MA2C03: TUTORIAL 15 PROBLEMS
MINIMAL SPANNING TREES

1) (Annual Exam Trinity Term 2017) Consider the connected undirected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L, and with edges listed with associated costs in the following table:

AD	BC	EI	CF	JK	IJ	BL	CE	HG	FH
2	2	2	3	4	5	6	6	6	7
AB	FJ	GK	BH	EJ	CD	DE	HL	AC	EF
8	8	9	9	10	10	11	11	12	13

(a) Draw the graph and label each edge with its cost.
(b) Determine the minimal spanning tree generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, list the edge that is added and draw the graph.
(c) Determine the minimal spanning tree generated by Prim's Algorithm, starting from the vertex I, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, list the edge that is added and draw the graph.
(d) Determine the minimal spanning tree generated by Prim's Algorithm, starting from the vertex F, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, list the edge that is added and draw the graph.

2) In the previous problem, how many distinct ways can the edges of the graph be ordered in non-decreasing order of cost, i.e. how many different non-decreasing queues are there for the edges of the graph? Justify your answer.

3) Would every non-decreasing queue from problem 2 give a different minimal spanning tree when Prim's Algorithm is applied starting from vertex I? Justify your answer by either a proof or a counterexample.

Solution: 1)(a) The graph is at the end of the solutions. (b) The edges are added in the following order: AD, BC, EI, CF, JK, IJ, BL, CE, HG, FH, and AB.
(c) The edges are added in the following order: EI, IJ, JK, CE, BC, CF, BL, FH, HG, AB, and AD.
(d) The edges are added in the following order: CF, BC, BL, CE, EI, IJ, JK, FH, HG, AB, and AD.

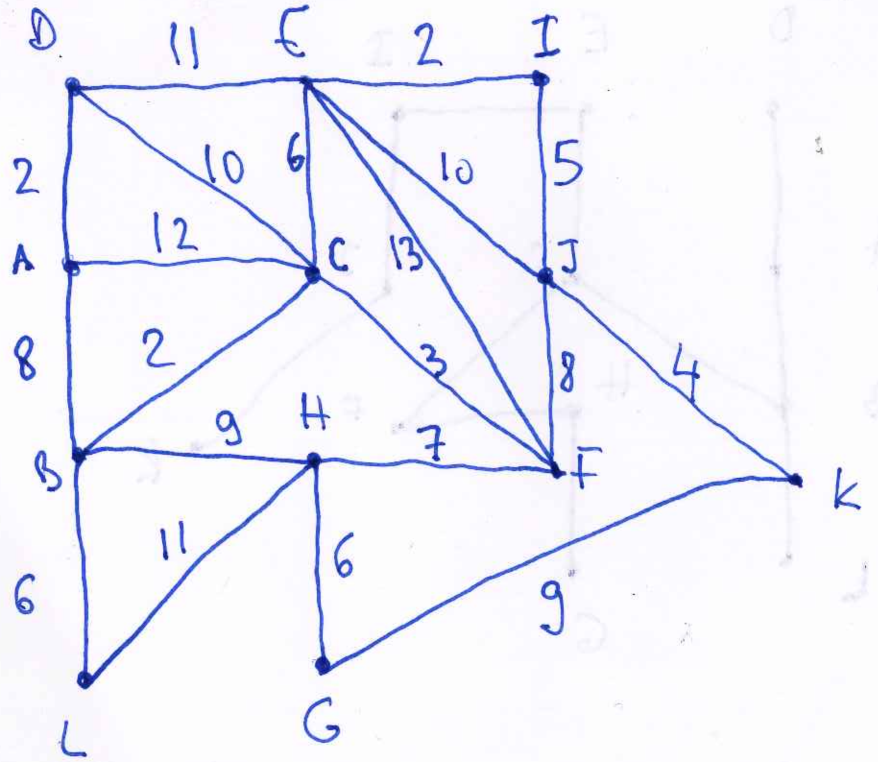
2) Three edges have cost 2, so they can be reshuffled (for a total of 3! possibilities), three edges have cost 6, two edges have cost 8, two edges have cost 9, two edges have cost 10, and two edges have cost 11. We thus have

$$3! \times 3! \times 2! \times 2! \times 2! \times 2! = 36 \times 16 = 576$$

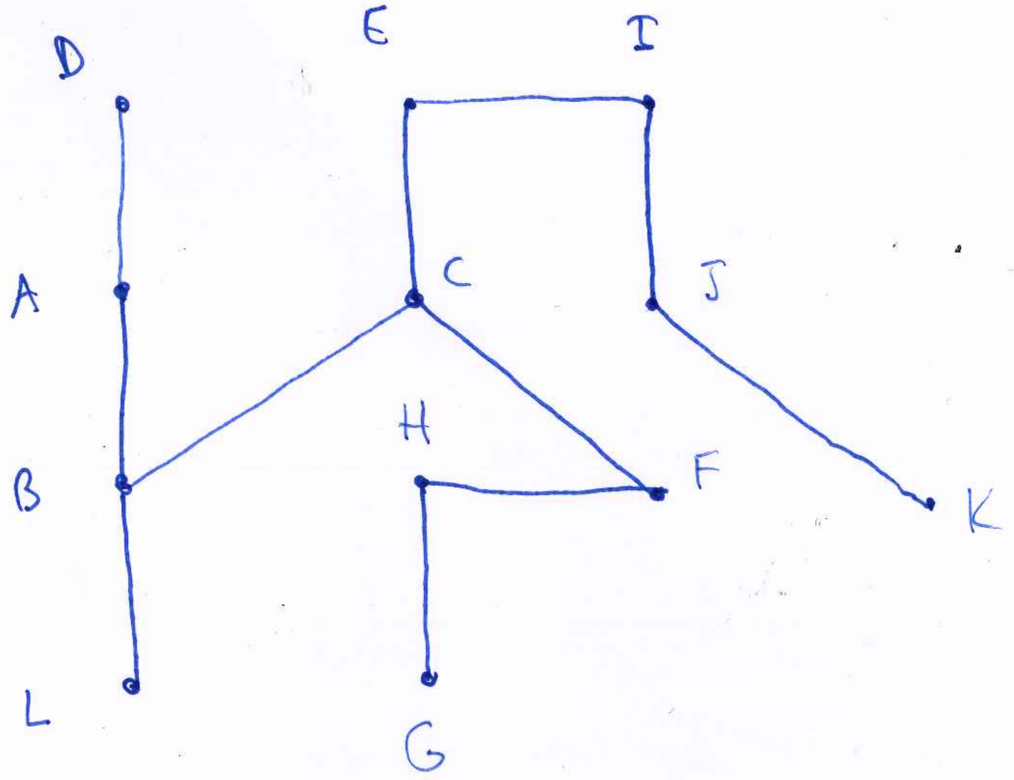
ways of obtaining a non-decreasing queue of edges.

3) Not necessarily! Here is a counterexample: Queue AD, BC, EI, CF, JK, IJ, BL, CE, HG, FH, AB, FJ, GK, BH, EJ, CD, DE, HL, AC, EF and queue BC, AD, EI, CF, JK, IJ, BL, CE, HG, FH, AB, FJ, GK, BH, EJ, CD, DE, HL, AC, EF with the first two edges exchanged give the same minimal spanning tree.

1(a)


D
11
E
2
I
2
10
6
10
5
A
12
C
13
J
8
2
3
8
4
9
H
7
B
F
K
11
6
6
9
L
G

1(b),(c),(d)


D
E
I
A
C
J
H
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K
L
G