Countability of Sets

Task:
Understand what it means for a set to be countable, countably infinite and uncountably infinite. Show that the set of all languages over a finite alphabet is uncountably infinite, whereas the set of all regular languages over a finite alphabet is countably infinite.

We want to understand sizes of sets. In the unit on functions last term, when we looked at functions defined on finite sets, we wrote down a set A with n elements as $A = \{a_1, a_2, \ldots, a_n\}$. This notation mimics the process counting: a_1 is the first element of A, a_2 is the second element of A and so on up to a_n is the n^{th} element of A.

In other words, another way of saying A is a set of n elements is that there exists a bijective function $f:A \to \{1, 2, \ldots, n\}$, let $J_n = \{1, 2, \ldots, n\}$.

Definition: A set A has n elements $\Leftrightarrow$ $\exists$ $f:A \to J_n$ a bijection.
NB: This definition works $\forall n \geq 1$, $n \in N^*$.

Notation: $\exists$ $f:A \to J_n$ a bijection is denoted as $A \sim J_n$.
More generally, $A \sim B$ means $\exists$ $f:A \to B$ a bijection, and it is a relation on sets. In fact, it is an equivalence relation (check!). $[J_n]$ is the equivalence class of all sets A of size n, i.e. $\#(A) = n$.

Definition: A set A is **finite** if $A \sim J_n$ for some $n \in N^*$ or $A = \varnothing$.
Definition: A set A is **infinite** if A is not finite.

Examples: N, Q, R, etc.
To understand sizes of infinite sets, generalise the construction above. Let $J = N^* = \{1, 2, \ldots\}$

Definition: A set A is **countably infinite** if $A \sim J$.
Definition: A set A is **uncountably infinite** if A is neither finite nor countably infinite.

In fact, we often treat together the cases A is finite or A is countably infinite since in both of these cases the counting mechanism that is so familiar to us works. Therefore, we have the following definition:

Definition: A set A is countable if A is finite ($A \sim J_n$ or $A = \varnothing$) or A is countably infinite.

There is a difference in terminology regarding countability between CS sources (textbooks, articles, etc.) and maths sources. Here is the dictionary:

CS	Maths
Countable	At Most Countable
Countably Infinite	Countable
Uncountably Infinite	Uncountable

It’s best to double check which terminology a source is using.

Goal: Characterise:

1. [N], the equivalence class of countably infinite sets, and
2. [R], the equivalence class of uncountably infinite sets the same size as R.

Bad News: Both [N] and [R] consist of infinite sets.
Good News: We only care about these two equivalence classes.

NB: There are uncountably infinite sets of size bigger than [R] that can be obtained from the power set construction, but it is unlikely you will see them in your CS coursework.

To characterise [N], we need to recall the notion of a sequence.

Sequences

Definition:
A **sequence** is a set of elements $\{x_1, x_2, \ldots\}$ indexed by J, i.e. $\exists\ f:J \to \{x_1, x_2, \ldots\}$ s.t. $f(n) = x_n\ \forall n \in J$.

Recall that sequences and their limits were used to define various notions in calculus (differentiation, integration, etc.) Also, calculators use sequences in order to compute with various rational and irrational numbers.

Examples:

1. $\pi \approx 3.14$, $x_3 = 3.141$, $x_4 = 3.1415$, … $\lim n \to \infty\ x_n = \pi$
   π is irrational. $\pi \in R \setminus Q$. (Real Numbers \ Rational Numbers)
2. ⅓ ≈ 0.333… means we can set up the sequence of rational numbers
   $x_1 = 0$, $x_2 = 0.3$, $x_3 = 0.33$, … such that $\lim n \to \infty\ x_n = ⅓$. Note that $⅓ \in Q$.

Restatement of the definition of countably infinite:
A set A is countably infinite if its elements can be arranged in a sequence $\{x_1, x_2, \ldots\}$ such that $A = \{x_1, x_2, \ldots\}$. This is another way of saying A is in bijective correspondence with J, i.e. $\exists\ f:A \to J$ is a bijection, namely $A \sim J$.

Application of this restatement: $Z \sim N$.
Indeed we can write Z as a sequence since $Z = \{0, 1, -1, 2, -1, \ldots\}$ so $Z \in [N]$, Z is countably infinite like N.

Big difference between finite and infinite sets:
Let A, B be finite sets such that $A \subseteq B$, but $A \neq B$. Then, A !~ B since $\#(A) < \#(B)$ and J_n !~ J_m if $n \neq m$.

Let A, B be infinite sets such that $A \subseteq B$, but $A \neq B$. It is possible that $A \sim B$.
We saw this behaviour in Hilbert's hotel problem (Hilbert's Paradox of the Grand Hotel): where $N^* \subseteq N$ and $N^* \neq N$, but $N \sim N^*$ via the bijection $f:N \to N^*$ given by $f(n) = n+1$ so $\{0, 1, 2, \ldots\} \sim \{1, 2, \ldots\}$.
In the same vein, we get the following result:

Theorem: Every infinite subset of a countably infinite set is itself countably infinite.

Proof:
Let $E \subseteq A$ be the subset in question, where E is infinite, and A is countably infinite.
A is countably infinite $\Leftrightarrow A \sim J \Leftrightarrow A = \{x_1, x_2, \dots\}$

To show E is countably infinite, we want to show we can represent $E = \{x_{n1}, x_{n2}, \dots\}$.
We construct this sequence of n_j's from the indices of the elements of A in the enumeration $\{x_1, x_2, \dots\}$.
Let n_1 be the smallest integer in J such that $x_{n1} \in E \subseteq A$. We construct the rest of the sequence of n_j's by induction.
Say we have constructed $n_1, n_2, \dots, n_{k-1} \in N^*$. Let n_k be the smallest integer greater than n_{k-1} such that $x_{nk} \in E$. By construction $n_1 < n_2 < \dots$ and $E = \{x_{n1}, x_{n2}, ...\}$ (*q.e.d*)

Remark: $\{x_{n1}, x_{n2}, ...\}$ is called a **subsequence** of $\{x_1, x_2, ...\}$.

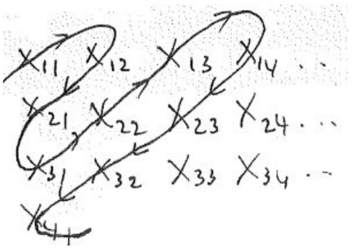
Algorithmic restatement of the previous proof:
Let $A = \{x_1, x_2, \dots\}$ be an enumeration of A (i.e. writing the countably infinite set A as a sequence). We process $\{x_1, x_2, \dots\}$ as a queue.
First look at x_1. If $x_1 \in E$, keep x_1 and let $n_1 = 1$. Otherwise, discard x_1.
Process each x_i in turn keeping only those that are in E.
Their indices from the subsequence $\{n_j\}_{j=1,2,...}$ where $E = \{x_{n1}, x_{n2}, ...\}$.

Next we want to show $Q \sim N$, the set of natural numbers is countably infinite.

Notation: A sequence $\{x_1, x_2, \dots\}$ can also be denoted by $\{x_i\}_{i=1,2,...}$

Theorem:
Let $\{A_n\}_{n=1,2,...}$ be a sequence of countably infinite sets. Let $S = \{A_1 \cup A_2 \cup \dots \cup A_\infty\}$.
Then S is countably infinite.

Proof:
Each A_n is countably infinite $\Leftrightarrow A_n \sim J\ \forall n \geq 1 \Leftrightarrow A_n = \{x_{nk}\}_{k=1,2,...} = \{x_{n1}, x_{n2}, ...\}$.
We use the indices like for the entries of a matrix. The first index tells us which A_n set the element belongs to, while the second index tells us where that element is in the enumeration (the counting) of A_n.


x_{11} x_{12} x_{13} x_{14} . .
x_{21} x_{22} x_{23} x_{24} . . .
x_{31} x_{32} x_{33} x_{34} . .
x_{41}

$\{x_{11}, x_{12}, x_{21}, x_{31}, x_{22}, x_{13}, x_{14}, x_{23}, x_{32}, x_{41}, ...\}$

$= \{A_1 \cup A_2 \cup \dots \cup A_\infty\} = S$ is countably infinite because even if some x_{ij}'s are the same.
$A_n \subseteq S\ \forall n \geq 1$ and $A_n \sim J$.

Corollary 1: Suppose an indexing set I is countable, and $\forall i \in I$, A_i is countable, then $T = \bigcup_{i \in I} A_i$ is countable.

Proof: The biggest set we can obtain here is when I is countably infinite and each A_i is countably infinite. By the previous theorem, T is countably infinite in that case. Therefore, T is at most countably infinite (may be finite if I is finite and each A_i is finite), so T is countable. (*q.e.d*)

Corollary 2: Let A be a countably infinite set, and let $A^n = A \text{ x } \dots \text{ x } A$ (n times) = $\{(a_1, a_2, \dots a_n) \mid a_1, a_2, \dots, a_n \in A\}$. Then A^n is countably infinite.

Proof: We use induction.

Base Case: n=1 $A^1 = A \sim J \Rightarrow A^1$ is countably infinite.

Inductive Step: Assume A^{n-1} is countably infinite.
$A^n = A^{n-1} \text{ x } A = \{(b, a) \mid b \in A^{n-1}, a \in A\}$
$\forall b \in A^{n-1}$ $S_b = \{(b, a) \in A^n \mid a \in A\} \sim J \sim A \Rightarrow S^b$ is countably infinite.
$A^n = \bigcup_{b \in A\text{^}(n-1)} S_b \sim J$ by Corollary 1, so A^n is indeed countably infinite as claimed. (*q.e.d*)

Corollary 3: N^n is countably infinite $\forall n \geq 1$.

Proof: N ~ J, so the result follows from Corollary 2. (*q.e.d*)

Corollary 4: Z^n is countably infinite $\forall n \geq 1$.

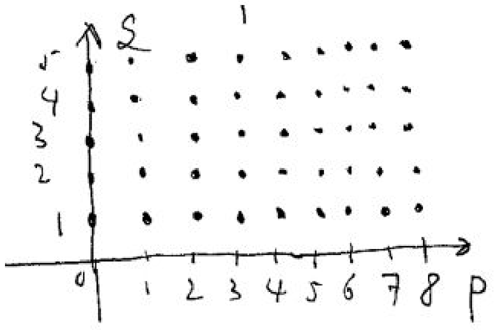
Proof: We showed Z ~ J, so the result follows from Corollary 2. (*q.e.d*)

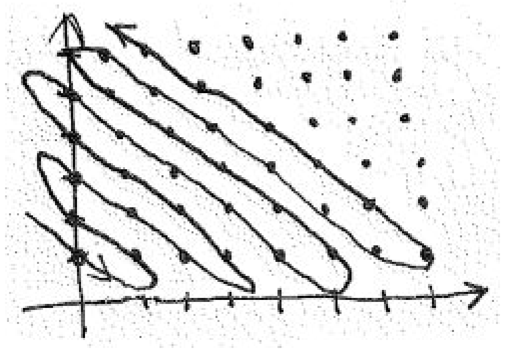
Corollary 5: Q is countably infinite.

Proof: $Q = \{p/q \mid q \neq 0, p,q \in Z, (p, q)=1$ i.e. no common factors$\}$ but we can represent Q as $\{(p, q) \mid q \neq 0, p,q \in Z\} / \sim \subseteq Z^2$ where $(p_1, q_1) \sim (p_2, q_2) \Leftrightarrow p_1/q_1 = p_2/q_2$ by cross multiplication.

We also know $Z \subseteq Q$ (let q = 1). Therefore Q is sandwiched between $Z = Z^1$ and Z^2, both of which are countably infinite ⇒ Q is countably infinite. (*q.e.d*)

Remark: We can give a visual representation of the previous argument as follows:





The dots are pairs (p, q) with $q \neq 0$, $p,q \in Z$ which form a lattice. We can use the snake trick from the theorem to show that the positive rationals $Q^+ = \{p/q \in Q \mid p/q > 0\}$ are countably infinite.

Similarly, we can show $Q^- = \{p/q \in Q \mid p/q < 0\}$ is countably infinite.

Then $Q = Q^- \cup \{0\} \cup \{Q^+\}$ is countably infinite by Corollary 1.

Next, show the set of sequences of 0s and 1s is uncountably infinite. We will use this result to show that other sets are uncountably infinite.

Theorem: Let A be the set of all sequences $s = \{x_1, x_2, ...\} = \{x_n\}_{n=1,2,...}$ such that $x_n \in \{0, 1\}$ $\forall n \geq 1$. Then A is uncountably infinite.

Remark: This result is proven via a clever construction, which is due to Georg Cantor (1845-1918), a very famous German mathematician who invented set theory. Cantor also came up with the diagonal argument (snake trick) which we used to prove that a countably infinite union of countably infinite sets is countably infinite, the idea that sizes of sets should be understood via bijections (A ~ B for A, B sets) as well as the notions of countably infinite and uncountably infinite.

Proof: Assume A is countable $\Leftrightarrow A = \{s_1, s_2, \ldots\}$ where $s_j = \{x^j_n\}_{n=1,2,...}$ for $x_n^j = 0$ or $x_n^j = 1$. We will now construct a sequence s_0 of 0s and 1s that cannot be in the enumeration $\{s_1, s_2, ...\}$. Let s_0 be such that $x_j^o = \{1 \text{ if } x_j^j = 0 \mid 0 \text{ if } x_j^j = 1\}$. In other words, s_0 differs from each s_j in the j^{th} element $\Rightarrow s_0 \notin \{s_1, s_2, ...\}$, but s_0 is a sequence of 0s and 1s $\Rightarrow s_0 \in A \Rightarrow\Leftarrow$ (*q.e.d*)

Corollary: The power set P(N) of N is uncountably infinite.

Remark:
Recall our proof that if B is a set with n elements, #(B) = n, then its power set P(B) has 2^n elements based on the "on/off" idea. For each element of B, we have the choice to include it in our subset ("on") or not to include it ("off"). Therefore we have 2 choices for each element and #(B) = n, so #P(B) = 2^n.

Proof: N ~ J, so we can write N = $\{x_1, x_2, \ldots\}$.
When we form a subset of N, for each i, we can include x_i or leave it out. Say we represent including x_i by 1 and leaving x_i out by 0. Then each subset of N can be represented uniquely as a sequence of 0s and 1s. In fact, there is a one-to-one correspondence between the subsets of N and the sequences of 0s and 1s. Therefore P(N) ~ A, where A is the set of all sequences of 0s and 1s, but we showed in the previous theorem that A is uncountably infinite ⇒ P(N) is uncountably infinite. (*q.e.d*)

We shall also use the one-to-one correspondence with the set of sequences of 0s and 1s in order to prove R is uncountably infinite. The argument proceeds in two steps:

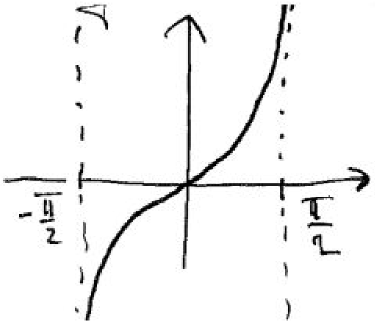
1. We show R ~ (0, 1) via a cleverly chosen bijection.
2. We set up a correspondence between (0, 1) and the set A of all sequences of 0s and 1s via a binary expansion.

Step 1 is the following proposition:

Proposition: R is in bijective correspondence with the interval (0, 1).

Remark:
(0, 1) ⊆ R, (0, 1) ≠ R, but we saw infinite sets can be in one-to-one correspondence with one of their proper subsets.

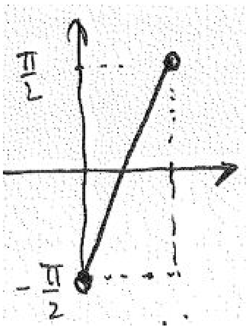
Proof: Recall from trigonometry that tan: $(-\pi/2, \pi/2) \to$ R is a bijection. Here is the graph:



x = -π/2, x = π/2 are asymptotes of the graph.

tan x = sin x / cos x
cos(-π/2) = cos(π/2) = 0

We now use a linear function, a bijection, to show $(0, 1) \sim (-\pi/2, \pi/2)$
$g(x) = \pi x -\pi/2$. Here is the graph:



The composition of two bijections is itself a bijection ⇒ $\tan(g(x)) = \tan(\pi x -\pi/2)$ is a bijection from $(0, 1)$ to R. The map we want $f:R \to (0, 1)$ is its inverse $f(x) = (\tan(\pi x -\pi/2))^{-1}$ as the inverse of a bijection is itself a bijection.

Step 2 is a bit more complicated. To each $x \in (0, 1)$, we want to associate $0.x_1x_2\ldots$ where after the decimal $\{x_1, x_2, ...\}$ is a sequence of 0s and 1s. In other words we are giving a binary expansion of every $x \in (0, 1)$ as $0.x_1x_2\ldots = 0 + x_1/2 + x_2/4 + x_3/8 + \ldots =$
$0 + x_1/2 + x_2/2^2 + x_3/2^3 + \ldots =$

$$0 + \sum_{n=1}^{\infty} 1/2^n . x_n = \sum_{n=1}^{\infty} 1/2^n x_n$$

Recall that ½ + ¼ + ⅛ + … = 1. This means that

$$1/2^k \sum_{n=1}^{\infty} 1/2^n . x_n = 1/2^{k+1} + 1/2^{k+2} + 1/2^{k+3} + \ldots = 1/2^k \quad \forall k \geq 1.$$

Thus 0.100000… and 0.0111111… both represent ½.
Similarly, and $x \in (0, 1)$ that is a sum of the form $1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk}$ for $p_1, \ldots, p_k \in N^*$, $p_1 < p_2 < \ldots < p_k$ has two binary representations.

Question: Can we represent $x = 1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk}$ in an easier to understand form?

Answer: Yes, we bring the fractions to the same denominator:
$x = 1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk}$
$= 2^{pk-p1}/(2^{pk-p1}.2^{p1}) + 2^{pk-p2}/(2^{pk-p2}.2^{p2}) + \ldots + 2^{pk-pk-1}/(2^{pk-pk-1}.2^{pk-1}) + 1/2^{pk}$
$= (2^{pk - p1} + 2^{pk - p2} + \ldots + 2^{pk - pk-1} + 1)/2^{pk}$
= odd natural number / power of 2
$= m/2^n$ for $m \in N$ odd and $n \in N^*$ as $p_1 < p_2 < \ldots < p_k$ so the differences p_k-p_1, p_k-p_2, …, p_k-p_{k-1} are all positive integers. So the sequence in (0, 1) that has two decimal binary expansions is {½, ¼, ¾, ⅛, ⅜, ⅝, ⅞, ...} = B.
Note that B is countably infinite as each set $B_n = \{0 < \text{odd}/2^n < 1\}$ is finite, $B = \bigcup_{n=1}^{\infty} B_n$ is countable by our corollary, and the countably infinite sequence {½, ¼, ⅛, …} ⊆ B, which means the countable set B must be countably infinite.

Now let us examine the binary expansions of the elements $y \in B$. $\forall y \in B$,
$y = 1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk}$ for $p_1, \ldots, p_k \in N^*$, $p_1 < p_2 < \ldots < p_k$. The two binary expansions corresponding to y, $b_{y,1}$ and $b_{y,2}$ are of the form $0.x_1x_2...x_{pk-1}x_{pk}x_{pk+1}$ where $x_1, \ldots, x_{pk-1}$ are common to $b_{y,1}$ and $b_{y,2}$, whereas x_{pk}, x_{pk+1}, … differ.
Now x_j = {1 if $j=p_1, p_2, \ldots, p_{k-1}$, 0 otherwise for $1 \leq j \leq p_k$ is the common part corresponding to $1/2^{p1} + 1/2^{p2} + \ldots + 1/2^{pk-1}$ whereas the difference comes from the two possible ways of representing the last term in the sum $1/2^{pk}$ namely 10000… or 011111…. Therefore, $b_{y,1}$ has $x_{pk} = 1$ and $x_j = 0$ $\forall j > p_k$ (corresponding to 1000....) whereas $b_{y,2}$ has $x_{pk} = 0$ and $x_j = 1$ $\forall j > p_k$ (corresponding to 01111…).
Let $s_{y,1} \in A$ be the sequence corresponding to $b_{y,1}$ in A, the set of all sequences of 0s and 1s, i.e. if $b_{y,1} = 0.x_1x_2x_3\ldots$ $s_{y,1} = \{x_1, x_2, x_3, ...\}$.
Let $s_{y,2} \in A$ be the sequence corresponding to $b_{y,2}$.
We now define $B_1 = \{b_{y,1} \mid y \in B\}$, $B_2 = \{b_{y,2} \mid y \in B\}$, $A_1 = \{s_{y,1} \mid y \in B\}$, $A_2 = \{s_{y,2} \mid y \in B\}$.
B is in one-to-one correspondence to B_1, B_2, A_1, A_2 by construction, so $B \sim B_1$, $B \sim B_2$, $B \sim A_1$, $B \sim A_2$, but B is countably infinite ⇒ A_1, A_2, B_1, B_2 are all countably infinite.

We have just one more observation to make regarding the correspondence between sequences of 0s and 1s in A and elements of (0, 1), namely that the zero sequence {0, 0, …} corresponds to the binary expansion $0.000\ldots = 0 \notin (0, 1)$ since $(0, 1) = \{x \in R \mid 0 < x < 1\}$ and the sequence {1, 1, 1, …} corresponds to the binary expansion 0.111... = ½ + ¼ + ⅛ + ...

$$= \sum_{n=1}^{\infty} 1/2^n = 1 \notin (0, 1).$$

Now we can finally prove that (0, 1) is uncountably infinite.

Proposition: (0, 1) is uncountably infinite.

Proof: We define a map $f:(0, 1) \to \{0.x_1x_2x_3\ldots \mid x_j \in \{0, 1\}\ \forall j \geq 1\}$ as follows:
$f(y) = \{$ $b_{y,1}$ if $y \in B$ (The first of the two possible binary expansions) or
$0.x_1x_2x_3\ldots$ if $y \notin B$ (The unique binary expansion)$\}$
By our previous discussion, f is a bijection as defined $\Rightarrow$ $(0, 1) \sim \{0.x_1x_2x_3\ldots \mid x_j \in \{0, 1\}\ \forall j \geq 1\}$
Also by our previous discussion $\{0.x_1x_2x_3\ldots \mid x_j \in \{0, 1\}\ \forall j \geq 1\} \sim A \backslash (A_2 \cup \{0, 0, \ldots\} \cup \{1, 1, \ldots\})$ where:

- A = set of all sequences of 1s and 0s
- $\{0, 0, \ldots\}$ = sequence of all 0s
- $\{1, 1, \ldots\}$ = sequence of all 1s

Therefore $(0, 1) \sim A \backslash (A_2 \cup \{0, 0, \ldots\} \cup \{1, 1, \ldots\})$ since ~ is transitive (it is an equivalence relation).
A_2 is countably infinite, so $A_2 \cup \{0, 0, \ldots\} \cup \{1, 1, \ldots\}$ is countably infinite (we've added two elements to A_2, so it stays countably infinite).

In a previous theorem we proved A is uncountably infinite.
Thus $A \backslash (A_2 \cup \{0, 0, \ldots\} \cup \{1, 1, \ldots\})$ is of the form {uncountably infinite set} \ {countably infinite set}. I claim $A \backslash (A_2 \cup \{0, 0, \ldots\} \cup \{1, 1, \ldots\})$ is uncountably infinite.
Indeed, let $\tilde{A} = A \backslash (A_2 \cup \{0, 0, \ldots\} \cup \{1, 1, \ldots\})$. Assume $\tilde{A}$ is countable, then A is the union of a countable set with a countably infinite set, hence A is countable $\Rightarrow\Leftarrow$
Therefore, $\tilde{A} = A \backslash (A_2 \cup \{0, 0, \ldots\} \cup \{1, 1, \ldots\})$ is uncountably infinite, but $\tilde{A} \sim (0, 1)$ (~ is asymmetric) $\Rightarrow$ (0, 1) is uncountably infinite. (*q.e.d*)

Theorem: R is uncountably infinite.

Proof:
By the previous proposition, (0, 1) is uncountably infinite. By the proposition before this one, $(0, 1) \sim R \Rightarrow R$ is uncountably infinite. (*q.e.d*)

Under the equivalence relation ~ of bijective correspondence, we have shown that
N, N^*, N^n $\forall n \geq 1$, Z^n $\forall n \geq 1$, Q^n $\forall n \geq 1$ $\in$ [N] are all countably infinite and
A (all sequences of 0s and 1s), P(N) and [R] are uncountably infinite.

Question: Is there some intermediate equivalence class in size between [N] and [R]?

Answer: The continuum hypothesis (CH) gives a negative answer to this question.

The continuum hypothesis:
There is no set whose cardinality is strictly between the cardinality of the integers and the cardinality of real numbers. Cardinality means size or number of elements.

Georg Cantor stated CH in 1878, believed it was true, but could not prove it. It became one of the crucial open problems in mathematics. Hilbert stated in 1900 first among the 23 problems that were supposed to hold the key for the advancement of mathematics. Everybody expected CH to be either true or false.

The answer is that CH is independent from the standard axiomatic system used in mathematics called ZFC (Zermelo–Fraenkel with The Axiom of Choice). In other words, CH **cannot** be proven either true or false when working within the axiomatic framework of ZFC. In 1940 Kurt Gödel showed CH cannot be proven false within ZFC. In 1963 Paul Cohen showed CH cannot be proven true within ZFC and won the Fields Medal (like the Nobel Prize for mathematics) for his work.

Applications of Countability of Sets to Formal Languages:

Task:
Figure out the size of the set of all languages over a finite alphabet and the size of all regular languages over a finite alphabet. Let A be a finite alphabet, i.e. A = $\{a_1, \ldots, a_n\}$.

Recall that $A^* = \bigcup_{j=0}^{\infty} A^i$ is the set of all possible words in the alphabet A.

A^j is the set of all words of length j in the alphabet A.

Question: What is $\#(A^j)$, the size (cardinality) of A^j?

Answer: If j=0, $A^0 = \{\varepsilon\}$ where ε is the empty word, so $\#(A^0) = 1$. In general, we have n choices of letters in the first position, n choices of letters $(a_1, \ldots, a_n)$ in the second position and so on up to the j^{th} position. In total we have n x n x … x n (j times) = n^j possibilities. Therefore, $\#(A^j) = n^j$. Note that when j=0 $n^0=1 = \#(A^0) = \#(\{\varepsilon\})$.

Theorem:
If A is a finite alphabet then the set of all words over A $A^* = \bigcup_{j=0}^{\infty} A^j$ is countably infinite.

Proof:
We showed A^j is a finite set for each j. In fact, $\#(A^j) = n^j$. $\bigcup_{j=0}^{\infty} A^j$ is therefore a countably infinite union of disjoint infinite sets.
Note that $A^j \cap A^k = \emptyset$ if $j \neq k$ as no words of length j can be of length k if $j \neq k$.
By Corollary 1 to the theorems that a countably infinite union of countably infinite sets is countably infinite,

$A^* = \bigcup_{j=0}^{\infty} A^j$ is countably infinite. (*q.e.d*)

Corollary 1: If A is a finite alphabet, the set of all languages over A is uncountably infinite.

Proof: Recall that a language L is any subset of words in the alphabet A, hence L is any subset of A*. Therefore the set of all languages over A is precisely P(A*), the power set of A*. We showed in the previous theorem that A* is countably infinite,
i.e. $A^* \sim N \Rightarrow P(A^*) \sim P(N)$, but we previously proved P(N) is uncountably infinite by putting it in one-to-one correspondence with the set of all sequences of 0s and 1s ⇒ P(A*) is uncountably infinite. (*q.e.d*)

Corollary 2: The set of all programs in any programming language is countably infinite.

Proof: For any programming language, a program is a finite string over a finite alphabet, the set of characters allowable in that programming language. Let us call this finite alphabet A. Then the set of all programs in the given programming language is A^*. Since $A^* \sim N$ as proven in the theorem, the set of all programs is countably infinite. (*q.e.d*)

Recall:
Theorem: A language over a finite alphabet is regular ⇔ it is given by a regular expression.

Recall the definition of a regular expression:
Definition: Let A be an alphabet.

1. Ø, ε, and all elements of A are regular expressions.
2. If w and w' are regular expressions then $w \circ w'$, $w \cup w'$ and w^* are regular expressions.

Note that regular expressions sometimes have parentheses in order to change the priority of operations $*$, $\circ$ (concatenation) and $\cup$ (union). Therefore, any regular expression over the alphabet A is a string over the enlarged alphabet $\tilde{A}$.

Theorem: The set of all regular languages over a finite alphabet A is countably infinite.

Proof:
Since the alphabet A is finite, the enlarged alphabet $\tilde{A} = A \cup \{$"Ø", "ε", "*", "o", "∪", "(", ")"$\}$ is also finite. By the theorem proven earler, $\tilde{A}^*$ is therefore countably infinite. A regular language then is given by a regular expression, which is a string over the enlarged alphabet $\tilde{A}$, hence an element of $\tilde{A}^*$. Therefore, the set of all regular languages over the alphabet A is countably infinite. (*q.e.d*)

Moral of the Story:
Given a finite alphabet A, the set of regular languages (which is countably infinite) is much smaller than the set of all languages over A (which is uncountably infinite). Therefore, regular languages constitute a special category within the set of all languages over a given alphabet.