**MA2C03: ASSIGNMENT 1SOLUTIONS**

1) For any sets $A$ and $B$, define $A \,\Delta\, B$, the **symmetric difference** of $A$ and $B$ to be the set $(A - B) \cup (B - A)$. Prove that intersection $\cap$ distributes over $\Delta$ : $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$ for all sets $A$, $B$, and $C$ using the proof methods employed in lecture. Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

**Solution:** Before we begin, let us recall several useful statements from set theory:

(1) For any $D$, $E$ sets, $D \setminus E = D \cap E^c$, where $E^c$ is the complement of $E$.

(2) For any $D$, $E$, and $F$ sets, $\cap$ distributes over $\cup$, namely
$$D \cap (E \cup F) = (D \cap E) \cup (D \cap F).$$

(3) For any set $D$, $D \cap D^c = \emptyset$.

(4) For any $D$, $E$ sets, $(D \cap E)^c = D^c \cup E^c$ (one of the de Morgan laws in set theory).

We will reduce the left-hand side to a certain expression, then prove that the right-hand side equals the same expression.

$$\begin{aligned} A \cap (B \,\Delta\, C) &= A \cap [(B \setminus C) \cup (C \setminus B)] = A \cap [(B \cap C^c) \cup (C \cap B^c)] \\ &= (A \cap B \cap C^c) \cup (A \cap C \cap B^c). \end{aligned}$$

We have used the definition of $\Delta$ for the first equality, statement (1) for the second, and statement (2) plus the associativity of $\cap$ proven in lecture for the third equality.

$$\begin{aligned} (A \cap B) \,\Delta\, (A \cap C) &= [(A \cap B) \setminus (A \cap C)] \cup [(A \cap C) \setminus (A \cap B)] \\ &= [(A \cap B) \cap (A \cap C)^c] \cup [(A \cap C) \cap (A \cap B)^c] \\ &= [(A \cap B) \cap (A^c \cup C^c)] \cup [(A \cap C) \cap (A^c \cup B^c)] \\ &= [(A \cap B \cap A^c) \cup (A \cap B \cap C^c)] \cup [(A \cap C \cap A^c) \cup (A \cap C \cap B^c)] \\ &= (A \cap B \cap C^c) \cup (A \cap C \cap B^c). \end{aligned}$$

We have used the definition of $\Delta$ for the first equality, statement (1) for the second, statement (4) for the third and statement (2) plus the associativity of $\cap$ proven in lecture for the fourth equality, and finally statement (3) to conclude $A \cap C \cap A^c = \emptyset$ and $A \cap C \cap A^c = \emptyset$ in the fifth equality. By the way, recall that we proved in lecture that $\emptyset$ is the identity element for the binary operation $\cup$.

We have thus shown that

$$A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C) = (A \cap B \cap C^c) \cup (A \cap C \cap B^c).$$

**Grading rubric:** 10 points total, 5 points for each of the directions if the statement is proven by inclusion in both directions. If the statement is proven as above by bringing the left-hand side and the right-hand side to the same expression, then 5 points for bringing each of the two sides to the same expression.

2) Let $\mathbb{R}$ be the set of real numbers. For $x, y \in \mathbb{R}$, $x \sim y$ iff $x - y \in \mathbb{Q}$, i.e., if the difference $x - y$ is a rational number. Determine:

(i) Whether or not the relation $\sim$ is *reflexive*;
(ii) Whether or not the relation $\sim$ is *symmetric*;
(iii) Whether or not the relation $\sim$ is *anti-symmetric*;
(iv) Whether or not the relation $\sim$ is *transitive*;
(v) Whether or not the relation $\sim$ is an *equivalence relation*;
(vi) Whether or not the relation $\sim$ is a *partial order*.

Justify your answers.

**Solution:** Here it helps to remember that $(\mathbb{Q}, +, 0)$ is a group.

(i) $\sim$ is reflexive because for all $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Q}$.

(ii) $\sim$ is symmetric because if $x \sim y$, then $x - y = q \in \mathbb{Q}$, so $y - x = -q \in \mathbb{Q}$, which means $y \sim x$.

(iii) $\sim$ is not anti-symmetric. **Counterexample:** Let $x = 1$ and $y = 0$. Then $x - y = 1 - 0 = 1 \in \mathbb{Q}$, so $x \sim y$, while $y - x = -1 \in \mathbb{Q}$, so $y \sim x$ also holds. Clearly, $x \neq y$.

(iv) $\sim$ is transitive because for any $x, y, z \in \mathbb{R}$, if $x \sim y$ and $y \sim z$, then $x - y = p \in \mathbb{Q}$ and $y - z = q \in \mathbb{Q}$, which implies $x - z = (x - y) - (y - z) = p - q \in \mathbb{Q}$.

(v) $\sim$ is an equivalence relation because it is reflexive, symmetric, and transitive.

(vi) $\sim$ is not a partial order because it fails to be anti-symmetric as we saw above.

**Grading rubric:** 10 points total, (i)-(iv) are 2 points each with 1 point for the answer and 1 point for the justification, while (v) and (vi) are 1 point each.

3) Let $A$ be a set, and let $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$, where $I$ is an indexing set, be any partition of the set $A$. Define a relation $R$ on $A$ as follows: $x, y \in A$ satisfy $xRy$ iff $x, y \in A_\alpha$ for some $\alpha \in I$. In other words, $xRy$ iff $x$ and $y$ belong to the same set of the partition. Prove that $R$ is an equivalence relation and that the partition $R$ defines on $A$ is precisely the given partition $\mathcal{A}$. (Hint: Recall we discussed in lecture the one-to-one correspondence between partitions and equivalence relations, and this is the proof direction I sketched in lecture without providing the details.)

**Solution:** First, let us prove $R$ is an equivalence relation:

**Reflexivity:** For any $x \in A$, since $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$ is a partition of $A$, there exists $\alpha \in I$ such that $x \in A_\alpha$. The element $x$ is in the same set $A_\alpha$ as itself, so $xRx$.

**Symmetry:** If $xRy$, then by definition $x, y \in A_\alpha$ for some $\alpha \in I$, i.e., $x$ and $y$ belong to the same set of the partition. Therefore, $yRx$ holds as well.

**Transitivity:** If $xRy$, then by definition $x, y \in A_\alpha$ for some $\alpha \in I$. If $yRz$, then $z$ belongs to the same set of the partition as $y$, namely $z \in A_\alpha$ for the same $\alpha$. Thus, $x, y, z \in A_\alpha$, which means $xRz$ holds as well.

**The partition determined by $R$ is exactly $\mathcal{A}$:** If $x \in A_\alpha$, then the equivalence class of $x$ given by $[x]_R = A_\alpha$ by the very definition of $R$. Since $\mathcal{A}$ is a partition of $A$ and it consists of the set of equivalence classes determined by the relation $R$, we conclude $R$ determines the partition $\mathcal{A}$ as needed.

**Grading rubric:** 10 points total, 6 points for proving $R$ is an equivalence relations (2 points each for reflexivity, symmetry, and transitivity) and 4 points for showing it defines the partition on $A$ that we started with.

4) Where is the fallacy in the following argument by induction? Justify your answer.

**Statement:** For every non-negative integer $k$, $2 \times k = 0$.

**"Proof:"** We give a proof using strong induction on $k$. Denote by $P(k)$ the statement "if $k$ is non-negative integer, then $2 \times k = 0$."

**Base case:** Show $P(0)$. Obviously, $2 \times 0 = 0$.

**Inductive step:** Assume $P(n)$ is true for every $n$ such that $0 \leq n \leq k$ (the strong induction hypothesis). We have to show that $P(k+1)$ also holds. We write $k+1 = i+j$, where $i$ and $j$ are non-negative integers. By the inductive hypothesis,

$$2(k+1) = 2(i+j) = 2i + 2j = 0 + 0 = 0.$$

Therefore, by induction, $P(k)$ is true for all $k \in \mathbb{N}$, so $2 \times 1 = 0$.

**Solution:** The argument fails at the level of the inductive step because for $k+1 = i+i$ it is not necessarily the case that both $i < k$ AND $j < k$, so the inductive hypothesis might not apply. For example, passing from $k = 0$ to $k+1 = 1$, we see that $i = 0$ and $j = 1 = k+1$, so the inductive hypothesis does not apply to $j$.

**Grading rubric:** 10 points total, 5 points for realizing that the failure is at the level of $k = 1$ and 5 points for the explanation.

5) Use mathematical induction to prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

(Hint: Recall we proved in lecture that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.)

**Solution:** We first prove the base case and then the inductive case.

**Base case:** For $n = 1$, $1^3 = 1^2 = 1$.

**Inductive step:** Assume $1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2$ and seek to prove that $1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = (1 + 2 + \cdots + n + n + 1)^2$.

By the inductive hypothesis and the hint,

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1 + 2 + \cdots + n)^2 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{2^2} + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + (n+1)\right) \\ &= (n+1)^2\left(\frac{n^2}{4} + \frac{4(n+1)}{4}\right) = (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\ &= (n+1)^2\frac{(n+2)^2}{2^2} = \left(\frac{(n+1)(n+2)}{2}\right)^2 \\ &= (1 + 2 + \cdots + n + n + 1)^2 \end{aligned}$$

as needed.

**Grading rubric:** 10 points total, 2 for the base case and 8 for the inductive step.

6) Let $f : [-2, 0] \to [0, 1]$ be the function defined by $f(x) = \dfrac{1}{x^2 + 6x + 9}$ for all $x \in [-2, 0]$. Determine whether or not this function is injective

and whether or not it is surjective. Justify your answers. Recall that $[-2,0]$ is the set of all real numbers between $-2$ and 0 with the end-points of $-2$ and 0 included in the set.

**Solution:** $f(x) = \dfrac{1}{x^2+6x+9} = \dfrac{1}{(x+3)^2}$. Since the function is a square, its graph is symmetric about the line $x=-3$, where it has a vertical asymptote. The entire interval $[-2,0]$, the domain of the function, lies on one side of the asymptote $x=-3$, and the function is decreasing as $x$ increases (since $(x+3)^2$ is in the denominator, and the numerator is the constant 1). As a result, the function $f$ must be injective as $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
As we remarked above, the function $f$ is decreasing. To see what values it assumes on its domain $[-2,0]$, it thus suffices to check what values it assumes at its endpoints as it is clear the function is continuous (the asymptote $x=-3$ is not in the domain). We compute that $f(-2) = \dfrac{1}{(-2+3)^2} = \dfrac{1}{1} = 1$ and $f(0) = \dfrac{1}{(0+3)^2} = \dfrac{1}{3^2} = \dfrac{1}{9}$. We conclude the function cannot assume any values in $\left[0, \dfrac{1}{9}\right)$, which means it is not surjective.

**Grading rubric:** 10 points total, 5 points for proving injectivity with 2 for the answer and 3 for the justification and 5 points for disproving surjectivity with 2 for the answer and 3 for the justification.

7) Let $A = \{(x,y) \in \mathbb{R}^2 \mid 2x-3y=0\}$ with the operation of addition given by $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$.

(a) Is $(A,+)$ a semigroup? Justify your answer.
(b) Is $(A,+)$ a monoid? Justify your answer.
(c) Is $(A,+)$ a group? Justify your answer.

**Solution:** $A = \{(x,y) \in \mathbb{R}^2 \mid 2x-3y=0\}$ consists of a line through the origin in $\mathbb{R}^2$ determined by the vector $(3,2)$, so

$$A = \{(3,2) \in \mathbb{R}^2 \mid 2x-3y=0\} = \{t(x,y) = (tx,ty) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}.$$

The given operation is the addition of vectors in $\mathbb{R}^2$ (similar to an example given in class in $\mathbb{R}^3$).
(a) Yes, $(A,+)$ is a semigroup. Vector addition $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$ is a binary operation on $A$ because if $2x_1-3y_1=0$ and $2x_2-3y_2=0$, then adding the two equations and rearranging terms we get $2(x_1+x_2)-3(y_1+y_2)=0$. Therefore, $(x_1,y_1)+(x_2,y_2) \in A$. Likewise, vector addition is associative because addition is associative

on $\mathbb{R}$, so it holds at the level of each of the two components of the vector.
(b) Yes, $(A, +)$ is a monoid. We immediately see that $2 \times 0 - 3 \times 0 = 0 - 0 = 0$, so $(0, 0) \in A$. Clearly, $(0, 0)$ is the identity element as $(x, y) + (0, 0) = (0, 0) + (x, y) = (x, y)$.
(c) Yes, $(A, +)$ is a group. For any $(x, y) \in A$, $2x - 3y = 0$. Multiply this equation by $-1$ and rearrange terms to obtain $2(-x) + 3(-y) = 0$, so $(-x, -y) \in A$. Clearly, $(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$, so every element of $A$ is invertible as needed.

**Grading rubric:** 10 points total: part (a) 4 points, 1 point for the answer, 1 point for proving the operation is a binary operation (hence closed), and 2 points for proving the binary operation is associative; part (b) 1 point for the answer and 2 points for showing $(0, 0)$ is the identity element of $(A, +)$; part (c) 1 point for the answer and 2 points for showing $(-x, -y)$ is the inverse of $(x, y) \in A$.