MA2C03: ASSIGNMENT 1
DUE BY FRIDAY, NOVEMBER 17
AT LECTURE OR IN THE MATHS OFFICE ROOM 0.6

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1) For any sets A and B, define $A \Delta B$, the **symmetric difference** of A and B to be the set $(A - B) \cup (B - A)$. Prove that intersection $\cap$ distributes over Δ : $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$ for all sets A, B, and C using the proof methods employed in lecture. Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

2) Let $\mathbb{R}$ be the set of real numbers. For $x, y \in \mathbb{R}$, $x \sim y$ iff $x - y \in \mathbb{Q}$, i.e., if the difference $x - y$ is a rational number. Determine:

(i) Whether or not the relation $\sim$ is *reflexive*;
(ii) Whether or not the relation $\sim$ is *symmetric*;
(iii) Whether or not the relation $\sim$ is *anti-symmetric*;
(iv) Whether or not the relation $\sim$ is *transitive*;
(v) Whether or not the relation $\sim$ is an *equivalence relation*;
(vi) Whether or not the relation $\sim$ is a *partial order*.

Justify your answers.

3) Let A be a set, and let $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$, where I is an indexing set, be any partition of the set A. Define a relation R on A as follows: $x, y \in A$ satisfy xRy iff $x, y \in A_\alpha$ for some $\alpha \in I$. In other words, xRy iff x and y belong to the same set of the partition. Prove that R is an equivalence relation and that the partition R defines on A is precisely the given partition $\mathcal{A}$. (Hint: Recall we discussed in lecture the one-to-one correspondence between partitions and equivalence relations, and this is the proof direction I sketched in lecture without providing the details.)

4) Where is the fallacy in the following argument by induction? Justify your answer.

Statement: For every non-negative integer k, $2 \times k = 0$.

"Proof:" We give a proof using strong induction on k. Denote by $P(k)$ the statement "if k is non-negative integer, then $2 \times k = 0$."

Base case: Show $P(0)$. Obviously, $2 \times 0 = 0$.

Inductive step: Assume $P(n)$ is true for every n such that $0 \leq n \leq k$ (the strong induction hypothesis). We have to show that $P(k+1)$ also holds. We write $k+1 = i+j$, where i and j are non-negative integers. By the inductive hypothesis,

$$2(k+1) = 2(i+j) = 2i + 2j = 0 + 0 = 0.$$

Therefore, by induction, $P(k)$ is true for all $k \in \mathbb{N}$, so $2 \times 1 = 0$.

5) Use mathematical induction to prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

(Hint: Recall we proved in lecture that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.)

6) Let $f : [-2, 0] \to [0, 1]$ be the function defined by $f(x) = \dfrac{1}{x^2 + 6x + 9}$ for all $x \in [-2, 0]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers. Recall that $[-2, 0]$ is the set of all real numbers between -2 and 0 with the end-points of -2 and 0 included in the set.

7) Let $A = \{(x, y) \in \mathbb{R}^2 \mid 2x - 3y = 0\}$ with the operation of addition given by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

(a) Is $(A, +)$ a semigroup? Justify your answer.
(b) Is $(A, +)$ a monoid? Justify your answer.
(c) Is $(A, +)$ a group? Justify your answer.