MA2C03 - TUTORIAL 1

1 Proofs in Propositional Logic

Example 1.1. Suppose we are told the following;
If turtles can sing then artichokes can fly. Artichokes can fly implies turtles can sing and dogs can't play chess. Dogs can play chess if and only if turtles can sing. Deduce that turtles can't fly.

Proof.
We convert these statments into a logical format;
P = "turtles can sing".
Q = "artichokes can fly".
R = "dogs can't play chess".

We can assume the following hypotheses from the above paragraph;

(a) $P \to Q$

(b) $Q \to (P \wedge R)$

(c) $\neg R \leftrightarrow P$

We wish to prove $\neg P$. We do so as follows:

(1) $Q \to (\neg R \wedge R)$ - substitution of (c) into (b).

(2) $\neg(\neg R \wedge R) \to \neg Q$ - contrapositive of (1) (tautology #24 on the list of tautologies posted in Course Documents)

(3) $R \vee \neg R$ - law of the excluded middle (tautology #1 on the list of tautologies)

(4) $\neg(\neg R \wedge R)$ - De Morgan's law applied to (3) (tautology #18) and substitution of $\neg(\neg R)$ by R by the law of double negation (tautology #3)

(5) $\neg Q$ - modus ponens (2, 4) (tautology #10)

(6) $\neg Q \to \neg P$ - contrapositive of (a) (tautology #24)

(7) $\neg P$ - modus ponens (5,6) (tautology #10)

■

2 Proofs in Set Theory

Example 2.1. Prove $A \setminus (A \setminus B) \subseteq B$.

Proof.
This is done by examining where the elements lie in the set to the left of $\subseteq$ and proving they also lie in B. To this end, take $x \in A \setminus (A \setminus B)$. By the definition of $X \setminus Y = X \cap Y^c$, we have

$$x \in A \setminus (A \setminus B) \Rightarrow x \in A \cap (A \setminus B)^c \Rightarrow x \in A \text{ AND } x \in (A \setminus B)^c$$

Applying the definition of $\setminus$ again we conclude $x \in A$ AND $x \in (A \cap B^c)^c$. Using De Morgan's laws for the later, we get $x \in A$ AND $x \in A^c \cup (B^c)^c$. Let's focus more on the later, with the knowledge that $x \in A$.

$$x \in A^c \cup (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in B$$

Since x cannot be in both A and A^c at the same time, we conclude $x \in B$ (now ignoring A). What we have shown:

$$\forall x (x \in A \setminus (A \setminus B) \Rightarrow x \in B)$$

So $A \setminus (A \setminus B) \subseteq B$ as required. ■

Remark. Veitch diagrams and/or Venn diagrams will **NOT** be accepted as a form of proof in set theory. Please bear in mind that only a solution of this kind is acceptable as a proof to an assertion in set theory.