3 general case - long method

Let $x \in \mathbb{R}^d, A \in \mathbb{R}^{n\times d}, b \in \mathbb{R}^d$, and $f(x) = \|Ax - b\|_2^2$. Our problem is:

$$\min_{\|x\|=1} f(x) = \min_{\|x\|=1} \|Ax - b\|_2^2$$

From the definition, note that:

$$f(x) = \sum_{i=1}^{n} (A_{i*}^{\top} x - b_i)^2$$

We now partition $\mathbb{R}^d$ into 2^d subspace in the following method, by covering all the options.

$$\mathbb{R}^d = \bigcup_{c\in\{-1,1\}^d} \{x : (Ax - b) \circ c \geq 0_d\} = \bigcup_{c\in\{-1,1\}^d} \{x : \forall i : (A_{i*}x - b_i)c_i \geq 0\}$$

Denote $S_c := \{x : (Ax - b) \circ c \geq 0_d\}$, So $\mathbb{R}^d = \cup_{c\in\{-1,1\}^d} S_c$. We can express $Im(f)$ by:

$$f(\mathbb{R}^d) = \bigcup_{c\in\{-1,1\}^d} f(S_c)$$

This formally implies that f is union of 2^d parabolas and the constraint of $\|x\| = 1$ won't change it.

Lets look back at the definition of f. The sign of $A_{i*}x - b_i$ may not affect the outcome, but under the constraint $\|x\| = 1$, for each possible case there is different set of values on the unit vector and hence, 2^d different parabolas. Using the decomposition of f to those 2^d subspace, we can solve each subspace separately, since $f(S_c)$ is still a parabola. Moreover, it is a convex hull, and thus we can solve the problem under $\|x\| \leq 1$.

The algorithm is:

- For each $c \in \{-1, 1\}^d$, get $m_c = \min_{\|x\|=1}\{f(x) : f \in S_c\}$.

So, $\min_{\|x\|=1} \|Ax - b\|_2^2 = \min_c\{m_c\}$. The same can be applied to max.

Simulation