# MA2C03: TUTORIAL 14 PROBLEMS
# GRAPH THEORY

1) Let $(V, E)$ be the graph with vertices $a$, $b$, $c$, $d$, and $e$ and edges $ab$, $bd$, $be$, $ac$, $cd$, and $ae$.

(a) Is this graph a tree? Justify your answer.

(b) If it is not a tree, how many distinct spanning trees does it have?

2) Consider the statement "A graph $(V, E)$ is a tree $\iff$ $\#(E) = \#(V) - 1$." What hypothesis is needed for this equivalence to be true? Give an example to show why this hypothesis is necessary.

Recall that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

read as " n choose k" gives the number of distinct combinations of k objects taken out of a possible n objects for $n \geq k \geq 0$ with the convention $0! = 1$.

3) Consider the complete graph $K_n$ for $n = 2, 3, 4, 5$. In each of the four cases

(a) Is this graph a tree? Justify your answer.

(b) If it is not a tree, how many distinct spanning trees does it have?

(Hint: How many edges does $K_n$ have?)

**Solution:** 1)(a) The graph is at the end of the solutions. It is not a tree as it contains circuits abea and abdca.

(b) There are 11 possibilities total as follows:

- Eliminate ab and one of ae, ac, cd, bd, or be (5 possibilities).
- Eliminate ae and one of ac, cd, or bd (eliminating ab gives a previously counted tree), so we have 3 possibilities in this case.
- Eliminate be and one of ac, cd, or bd (eliminating ab gives a previously counted tree), so we have 3 possibilities in this case as well.

2) The missing hypothesis is "connected." If the graph $(V, E)$ is not connected we could have something like the graph with vertices $a$, $b$, $c$, $d$, and $e$ and edges $ab$, $bc$, $cd$, and $da$, where the vertex $e$ is isolated. This graph has 5 vertices and 4 edges, but it contains the circuit abcda, so it is not acyclical, and it has two connected components, so it is not connected. Therefore, it cannot be a tree.

3) In a complete graph $K_n$ every vertex is connected to every other vertex, so the degree of every vertex is $n-1$. We have n vertices, so the number of edges in $K_n$ is $\frac{n(n-1)}{2}$ as each edge is counted twice. Out of $\frac{n(n-1)}{2}$ edges, we are supposed to choose $n-1$ to construct a spanning tree as we have $n$ vertices, so a tree connecting them has $n-1$ edges. Therefore, we first check whether our $K_n$ has any circuits. If it does not, it is a tree. If it does, then the count

$$\binom{\frac{n(n-1)}{2}}{n-1}$$

gives the number of ways $n-1$ edges can be chosen, but in certain configurations depending on $n$, we can get graphs $(V, E)$ satisfying $\#(E) = \#(V) - 1$ that are not connected (as we saw in the previous problem). We have to count those and subtract them from

$$\binom{\frac{n(n-1)}{2}}{n-1}$$

in order to get the number of distinct spanning trees.

$n = 2$ We have 2 vertices and 1 edge, so $K_2$ is a tree and hence its own spanning tree (1 choice of spanning tree).

$n = 3$ We have 3 vertices and 3 edges, $K_3$ contains a circuit, so it is not a tree. The number of distinct spanning trees is

$$\binom{3}{2} = \frac{3!}{1!\,2!} = 3$$

as it is not possible in this case to construct subgraphs of $K_3$ with 3 vertices and 2 edges that are disconnected.

$n = 4$ We have 4 vertices and $\frac{4 \cdot 3}{2} = 6$ edges, $K_4$ contains a number of circuits, so it is not a tree. The number of ways we can choose 3 edges out of 6 is

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20,$$

but there are

$$4 = \binom{4}{1}$$

different disconnected subgraphs of $K_4$ consisting of a triangle plus an isolated point. Those are not spanning trees of $K_4$, so the number of distinct spanning trees is

$$\binom{6}{3} - \binom{4}{1} = \frac{6!}{3!\,3!} - 4 = 20 - 4 = 16.$$

$n = 5$ We have 5 vertices and $\dfrac{5 \cdot 4}{2} = 10$ edges, $K_5$ contains a number of circuits, so it is not a tree. First of all,

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{7 \cdot 8 \cdot 8 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4} = 210.$$

Now, we have to figure out how many subgraphs of $K_5$ with 5 vertices and 4 edges are there that are disconnected. We could have a rectangle plus an isolated point. There are

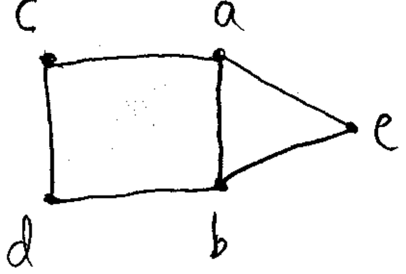
$$5 = \binom{5}{1}$$

of those. Also, we could have a triangle plus two vertices connected by an edge. There are

$$\binom{5}{2} = \frac{4 \cdot 5}{1 \cdot 2} = 10$$

such subgraphs of $K_5$. Therefore, the number of distinct spanning trees is

$$\binom{10}{4} - \binom{5}{1} - \binom{5}{2} = 210 - 5 - 10 = 195.$$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