# Random Variables and Probability Functions

***Definition.*** A **random variable** is a function that maps the sample space $S$ into the set of real numbers $\mathbb{R}$. In other words, we say $X$ is a random variable if

$$X : S \to \mathbb{R}.$$

Often "random variable" is abbreviated with "RV" or "rv". We will use capital letters, like $X, Y, \ldots,$ to denote rv's.

- Why do we introduce random variables?
  Because by defining a random variable we can extract from the sample space $S$ the information that is relevant to our problem. By doing this we usually end up with a simpler mathematical model.

***Definition.*** The set of values that a random variable can take is called the **range** of the random variable.

We shall denote the range of a rv $X$ by $range(X)$.

***Definition.*** We say that a random variable is **discrete** if its range is a discrete subset of $\mathbb{R}$ (a finite or countably infinite set).
A random variable is **continuous** if its range is an interval (e.g. $[0, 1], (0, \infty), \mathbb{R}$).

It is possible to define a random variables whose range is a mixture of discrete and continuous parts, but we will not consider such cases in this course.

**Examples:**

| Experiment | $X$ | Range |
|---|---|---|
| Flip a coin | # of Heads | $\{0, 1\}$ |
| Roll two 6 sided dice | sum of the rolls | $\{2, ..., 12\}$ |
| Monitor call center traffic | # of calls to the center | $\{0, 1, 2, ...\}$ |
| Measure travel time to school | time measured | $(0, \infty)$ |

In “real life” we cannot actually observe continuous random variables, so in that sense every random variable derived from “real life” situations is discrete.

However, in many instances the range of a random variable is so dense that it makes more sense to model it as a continuum than as a discrete set.

# Probability Function

***Definition.*** The **probability function** (p.f.) $f_X$ of a discrete random variable $X$ with range $range(X) = A$ is defined as

$$f_X(x) := P(X = x) \text{ for } x \in A.$$

Comments:

- Formally we can define $f_X$ as

$$f_X(x) := P(X = x),$$

  but the function is different from zero only when $x$ is in $A$.

- The set

$$\{(x, f_X(x)) : x \in A\}$$

  is called the **probability distribution** of $X$.

- In the definition of p.f. we use the short hand notation $f_X(x) = P(X = x)$ for

$$f_X(x) = P(\{s \in S : X(s) = x\}).$$

  This representation reminds us that $f_X$ is "implied" by the given probability $P$ on the sample space $S$.

- All probability functions of discrete rv's must have two properties:

  1. $0 \leq f_X(x) \leq 1, \; x \in range(X)$
  2. $\sum_{x \in range(X)} f_X(x) = 1$

- By specifying a probability distribution we are in fact defining a probability model.

  Often we can do this without defining first $S$ and $P$, but rather by selecting one of the “standard models”.

  This is possible because many problems share the same structure.

- Once the p.f. $f_X$ of $X$ is known, we can find any probability of the form $P(X \in A)$ by using

$$\boxed{P(X \in A) = \sum_{y \in A} f_X(y). \qquad (1)}$$

- We often graph a probability function $f_X$ using a probability **histogram**, which is a graph consisting of adjacent bars or rectangles constructed as follows:

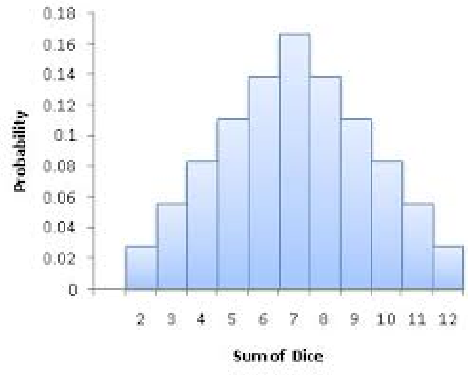
  at each $x$ from $range(X)$ we place a rectangle with

  base on $(x - 0.5; x + 0.5)$ and with height $f_X(x)$

  Notice that the area of each such bar is equal to $f_X(x)$.

  Therefore, the total area of all bars is equal to 1.

Histogram of the probability function corresponding to the sum of the outcomes of two fair six sided dice rolls.



**Example.** Suppose that $X$ is the sum of the outcomes of two fair **four**–sided dice rolls. Calculate the probability function of $X$.

**Clicker Question(s).**

# Cumulative Distribution Function

***Definition.*** The **Cumulative Distribution Function** (CDF or c.d.f.) of a random variable $X$ is

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

Again we use the short hand notation

$$P(X \leq x) = P(\{s \in S : X(s) \leq x\}).$$

- When $X$ is discrete, we can find $F_X$ from $f_X$ by using

$$F_X(x) = P(X \leq x) = \sum_{y:\ y \leq x} f_X(y) \ \text{ for each } x \in \mathbb{R}, \qquad (2)$$

which follows directly from (1).

- All CDF's must satisfy:
  1. $F_X(x) \leq F_X(y)$ for $x < y$
  2. $0 \leq F_X(x) \leq 1$ for all $x \in \mathbb{R}$.
  3. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

- Example: find the CDF of $X$ defined as

$$X = \text{ number of heads when we flip a fair coin once.}$$

- All CDF's of discrete r.v's share a particular structure. As functions of their argument, they are

  right-continuous step functions with jumps

  at points $x \in range(X)$

  For each $x \in range(X)$, the jump at $x$ is of the size

  $$F_X(x) - \lim_{y \to x^-} F_X(y) = f_X(x).$$

  If $X$ takes on integer values only, then for any $x \in range(X)$ we have

  $$F_X(x) - F_X(x-1) = f_X(x).$$

- The above facts show that if we know $F_X$ then we can easily find the corresponding p.f. $f_X$.

- Thus we can describe a probability model by using either p.f. or c.d.f. Typically we use the p.f., but for some problems working with the c.d.f. may be easier.

**Example.** Suppose that $N$ balls labelled $1, 2, \ldots, N$ are placed in a box, and $n$ balls ($n \leq N$) are randomly selected without replacement. Find the p.f. of the r.v. $X$ defined as

$$X = \text{smallest number selected.}$$

**Clicker Question(s).**

# Model Distributions

- We define a discrete probability distribution by specifying the set of pairs

$$\{(x, f(x)) : \ x \in A\}$$

  where the set $A$ is at most countable and the numbers $f(x), x \in A$, satisfy

  1. $0 \leq f(x) \leq 1, \ x \in A$
  2. $\sum_{x \in A} f(x) = 1$

- Equivalently, we can define the same probability distribution by using its c.d.f.

  In fact, by specifying either a p.d. or a c.d.f. we are defining a probability model.

- When we define a probability distribution in this way, we do not make reference to any particular random variable (or any particular "real life" situation).

  In fact there may be many random variables that have the same probability distribution.

  **Definition.** Two random variables $X$ and $Y$ are said to **have the same distribution** if

  $$F_X(x) = F_Y(x) \text{ for all } x \in \mathbb{R}.$$

  We shall denote this fact by

  $$X \sim Y.$$

- Consider:

| Experiment | |
|---|---|
| Flip a fair coin once | X= # of heads |
| Roll a fair six sided die | Y= 1 if die roll is even, 0 if die roll is odd |

Then,

$$X \sim Y.$$

- Other examples of problems that lead to the same type of distribution:
  (a) A fair coin is tossed 10 times and the "number of heads obtained" $X$ is recorded
  (b) Twenty seeds are planted in separate pots and the "number of seeds germinating" $X$ is recorded.

We shall consider the following model distributions:

- Discrete Uniform Distribution
- Hypergeometric Distribution
- Binomial Distribution
- Negative Binomial Distribution
- Geometric Distribution
- Poisson Distribution

We will also discuss

- Poisson Model (or Poisson Process )

For each model we will define a typical “physical setup” (or setting) that leads to a given distribution, and then will consider specific examples.

# Discrete Uniform Distribution

***Definition.*** We say that the random variable $X$ has a **discrete uniform distribution** on the set $\{a, a+1, \ldots, b\}$ if $X$ takes values

$$a, a+1, a+2, \ldots, b$$

with the same probability.

For this distribution,

$$f(x) = \begin{cases} \frac{1}{b-a+1} & \text{for } x = a, a+1, \ldots, b \\ 0 & \text{otherwise.} \end{cases}$$

Shorthand notation: $X \sim U_D(a, b)$.

Examples:

| Experiment | $X$ | Distribution |
|---|---|---|
| Roll a 6 sided die | # showing on the die | $U_D(1,6)$ |
| Draw an integer between 1 and 50 | # drawn | $U_D(1,50)$ |

**Clicker Question(s).**

# Hypergeometric Distribution

**Physical Setup:** Consider a population that consists of $N$ objects, of which

$r$ are considered "successes"

and

the remaining $N - r$ are considered "failures".

Suppose that a subset of size $n$ is drawn from the population **without replacement.** If we define

$$X = \text{the number of successes in this subset,}$$

then $X$ has a **hypergeometric distribution** (shorthand: $X \sim Hyp(N, r, n)$ or $X \sim hyp(N, r, n)$).

Examples of hypergeometric random variables:

| Experiment | $X$ | Distribution |
|---|---|---|
| Drawing 5 cards from a deck of 52 cards | # of Ace's | $Hyp(52, 4, 5)$ |
| Lotto 6/49 where 6 numbers are drawn from 49 | # Matches | $Hyp(49, 6, 6)$ |

**Problem:** Suppose $X \sim Hyp(N, r, n)$. Show that

$$f_X(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}, \quad \max\{0, n-(N-r)\} \le x \le \min\{r, n\}.$$

The fact that the p.f. of the hypergeometric distribution sums to 1 follows from the hypergeometric identity:

$$\binom{N}{n} = \sum_j \binom{r}{j}\binom{N-r}{n-j},$$

where the summation is taken over all $j$ that satisfy

$$\max\{0, n-(N-r)\} \le j \le \min\{r, n\}.$$

**Example.** Consider drawing a 5 card hand at random from a standard 52 card deck of playing cards (13 kinds: A,2,3,4,...,10,J,Q,K, in 4 suits: $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$).
By specifying proper hypergeometric distributions show that the probabilities of the following events

(i) the hand contains at least 3 K's
(ii) the hand contains 1 or fewer A's

are respectively

$$\frac{\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \quad \text{and} \quad \frac{\binom{4}{0}\binom{48}{5}+\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}.$$

**Exercise:** Find the probability of winning the jackpot prize in Lotto 6/49.

# Binomial Distribution

***Definition.*** A **Bernoulli trial** with probability of success $p$ is an experiment that results in either a success or failure, and the probability of success is $p$.

**Physical Setup:** Consider $n$ independent Bernoulli trials, where in each experiment the probability of success is $p, \ p \in [0, 1]$. Let

$$X = \text{ \# of successes obtained.}$$

Then $X$ has a **Binomial distribution** with parameters $n$ and $p$.

Shorthand: $X \sim Binomial(n, p)$ (or $X \sim Bin(n, p)$).

Examples of Binomial Random Variables:

1. Flip a fair coin independently 20 times, and let $X$ denote the number of heads observed. Then

$$X \sim Binomial(20, 0.5)$$

2. Consider writing digits $\{1, 2 \ldots, 9\}$ **with replacement** to form a 5 number sequence.
Let

$$X = \text{ the number of odd digits in the sequence.}$$

Then

$$X \sim Binomial(5, 5/9).$$

**Problem:** Show that the probability function of a Binomial random variable with parameters *n* and *p* is

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, ..., n.$$

**Example.** Suppose a tack when flipped has probability 0.6 of landing point up. If the tack is flipped 10 times, what is the probability it lands point up more than twice?

**Clicker Question(s).**

# Binomial approximation to the Hypergeometric Distribution

- Comparison of the distributions:

| | |
|---|---|
| Binomial: | independent repetitions with the same probability of success |
| Hypergeometric: | draws are without replacement (hence not independent) |

- Intuitively, if $n$ is relatively small compared with $N$ and $r$, then there should be little difference whether we draw with or without replacement.

- It can be shown that if $N$ and $r$ are large compared to $n$, and $p = r/N$, where $p \in (0, 1)$, then for

$$X \sim Hyp(N, r, n) \ \text{ and } \ Y \sim Binomial(n, p),$$

we have

$$P(X = k) \approx P(Y = k).$$

**Example.** In Overwatch there are 25 playable characters, of which 6 are considered “Tanks”. Suppose that three characters are drawn at random.

(i) What is the probability the selection contains exactly 2 tanks.

(ii) Approximate this probability using a binomial approximation

**Clicker Question(s).**

# Negative Binomial Distribution

**Physical Setup:** Consider an experiment in which Bernoulli trials are independently performed, each with probability of success $p$, until exactly $k$ successes are observed. Then the random variable

$$X = \# \text{ the number of failures before observing } k \text{ successes},$$

has a **Negative Binomial** distribution with parameters $k$ and $p$.

Shorthand notation:

$$X \sim NB(k, p).$$

Examples of Negative Binomial Random Variables:

1. Flip a fair coin until 5 heads are observed, and let $X$ denote the number of tails observed. Then
$$X \sim NB(5, .5).$$

2. Roll a fair die until the number 6 has appeared three times. Let $Y$ denote the number of die rolls required. Then
$$Y - 3 \sim NB(3, 1/6).$$

Key differences between the Negative Binomial and Binomial Distributions:

- The Binomial has a fixed number of trials while the Negative Binomial has an indefinite number of trials.
- The Binomial counts successes while the Negative Binomial counts failures, with the number of successes known.

**Problem.** Show that the probability function of a Negative Binomial random variable with parameters $k$ and $p$ is given by

$$f(x) = \binom{x+k-1}{k-1} p^k (1-p)^x, \quad x = 0, 1, 2, \ldots$$

To check that $\sum_x f(x) = 1$, we can follow two steps:

- first show

$$\binom{x+k-1}{x} = (-1)^x \frac{(-k)^{(x)}}{x!} = (-1)^x \binom{-k}{x},$$

- then use the Binomial Theorem

$$\sum_x f(x) = p^k \sum_{x=0}^{\infty} \binom{-k}{x} [(-1)(1-p)]^x = p^k [1 + (-1)(1-p)]^{-k} = 1$$

**Example.** The fraction of a large population that has a specific blood type T is 8%. For blood donation purposes it is necessary to find 5 people with this type blood. If randomly selected individuals from the population are tested one after another, then what is the probability y persons have to be tested to get 5 type T persons?

For the variable

$$Y = y \text{ persons have to be tested to get 5 type T persons}$$

we have $X := Y - 5 \sim NB(5, 0.08)$.
Thus

$$\begin{aligned} P(Y = y) &= P(Y - 5 = y - 5) = P(X = y - 5) \\ &= \binom{y - 5 + 5 - 1}{5 - 1} 0.08^5 0.92^{y-5} \\ &= \binom{y-1}{4} 0.08^5 0.92^{y-5}, \quad y = 5, 6, \ldots . \end{aligned}$$

**Clicker Question(s).**

**Exercise.** What is the p.f. of the number of failures before $k$ successes when we sample from a finite population without replacement?

See Problem 5.5.2. in the Notes.

# Geometric Distribution

The Geometric distribution is a special case of the Negative Binomial:

***Definition.*** We say the random variable $X$ has a **geometric** distribution with parameter $p$ if $X \sim NB(1, p)$ (shorthand $X \sim Geometric(p)$ or $X \sim Geo(p)$).

In this case, the p.f. of $X$ is

$$f_X(x) = (1-p)^x p, \quad x = 0, 1, 2, 3, ...$$

Check:

$$\sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty} (1-p)^x p = \frac{p}{1-(1-p)} = 1.$$

**Examples:**

(1) the number of a fair coin flips until it comes up heads follows a Geometric distribution with $p = 0.5$.

(2) the number of times we need to buy a lottery ticket before winning a prize follows a Geometric distribution with $p$ equal to the probability of winning in a single attempt.

Using the p.f. of the Geometric distribution, we can answer questions like:

(i) if our chance of winning a lottery prize is $p = 10^{-10}$, what is the probability that we will ever win?

(ii) what is the probability that if we buy 5 tickets every week we will need less than 5 weeks to win a lottery prize?

**Problem:** Suppose $X \sim Geo(p)$.

(a) Find the CDF of $X$.

(b) Show that

$$P(X \geq s + t | X \geq s) = P(X \geq t)$$

for all non-negative integers $s, t$.
Why this property of the Geometric distribution is called the “memoryless” property?

**Clicker Question(s).**

# Poisson Distribution

***Definition.*** We say the random variable $X$ has a **Poisson** distribution with parameter $\mu, \mu > 0$, if

$$f_X(x) = e^{-\mu}\frac{\mu^x}{x!}, \quad x = 0, 1, 2, 3, \ldots.$$

Shorthand: $X \sim Poisson(\mu)$ or $X \sim Poi(\mu)$.

Check:

$$\sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty} e^{-\mu}\frac{\mu^x}{x!} = e^{-\mu}e^{\mu} = 1.$$

# Poisson approximation of Binomial

- One way the Poisson distribution arises is as a limiting case of the Binomial distribution when $n \to \infty$ and $p \to 0$.

  **Fact:** If $n \to \infty$ and the probability of success $p_n$ in the $n$-th trial is such that

  $$np_n \to \mu, \ \ \mu > 0,$$

  then

  $$\binom{n}{x} p_n^x (1 - p_n)^{n-x} \to e^{-\mu} \frac{\mu^x}{x!}, \quad as \quad n \to \infty,$$

  for each $x = 0, 1, 2, \ldots$.

Suppose $p_n = \mu/n$. Then for each fixed $x$:

$$\begin{aligned}
f(x) &= \binom{n}{x}(p_n)^x(1-p_n)^{n-x} = \frac{n^{(x)}}{x!}(\frac{\mu}{n})^x(1-\frac{\mu}{n})^{n-x} \\
&= \frac{\mu^x}{x!}\frac{n(n-1)(n-2)\cdots(n-x+1)}{n\cdot n\cdots n}(1-\frac{\mu}{n})^{n-x} = \\
&= \frac{\mu^x}{x!}\underbrace{\frac{n}{n}\frac{n-1}{n}\cdots\frac{n-x+1}{n}}_{x \text{ terms}}(1-\frac{\mu}{n})^n(1-\frac{\mu}{n})^{-x} \\
&= \frac{\mu^x}{x!}(1)(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{x-1}{n})(1-\frac{\mu}{n})^n(1-\frac{\mu}{n})^{-x}.
\end{aligned}$$

By taking the limit when $n \to \infty$, we get

$$\frac{\mu^x}{x!}\underbrace{(1)\cdots(1)}_{x \text{ terms}}e^{-\mu}1^{-x} = \frac{\mu^x}{x!}e^{-\mu}.$$

- In practice, if $n$ is large and $p$ is close to zero, we can use the following approximation

$$\binom{n}{x} p^x (1-p)^{n-x} \approx e^{-\mu} \frac{\mu^x}{x!},$$

  where $\mu = np$.
  If $p$ is close to one, we can still use a similar approximation, but then we have to interchange the labels "success" and "failure".

- **Example:** A bit error occurs for a given data transmission method independently in one out of every 1000 bits transferred.
  Suppose a 64 bit message is sent using the transmission system. What is the probability that there are exactly 2 bit errors? Approximate this using a Poisson approximation.

# Poisson Process

Consider counting the number of occurrences of an event that happens at random points in time (or space).

Examples include:

(1) Counting emissions of radioactive particles from a radioactive substance
(2) The number of earthquakes occurring during a fixed time span.
(3) The number of deaths in a given time period of the policyholders of an insurance company.
(4) The number of cars entering a gas station during a day.

- Let us recall the “order” notation:

$$g(x) = o(x) \ \text{ as } \ x \to 0$$

means that the function $g$ approaches 0 faster than $x$ as $x$ approaches zero:

$$\frac{g(x)}{x} \to 0 \text{ as } \ x \to 0.$$

**Physical Setup:** Suppose the events we are counting satisfy the following three conditions:

1. **Independence:** the number of occurrences in non-overlapping time intervals are independent.

2. **Individuality:** for sufficiently short time periods of length $\Delta t$, the probability of 2 or more events occurring in the interval converges to zero at a rate faster than $\Delta t$:

$$P\,(2 \text{ or more events in } (t, t + \Delta t)) = o(\Delta t) \text{ as } \Delta t \to 0.$$

3. **Homogeneity or Uniformity:** events occur at a uniform (or homogeneous) rate $\lambda$ over time, meaning that for each $t$ the probability of one occurrence in $(t, t + \Delta t)$ is approximately $\lambda \Delta t$ in the following sense:

$$P\,(\text{one event in } (t, t + \Delta t)) = \lambda \Delta t + o(\Delta t) \text{ as } \Delta t \to 0.$$

- ***Definition.*** A process that satisfies the above three conditions on the occurrence of events is called a **Poisson process.**

  More precisely, if $X_t,\ t \geq 0$, denotes the number of events that have occurred up to time $t$, then the collection of random variables $\{X_t, t \geq 0\}$ is called a Poisson process.

- **Basic result for the Poisson process:**
  Let $X_t$ denote the number of events observed up to a fixed time $t,\ t > 0$. Then, assuming that the conditions (1)-(3) on the occurrence of events hold, we have

$$X_t \sim Poisson(\mu = \lambda \cdot t) \quad \text{for any} \quad t > 0.$$

- More generally:

  If $Y$ denotes the number of events observed during the time period from $s$ to $s+t$, $s, t > 0$, then, assuming that the conditions (1)-(3) on the occurrence of events hold, we have

  $$Y \sim Poisson(\mu = \lambda \cdot t).$$

  Note that the p.f. of $Y$ does not depend on $s$, only on the length of the time interval.

# Comments

- Interpretation of $\lambda$:

> $\lambda$ represents the average rate of occurrence of events per unit of time (or area or volume)

It is often referred to as the **intensity** or **rate of occurrence** parameter for the events.

- Interpretation of $\mu$:

> $\mu = \lambda t$ represents the average number of occurrences in t units of time.

- Important: the value of $\lambda$ (and $\mu$) depends on the units used to measure time (or area or volume).

**Example.** Suppose calls arrive at a telephone distress centre according to the conditions for a Poisson process, with the average of 2 calls per minute. Find an expression for the probability that there will be:

(a) no calls in a period of 10 minutes

$$e^{-20}$$

(b) 5 or more calls in a period of 10 minutes

$$\sum_{j=5}^{\infty} \frac{20^j}{j!} e^{-20} = 1 - \sum_{j=0}^{4} \frac{20^j}{j!} e^{-20}$$

(c) 5 calls in the first 5 minutes of a 10 minute period, given that 12 calls occur in the entire period.

$$\frac{e^{-10}\frac{10^5}{5!}\, e^{-10}\frac{10^7}{7!}}{e^{-20}\frac{20^{12}}{12!}}$$

(d) exactly 10 calls in a period of 10 minutes knowing that there were at least 5 calls

$$\frac{e^{-20}\frac{20^{10}}{10!}}{1-\sum_{j=0}^{4}\frac{20^j}{j!}e^{-20}}.$$

**Exercise.** Find an expression for the probability that:

(i) there will be less than 3 calls in the first 5 minutes of a 10 minute period, given that 12 calls occur in the entire period.

(ii) there will be $x$ calls in the first 5 minutes of a 10 minute period, given that $t$ calls occur in the entire period $(x = 0, 1, 2, \ldots, t)$.

(iii) all the calls in a 10 minute period occur in the first 5 minutes.

# Random Occurrence of Events in Space

- The Poisson process also applies when “events” occur randomly in space (either 2 or 3 dimensions).
- For example, the “events” might be bacteria in a volume of water or spruce budworms distributed through a forest.
- For the Poisson process to be a valid model in such cases, we have to assume that the conditions (1)-(3) still hold, with “time” replaced by “volume” or “area”.
- We interpret $\lambda$ as the average number of events per unit volume (or area). Then the random variable

  $$X = \text{number of events in a volume or area in space of size } v$$

  satisfies

  $$X \sim Poi(\mu = \lambda \cdot v).$$

# Clicker Question(s)

# Combining Other Models with the Poisson Process

- In order to find the probabilities of some events, we will need to use two or more of the standard distributions.
- To handle this type of problems we need to be clear about the characteristics of each model.

**Clicker Question(s)**

**Problem.** Website hits for a given website occur according to a Poisson process with a rate of 100 hits per minute. We say a second is a "break" if there are no hits in that second.

(i) What is the probability of a break in any given second?

(ii) Compute the probability of observing exactly 10 breaks in 60 consecutive seconds.

(iii) Compute the probability that one must wait for 30 seconds to get 2 breaks.

**Problem.** Suppose that car accidents occur within a given day according to a Poisson process. If it is raining, the rate parameter of the Poisson process is 3 accidents per hour. If it is not raining, the rate parameter is 1 accident per hour. The probability that it is raining on a given day is 0.05.

(i) What is the probability that there are no accidents in a 30 minute period on a given day?

(ii) Given that there are no accidents in a 30 minute period on a given day, calculate the probability that it was raining that day.

**Problem.** Shiny versions of Pokemon are possible to encounter and catch starting in Generation 2 (Pokemon Gold/Silver). Normal encounters with Pokemon while running in grass occur according to a Poisson process with rate 1 per minute on average. 1 in every 8192 encounters will be a Shiny Pokemon, on average.

(i) If you run around in grass for 15 hours, what is the probability you will encounter at least one Shiny pokemon?

(ii) How long would you have to run around in grass so that you have a better than 50 percent chance of encountering at least one Shiny pokemon?

## 5.10 Summary of Probability Functions for Discrete Random Variables

| Name | Probability Function |
|---|---|
| Discrete Uniform | $f(x) = \frac{1}{b-a+1}; x = a, a+1, a+2, \ldots, b$ |
| Hypergeometric | $f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}; x = \max(0, n-(N-r)), \ldots, \min(n, r)$ |
| Binomial | $f(x) = \binom{n}{x} p^x (1-p)^{n-x}; x = 0, 1, 2, \ldots, n$ |
| Negative Binomial | $f(x) = \binom{x+k-1}{x} p^k (1-p)^x; x = 0, 1, 2, \ldots$ |
| Geometric | $f(x) = p(1-p)^x; x = 0, 1, 2, \ldots$ |
| Poisson | $f(x) = \frac{e^{-\mu}\mu^x}{x!}; x = 0, 1, 2, \ldots$ |