MA2C03: TUTORIAL 16 PROBLEMS
COUNTABILITY OF SETS

For each of the following sets, determine whether it is finite, countably infinite, or uncountably infinite. Justify your answer.

1) $\left\{ \left(\frac{m}{2}, \frac{n}{3}\right) \in \mathbb{R}^2 \;\middle|\; m, n \in \mathbb{Z} \right\}$

2) $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \cap \mathbb{Z}^2$

3) $\bigcup_{q \in \mathbb{Q}} L_q$ where $L_q = \{(x, y) \in \mathbb{R}^2 \mid x = q\} \cap (\mathbb{Q} \times \mathbb{N})$.

4) $\{2^p \mid p \in \mathbb{Z}\}$

5) $\{x \in \mathbb{C} \mid x^8 - 1 = 0\}$

6) $\{x \in \mathbb{R} \mid \cos x = 0\}$

7) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\}$

Solution: 1) $\mathbb{Z} \times \mathbb{Z} \subset \left\{ \left(\frac{m}{2}, \frac{n}{3}\right) \in \mathbb{R}^2 \;\middle|\; m, n \in \mathbb{Z} \right\} \subset \mathbb{Q} \times \mathbb{Q}$. Since both $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ and $\mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2$ are countably infinite as proven in class, the set itself must be countably infinite.

2) $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \cap \mathbb{Z}^2$ is a subset of $\mathbb{Z}^2$ by definition, and $\mathbb{Z}^2$ is countably infinite as proven in class. It remains to figure out if the set is finite or countably infinite. We note that all pairs (x, x^2) for $x \in \mathbb{Z}$ are in our set, which is clearly countably infinite because $\{(x, x^2) \mid x \in \mathbb{Z}\} \sim \mathbb{Z}$. Therefore, $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \cap \mathbb{Z}^2$ is countably infinite.

3) $L_q = \{(x, y) \in \mathbb{R}^2 \mid x = q\} \cap (\mathbb{Q} \times \mathbb{N}) = \{q\} \times \mathbb{N} \sim \mathbb{N}$. Therefore, $\bigcup_{q \in \mathbb{Q}} L_q$ is a union of disjoint countably infinite sets and thus countably infinite by the theorem proven in class.

4) $\{2^p \mid p \in \mathbb{Z}\} \sim \mathbb{Z}$ via the bijection $f(p) = 2^p$ (check it is a bijection). Therefore, the set is countably infinite.

5) $\{x \in \mathbb{C} \mid x^8 - 1 = 0\}$ consists of all roots of the polynomial $x^8 - 1 = 0$, which has degree 8. Therefore, there are at most 8 roots over $\mathbb{R}$ and exactly 8 roots over $\mathbb{C}$ by the Fundamental Theorem of Algebra. It means our set must be finite.

6)

$$\{x \in \mathbb{R} \mid \cos x = 0\} = \left\{ \frac{\pi}{2} + n\pi \;\middle|\; n \in \mathbb{Z} \right\} \sim \mathbb{Z},$$

so the set must be countably infinite.

7) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\}$ is a finite set. Let $q = \dfrac{r}{s}$ for $r, s \in \mathbb{Z}$, $s \neq 0$, $(r, s) = 1$. Therefore, $a^p = e^{\frac{pr\pi i}{s}}$, which assumes one of s values $e^{\frac{\pi i}{s}}, e^{\frac{2\pi i}{s}}, \dots, e^{\frac{(s-1)\pi i}{s}}, e^{\frac{s\pi i}{s}}$ depending upon the value of p. We conclude that our set is finite

$$\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\} = \left\{e^{\frac{\pi i}{s}}, e^{\frac{2\pi i}{s}}, \dots, e^{\frac{(s-1)\pi i}{s}}, e^{\frac{s\pi i}{s}}\right\}.$$