Uniform Probability Model (CH3)

- Let us recall the definition of the *uniform probability model*.

  If the following two conditions are satisfied:

  (i) the sample space S has a **finite number** of outcomes

  (ii) simple events are **equally likely**,

  then the probability of any event $A \subseteq S$ is given by

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S} \equiv \frac{|A|}{|S|}.$$

- To use this model in practice we need to:
  - define a proper sample space
  - find an efficient way of counting the number of outcomes for the events of interest.

Counting Arrangements and Combinations

- Number of arrangements of n objects in a sequence (order matters):

$$n! = n \times (n-1) \times \cdots \times 1,$$

- Number of arrangements of k objects out of n without replacement:

$$n^{(k)} = \frac{n!}{(n-k)!}, \quad k = 1, 2, \ldots, n,$$

- Number of arrangements of length k out of n with replacement:

$$n^k.$$

- Number of ways we can select k objects out of n (order does not matters):

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

- Number of arrangements with repeated symbols

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Probability Rules (CH4)

- If $A_1, A_2, ...A_k$ is a finite sequence of mutually exclusive events, then

$$P(\cup_{i=1}^{k} A_i) = \sum_{i=1}^{k} P(A_i).$$

- For any event $A \in \mathcal{S}$, we have

$$P(\bar{A}) = 1 - P(A)$$

- For any two events A and B we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

- For any three events A, B, and C we have

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(AB) - P(BC) \\ &\quad -P(AC) + P(ABC). \end{aligned}$$

- If A and B are **independent** then

$$P(A \cap B) = P(A)P(B).$$

- In general we have to use one of

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A).$$

- **Law of total probability:** If $B_1, B_2, ..., B_k$ is a partition of S such that $P(B_i) > 0$ for each i, then for any event A

$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i).$$

- **Bayes Theorem:**

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}, \quad i = 1, \ldots, k.$$

Discrete Random Variables (CH5)

- The **probability function** (p.f.) f_X of a discrete random variable X with range $range(X) = A$ is defined as

$$f_X(x) := P(X = x) \text{ for } x \in A.$$

Once the p.f. f_X of X is known, we can find any probability of the form $P(X \in A)$:

$$P(X \in A) = \sum_{y \in A} f_X(y).$$

Discrete Random Variables (CH5)

- The **Cumulative Distribution Function** of a random variable X is

$$F_X(x) = P(X \leq x) = \sum_{y:\, y \leq x} f_X(y), \quad x \in \mathbb{R}.$$

- All CDF's:
  - are non-decreasing functions with values between 0 and 1
  - $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

We have considered several “standard” model distributions and models:

- Discrete Uniform Distribution
- Hypergeometric Distribution
- Binomial Distribution
- Negative Binomial Distribution
- Geometric Distribution
- Poisson Distribution
- Poisson Process

EXPECTED VALUE AND VARIANCE (CH7)

If X is a discrete random variable with probability function $f(x)$, then its **expected value** is given by

$$E(X) := \sum_{x \in range(X)} x \cdot f(x).$$

1. For any constants a and b we have
$$E(aX+b)=aE(X)+b.$$
and
$$E(aX+bY)=aE(X)+bE(Y).$$
2. If a r.v. X is such that $a \leq X \leq b$ for two constants a and b, then
$$a \leq E(X) \leq b.$$
3. If $X \geq 0$, then
$$E(X) \geq 0.$$
4. For a given function g
$$E[g(X)] = \sum_{x \in range(X)} g(x) f_X(x).$$
5. If g is non-linear then typically
$$E[g(X)] \neq g(E[X]).$$

- The **variance** of a random variable X, denoted by $Var(X)$, is defined as

$$Var(X) = E[(X - E(X))^2].$$

We have

$$Var(aX + b) = a^2 Var(X).$$

- Equivalent representations of variance

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ Var(X) &= E[X(X-1)] + E(X) - [E(X)]^2. \end{aligned}$$

- The **standard deviation** of a random variable X is

$$SD(X) := \sqrt{Var(X)}.$$

We have

$$SD(aX + b) = \sqrt{a^2 Var(X)} = |a| SD(X).$$

CONTINUOUS RANDOM VARIABLES (CH8)

- A random variable X is said to be continuous if its range is an interval $(a, b) \subseteq \mathbb{R}$, or a collection of intervals.
- We define a probability model for a continuous r.v. by defining its probability density function $f_X(x)$.

  For any a and b (including $a = -\infty$ and $b = \infty$) we have

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

- Any p.d.f. must satisfy:

$$f_X(x) \geq 0 \text{ for all } x.$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

- The distribution of a continuous rv X can also be defined by specifying its **Cumulative Distribution Function**:

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

By the fundamental theorem of calculus,

$$\frac{d}{dx} F_X(x) \equiv F'_X(x) = f(x).$$

Properties of the CDF of a continuous random variable:

1. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$.
2. $F(x)$ is continuous.
3. $F(x)$ is differentiable (possibly except a countable number of points).
4. $F(x)$ is non-decreasing.

Quantiles

- If X is a continuous random variable with the cumulative distribution function $F(x)$, then the pth quantile of X ($p \in (0,1)$) is the value $q(p)$, such that

$$P(X \leq q(p)) = p,$$

or, in terms of the CDF,

$$F(q(p)) = p.$$

If F is strictly increasing, then

$$q(p) = F^{-1}(p),$$

where F^{-1} is the inverse function of F,

Expectation and Variance

- If X is a continuous random variable with pdf $f(x)$ and $g : \mathbb{R} \to \mathbb{R}$ a given function, then

$$E[g(X)] := \int_{-\infty}^{\infty} g(x)f(x)dx.$$

- It follows that

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$Var(X) = E[(X - E(X))^2] = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx.$$

- Means and variances of continuous r.v.'s have the same properties as for discrete r.v.'s.

- We have considered the following standard continuous distributions:
  - Continuous Uniform Distribution
  - Exponential Distribution
  - Normal (or Gaussian) Distribution

Change of Variable

- Consider the transformation $Y = h(X)$ of a continuous random variable X, where $h : \mathbb{R} \to \mathbb{R}$.

  A possible strategy for computing the pdf and/or CDF of $Y = h(X)$ in terms of the analogous functions for X:

  (i) Determine the range of X, and from this deduce the range of $Y = h(X)$.

  (ii) Derive the CDF of Y.

  (iii) If desired, differentiate the CDF of Y to obtain the pdf f_Y as a function of f_X.

Final Exam

- Basic Info: Tuesday, Dec 11 4:00 - 6:30 pm
  PAC sections 1-6, 9-11 (check Odyssey for your seat after Dec 4)
- Format:
  - Similar style/format to the Sample Final in the Course Notes
  - 15 marks MC, 10 marks T/F, 10 marks identify the distribution, 65 marks long answer
  - More emphasis on material since the second midterm
- Coverage:
  - All Course Notes sections except: Chapter 6, 8.4, 9.3, 10.3
  - Slight bias towards material past midterm 2 beyond this.

- Study strategy:
  - Attend one of the review sessions
  - Do the Sample Final at least a few days before the final.
  - If you missed any assignments, test, or midterms, try those problems too
  - Don't ignore the conceptual aspects/proofs from the class.
  - Use office hours of the instructors and/or TA's.

- Review Sessions:
  - Wednesday Dec 5 from 10:00-12:00 in STC 1012 (Diana)
  - Friday Dec 7 from 12:00 to 2:00 in HH 1101 (Adam)
  - Email your questions!

Good Luck!

Thank You!