2 general case - short method

Let $x \in \mathbb{R}^d, A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^d$, and $f(x) = \|Ax - b\|_2^2$. Our problem is:

$$\min_{\|x\|=1} f(x) = \min_{\|x\|=1} \|Ax - b\|_2^2 = \min_{\|x\|=1} x^\top A^\top Ax - 2Ab^\top x + b^\top b$$

Denote $A_0 = A^\top A$, $b_0 = -A^\top b$, $c_0 = b^\top b$, rewrite the problem to:

$$\min_{\|x\|=1} x^\top A_0 x + 2b_0^\top x + c_0$$

The Lagrangian is:

$$L(x, \lambda) = \|Ax - b\|_2^2 + \lambda(x^\top x - 1) = x^\top (A_0 + \lambda I)x + 2b_0^\top x + (c_0 - 1 \cdot \lambda)$$

and the dual function is:

$$\begin{aligned} g(\lambda) &= \inf_x L(x, \lambda) \\ &= \begin{cases} c_0 - \lambda - b_0^\top (A_0 + \lambda I)^\top b_0 & A_0 + \lambda I \succeq 0, b_0 \in R(A_0 + \lambda I) \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Using a Schur complement, we can express the dual problem as:

$$\begin{gathered} \text{maximize } \gamma \\ \text{subject to } \lambda \geq 0 \\ \begin{bmatrix} A_0 + \lambda I & b_0 \\ b_0^\top & c_0 - \lambda - \gamma \end{bmatrix} \succeq 0 \end{gathered}$$

an SDP with two variables $\gamma, \lambda \in \mathbb{R}$. Boyd proves also strong duality.