MA2C03: TUTORIAL 4 PROBLEM SHEET

1) (From the 2016-2017 Annual Exam) Let $f : [-2,2] \to [-15,1]$ be the function defined by $f(x) = x^2 + 3x - 10$ for all $x \in [-2,2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

Injectivity: $f(x) = x^2 + 3x - 10 = (x-2)(x-5)$ This function is not injective on the interval $[-2,2]$. Acceptable justifications: drawing the graph, providing two values $x_1, x_2 \in [-2,2]$, $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$, applying Rolle's theorem (noticing that $f'(x) = 2x+3$ so $f'\left(-\frac{3}{2}\right) = 0$, and $\frac{3}{2} \in [-2,2]$), etc.

Surjectivity: $f(x) = x^2 + 3x - 10$ is not surjective on the interval $[-2,2]$. Acceptable justifications: drawing the graph, providing a value in $[-15,1]$ that $f(x)$ does not assume, showing the minimum value occurs at $\frac{3}{2}$, where $f\left(\frac{3}{2}\right) = -12.25 > -15$, etc.

2) Use mathematical induction to prove the geometric series formula, which states that for any $a, r \in \mathbb{R}$ with $r \neq 1$ and any $n \in \mathbb{N}^*$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{(1-r^n)}{(1-r)}.$$

Fix $a, r \in \mathbb{R}$, $r \neq 1$.
Base case: $n = 1$**.**
Then

$$a\frac{(1-r^1)}{(1-r)} = a(1) = a$$

as required.
Induction step: Assume true for $n = k$**.**
Prove true for $n = k+1$**.**

$$a + ar + ar^2 + \cdots + ar^{k-1} + ar^k = a\frac{(1-r^k)}{(1-r)} + ar^k$$

$$= a\left(\frac{(1-r^k)}{(1-r)} + \frac{(1-r)r^k}{(1-r)}\right) = a\left(\frac{(1-r^k) + (r^k - r^{k+1})}{(1-r)}\right)$$

$$= a\frac{(1 - r^{k+1})}{(1 - r)}$$

as required.

3) In the 1730's, the "Grande Loge" of Freemasons in Paris was a highly secretive society following some rather bizarre rules. Each of the freemasons in the lodge had shaved one other member. No freemason in the lodge had ever shaved himself. Furthermore, no freemason was ever shaved by more than one member of the lodge. There was one freemason who had never been shaved by any other member of the lodge. The number and identity of the freemasons in the lodge was kept secret. One rumour circulating in Paris at that time was that there were less than a hundred freemasons in the "Grande Loge." Another rumour put the number at over a hundred. Which one of the two rumours is true? Justify your answer.

Postponed to next week's tutorial.