MA2C03: TUTORIAL 5 PROBLEM SHEET

1) In the 1730's, the "Grande Loge" of Freemasons in Paris was a highly secretive society following some rather bizarre rules. Each of the freemasons in the lodge had shaved one other member. No freemason in the lodge had ever shaved himself. Furthermore, no freemason was ever shaved by more than one member of the lodge. There was one freemason who had never been shaved by any other member of the lodge. The number and identity of the freemasons in the lodge was kept secret. One rumour circulating in Paris at that time was that there were less than a hundred freemasons in the "Grande Loge." Another rumour put the number at over a hundred. Which one of the two rumours is true? Justify your answer.

Solution: This problem is an exercise in concept recognition. You're looking at Hilbert's hotel problem in disguise. Let x_0 be the freemason who has never been shaved by any member of the lodge. Let x_1 be the freemason x_0 shaves. Let x_2 be the freemason x_1 shaves and so on. We've constructed the map $x_{i-1} \to x_i$ for $i \geq 1$. If the number of freemasons were finite, we would have the scenario of musical chairs, which is ruled out by the problem. Therefore, the number of freemasons is infinite hence bigger than 100.

2) Where is the fallacy in the following argument by induction?

Statement: If p is an even number and $p \geq 2$, then p is a power of 2.

"Proof:" We give a proof using strong induction on the even number p. Denote by $P(n)$ the statement "if n is an even number and $n \geq 2$, then $n = 2^j$, where $j \in \mathbb{N}$."

Base case: Show $P(2)$. $2 = 2^1$, so 2 is indeed a power of 2.

Inductive step: Assume $p > 2$ and that $P(n)$ is true for every n such that $2 \leq n < p$ (the strong induction hypothesis). We have to show that $P(p)$ also holds. We consider two cases:

Case 1: p is odd, then there is nothing to show.

Case 2: p is even. Since $p \geq 4$ and p is an even number, we can write $p = 2n$ with $2 \leq n < p$. By the inductive hypothesis, $P(n)$ holds, so we conclude that $n = 2^j$ for some $j \in \mathbb{N}$. Since $p = 2n = 2 \times 2^j = 2^{j+1}$, we conclude that $P(p)$ also holds.

Solution: The argument fails at the inductive step as it is possible that $p = 2n$ and n is not even. For example, if $p = 6 = 2 \times 3$, the argument in the inductive step fails.

3) (Slight modification of a question from the 2016-2017 Annual Exam) Let $A = \{3^p \bigm| p \in \mathbb{N}\}$ with the operation of multiplication.

(a) Is $(A, \cdot)$ a semigroup? Justify your answer.
(b) Is $(A, \cdot)$ a monoid? Justify your answer.

Solution: (a) Yes, $(A, \cdot)$ is a semi-group. $A \subset \mathbb{Q}^*$, and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ is a monoid under the operation of multiplication. We proved in lecture that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$. Here $a = 3$ and since addition is a binary operation on $\mathbb{N}$ as we showed in class, multiplication is a binary operation on A. The associativity of multiplication on A follows from the associativity of addition on $\mathbb{N}$ and the theorem that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$.

(b) Yes, $(A, \cdot)$ is a monoid. $3^0 = 1$ is the identity element on A because any $b \in A$ is of the form 3^p, so $b \cdot 1 = a^p \cdot a^0 = a^{p+0} = a^{0+p} = 1 \cdot b = a^p = b$.