MA2C03: ASSIGNMENT 4 SOLUTIONS

1) (20 points)

(a) Is $\{(x, \sin(\pi x)) \mid x \in \mathbb{R}\} \cap \{\mathbb{Q} \times \mathbb{Z}\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(b) Is $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(c) Is $\left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} = 1\right\} \cap \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{9} + \frac{y^2}{4} = 1\right\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(d) Let $A = \{0, 1\}$. Is $(0^* \circ 1^*) \cap \{A^* \circ 11 \circ A^*\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

2) (20 points)

(a) Let L be the language consisting of the binary representations of all odd natural numbers. Write down the algorithm of a Turing machine that recognizes L. Process the following strings according to your algorithm: ϵ, 0, 11, and 100.

(b) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

3) (10 points) Write down the algorithm of an enumerator that prints out EXACTLY ONCE every odd natural number divisible by 5. (Hint: Order the words on the input tape.)

Solutions: 1 (a) $\sin(\pi x)$ is integer valued if $\sin(\pi x) \in \{-1, 0, 1\}$, which happens precisely when $x \in \left\{\frac{m}{2} \mid m \in \mathbb{Z}\right\}$. Therefore,

$$\{(x, \sin(\pi x)) \mid x \in \mathbb{R}\} \cap \{\mathbb{Q} \times \mathbb{Z}\} \sim \left\{\frac{m}{2} \mid m \in \mathbb{Z}\right\}$$

is countably infinite.

1 (b) This hyperbola is uncountably infinite. Restrict x to the interval $(1, 2)$ to get a subset of it, which is in bijective correspondence to $(1, 2)$ and hence to $(0, 1)$.

1 (c) The intersection of these two ellipses consists of four points $\left(\pm\frac{6}{\sqrt{13}}, \pm\frac{6}{\sqrt{13}}\right)$ as you can see by simultaneously solving the two equations, hence the set is finite.

1 (d) $A = \{0, 1\}$. $(0^* \circ 1^*) \cap \{A^* \circ 11 \circ A^*\} = 0^* \circ 11 \circ 1^*$. Each of 0^* and 1^* is countably infinite, so the given set is countably infinite.

Grading rubric: 5 points for each of the four parts. Out of the 5 points, 2 points are for the answer and 3 points are for the justification.

2 (a) Note that an odd natural number has a binary representation whose last digit is 1. We thus write down an algorithm that rejects ϵ and checks that a non-empty binary string ends in 1:

INPUT: string of 0's and 1's (the binary representation of a natural number)

1. If the entry in the first cell is ϵ, then REJECT; otherwise, move right.
2. If the entry in the current cell is 0 or 1, then move right. If the entry in the current cell is ϵ, then go to step 4.
3. Go to step 2.
4. Move left and examine the contents of the cell. If 0 is in that cell, then REJECT. If 1 is in that cell, then ACCEPT.

Processed strings are in the image at the end of the solutions.

Grading rubric: 10 points total, 8 points for the algorithm and 2 points for processing the strings. Points were taken off for incorrect or missing steps in the algorithm.

2 (b) The transition diagram is at the very end of the solutions.

Grading rubric: 10 points total. Points were taken off for incorrect or missing transitions or states.

3) Let us order the input tape as $\{w_1, w_2, w_3, \dots\} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, \dots\}$.
$E =$ Given the input tape $\{w_1, w_2, w_3, \dots\}$

(1) If $i = 5m + 3$ for $m \in \mathbb{N}$, then print w_i.

Grading rubric: 10 points total. The solution above is not the only correct one. Any correct solution was awarded full points. For incorrect solutions, points were taken off depending on whether the idea could lead to a possible correct solution and how proportionally incorrect the given solution was.

ε

$\boxed{\varepsilon}\boxed{}$ REJECT at step 1

0

$\boxed{0}\boxed{\varepsilon}$ start

$\boxed{0}\boxed{\varepsilon}$ step 1

$\boxed{0}\boxed{\varepsilon}$ REJECT at step 4

11

$\boxed{1}\boxed{1}\boxed{\varepsilon}$ start

$\boxed{1}\boxed{1}\boxed{\varepsilon}$ step 1

$\boxed{1}\boxed{1}\boxed{\varepsilon}$ step 2

$\boxed{1}\boxed{1}\boxed{\varepsilon}$ ACCEPT at step 4

100

$\boxed{1}\boxed{0}\boxed{0}\boxed{\varepsilon}$ start

$\boxed{1}\boxed{0}\boxed{0}\boxed{\varepsilon}$ step 1

$\boxed{1}\boxed{0}\boxed{0}\boxed{\varepsilon}$ step 2

$\boxed{1}\boxed{0}\boxed{0}\boxed{\varepsilon}$ step 3

$\boxed{1}\boxed{0}\boxed{0}\boxed{\varepsilon}$ step 2

$\boxed{1}\boxed{0}\boxed{0}\boxed{\varepsilon}$ REJECT at step 4

2(b)