MA2C03: TUTORIAL 3 PROBLEM SHEET

1) Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ for all sets A, B, and C.

Solution: We prove inclusion in both directions by the criterion for proving equality of sets that we stated at the beginning of the course.

"$\subseteq$" For every $x \in A \setminus (B \setminus C)$, $x \in A$ and $x \notin (B \setminus C)$. The second condition amounts to $x \notin B$ or $x \in C$. Therefore, we have $(x \in A)$ *and* $((x \notin B)$ *or* $(x \in C))$. We know the connective *and* distributes with respect to the connective *or*, so we get $(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in C)$. In other words, we have gotten $x \in (A \setminus B)$ or $x \in A \cap C$. The connective *or* translates to union, so we have $x \in (A \setminus B) \cup (A \cap C)$. We have thus proven that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$ as needed.

"$\supseteq$" For every $x \in (A \setminus B) \cup (A \cap C)$, $x \in (A \setminus B)$ or $x \in (A \cap C)$. Therefore, $(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in C)$. We thus have $x \in A$ and $(x \notin B$ or $x \in C)$. The second condition amounts to $x \notin (B \setminus C)$ by the de Morgan laws. We have shown $x \in A \setminus (B \setminus C)$ for every $x \in (A \setminus B) \cup (A \cap C)$. Therefore, $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$.

Please note that **NO Veitch or Venn diagrams** will be accepted as valid solutions. On a homework set or exam you are advised to follow this procedure when solving a problem that asks you to prove some statement in set theory.

2) Let A be the set of all people who have ever lived. For $x, y \in A$, xRy if and only if x and y were born less than one week apart. Determine:

(i) Whether or not the relation R is *reflexive*: Yes, it is reflexive as xRx must hold. A person is born less than a week from herself or himself.

(ii) Whether or not the relation R is *symmetric*: Yes, R is symmetric since xRy means x was born less than a week apart from y, which in turn means y was born less than a week apart from x, i.e. yRx holds.

(iii) Whether or not the relation R is *anti-symmetric*: No, R is symmetric, so $xRy \Rightarrow yRx$, which means xRy and yRx are both true at the same time without necessarily implying that $x = y$. Any two people born less than a week apart assigned to x and y provides a counterexample.

(iv) Whether or not the relation R is *transitive*: No, use as a counterexample a set of three people x, y, and z, where x and y are born five days apart, y and z are also born five days apart, but x and z are born ten days apart. Therefore, xRy and yRz both hold, but xRz is false contradicting transitivity.

(v) Whether or not the relation R is an *equivalence relation*: No, since R is not transitive.

(vi) Whether or not the relation R is a *partial order*: No, since R is neither anti-symmetric nor transitive.

3) (From the 2016-2017 Annual Exam) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = (x - y)(x + 2y).$$

Determine the following:

(i) Whether or not the relation R is *reflexive*;
(ii) Whether or not the relation R is *symmetric*;
(iii) Whether or not the relation R is *transitive*;
(iv) Whether or not the relation R is an *equivalence relation*;
(v) Whether or not the relation R is *anti-symmetric*;
(vi) Whether or not the relation R is a *partial order*.

Justify your answers.

Solution: $x, y \in \mathbb{Z}$ satisfy xRy iff $x - y = (x - y)(x + 2y)$, which is equivalent to $(x - y)(x + 2y - 1) = 0$, i.e., $x = y$ or $x + 2y - 1 = 0$.

(i) **Reflexivity:** The relation R is reflexive because xRx holds for all $x \in \mathbb{Z}$ as $x - x = (x - x)(x + 2x) = 0$.

(ii) **Symmetry:** The relation R is not symmetric because if $x \neq y$, then xRy holds if $x + 2y = 1$, thus for yRx we would need $y + 2x = 1$, which only holds at the same time with $x+2y = 1$ when $x = y = \frac{1}{3} \notin \mathbb{Z}$.

(iii) **Anti-symmetry:** The relation R is anti-symmetric. Having xRy and yRx when $x \neq y$ would imply $x+2y = 1$ and $y+2x = 1$ hold simultaneously, which gives $x = y = \frac{1}{3} \notin \mathbb{Z}$. Therefore, xRy and yRx can both be true only if $x = y$.

(iv) **Transitivity:** The relation R is not transitive. Assume xRy and yRz hold for $x, y, z \in \mathbb{Z}$. There are 4 cases to consider:
Case 1: $x = y$ and $y = z$, then $x = z$, so xRz as needed.
Case 2: $x = y$ and $y + 2z = 1$, then $x + 2z = 1$, so xRz as needed.
Case 3: $x + 2y = 1$ and $y = z$, then $x + 2z = 1$, so xRz as needed.

Case 4: $x + 2y = 1$ and $y + 2z = 1$, then $x + 2(1 - 2z) = 1$, so $x + 2 - 4z = 1$, i.e., $x - 4z = -1$. This last equation is satisfied for example for $x = 3$, $z = 1$. Take $y = -1$ in order to satisfy $x + 2y = 1$. We see that $x + 2z = 3 + 2 = 5 \neq 1$, so xRz fails. We have constructed a counterexample.

(v) **Equivalence relation:** The relation R is not an equivalence relation because while reflexive, it fails to be symmetric and transitive.

(vi) **Partial order:** The relation R is not a partial order because while reflexive and anti-symmetric, it fails to be transitive.

4) (From the 2016-2017 Annual Exam) Let $f : [-2, 2] \to [-15, 1]$ be the function defined by $f(x) = x^2 + 3x - 10$ for all $x \in [-2, 2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

There was no time to discuss problem #4, which is deferred to next week's tutorial.