MA2C03: TUTORIAL 17 PROBLEMS
COUNTABILITY OF SETS

For each of the following sets, determine whether it is finite, countably infinite, or uncountably infinite. Justify your answer.

1) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\}$

2) $\{(x, y, z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x, y, z \in \mathbb{N}^*\}$, the Pythagorean triplets that give the lengths of the legs and the hypothenuse of a right triangle.

3) $\bigcup_{q \in \mathbb{Q}} L_q$ where $L_q = \{(x, y) \in \mathbb{R}^2 \mid x = q\}$.

4) $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

5) $\mathcal{P}(J_n) \times \mathcal{P}(\mathbb{N})$, where $J_n = \{1, \ldots, n\}$ and $\mathcal{P}(A)$ is the power set of a set A.

6) $\mathbb{R}^n$ for $n \geq 1$.

Solution: 1) $A = \{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\}$ is countably infinite. Since $q \in \mathbb{R} \setminus \mathbb{Q}$, $a^{p_1} \neq a^{p_2}$ if $p_1 \neq p_2$, so the map $f : \mathbb{N} \to A$ given by $f(p) = a^p$ is a bijection. Therefore,

$$A = \{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\} \sim \mathbb{N}.$$

2) $\{(x, y, z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x, y, z \in \mathbb{N}^*\} \subset \mathbb{N}^3$, and we know from class that $\mathbb{N}^3$ is countably infinite. Therefore, our set can be finite or countably infinite. We remark that

$$(3, 4, 5) \in \{(x, y, z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x, y, z \in \mathbb{N}^*\}$$

as $3^2 + 4^2 = 9 + 16 = 25 = 5^2$. Furthermore,

$$(3p, 4p, 5p) \in \{(x, y, z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x, y, z \in \mathbb{N}^*\}$$

for every $p \in N^*$ as $3^2p^2 + 4^2p^2 = 9p^2 + 16p^2 = 5^2p^2$. Since $\mathbb{N}^* \sim \mathbb{N}$ is countably infinite, our set must likewise be countably infinite.

3) $L_q = \{(x, y) \in \mathbb{R}^2 \mid x = q\} = \{q\} \times \mathbb{R} \sim \mathbb{R}$. Therefore, $\bigcup_{q \in \mathbb{Q}} L_q$ is a union of disjoint uncountably infinite sets, so it must itself be uncountably infinite.

4) Consider the subset A of $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ given by

$$A = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \cap [(0, 1) \times \mathbb{R}].$$

The function $f(x) = x^2 = y$ is a bijection on $(0,1)$ (easy to check). Therefore, $\{(x,y) \in \mathbb{R}^2 \mid y = x^2\} \cap [(0,1) \times \mathbb{R}] \sim (0,1)$, so the set A is uncountably infinite as we proved in class that $(0,1)$ was uncountably infinite. Since $A \subset \{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$, the set $\{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$ must itself be uncountably infinite. Note that we have employed here a very standard technique for showing a set is uncountably infinite. It suffices to show it has an uncountably infinite subset.

5) We proved in Michaelmas term that the number of elements of a set with n elements is 2^n, so $\mathcal{P}(J_n)$ is a finite set with 2^n elements. By contrast, we proved in class that $\mathcal{P}(\mathbb{N})$ is uncountably infinite. Since $1 \in J_n$, the set consisting of a single element $\{1\} \in \mathcal{P}(J_n)$. Therefore, $\{1\} \times \mathcal{P}(\mathbb{N}) \subset \mathcal{P}(J_n) \times \mathcal{P}(\mathbb{N})$, but $\{1\} \times \mathcal{P}(\mathbb{N}) \sim \mathcal{P}(\mathbb{N})$. We conclude that $\mathcal{P}(J_n) \times \mathcal{P}(\mathbb{N})$ has an uncountably infinite subset, so it itself must be uncountably infinite.

6) For $n = 1$, we have already shown in class that $\mathbb{R}^1 = \mathbb{R}$ was uncountably infinite. Now for $n \geq 2$ consider

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\ \forall i\}.$$

The set

$$\mathbb{R} \times \{0\} \cdots \{0\} = \{(x_1, 0 \ldots, 0) \mid x_1 \in \mathbb{R}\} \subset \mathbb{R}^n,$$

but $\mathbb{R} \times \{0\} \cdots \{0\} \sim \mathbb{R}$, which is uncountably infinite. Therefore, $\mathbb{R}^n$ has an uncountably infinite subset, which means it must itself be uncountably infinite.