MA2C03 Assignment 2
Solutions and Marking

1) (a) (5 points) The language is $K = \{(ab)^n c : n \geq 0\}$. We can prove by induction on string length that a string w generated by these production rules is of one of the following forms:

- $w = (ab)^n \langle S \rangle$,
- $w = (ab)^n a \langle A \rangle$,
- $w = (ab)^n c$.

Here $n \geq 0$. Of these, only those of the form $w = (ab)^n c$ contain only terminals.

Grading rubric: 5 points total: 2 for realizing the form of words in this language is $(ab)^n c$, 1 point for realizing $n \geq 0$, and 2 points for the justification.

(b) This grammar is regular. Using the designations from the notes, rules (1) and (2) are of form (i) and rule (3) is of form (ii).

Grading rubric: 5 points total: 2 for the answer and 3 for the justification.

(c) This language is not in normal form, as rule (3) is of form (ii). To bring it into normal form, introduce a new non-terminal $\langle B \rangle$ and use the production rules

(1') $\langle S \rangle \to a \langle A \rangle$,
(2') $\langle A \rangle \to b \langle S \rangle$,
(3') $\langle S \rangle \to c \langle B \rangle$,
(4') $\langle B \rangle \to \epsilon$.

As before, we can show by induction that a string w generated by these production rules is of is of one of the following forms:

- $w = \langle S \rangle$,
- $w = (ab)^n \langle S \rangle$,
- $w = (ab)^n a \langle A \rangle$,
- $w = (ab)^n c \langle B \rangle$,
- $w = (ab)^n c$.

Only those of the form $w = (ab)^n c$ contain only terminals, so that the language we generate is the same.

Grading rubric: 5 points total: 2 for the answer, 2 for the modification, and 1 point for the justification.

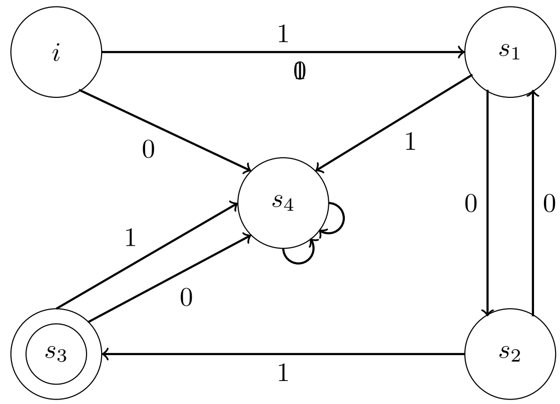
(d) The regular expression $(ab)^* \circ c$ gives the language $\{(ab)^n c : n \geq 0\}$, which is exactly our language K.

Grading rubric: 5 points total: 3 for getting the $(ab)^*$ part of the answer and 2 for getting the final concatenation with c.

2) (a) Let $S = \{i, s_1, s_2, s_3, s_4\}$ be the set of states and $F = \{s_3\}$ be the set of finishing states. Let the transitions be given by

$$\begin{array}{lllll} t(i,0) = s_4 & t(s_1,0) = s_2 & t(s_2,0) = s_1 & t(s_3,0) = s_4 & t(s_4,0) = s_4 \\ t(i,1) = s_1 & t(s_1,1) = s_4 & t(s_2,1) = s_3 & t(s_3,1) = s_4 & t(s_4,1) = s_4. \end{array}$$

The corresponding diagram is



Note that if a string causes the acceptor to enter state s_4, that string will not be accepted. Take a string w and suppose $w = 10^{2m+1}1 \in L$. Then the acceptor goes from state i to state s_1. It will then alternate between states s_1 and s_2, before going to state s_3 and being accepted.

Otherwise, if w fails to start with '1' it will go to state s_4. If w starts with '1' but fails to have an odd number of zeroes afterwards, it will alternate between states s_1 and s_2, before going to state s_4. If w starts with '1' and has an odd number of zeroes afterwards, but fails to end after the second '1', then it will reach state s_3 before going to state s_4. In this way, any w not in L will not be accepted.

Grading rubric: 5 points total: 4 for the acceptor and 1 for the justification. Points decked for wrong or missing transitions in the acceptor.

(b) Consider the regular grammar with production rules

(1) $\langle S \rangle \to 1 \langle A \rangle$,
(2) $\langle A \rangle \to 0 \langle B \rangle$,
(3) $\langle B \rangle \to 0 \langle A \rangle$,
(4) $\langle B \rangle \to 1 \langle C \rangle$,
(5) $\langle C \rangle \to \epsilon$.

We can show by induction that a string w generated by these production rules is of one of the following forms:

- $w = \langle S \rangle$,
- $w = 1 \langle A \rangle$,
- $w = 10^{2m+1} \langle B \rangle$,
- $w = 10^{2m+2} \langle A \rangle$,
- $w = 10^{2m+1}1 \langle C \rangle$,
- $w = 10^{2m+1}1$.

Here $m \geq 0$. It's clear then that this grammar generates L.

Grading rubric: 5 points total: 4 for the grammar and 1 for the justification. Points decked for wrong or missing production rules in the grammar.

3) (a) Let $w = 0^n10^n1 \in M$, where $n \geq 1$, and write $w = xuy$ with $|u| \geq 1$. Suppose u contains an occurrence of '1'. Then xu^2y will have more than two occurrences of '1', and hence will not be in M.

Suppose instead that u does not contain an occurrence of '1' and let $u = 0^{n_1}$, for some $n_1 \geq 1$. Then we can write w as $w = 0^{n_2}u0^{n-(n_1+n_2)}10^n1$ or as $w = 0^n10^{n_2}u0^{n-(n_1+n_2)}1$.

In the first case, we have $xu^2y = 0^{n+n_1}10^n1$, which is evidently not in M. Similarly, in the second case $xu^2y \notin M$. Thus, it is not possible to find any $w \in M$ which satisfies the properties listed in the Pumping Lemma, and therefore M is not regular.

Alternative solution: If M is regular, then it has a pumping length p. Consider $w = 0^p10^p1 \in M$ and the decomposition $w = xuy$ with $|u| \geq 1$ and $|xu| \leq p$. Since $|xu| \leq p$, u can only consist of zeroes. Let $u = 0^{n_1}$, for some $n_1 \geq 1$. Clearly, $xu^2y \notin M$ as $xu^2y = 0^{p+n_1}10^p1$, so the length of the first sequence of zeroes is greater than that of the second sequence of zeroes violating the pattern of the language.

Grading rubric: 5 points total: 1 point for the set-up (either the division into cases in the first solution or figuring out which word to look at in the second solution), 2 points each case in the first solution for a total of 4 or 4 points for the rest of the argument in the second solution.

(b) Consider the following production rules:

(1) $\langle S\rangle \to 0\,\langle A\rangle\,01$,
(2) $\langle A\rangle \to 0\,\langle A\rangle\,0$,
(3) $\langle A\rangle \to 1$.

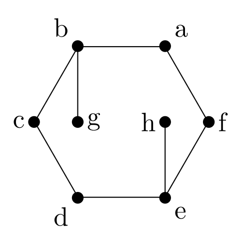
We can show by induction that a string w generated by these production rules is of is of one of the following forms:

- $w = \langle S\rangle$,
- $w = 0^n\,\langle A\rangle\,0^n1$,
- $w = 0^n10^n1$.

Here $n \geq 1$. These rules will then generate exactly M.

Grading rubric: 5 points total: 3 points for the production rules and 2 for the justification. Points decked for wrong or missing production rules.

4) (a) One way of drawing this graph is



Grading rubric: 2 points

(b) The incidence table is

	ab	bc	cd	de	ef	af	bg	eh
a	1	0	0	0	0	1	0	0
b	1	1	0	0	0	0	1	0
c	0	1	1	0	0	0	0	0
d	0	0	1	1	0	0	0	0
e	0	0	0	1	1	0	0	1
f	0	0	0	0	1	1	0	0
g	0	0	0	0	0	0	1	0
h	0	0	0	0	0	0	0	1

and the incidence matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Grading rubric: 2 points: 1 point each the table and the matrix

(c) The adjacency table is

	a	b	c	d	e	f	g	h
a	0	1	0	0	0	1	0	0
b	1	0	1	0	0	0	1	0
c	0	1	0	1	0	0	0	0
d	0	0	1	0	1	0	0	0
e	0	0	0	1	0	1	0	1
f	1	0	0	0	1	0	0	0
g	0	1	0	0	0	0	0	0
h	0	0	0	0	1	0	0	0

and the adjacency table is

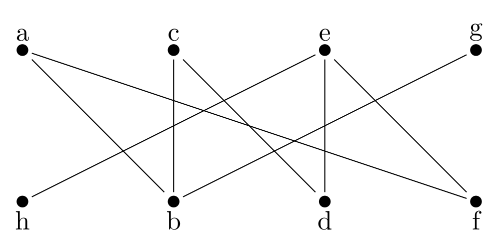
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Grading rubric: 2 points: 1 point each the table and the matrix

(d) This graph isn't complete as there are vertices not connected by an edge. For example, a and c are not connected.

Grading rubric: 2 points: 1 for the answer and 1 for the justification

(e) This graph is bipartite, with $V_1 = \{a, c, e, g\}$ and $V_2 = \{b, d, f, h\}$. This can be seen easily when we draw the graph as



Grading rubric: 2 points: 1 for the answer and 1 for the justification

(f) This graph is not regular. The degree of a is 2, while the degree of b is 3.

Grading rubric: 2 points: 1 for the answer and 1 for the justification

(g) The subgraph (V', E') with $V' = \{a, b, c, d, e, f\}$ and $E' = \{ab, bc, cd, de, ef, af\}$ is 2-regular.

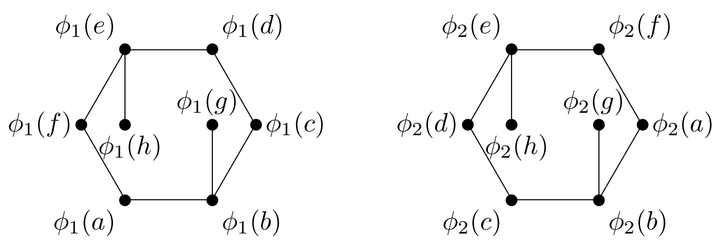
Grading rubric: 2 points: 1 for the answer and 1 for the justification

(h) Consider the maps $\phi_1 : V \to V$ and $\phi_2 : V \to V$ defined by

$$\begin{array}{llll} \phi_1(a) = d & \phi_1(b) = e & \phi_1(c) = f & \phi_1(d) = a \\ \phi_1(e) = b & \phi_1(f) = c & \phi_1(g) = h & \phi_1(h) = g, \end{array}$$

$$\begin{array}{llll} \phi_2(a) = f & \phi_2(b) = e & \phi_2(c) = d & \phi_2(d) = c \\ \phi_2(e) = b & \phi_2(f) = a & \phi_2(g) = h & \phi_2(h) = g. \end{array}$$

These are graph isomorphisms, as evidenced by the diagrams



The map ϕ_1 is seen to correspond to a rotation by π, and ϕ_2 is seen to correspond to a rotation by π and a reflection about the line joining b and e.

Grading rubric: 4 points: points decked if not all vertex assignments in the map lead to an isomorphism.

(i) Such an isomorphism is not unique, and we gave two in the last part. Note however that these are the only possibilities.

If ϕ is such an isomorphism, then $\phi(g)$ has degree 1 and is connected to $\phi(b) = e$. It follows that $\phi(g) = h$ and by similar reasoning that $\phi(h) = g$. The vertex a must be mapped to a vertex of degree 2 connected to $\phi(b) = e$, so that $\phi(a) \in \{d, f\}$. In the same way, $\phi(c) \in \{d, f\}$ and $\phi(d), \phi(f) \in \{a, c\}$. If we suppose that $\phi(a) = d$, then we find that $\phi = \phi_1$. On the other hand, if $\phi(a) = f$ then $\phi = \phi_2$.

Grading rubric: 2 points: 1 for the answer and 1 for the justification