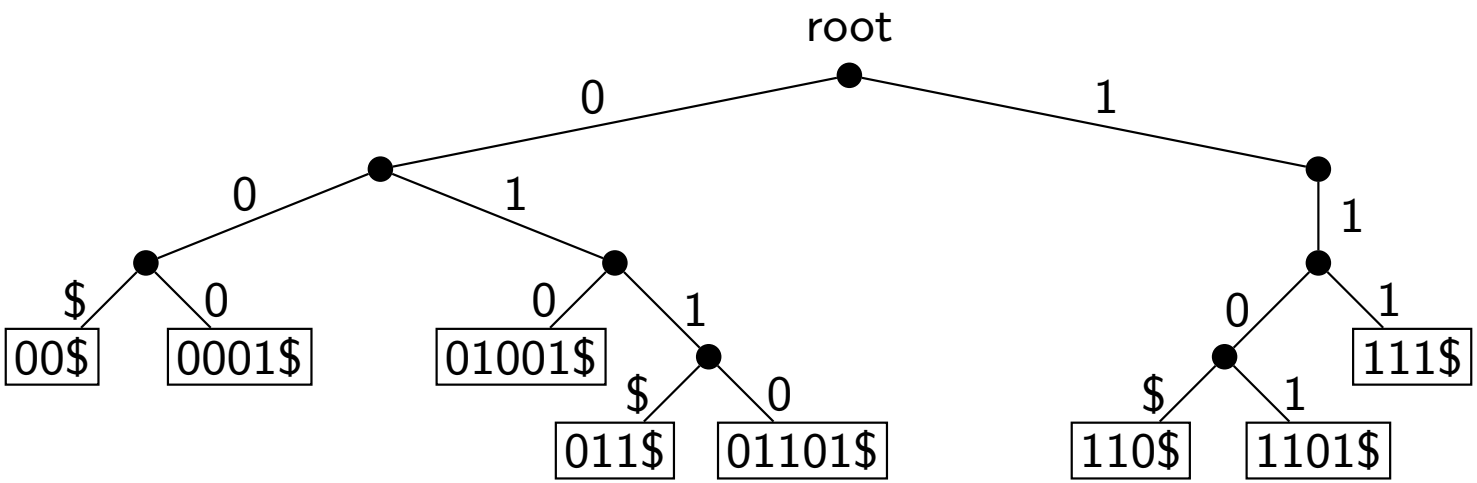


Variations 2 of Tries: Remove Chains to Labels

Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Saves space if there are only few bitstrings that are long.
- Note that this variation *cannot* be combined with the previous one (why not?)

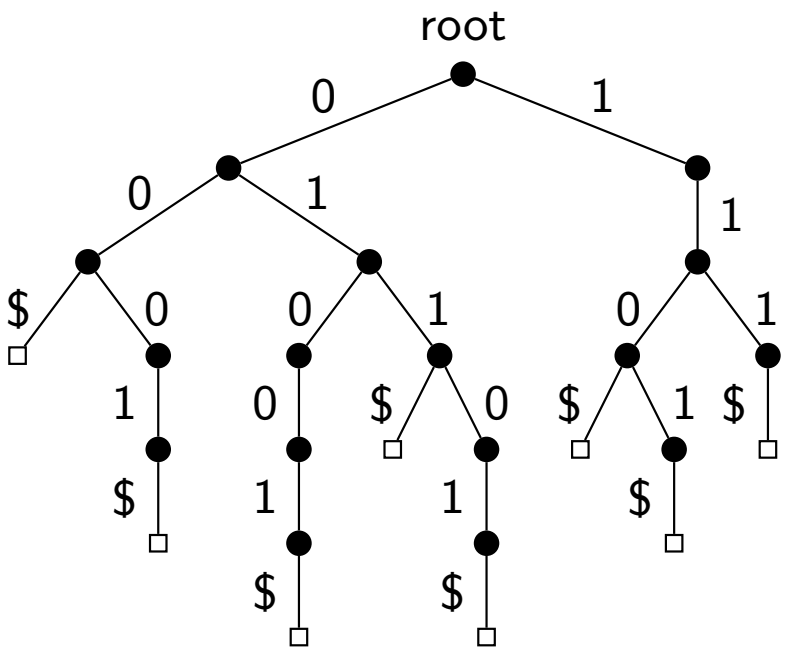
This variation is the one presented in Sedgewick.

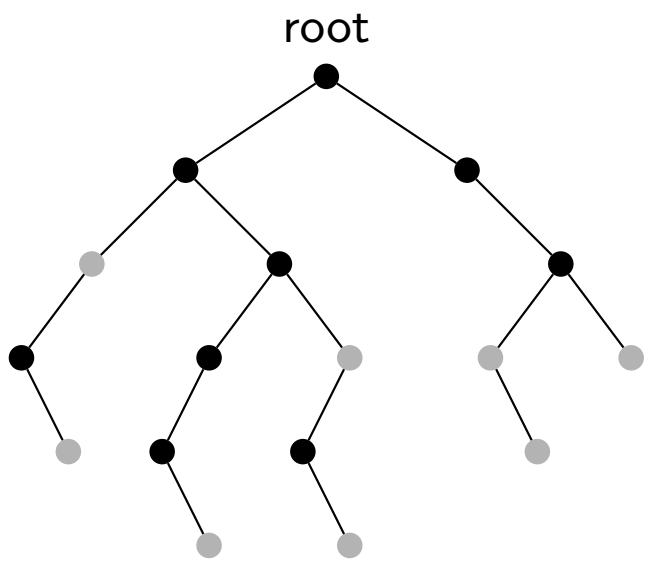


Variation 3 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Then internal nodes may also represent keys. We then use a *flag* at each node to indicate whether this represents a key of the dictionary.
- This replaces the reference to the \$-child with a flag (one bit) and saves space.
- For bitstrings, we then do not need \$, and so every node has at most two children. Can express 0-child and 1-child implicitly via left and right child.





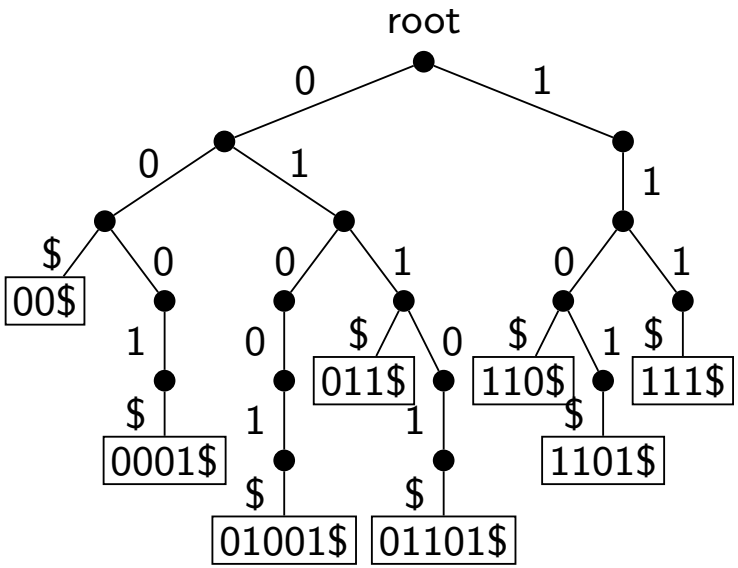
Outline

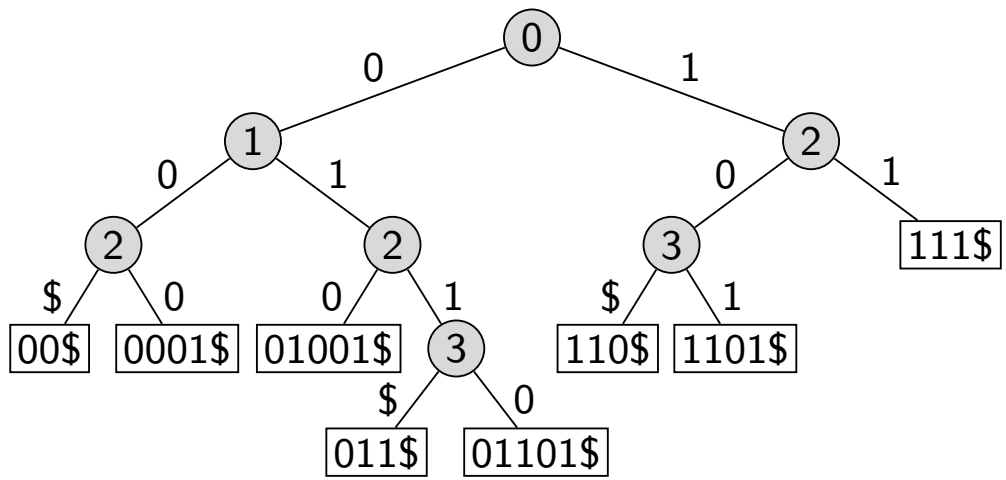
Compressed Tries (Patricia Tries)

- Morrison (1968):
  Patricia: Practical Algorithm to Retrieve Information Coded in Alphanumeric
- **Idea:** eliminate unflagged nodes with only one child to reduce space
- Every path of one-child unflagged nodes is compressed to a single edge
- Each node stores an *index* indicating the next bit to be tested during a search (index $= 0$ for the first bit, index $= 1$ for the second bit, etc.)
- A compressed trie storing n keys always has at most $n - 1$ internal (non-leaf) nodes

Compressed Tries Example

Example: A trie and the equivalent compressed trie.





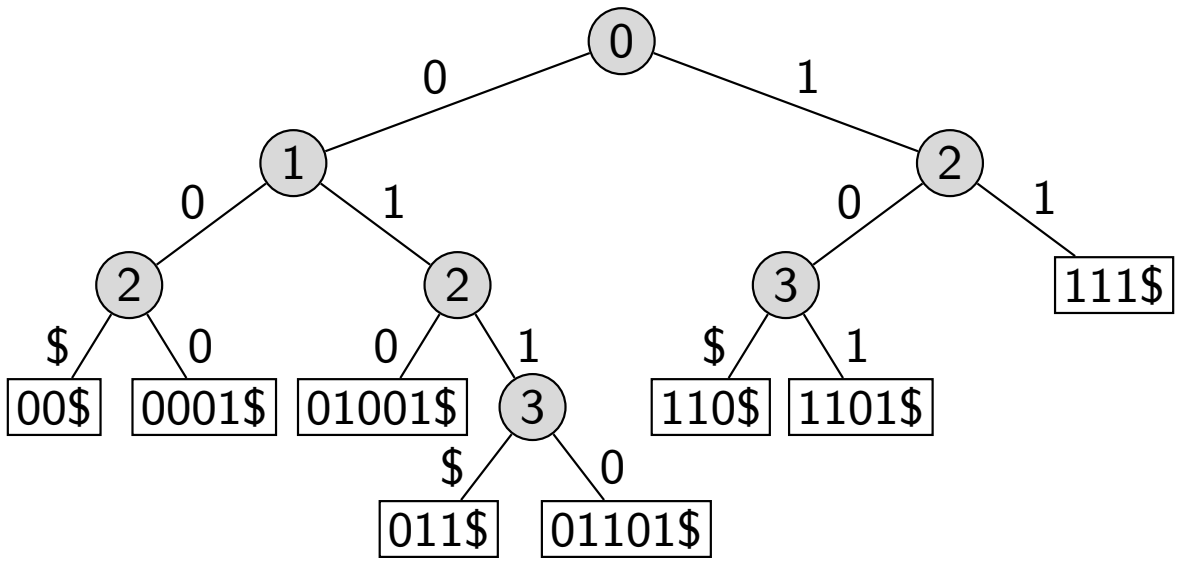
Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in x;
  return failure if the link is missing
- if we reach a leaf, expicitly check whether word stored at leaf is x
- else recurse on the new node and the next bit of x

```
Patricia-Trie-search(v ← root, x)
v: node of trie; x: word
 1.    if v is a leaf
 2.        return strcmp(x, key(v))
 3.    else
 4.        let d be the bit stored at v
 5.        let c be child of v labelled with x[d]
 6.        if there is no such child
 7.            return "not found"
 8.        else Patricia-Trie-search(c, x)
```

Compressed Tries: Search Example

Example: Search(10$) **unsuccessful**



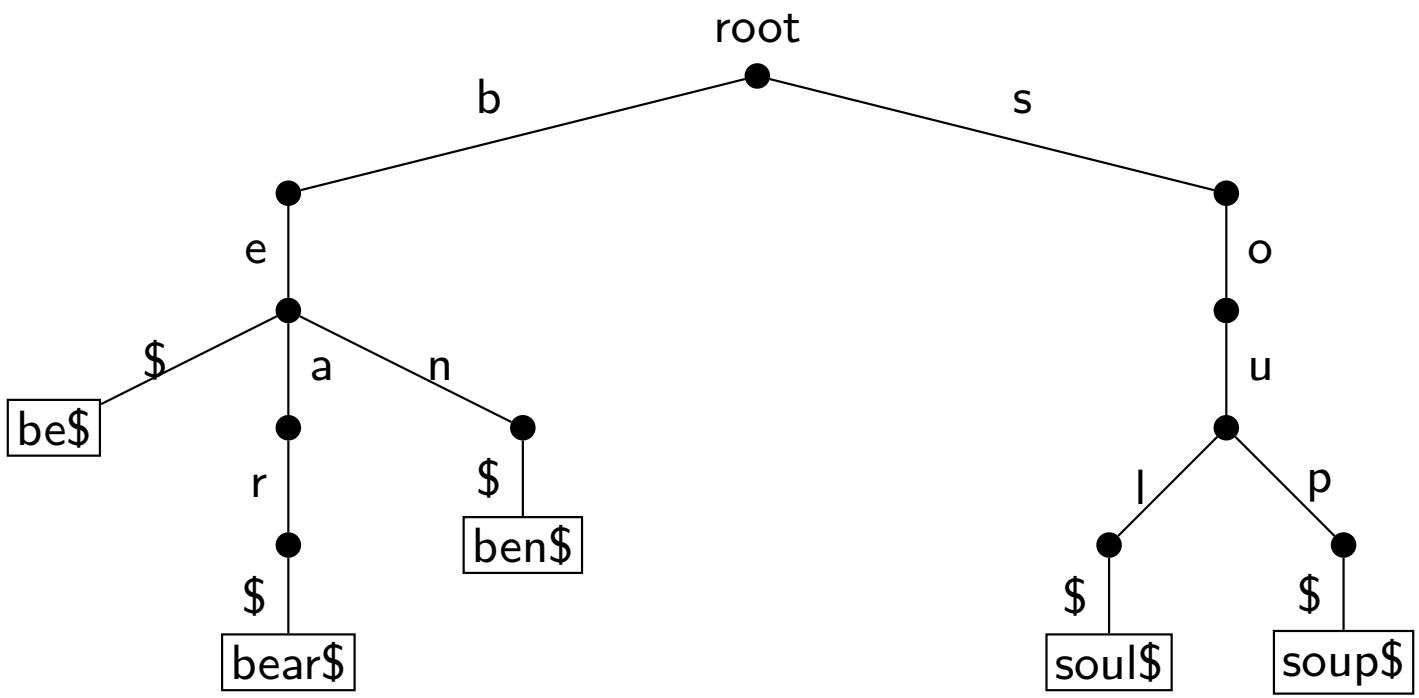
Compressed Tries: Insert & Delete

- **Delete(x)**:
  - Perform Search(x)
  - Remove the node v that stored x
  - Compress along path to v whenever possible.
- **Insert(x)**:
  - Perform Search(x)
  - Let v be the node where the search ended.
  - Conceptually simplest approach:
    - ★ Uncompress path from root to v.
    - ★ Insert x as in an uncompressed trie.
    - ★ Compress paths from root to v and from root to x.

    But it can also be done by only adding those nodes that are needed, see the textbook for details.
- All operations take $O(|x|)$ time.

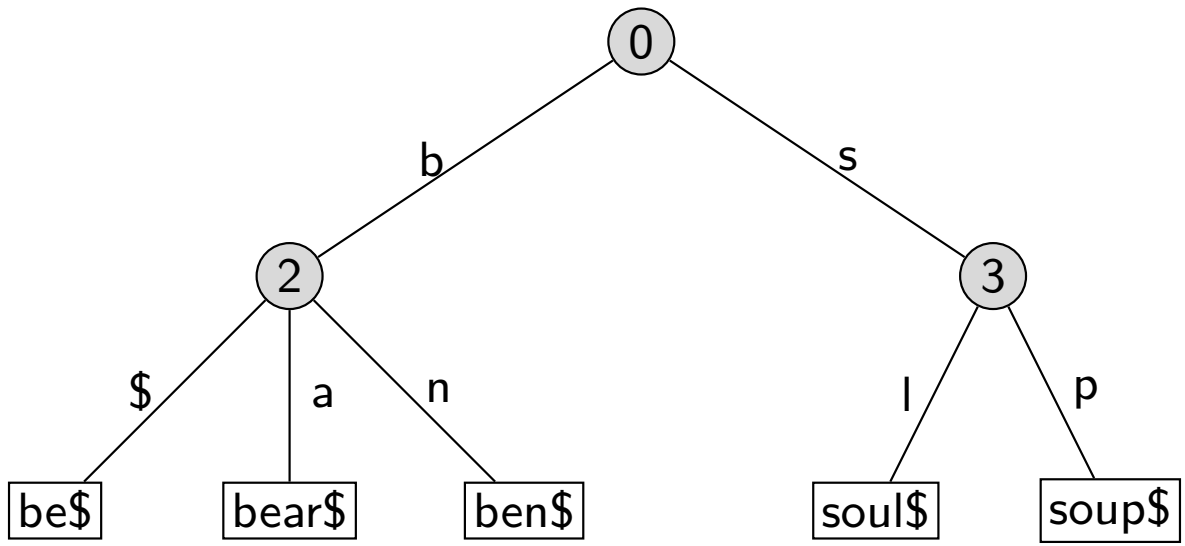
Multiway Tries: Larger Alphabet

- To represent **Strings** over any **fixed alphabet** Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



Compressed Multiway Tries

- **Compressed** multi-way tries
- Example: A compressed trie holding strings {bear$, ben$, be$, soul$, soup$}



Multiway Tries: Summary

- Operations Search(x), Insert(x) and Delete(x) are exactly as for tries for bitstrings.
- Run-time $O(|x| \cdot (\text{time to find the appropriate child}))$
- Each node now has up to $|\Sigma| + 1$ references to children. How should they be stored?
  - Could store array of size $|\Sigma| + 1$ at each node. $O(1)$ time to find the appropriate child, but then for n nodes the total space is $O(|\Sigma|n)$.
  - Could store list of children at each node. Then total space is $O(n)$, but time to find child increases to $O(|\Sigma|)$. Use MTF!
  - Could use some good dictionary implementation (AVL-tree? Skip list?) at each node. Then the total space is $O(n)$ and the time to find a child is $O(\log |\Sigma|)$ or better. Best in theory, but in practice not worth it unless the alphabet is huge.