Discrete Fourier Methods.

~~Background~~ General definition:

The process of approximating periodic functional data (from observations or a known function) with a possibly infinite combination of sine and cosine waves is called Fourier Analysis.

This process involves the conversion of time or spatial information into frequency information.

For example: Fourier transform (FT) of a signal informs us what frequencies are present in the signal and in what proportion

If you press the phone buttons, they produce different sounds because they are composed of two different frequencies that combine to produce the sound. Fast Fourier Transformation (FFT) is used to identify the number dialed.

Fourier analysis is used in some ways in storing images, optical information processing and Mp3s.

outline: what we will study in this chapter:

- consider the continuous function $f(t)$.
- Any periodic function can be written as a combination of trig functions:

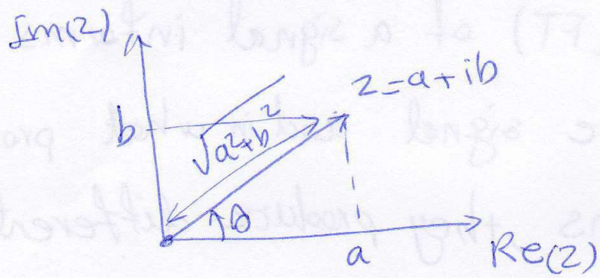
$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi k t}{T}\right)$$

- How to determine a_0, a_k, b_k from a function $f(t)$?
- How to determine a_0, a_k, b_k from a discrete data?
- How to do this computation quickly?

background:

- complex number:

we define a complex number z as $z = a + ib$, with a the real part of z and b the imaginary part, and $i = \sqrt{-1}$. Then, a complex number $z = a + ib$ is depicted as a vector in the complex plane:



For any complex number $z = a + ib$ we have:

$|z|^2 = z.\bar{z} = a^2 + b^2$

complex conjugate : $\bar{z} = a - ib$

real part : $Re(z) = a$

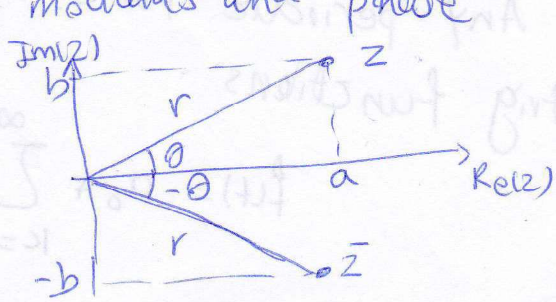
imaginary part : $Im(z) = b$

Modulues : $r = |z| = \sqrt{a^2 + b^2}$

phase angle : $\theta = \tan^{-1}(b/a)$

A complex number can be represented by modulus and phase angle: $z = re^{i\theta}$. In this polar form

$z_1 . z_2 = r_1 e^{i\theta_1} . r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$, $\bar{z} = re^{-i\theta}$



② Euler formula: $\begin{cases} e^{i\theta} = \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} = \cos(\theta) - i\sin(\theta) \end{cases} \longrightarrow \begin{matrix} \cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{matrix}$

periodic function:

$f(t)$ is a periodic function if $\exists$ a positive T s.t $f(t+T) = f(t)$

in which T is period of $f(t)$. A period is not unique and every integer-multiple of a period is another period.

The smallest T is called the fundamental period.

Ex: $f(t) = \cos(2\pi t) \longrightarrow T = 1$, $f(t) = \sin(\frac{2\pi}{T} t) \longrightarrow T = 1$

In general: $\left.\begin{matrix} f(t) = \sin(kt) \\ f(t) = \cos(kt) \end{matrix}\right\} \longrightarrow T = \frac{2\pi}{k}$ | $\left.\begin{matrix} f(t) = \sin(\frac{2\pi}{T} t) \\ f(t) = \cos(\frac{2\pi}{T} t) \end{matrix}\right\} \longrightarrow T$: period

Ex: what is the period of $f(t) = \underbrace{\sin(\frac{2\pi}{7} t)}_{} + \frac{1}{5}\underbrace{\sin(\frac{24}{7}\pi t)}_{}$

$T = \frac{2\pi}{\frac{2\pi}{7}} = 7$, $T = \frac{2\pi}{\frac{24\pi}{7}} = \frac{7}{12}$

any inter-multiple of period is period $\longrightarrow$ $12 \times \frac{7}{12} = 7$ is a period of the second term $\longrightarrow T = 7$

$f(t) = \sin(\frac{2\pi k t}{T}) \longrightarrow$ period $\frac{T}{k}$

$f(t) = \cos(\frac{2\pi k}{T} t)$

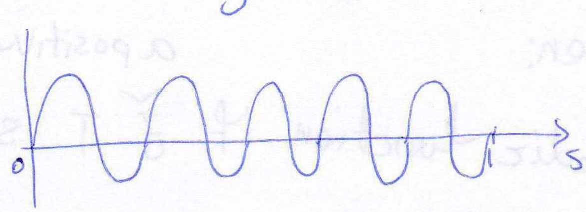
result: $\sin(\frac{2\pi t k}{T})$ and $\cos(\frac{2\pi t k}{T})$ are $\frac{T}{k}$ periodic. If k is an integer they are also T-periodic.

frequency of the wave is defined as $f = \frac{1}{T}$ and its dimension is oscillation per second or (HZ).

Angular frequency (ω) is related to the frequency by $\omega = 2\pi f$ with the dimension of radians per second.

For example: $f(t) = \sin(10\pi t) \rightarrow T = \frac{1}{5}$ s , $f = 5$ $\frac{\text{oscillation}}{\text{s}}$ (Hz)

$\omega = 2\pi f = 10\pi$ radians/s



Any linear combination of $\sin(\frac{2\pi k}{N} t)$ and $\cos(\frac{2\pi k}{N} t)$ which are N-periodic for integer k is N-periodic as well. In fact, the set $\{\sin(\frac{2\pi k}{N} t), \cos(\frac{2\pi k}{N} t)\}$ for $k = 0, \cdots, \infty$ forms a basis for any periodic continious function, i.e. $\exists\, a_k, b_k$ s.t on $[a,b]$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{2\pi k}{N} t) + \sum_{k=1}^{\infty} b_k \sin(\frac{2\pi k}{N} t)$$

This is called Fourier series of $f(t)$ with [note that $b-a$ is a period of $f(t)$]

$$a_k = \frac{2}{b-a} \int_a^b f(t) \cos(\frac{2\pi k}{N} t) dt \quad , \quad b_k = \frac{2}{b-a} \int_a^b f(t) \sin(\frac{2\pi k}{N} t) dt$$

useful information to derive these coefficients and to show that basis are orthogonal:

If $N = 2\pi$ and $t \in [0, 2\pi]$

1) $\int_0^{2\pi} \cos(kt)\sin(kt) dt = 0$

2) $\int_0^{2\pi} \cos(kt)\cos(jt) dt = 0$, $k \neq j$

3) $\int_0^{2\pi} \sin(kt)\sin(jt) dt = 0$, $k \neq j$

4) $\int_0^{2\pi} \sin(kt) dt = 0$

5) $\int_0^{2\pi} \cos(kt) dt = 0$

So, $\{\cos(kt), \sin(kt)\}$ are orthogonal on $[0, 2\pi]$

The basis in V is $B=\{1, \cos(kt), \sin(kt)\}$ $1 \leq k \leq \infty$

$$\rightarrow f(t) = a_0 \times 1 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

and scalar product over this vector space is defined as

$$\langle f(t), g(t) \rangle = \int_{-\pi}^{\pi} f(t) g(t) dt$$

which gives the standard norm2 as

$$\|f(t)\|_2 = \sqrt{\langle f(t), f(t) \rangle} = \sqrt{\int_{-\pi}^{\pi} f(t)^2 dt}$$

we can show that (by the given property of sin and cos) that B is an orthogonal basis for V.

$\langle b_i, b_j \rangle = 0$ $i \neq j$ and $\langle \sin(kt), \sin(lt) \rangle = 0$ $(k \neq l)$

$\langle b_i, b_i \rangle \neq 0$ $\langle \cos(kt), \cos(lt) \rangle = 0$ $(k \neq l$

how to use orthogonality to find the coefficient:

In 2D space, the standard basis is $B=\{e_1, e_2\}$, this is orthogonal because $\langle e_1, e_1 \rangle = 1$, $\langle e_2, e_2 \rangle = 1$, $\langle e_1, e_2 \rangle = 0$

$$\rightarrow a = a_1 e_1 + a_2 e_2 \rightarrow \langle a, e_1 \rangle = \cancel{a_1 e_1 e_1} \langle a_1 e_1 + a_2 e_2, e_1 \rangle$$

$$= a_1 \langle e_1, e_1 \rangle + a_2 \langle e_2, e_1 \rangle = a_1 \langle e_1, e_1 \rangle = a_1$$

$$\Rightarrow a_i = \frac{\langle a, e_i \rangle}{\langle e_i, e_i \rangle}$$

by this projection formula, we can use the basis defined over V to find a_k, b_k:

$$a_k = \frac{\langle f(t), \cos(kt) \rangle}{\langle \cos(kt), \cos(kt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt$$

$$b_k = \frac{\langle f(t), \sin(t) \rangle}{\langle \sin(kt), \sin(kt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt)\,dt$$

$$\frac{a_0}{2} = \frac{\langle f(t), 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt$$

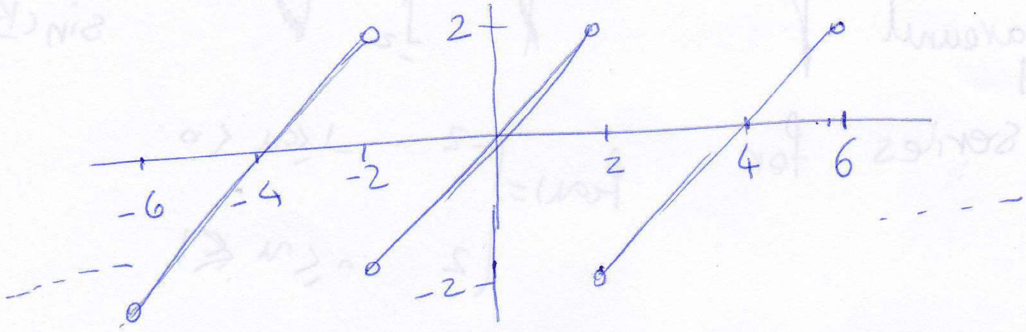
Since we find a_0 from a_k for $k=0$, we use $\frac{a_0}{2}$

go to page 4 for the definition of space V

(3)

Note. the function f(t) and its Fourier series are equal to each other if f(t) is continuous, but if f is piecewise continuous, then it disagrees with its Fourier Series at every discontinuity.

Ex: Find a fourier series for $f(x)=x \quad -2<x<2 \,, \; f(x+4)=f(x)$



$N=4 \qquad a_k = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2\pi k}{N}x\right)dx$

$\rightarrow a_0 = \frac{2}{4}\int_{-2}^{2} x\,dx = \frac{1}{2}\times\frac{1}{2}x^2\Big|_{-2}^{2} = 0$

$a_k = \frac{1}{2}\int_{-2}^{2} x\cos\left(\frac{2\pi}{N}kx\right)dx = \frac{1}{2}\left[\frac{x\times N}{2\pi k}\sin\left(\frac{2\pi}{N}kx\right)\Big|_{-2}^{2} - \frac{N}{2\pi k}\int_{-2}^{2}\sin\left(\frac{2\pi}{N}kx\right)dx\right.$

$\underline{N=4} \rightarrow \frac{x}{\pi k}\sin\left(\frac{\pi}{2}kx\right)\Big|_{-2}^{2} - \frac{2}{\pi k}\int_{-2}^{2}\sin\left(\frac{\pi}{2}kx\right)dx$

$= \cancel{\frac{2}{\pi k}\sin(k\pi)} + \cancel{\frac{2}{\pi k}\sin(-k\pi)} - \frac{2}{\pi k}\left[-\frac{2}{\pi k}\cos\left(\frac{\pi}{2}kx\right)\Big|_{-2}^{2}\right] = 0$

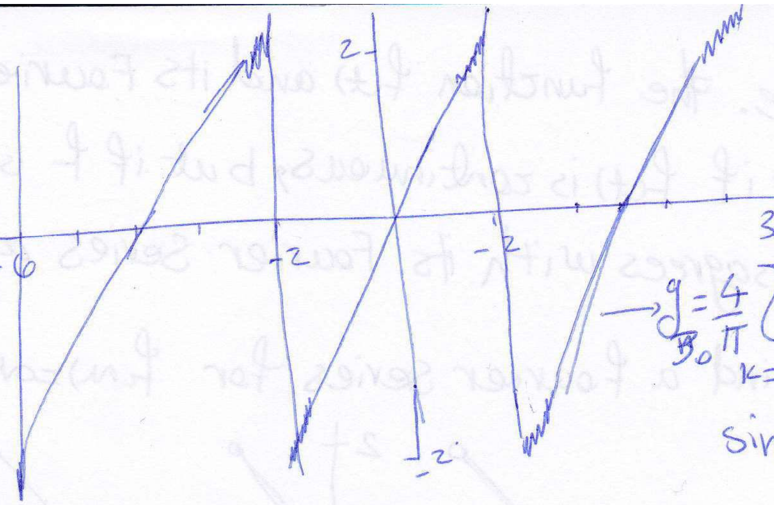
$b_k = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2\pi k}{N}x\right)dx = \frac{1}{2}\int_{-2}^{2} x\sin\left(\frac{\pi}{2}kx\right)dx$

$= \frac{1}{2}\left[\frac{-2x}{\pi k}\cos\left(\frac{\pi}{2}kx\right)\Big|_{-2}^{2} + \frac{2}{\pi k}\int_{-2}^{2}\cos\left(\frac{\pi}{2}kx\right)dx\right] = \frac{1}{2}\left[-\frac{4}{\pi k}\cos(k\pi) - \frac{4}{k\pi}\cos(k\pi)\right]$

$+\frac{2}{\pi k}\times\frac{2}{\pi k}\sin\frac{\pi k}{2}x\Big|_{-2}^{2} = \frac{-4}{k\pi}\cos(k\pi) = \begin{cases} \frac{-4}{k\pi} & k \text{ even} \\ \frac{4}{k\pi} & k \text{ odd} \end{cases}$

$= 0$

$$f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin\left(\frac{k\pi x}{2}\right)$$



$\longrightarrow g_{30} = \frac{4}{\pi} \sum_{k=1}^{30} \frac{(-1)^{k+1}}{k} \sin\left(\frac{k\pi x}{2}\right)$

for $n=3$. $\rightarrow$ convergence is not uniform around the ~~dis~~ jumping

Ex. find a Fourier series for $f(x) = \begin{cases} -2 & -1 \le x < 0 \\ 2 & 0 \le x \le 1 \end{cases}$

$f(x+2) = f(x)$

$N=2 \qquad a_k = \frac{2}{N}\int_a^b f(x)\cos\left(\frac{2\pi k}{N}x\right)dx = \int_{-1}^{1} f(x)\cos(\pi k x)\,dx$

$k=0 \rightarrow \quad a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} -2\,dx + \int_0^1 2\,dx = -2(0+1) + 2(1-0) = 0$

$a_k = \int_{-1}^{1} f(x)\cos(k\pi x)\,dx = \int_{-1}^{0} -2\cos(k\pi x)\,dx + \int_0^1 2\cos(k\pi x)\,dx$

$= \frac{-2}{k\pi}\sin(k\pi x)\Big|_{-1}^{0} + \frac{2}{k\pi}\sin(k\pi x)\Big|_0^1 = 0$

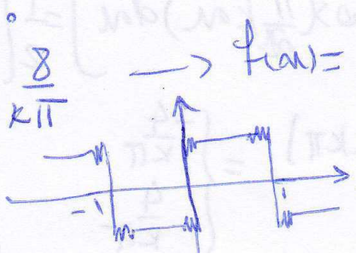
$b_k = \frac{2}{N}\int_a^b f(x)\sin\left(\frac{2\pi k}{N}x\right)dx = \int_{-1}^{1} f(x)\sin(k\pi x)\,dx = -2\int_{-1}^{0}\sin(k\pi x)\,dx$

$+ 2\int_0^1 \sin(k\pi x)\,dx = \frac{+2}{k\pi}\cos(k\pi x)\Big|_{-1}^{0} - \frac{2}{k\pi}\cos(k\pi x)\Big|_0^1$

$= \frac{2}{k\pi}(1 - \cos(k\pi)) - \frac{2}{k\pi}(\cos(k\pi) - 1) = \frac{4}{k\pi} - \frac{4}{k\pi}\cos(k\pi)$

$k =$ even $\rightarrow b_k = 0 \longrightarrow f(x) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin(\pi(2k-1)x)$

$k =$ odd $\rightarrow \quad b_k = \frac{8}{k\pi}$



$\longrightarrow g_{30}(x) = \frac{8}{\pi} \sum_{k=1}^{30} \frac{1}{(2k-1)} \sin(\pi(2k-1)x)$

④ proposition: The Fourier coefficients of a function $f(t)$ satisfy

$f(t)$ is even $\Rightarrow$ $b_k = 0 \quad \forall K$

$f(t)$ is odd $\Rightarrow$ $a_K = 0 \quad \forall K$

Q: what functions can be expressed in terms of a Fourier series?

Theorem: Fundamental convergence Theorem for Fourier series.

Let $V = \{ f(x) \mid \sqrt{\int_a^b f(x)^2 dx} < \infty \}$. Then for all $f(x) \in V$, there exists coefficients a_0, a_K, b_K $(K = 1, \dots)$ s.t

$$g_n(x) = \frac{a_0}{2} + \sum_{K=1}^{n} \left[a_k \cos\left(\frac{2\pi k}{N} x\right) + b_k \sin\left(\frac{2\pi k}{N} x\right) \right]$$

Fourier series of $f(x)$

converges to $f(x)$ for $n \to \infty$ in the sense that $\sqrt{\int_a^b (f(x) - g(x))^2 dx} = 0$

with $g(x) = \lim_{n \to \infty} g_n(x)$

This theorem holds for any $[a,b]$, but for simplicity we generally consider $[-\pi, \pi]$

V is called the set of square integrable functions and contains polynomials Sinusoids and other nicely behaved bounded functions.

V is also a vector space of functions. So, if $f_1(x) \in V$ and $f_2(x) \in V \Rightarrow$ $c_1 f_1(x) + c_2 f_2(x) \in V$ $(\forall c_1, c_2 \in \mathbb{R})$. Norm on V is defined by

$$\| h(x) \|_2 = \sqrt{\int_a^b h(x)^2 dx} \qquad (L_2 \text{ norm})$$

This norm measures the size of $h(x)$. Therefore, the a measure of distance between $f(x)$ and $g(x)$ both in V is defined by

$\|f(x)-g(x)\|_2 = \sqrt{\int_a^b (f(x)-g(x))^2 dx}$. is called

The set of functions V that are defined on $[a,b]$ $L_2[a,b]$, i.e.

$$L_2([a,b]) = \{ f(x) \mid \|f(x)\|_2 < \infty \}$$

where $\|\cdot\|_2$ is the L_2 norm

Complex form of the Fourier Series:

representing the Fourier series in the complex form is more natural. In this form only one sequence of coefficient should be found which is complex-valued.

Definition. The complex Fourier series of a function $f(t)$ is

$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$ with $f(t)$ defined on $[-\pi,\pi]$

and period 2π, $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt} dt$, $B = \{e^{ikx}\}_{-\infty<k<\infty}$

$\langle f(t), g(t)\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$

$\langle e^{ikt}, e^{ilt}\rangle = 2\pi \boxed{\delta_{kl}}$

$c_k = \frac{\langle f(t), e^{ikt}\rangle}{\langle e^{ikt}, e^{ikt}\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt} dt$

its relation between c_k and a_k, b_k:

using Euler's identity $\rightarrow$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)[\cos(kt) + i\sin(-kt)]\,dt =$$

$$\frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt - i\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt\right)$$

$$= \frac{1}{2}(a_k - ib_k) \qquad k \geq 0 \rightarrow \text{(can be extended to } -\infty \leq k < \infty)$$

note: $\delta_{kl} = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}$

⑤ ~~The~~ proposition: The real and complex Fourier coefficient of a real function $f(t)$ have the following properties:

1- $\bar{c}_k = c_{-k}$

2- $a_{-k} = a_k$, $b_{-k} = -b_k$

3- $a_k = 2Re(c_k)$, $b_k = -2Im(c_k)$

4- $f(t)$ even $\longrightarrow Im(c_k) = 0$,
$f(t)$ odd $\longrightarrow Re(c_k) = 0$

5- $b_0 = 0$, $c_0 = \frac{1}{2}a_0$

theorem. The complex and real forms of the Fourier series are equivalent.

use the Euler's identity to verify that.

EX: find the fourier series of the following function, which is assumed to have period 2π:
$$f(x) = \begin{cases} x + \pi & -\pi \le x \le 0 \\ \pi & 0 < x \le \pi \end{cases}$$
using the complex form:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (x+\pi) e^{-ikx}\,dx + \int_{0}^{\pi} \pi e^{-ikx}\,dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{x}{-ik} e^{-ikx}\Big|_{-\pi}^{0} + \frac{1}{(-ik)^2} e^{-ikx}\Big|_{-\pi}^{0} + \frac{\pi}{-ik} e^{-ikx}\Big|_{-\pi}^{0} + \frac{\pi}{-ik} e^{-ikx}\Big|_{0}^{\pi}\right] \quad \text{if } k \neq 0$$

$$= \frac{1}{2\pi}\left[\frac{-\pi}{ik} e^{ik\pi} + \frac{1}{k^2}(1 - e^{ik\pi}) - \frac{\pi}{ik}(1 - e^{ik\pi}) - \frac{\pi}{ik}(-1 + e^{-ik\pi})\right]$$

for $k=0 \longrightarrow c_k = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (x+\pi)\,dx + \int_{0}^{\pi} \pi\,dx\right] = \frac{1}{2\pi}\left[\frac{-1}{2}\pi^2 + \pi^2 + \pi^2\right] = \frac{3}{4}\pi$

$$\longrightarrow c_k = \begin{cases} \frac{i}{2k} & k \neq 0, \text{ even} \\ \frac{1}{k^2\pi} - \frac{i}{2k} & k \neq 0, \text{ odd} \\ \frac{3\pi}{4} & k = 0 \end{cases}$$

$\rightarrow f(x) = \frac{3\pi}{4} + \sum_{k \neq 0} \frac{i}{2k}(-1)^k e^{ikx} + \sum_{k \text{ odd}} \frac{1}{k^2\pi} e^{ikx}$

Discrete Fourier transferm:

The fourier series is used to describe a continuous time signal f(t) with $t \in [a,b]$, or f(t) ~~pre~~ periodic. Question is ~~how~~ how ~~to~~ can we de that if f is a discrete time signal $f[n]$ over N points in $0 \leq n \leq N-1$. This type of signal may arise from sampling or digital recording of a continuous time signal ~~us~~ such as music, image ~~ore~~ weather data. applying Discrete Fourier Transferm (DFT) to a discrete signal yields the frequencies present in the signal.

Background:

The N^{th} roots of unity are the Eq. solutions to the complex solution to the equation $x^N = 1$, and there are N we may write the N distinct N^{th} roots of unity as $W_N^k = e^{i\frac{2\pi k}{N}}$

$0 \leq k \leq N$

$$x^N = 1 \rightarrow e^{2\pi i} = e^{4\pi i} = e^{6\pi i} = \cdots = e^{2k\pi i}$$

$$\rightarrow (x^N)^{1/N} = x = e^{2\pi i/N} = e^{4\pi i/N} = \cdots = e^{2\pi k i/N}$$

$\rightarrow$ solution of the given equation is $x = e^{\frac{2\pi k}{N} i}$

$k = 0, \ldots, N-1$ ~~we sho~~ that is represented by $W_N^k = e^{\frac{2\pi k}{N} i}$

EX: solve the equation $x^3 = 1$ (find 3rd roots of unity)

$$W_3^k = e^{\frac{2\pi k}{3} i} \quad k = 0, 1, 2,$$

⑥ $\underbrace{K=0 \quad x=1}_{\text{trivial}}$, $K=1 \quad x=e^{\frac{2\pi}{3}i}$, $K=2 \quad x=e^{\frac{4\pi}{3}i}$,

$x=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)=\frac{-1}{2}+i\frac{\sqrt{3}}{2}$

$x=\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$

proposition: The N^{th} roots of unity satisfy the following property:

1) $\left(W_N^K\right)^N=1$ 2) $W_N^{-K}=W_N^{N-K}$

Definition: The discrete Fourier transform of a discrete time signal $f[n]$ with $0\leq n\leq N-1$ is

$$F[K]=DFT\{f[n]\}=\frac{1}{N}\sum_{n=0}^{N-1} f[n]\,W_N^{-Kn} \qquad 0\leq K\leq N-1$$

Definition: The inverse discrete Fourier transform of a discrete frequency signal $F[K]$ $0\leq K\leq N-1$ is

$$f[n]=IDFT\{F[K]\}=\sum_{K=0}^{N-1} F[K]\,W_N^{Kn} \qquad 0\leq n\leq N-1$$

Note: we assume ~~th~~ implicitly that the time signal $f[n]$ is periodic when applying the discrete Fourier transform, which necessarily implies that the frequency signal is also periodic

proposition. The frequency signal $F[K]$ is periodic with period N:

$$F[K]=F[K+SN] \quad \text{with } S\in\mathbb{Z}$$

proposition: For any real time signal $f[n]$, the frequency signal $F[k]$ satisfies

1) $Re(F[k])$ is even in k

2) $Im(F[k])$ is odd in k

3) $\overline{F[k]} = F[-k]$

4) $f[n]$ is even in $n \Rightarrow Im(F[k]) = 0 \rightarrow$ (DFT is real)

5) $f[n]$ is odd in $n \rightarrow Re(F[k]) = 0 \rightarrow$ DFT is purely imaginary

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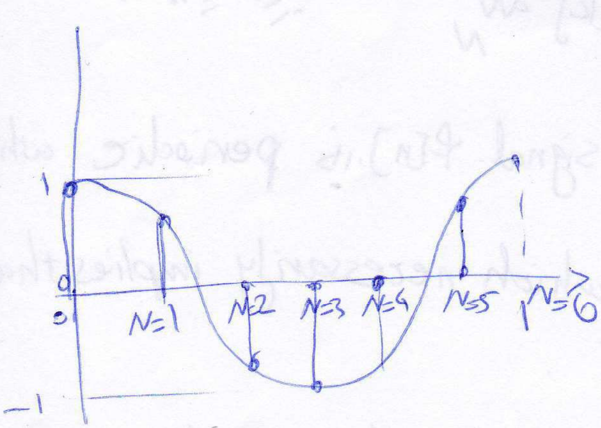
Ex: consider a cosine wave $f(t) = \cos(2\pi t)$, $t \in [0,1]$. We sample at $N=6$ points $f[n] = \cos(2\pi t_n)$, $t_n = \frac{n}{N}$, $0 \le n \le N-1$



$$f[n] = \cos\left(\frac{2\pi n}{N}\right) = \cancel{\cos\left(\frac{\pi}{3} n\right)}$$

$$= \frac{1}{2} e^{\frac{2\pi n}{N} i} + \frac{1}{2} e^{-\frac{2\pi n}{N} i}$$

$$= \frac{1}{2} W_N^{n} + \frac{1}{2} W_N^{-n}$$

by definition of IDFT, $F[1] = 1/2$ and $F[-1] = 1/2$

and others are zero

(7) Find the DFT of $(x_0, x_1, x_2, x_3) = (0,1,0,0)$

$$X[k] = \frac{1}{N}\sum x[n]\, e^{\frac{-2\pi i k n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, W_N^{-kn} \qquad 0 \le k \le N-1$$

$$\to X[k] = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{\frac{-2\pi i k n}{4}} = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{\frac{-\pi i k n}{2}} = e^{\frac{-\pi i k}{2}} \qquad 0 \le k \le N-1$$

$$X[0] = \frac{1}{4}, \quad X[1] = \frac{1}{4} e^{-\frac{\pi}{2} i} = -\frac{i}{4}, \quad X[2] = e^{-\pi i} = -\frac{1}{4}, \quad X[3] = e^{-\frac{3\pi i}{2}} = \frac{i}{4}$$

$$\to x[n] = IDFT\{x[k]\} = \sum_{k=0}^{3} X[k]\, e^{\frac{2\pi i k n}{4}} \qquad 0 \le n \le 3$$

$$= \sum_{k=0}^{3} X[k]\, e^{\frac{\pi i k n}{2}}$$

$$x[n] = X[0] + X[1]\, e^{\frac{\pi i n}{2}} + X[2]\, e^{\pi i n} + X[3]\, e^{\frac{3\pi i n}{2}}$$

EX: Find the 8-point DFT of the signal $x[n] = 6\cos^2(\frac{\pi}{4} n)$

$$x[n] = 6\left(\frac{1+\cos(\frac{\pi}{2} n)}{2}\right) = 3 + 3\cos(\frac{\pi}{2} n)$$

$$3 + \frac{3}{2}\left[e^{\frac{+i\pi n}{2}} + e^{\frac{-i\pi n}{2}}\right] = 3 + \frac{3}{2}\left[e^{\frac{2\pi i(2n)}{8}} + e^{-\frac{2\pi i(2n)}{8}}\right]$$

by matching the coefficient with $x[n]$ obtained by IDFT $\to$ $X[0] = 3$, $X[1] = 0$, $X[2] = \frac{3}{2}$, $X[3] = 0$

$X[4] = 0$, $X[5] = 0$, $X[6] = X[-2] = \frac{3}{2}$, $X[7] = 0$

Ex: consider the continuous signal $f(t) = 5 + 2\cos(2\pi t - \frac{\pi}{2}) + 3\cos(4\pi t)$. find 4-point DFT of $f(t)$.

$f_s = 4\,Hz \rightarrow T_s = 1/4$

$$f(t) = 5 + 2\left[\cos(2\pi t)\cos(\tfrac{\pi}{2}) + \sin(2\pi t)\sin(\tfrac{\pi}{2})\right] + 3\cos(4\pi t)$$

$$= 5 + 2\sin(2\pi t) + 3\cos(4\pi t) \xrightarrow{t = \frac{n}{N}} = 5 + 2\sin(\tfrac{2\pi n}{N}) + 3\cos(\tfrac{4\pi n}{N})$$

$$= 5 + \frac{2}{2i}\left[e^{\frac{2\pi n}{N} i} + e^{-\frac{2\pi n}{N} i}\right] + \frac{3}{2}\left[e^{\frac{4\pi n}{N} i} + e^{-\frac{4\pi n}{N} i}\right]$$

$$= 5 + i\left[e^{\frac{2n\pi}{N} i} + e^{-\frac{2n\pi}{N} i}\right] + \frac{3}{2}\left[e^{\frac{2n\pi i}{N}(2)} + e^{\frac{2n\pi i}{N}(-2)}\right]$$

by matching the coefficient $\frac{3}{2} \times 2 e^{\frac{2n\pi i}{N}(2)} = 3e^{\frac{2n\pi i}{N}}$

$F[0] = 5$, $F[1] = -i$, $F[-1] = F[3] = +i$, $F[2] = \frac{3}{2}$ because

$F[-2] = F[2] = \frac{3}{2}$

Aliasing:

consider the signals $f(t) = \cos(2\pi(2)t)$ and $f(t) = \cos(2\pi(4)(t))$

sampled at 6 points: $0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$

Applying DFT, we identify nonzero Fourier coefficient

$F_2 = F_{1/4} = 1/2$ for both signals.

⑧ ~~problem~~ for illustration consider $f(t)=\cos(2\pi l t)$, $T=\frac{1}{l} \to f=l\,Hz$

If we sample $f(t)$ with $N=6$ at $t_n=\frac{n}{N}=\frac{n}{6}$ with $\le n \le 6$ to obtain a discrete time signal $f[n]=\cos(2\pi l \frac{n}{N})$. then:

$l=0 \to f[n]=1 = $ ~~add~~ $1\times e^{\frac{2n\pi i}{N}(0)} \to F[0]=1$, $F[6]=1$

$l=1 \to f[n]=\cos(\frac{2\pi n}{N})=\frac{1}{2}[e^{\frac{2n\pi i}{N}}+e^{\frac{-2n\pi i}{N}}] \to F[1]=\frac{1}{2}$, $F[-1]=F[5]=\frac{1}{2}$

$l=2 \to f[n]=\cos(\frac{4n\pi}{N})=\frac{1}{2}[e^{\frac{2n\pi i}{N}(2)}+e^{\frac{-2n\pi i}{N}(2)}] \to F[2]=\frac{1}{2}$, $F[-2]=F[4]=\frac{1}{2}$

$l=3 \to f[n]=\cos(\frac{6n\pi}{N})=\frac{1}{2}[e^{\frac{2n\pi i}{N}(3)}+e^{\frac{-2n\pi i}{N}(3)}] \to F[3]=1$

$l=4 \to f[n]=\cos(\frac{8n\pi}{N})=\frac{1}{2}[e^{\frac{2n\pi i}{N}(4)}+e^{\frac{-2n\pi i}{N}(4)}] \to F[4]=\frac{1}{2}$, $F[2]=\frac{1}{2}$

we see that for $l=4$ and $l=2$, F is the same.

For small numbers of sampled values, there are different periodic functions that could fit the data, and we don't ~~know~~ have enough information to distinguish them. As we showed, sampled high frequency signals show up as low-frequency discrete signals. this causes, for example, poor picture quality and noise in reconstructed sounds.

~~To aveid~~ How to aveide this phenomenon?

Theorem: In order to aveid aliasing errors, the sampling frequency f_s should be at least ~~or~~ twice the largest frequency present in the continuous signal

In the above example, f_s is 6Hz, then the critical frequency for aliasing to occure is $f=3Hz$. for $f_s \geq 2f$ there is no aliasing and for $f_s < 2f$ aliasing occurs.

- ~~For~~ Human audible sound falls in the range 20Hz – 20KHz.

So, to avoid aliasing $f_s \geq 2f = 2\times 20\times 10^3 = 40KHz$. As a result of this requirement, digital music CDs have a sampling frequency of $f_s = 44.1$ KHz.

[if the signal reconstructed from samples is different from the original continuous signal, we have aliasing]

This is an effect that causes different signals to become indistinguishable.

Fast Fourier Transform (FFT)

To find the coefficients of $F[k]$ using $F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n] W_N^{-kn}$, $0 \leq k \leq N-1$

~~for each k~~ assuming W_N^{-kn} are given, for each k, $(N-1)$ addition and $(N+1)$ multiplication are required. So, per coefficient

$$W = (N+1)M + (N-1)A = \cancel{O(N^2)}\ 2N \text{ flops}$$

and totally for all coefficients $(N)(2N) = 2N^2$ flops are required, which is not efficient.

Q. How to compute ~~DFT coefficients~~ DF more efficiently?

To answer this question we need a theorem:

Theorem: If $N = 2^m$ for some integers m, then the length N DFT $F[k]$ of discrete time signal $f[n]$ can be calculated by combining two lengths $\frac{N}{2}$ discrete Fourier transforms.

(9) We define $g[n] = f[n] + f[n+\frac{N}{2}] \quad 0 \le n \le N/2 - 1$

$h[n] = (f[n] + f[n+\frac{N}{2}]) W_N^{-n} \quad 0 \le n \le \frac{N}{2} - 1$

Then $F[2l] = \frac{1}{2} DFT\{g[n]\}$ even indices

$F[2l+1] = \frac{1}{2} DFT\{h[n]\}$ odd indices

proof:
$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n] W_N^{-kn} = \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} f[n] W_N^{-kn} + \frac{1}{N}\sum_{n=\frac{N}{2}}^{N-1} f[n] W_N^{-kn}$$

$$= \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} f[n] W_N^{-kn} + \frac{1}{N}\sum_{l=0}^{\frac{N}{2}-1} f[l+\frac{N}{2}] W_N^{-k(l+\frac{N}{2})}$$

$$= \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1}\left(f[n] + f[n+\frac{N}{2}] W_N^{-\frac{kN}{2}}\right) W_N^{-kn}$$

$$W_N^{-\frac{kN}{2}} = e^{\frac{-2\pi i k(N/2)}{N}} = e^{-\pi i k} = (e^{-\pi i})^k = (-1)^k$$

case 1 $\quad k = 2l \quad 0 \le l \le \frac{N}{2} - 1$

$$\to F[2l] = \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1}\left(f[n] + f[n+\frac{N}{2}]\right) W_N^{-2ln} = \frac{1}{2}\left(\frac{1}{N/2}\sum_{n=0}^{\frac{N}{2}-1} g[n] W_{\frac{N}{2}}^{-ln}\right) = \frac{1}{2} DFT\{g[n]\}$$

case 2 $\quad k = 2l+1 \quad 0 \le l \le \frac{N}{2} - 1$

$$F[2l+1] = \frac{1}{N}\sum\left(f[n] - f[n+\frac{N}{2}]\right) W_N^{-2ln-n} = \frac{1}{2}\left(\frac{1}{\frac{N}{2}}\sum_{n=0}^{\frac{N}{2}-1} h[n] W_{\frac{N}{2}}^{-ln}\right) =$$

$$\frac{1}{2} DFT\{h[n]\}$$

By splitting the DFT into two transforms (each of length $\frac{N}{2}$), the total work will be

$$W_{tot} = 2 \cdot 2\left(\frac{N}{2}\right)^2 \text{flops} = N^2 \text{ flops}$$

while the direct ~~DFT~~ method requires $2N^2$ flops.

In splitting up the Fourier transform, only half of the original computational work is required, but we need to apply the splitting method recursively. In order to compute the splitting, we require $\frac{N}{2}$ additions for $g[n]$, $\frac{N}{2}$ additions and multiplications for $h[n]$, and N multiplications for the transform. In total $\frac{5}{2}N$ flops are required at each level (where N is the length of each level).

Theorem. The fast Fourier Transform (FFT) requires $O(N \log_2 N)$ flops.

EX: using FFT to calculate DFT([1, 2, 3, 4, 5, 6, 7, 8])

$f_0 = 1$ $\qquad g_0 = f_0 + f_4 = 6 \longrightarrow \frac{1}{2} g_0 = 3$

$f_1 = 2$ $\qquad g_1 = f_1 + f_5 = 8 \longrightarrow \frac{1}{2} g_1 = 4$

$f_2 = 3$ $\qquad g_2 = f_2 + f_6 = 10 \longrightarrow \frac{1}{2} g_2 = 5$

$f_3 = 4$ $\qquad g_3 = f_3 + f_7 = 12 \longrightarrow \frac{1}{2} g_3 = 6$

$f_4 = 5$ $\qquad h_0 = (f_0 - f_4) W_8^0 = -4 \longrightarrow \frac{1}{2} h_0 = -2$

$f_5 = 6$ $\qquad h_1 = (f_1 - f_5) W_8^{-1} = -2\sqrt{2} + i2\sqrt{2} \longrightarrow \frac{1}{2} h_1 = -\sqrt{2} + i\sqrt{2}$

$f_6 = 7$ $\qquad h_2 = (f_2 - f_6) W_8^{-2} = 4i \longrightarrow \frac{1}{2} h_2 = 2i$

$f_7 = 8$ $\qquad h_3 = (f_3 - f_7) W_8^{-3} = 2\sqrt{2} + 2i\sqrt{2} \longrightarrow \frac{1}{2} h_3 = \sqrt{2} + i\sqrt{2}$

① Next step: calculate $DFT([g_0, g_1, g_2, g_3])$: $\cancel{\frac{N}{2}=4}$ length: $4 \cancel{\frac{N}{2}}2$

$g_0 = 3$ $\quad gg_0 = \frac{1}{2}[g(0)+g_2] = 4$

$g_1 = 4$ $\quad gg_1 = \frac{1}{2}[g_1 + g_3] = 5$

$g_2 = 5$ $\quad hg_0 = \frac{1}{2}[g_0 - g_2]W_4^0 = \frac{1}{2}[3-5] = -1$

$g_3 = 6$ $\quad hg_1 = \frac{1}{2}[g_1 - g_3]W_4^{-1} = \frac{1}{2}(4-6)W_4^{-1} = i$

Next recursive step: length $= 2$

$gg_0 = 4$ $\quad ggg_0 = \frac{1}{2}[gg_0 + gg_1] = \frac{1}{2}(4+5) = 9/2$

$gg_1 = 5$ $\quad hgg_0 = \frac{1}{2}[gg_0 - gg_1]W_2^0 = -1/2$

$hg_0 = -1$ $\quad ghg_0 = \frac{1}{2}[hg_0 + hg_1] = -\frac{1}{2} + \frac{1}{2}i$

$hg_1 = i$ $\quad hhg_0 = \frac{1}{2}[hg_0 - hg_1]W_2^0 = \frac{1}{2}(-1-i) = -\frac{1}{2} - \frac{1}{2}i$

$h_0 = -2$ $\quad gh_0 = \frac{1}{2}[h_0 + h_2] = -1+i$

$h_1 = -\sqrt{2} + i\sqrt{2}$ $\quad gh_1 = \frac{1}{2}[h_1 + h_3] = i\sqrt{2}$

$h_2 = 2i$ $\quad hh_0 = \frac{1}{2}[h_0 - h_2]W_4^0 = -1-i$

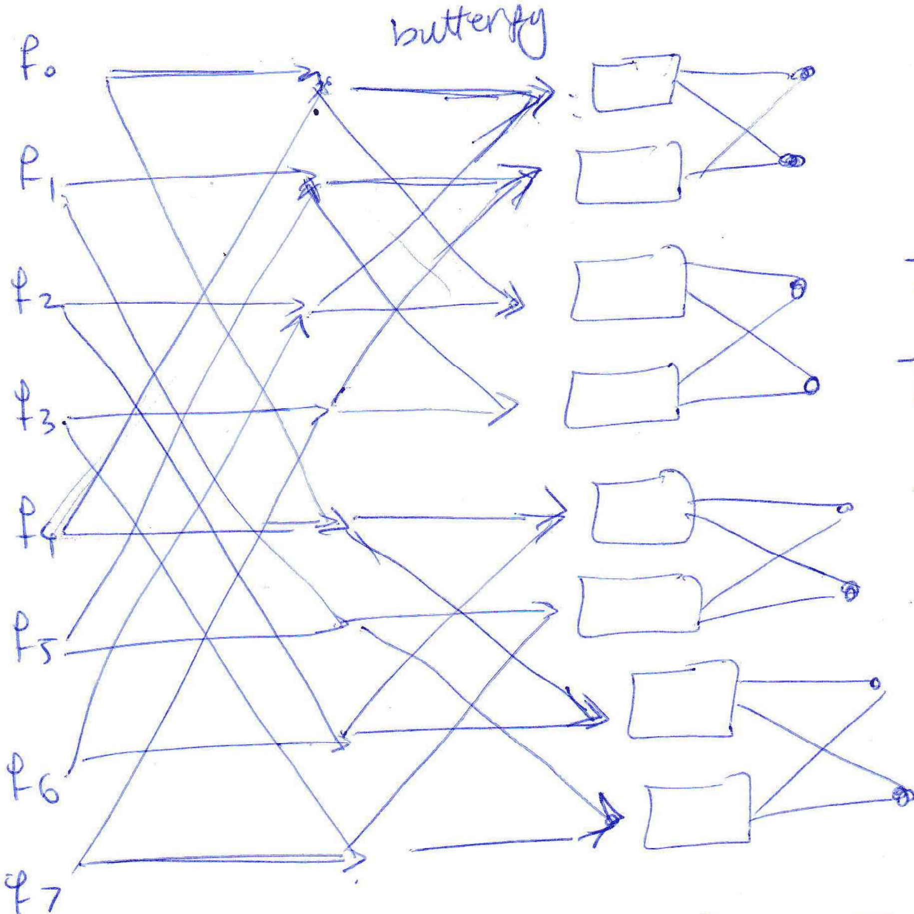
$h_3 = \sqrt{2} + i\sqrt{2}$ $\quad hh_1 = \frac{1}{2}(h_1 - h_3)W_4^{-1} = i\sqrt{2}$

$gh_0 = -1+i$ $\quad ggh_0 = \frac{1}{2}[gh_0 + gh_1] = -\frac{1}{2} + (\frac{1+\sqrt{2}}{2})i$

$gh_1 = i\sqrt{2}$ $\quad hgh_0 = \frac{1}{2}[gh_0 - gh_1]W_2^0 = -\frac{1}{2} + (\frac{1-\sqrt{2}}{2})i$

$hh_0 = -1-i$ $\quad ghh_0 = \frac{1}{2}(hh_0 + hh_1) = -\frac{1}{2} + (\frac{-1+\sqrt{2}}{2})i$

$hh_1 = i\sqrt{2}$ $\quad hhh_0 = \frac{1}{2}[hh_0 - hh_1]W_2^0 = -\frac{1}{2} + (\frac{-1-\sqrt{2}}{2})i$



$\frac{1}{2}$ ggg_0 $\longrightarrow$ F_0

$\frac{-1}{2}$ hgg_0 $\longrightarrow$ F_4

$-\frac{1}{2}+\frac{1}{2}i$ ghg_0 $\longrightarrow$ F_2

$-\frac{1}{2}-\frac{1}{2}i$ hhg_0 $\longrightarrow$ F_6

$-\frac{1}{2}+(\frac{1+\sqrt{2}}{2})i$ ggh_0 $\longrightarrow$ F_1

$-\frac{1}{2}+(\frac{1-\sqrt{2}}{2})i$ hgh_0 $\longrightarrow$ F_5

$-\frac{1}{2}+(\frac{-1+\sqrt{2}}{2})i$ ghh_0 $\longrightarrow$ F_3

$-\frac{1}{2}+(\frac{-1-\sqrt{2}}{2})i$ hhh_0 $\longrightarrow$ F_7

To unpack The results, we can use Bit reverse

calculated position	Binary	reverse bit	Actual coeffizien	
0	000	000	0	F_0
1	001	100	4	F_4
2	010	010	2	F_2
3	011	110	6	F_6
4	100	001	1	F_1
5	101	101	5	F_5
6	110	011	3	F_3
7	111	111	7	F_7

11) Or we can use reordering and decomposition:

| 0 1 2 3 4 5 6 7 |

| 0 2 4 6 | | 1 3 5 7 | (Interlaced decomposition)

| 0 4 | | 2 6 | | 1 5 | | 3 7 |

| 0 | | 4 | | 2 | | 6 | | 1 | | 5 | | 3 | | 7 |

Bit reveral explanation: In this algorithm, $\frac{N}{2}$ is added to the ~~at ea~~ present index from left to righ

EX: $N=8 \rightarrow \frac{N}{2}=4 \equiv 100$ (Binary)

$$
\begin{array}{lll}
① & 0\;0\;0 & \longrightarrow 0 \\
 & +1\;0\;0 & \\ \hline
② & 1\;0\;0 & \longrightarrow 4 \\
 & +1\;0\;0 & \\ \hline
③ & 0\;1\;0 & \longrightarrow 2 \\
 & +1\;0\;0 & \\ \hline
④ & 1\;1\;0 & \longrightarrow 6 \\
 & +1\;0\;0 & \\ \hline
⑤ & 0\;0\;1 & \longrightarrow 1 \\
 & +1\;0\;0 & \\ \hline
⑥ & 1\;0\;1 & \longrightarrow 5 \\
 & +1\;0\;0 & \\ \hline
⑦ & 0\;1\;1 & \longrightarrow 3 \\
 & +1\;0\;0 & \\ \hline
⑧ & 1\;1\;1 & \longrightarrow 7
\end{array}
$$

Another approach which is not very efficient in terms of occupying the memory is: $(N=8)$

1- start with 0

2- add $\frac{N}{2}$ to get the second index

3- add $\frac{N}{4}$ to the previous two indeces to the the next 2 indicise

4- add $\frac{N}{8}$ to the four previous indices to get the next 4 indices

1- 0 ✓

2- $0+\left[\frac{8}{2}\right]=4$ ✓

3- $\begin{cases} 0+\left[\frac{8}{4}\right]=2 & ✓ \\ 4+\left[\frac{8}{4}\right]=6 & ✓ \end{cases}$

4- $\begin{cases} 0+\left[\frac{8}{8}\right]=1 & ✓ \\ 4+\left[\frac{8}{8}\right]=5 & ✓ \\ 2+\left[\frac{8}{8}\right]=3 & ✓ \\ 6+\left[\frac{8}{8}\right]=7 & ✓ \end{cases}$

Notes:

convegence in L_2 norm implies convegence in L^1 norm

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we use complex numbers in Fourier transform because many information are in the phase not only in the amplitude. For example, for images most of the information is contained in phase.

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requirements of FFT:

1- N should be power of 2 $(N=2^m)$

2- inputs or outputs must be arranged according to a certain order to obtain the correct output. In cooley-Tukey algorithm, Bit reversal method is used

3- The main reason ~~is~~ ~~f~~ of Bit-reversal is ~~de~~ reducing the use of temporary memory when using a sequential processing engine.