Norm 2 optimization under unit vectors

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It is relatively easy to optimize convex functions without any constraints, but in CS, we need many times to solve problems using the constraint of $\|x\| = 1$. We will see how to solve two fundametal problems.

1 simple case

Let $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{n\times 2}$, $b \in \mathbb{R}^2$ and $f(x) = \|Ax - b\|_2^2$. Our problem is:

$$\min_{\|x\|=1} f(x) = \min_{\|x\|=1} \|Ax - b\|_2^2 = \min_{\|x\|=1} \sum_{i=1}^{n} (A_{i1}x_1 + A_{i2}x_2 - b_i)^2$$

Using polar representation, we get rid of $\|x\| = 1$ by choosing $x_1 \leftarrow \cos(x)$, $x_2 \leftarrow \sin(x)$.

$$\min_{\|x\|=1} f(x) = \min_{x} \sum_{i=1}^{n} (A_{i1}\cos(x) + A_{i2}\sin(x) - b_i)^2$$

Lets derive!

$$f'(x) = 2\sum_{i=1}^{n} (A_{i1}\cos(x) + A_{i2}\sin(x) - b_i)(A_{i2}\cos(x) - A_{i1}\sin(x)) = 0$$

For each inner expression:

$$\begin{aligned} E &= (A_{i1}\cos(x) + A_{i2}\sin(x) - b_i)(A_{i2}\cos(x) - A_{i1}\sin(x)) \\ &= A_{i1}A_{i2}\cos(2x) + 0.5(A_{i2}^2 - A_{i1}^2)\sin(2x) - A_{i2}b_1\cos(x) - A_{i1}b_i\sin(x) \end{aligned}$$

And totally:

$$f'(x) = \cos(2x)\sum_{i=1}^{n} A_{i1}A_{i2} + 0.5\sin(2x)\sum_{i=1}^{n}(A_{i2}^2 - A_{i1}^2) - \cos(x)\sum_{i=1}^{n} A_{i2}b_i - \sin(x)\sum_{i=1}^{n} A_{i1}b_i$$

Denote the following:

$$A = \tfrac{1}{2}\textstyle\sum_{i=1}^{n} A_{i1}A_{i2} \qquad B = \tfrac{1}{4i}\textstyle\sum_{i=1}^{n}(A_{i2}^2 - A_{i1}^2) \qquad C = \tfrac{1}{2}\textstyle\sum_{i=1}^{n} A_{i2}b_i \qquad D = \tfrac{1}{2i}\textstyle\sum_{i=1}^{n} A_{i1}b_i$$

And using Euler expressions, we get:

$$f'(x) = A(e^{2ix} + e^{-2ix}) + B(e^{2ix} - e^{-2ix}) - C(e^{ix} + e^{-ix}) + D(e^{ix} - e^{-ix}) = 0$$

Let $\theta \leftarrow e^{ix}$, we are left with

$$\theta^4(A + B) + \theta^3(D - C) - \theta(C + D) + (A - B) = 0$$

This gives ap to four solutions and once filtering solutions which are not on the unit vectors, solution to x will be the real and imaginary coordinated of $\theta_i = \mathrm{cis}(x)$.