The eigenvalue problem solution:

$$\max_{\|x\|=1} \|Ax\|$$

But we can say that $\lambda_{\max} = \max_{\|x\|=1} x^\top Ax$.

Note that:

$$x^\top Ax = \sum x_i x_j A_{ij} = \sum X_{ij} A_{ij}$$

Where $X = x^\top x$. So we get:

$$x^\top Ax = \mathrm{tr}(AX)$$

And our constrains are $\mathrm{tr}(X) = \sum x_i^2 = \|x\|^2 = 1$.

Moreover, X has to be semi positive definite.

So:

$$\max_{\|x\|=1} \|Ax\| = \max_{\substack{\mathrm{tr}(X)=1 \\ X \succcurlyeq 0}} \mathrm{tr}(AX)$$

Which is convex in X.

My real goal however, is:

$$\max_{\|q\|=1} \|A \cdot f(q)\|^2$$

Donate $f(q)$ as x. I have proved on the next page that $\|x\| = \|q\|^2\sqrt{3}$.

The problem is now:

$$\max_{\|x\|=\sqrt{3}} \|Ax\|^2$$

Replace $\|x\|$ with $y = x/\sqrt{3}$, so our problem becomes:

$$3 \max_{\|y\|=1} \|Ay\|^2$$

$$\|Ay\|^2 = y^\top A^\top Ay = y^\top By = \sum y_i y_j B_{ij} = \sum Y_{ij} B_{ij} = \operatorname{tr}(BY)$$

Where $Y = y^\top y = \frac{1}{3}x^\top x, B = A^\top A$. So we get:

$$\|Ay\|^2 = \operatorname{tr}(BY)$$

And our constrains are

$$\operatorname{tr}(Y) = \sum Y_{ii} = \|y\|^2 = 1$$

So:

$$\max_{\|q\|=1} \|A \cdot f(q)\|^2 = 3 \max_{\|y\|=1} \|Ay\|^2 = 3 \max_{\substack{\operatorname{tr}(Y)=1 \\ Y\succcurlyeq 0}} \operatorname{tr}(BY)$$

This is not enough, however. We also need that Y to be decomposed later to valid rotation matrix. We can tell that $\operatorname{rank}(Y) = 1$.

$$3 \max_{\substack{\operatorname{tr}(Y)=1 \\ Y\succcurlyeq 0 \\ \operatorname{rank}(Y)=1}} \operatorname{tr}(BY)$$

Suppose we found c, so $3\lambda_{\max}^2 = c \Longrightarrow \lambda_{\max} = \sqrt{c/3}$.

y will be the eigenvector matches to this eigenvalue and we can easily restore x.

I need vector x such that when reshaped to 9x9 will be a valid rotation matrix.

I will attach proof that $\|q\| = 1 \Leftrightarrow \|f(q)\| = 3$:

the expressions in $f(q)_i$ are:

$$(a^2 + b^2 - c^2 - d^2)^2 = a^4 + b^4 + c^4 + d^4 + \\ +2a^2b^2 - 2a^2c^2 - 2a^2d^2 - 2b^2c^2 - 2b^2d^2 + 2c^2d^2$$

And

$$(2ab \pm 2cd)^2 = 4a^2b^2 + 4c^2d^2 \pm 8abcd$$

Let's start with none squares sum:

$$(2q_2q_3 + 2q_1q_4)^2 + (2q_2q_3 - 2q_1q_4)^2 = 2q_2^2q_3^2 + 6q_2^2q_3^2 + 2q_1^2q_4^2 + 6q_1^2q_4^2$$

$$(2q_2q_4 + 2q_1q_3)^2 + (2q_2q_4 - 2q_1q_3)^2 = 2q_2^2q_4^2 + 6q_2^2q_4^2 + 2q_1^2q_3^2 + 6q_1^2q_3^2$$

$$(2q_3q_4 + 2q_1q_2)^2 + (2q_3q_4 - 2q_1q_2)^2 = 2q_3^2q_4^2 + 6q_3^2q_4^2 + 2q_1^2q_2^2 + 6q_1^2q_2^2$$

And continue with the squares:

$$(q_1^2 + q_2^2 - q_3^2 - q_4^2)^2 = q_1^4 + q_2^4 + q_3^4 + q_4^4 + 2q_1^2q_2^2 + 2q_3^2q_4^2 \\ -2q_1^2q_3^2 - 2q_1^2q_4^2 - 2q_2^2q_3^2 - 2q_2^2q_4^2$$

$$(q_1^2 + q_3^2 - q_2^2 - q_4^2)^2 = q_1^4 + q_2^4 + q_3^4 + q_4^4 + 2q_1^2q_3^2 + 2q_2^2q_4^2 \\ -2q_1^2q_2^2 - 2q_1^2q_4^2 - 2q_3^2q_2^2 - 2q_3^2q_4^2$$

$$(q_1^2 + q_4^2 - q_2^2 - q_3^2)^2 = q_1^4 + q_2^4 + q_3^4 + q_4^4 + 2q_1^2q_4^2 + 2q_2^2q_3^2 \\ -2q_1^2q_2^2 - 2q_1^2q_3^2 - 2q_4^2q_2^2 - 2q_4^2q_3^2$$

Moreover, we know that $\|q\|^4 = q_1^4 + q_2^4 + q_3^4 + q_4^4 +$
$$+2q_1^2q_2^2 + 2q_1^2q_3^2 + 2q_1^2q_4^2 + 2q_2^2q_3^2 + 2q_2^2q_4^2 + 2q_3^2q_4^2$$

Note that the yellow part summarizes up to $3 \cdot \|q\|^4$

Moreover, all the other expressions having the same color sum up to 0.

Therefore, $\|f(q)\|^2 = 3 \cdot \|q\|^4 \Longrightarrow \|f(q)\| = \|q\|^2\sqrt{3}$

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