Uniform Probability Model

- Let us recall the definition of the *uniform probability model*.

  If the following two conditions are satisfied:

  (i) the sample space S has a **finite number** of outcomes

  (ii) simple events are **equally likely**,

  then the probability of any event $A \subseteq S$ is given by

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S} \equiv \frac{|A|}{|S|}.$$

- To use this model in practice we need to:
  - define a proper sample space
  - find an efficient way of counting the number of outcomes for the events of interest.

Example. Suppose the letters of the word STATISTICS are arranged at random.
Find the probability of the event that the arrangement begins and ends with S.

Example. Suppose we make a four digit number by randomly selecting and arranging four digits from the set

$$1, 2, 3 \ldots, 7$$

without replacement.

Find the probability that the number formed is an even number over 3000.

Addition and Multiplication Rules

The Addition Rule:

- if we can do job 1 in p ways and job 2 in q ways, then we can do either job 1 **OR** job 2 (but not both) in $p + q$ ways.

Examples:

- in how many ways can we select a person from a group of 80 women and 60 men?
  Answer: in $80 + 60 = 140$ ways
- in how many ways can we pick a number from the set $\{1, 2, 3, \ldots, 19, 20\}$ that is either divisible by 5 or by 7?
  Answer: in $4 + 2 = 6$ ways.

The Multiplication Rule:

- if we can do job 1 in p ways and, **for each of these ways**, job 2 in q ways, then we can do both job 1 **AND** job 2 in $p \times q$ ways.

Examples:

- in how many ways can we form a pin number of length two by randomly selecting two digits from $\{0, 1, 2, 3, \ldots, 9\}$ **with replacement**[1]?

  Answer: in $10 \times 10 = 100$ ways

- the same but **without replacement.**

  Answer: in $10 \times 9 = 90$ ways.

[1] "with replacement" means that after the first number is picked it is "replaced" in the set of numbers so it could be picked again

Counting Arrangements

In some problems, the sample space will be a set of arrangements or sequences (also called *permutations*).

Suppose we have n **different** objects.

Q1: In how many ways can we arrange them in a sequence? (thus the order matters)

Idea: count the number of ways we can fill the n positions in the sequence. This gives us

$$n \times (n-1) \times \cdots \times 1 = n! \quad (\text{“}n \text{ factorial”}).$$

Thus, here we are using the Multiplication Rule!

Note that the result is the same regardless of the order in which we fill the positions.

To get some sense how quickly $n!$ increases for large n, we can use the following Stirling's approximation

$$n! \approx (\frac{n}{e})^n \sqrt{2\pi n},$$

where "$\approx$" means that the two sequences are *asymptotically equiavlent*[2].

For example, for $n = 10$ we have

$$10! = 3628800$$

while the Stirling's approximation gives

$$3598695.6.$$

The relative error is less than 0.01.

[2] $\{a_n\}$ is asymptotically equivalent to $\{b_n\}$ if $\lim_{n\to\infty} a_n/b_n = 1$.

Q2: In how many ways can we form arrangements of length k ($k \leq n$) using each object at most once?
Using the same idea as before, the answer is

$$n \times (n-1) \times \cdots \times (n-k+1).$$

We shall denote this product by $n^{(k)}$ ("n to k factors").
Note

$$n^{(k)} = \frac{n!}{(n-k)!}, \quad k = 1, 2, \ldots, n,$$

where we use the convention $0! = 1$.

Q3: In how many ways can we form arrangements of length k ($k \leq n$) using each object as often as we wish?
The answer

$$n \times n \times \cdots \times n = n^k.$$

Example. A pin number of length three is formed by randomly selecting three digits from $\{0, 1, 2, 3, \ldots, 9\}$ **with replacement.** Find the probabilities of the following events:

- A: the pin number is even
- B: the pin number has only odd digits
- C: all the digits are unique
- D: the pin number contains at least two 1.
- E: all the digits are the same

Example. A pin number of length three is formed by randomly selecting three digits from $\{0, 1, 2, 3 \ldots, 9\}$ **without replacement.** Find the probabilities of the following events:

A: the pin number is even

B: the pin number has only odd digits

C: all the digits are unique

D: the pin number contains at least two 1.

E: all the digits are the same

Clicker Question(s).

Combinations (counting subsets)

- In some problems, the outcomes in the sample space will be subsets of a given set.

For example, when we randomly select a subset of two digits from the set $\{1, 2, 3, 4, 5\}$, then

$$S = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \dots, \{4, 5\}\}.$$

- Suppose we have n different objects $o_1, \dots, o_n$.

Q: In how many different ways can we select a <u>subset</u> of k objects from the set $\{o_1, \dots, o_n\}$?

Idea: use the known number of different arrangements of k objects out of n (given by $n^{(k)}$).

Let

m = the number of ways we can select a subset of k objects from the set $\{o_1, \ldots, o_n\}$.

Then

$$m \times k! = n^{(k)}.$$

Therefore, the answer to our question is

$$m = \frac{n^{(k)}}{k!} = \binom{n}{k},$$

where the combinatorial symbol $\binom{n}{k}$ ("n choose k") is

$$\binom{n}{k} := \frac{n^{(k)}}{k!} \equiv \frac{n!}{(n-k)!k!}.$$

- **Example.** Suppose a box contains 8 balls of which 3 are red, 2 are white and 3 are green. A sample of 4 balls is selected at random **without replacement.** Verify that the probabilities of the following events:

  A = the sample contains 2 red balls

  B = the sample contains 2 or more red balls

  are respectively

$$P(A) = \frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} = \frac{3}{7} \quad \text{and} \quad P(B) = \frac{\binom{3}{2}\binom{5}{2} + \binom{3}{3}\binom{5}{1}}{\binom{8}{4}} = \frac{1}{2}.$$

Clicker Question(s).

Arrangements with Repeated Symbols

Suppose we have n_i symbols/objects of type $i, \ i = 1, 2, \ldots, k$, with

$$n_1 + n_2 + \cdots + n_k = n.$$

Q: What is the number of arrangements using all the symbols?

Example. A box contains 8 balls of which 3 are red, 2 are white and 3 are green. Suppose we randomly draw one ball at a time **without replacement** and after each draw we record the color.
How many different outcomes are possible?

Here we need to count arrangements when some elements are the same.

Method 1: construct arrangements by filling 8 boxes corresponding to the eight positions in the arrangement.

Method 2: treat the balls as different, but then divide the outcome by the number of permutations in each group with the same color.

Suppose we have n_i symbols/objects of type i, $i = 1, 2, \ldots, k$, with

$$n_1 + n_2 + \cdots + n_k = n.$$

Q: What is the number of arrangements using all the symbols?

Answer:

$$\underbrace{\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k}}_{\text{Method 1}} = \underbrace{\frac{n!}{n_1!n_2!\cdots n_k!}}_{\text{Method 2}}.$$

Note: the symbol

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

if often referred to as **multinomial coefficient**.

Example. In how many ways can we arrange the letters of the word STATISTICS?

Here we have $n = 10$, but only 5 different symbols: S, T, A, I, and C.
Among them:

S appears $n_1 = 3$ times,
T appears $n_2 = 3$ times,
I appears $n_3 = 2$ times,
A appears once ($n_4 = 1$),
C appears once ($n_5 = 1$).

Using the general formula, we get the answer

$$\frac{10!}{3!3!2!1!1!} = 50400.$$

Suppose that the letters of the word STATISTICS are arranged at random. What is the probability of the event A that the arrangement begins and ends with T?

The number of equally likely outcomes in our sample space S is

$$|S| = \frac{10!}{3!3!2!} = 50400.$$

We need to count outcomes in

$$A = \{TSATISICST, \ldots\}.$$

By counting the number of ways we can fill the remaining 8 positions we get

$$|A| = \frac{8!}{3!2!1!1!1!} = 3360.$$

Thus, the answer is

$$P(A) = \frac{3360}{50400} = \frac{1}{15}.$$

Additional Examples

Example. Suppose we make a random arrangement of length 3 using the letters form

$$\{a, b, c, d, e, f\}.$$

What is the probability of the event

$$B = \text{"letters are in alphabetic order"}$$

if

(a) letters are selected without replacement?
(b) letters are selected with replacement?

- For part (a):

$$S = \{abc, acb, bca, \ldots\}$$

and the answer is

$$P(A) = \frac{\binom{6}{3}}{6^{(3)}}.$$

- For part (b):

$$S = \{aaa, aab, baa, \ldots\}$$

and the answer is

$$P(A) = \frac{6 + 2\binom{6}{2} + \binom{6}{3}}{6^3}.$$

Example. Suppose we make a four digit number by randomly selecting and arranging four digits from the set

$$1, 2, 3 \ldots, 7$$

without replacement.

Find the probability that the number formed is

(a) even
(b) over 3000
(c) an even number over 3000.

Answers:

- For part (a):

$$\frac{3 \times 6^{(3)}}{7^{(4)}} = \frac{3}{7}.$$

- For part (b):

$$\frac{5 \times 6^{(3)}}{7^{(4)}} = \frac{5}{7}.$$

- For part (c):

$$\frac{2 \times 2 \times 5^{(2)} + 3 \times 3 \times 5^{(2)}}{7^{(4)}} = \frac{5 \times 5^{(2)} + 4 \times 5^{(2)} + 4 \times 5^{(2)}}{7^{(4)}} = \frac{13}{42}.$$

Example. In a race, the 15 runners are randomly assigned the numbers $1, 2, \ldots, 15$. Find the probability that[3]

(a) 3 of the last 4 runners have single digit numbers

(b) the fifth runner to finish is the 3rd finisher with a single digit number.

Solutions:

(a) $\frac{\binom{9}{3} \cdot 6}{\binom{15}{4}}$

(b) $\frac{\binom{9}{3}\binom{6}{2} \cdot 3 \cdot 4!}{15^{(5)}}$

[3] See Problem 3.5.2 in the Notes.

Clicker Question(s).

Properties of $\binom{n}{k}$

Define $0! = 1$. Then $\binom{n}{k}$ has the following properties for n and k being non-negative integers with $n \geq k$:

1. $n^{(k)} = \frac{n!}{(n-k)!} = n(n-1)^{(k-1)},\ k \geq 1$
2. $\binom{n}{k} = \binom{n}{n-k} = \frac{n^{(k)}}{k!},\ k = 0, 1, \ldots, n.$
3. $\binom{n}{0} = \binom{n}{n} = 1$
4. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
5. Binomial Theorem:

$$\begin{aligned}(1+x)^n &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k.\end{aligned}$$

6. Note that

$$n^{(k)} = n(n-1)\cdots(n-k+1)$$

is still well defined when n is a real number and k is a non-negative integer.

We will use this fact later when we discuss an extension of the Binomial Theorem.