Module 9: String Matching

CS 240 - Data Structures and Data Management

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Based on lecture notes of many previous cs240 instructors

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Pattern Matching

- Search for a string (pattern) in a large body of text
- $T[0..n-1]$ – The **text** (or **haystack**) being searched within
- $P[0..m-1]$ – The **pattern** (or **needle**) being searched for
- Strings over **alphabet** Σ
- Return the first i such that

$$P[j] = T[i+j] \quad \text{for} \quad 0 \le j \le m-1$$

- This is the first **occurrence** of P in T
- If P does not **occur** in T, return `FAIL`
- Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining

Pattern Matching

Example:

- $T =$ "Where is he?"
- $P_1 =$ "he"
- $P_2 =$ "who"

Definitions:

- **Substring** $T[i..j]$ $0 \leq i \leq j < n$: a string of length $j - i + 1$ which consists of characters $T[i], \ldots T[j]$ in order
- A **prefix** of T:
  a substring $T[0..i]$ of T for some $0 \leq i < n$
- A **suffix** of T:
  a substring $T[i..n-1]$ of T for some $0 \leq i \leq n-1$

General Idea of Algorithms

Pattern matching algorithms consist of **guesses** and **checks**:

- A **guess** is a position i such that P might start at $T[i]$. Valid guesses (initially) are $0 \leq i \leq n - m$.
- A **check** of a guess is a single position j with $0 \leq j < m$ where we compare $T[i + j]$ to $P[j]$. We must perform m checks of a single **correct** guess, but may make (many) fewer checks of an **incorrect** guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.

Brute-force Algorithm

Idea: Check every possible guess.

BruteforcePM($T[0..n-1], P[0..m-1]$)
T: String of length n (text), P: String of length m (pattern)

```
for i ← 0 to n − m do
    match ← true
    j ← 0
    while j < m and match do
        if T[i + j] = P[j] then
            j ← j + 1
        else
            match ← false
    if match then
        return i
return FAIL
```

Example

- Example: $T =$ `abbbababbab`, $P =$ `abba`

a	b	b	b	a	b	a	b	b	a	b
a	b	b	**a**							
	a									
		a								
			a							
				a	b	**b**				
					a					
						a	b	b	a	

- What is the worst possible input?
  $P = a^{m-1}b, \ T = a^n$
- Worst case performance $\Theta((n-m+1)m)$
- $m \leq n/2 \Rightarrow \Theta(mn)$

Pattern Matching

More sophisticated algorithms

- **KMP** and **Boyer-Moore**
- Do extra **preprocessing** on the pattern P
- We **eliminate guesses** based on completed matches and mismatches.

KMP Algorithm

- Knuth-Morris-Pratt algorithm (1977)
- Compares the pattern to the text in **left-to-right**
- **Shifts** the pattern more **intelligently** than the brute-force algorithm
- When a mismatch occurs, what is the **most** we can shift the pattern (reusing knowledge from previous matches)?

$T =$	a	b	c	d	c	a	b	c	?	?	?
	a	b	c	d	c	a	b	a			
						a	b	c	d	c	a

- **KMP Answer**: the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$

KMP Failure Array

- Preprocess the pattern to find matches of prefixes of the pattern with the pattern itself
- The **failure array** F of size m: $F[j]$ is defined as the length of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- $F[0] = 0$
- If a **mismatch** occurs at $P[j] \neq T[i]$ we set $j \leftarrow F[j-1]$
- Consider $P =$ `abacaba`

j	$P[1..j]$	P	$F[j]$
0	—	`abacaba`	0
1	`b`	`abacaba`	0
2	`ba`	`abacaba`	1
3	`bac`	`abacaba`	0
4	`baca`	`abacaba`	1
5	`bacab`	`abacaba`	2
6	`bacaba`	`abacaba`	3

KMP Algorithm

KMP(T, P)
T: String of length n (text), P: String of length m (pattern)

```
F ← failureArray(P)
i ← 0
j ← 0
while i < n do
    if T[i] = P[j] then
        if j = m − 1 then
            return i − j //match
        else
            i ← i + 1
            j ← j + 1
    else
        if j > 0 then
            j ← F[j − 1]
        else
            i ← i + 1
return −1 // no match
```

KMP: Example

$P = \texttt{abacaba}$

$T = \underline{\texttt{abaxyabacabb}}\texttt{aababacaba}$

0	1	2	3	4	5	6	7	8	9	10	11
a	b	a	x	y	a	b	a	c	a	b	b
a	b	a	c								
		(a)	b								
			a								
				a							
					a	b	a	c	a	b	a
									(a)	(b)	a

Exercise: continue with $T = \texttt{abaxyabacabb}\underline{\texttt{aababacaba}}$

Computing the Failure Array

failureArray(P)
P: String of length m (pattern)

```
F[0] ← 0
i ← 1
j ← 0
while i < m do
    if P[i] = P[j] then
        F[i] ← j + 1
        i ← i + 1
        j ← j + 1
    else if j > 0 then
        j ← F[j − 1]
    else
        F[i] ← 0
        i ← i + 1
```

KMP: Analysis

failureArray

- At each iteration of the while loop, either
  1. i increases by one, or
  2. the **guess index** $i - j$ increases by at least one ($F[j-1] < j$)
- There are no more than $2m$ iterations of the while loop
- Running time: $\Theta(m)$

KMP

- failureArray can be computed in $\Theta(m)$ time
- At each iteration of the while loop, either
  1. i increases by one, or
  2. the **guess index** $i - j$ increases by at least one ($F[j-1] < j$)
- There are no more than $2n$ iterations of the while loop
- Running time: $\Theta(n)$

KMP: Another Example

- $T =$abacaabaccabacabaabb
- $P =$abacab

Boyer-Moore Algorithm

Based on three key ideas:

- **Reverse-order searching**: Compare P with a subsequence of T moving backwards
- **Bad character jumps**: When a mismatch occurs at $T[i] = c$
  - If P contains c, we can shift P to align the last occurrence of c in P with $T[i]$
  - Otherwise, we can shift P to align $P[0]$ with $T[i+1]$
- **Good suffix jumps**: If we have already matched a suffix of P, then get a mismatch, we can shift P forward to align with the previous occurence of that suffix (with a mismatch from the actual suffix). Similar to failure array in KMP.
- When a mismatch occurs, Boyer-Moore chooses whichever of **bad character** or **good suffix** shifts the pattern further to the right.
- Can skip large parts of T

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e

T = b o y e r m o o r e

				e					
				[r]	**e**				
					[m]	o	o	r	e

7 comparisons (checks)

Good suffix examples

$P =$ sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							$\times$	(e)	(l)	(l)	(s)							

$P =$ odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
							(o)	(d)										

- Crucial ingredient: longest suffix of $P[i+1..m-1]$ that occurs in P.

Last-Occurrence Function

- **Preprocess** the pattern P and the alphabet Σ
- Build the **last-occurrence function** L mapping Σ to integers
- $L(c)$ is defined as
  - the largest index i such that $P[i] = c$ or
  - -1 if no such index exists
- Example: $\Sigma = \{a, b, c, d\}, P = abacab$

c	a	b	c	d
$L(c)$	4	5	3	-1

- The last-occurrence function can be computed in time $O(m + |\Sigma|)$
- In practice, L is stored in a size-$|\Sigma|$ array.

Suffix skip array

- Again, we **preprocess** P to build a table.
- **Suffix skip array** S of size m: for $0 \leq i < m$, $S[i]$ is the largest index j such that $P[i+1..m-1] = P[j+1..j+m-1-i]$ **and** $P[j] \neq P[i]$.
- **Note**: in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.

Example: $P =$ bon**obobo**

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$	-6	-5	-4	-3	2	-1	2	6

- Computed similarly to KMP failure array in $\Theta(m)$ time.

Boyer-Moore Algorithm

boyer-moore(T,P)
1. $L \leftarrow$ last occurrance array computed from P
2. $S \leftarrow$ suffix skip array computed from P
3. $i \leftarrow m-1, \quad j \leftarrow m-1$
4. **while** $i < n$ **and** $j \geq 0$ **do**
5.   **if** $T[i] = P[j]$ **then**
6. $i \leftarrow i-1$
7. $j \leftarrow j-1$
8.   **else**
9. $i \leftarrow i+m-1-\min(L[T[i]], S[j])$
10. $j \leftarrow m-1$
11. **if** $j = -1$ **return** $i+1$
12. **else return** `FAIL`

Exercise: Prove that $i - j$ always increases on lines 9–10.

Boyer-Moore algorithm conclusion

- Worst-case running time $\in O(n + |\Sigma|)$
- This complexity is difficult to prove.
- What is the worst case?
- On typical **English text** the algorithm probes approximately 25% of the characters in T
- Faster than KMP in practice on English text.

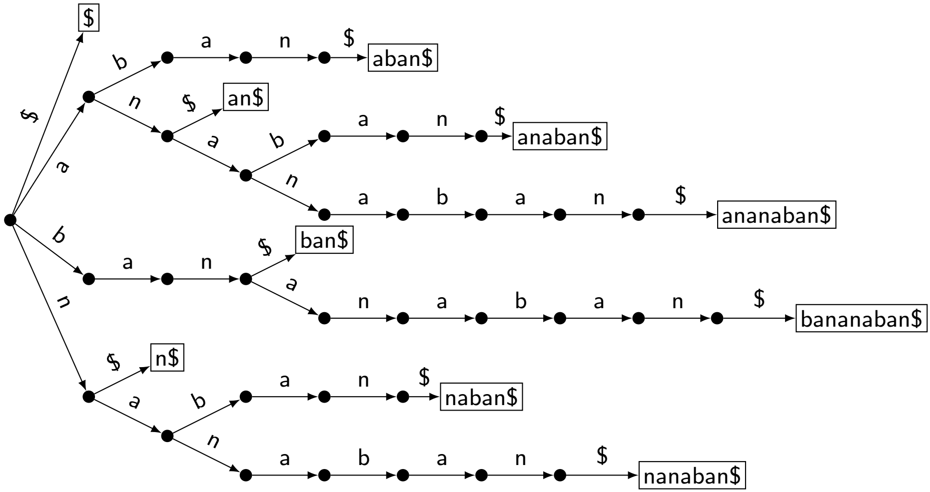
Tries of Suffixes and Suffix Trees

- What if we want to search for **many patterns** P within the same **fixed text** T?
- Idea: Preprocess the text T rather than the pattern P
- Observation: P is a substring of T if and only if P is a prefix of some suffix of T.
- So want to store all suffixes of T in a trie.
- To save space:
  - Use a compressed trie.
  - Store suffixes implicitly via indices into T.
- This is called a **suffix tree**.

Trie of suffixes: Example

$T =$ `bananaban` has suffixes

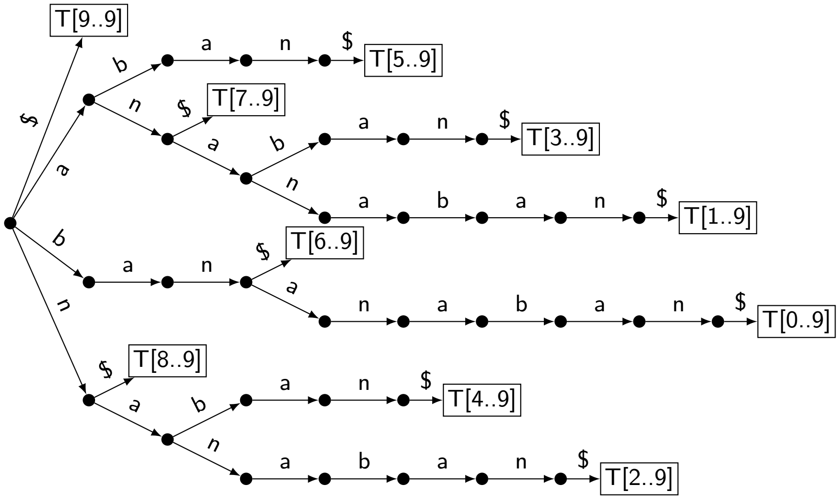
$\{$`bananaban`, `ananaban`, `nanaban`, `anaban`, `naban`, `aban`, `ban`, `an`, `n`, $\Lambda\}$



Tries of suffixes

Store suffixes via indices:

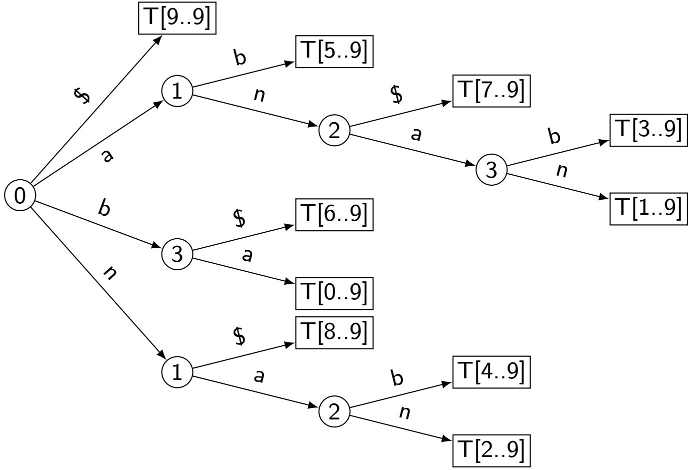
	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$



Suffix tree

Suffix tree: Compressed trie of suffixes

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	$



Building Suffix Trees

- Text T has n characters and $n+1$ suffixes
- We can build the suffix tree by inserting each suffix of T into a compressed trie.
  This takes time $\Theta(n^2)$.
- There *is* a way to build a suffix tree of T in $\Theta(n)$ time.
  This is quite complicated and beyond the scope of the course.

Suffix Trees: String Matching

Assume we have a suffix tree of text T.

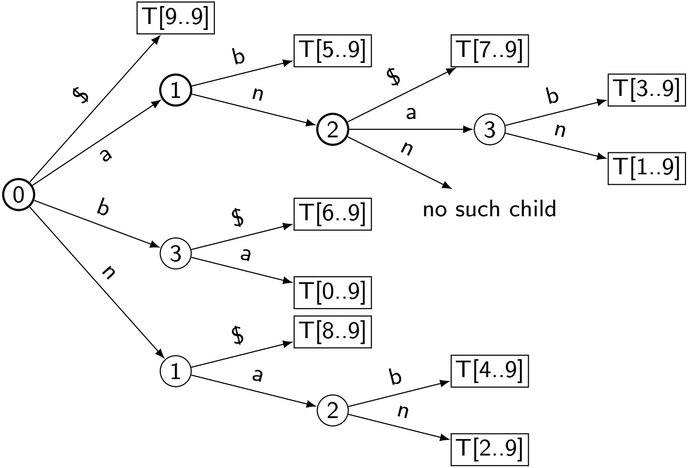
To search for pattern P of length m:

- We assume that P does not have the final \$.
- P is the prefix of some suffix of T.
- In the *uncompressed* trie, searching for P would be easy: P exists in T if and only search for P reaches a node in the trie.
- In the suffix tree, search for P until one of the follow occurs:
  1. If search fails due to *"no such child"* then P is not in T
  2. If we reach end of P, say at node v, then jump to leaf ℓ in subtree of v. (We presume that suffix trees stores such shortcuts.)
  3. Else we reach a leaf $\ell = v$ while characters of P left.
- Either way, left index at ℓ gives the shift that we should check.
- This takes $O(|P|)$ time.

Pattern Matching in Suffix Tree: Example 1

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$

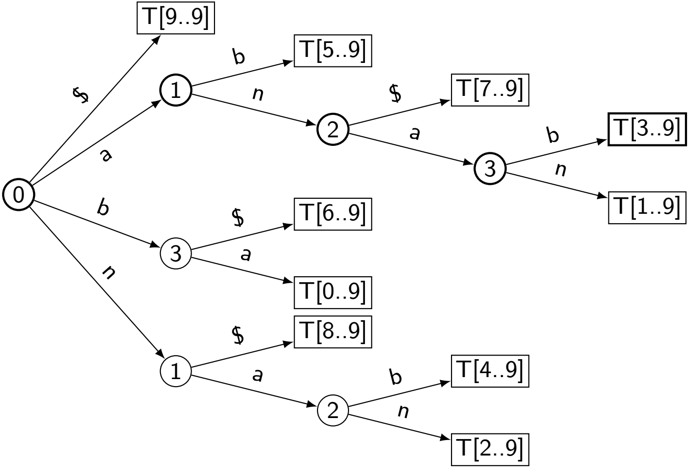
$P =$ `ann`



Pattern Matching in Suffix Tree: Example 2

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	$

$P =$ `ana`

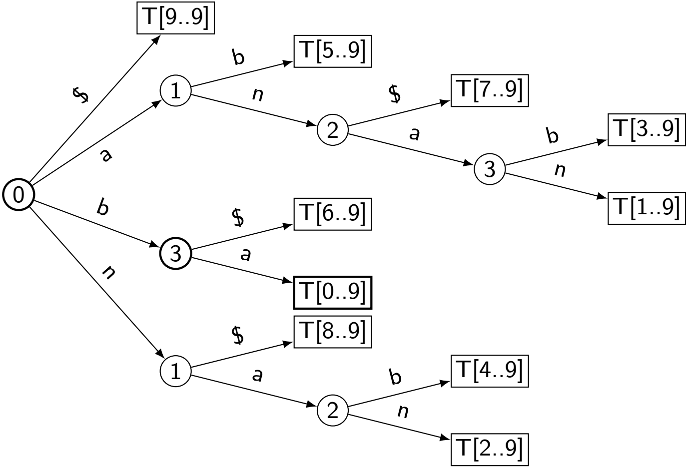


Pattern Matching in Suffix Tree: Example 3

$T =$

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$

$P =$ `briar`



Pattern Matching Conclusion

	Brute-Force	KMP	Boyer-Moore	Suffix trees
Preprocessing:	–	$O(m)$	$O(m + \|\Sigma\|)$	$O(n^2)$
Search time:	$O(nm)$	$O(n)$	$O(n)$ (often better)	$O(m)$
Extra space:	–	$O(m)$	$O(m + \|\Sigma\|)$	$O(n)$